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FINAL REPORT

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2. PERIOD COVERED BY REPORT: 8/1/79 - 12/31/82

3. TITLE OF PROPOSAL: Locally one dimensional numerical methods for multi-dimensional free surface problems

4. CONTRACT OR GRANT NUMBER: DAAG 29-79-C-0145

5. NAME OF INSTITUTION: Georgia Institute of Technology

6. AUTHOR(S) OF REPORT: Gunter H. Meyer

7. LIST OF MANUSCRIPTS SUBMITTED OR PUBLISHED UNDER ARO SPONSORSHIP DURING THIS PERIOD, INCLUDING JOURNAL REFERENCES:

   Attached

8. SCIENTIFIC PERSONNEL SUPPORTED BY THIS PROJECT AND DEGREES AWARDED DURING THIS REPORTING PERIOD:

Final Report on research performed under Contract DAAG29-79-C-0145 on "Locally one dimensional numerical methods for multi-dimensional free surface problem."

Research on the numerical solution of free boundary problems for partial differential equations with locally or sequentially one-dimensional methods has been supported by the U.S. Army Research Office through two consecutive 3-year research contracts. The time and resources provided have made it possible to develop a reasonably comprehensive mathematical theory and flexible numerical algorithms on which to base current computational methods and future research.

During the first 3-year period, the method of fractional steps and the method of lines were applied to elliptic and parabolic free boundary problems. Both methods lead to sequentially one-dimensional algorithms for multi-dimensional partial differential equations. They are judged to be conceptually simple, easily coded, and little affected by complex free boundary conditions. A more detailed account of this phase of the research is contained in the final report on Grant DAAG-29-76-G-0261 (copy attached).

During the past three years work was directed toward demonstrating the flexibility of the method of lines for increasingly complex problems, on examining the behavior of certain ill-posed elliptic free boundary problems, and on establishing a mathematical theory for sequentially one-dimensional algorithms. In particular, the locally one-dimensional solver was applied to coupled heat and man transport during alloy solidification [4]. This work is preparatory to applying locally one-dimensional methods to multi-dimensional reaction-diffusion systems. With the same goal in mind monotone iterations were applied to elliptic free boundary problems with nonlinear source terms. In this setting a complete convergence analysis can be given for the method of lines when applied to free boundary problems which are equivalent to variational inequalities.

At the same time Hele-Shaw flow in a porous medium was examined. The Hele-Shaw injection problem is readily solvable and completely analyzable with variational inequalities. In contrast, the suction problem is ill-posed, has no variational structure and has not yielded to analysis. Our numerical method, on the other hand, remains applicable but requires careful attention to implementation [5]. An attack on the suction problem with conformal mapping [6] yielded some special analytic solutions against which numerical results can be compared.

Among our current projects we are examining whether surface tension effects can stabilize the suction problem. This work requires extensive numerical experiments on a Cyber-type computer. At the same time, we have begun work on implementing locally one-dimensional methods on a micro-computer.

Reports and papers prepared during the contract period:

G. H. Meyer, On the computational solution of elliptic and parabolic free boundary problems, in Free Boundary Problems, E. Magenes, edt., Istituto Francesco Severi, Rom 1980.

_____, The method of lines and invariant imbedding for elliptic and parabolic free boundary problems, SIAM J. Num. Anal. 18 (1981), 150-164.

_____, An analysis of the method of lines for the Reynolds equation in hydrodynamic lubrication, SIAM J. Num. Anal. 18 (1981), 165-177.

_____, A numerical method for the solidification of a binary alloy, Int. J. Heat Mass Transfer 24 (1981), 778-781.

_____, Hele-Shaw flow with a cusping free boundary, J. Comp. Phys. 44 (1981), 262-276.

_____, Test solutions for a one-phase problem through conformal transformations, in Numerical Treatment of Free Boundary Problems, T. Albrecht et al., edts., Birkhäuser, Basel, 1982.

_____, Anfangs-und Randwertaufgaben bei der Linienmethode fur elliptische Gleichungen, Seminar uber die Linienmethode und Anwendunzen, K. J. Schwenzfeger, edt., Hochschule der Bundeswehr Munich, 1982.

_____, Numerical methods for free boundary problems -- 1981 survey, Proceedings of the 1981 Montecatini Conference, to appear.

_____, Free boundary problems with nonlinear source terms, submitted to Numerische Mathematik.

GUNTER H. MEYER

Final report on work performed under Grant DAAG 29-76-G-0261 on "Locally one dimensional numerical methods for multidimensional free surface problems"

The applicability of locally one dimensional methods to multidimensional free boundary problems was examined. Such problems require the solution of an elliptic or parabolic field equation over a domain which is unknown a priori and which must be determined from over-prescribed boundary data. Our main objective was to identify a numerical method for such problems which is widely and easily applicable, but which at the same time has a solid mathematical foundation.

Initially, we examined the method of fractional steps which for a multidimensional diffusion equation leads to a sequence of loosely coupled one dimensional free boundary problems in alternating orthogonal directions. In a report prepared just prior to the grant period this method is introduced and used together with invariant imbedding to solve a variety of Stefan type free boundary problems. The method works well and allows comparable numerical resolution in all space directions. However, it does impose certain restrictions on the shape of the free boundary which serverely limit the applicability of the ADI method.

To overcome this limitation we began working with the method of lines based on a simple finite difference discretization of time and of all but one space variable. The original multidimensional equation is thus replaced by a system of one dimensional problems along parallel lines. This system is solved in line SOR fashion so that again a sequence of uncoupled one dimensional problems results. As above we apply invariant imbedding to each one dimensional problem. A description of this approach appears in [1]. A comparison with the fractional step method for a two dimensional ablation problem [2] shows that only relatively few lines compared to the number of such points along each line should be used and hence that the numerical resolution is not equally fine in all directions. Nonetheless, even in this severe test problem the method of lines performs as well as the ADI method. Because free surfaces of more general shape are admissible further work has concentrated on the method of lines/SOR/invariant imbedding.

The success of this method, even in the crude implementation employed as far, was good motivation to establish the mathematical validity of the algorithm. As a first step convergence of the iterative numerical method and the discretization was established for Poisson's equation subject to nonlinear boundary data on a fixed domain [3]. Recently, the arguments were extended to establish convergence for an elliptic free boundary problem arising in hydrodynamic lubrication [4]. As far as we are aware this is the first time that a multidimensional front tracking algorithm can be completely analyzed. Current work is concerned with convergence for the equations arising in fluid flow through a porous dam. This problem is characterized by considerably more complicated boundary conditions than present in the lubrication problem.

Although we are primarily concerned with multidimensional problem the one dimensional solver is at the heart of our method and we continue to gather experience with the invariant imbedding method. An application to a problem with distributed boundary conditions is described in [7].

During the grant period we particapted in two conferences dedicated to free and moving boundary problems. A survey over numerical methods for Stefan problems and a comparison with the method of lines was presented in [5], while an application of our methods to a variety of practical problems including heat and mass transfer, Hele-Shaw flow, and three dimensional ablation is described in [6].

Two additional publications were prepared for the 1977 and 1978 Army Numerical Analysis Conferences. A revision of the first report was published as [2] while the material of the second is contained in [6].

Colloquium talks on our work were presented during the past three years of several universities and laboratories in the U.S., Canada, Austria, Germany and Great Britain.

In summary, the basic premise that sequential one dimensional methods provide a useful tool for multidimensional free boundary problems has proved to be correct. It remains to establish firmly the limits of their applicability and to provide an analysis for a wider class of problems than was possible so far. First and foremost, however, the software aspects of our methods deserve attention in order to improve the efficiency of our methods.

During the grant period the following graduate student assisted in our project.

L. Kramarz, Ph.D. from Georgia Tech, 1977, [8]
F. Gerwig, M.S. from Georgia Tech, 1977
D. Dvorak, graduate student in mathematics

Papers and reports prepared during the grant period

1. G. H. Meyer, An application of the method of lines to multi-dimensional free boundary problems, J. Inst. Maths. Applcs. 20 (1977), 317-329.

2. ______, Direct and iterative one dimensional front tracking methods for the two dimensional Stefan problem, Numerical Heat Transfer 1 (1978), 351-369.

3. ______, The method of lines for Poisson's equation with non-linear or free boundary conditions, Numer. Math. 29 (1978), 329-344.

4. ______, The method of lines for the Reynolds equation in hydro-dynamic lubrication, Brunel Inst. of Comp. Maths. Tech. Report TR 21, 1978, submitted for publication.

5. ______, The numerical solution of multidimensional Stefan problems - a survey, in Moving Boundary Problems, D. G. Wilson, A. D. Solomon, P. T. Boggs, edts., Academic Press, N. Y. 1978.

6. ______, Invariant imbedding for elliptic and parabolic free boundary problems, Brunel Inst. of Comp. Maths. Tech Report TR 13, 1978, submitted for publication.

7. ______, A numerical method for heat transfer in an expanding rod, Int. J. Heat Mass Transfer 21 (1978), 824-826.

8. L. Kramarz, Global approximations to solutions of initial value problems, Math. Comp. 32 (1978), 35-59.