# COUNTING CLIQUES IN GRAPHS WITH EXCLUDED MINORS

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**COUNTING CLIQUES IN GRAPHS WITH EXCLUDED MINORS**

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# SUMMARY

In this thesis, we study problems at the intersection of extremal and structural graph theory, focusing on the maximum number of $k$-cliques in graphs excluding certain minors.

Chapters 2–4 are devoted to counting the maximum number of $K_2$'s (edges) in planar graphs. Euler's classical formula (Theorem 1.2.1) implies that any planar graph has at most $3|V(G)| - 6$ edges. A well-known consequence, derived using Euler's formula together with additional structural arguments, is that any triangle-free planar graph has at most $2|V(G)| - 4$ edges.

This naturally leads to the study of the maximum number of edges in a planar graph without a copy of $C_k$ (a cycle of length $k$), known as the planar Turán number $\mathrm{ex}_{\mathcal{P}}(n, C_k)$. More generally, for a graph $H$, the planar Turán number $\mathrm{ex}_{\mathcal{P}}(n, H)$ denotes the maximum number of edges in an $n$-vertex planar graph that contains no subgraph isomorphic to $H$. The values of $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ are known for $\ell \in \{3, 4, 5, 6\}$ and are conjectured to behave differently when $\ell \geq 11$. In Chapters 2 and 3, we prove that $\mathrm{ex}_{\mathcal{P}}(n, C_7) \leq \frac{18n}{7} - \frac{48}{7}$ for all $n \geq 39$, with equality attained for infinitely many $n$.

Chapter 4 investigates $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for $\ell \geq 11$. We first show that dense planar graphs satisfying a certain connectivity condition (known as circuit graphs) contain large near-triangulations. Using this, we derive new upper bounds for planar Turán numbers. In particular, we show there exists a constant $D$ such that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - 6 - Dn/k^{\log_2 3}$ for all $k \geq 4$ and $n \geq k^{\log_2 3}$. For $k \geq 11$, this bound is tight up to the constant $D$, confirming a conjecture of Cranston, Lidický, Liu, and Shantanam. In fact, we prove a stronger result: $\mathrm{ex}_{\mathcal{P}}(n, \theta_k) \leq 3n - 6 - n/(4k^{\log_2 3})$, where $\theta_k$ denotes the graph obtained by adding a triangle-forming edge to $C_k$.

Chapters 5 focus on extremal problems in graphs excluding clique minors. Alon and Shikhelman initiated the study of generalized extremal functions, and in particular, we consider ex($n, K_k, K_t$-minor), the maximum number of $k$-cliques in $K_t$-minor-free graphs

on $n$ vertices. Motivated by algorithmic applications, this problem has received significant attention. We determine nearly sharp bounds on this function: for each $k < t$ with $t - k \gg \log_2 t$, every $K_t$-minor-free graph on $n$ vertices contains at most $n \cdot C(k,t)^{1+o_t(1)}$ cliques of size $k$, for an explicit function $C(k,t)$. This bound is asymptotically tight, as we construct $K_t$-minor-free graphs with $C(k,t)n$ such cliques. Our result answers a question of Wood and Fox–Wei up to the $o_t(1)$ factor in the exponent, except in extreme cases where $k$ is very close to $t$.

This work is based on the following journal publications and preprints: [40], and [41] (with Xingxing Yu and Zachary Walsh), and [42] (with Fan Wei).

# CHAPTER 1

# INTRODUCTION

In this chapter, I first provide an overview of the thesis and then introduce standard graph-theoretic notations, definitions, and well-known results.

This thesis investigates Turán-type problems in graphs with excluded minors. In particular, we are interested in the maximum number of $k$-cliques in such graphs for any fixed $k \geq 2$.

Our first focus is on Turán-type problems in planar graphs, which exclude both $K_5$ and $K_{3,3}$ as minors. Since any planar graph excludes cliques of size at least 5 as subgraphs, we study the maximum number of 2-cliques (i.e., edges) in planar graphs that also exclude a cycle of length $k$.

A well-known consequence of Euler's formula—derived with additional structural arguments—is that any triangle-free planar graph has at most $2|V(G)| - 4$ edges. This leads to the study of the planar Turán number $\mathrm{ex}_{\mathcal{P}}(n, C_k)$, the maximum number of edges in an $n$-vertex planar graph that does not contain a copy of $C_k$. The values of $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ are known for $\ell \in \{3, 4, 5, 6\}$. In this thesis, we will study $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ for $\ell = 7$ and $\ell \geq 11$.

In Chapters 2 and 3, we prove that $\mathrm{ex}_{\mathcal{P}}(n, C_7) \leq \frac{18n}{7} - \frac{48}{7}$ for all $n \geq 39$, with equality attained for infinitely many values of $n$. Section 2.2 introduces our proof strategy, building on the approach of Ghosh et al. [16]. We first partition a plane graph $G$ into edge-disjoint *triangular blocks*, which are unions of adjacent triangular faces. For a connected $n$-vertex plane graph $G$ with $e$ edges and $f$ faces, we define a charge function $g(B)$ for each triangular block $B$ such that the total sum over all blocks is $24f - 17e + 6n$. We then aim to prove that $g(B) \leq 0$ for each block, implying the inequality $24f - 17e + 6n \leq 0$, and derive the main result of Chapter 2 using Euler's formula.

Section 2.3 characterizes near triangulations with at most 6 vertices where there exist two vertices on the outer cycle with no Hamiltonian path between them. This leads to a full classification of all possible triangular blocks in $C_7$-free planar graphs.

In Chapter 3, we prove Theorem 2.1.1 first for "good" planar graphs—those that are 2-connected and where every small vertex set $S$ is incident to at least $18|S|/7$ edges. In Section 3.5, we extend the result to general planar graphs by showing that every planar graph can be obtained from good graphs via operations that only decrease edge density. In Sections 3.1–3.3, we show that in a good graph, all triangular blocks $B$ satisfy $g(B) \leq 0$ except for two special types, which we group with adjacent blocks of small charge to ensure the total sum of $g(B)$ remains non-positive.

Chapter 4 addresses the planar Turán number for large cycles. We prove that there exists a constant $D$ such that $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - 6 - Dn/k^{\log_2 3}$ for all $k \geq 4$ and $n \geq k^{\log_2 3}$, and focus on the case $k \geq 11$. We proceed by contradiction, assuming a $C_k$-free planar graph $G$ with at least $3n - 6 - n/(4k^{\log_2 3})$ edges.

In Section 4.3, we reduce to a special class of nearly 3-connected graphs known as *circuit graphs*, defined as pairs $(G, C)$ where $G$ is a 2-connected plane graph and $C$ is a facial cycle such that every 2-cut in $G$ leaves each component containing a vertex from $C$.

Section 4.2 shows that such a circuit graph counterexample cannot exist. Assuming the existence of a minimal counterexample $G$ with respect to a potential function $m(G) = 3v(G) - 6 - e(G) - (|C| - 3)$, we derive a contradiction by analyzing interactions between the outer face and non-triangular faces, ultimately showing that a smaller counterexample $G'$ exists—contradicting the minimality of $G$.

In Chapters 5, we turn to a broader setting: extremal problems in graphs with forbidden clique minors. A fundamental result in this area is that the maximum number of edges in an $n$-vertex $K_t$-minor-free graph is $\Theta(t\sqrt{\log_2 t})n$, as shown independently by Kostochka [26, 27] and Thomason [44].

Motivated by algorithmic applications, we study $\mathrm{ex}(n, K_k, K_t\text{-minor})$, the maximum number of $k$-cliques in a $K_t$-minor-free graph on $n$ vertices. We determine nearly sharp bounds: for $k < t$ and $t - k \gg \log_2 t$, every such graph has at most $n \cdot C(k,t)^{1+o_t(1)}$ cliques of size $k$, for an explicit function $C(k,t)$. We also construct matching lower-bound examples, answering a question of Wood [47] and Fox–Wei [14] up to a $o_t(1)$ term in the exponent (except in extreme cases where $k$ is very close to $t$).

In Section 5.1, we present background and applications of clique counting in graphs with excluded minors. Section 5.2 introduces a modified peeling process based on Fox–Wei [14], with a refined stopping condition. We prove two key lemmas: one reduces the problem to dense graphs (Key Lemma 2), and the other lower bounds the clique minor size in such graphs using the peeling parameters (Key Lemma 1). A graph $G = (V, E)$ is considered dense if $|V| \geq \omega(G) + 2\bar{\Delta}^2 + 2$ or $\bar{\Delta} \leq 1$, where $\omega(G)$ is the clique number and $\bar{\Delta}$ is the maximum degree of the complement of $G$.

Fox and Wei showed that for a dense graph $G$, the maximum clique minor size is $\lfloor(|V| + \omega(G))/2\rfloor$. The family of dense graphs excluding a $K_s$-minor is denoted by $\mathcal{G}_s$.

In Section 5.3, we study the optimal graph in $\mathcal{G}_t$ maximizing the number of $k$-cliques. For $k \geq 2t/3$, we will first proves that the optimum is $K_t^-$, the complete graph missing one edge. When $k < 2t/3$, then we will show that the extremal graph is a Turán graph $T(n, \omega)$ with $n + \omega = 2t - 1$ and $\sqrt{tk}/4 \leq \omega \leq 10\sqrt{tk}$.

In Section 5.4. we will prove our main Theorem 5.1.5 of Chapter 5 for $k$ in three different ranges. The first two ranges are for $k$ very large range, i.e., $k \geq 2t/3 + 2\sqrt{t}{\log_2}^{1/4} t$ (Theorem 5.4.1); and for $k$ moderately large, i.e., $\min(k, t-k) \gg O(t^{1/2}{\log_2}^{5/4} t)$ (Theorem 5.4.4). The last range is for $k$ small, where we will apply Theorem 5.1.3.

Finally, Appendix A disproves Wood's Conjecture 5.1.4 for $\lambda < 0.553$ (Theorem A.0.1) by analyzing the number of $k$-cliques in disjoint unions of Turán graphs $T((4t-4)/3, (2t-2)/3)$.

## 1.1 Definitions

A *graph* is an ordered pair $G = (V, E)$, where $V$ is a set of *vertices* and $E$ is a set of unordered pairs of distinct elements of $V$, called *edges*. A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$, and every edge in $E'$ has both endpoints in $V'$. A graph $G$ is said to be $H$*-free* if $G$ does not contain a subgraph isomorphic to $H$.

A *graph* is an ordered pair

$$G = (V, E),$$

where $V$ is a set of *vertices* and $E$ is a set of unordered pairs of distinct elements of $V$, called *edges*.

For a graph $G = (V, E)$ and a vertex $v \in V$, the *neighborhood* of $v$, denoted by $N_G(v)$, is the set of vertices adjacent to $v$. That is,

$$N_G(v) = \{u \in V : \{u, v\} \in E\}.$$

Each vertex in $N_G(v)$ is called a *neighbor* of $v$ in $G$.

The *degree* of a vertex $v$ in a graph $G$, denoted by $\deg_G(v)$ or simply $\deg(v)$ when the graph is clear from context, is the number of neighbors of $v$ in $G$. In other words,

$$\deg_G(v) = |N_G(v)|.$$

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where

$$\overline{E} = \{\{u, v\} \subseteq V : u \neq v \text{ and } \{u, v\} \notin E\}.$$

That is, $\overline{G}$ has an edge between two distinct vertices if and only if $G$ does not.

A *minor* of a graph $G$ is a graph that can be obtained from $G$ by performing a sequence of the following operations:

- *Vertex deletion*: removing a vertex $v$ from $G$ along with all edges incident to $v$.
- *Edge deletion*: removing an edge $e$ from $G$ while keeping its endpoints.
- *Edge contraction*: replacing an edge $e = \{u, v\}$ by a single vertex, say $w$, and making $w$ adjacent to all vertices that were adjacent to either $u$ or $v$ (except for $u$ and $v$ themselves), and then deleting $u$ and $v$.

A graph $H$ is a *minor* of a graph $G$ if $H$ can be obtained from $G$ by applying a sequence of these operations.

Let $G = (V, E)$ be a graph. For a vertex $v \in V$, the graph $G - v$ denotes the subgraph obtained by removing $v$ and all edges incident to $v$ from $G$. More generally, for a subset $S \subseteq V$, the graph $G - S$ is the subgraph obtained by removing all vertices in $S$ and all edges incident to any vertex in $S$. That is,

$$G - S = (V \setminus S, E'),$$

where $E'$ consists of all edges in $E$ whose endpoints are both in $V \setminus S$.

A *path* in a graph $G = (V, E)$ is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ such that for every pair of consecutive vertices $v_i$ and $v_{i+1}$ in the sequence, the edge $\{v_i, v_{i+1}\} \in E$. In other words, a path is a walk where all the vertices are distinct. The *length* of a path is the number of edges in the path, which is given by $k - 1$ for a path with $k$ vertices. The *size* of a path refers to the number of vertices in the path, denoted by $k$.

A *connected graph* is a graph in which there is a path between any pair of distinct vertices. In other words, for every pair of vertices $u, v \in V$, there exists a sequence of edges in $E$ that connects $u$ and $v$. A graph that is not connected is called *disconnected*. A graph is said to be *$k$-connected* if it remains connected whenever fewer than $k$ vertices are removed. More formally, a graph $G = (V, E)$ is $k$-connected if for any subset of fewer than $k$ vertices

$S \subset V$, the subgraph induced by the remaining vertices is connected. The minimum number of vertices whose removal disconnects the graph is called its *connectivity*, denoted by $\kappa(G)$.

A *clique* in a graph $G = (V, E)$ is a subset of vertices $S \subseteq V$ such that every pair of distinct vertices in $S$ is adjacent in $G$. That is, for all $u, v \in S$ with $u \neq v$, we have $\{u, v\} \in E$. A $k$-*clique* is a clique on $k$ vertices; in other words, a set of $k$ vertices that induces a complete subgraph. The *clique number* of a graph $G = (V, E)$, denoted by $\omega(G)$, is the size of the largest clique in $G$. That is, it is the maximum number of vertices in a subset of $V$ such that every pair of distinct vertices in the subset is adjacent in $G$. In other words, the clique number is the order of the largest complete subgraph that can be found in $G$.

A *cycle* in a graph is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ with $k \geq 3$ such that $\{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq k-1$ and $\{v_k, v_1\} \in E$. A $k$-*cycle* is a cycle consisting of exactly $k$ vertices. A 3-cycle is also called a *triangle*.

A *complete graph*, denoted by $K_n$, is a graph on $n$ vertices in which every pair of distinct vertices is connected by an edge. In other words, the edge set of $K_n$ contains all possible edges between the $n$ vertices. A *complete bipartite graph*, denoted by $K_{m,n}$, is a graph whose vertex set can be partitioned into two disjoint sets $V_1$ and $V_2$ such that every vertex in $V_1$ is adjacent to every vertex in $V_2$, and there are no edges within each set. That is, the edge set of $K_{m,n}$ consists of all edges between the sets $V_1$ and $V_2$, where $|V_1| = m$ and $|V_2| = n$.

A *multipartite graph* is a graph whose vertex set can be partitioned into $r \geq 2$ disjoint subsets $V_1, V_2, \ldots, V_r$ such that no two vertices within the same subset are adjacent. That is, for all $i$, the subset $V_i$ is an independent set, and edges may only occur between vertices in different subsets. A *balanced multipartite graph* is a multipartite graph in which the vertex partition is as equal in size as possible. That is, the sizes of the parts $V_1, \ldots, V_r$ differ by at most one. A *Turán graph*, denoted by $T(n, r)$, is the complete $r$-partite graph on $n$ vertices in which the vertex classes are as equal in size as possible.

A graph is called *planar* if it can be drawn in the plane without any edges crossing, except possibly at their endpoints. That is, there exists a drawing of the graph in which its vertices are represented as distinct points in the plane and its edges as continuous curves connecting the corresponding pairs of points, such that no two edges intersect except at a common endpoint. By Wagner's Theorem, a graph is *planar* if and only if it does not contain $K_5$ or $K_{3,3}$ as a minor. Here, $K_5$ denotes the complete graph on five vertices, and $K_{3,3}$ denotes the complete bipartite graph with two parts of size three.

Let $G$ be a planar graph with a fixed embedding in the plane. A *facial cycle* is a cycle in $G$ that forms the boundary of a face in the embedding. In particular, each bounded face, as well as the unbounded outer face, is enclosed by a facial cycle. In a 2-connected planar graph, every face is bounded by a unique facial cycle.

Let $G$ be a 2-connected planar graph with a fixed embedding in the plane. The *outercycle* of $G$ is the cycle that bounds the unbounded (outer) face of the embedding. Since $G$ is 2-connected, every face of $G$ is bounded by a cycle, and in particular, the boundary of the outer face forms a well-defined cycle in $G$.

The *little o notation*, denoted by $o(f(n))$, is used to describe an upper bound that is not tight. We say that $g(n) = o(f(n))$ if for every constant $c > 0$, there exists a constant $n_0$ such that for all $n \geq n_0$, we have

$$|g(n)| < c \cdot |f(n)|.$$

In other words, $g(n)$ grows strictly slower than $f(n)$ as $n \to \infty$.

The *big O notation*, denoted by $O(f(n))$, is used to describe an upper bound for a function. We say that $g(n) = O(f(n))$ if there exist constants $c > 0$ and $n_0$ such that for all $n \geq n_0$, we have

$$|g(n)| \leq c \cdot |f(n)|.$$

In other words, $g(n)$ grows at most as fast as $f(n)$ asymptotically.

The *Theta notation*, denoted by $\Theta(f(n))$, is used to describe a tight bound for a function. We say that $g(n) = \Theta(f(n))$ if there exist positive constants $c_1$, $c_2$, and $n_0$ such that for all $n \geq n_0$, we have

$$c_1 \cdot |f(n)| \leq |g(n)| \leq c_2 \cdot |f(n)|.$$

In other words, $g(n)$ grows at the same rate as $f(n)$ asymptotically.

### 1.2 Theorems

In this section, we list some well-known results that will be used in this thesis.

**Theorem 1.2.1** (Euler's Formula)**.** *Let $G$ be a connected planar graph drawn in the plane without edge crossings. If $G$ has $n$ vertices, $m$ edges, and $f$ faces (including the outer, unbounded face), then*

$$n - m + f = 2.$$

**Theorem 1.2.2** (Turán's Theorem)**.** *Let $G$ be a graph on $n$ vertices that does not contain a complete subgraph on $r$ vertices, i.e., $G$ is $K_r$-free. Then the number of edges in $G$ satisfies*

$$|E(G)| \leq |E(T(n, r-1))|,$$

*where $T(n, r-1)$ is the Turán graph, the complete $(r-1)$-partite graph on $n$ vertices with parts as equal in size as possible. Equality holds if and only if $G \cong T(n, r-1)$.*

# CHAPTER 2

# PLANAR TURÁN PROBLEM AND TRAINGULAR-BLOCKS

## 2.1 Introduction

The *Turán number* $\mathrm{ex}(n, H)$ of a graph $H$ is the maximum number of edges in an $n$-vertex graph without $H$ as a subgraph. Turán's theorem [46], a cornerstone of extremal graph theory, states that $\mathrm{ex}(n, K_t) \leq (1 - \frac{1}{t-1})\binom{n}{2}$ for all integers $t$ and $n$ with $n, t \geq 3$, with equality for balanced complete $(t-1)$-partite graphs. This has led to a huge amount of related work, including the Erdős–Stone theorem [11], generalized Turán numbers (see [1]), and Turán problems for hypergraphs (see [24]).

Another well-studied variant of Turán's theorem involves placing restrictions on the host graph; for example, forcing the host graph to be a hypercube (see [36]) or an Erdős–Rényi random graph (see [38]). In 2015, Dowden considered the variant in which the host graph is planar, and defined the *planar Turán number* $\mathrm{ex}_{\mathcal{P}}(n, H)$ of a graph $H$ to be the maximum number of edges in an $n$-vertex planar graph without $H$ as a subgraph [8]. For example, it follows from Euler's formula that $\mathrm{ex}_{\mathcal{P}}(n, C_3) = 2n - 4$ for all $n \geq 3$, where $C_\ell$ denotes the cycle of length $\ell$. Since cycles are arguably the most natural choice for $H$ in the planar setting, Dowden proved that $\mathrm{ex}_{\mathcal{P}}(n, C_4) \leq \frac{15(n-2)}{7}$ for all $n \geq 4$ and $\mathrm{ex}_{\mathcal{P}}(n, C_5) \leq \frac{12n-33}{5}$ for all $n \geq 11$, and showed that in both cases equality holds infinitely often [8]. This initiated a flurry of research on planar Turán numbers. We direct the reader to the survey paper of Lan, Shi, and Song [29] for details regarding recent related work, while we continue to focus on the planar Turán number of $C_\ell$.

In [16], Ghosh, Győri, Martin, Paulos, and Xiao took the next step and proved that $\mathrm{ex}_{\mathcal{P}}(n, C_6) \leq \frac{5n}{2} - 7$ with equality holding infinitely often, improving upon a result of Yan, Shi, and Song [28]. They also conjectured that $\mathrm{ex}_{\mathcal{P}}(n, C_\ell) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$ for all $\ell \geq 7$

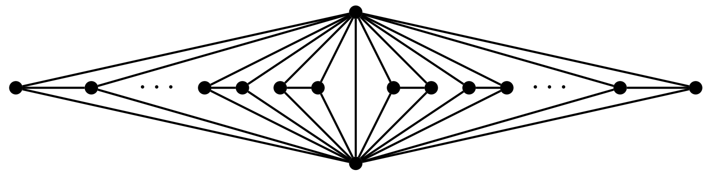

Figure 2.1: The planar graph obtained by gluing $18$ copies of $K_4$ along a common edge has $38$ vertices and more than $\frac{18}{7} \cdot 38 - \frac{48}{7}$ edges.

and all sufficiently large $n$, as this is the correct bound for $\ell \in \{5, 6\}$. If true, this bound would be tight infinitely often for every $\ell \geq 7$. However, this conjecture was disproved for all $\ell \geq 11$ by Cranston, Lidický, Liu, and Shantanam [6], and later also by Lan and Song [30], using the fact that planar triangulations with at least $11$ vertices need not be Hamiltonian. While the correct value of $\mathrm{ex}_{\mathcal{P}}(n, C_\ell)$ is rather mysterious for $\ell \geq 11$ (see [6] and [30]), we prove the conjecture of Ghosh et al. in the case that $\ell = 7$ (writing $e(G)$ for the number of edges of a graph $G$).

**Theorem 2.1.1.** *Let $n$ be an integer with $n \geq 39$ and let $G$ be an $n$-vertex $C_7$-free planar graph. Then $e(G) \leq \frac{18}{7}n - \frac{48}{7}$, and equality holds infinitely often.*

We comment that this result has been obtained independently by Győri, Li, and Zhou [19]. The condition that $n \geq 39$ is necessary, because the $38$-vertex graph obtained by gluing $18$ copies of $K_4$ together on a common edge (as illustrated in Figure 2.1) has $91$ edges, and $91 > \frac{18}{7} \cdot 38 - \frac{48}{7}$.

We next describe, for each $\ell \geq 5$ and for infinitely many $n$, a $C_\ell$-free $n$-vertex planar graph with $\frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$ edges, following Cranston et al. [6]. Let $G$ be an $n$-vertex planar graph with girth $\ell+1$ and with $\frac{\ell+1}{\ell-1}(n-2)$ edges, and with each vertex having degree 2 or 3; such a graph exists for infinitely many integers $n$ [6, Lemma 2]. Let $G'$ be obtained from $G$ by *substituting* an $(\ell-1)$-vertex planar triangulation $B$ for each vertex of $G$, which means that each vertex $v$ of $G$ is replaced by a copy of $B$ and $\deg_G(v)$ vertices of $B$ on a facial triangle are identified with the neighbors of $v$ in $G$. Then $G'$ is a $C_\ell$-free $n'$-vertex graph with $\frac{3(\ell-1)}{\ell}n' - \frac{6(\ell+1)}{\ell}$ edges [6, Corollary 5]. In particular, if $\ell = 7$ then $G'$ is a $C_7$-free

$n'$-vertex graph with $\frac{18}{7}n' - \frac{48}{7}$ edges, so the bound in Theorem 2.1.1 holds infinitely often.

To prove Theorem 2.1.1 it suffices to consider graphs without small separators, and without small sets of vertices with only a few incident edges. For a graph $G$ and constant $\alpha > 0$, a set $S \subseteq V(G)$ is $\alpha$-*sparse* if $G$ has at most $\alpha|S|$ edges with at least one incident vertex in $S$. For each positive integer $n$, we write $\mathcal{P}_n$ for the class of $n$-vertex, 2-connected, $C_7$-free plane graphs with no (18/7)-sparse set of order at most 4. We will obtain Theorem 2.1.1 as a consequence of the following result, by generating all $C_7$-free planar graphs from $\cup_{i \geq 1} \mathcal{P}_i$ through several basic operations.

**Theorem 2.1.2.** *Let* $G \in \mathcal{P}_n$ *with* $n \geq 7$. *Then* $e(G) \leq \frac{18}{7}n - \frac{48}{7}$.

Our proof uses the same strategy employed by Ghosh et al. in [16] to find $\mathrm{ex}_{\mathcal{P}}(n, C_6)$, as we believe it is the natural approach. We also use their terminology whenever possible, for consistency. This terminology and the proof strategy are both described in Section 2.2. We also enhance their strategy with a result (Lemma 2.3.1) about long paths in near triangulations which we believe will be useful for other planar Turán problems.

For a graph $G$, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively, and often write $|G|$ for $|V(G)|$ and $e(G)$ for $|E(G)|$. For any $S \subseteq V(G)$, we use $G[S]$ to denote the subgraph of $G$ induced by $S$ and write $G - S$ for $G[V(G) \setminus S]$ (and write $G - s$ instead of $G - \{s\}$). For any subgraph $H$ of $G$, we write $G - H$ for $G - V(H)$. For a subgraph $B$ of a graph of $G$, a $B$-*path* is a path in $G$ that has both ends in $B$ and is internally disjoint from $B$. A $B$-path of length 1 is also known as a *chord* of $B$. For two graphs $G_1$ and $G_2$, we write $G_1 \cup G_2$ for the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. For any positive integer $k$, we let $[k] = \{1, \ldots, k\}$.

If $G$ is a plane graph, then $F(G)$ denotes its set of faces. It is well-known that if $G$ is a 2-connected plane graph then each member of $F(G)$ is bounded by a cycle. The *interior* of a cycle $C$ in a plane graph is defined to be the subgraph of $G$ consisting of all edges and vertices of $G$ contained in the closed disc of the plane bounded by $C$. Paths and cycles will be represented as a sequence of vertices such that consecutive vertices in the sequence are adjacent. For instance, $x_1x_2x_3 \dots x_k$ is a path of length $k-1$, and $x_1x_2x_3 \dots x_kx_1$ is a cycle of length $k$. For any distinct vertices $x, y$ in a graph $G$, we use $d(x, y)$ to denote the distance between $x$ and $y$ in $G$, and if $x, y$ are on a path $P$ then $xPy$ denotes the subpath of $P$ between $x$ and $y$. For any cycle $C$ in a plane graph and any two distinct vertices $x, y$ on $C$, we use $xCy$ to denote the subpath of $C$ from $x$ to $y$ in clockwise order. For vertices $x, y \in V(G)$, an $x$-$y$ path is a path $x_1x_2x_3 \dots x_k$ with $x = x_1$ and $y = x_k$. An $x$-$y$ Hamiltonian path is an $x$-$y$ path that contains every vertex of $G$.

### 2.2 Proof strategy and triangular-blocks

We now describe the proof strategy for Theorem 2.1.2, which is based on that of Ghosh et al. [16]. If $G$ is a connected $n$-vertex plane graph with $e$ edges and $f$ faces, then $e \leq \frac{18n}{7} - \frac{48}{7}$ is equivalent via Euler's formula to

$$24f - 17e + 6n \leq 0,$$

by replacing the 48 with $24(n - e + f)$. We will decompose the graph $G$ into edge-disjoint subgraphs, and show that no subgraph contributes too much towards the left-hand side of the inequality. Due to the extremal construction, it is natural to decompose $G$ into unions of facial triangles. For facial triangles $F$ and $F'$ of $G$ we say that $F \sim F'$ if there is a sequence $F = F_1, F_2, \dots, F_k = F'$ of facial triangles of $G$ so that for each $i \in [k-1]$, $F_i$ and $F_{i+1}$ share an edge. Clearly $\sim$ is an equivalence relation on the facial triangles of $G$. This motivates the following key definition [16].

**Definition 2.2.1.** Let $G$ be a plane graph. For $e \in E(G)$, if $e$ is not contained in any facial triangle, then let $B(e)$ be the 2-vertex subgraph of $G$ induced by $e$; otherwise, let $B(e)$ denote the union of all facial triangles equivalent to some facial triangle containing $e$. We call $B(e)$ the *triangular-block* containing $e$, and it is *trivial* if $B(e)$ has just two vertices. We use $\mathcal{B}(G)$ to denote the collection of all triangular-blocks of $G$.

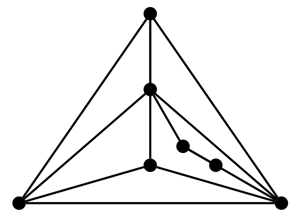

Figure 2.2: Let $G$ be the graph shown above, and let $B$ be the subgraph of $G$ obtained by deleting the two degree-2 vertices. Then $B$ is a triangular-block of $G$, even though not every face of $B$ is a face of $G$.

It is clear from the definition that if $B$ is a triangular-block of $G$ and $e_1$ and $e_2$ are distinct edges of $B$, then $B = B(e_1) = B(e_2)$. In particular, any two distinct triangular-blocks of $G$ are edge-disjoint. Also, note that any redrawing of $G$ with the same set of faces has the same triangular-blocks. However, note that a face of a triangular-block $B$ is not necessarily a face of $G$; see Figure 2.2.

**Definition 2.2.2.** Let $B$ be a triangular-block of a plane graph $G$. A *hole* of $B$ is a face of $B$ that is not a face of $G$.

Not every face of a triangular-block can be a hole. We next make two useful observations about faces of triangular-blocks.

**Lemma 2.2.3.** *Let $G$ be a 2-connected plane graph and let $B$ be a triangular-block of $G$ whose outer face is not a triangular face of $G$. Let $C$ be the outer cycle of $B$ and let $P$ be a $C$-path in $B$. Then some edge of $P$ is in two facial triangles of $G$.*

*Proof.* Let $P_1$ and $P_2$ be the two subpaths of $C$ for which $P_1 \cup P$ and $P_2 \cup P$ are cycles. Since the outer face of $B$ is not a triangular face of $G$, each facial triangle of $B$ that is a facial triangle of $G$ is in the interior of either $P_1 \cup P$ or $P_2 \cup P$. If no edge of $P$ is in two facial triangles, then no facial triangle of $G$ in the interior of $P_1 \cup P$ is equivalent under $\sim$ to a facial triangle of $G$ in the interior of $P_2 \cup P$, a contradiction. $\square$

The following is a useful consequence of Lemma 2.2.3.

**Lemma 2.2.4.** *Let $B$ be a triangular-block of a $2$-connected plane graph $G$, and let $F_1$ and $F_2$ be facial cycles of $B$ that each have length at least four. Then $F_1$ and $F_2$ share at most one vertex.*

*Proof.* By redrawing $G$, we may assume that $F_1$ is the outer cycle of $G$. If the statement is false, then there are vertices $a$ and $b$ in $V(F_1) \cap V(F_2)$ and an $a$-$b$ path $P$ in the interior of $F_1$ so that $E(P) \subseteq E(F_2)$. But by Lemma 2.2.3 this is a contradiction. □

The proof strategy for Theorem 2.1.2 is to show that no triangular-block of $G$ contributes too much towards the sum $24f - 17e + 6n$. We next make several definitions, following Ghosh et al. [16], to describe how a triangular-block contributes to $24f - 17e + 6n$. This is easy for edges: each triangular-block $B$ contributes $e(B)$ towards $e$. Then

$$\sum_{B \in \mathcal{B}(G)} e(B) = e,$$

since triangular-blocks are pairwise edge-disjoint. We next describe the contribution of a triangular-block $B$ to the number of vertices of $G$.

**Definition 2.2.5.** Let $G$ be a plane graph and $v \in V(G)$. We write $\mathcal{B}(G)_v = \{B \in \mathcal{B}(G) : v \in V(B)\}$ and say that $v$ is a *junction vertex* if $|\mathcal{B}(G)_v| > 1$. We define $n(v) = 1/|\mathcal{B}(G)_v|$, and $n(B) = \sum_{v \in V(B)} n(v)$.

Thus, for an $n$-vertex plane graph $G$,

$$\sum_{B \in \mathcal{B}(G)} n(B) = \sum_{B \in \mathcal{B}(G)} \sum_{v \in V(B)} n(v) = \sum_{v \in V(G)} \sum_{B \in \mathcal{B}(G)_v} n(v) = n.$$

We next define the contribution of a triangular-block to the number of faces of a plane graph $G$. To do this we must develop some notation concerning the faces incident with edges of a triangular-block $B$.

**Definition 2.2.6.** Let $B$ be a triangular-block of a $2$-connected plane graph $G$. A facial cycle $F$ of $G$ that is not a facial cycle of $B$ but shares edges with $B$ is a *petal* of $B$. The petal $F$ is said to be *leaky* if the graph $F \cap B$ is disconnected. The subgraph of $G$ consisting of $B$ and all petals of $B$ is the *flower* of $B$.

Our next definition will help deal with a facial cycle that intersects a triangular-block in a path with at least two edges.

**Definition 2.2.7.** Let $G$ be a 2-connected plane graph and let $F$ be a facial cycle of $G$. A triple $x_1x_2x_3$ of consecutive vertices on $F$ is a *bad cherry* if $x_1x_3 \in E(G) \setminus E(F)$ and $x_1$ and $x_3$ are junction vertices of a triangular-block of $G$ that also contains the triangle $x_1x_2x_3x_1$. Let $F'$ be the cycle obtained from $F$ in the following manner: for each bad cherry $x_1x_2x_3$ of $F$, replace the path $x_1x_2x_3$ in $F$ with the path $x_1x_3$. We say that $F'$ is the *refinement* of $F$. For convenience, let $F' = F$ if $F$ has no bad cherry.

Note that the refinement $F'$ of a facial cycle $F$ in a plane graph $G$ is a cycle which has length at most $|F| - 1$ except when $F' = F$ (i.e., $F$ has no bad cherry). Also, it is straightforward to check that if $G$ is $C_7$-free and $|F| \geq 8$, then $|F'| \geq 8$ as well; we freely use this fact. We can now define the contribution of a triangular-block of a plane graph $G$ to the number of faces of $G$.

**Definition 2.2.8.** Let $G$ be a 2-connected plane graph. Let $B$ be a triangular-block of $G$, let $\mathcal{P}(B)$ be the set of petals of $B$, and let $\mathcal{H}(B)$ be the set of non-hole faces of $B$. Let $F \in \mathcal{P}(B)$ and let $F'$ be the refinement of $F$. Define $f_F(B) = e(F' \cap B)/e(F')$, and define

$$f(B) = |\mathcal{H}(B)| + \sum_{F \in \mathcal{P}(B)} f_F(B).$$

We briefly show that $\sum_{B \in \mathcal{B}(G)} f(B) = |F(G)|$. For a plane graph $G$, we write $\overline{F}_1(G)$ for the set of facial cycles of $G$ that are also a facial cycle of a triangular-block of $G$, and we write $\overline{F}_2(G)$ for the set of facial cycles of $G$ that are not a facial cycle of any triangular-block of $G$. Note that every facial cycle in $\overline{F}_2(G)$ is a petal of some triangular-block of $G$. For a facial cycle $F \in \overline{F}_2(G)$, let $\mathcal{B}(G)_F$ be the set of triangular-blocks of $G$ that have $F$

as a petal. Note that every edge of the refinement of $F$ is in exactly one triangular-block in $\mathcal{B}(G)_F$. If $G$ is 2-connected, then

$$\begin{aligned}
\sum_{B\in\mathcal{B}(G)} f(B) &= \sum_{B\in\mathcal{B}(G)} \left(|\mathcal{H}(B)| + \sum_{F\in\mathcal{P}(B)} f_F(B)\right) \\
&= \left(\sum_{B\in\mathcal{B}(G)} |\mathcal{H}(B)|\right) + \sum_{B\in\mathcal{B}(G)} \sum_{F\in\mathcal{P}(B)} f_F(B) \\
&= |\overline{F}_1(G)| + \sum_{F\in\overline{F}_2(G)} \sum_{B\in\mathcal{B}(G)_F} f_F(B) \\
&= |\overline{F}_1(G)| + \sum_{F\in\overline{F}_2(G)} \sum_{B\in\mathcal{B}(G)_F} \frac{e(F'\cap B)}{e(F')} \\
&= |\overline{F}_1(G)| + \sum_{F\in\overline{F}_2(G)} 1 \\
&= |\overline{F}_1(G)| + |\overline{F}_2(G)| = |F(G)|,
\end{aligned}$$

as desired.

Now that we have defined the contribution of a triangular-block to edges, vertices, and faces, we can define the contribution of a triangular-block to $24f - 17e + 6n$.

**Definition 2.2.9.** Let $G$ be a 2-connected plane graph, and let $B$ be a triangular-block of $G$. We define

$$g(B) = 24f(B) - 17e(B) + 6n(B).$$

We comment that $g(B)$ will be independent of the planar drawing of $G$ we choose as all planar drawings of $G$ we use will have the same collection of faces and facial cycles. So when we compute an estimate for $g(B)$ we may work with a particular planar drawing of $G$ that is most convenient and therefore of $B$ as well. Also note that if $G$ has $n$ vertices, $e$ edges, and $f$ faces, then $\sum_{B\in\mathcal{B}(G)} g(B) = 24f - 17e + 6n$. This leads to the following proof strategy for Theorem 2.1.2: show that each triangular-block $B$ of $G$ satisfies $g(B) \leq 0$. However, there are two exceptional cases, shown in 3.1, for which $g(B) > 0$. To deal with these cases we will instead find a partition of $\mathcal{B}(G)$ so that the sum of $g(B)$ for the triangular-blocks in each part of the partition is at most zero. Then Theorem 2.1.2 will follow from the following lemma, which was also proved in [16].

**Lemma 2.2.10.** *Let $G$ be a 2-connected $n$-vertex plane graph. If there is a partition $\mathcal{P} = \{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_m\}$ of $\mathcal{B}(G)$ so that $\sum_{B \in \mathcal{B}_i} g(B) \leq 0$ for all $i \in [m]$, then $e(G) \leq \frac{18n}{7} - \frac{48}{7}$.*

*Proof.* The statement follows from the facts that $\sum_{B \in \mathcal{B}(G)} g(B) = 24f(G) - 17e(G) + 6n$, and $e(G) \leq \frac{18n}{7} - \frac{48}{7}$ if and only if $24f(G) - 17e(G) + 6n \leq 0$. □

In order to apply this lemma we must characterize the possible triangular-blocks of a $C_7$-free plane graph and then estimate $g(B)$ for each such triangular-block $B$. In Section 2.3 we prove some properties about paths in triangular-blocks, and in Section 2.4 we characterize the possible triangular-blocks of a $C_7$-free plane graph. Then in Sections 3.1–3.3 we bound $g(B)$ for each possible $C_7$-free triangular-block $B$. We prove Theorem 2.1.2 in Section 3.4 and then prove Theorem 2.1.1 in Section 3.5. We finish by discussing some related open problems in Section 3.6.

## 2.3 Near Triangulations

A *near triangulation* is a 2-connected plane graph in which every face is bounded by a triangle except possibly the outer face. Note that any near triangulation is an iterative union of facial triangles, and could therefore be a triangular-block of a plane graph. For convenience, we allow any planar embedding of $K_2$ to be a near triangulation with itself as its outer cycle. In this section we prove two lemmas that will help us characterize all possible $C_7$-free triangular-blocks of a plane graph, and also help us bound $g(B)$ for a given triangular-block $B$.

**Lemma 2.3.1.** *Let $G$ be a near triangulation with outer cycle $C$, and let $x, y \in V(C)$ be distinct. Suppose $G$ contains an $x$-$y$ Hamiltonian path. Then $G$ also contains an $x$-$y$ path of length $\ell$ for all $\ell$ with $d(x, y) \leq \ell \leq |V(G)| - 1$.*

*Proof.* We apply induction on $n := |V(G)| \geq 2$. The assertion is easily seen to be true for $n = 2$ and $n = 3$. So assume $n \geq 4$.

*Case* 1. $G - x$ is 2-connected.

Then $G - x$ is also a near triangulation. Let $D$ denote the outer cycle of $G - x$. Let $x_1, x_2, \ldots, x_k$ be the neighbors of $x$ and assume that $x_1, x_2, \ldots, x_k, y$ occur on $D$ clockwise in the order listed, with $y = x_k$ if $x$ and $y$ are adjacent in $G$. Then $x_1, x_k \in V(C)$, $x_1 D x_k = x_1 x_2 \ldots x_k$, and $x_k C x_1 = x_k D x_1$.

Let $t \in [k]$ so that some $x$-$y$ Hamiltonian path in $G$ uses the edge $xx_t$. Then $G - x$ has an $x_t$-$y$ Hamiltonian path. Hence, by induction, $G - x$ contains an $x_t$-$y$ path of any length between $d_{G-x}(x_t, y)$ and $n-2$. So by adding the edge $xx_t$, we see that $G$ has an $x$-$y$ path of any length between $d_{G-x}(x_t, y) + 1$ and $n - 1$. We are done if $d_{G-x}(x_t, y) \leq d_G(x, y) - 1$. So assume $d_{G-x}(x_t, y) \geq d_G(x, y)$.

Let $P$ be a shortest $x$-$y$ path in $G$, i.e., the length of $P$ is $d_G(x, y)$, and let $xx_s \in E(P)$, where $s \in [k]$. Then for all $i \in [k] \setminus \{s\}$, $x_i \notin V(P)$; otherwise, $xx_i \cup x_i P y$ would be an $x$-$y$ path in $G$ shorter than $P$. Let $Q$ denote the subpath of $x_1 D x_k$ between $x_s$ and $x_t$. Then $Q \cup x_s P y$ is an $x_t$-$y$ path in $G - x$ of length $d_G(x, y) - 1 + |s - t|$. Hence, $d_{G-x}(x_t, y) \leq d_G(x, y) - 1 + |s - t|$. Thus, $G$ has an $x$-$y$ path of any length between $d_G(x, y) + |s - t|$ and $n - 1$.

Therefore, it remains to show that $G$ has $x$-$y$ paths of any length between $d_G(x, y)$ and $d_G(x, y) - 1 + |s - t|$. For each $x_i \in V(Q)$ we see that $P_i := xx_i \cup x_i Q x_s \cup (P - x)$ is an $x$-$y$ path in $G$ of length $d_G(x, y) + |i - s|$. Thus, the paths $P_i$, for all $x_i \in V(Q)$, are $x$-$y$ paths of all possible lengths between $d_G(x, y)$ and $d_G(x, y) + |s - t|$.

*Case* 2. $G - x$ is not 2-connected.

Then, since $G$ is a near triangulation, $C$ has chords from $x$ to $C - x$. Since $G$ has an $x$-$y$ Hamiltonian path, the chords of $C$ from $x$ must all end on $xCy - \{x, y\}$ or all end on $yCx - \{x, y\}$. By symmetry, we may assume that all chords of $C$ from $x$ end on $xCy$, and let $xz$ be the chord of $C$ with $zCy$ minimal.

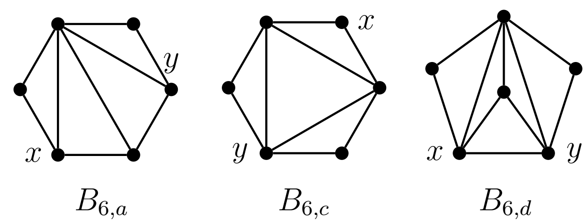


Figure 2.3: The three exceptional cases for Lemma 2.3.2.

Let $C_1 := xCz \cup zx$ and $C_2 := zCx \cup xz$, which are two cycles in $G$. For $i \in [2]$, let $G_i$ denote the interior of $C_i$. Then each $G_i$ is a near triangulation and $C_i$ is its outer cycle. Observe that any $x$-$y$ Hamiltonian path in $G$ gives rise to an $x$-$z$ Hamiltonian path in $G_1$, as well as a $z$-$y$ Hamiltonian path in $G_2 - x$ (and hence a $x$-$y$ Hamiltonian path in $G_2$ by adding the edge $xz$). Moreover, $d_{G_2}(x, y) = d_G(x, y)$. Let $n_i := |V(G_i)|$ for $i \in [2]$.

By induction, $G_1$ has an $x$-$z$ path of length $\ell_1$ for every $\ell_1$ between $d_{G_1}(x, z) = 1$ and $n_1 - 1$, and $G_2$ has an $x$-$y$ path of length $\ell_2$ for every $\ell_2$ between $d_{G_2}(x, y) = d_G(x, y)$ and $n_2 - 1$. It remains to show that $G$ has $x$-$y$ paths of any length between $n_2$ and $n - 1$.

Let $P$ be a $z$-$y$ Hamiltonian path in $G_2 - x$. Thus, the length of $P$ is $n_2 - 2$. Now take an $x$-$z$ path $P_m$ in $G_1$ of length $m$ for each $m \in [n_1 - 1]$. Then for $m \in [n_1 - 1]$, $P \cup P_m$ is an $x$-$y$ path of length $(n_2 - 2) + m$. Noting that $n = n_1 + n_2 - 2$, we see that $G$ has an $x$-$y$ path of any length between $n_2$ and $n - 1$. □

Next, we consider near triangulations with at most six vertices. Such a graph is certainly $C_7$-free, and is therefore relevant for the proof of Theorem 2.1.1.

**Lemma 2.3.2.** *Let $G$ be a near triangulation with outer cycle $C$, and let $x, y \in V(C)$ be distinct. Suppose $|V(G)| \leq 6$. Then $G$ contains an $x$-$y$ Hamiltonian path or one of the following holds:*

$(i)$ *$xy$ is a chord of $C$.*

$(ii)$ *$G$ has a path $xzy$ such that both $xz$ and $zy$ are chords of $C$, and $G - \{x, y, z\}$ consists of 3 isolated vertices (so $G \cong B_{6,a}$ or $G \cong B_{6,d}$ with $x, y$ as in Figure 1).*

*(iii)* $G \cong B_{6,c}$ *with* $x$, $y$ *as in Figure 1.*

*Proof.* Suppose for a contradiction that the assertion is false, and let $G$ be a counterexample with $|V(G)|$ minimum. Then $|V(G)| \geq 4$, as otherwise $G$ contains an $x$-$y$ Hamiltonian path. Moreover, $xy$ is not a chord of $C$; as otherwise we have $(i)$.

Observe that if $G$ contains a chord from $x$ to $xCy - x$ and a chord from $x$ to $yCx - x$, then since $|V(G)| \leq 6$ (and because of planarity), $G$ has no chord from $y$ to $xCy - y$ or $yCx - y$. Hence, we may assume by symmetry (between $x$ and $y$ and between $xCy$ and $yCx$) that $G$ has no chord from $x$ to $yCx - x$.

Since $xy$ is not a chord of $C$ and $G$ is a near triangulation, $y$ is not a cut vertex of the connected graph $G - x$. So let $Y$ denote the unique block (maximal 2-connected component) of $G-x$ containing $y$. Let $x_1, x_2 \in V(Y) \cap V(C-x)$ such that $x, x_1, y, x_2$ occur in $C$ in clockwise order, and $x_1Cx_2 = Y \cap C$. Note that $y$ might be the same as $x_1$ or $x_2$. Since $G$ has no chord from $x$ to $yCx-x$, we see that $x_2x \in E(C)$. Let $X := G-(Y-x_1)$. Note that if $|V(X)| \geq 3$ then $X$ is a near triangulation with outer cycle $C_X := xCx_1 \cup x_1x$, and that if $|V(Y)| \geq 3$ then $Y$ is a near triangulation and its outer cycle, denoted $C_Y$, satisfies $x_1Cx_2 = x_1C_Yx_2$.

We may assume $y \neq x_1$. For otherwise, $V(X) = \{x, y\}$ as $xy$ is not a chord of $C$. But then $x_1$ and $x_2$ are symmetric, and we can relabel $x_1, x_2$ and flip the embedding of $G$.

Now suppose $|V(G)| \leq 5$. Then $|V(X)| \leq 4$ and $|V(Y)| \leq 4$. Hence, $X$ has an $x$-$x_1$ Hamiltonian path (since $xx_1$ is not a chord of $C_X$) and, thus, $Y$ has no $x_1$-$y$ Hamiltonian path. Hence, $Y$ consists of a 4-cycle $x_1aybx_1$ and chord $x_1y$, and we have $V(X) = \{x, x_1\}$. Choose the labeling so that $x_1ay \subseteq x_1Cx_2$. Since $x_2x \in E(C)$ and $G$ is a near triangulation, $xb \in E(G)$. But then $xbx_1ay$ is an $x$-$y$ Hamiltonian path in $G$, a contradiction.

Thus, $|V(G)| = 6$. Then $|V(X)| \leq 5$. Hence, by the minimality of $G$, $X$ has an $x$-$x_1$ Hamiltonian path (since $xx_1$ is not a chord of $C_X$). Therefore, $Y$ has no $x_1$-$y$ Hamiltonian path. Thus $|V(X)| = 3$ and $|V(Y)| = 4$, or $|V(X)| = 2$ and $|V(Y)| = 5$. Moreover, by the minimality of $G$, $x_1y$ is a chord of $C_Y$, and is therefore a chord of $C$ as well since

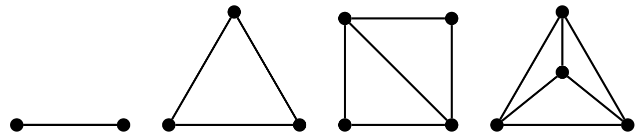


Figure 2.4: Possible triangular-blocks on at most four vertices (up to redrawing).

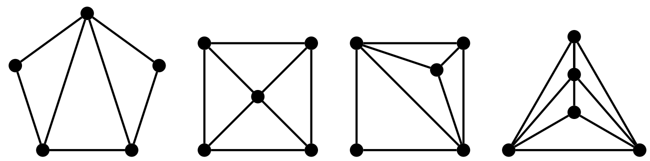


Figure 2.5: Possible 5-vertex triangular-blocks (up to redrawing).

$x_1, y \in V(C)$.

Suppose that $|V(X)| = 3$ and $|V(Y)| = 4$. Then $xx_1 \notin E(C)$, or else $|V(Y)| \geq 5$. Since $xx_1 \in E(X)$ and $x, x_1 \in V(C)$, it follows that $xx_1$ is a chord of $C$. So $xx_1$ and $x_1y$ are both chords of $C$. In $G - \{x, x_1, y\}$ there are no edges between $V(Y) - \{x_1, y\}$ and $X - \{x, x_1\}$ because $x_1$ is a cut-vertex of $G - x$. There is also no edge in $G - \{x, x_1, y\}$ between the vertices in $V(Y) - \{x_1, y\}$ because $Y$ is a 4-vertex near triangulation with chord $x_1y$. So $G - \{x, x_1, y\}$ consists of 3 isolated vertices, and therefore $(ii)$ holds with $z = x_1$.

So $|V(Y)| = 5$ and $|V(X)| = 2$. Note that, for any $w \in N(x)$, there is no $w$-$y$ Hamiltonian path in $Y$, as extending such a path with the edge $xw$ would give an $x$-$y$ Hamiltonian path in $G$. Thus, since $x_1y$ is a chord of $C_Y$, we have $|V(C_Y)| = 5$ and $yC_Yx_1 = yvx_2x_1$ is a path of length 3. Now $vx_1 \notin E(G)$, as otherwise $xx_2vx_1 \cup x_1C_Yy$ is an $x$-$y$ Hamiltonian path in $G$, a contradiction. Hence, $x_2y \in E(G)$, which shows that $(iii)$ holds. $\square$

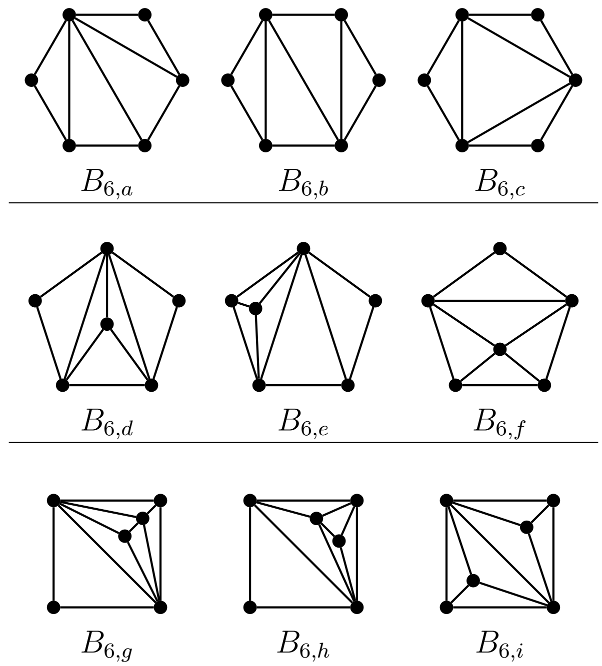


Figure 2.6: Possible $6$-vertex triangular-blocks with a chord (up to redrawing).

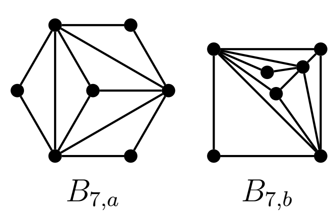


Figure 2.7: Possible 7-vertex $C_7$-free triangular-blocks (up to redrawing).

## 2.4 $C_7$-Free Triangular-Blocks

In this section we characterize all possible triangular-blocks in a $C_7$-free plane graph, up to redrawings that preserve all facial cycles. Certainly any near triangulation with at most six vertices is a candidate. In fact, the following corollary of Lemma 2.2.4 shows that these are the only possible triangular-blocks with at most six vertices.

**Lemma 2.4.1.** *Let $G$ be a 2-connected plane graph and let $B$ be a triangular-block of $G$ with at most six vertices. Then $B$ is a near triangulation, up to redrawings that preserve all facial cycles.*

Figures 2.4, 2.5, and 2.6 list all near triangulations with at most five vertices and all six-vertex near triangulations with a chord. We next prove that there are only two possible $C_7$-free triangular-blocks with more than six vertices; these are shown in 2.7.

**Lemma 2.4.2.** *Every $C_7$-free triangular-block of a 2-connected plane graph $G$ has at most six vertices, with the exceptions of $B_{7,a}$ and $B_{7,b}$ as in 2.7.*

*Proof.* Let $B$ be a $C_7$-free triangular-block of $G$ with at least seven vertices. Since $B$ is a triangular-block of $G$, there is a sequence $B_1, B_2, \ldots, B_m$ of subgraphs of $G$ so that $B_1$ is a facial triangle of $G$ and $B_m = B$, and for each $i \in \{2, 3, \ldots, m\}$ the graph $B_i$ is the union of $B_{i-1}$ and a facial triangle of $G$ that shares an edge with $B_{i-1}$. Let $t$ be minimum so that $|V(B_t)| = 7$. Let $H = B_t$ and let $B' = B_{t-1}$.

We first show that $H \in \{B_{7,a}, B_{7,b}\}$. Note that $B'$ is a triangular-block of itself, so by Lemma 2.4.1 applied with $G = B = B'$ we know that $B'$ is a near triangulation, up to redrawings that preserve all facial cycles. Also, $H$ is obtained from $B'$ by adding a vertex $z$ that is adjacent to both vertices of an edge $xy$ of some facial cycle of $B'$ bounding a hole of $B'$. Since $B$ is $C_7$-free, $B'$ has no $x$-$y$ Hamiltonian path. Then Lemma 2.3.2 implies that $B'$ is not a triangulation. By redrawing $G$ we may assume that the outer cycle $F_1$ of $B'$ has length at least four. Lemma 2.3.2 implies that if $xy$ is an edge on the outer cycle of $B'$, then $B' \cong B_{6,d}$ and $H = B_{7,a}$. So we may assume that $xy$ is incident to an interior triangular hole of $B'$. This implies that $H$ has another facial cycle, $F_2$, of length at least four. By Lemma 2.2.4, $|F_1| = |F_2| = 4$ and $|V(F_1) \cap V(F_2)| = 1$. Let $F_1 = abcda$ and $F_2 = auvwa$.

Suppose that $F_1$ has no chord in its interior. Note that each interior face of $H$ is triangular except for $F_2$. If $ab$ and $bc$ are incident with the same interior triangular face of $H$, then $ac$ is a chord of $H$, a contradiction. Since $ab$ and $bc$ are incident with different interior faces, there is an interior edge of $H$ with $b$ as an end and with the other end in the set $\{u, v, w\}$. However, neither $bu$ nor $bw$ is an edge of $H$; otherwise, $H$ (and hence $G$) would contain $C_7$. So $bv$ is an edge of $H$. By symmetry, $dv$ is an edge of $H$ and $du$ and $dw$ are not edges of $H$. The 4-cycles $abvua$ and $advwa$ each bound two triangular faces of $H$, which forces $av$ to be an edge in the interior of both $abvua$ and $advwa$. This is a contradiction.

Thus, we may assume that $F_1$ has a chord in its interior. 2.6 shows all possible 6-vertex triangular-blocks whose outer cycle has length four and has a chord in its interior, and only $B_{6,g}$ and $B_{6,h}$ could have an interior hole. This hole is bounded by a facial cycle $F$ of $B'$ containing both interior vertices of $B'$. It is not hard to check that for any distinct $s, t \in V(F)$, $B'$ has an $s$-$t$ Hamiltonian path, with one exception (when $B' \cong B_{6,g}$, $s \in V(F_1)$, and $t$ is of degree 4 in $B'$) that results in $H = B_{7,b}$.

So we have shown that $H \in \{B_{7,a}, B_{7,b}\}$. It is straightforward to check that every plane graph obtained by adding an edge to $B_{7,a}$ or $B_{7,b}$ has a $C_7$-subgraph. This implies that $B_{7,a}$ and $B_{7,b}$ are the only possible $C_7$-free triangular-blocks with exactly 7 vertices. Now suppose that $|V(B)| \geq 8$, and let $k$ be minimum so that $|V(B_k)| = 8$. Note that $B_{k-1}$ is a $C_7$-free triangular-block, so $B_{k-1} \in \{B_{7,a}, B_{7,b}\}$. So $B_k$ is obtained by adding a vertex adjacent to both ends of an edge of a facial cycle $F$ of $B_{7,a}$ or $B_{7,b}$ bounding a hole of $B_{7,a}$ or $B_{7,b}$. Since Lemma 2.2.3 implies that the triangular faces of $B_{7,a}$ and $B_{7,b}$ are not holes, $|F| \geq 4$. But by checking cases we see that every graph obtained from $B_{7,a}$ or $B_{7,b}$

by adding a vertex adjacent to both ends of an edge of a non-triangular facial cycle has a $C_7$-subgraph, a contradiction. So $|V(B)| = 7$, and therefore $B \in \{B_{7,a}, B_{7,b}\}$. $\square$

We will also need the following characterization of bad cherries in facial cycles with length at most six.

**Lemma 2.4.3.** *Let $G$ be a 2-connected plane graph, let $F$ be a petal of some triangular-block $B$ of $G$ with $4 \leq |F| \leq 6$ and $|B| \geq 3$, and let $F'$ be the refinement of $F$. Then $F = F'$, unless*

*$(i)$ $B \in \{B_{5,d}, B_{6,g}, B_{6,h}, B_{7,b}\}$,*

*$(ii)$ $F = wx_1x_2x_3$ where $x_1x_2x_3$ is a bad cherry on the outer cycle of $B$, and*

*$(iii)$ $wx_1$ and $wx_3$ are trivial triangular-blocks incident with exactly one face of $G$ with length less than eight.*

*Proof.* Assume that $F \neq F'$. Then $F$ has a bad cherry $x_1x_2x_3$. By the definition of bad cherry, $x_1, x_2, x_3$ are in a common triangular-block $B$ of $G$. We may assume that $G$ is drawn so that $x_1x_2x_3$ is contained in the outer cycle $C$ of $B$ and that $F$ is not in the interior of $C$. Let $B'$ be the interior of the triangle $x_1x_2x_3x_1$ in $B$, so $B'$ is a subgraph of $B$. Since $G$ has no (18/7)-sparse sets of order at most two, there are at least two vertices in the interior of $x_1x_2x_3x_1$ in $B$, and therefore $B'$ has at least five vertices. Thus, by Lemma 2.4.2, $B \in \{B_{5,d}, B_{6,g}, B_{6,h}, B_{7,b}\}$ and $(i)$ holds. Note that $B' \in \{B_{5,d}, B_{6,g}\}$, up to redrawing.

We claim that $|F| = 4$. Since $B' \in \{B_{5,d}, B_{6,g}\}$, by checking cases, there is an $x_1$-$x_3$ path $P_i$ of length $i$ in $B'$ for each $i \in [4]$. Then $|F| \notin \{5, 6\}$, or else $(F - x_2) \cup P_i$ with $i \in \{3, 4\}$ is a $C_7$ in $G$. Therefore $|F| = 4$.

Let $w$ be the vertex in $V(F) \setminus \{x_1, x_2, x_3\}$. We see that $(ii)$ holds. If $wx_1$ or $wx_3$ is in a facial cycle of length less than eight other than $F$, then such a facial cycle contains a $w$-$x_1$ path or a $w$-$x_3$ path of length $\ell \in \{3, 4, 5, 6\}$ that is internally disjoint from $B'$. The concatenation of this path with a length-$(7 - \ell)$ $w$-$x_1$ path or $w$-$x_3$ path of $B' \cup F$ forms a

$C_7$ in $G$, a contradiction. Therefore $wx_1$ and $wx_3$ are trivial triangular-blocks of $G$ that are incident with exactly one face of $G$ with length less than eight and $(iii)$ holds. $\square$

# CHAPTER 3

# COMPUTE DISTRIBUTION FOR EVERY TRIANGULAR-BLOCK

## 3.1 Large Triangular-Blocks

Now that we have a complete characterization of possible triangular-blocks in a $C_7$-free plane graph, we proceed by estimating

$$g(B) = 24f(B) - 17e(B) + 6n(B)$$

for each possible triangular-block $B$. In this section we analyze the possible triangular-blocks with at least six vertices for a graph $G \in \mathcal{P}_n$. We first consider chordless near triangulations, and then divide the remaining cases into three groups based on the length of the outer cycle.

**Lemma 3.1.1.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of $G$. If $B$ is a $6$-vertex near triangulation whose outer cycle has no chord and whose outer face is its only hole, then $g(B) \leq 0$.*

*Proof.* Let $C$ be the outer cycle of $B$; this is the only possible non-triangle facial cycle of $B$. Since $C$ has no chord, $|C| \leq 5$. It follows from Lemmas 2.3.2 and 2.3.1 that for any distinct $x, y \in V(C)$, there is an $x$-$y$ path in $B$ of length $\ell$ for each $\ell$ between $d_B(x, y)$ (at most 2) and $|B| - 1 = 5$. Thus, to avoid $C_7$, each petal $F$ of $B$ has length at least $8$ and, hence, the refinement of each petal has length at least $8$.

Note that at least two vertices on $C$ are junction vertices since $G$ is 2-connected, and for each junction vertex $v$ we have $n(v) \leq 1/2$. Moreover, from Euler's formula, we have $|E(B)| = 15 - |C|$ and $|F(B)| = 11 - |C|$.

If $|C| = 4$, then

$$g(B) \leq 24(6 + 4/8) - 17 \cdot 11 + 6(4 + 2/2) = -1;$$

and if $|C| = 5$, then

$$g(B) \leq 24(5 + 5/8) - 17 \cdot 10 + 6(4 + 2/2) = -5.$$

Thus we may assume that $B$ is a triangulation. If all vertices of $C$ are junction vertices of $B$ then

$$g(B) \leq 24(7 + 3/8) - 17 \cdot 12 + 6(3 + 3/2) = 0.$$

So assume that only two vertices of $C$ are junction vertices of $B$. Then $B$ has a bad cherry $x_1x_2x_3$ on a facial cycle $F$ of $B$ such that $x_1x_2, x_2x_3 \in E(F)$. Thus,

$$g(B) \leq 24(7 + 1/8 + 1/8) - 17 \cdot 12 + 6(4 + 2/2) = 0,$$

as desired. □

We next consider 6-vertex and 7-vertex triangular-blocks whose outer cycle has length 4 and a chord.

**Lemma 3.1.2.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of $G$ with no triangular holes and at most one hole. If $B \in \{B_{6,g}, B_{6,h}, B_{6,i}, B_{7,b}\}$, then $g(B) < 0$.*

*Proof.* We may assume that $G$ is drawn in the plane such that the outer face of $B$ is its hole. Let $C$ be the outer cycle of $B$. Let $x$ and $y$ be the ends of the unique chord of $C$, and let $a$ and $b$ be the other two vertices on the outer cycle of $B$, where $d_B(a) = 2$ if $B \neq B_{6,i}$. If $B = B_{7,b}$, let $c$ be the interior degree-2 vertex of $B$. Note that when $|B| = 6$ then $|E(B)| = 11$ and $|F(B)| = 7$, and when $|B| = 7$ then $|E(B)| = 13$ and $|F(B)| = 8$.

Suppose for each petal $F$ of $B$, $|F| \geq 8$. Since $B$ has at least two junction vertices, if $|B| = 6$ then

$$g(B) \leq 24(6 + 4/8) - 17 \cdot 11 + 6(4 + 2/2) = -1,$$

and if $|B| = 7$ then

$$g(B) \leq 24(7 + 4/8) - 17 \cdot 13 + 6(5 + 2/2) = -5$$

as desired.

Thus, we may assume that there is some petal $F$ of $B$ with $|F| < 8$. We now show that $|F| = 4$ and $F = xbytx$ for some $t \notin V(B)$. To see this, let $P$ be a subpath of $F$ that is also a $B$-path, and let $v$ and $w$ be the ends of $P$ in $B$. If $\{v, w\} \neq \{x, y\}$, then Lemmas 2.3.1 and 2.3.2 (applied to $B - c$ if $B = B_{7,b}$) imply that $B$ and $B - c$ have $v$-$w$ paths of any length between $d_B(v, w)$ and 5. Since $G$ is $C_7$-free, this implies that $P$ consists of a single edge. So $v$ and $w$ are not adjacent in $B$, and therefore $\{v, w\} = \{a, b\}$ and $P = ab$. But neither $abx$ nor $aby$ is a facial cycle because $B$ is a triangular-block, so $F$ has a different $B$-path $P'$ as a subpath, and $P'$ must have ends $x$ and $y$. But $P = ab$ and $P'$ cannot both be subpaths of $F$, so $P$ is not a subpath of $F$, and therefore every $B$-path contained in $F$ has ends $x$ and $y$. Since $F$ is a cycle that contains an edge of $B$, this implies that $P'$ is the unique $B$-path contained in $F$, and that $F = P' \cup xby$ or $F = P' \cup xay$. Then $B \neq B_{6,i}$ or else $G$ has an (18/7)-sparse set consisting of adjacent degree-3 vertices, and $F \neq P' \cup xay$ or else $a$ is a degree-2 vertex of $G$. So $F = P' \cup xby$. Since $B \neq B_{6,i}$ there are $x$-$y$ paths of length 1, 2, 3, and 4 in $B$, and so $P' = xty$ where $t \in V(F) \setminus V(B)$. Hence, $|F| = 4$ and $F = xbytx$.

Note that $F$ is the unique petal of $B$ with length less than eight, $b$ is not a junction vertex, and $xby$ is a bad cherry (thus $f_F(B) = 1/3$). Therefore, if $B \in \{B_{6,g}, B_{6,h}\}$, we have

$$g(B) \leq 24(6 + 2/8 + 1/3) - 17 \cdot 11 + 6(3 + 3/2) = -2,$$

and if $B = B_{7,b}$ we have

$$g(B) \leq 24(7 + 2/8 + 1/3) - 17 \cdot 13 + 6(4 + 3/2) = -6,$$

as desired. □

To deal with other 6-vertex and 7-vertex triangular-blocks, we need a technical lemma.

**Lemma 3.1.3.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$ with no triangular holes and $B \in \{B_{6,a}, B_{6,b}, B_{6,c}, B_{6,d}, B_{6,e}, B_{6,f}, B_{7,a}\}$. Let $C$ be the outer cycle of $B$, $x_1x_2x_3$ be a subpath of $C$, and let $F_1, F_2$ be the petals of $B$ containing $x_1x_2, x_2x_3$, respectively. Suppose $G$ has a $B$-path $P$ between $x_1$ and $x_3$ of length at most 2. Then $|F_i| \geq 8$ for $i = 1, 2$, unless*

(1) *$B = B_{6,a}$ with $x_1, x_3$ corresponding to $x, y$ as in 2.3 and $F_1 = F_2$ is a 4-cycle, or*

(2) *$B = B_{6,d}$ with $x_1, x_2$ corresponding to $x, y$ as in 2.3 and $|F_1| \geq 8$.*

*Proof.* First, suppose $F_1 \neq F_2$. Then, because of $P$ we see that each $F_i$ contains an $x_{s_i}$-$x_{s_i+1}$ $B$-path for some $s_i \in [2]$. If $B \notin \{B_{6,d}, B_{7,a}\}$, then Lemmas 2.3.1 and 2.3.2 imply that $B$ has an $x_{s_i}$-$x_{s_i+1}$ path of length $\ell$ for all $\ell \in [5]$. Thus, since $G$ is $C_7$-free, $|F_i| \geq 8$ for $i = 1, 2$. If $B = B_{7,a}$, then it is straightforward to check that $B$ has an $x_{s_i}$-$x_{s_i+1}$ path of length $\ell$ for all $\ell \in [5]$. Again, this implies that $|F_i| \geq 8$ for $i = 1, 2$. If $B = B_{6,d}$, then by checking cases we see that $B$ has an $x_{s_i}$-$x_{s_i+1}$ path of length $\ell$ for all $\ell \in [4]$. Since $|F_i| \geq 4$ (as $B$ is a triangular-block), if $|F_i| \leq 6$ then by case analysis $B \cup F_i \cup P$ contains a $C_7$ unless $B = B_{6,d}$ with $x_1, x_2$ corresponding to $x, y$ as in 2.3, $|F_2| = 4$, and $x_2x_3x_1 \subseteq F_2$. Hence, $|F_1| \geq 8$.

So we may assume $F_1 = F_2$. Again, because of the path $P$ and planarity, $F_1 \cap C = x_1x_2x_3$. This implies that $x_2$ is not a junction vertex of $B$. Also, $|F_1| \geq 4$ as $B$ is a triangular-block.

Suppose $B$ has an $x_1$-$x_3$ Hamiltonian path. Then by, Lemma 2.3.1, $B$ has an $x_1$-$x_3$ path of length $\ell$ for every $\ell$ between 2 and 5. Therefore, $|F_1| \geq 8$ to avoid a $C_7$ in $G$.

Now assume $B$ has no $x_1$-$x_3$ Hamiltonian path. Then by Lemma 2.3.2, either $x_1x_3$ is a chord of $C$, or $B = B_{6,a}$ with $x_1, x_3$ corresponding to $x, y$ in Figure 1. If $x_1x_3$ is a chord of $C$, then $G$ has an (18/7)-sparse set with order at most 2 because $x_2$ is not a junction vertex. So $B = B_{6,a}$, and therefore $B$ has an $x_1$-$x_3$ path of any length between 2 and 4. So if $|F_1| < 8$ then $|F_1| = 4$. □

We next consider 6-vertex triangular-blocks with five vertices on the outer cycle.

**Lemma 3.1.4.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of $G$ with no triangular hole. If $B \in \{B_{6,d}, B_{6,e}, B_{6,f}\}$, then $g(B) \leq 0$.*

*Proof.* By Lemma 2.4.3, each petal of $B$ with length at most six is equal to its refinement. We may assume that $B$ is drawn as shown in 2.6. Let $C$ be the outer cycle of $B$, so $|C| = 5$. Since no triangular face of $B$ is a hole, all junction vertices of $B$ are contained in $C$ and there are at least two (as $G$ is 2-connected) and must include all degree-2 vertices of $B$ (to avoid an (18/7)-sparse set of order 1).

In fact, $C$ has at least three junction vertices of $B$. For, otherwise, let $S$ be obtained from $V(B)$ by removing the junction vertices of $B$ (so $|S| = 4$). Then one can check that $S$ is an (18/7)-sparse set, a contradiction.

Suppose there is a petal $F$ of $B$ with $|F| < 8$ that contains a $B$-path $P$ of length 3. It is straightforward to check that $B = B_{6,e}$ and the ends of $P$ are the degree-4 vertices of $B$, or else $G$ has a $C_7$-subgraph. Then $B$ has at least 4 junction vertices, to avoid an (18/7)-sparse set of order 1 or 2. Also, to avoid an (18/7)-sparse set of order 1 or 2, the petal $F$ contains another $B$-path $P'$. Note that $|P'| \leq 2$ since $|F| < 8$ and $F$ contains $P$ and at least one edge of $B$ and $G$ is $C_7$-free. Since $|C| = 5$, the ends of $P'$ are at distance 2 on $C$. Applying Lemma 5.3 we see that outcomes (1) and (2) do not hold since $B \notin \{B_{6,a}, B_{6,d}\}$. Therefore

$C$ has two edges in a petal with length at least 8, and so

$$g(B) \leq 24(5 + 2/8 + 3/4) - 17 \cdot 10 + 6(2 + 4/2) = -2.$$

So we may assume that no petal $F$ of $B$ with $|F| < 8$ contains a length-3 $B$-path.

It is straightforward to check that for each pair $\{x, y\}$ of vertices on $C$, there is an $x$-$y$ path in $B$ of length $\ell$ for all $\ell \in [2, 3]$. This implies that there are no $B$-paths with length in $\{4, 5\}$. It follows, using the previous paragraph, that if there are no $B$-paths of length at most 2, then each petal of $B$ has length at least 8, and so

$$g(B) \leq 24(5 + 5/8) - 17 \cdot 10 + 6(3 + 3/2) < 0.$$

So we may assume that $B$ has a petal with length less than 8. If there are two $B$-paths of length at most 2 that have different sets of ends on $C$, then by Lemma 3.1.3, at least three petals of $B$ have length at least 8; so

$$g(B) \leq 24(5 + 3/8 + 2/4) - 17 \cdot 10 + 6(3 + 3/2) = -2.$$

So we may assume that all $B$-paths of length at most 2 in $G$ have the same ends on $B$, say $x$ and $y$, and that there is a petal $F$ of $B$ with $|F| < 8$. Since all $B$-paths of length at most 2 have ends $x$ and $y$, it follows that $F$ is the union of a $B$-path $P_1$ with ends $x$ and $y$ and a subpath $P_2$ of $C$ with ends $x$ and $y$. Note that $xy$ is not a chord of $C$ in $B$, or else it is straightforward to check that $G$ has an (18/7)-sparse set consisting of 1 or 2 internal vertices of $P_2$. Similarly, $P_2$ has length at most 3, or else $G$ has an (18/7)-sparse set consisting of 1 or 2 internal vertices of $P_2$. Suppose that $P_2 = xy$. Then $P_1$ has length at least 3 because $B$ is a triangular-block. But since there are no $B$-paths with length in $\{3, 4, 5\}$ (as previously observed) and $G$ is $C_7$-free, it follows that $|F| \geq 8$, a contradiction. So $P_2$ has length 2 or 3, which means that $x$ and $y$ are at distance 2 on $C$ since $|C| = 5$.

Since $xy$ is not a chord of $C$ in $B$, by checking cases we see that $P_1$ has length 1, or else $B \cup P_1$ has a $C_7$-subgraph. Since $P_1$ has length 1 and $B$ is a triangular-block, it follows that $P_2$ has length 3, and so $|F| = 4$.

Since $x$ and $y$ are at distance 2 on $C$, there is a vertex $z$ so that $xzy$ is a subpath of $C$. Applying Lemma 3.1.3 with $(x_1, x_2, x_3) = (x, z, y)$ we see that outcomes (1) and (2) do not hold, and so the edges $xz$ and $yz$ are each in a petal with length at least 8. Note that $z$ is a junction vertex because $B$ has at least 3 junction vertices, so the petals of $B$ containing $xz$ and $yz$ are distinct, and one of them contains a $B$-path with ends $\{x, z\}$ or $\{y, z\}$. This implies that $x$ or $y$ has an incident edge that is not in $B \cup xy$; assume without loss of generality that $x$ has an incident edge $e$ not in $B \cup xy$. We will show that $x$ is in at least 3 triangular-blocks. If $xy$ is in a non-trivial triangular-block, then there is a vertex $w$ so that $xywx$ is a facial triangle of $G$. We cannot have $w = z$ because $B$ is a triangular-block, so $w \notin V(B)$. But then $xwy$ is a $B$-path of length 2 with ends $x$ and $y$, which forms a $C_7$ with $B$ (as previously discussed), a contradiction. So $xy$ is a trivial triangular-block, and therefore $x$ is in at least three triangular-blocks: $B$, $xy$, and the block containing $e$. Then we have

$$g(B) \leq 24(5 + 2/8 + 3/4) - 17 \cdot 10 + 6(3 + 2/2 + 1/3) = 0,$$

as desired. □

Finally, we consider triangular-blocks with six vertices on the outer cycle.

**Lemma 3.1.5.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of $G$ with no triangular hole. If $B \in \{B_{6,a}, B_{6,b}, B_{6,c}, B_{7,a}\}$, then $g(B) \leq 0$.*

*Proof.* By Lemma 2.4.3, each petal of $B$ with length at most six is equal to its refinement. We may assume that $B$ is drawn as shown in 2.6 or 2.7. Let $C$ be the outer cycle of $B$, so $|C| = 6$. Note that every degree-2 vertex in $B$ must be a junction vertex, to avoid an (18/7)-sparse set of order 1. Thus, if $B$ has two junction vertices then $B \in \{B_{6,a}, B_{6,b}\}$;

but then $\{v \in V(B) : d_B(v) \geq 3\}$ is an (18/7)-sparse set in $G$, a contradiction. So $B$ has at least three junction vertices.

Let $\mathcal{P}$ be the collection of $B$-paths of $G$ with at most two edges. We will show that if $F$ is a petal of $B$ with $|F| < 8$, then $F$ contains a $B$-path in $\mathcal{P}$. Let $P$ be a $B$-path contained in a petal $F$ with $|F| < 8$. It is straightforward to check that $B \cup F$ contains $C_7$ if $|P| > 3$, so we may assume that $|P| = 3$. Then it is straightforward to check that $B \cup F$ contains $C_7$ unless $B \in \{B_{6,a}, B_{6,b}\}$ and $P$ has ends $x$ and $y$ where $xy$ is a chord of $C$ in $B$ and $x$ and $y$ are at distance three on $C$. To avoid an (18/7)-sparse set of order 1, $F$ contains another $B$-path $P'$. Note that $P'$ has different ends from $P$, or else $F$ contains a cycle as a proper subgraph. Using the previous argument, this implies that $|P'| < 3$, so $P' \in \mathcal{P}$.

If $B \neq B_{7,a}$ (so $|B| = 6$, $|F(B)| = 5$, and $|E(B)| = 9$) and at least two edges of $C$ are in petals of $B$ of length at least 8, then

$$g(B) \leq 24(4 + 2/8 + 4/4) - 17 \cdot 9 + 6(3 + 3/2) = 0. \tag{a}$$

We now consider three cases.

First suppose that $B = B_{6,a}$. Let $v$ be a degree-2 vertex of $B$. Since $v$ is a junction vertex, it is in at least two petals of $G$. If both petals have length at least 8, then $g(B) \leq 0$ by (a), so we may assume that $v$ is in a petal $F$ with $|F| < 8$. Since $G$ has no degree-2 vertices, $F$ contains a path $P$ in $\mathcal{P}$ with $v$ as an end. Since $v$ is not the end of a chord of $C$ in $B$ and $G$ contains no $C_7$, it follows from Lemma 2.3.2 that $P$ consists of a single edge. Let $w$ be the other end of $P$. Suppose that $w$ is at distance two from $v$ on $C$. Let $vzw$ be a subpath of $C$. Applying Lemma 3.1.3 with $(x_1, x_2, x_3) = (v, z, w)$ we see that outcomes (1) and (2) do not hold, so both petals of $B$ containing $v$ have length at least 8, and $g(B) \leq 0$ by (a). So we may assume that $w$ is at distance three from $v$ on $C$. Let $u$ be the other degree-2 vertex of $B$, and let $t$ be the vertex at distance three from $u$ on $C$. By applying the same reasoning to $u$ that we applied to $v$, either $ut$ is an edge of $G$, or

$g(B) \leq 0$. Since $vw$ is an edge, $ut$ cannot be an edge by planarity, so we conclude that $g(B) \leq 0$.

Next suppose that $B \in \{B_{6,b}, B_{6,c}\}$. We may assume that $B$ has a petal $F$ with $|F| < 8$ or else (a) implies that $g(B) \leq 0$. Suppose that there is some $P \in \mathcal{P}$ with ends at distance two on $C$. Let $u$ and $v$ be the ends of $P$, and let $uzv$ be a subpath of $C$. Applying Lemma 3.1.3 with $(x_1, x_2, x_3) = (u, z, v)$ we see that outcomes (1) and (2) do not hold because $B \notin \{B_{6,a}, B_{6,d}\}$. Then $B$ has two petals with length at least 8, and (a) implies that $g(B) \leq 0$. So we may assume that the ends of each path in $\mathcal{P}$ are at distance three on $C$. If a petal $F$ with $|F| < 8$ contains a $B$-path $P'$ with length greater than 2, then the argument of the second paragraph shows that $B = B_{6,b}$ and $P'$ has ends $x$ and $y$ where $xy$ is a chord of $C$ in $B$ and $x$ and $y$ are at distance three on $C$. Since all $B$-paths contained in $F$ have ends at distance three on $C$, it follows that $F$ is not leaky, and is therefore the union of a path $P \in \mathcal{P}$ and a length-3 subpath of $C$. Then it is straightforward to check that $G$ has an (18/7)-sparse set unless $B = B_{6,b}$ and the ends of each path in $\mathcal{P}$ are the degree-2 vertices of $B$. Since the degree-2 vertices of $B$ are not the ends of a chord of $C$, it follows that each $P \in \mathcal{P}$ consists of a single edge to avoid a $C_7$ in $B \cup P$, and so $|\mathcal{P}| = 1$. Therefore there is a unique petal $F$ of $B$ with $|F| < 8$. Since only three edges of $C$ are contained in $F$, there are three edges of $C$ contained in petals of length at least 8. Then (a) implies that $g(B) \leq 0$.

Finally, suppose that $B = B_{7,a}$. It is straightforward to check that if $u$ and $v$ have degree 2 in $B$, then there is a $u$-$v$ path in $B$ of length $\ell$ for each $\ell \in \{5, 6\}$. It follows that there is no path $P$ in $\mathcal{P}$ with both ends having degree 2 in $B$, or else $B \cup P$ contains $C_7$, a contradiction. Suppose there is a path in $\mathcal{P}$ with ends $u$ and $v$, neither of which has degree 2 in $B$. Let $uzv$ be a subpath of $C$. Applying Lemma 3.1.3 with $(x_1, x_2, x_3) = (u, z, v)$ we see that outcomes (1) and (2) do not hold, and so $B$ has two petals with length at least 8.

Since $B$ also has at least 5 junction vertices ($u$, $v$, and each degree-2 vertex) we see that

$$g(B) \leq 24(6 + 2/8 + 4/4) - 17 \cdot 12 + 6(2 + 5/2) = -3.$$

So we may assume that every path in $\mathcal{P}$ has exactly one end with degree 2 in $B$. Then no path $P \in \mathcal{P}$ has length 2 or else $B \cup P$ contains $C_7$, a contradiction. So each path in $\mathcal{P}$ consists of a single edge between vertices at distance 3 on $C$, and therefore $|\mathcal{P}| = 1$ by planarity. Suppose $F$ is a petal of $B$ with $|F| < 8$. It is straightforward to check that $F$ contains a path in $\mathcal{P}$, or else $B \cup F$ has a $C_7$-subgraph. Then $F$ is the union of the unique path in $\mathcal{P}$ and a subpath of $C$. Since the unique path in $\mathcal{P}$ has ends at distance three on $C$, this is a length-3 subpath of $C$ between a degree-2 vertex and a degree-3 vertex of $B$. But then the other degree-2 vertex of $B$ on $F$ is not a junction vertex and therefore forms an (18/7)-sparse set in $G$, a contradiction. So every petal of $B$ has length at least 8, and therefore

$$g(B) \leq 24(6 + 6/8) - 17 \cdot 12 + 6(4 + 3/2) = -9,$$

as desired. □

### 3.2 Five-vertex triangular-blocks

In this section we analyze the possible 5-vertex triangular-blocks of a graph $G \in \mathcal{P}_n$.

**Lemma 3.2.1.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of G, and assume that $G$ is drawn in the plane such that $B = B_{5,a}$ is drawn as in 2.5. Then $g(B) \leq 0$.*

*Proof.* By Lemma 2.4.3, each petal of $B$ with length at most six is equal to its refinement. Lemma 2.2.3 shows that no triangular face of $B$ is a hole. Let $C = abcdea$ be the outer cycle of $B$ with $d_B(a) = 4$. Note that $B$ has at least three junction vertices, or else $\{b\}$ or $\{e\}$ or $\{c, d\}$ would be an (18/7)-sparse set in $G$.

If at least two edges of $C$ are contained in petals of $B$ of length at least 8, then

$$g(B) \leq 24(3 + 2/8 + 3/4) - 17 \cdot 7 + 6(2 + 3/2) = -2.$$

So we may assume that at most one edge of $C$ is contained in a petal of $B$ of length at least 8. This implies, up to relabeling vertices, that the edges $ab$ and $bc$ are not in a petal of length at least 8. Let $F_1$ and $F_2$ be petals of $B$ that contain $ab$ and $bc$, respectively.

Suppose $F_1$ and $F_2$ are both leaky. It is straightforward to check that there is a pair $(P_1, P_2)$ of length-2 $B$-paths where $P_i \subseteq F_i$, the ends of $P_i$ are not $\{a, c\}$ or $\{a, d\}$, and $P_1$ and $P_2$ have distinct sets of ends. But then $B \cup P_1 \cup P_2$ contains $C_7$, unless $P_1$ has ends $\{a, e\}$ and $P_2$ has ends $\{c, e\}$. In this case the petal containing $ae$ has length at least 8 to avoid a $C_7$ and $B$ has at least 4 junction vertices because $a$ is a junction vertex, and so

$$g(B) \leq 24(3 + 1/8 + 4/4) - 17 \cdot 7 + 6(1 + 4/2) = -2,$$

as desired. So we may assume that either $F_1$ or $F_2$ is not leaky.

We claim that if $F_1$ is not leaky then $F_1 \cap C = eab$ and if $F_2$ is not leaky then $F_2 \cap C = bcd$. Suppose $F_1$ is not leaky. First, $F_1 \cap C$ does not contain $deab$ or $abc$, as otherwise $\{e\}$ or $\{b\}$ would be an (18/7)-sparse set in $G$. Now suppose $F_1 \cap C = ab$. Then since $B$ has an $a$-$b$ path of every length between 1 and 4 and $|F| < 8$, $G$ has a $C_7$, a contradiction. Therefore $F_1 \cap C = eab$, and similar argument shows that if $F_2$ is not leaky then $F_2 \cap C = bcd$.

Suppose $F_1$ and $F_2$ are both non-leaky. Then by the above claim, $|F_1| = |F_2| = 4$ (to avoid $C_7$ in $G$). Let $F_1 = veabv$ and $F_2 = wbcdw$. If $v \neq w$, we see that $B \cup F_1 \cup F_2$ contains a $C_7$, a contradiction. So $v = w$. But then $\{b, c\}$ is an (18/7)-sparse set in $G$, a contradiction.

We may thus assume that $F_1$ is not leaky and $F_2$ is leaky, as the same argument applies to the case when $F_1$ is leaky and $F_2$ is not leaky. Hence, $F_1 \cap C = eab$ and $|F_1| = 4$.

Since $F_2$ is leaky, it contains two $B$-paths. One of these $B$-paths has length at least 2 since $|F_2| \geq 4$. It is straightforward to check that if $F_2$ contains a $B$-path with length at least 3 then $B \cup F_2$ has a $C_7$-subgraph, a contradiction. Also, to avoid a $C_7$ in $B \cup F_1 \cup F_2$, the only possible length-2 $B$-path contained in $F_2$ has ends $\{b, e\}$, so $F_2$ contains such a $B$-path. Then $F_2$ also contains a length-1 $B$-path, and since $F_2$ contains $bc$ the only option is the edge $ce$. So $F_2 = wbcew$ for some vertex $w \notin V(B)$. Let $F_1 = veabv$. If $v = w$ then $d_G(v) = 2$ and $\{v\}$ is an (18/7)-sparse set in $G$, a contradiction. So $v \neq w$.

We now consider two cases depending on whether or not $d$ is a junction vertex. First suppose that $d$ is not a junction vertex. If the petal containing $d$ has length at least 8, then

$$g(B) \leq 24(3 + 2/8 + 3/4) - 17 \cdot 7 + 6(2 + 3/2) = -2,$$

so we may assume that it has length less than 8. Then it contains a $(B \cup F_1)$-path with ends $c$ and $e$ and length between 2 and 5, which forms a $C_7$ with a $c$-$e$ path in $B \cup F_1$, a contradiction. Next suppose that $d$ is a junction vertex. If the petal containing $cd$ has length at least 8, then

$$g(B) \leq 24(3 + 1/8 + 4/4) - 17 \cdot 7 + 6(1 + 4/2) = -2,$$

so we may assume that it has length less than 8. Then it contains a $(B \cup F_1)$-path with both ends in $\{c, d, e\}$ and with length between 2 and 5, which forms a $C_7$ with a path in $B \cup F_1$, a contradiction. □

**Lemma 3.2.2.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$. Suppose that $G$ is drawn in the plane such that $B = B_{5,d}$ is drawn as in 2.5 and its only hole is its outer face. Then $g(B) \leq 0$.*

*Proof.* First, we observe that every $B$-path in $G$ has length $\ell$ with $\ell = 2$ or $\ell \geq 7$. For, otherwise, suppose $P$ is a $B$-path in $G$ of length $\ell$ with $3 \leq \ell \leq 6$. Note that $B$ contains a path $Q$ of length $7 - \ell$ between the ends of $P$. Now $P \cup Q$ is a $C_7$ in $G$, a contradiction.

Let $uvwu$ be the outer cycle of $B$. (Note that we do not specify the vertices $u, v, w$.) Thus, all junction vertices of $B$ are contained in $\{u, v, w\}$ and, since $G$ is 2-connected, at least two of $\{u, v, w\}$ are junction vertices of $B$. Hence, if each edge of $uvwu$ is in a petal of $B$ of length at least 8, then

$$g(B) \leq 24(5 + 3/8) - 17 \cdot 9 + 6(3 + 2/2) = 0,$$

so we may assume that there is a petal $F$ of $B$ with $|F| < 8$.

We claim that $|F| = 4$ and $F$ has a bad cherry (so $f_F(B) = 1/3$). To see this, let $P$ be a longest $B$-path contained in $F$. By the observation above, $P$ has length 2 (as $|F| < 8$). Since $B$ is a triangular-block, $P$ and two edges of $uvwu$ bounds the face $F$. So $|F| = 4$, and $F \cap B$ is a bad cherry. Without loss of generality, assume that $F \cap B = vwu$.

We next consider the petal $F_1$ of $B$ containing $uv$. Note that $F_1 - uv$ is a $B$-path, because $w$ is in the interior of $F$. Hence, since $|F_1| \geq 4$, we see that $F_1 - uv$ has length at least 7 (by the above observation). So $|F_1| \geq 8$ (and $f_{F_1}(B) \leq 1/8$).

We now show that $u$ and $v$ are each contained in at least three triangular-blocks of $G$. For, suppose, without loss of generality, that $v$ is in exactly two triangular-blocks of $G$. Let $x$ be the vertex in $F - \{u, v, w\}$. If $N_G(v) \subseteq V(B \cup F)$, then $V(B) \setminus \{u\}$ is an (18/7)-sparse set in $G$, a contradiction. So $v$ has a neighbor outside $B \cup F$, say $y$. Then $vy$ and $vx$ are in some triangular-block $B'$ of $G$, and $B'$ is nontrivial because $|V(B')| \geq 3$. Since $B'$ is a nontrivial triangular-block, the edge $xv$ is in a triangular face of $G$. The third vertex $z$ incident with this face cannot be $u$ or $w$, because $B$ is a triangular-block and $x \notin V(B)$. So $z$ is not in $B \cup F$. (It is possible that $z = y$.) But then the concatenation of the $B$-path $vzxu$ with a $u$-$v$ path of length 4 in $B$ gives a $C_7$-subgraph of $G$, a contradiction. Therefore,

$$g(B) \leq 24(5 + 1/8 + 1/3) - 17 \cdot 9 + 6(3 + 2/3) = 0,$$

as desired. □

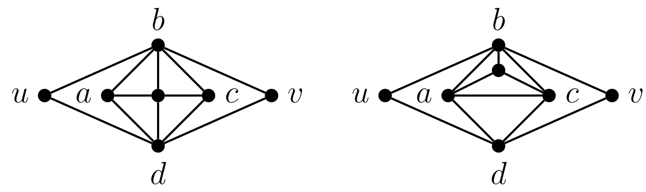


Figure 3.1: The two exceptional flowers when $B \in \{B_{5,b}, B_{5,c}\}$.

**Lemma 3.2.3.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$. Suppose that $G$ is drawn in the plane such that $B \in \{B_{5,b}, B_{5,c}\}$ is drawn as in 2.5. Then $g(B) \leq 0$ unless the flower of $B$ is one of the two graphs shown in 3.1, up to redrawings that preserve facial cycles.*

*Proof.* By Lemma 2.4.3, each petal of $B$ with length at most six is equal to its refinement. Lemma 2.2.3 shows that the only hole of $B$ is its outer face. Let $C = abcda$ be the outer cycle of $B$, with $d_B(d) = 2$ if $B = B_{5,c}$. Since $G$ is 2-connected, $B$ has at least 2 junction vertices. Thus, if each edge of $C$ is in a petal of $B$ of length at least 8, then

$$g(B) \leq 24(4 + 4/8) - 17 \cdot 8 + 6(3 + 2/2) = -4.$$

So we may assume that there is a petal $F$ of $B$ with $|F| < 8$.

Thus, $F$ contains a $B$-path $P$ of length between 2 and 5. To avoid forming $C_7$ with a path in $B$, $P$ has length 2 if the ends of $P$ do not induce a chord of $C$, and $P$ has length 2 or 3 otherwise.

Suppose that we may choose $P$ so that the ends of $P$ are adjacent on $C$. Then $P$ has length 2 to avoid forming a $C_7$ with $B$, and, since $B$ is a triangular-block, $F$ is a leaky petal. Therefore, $F$ contains a second $B$-path $P'$ which does not have the same two ends as $P$, and hence $B$ has at least three junction vertices. Note that $B \cup P$ is a near triangulation on 6 vertices, and also note by the above paragraph that $P'$ has length at most 3. Since $G$ has no $C_7$-subgraph, it is straightforward to check that either $P'$ has length 1, or $B = B_{5,c}$ and $P'$ has length 2 and ends $a$ and $c$. We consider these two cases separately. First suppose that

$P'$ has length 1. If $B = B_{5,c}$ then $P' \neq ac$, so without loss of generality we may assume that $P' = bd$, and that the triangle $abda$ separates $B$ from some vertex of $G$. Then the petal of $B$ containing $ab$ either contains an $a$-$b$ path of length at least 7 or a $b$-$d$ path of length at least 6 to avoid forming a $C_7$ with $B \cup F$, and in either case the petal has length at least 8. Similarly, the petal of $B$ containing $ad$ has length at least 8. Therefore

$$g(B) \leq 24(4 + 2/8 + 2/4) - 17 \cdot 8 + 6(2 + 3/2) = -1,$$

as desired. Next suppose that $B = B_{5,c}$ and $P'$ has length 2 and ends $a$ and $c$. Let $x \in \{b, d\}$ so that $x \notin V(F)$ and let $y \in \{b, d\}$ so that $y \in V(F)$. By symmetry we may assume that $P$ has ends $a$ and $y$. To avoid an (18/7)-sparse set of order 1 or 2, the petal of $B$ containing $ax$ contains an $a$-$x$ $B$-path or a $c$-$x$ $B$-path. In either case, this path has length at least 7 to avoid forming a $C_7$ with $B \cup P$, and so the petal has length at least 8. Similarly, the petal of $B$ containing $cx$ has length at least 8. Therefore

$$g(B) \leq 24(4 + 2/8 + 2/4) - 17 \cdot 8 + 6(2 + 3/2) = -1,$$

as desired.

So we may assume that $F$ contains no $B$-path with length between 2 and 5 whose ends are adjacent on $C$. Then $F$ is not a leaky petal. If $B = B_{5,c}$ and $P$ is an $a$-$c$ path, then $G$ has a sparse set consisting of 1 or 2 vertices from $V(B) \setminus \{a, c\}$, a contradiction. So we may assume without loss of generality that $P$ is a $b$-$d$ path (so $P$ has length 2 to avoid forming a $C_7$ with $B$) and that $F$ is the cycle $bcdvb$, where $v$ is the internal vertex of $P$.

If $B$ has three junction vertices, then the petal $F_1$ of $B$ containing $ab$ also contains a $B$-path of length at least 2 with ends $a$ and $b$ or $a$ and $d$. Since such a path has length at least 7 (to avoid forming a $C_7$ with $B$), $|F_1| \geq 8$. Similarly, the petal of $B$ containing $ad$

has length at least 8. Then

$$g(B) \le 24(4 + 2/8 + 2/4) - 17 \cdot 8 + 6(2 + 3/2) = -1.$$

So we may assume that $B$ has exactly two junction vertices, namely $b$ and $d$. Then the petal $F_1$ of $B$ containing the path $dab$ is the union of $dab$ and a $B$-path $Q$ in $G$ between $d$ and $b$. Note that $Q \neq P$ because the petal $F_1$ contains the path $dab$. As with $P$ and $F$, the path $Q$ has length 2 or at least 6 to avoid creating $C_7$ with $B$. If $Q$ has length 2, then since $Q \neq P$, the flower of $B$ is shown in 3.1.

So assume $Q$ has length at least 6. Then $|F_1| \ge 8$. If $b$ and $d$ are each in at least three triangular-blocks of $G$ then

$$g(B) \le 24(4 + 2/8 + 2/4) - 17 \cdot 8 + 6(3 + 2/3) = 0.$$

So assume by symmetry that $b$ is in exactly two triangular-blocks of $G$. If the triangular-block $B' \neq B$ containing $b$ is trivial, then $V(B - d)$ is an (18/7)-sparse set in $G$, a contradiction. So $B'$ has at least three vertices, and Lemma 2.4.2 implies that $B'$ contains a $b$-$v$ path with length between 2 and 6, which is also a $(B \cup P)$-path. However, this path and a $b$-$v$ path in $B \cup P$ form a $C_7$ in $G$, a contradiction. □

Finally, for the triangular-blocks with flower shown in 3.1, we note that each edge of its outer cycle is contained in a facial cycle of length at least 5, to avoid a $C_7$ in $G$. Thus, we have the following.

**Lemma 3.2.4.** *Let $G \in \mathcal{P}_n$ with $n \ge 7$. Let $B$ be a triangular-block of $G$ with flower shown in Figure 3.1, and let $e$ be an edge of the outer cycle of this flower. Then $e$ is incident with exactly one face of $G$ of length less than five. In particular, $e$ is a trivial triangular-block.*

### 3.3 Small Triangular-Blocks

In this section we analyze the triangular-blocks with at most four vertices in a graph $G \in \mathcal{P}_n$. We first consider trivial triangular-blocks.

**Lemma 3.3.1.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be trivial triangular-block of $G$. Then $g(B) \leq 0$.*

*Proof.* Let $x_1$ and $x_2$ be the two vertices of $B$, and let $F_1$ and $F_2$ be the two petals of $B$. Note that $|F_i| \geq 4$ for each $i \in [2]$ since $B$ is a triangular-block. Each of $x_1$ and $x_2$ is in at least two triangular-blocks, so $n(B) \leq \frac{1}{2} + \frac{1}{2} = 1$.

We may assume that for each $i \in [2]$, $|F_i| = 4$, $F_i$ equals its refinement, and $x_i$ is contained in exactly two triangular-blocks of $G$ (including $B$). First, if the refinement of $F_1$ is not $F_1$ then Lemma 2.4.3 implies that $|F_1| \geq 8$ and $|F_2| \geq 4$, or $|F_1| = 4$ and $|F_2| \geq 8$; so

$$g(B) \leq 24(1/8 + 1/4) - 17 + 6(1/2 + 1/2) = -2.$$

So we may assume that $F_1$ and $F_2$ are both equal to their refinements. Next, if $|F_1| \geq 5$ or $|F_2| \geq 5$ then

$$g(B) \leq 24(1/4 + 1/5) - 17 + 6(1/2 + 1/2) = -1/5,$$

so we may assume that $|F_1| = |F_2| = 4$. Now assume $x_1$ or $x_2$ is in at least three triangular-blocks of $G$. Then $n(B) \leq \frac{1}{2} + \frac{1}{3}$ and again

$$g(B) \leq 24(1/4 + 1/4) - 17 + 6(1/2 + 1/3) = 0.$$

So both $x_1$ and $x_2$ are each contained in exactly 2 triangular-blocks. Let $x_3$ and $x_4$ be the vertices of $F_1$ other than $x_1$ and $x_2$, such that $F_1 = x_1x_2x_3x_4x_1$.

*Case* 1. $V(F_1 \cap F_2) = \{x_1, x_2\}$.

Let $x_5$ and $x_6$ be the other two vertices of $F_2$, such that $F_2 = x_1x_2x_5x_6x_1$. Now $d_G(x_i) \geq 4$ for some $i \in [2]$, as $\{x_1, x_2\}$ is not an (18/7)-sparse set. By symmetry, assume $d_G(x_2) \geq 4$. Consider the triangular-block $B'$ of $G$ such that $B'$ contains $x_3x_2x_5$, and let $F_3$ and $F_5$ be the facial triangles of $B'$ containing $x_2x_3$ and $x_2x_5$, respectively. If $F_3$ contains a third vertex in $\{x_1, x_2, x_3, x_4, x_5, x_6\}$, it must be $x_6$ to avoid an (18/7)-sparse set of order 1. But then $F_5$ contains a vertex $v$ with $v \notin \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and $x_2vx_5x_6x_1x_4x_3x_2$ is a $C_7$ in $G$, a contradiction.

*Case* 2. $V(F_1 \cap F_2) \neq \{x_1, x_2\}$

Then $|V(F_1 \cap F_2)| = 3$. Without loss of generality we may assume that $x_4 \in V(F_2)$. Let $x_5$ be the fourth vertex of $F_2$; then $F_2 = x_1x_2x_4x_5x_1$. Note that $x_4x_1x_5x_4$ bounds a triangular-block $B_1$ of $G$ since $x_1$ is in exactly 2 triangular-blocks of $G$. Since $F_2$ is a facial cycle and $\{x_5\}$ cannot be a sparse set in $G$, $|B_1| \geq 4$. Since $x_1x_4$ is an edge on the outer cycle of $B_1$, Lemmas 2.3.2 and 2.3.1 imply that $B_1$ contains an $x_1$-$x_4$ path $P$ of length 3. Note that the path $x_3x_2x_4$ is contained in a triangular-block $B_2$ of $G$, and the facial triangle of $B$ containing $x_2x_3$ contains a vertex $v \notin \{x_1, x_2, x_3, x_4, x_5\}$ as $\{x_3\}$ cannot be an (18/7)-sparse set of $G$. But then $P \cup x_1x_2vx_3x_4$ is a $C_7$ in $G$, a contradiction. □

**Lemma 3.3.2.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$. If $B = B_3$ as in 2.4, then $g(B) \leq 0$.*

*Proof.* Note that each vertex of $B$ is a junction vertex, since $G$ has no degree-2 vertex. Hence,

$$g(B) \leq 24(1 + 3/4) - 17 \cdot 3 + 6(3/2) = 0,$$

as desired. □

**Lemma 3.3.3.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$. Suppose $G$ is drawn in the plane such that $B = B_{4,b}$ is drawn as in 2.4 with its outer face as its only hole. Then $g(B) \leq 0$.*

*Proof.* Let $a, b, c$ be the vertices of the outer cycle of $B$, which are all junction vertices (to avoid an (18/7)-sparse set consisting of two adjacent vertices). If there is a petal of $B$ with length at least 8, then

$$g(B) \leq 24(3 + 1/8 + 2/4) - 17 \cdot 6 + 6(1 + 3/2) = 0.$$

So we may assume that each petal of $B$ has length less than 8.

Suppose $B$ has a leaky petal, say $F_1$, and assume by symmetry that $bc \in E(F_1)$. Then $F_1$ is the union of $bc$, a $B$-path $P_1$ between $a$ and $b$, and a $B$-path $P_2$ between $a$ and $c$. Note that for each $i \in [2]$, $P_i$ has length 2 (or else $B \cup P_1 \cup P_2$ would contain $C_7$ or $F_1$ would have length at least 8), and let $v_i$ be the internal vertex of $P_i$. Now, the petal $F_2$ of $B$ containing $ac$ contains a $B$-path $Q$ between $a$ and $c$. Note that the length of $Q$ is between 3 and 5, and hence $Q \cup P_1 \cup B$ contains a $C_7$, a contradiction.

So we may assume that $B$ has no leaky petal. Since $a, b, c$ are all junction vertices of $G$ and there is no $B$-path of length 4 or 5 in $G$ (to avoid forming a $C_7$ with $B$), each petal of $B$ has length 4. Let $F_1, F_2, F_3$ be the petals containing $ac, ab, bc$, respectively.

Let $x$ and $y$ be the vertices of $F_1$ that are not in $B$ so that $F_1 = axyca$. Then $F_2$ contains a vertex $z$ that is not in $B \cup F_1$ (to avoid an (18/7)-sparse set of order at most 2 in $G$) and such $z$ is unique to avoid a $C_7$ in $B \cup F_1 \cup F_2$. So $z$ has two neighbors on $B \cup F_1$ that are in $F_2$: $b$ and $x$, $a$ and $c$, or $a$ and $y$. If these neighbors of $z$ are $b$ and $x$ then $B \cup F_1 \cup F_2$ contains $C_7$, a contradiction. If the neighbors are $a$ and $c$ then $b$ is not a junction vertex, a contradiction. So the neighbors of $z$ are $a$ and $y$, and the fourth edge of $F_2$ is $by$.

By applying the same reasoning to the petals $F_1$ and $F_3$, we deduce that $bx \in E(F_3)$, and this is a contradiction because $b$ and $x$ are separated by the cycle $azyca$. □

**Lemma 3.3.4.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$ and let $B$ be a triangular-block of $G$. Suppose $G$ is drawn in the plane such that $B = B_{4,a}$ is drawn as in 2.4 with its outer face as its only hole. Then $g(B) \leq 0$.*

*Proof.* Lemma 2.2.3 shows that $B$ has no triangular hole. Let $C = abcda$ be the outer cycle of $B$, with $a, c$ of degree 3 in $B$. Note that $b, d$ must be junction vertices as $\{b\}$ and $\{d\}$ cannot be an (18/7)-sparse set. If both $a$ and $c$ are junction vertices, then

$$g(B) \leq 24(2 + 4/4) - 17 \cdot 5 + 6(4/2) = -1.$$

So we may assume by symmetry that $c$ is not a junction vertex. Then $a$ is a junction vertex (as $\{a, c\}$ is not an (18/7)-sparse set), and the path $bcd$ is contained a petal of $B$, say $F_1$.

We may assume $|F_1| = 4$, or else, $|F_1| \geq 5$ and

$$g(B) \leq 24(2 + 2/4 + 2/5) - 17 \cdot 5 + 6(1 + 3/2) = -2/5.$$

Then $F_1$ is not leaky. Let $F_1 = bcdvb$, where $v \notin V(B)$.

Let $F_2, F_3$ denote the petals of $B$ containing $ab, ad$, respectively. Then $F_2 \neq F_3$, or else $\{a, c\}$ would be an (18/7)-sparse set in $G$. We may assume $|F_2| = 4$; for otherwise, $|F_2| \geq 5$ and $|F_3| \geq 5$, and

$$g(B) \leq 24(2 + 2/4 + 2/5) - 17 \cdot 5 + 6(1 + 3/2) = -2/5.$$

Note that $F_2$ contains a unique vertex, say $w$, not contained in $B \cup F_1$, to avoid a $C_7$ in $B \cup F_1 \cup F_2$. Let $e_1, e_2$ be the edges of $F_2 - ab$ incident with $a, b$, respectively. Then $e_1, e_2 \notin E(B \cup F_1)$, or else $\{a, c\}$ or $\{b, c\}$ would be an (18/7)-sparse set in $G$. Since $e_1$ and $e_2$ cannot both be incident to $w$, either $e_1 = aw$ and $e_2 = bd$, or $e_1 = av$ and $e_2 = bw$.

If $e_1 = aw$ and $e_2 = bd$ then $w$ is incident to $a$ and $d$. Then $F_3$ consists of $ad$ and a $(B \cup F_1)$-path between $a$ and $d$. This path has length at least 3 because $B$ is a triangular-block, and it therefore has length at least 7 to avoid forming a $C_7$ with $B \cup F_1$. But then

$|F_3| \geq 8$, and

$$g(B) \leq 24(2 + 3/4 + 1/8) - 17 \cdot 5 + 6(1 + 3/2) = -1.$$

So $e_1 = av$ and $e_2 = bw$, which implies that $F_2 = abwva$. Since $\{c, d\}$ is not an (18/7)-sparse set in $G$, $F_3$ contains a $(B \cup F_1 \cup F_2)$-path between $a$ and $d$ or between $d$ and $v$. But then $|F_3| \geq 8$ to avoid a $C_7$ in $B \cup F_1 \cup F_3$ or $B \cup F_1 \cup F_2 \cup F_3$, and so

$$g(B) \leq 24(2 + 3/4 + 1/8) - 17 \cdot 5 + 6(1 + 3/2) = -1,$$

as desired. □

### 3.4 The 2-Connected Case

The lemmas in the previous sections combine to show the following.

**Lemma 3.4.1.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$, and let $B$ be a triangular-block of $G$ with only one hole, and no triangular hole unless $B$ is a triangulation. Then $g(B) \leq 0$, with the exception of the cases shown in 3.1.*

We next show that this is true even for triangular-blocks with more than one hole.

**Lemma 3.4.2.** *Let $G \in \mathcal{P}_n$ with $n \geq 7$. If $B$ is a triangular-block of $G$ then $g(B) \leq 0$, with the exception of the cases shown in 3.1.*

*Proof.* We draw $G$ so that $B$ is drawn as shown in Figures 2.4–2.7 and the outer face of $B$ is a hole. If $B$ has no other holes, then the statement follows from Lemma 3.4.1. So let $\mathcal{F}$ be the set of facial cycles of $B$ bounding a hole other than the outer face of $B$. Note that each $F \in \mathcal{F}$ satisfies $|F| \leq 4$.

Let $G'$ be obtained from $G$ by deleting all vertices and edges in the interior of $F$ for all $F \in \mathcal{F}$. Let $e'(B)$, $n'(B)$, $f'(B)$, and $g'(B)$ be the values of $e(B)$, $n(B)$, $f(B)$, and $g(B)$

calculated with respect to $G'$. By Lemma 3.4.1, we have $g'(B) \leq 0$, unless the flower of $B$ is an exceptional one shown in 3.1. Note that $e(B) = e'(B)$, and $n(B) \leq n'(B)$ because for each vertex $v$ of $B$, the number of triangular-blocks in $G$ containing $v$ is at least the number of triangular-blocks in $G'$ containing $v$. Also, each $F \in \mathcal{F}$ contributes $1$ towards $f'(B)$ and at most $4/4$ towards $f(B)$, because $|F| \leq 4$, and each petal of $B$ has length at least four since $B$ is a triangular-block. So $f(B) \leq f'(B)$. Therefore, $g(B) \leq g'(B) \leq 0$, unless the flower of $B$ is an exceptional one shown in 3.1. □

We can now combine Lemma 2.2.10 with Lemma 3.4.2 to show that Theorem 2.1.1 holds for graphs in $\mathcal{P}_n$ with $n \geq 7$.

**Theorem 3.4.3.** *Let $G \in \mathcal{P}_n$ with $n$ vertices, where $n \geq 7$. Then $e(G) \leq \frac{18n}{7} - \frac{48}{7}$.*

*Proof.* Let $\mathcal{B}$ be the collection of triangular-blocks of $G$, and let $\mathcal{B}_1 \subseteq \mathcal{B}$ be the collection of all exceptional triangular-blocks of $G$ with flower shown in 3.1.

We now define a suitable partition $\mathcal{P}$ of $\mathcal{B}$. Let $B \in \mathcal{B}$. If $B \in \mathcal{B}_1$, then (by Lemma 3.2.4) let $e_1, e_2, e_3, e_4$ be the four trivial blocks in $G$ that are contained in the flower of $B$, and let $\{B, e_1, e_2, e_3, e_4\} \in \mathcal{P}$. If $B \notin \mathcal{B}_1$ and is not a trivial block in $G$ contained in the flower of a triangular-block in $\mathcal{B}_1$, let $\{B\} \in \mathcal{P}$. To show that $\mathcal{P}$ is a partition of $\mathcal{B}$, it suffices to show that there is no trivial triangular-block $e$ that is in the flowers of two blocks in $\mathcal{B}_1$. This follows from Lemma 3.2.4, because $e$ would be incident with two faces of length at most four.

By Lemma 2.2.10, it suffices to show that each set $\mathcal{B}' \in \mathcal{P}$ satisfies $\sum_{B \in \mathcal{B}'} g(B) \leq 0$. If $\mathcal{B}' = \{B\}$ for some triangular-block $B$ in $G$, then $B$ is not exceptional and it follows from Lemma 3.4.2 that $\sum_{B \in \mathcal{B}'} g(B) = g(B) \leq 0$. Otherwise $\mathcal{B}' = \{B, e_1, e_2, e_3, e_4\}$ for some exceptional block $B$ and the four trivial blocks $e_1, e_2, e_3, e_4$ in the flower of $B$. For each $i \in [4]$, the edge $e_i$ is incident with a face of length at least five by Lemma 3.2.4, and both its ends are in at least three triangular-blocks (using the assumption that $G$ has no degree-2 vertices), and so

$$g(e_i) \leq 24(1/4 + 1/5) - 17 + 6(2/3) = -11/5.$$

Since

$$g(B) \leq 24(4 + 4/4) - 17 \cdot 8 + 6(3 + 2/2) = 8$$

using Lemma 2.4.3, we calculate that $\sum_{B \in \mathcal{B}'} g(B) \leq 8 + 4 \cdot (-11/5) = -4/5 < 0$, as desired. Therefore each $\mathcal{B}' \in \mathcal{P}$ satisfies $\sum_{B \in \mathcal{B}'} g(B) \leq 0$, and Lemma 2.2.10 implies that $e(G) \leq \frac{18n}{7} - \frac{48}{7}$. □

### 3.5 The proof of Theorem 2.1.1

We now obtain Theorem 2.1.1 from Theorem 2.1.2 by considering small separations and sparse sets.

*Proof of Theorem 2.1.1.* Let $\mathcal{G}_0$ be the class consisting of all planar graphs with at most six vertices or with a planar drawing in $\mathcal{P}_n$ for some $n \geq 7$. For each positive integer $i$, let $\mathcal{G}_i$ be the union of $\mathcal{G}_{i-1}$ and the class of $C_7$-free planar graphs obtained from any of the following operations:

(1) Add a set $S$ of vertices, and edges with at least one vertex in $S$ to a graph in $\mathcal{G}_{i-1}$ such that $|S| \leq 4$ and $S$ is an (18/7)-sparse set in the new graph.

(2) Take the disjoint union of two graphs in $\mathcal{G}_{i-1}$.

(3) Glue two graphs in $\mathcal{G}_{i-1}$ together at a vertex.

Let $\mathcal{G} = \cup_{i \geq 0} \mathcal{G}_i$, and note that $\mathcal{G}$ is precisely the class of $C_7$-free planar graphs. We will show that operations (1)–(3) can only decrease density.

**Claim 3.5.1.** *Let $G_1$ be an $n_1$-vertex $C_7$-free planar graph with $e(G_1) \leq \frac{18n_1}{7} - c_1$ for some constant $c_1$. Let $G$ be an $n$-vertex $C_7$-free planar graph obtained from $G_1$ by operation $(1)$ by adding an $(18/7)$-sparse set $S$.*

$(i)$ *If $|S| = 1$, then $e(G) \leq \frac{18n}{7} - c_1 - \frac{4}{7}$.*

$(ii)$ *If $|S| = 2$, then $e(G) \leq \frac{18n}{7} - c_1 - \frac{1}{7}$.*

$(iii)$ *If $|S| = 3$, then $e(G) \leq \frac{18n}{7} - c_1 - \frac{5}{7}$.*

$(iv)$ *If $|S| = 4$, then $e(G) \leq \frac{18n}{7} - c_1 - \frac{2}{7}$.*

*Proof.* When $|S| = 1$ there are at most 2 edges incident with the vertex in $S$, and $n = n_1+1$, so we have

$$e(G) \leq e(G_1) + 2 \leq \frac{18n_1}{7} - c_1 + 2 = \frac{18n}{7} - c_1 - \frac{4}{7}.$$

We omit the calculations for $|S| \in \{2, 3, 4\}$, which are very similar. □

We perform a similar calculation for operations (2) and (3).

**Claim 3.5.2.** *For each $i \in [2]$, let $G_i$ be an $n_i$-vertex $C_7$-free planar graph with $e(G_i) \leq \frac{18n_i}{7} - c_i$ for a constant $c_i$. Let $G$ be an $n$-vertex $C_7$-free planar graph obtained from $G_1$ and $G_2$ via operation $(2)$ or $(3)$.*

$(i)$ *If $G$ is obtained via operation $(2)$, then $e(G) \leq \frac{18n}{7} - c_1 - c_2$.*

$(ii)$ *If $G$ is obtained via operation $(3)$, then $e(G) \leq \frac{18n}{7} - c_1 - c_2 + \frac{18}{7}$.*

*Proof.* If $G$ is obtained via operation (3), then

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) \\ &\leq \frac{18n_1}{7} - c_1 + \frac{18n_2}{7} - c_2 \\ &\leq \frac{18(n+1)}{7} - c_1 - c_2 \\ &= \frac{18n}{7} - c_1 - c_2 + \frac{18}{7}. \end{aligned}$$

If $G$ is obtained via operation (2), then $n_1 + n_2 = n$ and the above calculation gives $e(G) \le \frac{18n}{7} - c_1 - c_2$. □

Let $\mathcal{H}_0$ be the class of planar graphs with at least 3 and at most 6 vertices, and for each positive integer $i$, let $\mathcal{H}_i$ be the union of $\mathcal{H}_{i-1}$ and the class of $C_7$-free planar graphs obtained from applying any of operations (1)–(3) to graphs in $\mathcal{H}_{i-1}$. Let $\mathcal{H} = \bigcup_{i \ge 0} \mathcal{H}_i$. Since Claims 3.5.1–3.5.2 show that operations (1)–(3) only decrease density, it follows from Theorem 2.1.2 that every $n$-vertex planar graph $G$ with $n \ge 7$ in $\mathcal{G} \setminus \mathcal{H}$ satisfies $e(G) \le \frac{18n}{7} - \frac{48}{7}$. We now inductively bound the number of edges of graphs in $\mathcal{H}$.

**Claim 3.5.3.** *Let $k$ be non-negative integer, and let $G$ be a $C_7$-free $n$-vertex planar graph in $\mathcal{H}$ with $n \ge 3 + 2k$. Then either*

$(a)$ $e(G) \le \frac{18n}{7} - \frac{30}{7} - \frac{k}{7}$, *or*

$(b)$ *$G$ is connected and has at most two blocks (maximal connected subgraph without a cut vertex), each of which is a triangulation on five or six vertices.*

*Proof.* Note that $n \ge 3$. We apply induction on $n$. First, suppose $n \le 6$. Then $k = 0$ or $k = 1$, and $G \in \mathcal{H}_0$. If $e(G) \le 3n - 7$ then $n \le 6$ implies $e(G) \le 3n - 7 \le \frac{18n}{7} - \frac{31}{7}$ and $(a)$ holds. So $e(G) = 3n - 6$ and, hence, $G$ is a triangulation. If $n \le 4$ then $k = 0$ and $e(G) \le \frac{18n}{7} - \frac{30}{7}$. If $n \in \{5, 6\}$ then $(b)$ holds (with one block).

So we may assume that $n \ge 7$. Then $G \notin \mathcal{H}_0$, and thus, either $G$ is obtained from a graph $G_1 \in \mathcal{H}$ by applying a single operation (1), or $G$ is obtained by applying a single operation (2) or (3) to graphs $G_1$ and $G_2$ in $\mathcal{H}$.

First suppose that $G$ is obtained from an $n_1$-vertex graph $G_1 \in \mathcal{H}$ by applying a single operation (1), and that $G$ cannot be obtained from applying operation (2) or (3) to graphs in $\mathcal{H}$. Since $n \ge 7$ it follows that $n_1 \ge 3$, so by induction hypothesis, either $(a)$ or $(b)$ holds for $G_1$. Suppose $(a)$ holds for $G_1$. Thus, since $n_1 \ge 3 + 2(k - |S|/2)$, we have $e(G_1) \le \frac{18n_1}{7} - \frac{30}{7} - \frac{(k-|S|/2)}{7}$. If $|S| \in \{1, 2\}$, then Claim 3.5.1 implies that $e(G) \le \frac{18n}{7} - \frac{30}{7} - \frac{(k-|S|/2)}{7} - \frac{1}{7} \le \frac{18n}{7} - \frac{30}{7} - \frac{k}{7}$. Similarly, if $|S| \in \{3, 4\}$, then Claim 3.5.1

implies that $e(G) \leq \frac{18n}{7} - \frac{30}{7} - \frac{(k-|S|/2)}{7} - \frac{2}{7} \leq \frac{18n}{7} - \frac{30}{7} - \frac{k}{7}$. In either case, $(a)$ holds for $G$. So assume $(b)$ holds for $G_1$. Then $n_1 \leq 11$. It is a straightforward calculation to check that $e(G_1) \leq \frac{18n_1}{7} - \frac{24}{7}$ (with equality when $G_1$ is a 6-vertex triangulation). We may assume that the (18/7)-sparse set $S$ induces a connected graph, or else we may replace $S$ with one of its subsets. We consider two cases. First suppose that $|S| > 1$. If $G$ is connected, then $G$ is 2-connected because $G$ cannot be obtained from applying operation (2) or (3) to graphs in $\mathcal{H}$. By Menger's theorem applied to $S$ and $V(G_1)$, there are two vertex-disjoint paths from $S$ to $G_1$. Let $x$ and $y$ be the ends of these paths in $G_1$. Since $G[S]$ is connected and $2 \leq |S| \leq 4$, the graph induced by $S \cup \{x, y\}$ has an $x$-$y$ path of length $m$ for some $m \in \{3, 4, 5\}$. Since $d_{G_1}(x, y) \leq 2$ and $(b)$ holds for $G_1$ we know that $G_1$ has an $x$-$y$ path of length $7 - m$ by applying Lemmas 2.3.2 and 2.3.1 to each block of $G_1$. But then $G$ has a $C_7$-subgraph, a contradiction. So $G$ is disconnected. Since $|S| > 1$ and $G$ cannot be obtained from applying operation (2) or (3) to graphs in $\mathcal{H}$, it follows that $|S| = 2$ and $S$ has no neighbors in $G_1$. Let $G_2$ be the graph induced by $S$, which is just a single edge. Then $e(G_2) = \frac{18n(G_2)}{7} - \frac{29}{7}$, and $G$ is obtained by applying operation (3) to $G_1$ and $G_2$. So $n \leq 13$ and $k \leq 5$, and Claim 3.5.2(i) with $c_1 = \frac{24}{7}$ and $c_2 = \frac{29}{7}$ implies that $e(G) \leq \frac{18n}{7} - \frac{53}{7}$, so $(a)$ holds for $G$.

So we may assume that $|S| = 1$, which implies that $S$ has at most two neighbors in $G_1$. Suppose $S$ has one neighbor in $G_1$. Then $n \leq 12$ so $k \leq 4$, and Claim 3.5.2 with $G_2$ as a single edge implies that $e(G) \leq \frac{18n}{7} - \frac{24+29-18}{7} \leq \frac{18n}{7} - \frac{30+k}{7}$, so $(a)$ holds for $G$. So we may assume that $S$ has two neighbors in $G_1$. Both neighbors of $S$ are in a block of $G_1$ that is a 5-vertex triangulation, or else Lemmas 2.3.2 and 2.3.1 and the assumption that $(b)$ holds for $G_1$ imply that $G_1$ has a length-5 path between the two neighbors of $S$, which gives a $C_7$-subgraph of $G$ when concatenated with the two edges incident with $S$. Since $n \geq 7$ and $G_1$ has a 5-vertex block, it follows that $G_1$ has two blocks, one of which is a 5-vertex triangulation. Then either $n = 10$ and $e(G) = 20$ (if both blocks of $G_1$ have 5 vertices) or $n = 11$ and $e(G) = 23$ (if one block of $G_1$ has 6 vertices). In the former case,

$k \leq 3$ and $e(G) \leq \frac{18n}{7} - \frac{40}{7}$, and in the latter case $k \leq 4$ and $e(G) \leq \frac{18n}{7} - \frac{37}{7}$, so in either case $(a)$ holds for $G$.

Next suppose that $G$ is obtained by applying a single operation (2) or (3) to graphs $G_1$ and $G_2$ in $\mathcal{H}$. For each $i \in [2]$, let $n_i = |G_i|$ and let $k_i$ be a non-negative integer such that $n_i \geq 3 + 2k_i$. Note that we may choose $k_1, k_2$ such that $k_1 + k_2 \in \{k-1, k\}$. In particular, $k \leq k_1 + k_2 + 1$. Also note that

$$n \geq n_1 + n_2 - 1 \geq (3 + 2k_1) + (3 + 2k_2) - 1 = 3 + 2(k_1 + k_2 + 1).$$

First suppose that outcome $(a)$ holds for both $G_1$ and $G_2$; so for $i = 1, 2$, $e(G_i) \leq \frac{18n_i}{7} - \frac{30}{7} - \frac{k_i}{7}$. Then Claim 3.5.2 with $c_i = \frac{30+k_i}{7}$ for $i = 1, 2$ shows that

$$e(G) \leq \frac{18n}{7} - \frac{30}{7} - \frac{k_1 + k_2 + 12}{7} \leq \frac{18n}{7} - \frac{30}{7} - \frac{k}{7},$$

where the second inequality holds because $k \leq k_1 + k_2 + 1$. So $(a)$ holds for $G$.

Now suppose $(a)$ holds for $G_1$ and $(b)$ holds for $G_2$. Then Claim 3.5.2 shows that $(a)$ holds for $G$ with $k = k_1 + 2$ if $G_2$ has one block and $k = k_1 + 4$ if $G_2$ has two blocks.

So we may assume that $(b)$ holds for both $G_1$ and $G_2$. Suppose $(b)$ does not hold for $G$. Then $G$ consists of 2, 3 or 4 blocks, each of which is a triangulation with five or six vertices. Note that a triangulation $B$ with 5 or 6 vertices satisfies $e(B) \leq \frac{18n(B)}{7} - \frac{24}{7}$. If $G$ has 2 blocks, then $G$ is disconnected because $(b)$ does not hold for $G$. Then $k \leq 4$, and Claim 3.5.2(i) with $c_1 = c_2 = \frac{24}{7}$ implies that $e(G) \leq \frac{18n}{7} - \frac{48}{7}$, so $(a)$ holds for $G$. If $G$ has 4 blocks, then $n \leq 22$ and so $k \leq 9$. Then applying Claim 3.5.2 three times with $c_2 = \frac{24}{7}$ each time implies that $e(G) \leq \frac{18n}{7} - \frac{24}{7} - 3 \cdot \frac{24}{7} + 3 \cdot \frac{18}{7} = \frac{18n}{7} - \frac{42}{7}$, and since $k \leq 9$ it follows that $(a)$ holds for $G$. If $G$ is connected and has 3 blocks, then $n \leq 16$ and so $k \leq 6$. Applying Claim 3.5.2(ii) twice with $c_2 = \frac{24}{7}$ each time implies that $e(G) \leq \frac{18n}{7} - \frac{24}{7} - 2 \cdot \frac{24}{7} + 2 \cdot \frac{18}{7} = \frac{18n}{7} - \frac{36}{7}$, and $(a)$ holds for $G$ since $k \leq 6$. So $G$ is disconnected and has 3 blocks, so $n \leq 18$ and $k \leq 7$. Applying Claim 3.5.2(i) once with $c_2 = \frac{24}{7}$ and Claim 3.5.2(ii) once with $c_2 = \frac{24}{7}$

implies that $e(G) \leq \frac{18n}{7} - \frac{24}{7} - 2 \cdot \frac{24}{7} + \frac{18}{7} = \frac{18n}{7} - \frac{54}{7}$, and $(a)$ holds for $G$ since $k \leq 7$. $\square$

Let $G$ be an $n$-vertex $C_7$-free planar graph with $n > 38$. If $G \in \mathcal{G} \setminus \mathcal{H}$ then $e(G) \leq \frac{18n}{7} - \frac{48}{7}$. If $G \in \mathcal{H}$, then Claim 3.5.3 applies with $k = 18$ and outcome $(a)$ holds, so $e(G) \leq \frac{18n}{7} - \frac{48}{7}$, as desired. $\square$

### 3.6 Future Work

As discussed in Section 2.1, the conjecture of Ghosh et al. that $\mathrm{ex}_{\mathcal{P}}(n, C_\ell) \leq \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$ for all $\ell \geq 7$ and all sufficiently large $n$ is false for all $\ell \geq 11$, but true for $\ell = 7$ by Theorem 2.1.1. Is this conjecture true for $\ell \in \{8, 9, 10\}$? We believe that the answer is yes for $\ell = 8$, and that this can be proven using existing techniques and extensive case work. A complete draft was made, but is still in revision. However, when $\ell = 9$, $\mathrm{ex}_{\mathcal{P}}(n, C_\ell) > \frac{3(\ell-1)}{\ell}n - \frac{6(\ell+1)}{\ell}$ for infinitely many $n$. Instead, we plan to prove $\mathrm{ex}_{\mathcal{P}}(n, C_9) \leq \frac{8}{3}n - \frac{13}{3}$ for every sufficiently large $n$ in a forthcoming paper. For $\ell = 10$ the correct answer is unclear.

We mention one other direction that could be approached using techniques from this dissertation. For each $\ell \geq 4$, let $\Theta_\ell$ denote the family of graphs (called *theta graphs*) obtained by adding an edge between two non-adjacent vertices of $C_\ell$. Sharp upper bounds for $\mathrm{ex}_{\mathcal{P}}(n, \Theta_\ell)$ are known for $\ell \in \{4, 5, 6\}$ by results of Lan, Shi and Song [28] and Ghosh, Győri, Paulos, Xiao, and Zamora [17], but no sharp upper bound is known for $\ell \geq 7$. Due to the importance of triangular-blocks with a chord in our proof, we believe that our techniques could make progress towards a sharp upper bound on $\mathrm{ex}_{\mathcal{P}}(n, \Theta_7)$.

# CHAPTER 4

# PLANAR TURAN NUMBER OF LONG CYCLES

## 4.1 Introduction

In this chapter, we continue to analyze the planar Tur/'an number of $C_k$, the cycle of length $k$ and $k \geq 11$. In [16], Ghosh, Győri, Martin, Paulos, and Xiao took the made the following conjecture.

**Conjecture 4.1.1** ([16]). $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq \frac{3(k-1)}{k}n - \frac{6(k+1)}{k}$ *for all* $k \geq 7$ *and all sufficiently large* $n$.

It was recently shown independently by Győri, Li, and Zhou [19] and the present authors [40] that Conjecture 4.1.1 holds for $k = 7$. However, the conjecture was disproved for all $k \geq 11$ by Cranston, Lidický, Liu, and Shantanam [6], and later also by Lan and Song [30] and Győri, Varga, and Zhu [20], using the fact that planar triangulations with at least $11$ vertices need not be Hamiltonian. Specifically, Moon and Moser [34] showed that for each $k \geq 11$ there exists a $C_k$-free planar triangulation with $(2k/7)^{\log_2 3}$ vertices; this is tight up to a constant factor by a result of Chen and Yu [5].

We next describe the construction that disproves Conjecture 4.1.1, following Cranston et al. [6]. For a graph $G$, we use $V(G)$ and $E(G)$ to denote its vertex set and edge set, respectively, and we write $v(G)$ for $|V(G)|$ and $e(G)$ for $|E(G)|$. Let $k$ and $n$ be integers with $k \geq 5$ and $n \geq k+1$. Let $G$ be an $n$-vertex planar graph with girth $k+1$, each vertex having degree $2$ or $3$, and $\frac{k+1}{k-1}(n-2)$ edges; such a graph exists for infinitely many integers $n$ [6, Lemma 2]. Let $G'$ be obtained from $G$ by sudividing each edge and then *substituting* for each vertex of $G$ a $C_k$-free planar triangulation $B$ on as many vertices as possible, which means that each vertex $v$ of $G$ is replaced by a copy of $B$ and that $\deg_G(v)$ vertices of $B$ on a facial triangle are identified with the neighbors of $v$ in $G$. Then $G'$ is a $C_k$-free graph.

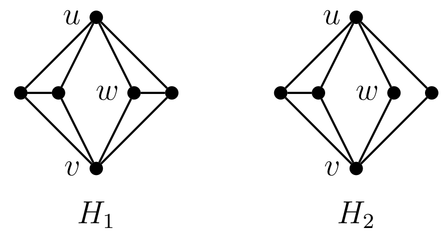


Figure 4.1: For $i = 1, 2$ let $C_i$ be the outer cycle of $H_i$. Then $(H_1, C_1)$ is a circuit graph, but $(H_2, C_2)$ is not.

Cranston et al. conjecture that $G'$ has $\mathrm{ex}_{\mathcal{P}}(v(G'), C_k)$ edges [6], but acknowledge that this will be difficult to prove, in part because the number of vertices of $B$ is only known up to a constant factor. Because of this, they also made the following weaker conjecture.

**Conjecture 4.1.2** ([6])**.** *There is a constant $D$ so that for all $k$ and sufficiently large $n$ we have* $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - \frac{Dn}{k^{\log_2 3}}$.

While this might not provide an exact upper bound, it is tight up to the constant $D$ when $k \geq 11$. This follows from Cranston et al.'s construction, and also a more recent simpler construction of Győri, Varga, and Zhu [20] which shows that $D$ can be at most $12$. We prove this conjecture with $D = 1/4$.

**Theorem 4.1.3.** $\mathrm{ex}_{\mathcal{P}}(n, C_k) \leq 3n - 6 - \frac{n}{4k^{\log_2 3}}$ *for all $k \geq 4$ and $n \geq k^{\log_2 3}$.*

To prove this, we first show that it suffices to consider graphs that are close to being 3-connected. A *circuit graph* is a pair $(G, C)$ where $G$ is a 2-connected plane graph and $C$ is a facial cycle of $G$ so that for any 2-cut $S$ of $G$, each component of $G - S$ contains a vertex of $C$. For example, for $i = 1, 2$, let $C_i$ be the outer cycle of the plane graph $H_i$ shown in Figure 4.1. Then $(H_1, C_1)$ is a circuit graph (even though $H_1$ is not 3-connected), while $(H_2, C_2)$ is not a circuit graph because $H_2 - \{u, v\}$ has a component (the vertex $w$) that does not contain a vertex of $C_2$. We show that if there is a counterexample to Theorem 4.1.3, then there is a circuit graph counterexample.

We then show that a dense circuit graph has a large near triangulation as a subgraph, where a *near triangulation* is a plane graph in which every face is bounded by a triangle except possibly the outer face. For a 2-connected plane graph $G$ with outer cycle $C$ we write $m(G)$ for the number of interior edges needed to make $G$ a near triangulation, so

$$m(G) = 3v(G) - 6 - e(G) - (|C| - 3).$$

We will apply the following theorem with $t = \lceil k^{\log_2 3} \rceil$.

**Theorem 4.1.4.** *For all $t \geq 4$, if $(G, C)$ is a circuit graph with $v(G) \geq t$ and with outer cycle $C$ so that $m(G) < \frac{v(G)-(t-1)}{3t-7}$, then $G$ has a near triangulation subgraph $T$ with $v(T) \geq t$.*

This bound is sharp, as shown via an iterative construction. The starting graph $G_1$, shown in Figure 4.2(a), consists of four copies of a $(t-1)$-vertex triangulation arranged in a cyclic order. Then $m(G_1) = \frac{v(G_1)-(t-1)}{3t-7} = 1$, but $G_1$ has no near triangulation subgraph with at least $t$ vertices. For each $i \geq 2$, let $G_i$ be obtained from $G_{i-1}$ by adding three copies of a $(t-1)$-vertex triangulation arranged in a cyclic order; $G_2$ is shown in Figure 4.2(b). This operation adds $3t-7$ vertices and one new interior non-triangular face of size four, so $m(G_k) = \frac{v(G_k)-(t-1)}{3t-7} = k$ for all $k \geq 1$, but $G_k$ has no near triangulation subgraph with at least $t$ vertices.

Since every near triangulation is a circuit graph we can apply the following theorem of Chen and Yu [5].

**Theorem 4.1.5** ([5])**.** *For all $k \geq 3$, if $(G, C)$ is a circuit graph with at least $k^{\log_2 3}$ vertices, then $G$ has a cycle of length at least $k$.*

Finally, it is straightforward to show (see Lemma 4.3.1) that every near triangulation with a cycle of length at least $k$ also has a cycle of length exactly $k$, and therefore Theorem 4.1.3 follows from Theorems 4.1.4 and 4.1.5.

In fact, we will prove a result that is stronger than Theorem 4.1.3. For each integer $k \geq 4$, we write $\Theta_k$ for the family of graphs (called *theta graphs*) obtained from $C_k$ by

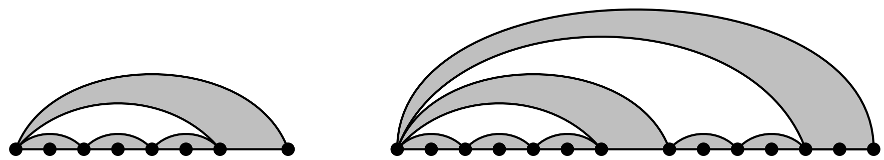


Figure 4.2: Each shaded region is a triangulation with $t-1$ vertices.

adding a single edge, and we write $\theta_k$ for the graph obtained from $C_k$ by adding an edge that forms a triangle with two consecutive edges of $C_k$. The number $\mathrm{ex}_{\mathcal{P}}(n, \Theta_k)$ is known only when $k \in \{4, 5, 6\}$, by results of Lan, Shi, and Song [28] and Ghosh, Győri, Paulos, Xiao, and Zamora [17]. We prove the following, which again is tight up to the constant $1/4$ by the construction of Cranston et al [6].

**Theorem 4.1.6.** $\mathrm{ex}_{\mathcal{P}}(n, \theta_k) \leq 3n - 6 - \frac{n}{4k^{\log_2 3}}$ *for all* $k \geq 4$ *and* $n \geq k^{\log_2 3}$.

This also holds with $\Theta_k$ in place of $\theta_k$ because $\theta_k \in \Theta_k$. Interestingly, this bound may not hold if we replace $\theta_k$ with another graph in $\Theta_k$; we discuss this in more detail after Lemma 4.3.1.

After describing all relevant notation, we will prove Theorem 4.1.4 in Section 4.2 and then prove Theorem 4.1.6 in Section 4.3. In Section 4.4 we will discuss some consequences of Theorems 4.1.3 and 4.1.4 and some related questions.

For any positive integer $k$, we let $[k] = \{1, \ldots, k\}$. It is well known that if $G$ is a 2-connected plane graph then each face of $G$ is bounded by a cycle of $G$ called a *facial cycle*. The *interior* of a cycle $C$ in a plane graph is defined to be the subgraph of $G$ consisting of all edges and vertices of $G$ contained in the closed disc of the plane bounded by $C$. A path or cycle will be represented as a sequence of vertices such that consecutive vertices in the sequence are adjacent. For instance, $x_1x_2x_3\ldots x_k$ represents a path of length $k-1$ in which $x_ix_{i+1}$, $i \in [k-1]$, are the edges of the path, and $x_1x_2x_3\ldots x_kx_1$ represents a cycle of length $k$ with edges $x_kx_1$ and $x_ix_{i+1}$, $i \in [k-1]$. For any distinct vertices $x, y$ in a graph $G$, if $x, y$ are on a path $P$ then $xPy$ denotes the subpath of $P$ between $x$ and $y$. For any cycle $C$ in a plane graph and any two distinct vertices $x, y$ on $C$, we use $xCy$ to denote the subpath of $C$ from $x$ to $y$ in clockwise order.

## 4.2 Finding a large near triangulation

In this section we will prove Theorem 4.1.4.

*Proof of Theorem 4.1.4.* Suppose that the theorem statement is false, and let $G$ be a counterexample with $m(G)$ as small as possible. This means that $v(G) \geq t$ and $m(G) < \frac{v(G)-(t-1)}{3t-7}$ but $G$ contains no near triangulation on at least $t$ vertices. So $m(G) > 0$, as otherwise $G$ is a near triangulation on at least $t$ vertices, a contradiction. Moreover, $G$ is not a cycle as otherwise $m(G) = v(G)-3$, contradicting the assumption that $m(G) < \frac{v(G)-(t-1)}{3t-7}$. Note that if $C'$ is a cycle in $G$ and $H$ is the interior of $C'$, then $(H, C')$ is also a circuit graph. Step-by-step, we will show that the behavior of the interior non-triangular faces of $G$ is quite restricted.

**Claim 4.2.1.** *If $F$ is a non-triangular interior facial cycle of $G$, then $F \cap C$ is either empty or is a path.*

*Proof.* Suppose that there is a non-triangular interior facial cycle $F$ so that $F \cap C$ has more than one component. Note that $m(G) \geq |F|-3$. Let $x_1, x_2, \ldots, x_s$ be the vertices in $F \cap C$ listed in clockwise order on $C$. Note that $s \geq 2$ since $F \cap C$ has more than one component. For each $i \in [s]$, let $C'_i = x_iCx_{i+1} \cup x_iFx_{i+1}$, where $x_{s+1} = x_1$. If $C'_i$ is not a cycle then let $T_i = C'_i$; in this case $T_i$ has just one edge. If $C'_i$ is a cycle then let $T_i$ be the interior of $C'_i$. Let $J \subseteq [s]$ so that $C'_i$ is a cycle if and only if $i \in J$. Then $J$ is non-empty, as $G$ is not a cycle. Note that $(T_i, C'_i)$ is a circuit graph for all $i \in J$, and that $m(T_i) < m(G)$ for all $i \in J$ because $F$ is not a facial cycle of $T_i$. We will argue that some $T_i$ with $i \in J$ contradicts the choice of $G$.

We first claim that there is some $i \in J$ so that $v(T_i) \geq t$. Suppose not. Then $v(G) \leq$

$s(t-2)$. We will show that

$$(3t-7)\cdot(|F|-3)+(t-1)\geq s(t-2). \tag{4.1}$$

If $s\leq 4$, then

$$(3t-7)\cdot(|F|-3)+(t-1)\geq 4(t-2)\geq s(t-2),$$

and if $s\geq 5$, then

$$(3t-7)\cdot(|F|-3)+(t-1)\geq(3t-7)(s-3)+(t-1)\geq s(t-2).$$

So (1) holds. Then

$$(3t-7)\cdot m(G)+(t-1)\geq(3t-7)\cdot(|F|-3)+(t-1)\geq v(G),$$

a contradiction. So $v(T_i)\geq t$ for some $i\in J$.

Let $J'\subseteq J$ so that $v(T_i)\geq t$ if and only if $i\in J'$. Note that $m(T_i)>0$ for all $i\in J'$ or else $T_i$ is a near triangulation with at least $t$ vertices, a contradiction. Also note that

$$m(G)=(|F|-3)+\sum_{i\in[s]}m(T_i)\geq(|F|-3)+\sum_{i\in J'}m(T_i) \tag{4.2}$$

$$v(G)\leq\sum_{i\in[s]}v(T_i)-s\leq(s-|J'|)(t-1)+\Big(\sum_{i\in J'}v(T_i)\Big)-s, \tag{4.3}$$

where we subtract $s$ in line (3) because $x_i$ is counted twice for each $i\in[s]$. Choose $j\in J'$ so that $\frac{v(T_j)-(t-1)}{m(T_j)}$ is maximized; this is well-defined because $m(T_j)>0$ when $j\in J'$.

Then we have

$$\frac{v(T_j)-(t-1)}{m(T_j)} \geq \frac{\sum_{i\in J'}\big(v(T_i)-(t-1)\big)}{\sum_{i\in J'} m(T_i)} \tag{4.4}$$

$$= \frac{\Big(\sum_{i\in J'} v(T_i)\Big) - |J'|(t-1)}{\sum_{i\in J'} m(T_i)} \tag{4.5}$$

$$\geq \frac{v(G)-|J'|(t-1)-(s-|J'|)(t-1)+s}{m(G)-(|F|-3)} \tag{4.6}$$

$$= \frac{\big(v(G)-(t-1)\big)-s(t-2)+t-1}{m(G)-(|F|-3)} \tag{4.7}$$

$$\geq \frac{v(G)-(t-1)}{m(G)}, \tag{4.8}$$

where line (6) follows from (2) and (3), and line (8) follows from (1) and the assumption that $\frac{v(G)-(t-1)}{m(G)} > 3t-7$. Hence $\frac{v(T_j)-(t-1)}{m(T_j)} > 3t-7$. But since $m(T_j) < m(G)$ and $v(T_j) \geq t$, this contradicts that $m(G)$ is as small as possible. □

Now every interior non-triangular face is either disjoint from $C$ or meets $C$ in a path. We next show that such a path has no edges. Recall the assumption that $v(G) > (3t-7)\cdot m(G)+(t-1)$.

**Claim 4.2.2.** *No interior non-triangular face is incident with an edge of $C$.*

*Proof.* Suppose there is an interior non-triangular facial cycle $F$ so that $F\cap C$ has at least one edge. Then $F\cap C$ is a subpath $P$ of $C$ for which each internal vertex has degree 2 in $G$. Let $r$ be the number of internal vertices of $P$, and let $H$ be the subgraph of $G$ obtained from deleting all internal vertices of $P$, or deleting the edge of $P$ if $r=0$. Note that the outer face of $H$ is bounded by a cycle $C'$ since $F\cap C$ is a path, and therefore $(H,C')$ is a circuit graph. Also, $m(H) < m(G)$ because $F$ is not a facial cycle of $H$.

We claim that $v(H)\geq t$. Suppose not. Since $v(G)\geq t$ this implies that $r\geq 1$. If $r\leq 4$ then $v(G)\leq t+3\leq 4(t-2)\leq (3t-7)\cdot m(G)+(t-1)$, which contradicts the assumption that $(3t-7)\cdot m(G)+(t-1) < v(G)$. So $r\geq 5$. Using the estimate

$m(G) \geq |F| - 3 \geq r - 1$, we have

$$\begin{aligned} v(G) &\leq t - 1 + r \\ &\leq t - 1 + r + \big(3rt - 3t - 8r + 8\big) \\ &= (3t-7)(r-1) + (t-1) \\ &\leq (3t-7)\cdot m(G) + (t-1), \end{aligned}$$

a contradiction. Therefore $v(H) \geq t$.

Since $v(H) \geq t$, it follows that $(3t-7)\cdot m(H) + (t-1) \geq v(H)$ or else $(H, C')$ contradicts the choice of $(G, C)$ with $m(G)$ minimum. But if $r \leq 4$ then

$$\begin{aligned} (3t-7)\cdot m(H) + (t-1) &\leq (3t-7)\cdot(m(G)-1) + (t-1) \\ &< v(G) - (3t-7) \\ &\leq v(G) - 4 \\ &\leq v(G) - r = v(H), \end{aligned}$$

which is a contradiction. If $r \geq 5$ then $m(H) \leq m(G) - (r-1)$ because $F$ is not a facial cycle of $H$, and so

$$\begin{aligned} (3t-7)\cdot m(H) + (t-1) &\leq (3t-7)\cdot\big(m(G)-(r-1)\big) + (t-1) \\ &< v(G) - (3t-7)(r-1) \\ &= v(G) - r - \big(3tr - 8r - 3t + 7\big) \\ &\leq v(G) - r \\ &= v(H), \end{aligned}$$

again, a contradiction. □

Now every interior non-triangular face meets $C$ in at most one vertex. In fact, something

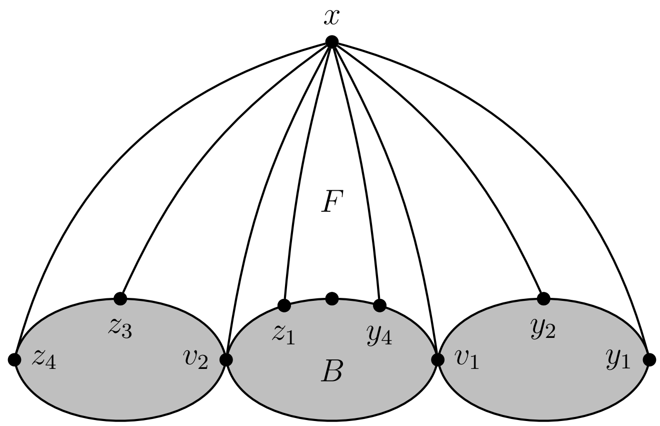


Figure 4.3: This example illustrates the definitions of $C_1$ and $C_2$ in Claim 4.2.3.

stronger is true.

**Claim 4.2.3.** *No interior non-triangular face of $G$ meets $C$.*

*Proof.* Suppose that an interior non-triangular facial cycle $F$ of $G$ meets $C$. Then $|F \cap C| = 1$ by the previous two claims; let $x \in V(F \cap C)$. Let $y_1, y_2, \ldots, y_s$, $z_1, z_2, \ldots, z_r$ be the neighbors of $x$ listed in clockwise order around $x$ where $xy_1$ and $xz_r$ are edges of $C$ and $xy_s$ and $xz_1$ are edges of $F$. Let $Y = \{xy_i : i \in [s]\}$ and let $Z = \{xz_i : i \in [r]\}$.

Since $m(G - Y) < m(G)$ (because $F$ is not a facial cycle of $G - Y$) and $v(G - Y) = v(G)$ it follows from the minimality of $m(G)$ that $G - Y$ is not 2-connected. Similarly, $G - Z$ is not 2-connected. Both statements imply that $G - x$ is not 2-connected. Since $x$ is on $C$, every cut vertex of $G - x$ is also on $C$ because $(G, C)$ is a circuit graph. Since every 2-cut of $G$ has both vertices on $C$ but $F$ has only one vertex on $C$, there is a unique block $B$ of $G - x$ that contains $F - x$ and satisfies $B \cap (C - x) \neq \varnothing$. Since $G - Y$ is not 2-connected, there is a cut vertex $v_1$ of $G - x$ in $B$. Similarly, there is a cut vertex $v_2$ of $G - x$ in $B$. Note that $v_1, v_2$ are on $C$ and that $x, y_1, v_1, v_2, z_r$ appear in this order clockwise around $C$, by the definitions of $Y$ and $Z$. Also, $v_1 \neq v_2$ or else the cut $\{x, v_1\}$ contradicts that $(G, C)$ is a circuit graph.

Let $C_B$ be the outer cycle of $B$. Then $C_1 := y_s C_B v_1 \cup xCv_1 \cup xy_s$ is a cycle of $G$; let

$H_1$ be the interior of $C_1$. Similarly, $C_2 := v_2C_Bz_1 \cup v_2Cx \cup xz_1$ is a cycle of $G$; let $H_2$ be the interior of $C_2$. (See Figure 4.3 for an example illustrating $C_1$ and $C_2$. In this example, $s = r = 4$, and $v_1 = y_3$ and $v_2 = z_2$.) Note that $H_1$ and $H_2$ intersect only at $x$ and that each of $(H_1, C_1)$, $(H_2, C_2)$, $(B, C_B)$ is a circuit graph. Also, $V(G) = V(H_1) \cup V(H_2) \cup V(B)$ and $m(G) = m(H_1) + m(H_2) + m(B) + |F| - 3$, because every interior face of $G$ other than the face bounded by $F$ is an interior face of $H_1$, $H_2$, or $B$.

Note that $\max\{v(H_1), v(H_2), v(B)\} \geq t$, or else $v(G) \leq 3(t-2) \leq (3t-7) \cdot m(G) + (t-1)$, a contradiction. Let $H_3 = B$, and let $S \subseteq \{1, 2, 3\}$ so that $v(H_i) \geq t$ if and only if $i \in S$. Then $m(H_i) > 0$ for all $i \in S$, or else $H_i$ is a near triangulation in $G$ with at least $t$ vertices. Choose $j \in S$ so that $\frac{v(T_j)-(t-1)}{m(T_j)}$ is maximized. Then

$$\begin{aligned}
\frac{v(H_j) - (t-1)}{m(H_j)} &\geq \frac{\sum_{i \in S} \big(v(H_i) - (t-1)\big)}{\sum_{i \in S} m(H_i)} \\
&\geq \frac{v(G) - 3(t-2)}{m(G) - 1} \\
&= \frac{v(G) - (t-1) - (2t-5)}{m(G) - 1}.
\end{aligned}$$

Since $(3t-7) \cdot m(G) < v(G) - (t-1)$ by assumption, we have

$$(3t-7) \cdot (m(G) - 1) < v(G) - (t-1) - (3t-7) \leq v(G) - (t-1) - (2t-5),$$

and it follows that $\frac{v(H_j)-(t-1)}{m(H_j)} > 3t - 7$. But since $v(H_j) \geq t$ and $m(H_j) < m(G)$, this is a contradiction. $\square$

We now finish the proof. For each interior non-triangular facial cycle $F$ of $G$, choose vertices $x_1, x_2$ on $C$ and $y_1, y_2$ on $F$ and an $x_i$-$y_i$ path $P_i$ for each $i \in [2]$ so that $P_1$ and $P_2$ are vertex-disjoint and both internally disjoint from $C \cup F$, and the graph $H_F$ bounded by the cycle $C_F := P_1 \cup y_1Fy_2 \cup P_2 \cup x_1Cx_2$ is minimal (with respect to subgraph containment). Let $F$ be an interior non-triangular facial cycle of $G$ so that $H_F$ is minimal over all choices of $F$. Note that $H_F$ is 2-connected because every face of $H_F$ is bounded by a cycle, and so

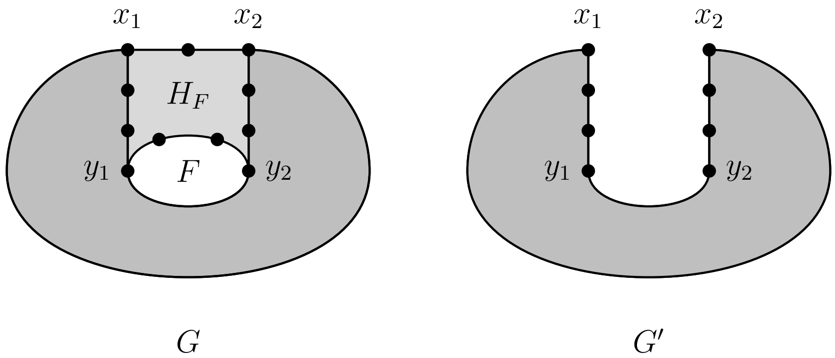


Figure 4.4: This example illustrates the definitions of $C_F$, $H_F$, and $G'$ at the end of the proof of Theorem 4.1.4.

$(H_F, C_F)$ is a circuit graph.

Suppose $H_F$ has an interior non-triangular facial cycle $F'$. Since $H_F$ is 2-connected, there are vertex-disjoint paths $Q_1$ and $Q_2$ in $H_F$ from $F'$ to $z_1, z_2 \in V(C_F)$, respectively, that are internally disjoint from $C_F \cup F'$. Then $Q_1, Q_2$ can be extended along $C_F$ to disjoint paths from $F'$ to $x_1 C x_2$ giving rise to an $H_{F'}$ that is a proper subgraph of $H_F$, a contradiction. Therefore $H_F$ is a near triangulation.

Then $v(H_F) < t$, as $G$ contains no near triangulation on at least $t$ vertices. Let $G'$ be obtained from $G$ by deleting all vertices and edges of $H_F$ that are not on $P_1 \cup P_2$; see Figure 4.4 for an illustration of the relationship between $G$ and $G'$. The outer face of $G'$ is bounded by the cycle $C' = P_1 \cup y_2 F y_1 \cup P_2 \cup x_2 C x_1$ and so $(G', C')$ is a circuit graph. If $v(G') < t$, then $v(G) < 2t$ and $m(G) \geq \frac{v(G)-(t-1)}{3t-7}$, a contradiction. So $v(G') \geq t$. Also, $m(G') < m(G)$ because $F$ is not a face of $G'$, and it follows from $(3t-7) \cdot m(G) + (t-1) < v(G)$ and $v(H_F) < t$ that $(3t-7) \cdot m(G') + (t-1) < v(G')$. But then $G'$ contradicts the choice of $G$. $\square$

### 4.3 The proof of Theorem 4.1.6

In this section we will prove Theorem 4.1.6, which directly implies Theorem 4.1.3. We need the following lemma.

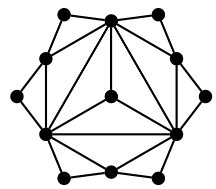

Figure 4.5: A near triangulation with a cycle of length 12 but not every graph in $\Theta_{12}$ as a subgraph.

**Lemma 4.3.1.** *If $G$ is a near triangulation with a cycle of length at least $k$, then $G$ has a $\theta_k$-subgraph.*

*Proof.* Let $C$ be a cycle of $G$ with length at least $k$ so that the interior of $C$ is as small as possible. Then $C$ bounds a near triangulation $H$. It is straightforward to prove by induction on $v(H)$ that either

(i) there is an edge $e = uv$ of $C$ in a facial triangle that also contains an interior vertex $x$ of $H$, or

(ii) there are edge-disjoint subpaths of $C$, each of length 2 and in a facial triangle, which gives two non-adjacent vertices of degree 2 in $H$.

If (i) holds, then the cycle obtained from $C - uv$ by adding the path $uxv$ contradicts the minimality of the interior of $C$. So (ii) holds. If $|C| > k$, then the cycle obtained from $C$ by deleting a vertex of degree 2 and adding the edge between its neighbors contradicts the minimality of the interior of $C$. Therefore $|C| = k$, and (ii) implies that $H$ has a $\theta_k$-subgraph. □

We comment that Lemma 4.3.1 is not true for all graphs in $\Theta_k$, as shown by the graph in Figure 4.5. This is the reason why Theorem 4.1.6 may not hold for all graphs in $\Theta_k$. In particular, the graph in Figure 4.5 has no subgraph consisting of a 12-cycle with a chord joining two vertices of distance 6.

We are now ready to prove our main result of this chapter.

*Proof of Theorem 4.1.6.* Fix $k \geq 4$, and let $t = \lceil k^{\log_2 3} \rceil$. We will show that $\mathrm{ex}_{\mathcal{P}}(n, \theta_k) \leq 3n - 6 - \frac{n}{4(t-2)}$ for all $n \geq t$, which implies that $\mathrm{ex}_{\mathcal{P}}(n, \theta_k) \leq 3n - 6 - \frac{n}{4k^{\log_2 3}}$ for all $n \geq t$. Suppose there is a $\theta_k$-free plane graph $G$ with $v(G) \geq t$ and $e(G) > 3v(G) - 6 - \frac{v(G)}{4(t-2)}$, and assume that $G$ has no proper subgraph with these properties. We will show that $G$ has a $\theta_k$-subgraph. We first reduce to the 3-connected case.

**Claim 4.3.2.** *$G$ is 3-connected.*

*Proof.* Suppose $G$ is not 3-connected. Then $G$ has edge-disjoint subgraphs $G_1$ and $G_2$ such that $G = G_1 \cup G_2$ and $|V(G_1) \cap V(G_2)| = j$ with $j \leq 2$. By the minimality of $v(G)$, for each $i \in [2]$ either $v(G_i) < t$ or $e(G_i) \leq 3v(G_i) - 6 - \frac{v(G_i)}{4(t-2)}$. We consider the three possibilities separately.

First, assume $v(G_i) < t$ for $i \in [2]$. Then $v(G) < 2t$. Therefore, since $G$ is not a triangulation, $e(G) \leq 3v(G) - 6 - \frac{v(G)}{4(t-2)}$, a contradiction.

Now assume $e(G_i) \leq 3v(G_i) - 6 - \frac{v(G_i)}{4(t-2)}$ for $i \in [2]$. Then $v(G_1) + v(G_2) \leq v(G) + 2$, and so

$$
\begin{aligned}
e(G) &\leq e(G_1) + e(G_2) \\
&\leq \Big(3v(G_1) - 6 - \frac{v(G_1)}{4(t-2)}\Big) + \Big(3v(G_2) - 6 - \frac{v(G_2)}{4(t-2)}\Big) \\
&\leq 3v(G) - 6 - \frac{v(G)}{4(t-2)},
\end{aligned}
$$

a contradiction.

So, up to relabeling, we may assume that $e(G_1) \leq 3v(G_1) - 6 - \frac{v(G_1)}{4(t-2)}$ and $v(G_2) < t$.

If $V(G_1) \cap V(G_2)$ is not an independent set in $G$, then $v(G_1) + v(G_2) = v(G) + 2$, and

$$\begin{aligned} e(G) &= e(G_1) + e(G_2) - 1 \\ &\le \Big(3v(G_1) - 6 - \frac{v(G_1)}{4(t-2)}\Big) + \Big(3v(G_2) - 6\Big) - 1 \\ &= 3v(G) - 7 - \frac{v(G_1)}{4(t-2)} \\ &\le 3v(G) - 7 - \frac{v(G) - (t-2)}{4(t-2)} \\ &= 3v(G) - 6 - \frac{v(G)}{4(t-2)} - \Big(1 - \frac{t-2}{4(t-2)}\Big), \end{aligned}$$

a contradiction. If $V(G_1) \cap V(G_2)$ is independent in $G$, then either $G_2$ is not a triangulation or $v(G_1) + v(G_2) \le v(G) + 1$. In either case a very similar calculation applies. So in all cases we arrive at a contradiction, and therefore $G$ is 3-connected. $\square$

Let $C$ be the outer cycle of $G$, and note that $(G, C)$ is a circuit graph. Since $e(G) > 3v(G) - 6 - \frac{v(G)}{4(t-2)}$ we have $m(G) < \frac{v(G)}{4(t-2)} \le \frac{v(G)-(t-1)}{3t-7}$. By applying Theorem 4.1.4 to $(G, C)$ with $t = \lceil k^{\log_2 3} \rceil$, $G$ has a near triangulation subgraph $T$ with $v(T) \ge k^{\log_2 3}$. Since every near triangulation is a circuit graph, Theorem 4.1.5 and Lemma 4.3.1 imply that $T$ contains $\theta_k$ as a subgraph. $\square$

### 4.4 Related problems

Theorem 4.1.3 has interesting consequences related to circumference. The *circumference* of a graph (with a cycle) is the length of a longest cycle in the graph. Erdős and Gallai proved that every $n$-vertex graph with circumference less than $k$ has at most $\frac{(n-1)(k-1)}{2}$ edges, and this is tight for infinitely many integers $n$ [10]. Theorem 4.1.3 gives an analogue for planar graphs, because graphs with circumference less than $k$ are certainly $C_k$-free.

**Theorem 4.4.1.** *Let $k \ge 4$ and $n \ge k^{\log_2 3}$. If $G$ is an $n$-vertex planar graph with circumference less than $k$, then $e(G) \le 3n - 6 - \frac{n}{4k^{\log_2 3}}$.*

Again, this is tight up to the constant $1/4$ because the construction of Győri, Varga, and Zhu [20] gives a lower bound of $3n - 6 - \frac{12n}{k^{\log_2 3}}$.

We believe that analogues of the Erdős-Gallai theorem are interesting for other classes of graphs (such as graphs with bounded genus, or without a fixed subgraph $H$), and also more generally for classes of matroids. The *circumference* of a matroid (with a circuit) is the number of elements in a largest circuit in the matroid. We mention one question in this direction that is closely related to Theorem 4.1.3. A matroid is *cographic* if it is the dual of the cycle matroid of a graph. It is well-known that a rank-$n$ cographic matroid has at most $3n - 3$ elements; this bound is tight for planar-graphic matroids.

**Problem 4.4.2.** *Show that there is a constant $D$ so that for all $k \geq 4$ and $n \geq k^{\log_2 3}$, if $M$ is a rank-$n$ cographic matroid with circumference less than $k$, then the number of elements of $M$ is at most $3n - 3 - \frac{Dn}{k^{\log_2 3}}$.*

This would follow from Theorem 4.1.3 with $D = 1/4$ if every rank-$n$ cographic matroid with circumference less than $k$ and greater than $3n - 3 - \frac{Dn}{k^{\log_2 3}}$ elements is in fact planar-graphic.

Finally, we point out that Theorem 4.1.4 may be true for all 2-connected plane graphs, not just circuit graphs.

**Problem 4.4.3.** *Does Theorem 4.1.4 hold for all $2$-connected plane graphs?*

The structure of 2-cuts in plane graphs that are not circuit graphs can be quite complicated, and it is unclear how to reduce from general 2-connected plane graphs to circuit graphs. However, while we do make use of the connectivity properties of circuit graphs in Claim 4.2.3, there may be other arguments that hold for all 2-connected plane graphs.

# CHAPTER 5

# COUNTING CLIQUES IN THE GRAPH WITH FORBIDDEN CLIQUE MINORS

## 5.1 Introduction

A clique is a set of vertices where there are edges between any two vertices. We use $K_t$ to denote a clique on $t$ vertices, i.e., of order $t$. We also call it a $t$-clique.

A cornerstone result in extremal combinatorics is Turán's theorem [46], which asks the maximum number of edges in a graph on $n$ vertices that do not have $K_t$ as a subgraph. The answer is obtained by the Turán graph $T(n, t-1)$, which is the complete multipartite graph where each part has order $\lfloor n/(t-1) \rfloor$ or $\lceil n/(t-1) \rceil$. A natural question to ask is: for each positive integer $k < t$, what is the maximum number of cliques of order $k$ in a graph on $n$ vertices without $K_t$ as a subgraph? This is answered by Zykov [48]; the same Turán graph $T(n, t-1)$ also maximizes the number of $k$-cliques, i.e., cliques of order $k$.

Alon and Shikhelman [1] initiated the systematic study of a generalization of this question. Let ex$(n, T, H)$ be the maximum possible number of copies of $T$ in an $H$-free graph on $n$ vertices. Thus Turán's theorem gives an answer to ex$(n, K_2, K_t)$ and Zykov's theorem provides an answer to ex$(n, K_k, K_t)$ and furthermore ex$(n, \text{clique}, K_t)$. Some other examples of results in this trend can be found in [12, 3, 4, 21, 1].

Analogous questions for forbidding minors have also been studied for a long time, where minors can be considered as a generalization of subgraphs. A graph $H$ is a *minor* of a graph $G$ if it can be obtained from $G$ by contracting edges and deleting vertices and edges. A natural generalization asks: what is the maximum possible number of cliques (of possibly fixed sizes) a graph on $n$ vertices could have?

The study of bounding the number of cliques in $K_t$-minor free graphs, i.e., understanding the extremal functions ex$(n, \text{clique}, K_t\text{-minor})$ and ex$(n, K_k, K_t\text{-minor})$, also have applications in theoretical computer science such as designing linear-time algorithms (e.g., see [37, 9] and the references therein). The bounds on the function ex$(n, \text{clique}, K_t\text{-minor})$ have been studied through works such as by Norine, Seymour, Thomas, and Wollan [35], Reed and Wood [37], Fomin, Oum, and Thilikos [13], Lee and Oum [31], and Wood [47].

The paper [35] showed a classical result that the number of $n$-vertex graphs in a proper minor-closed family $\mathcal{I}_n$ is most $c^n n!$ for some constant $c$. The proof is through induction by showing that by deleting a twin vertex or by contracting two adjacent vertices with small degrees, there is a mapping from $\mathcal{I}_n$ to $\mathcal{I}_{n-1}$ where the size of pre-image is small. To show this, one key step is to upper bound the number of cliques in $K_t$-minor free graphs. The bound on the number of cliques in $K_t$-minor free graphs is later improved to $2^{ct\sqrt{\log t}}n$ by Reed and Wood [37] by showing that the number of $k$-cliques in $d$-degenarate graph is at most $d^{k-1}n$.

Fomin, Oum, and Thilikos [13] showed more applications of counting cliques in $K_t$-minor free graphs. They bounded the tree-width and clique-width of $G$ by the rank-width of $G$ and the number of cliques in $G$, and showed that numbers of many important structures are highly related to the number of cliques such as the number of hyperedges in a hypergraph and the number of distinct columns in a binary matrix. Notice that they improved the bound of the number of cliques to $2^{ct \log\log t}n$ by bounding the number of $k$-cliques for each $k \leq t-1$.

Lee and Oum [31] considered the number of cliques in $K_t$-subdivision free graphs, and improved the bound to $2^{5t+o(t)}$. Wood [47] counted the exact numbers of cliques in the $K_t$-minor free graphs for every $3 \leq t \leq 9$. More precisely, he counted numbers of $k$-cliques in the $K_t$-minor free graphs for every $3 \leq k < t \leq 9$ and gave an upper bound for ex$(n, K_k, K_t)$. He also made several conjectures about this bound which inspired this dissertation.

The question about the *total* number of cliques in $K_t$-minor free graphs was answered by Fox and Wei [14] where the asymptotically sharp bound is obtained.

**Theorem 5.1.1** (Theorem 1.1 [14] 2016)**.** *Every graph on $n$ vertices with no $K_t$-minor has at most $3^{2t/3+o(t)}n$ cliques. This bound is tight for $n \geq 4t/3$.*

Note the bound above is adding up the number of cliques of *all possible* sizes. This bound is asymptotically sharp for $n \geq 4t/3$ by considering a disjoint union of copies of the graph which is the complement of a perfect matching on $2\lceil 2t/3 \rceil - 2$ vertices. Counting the number of cliques was also studied in other graph families that can be found in [23, 15, 18].

When we fix the clique size $k$, counting the number of $k$-cliques instead of the total number of cliques in graphs on $n$ vertices with no $K_t$-minor, i.e., to understand ex$(n, K_k, K_t$-minor$)$, has received much attention. Clearly, when $n < t$, the maximum number of cliques of order $k$ is at most $\binom{n}{k}$; this bound is exact and sharp by considering a clique on $n$ vertices, which has no $K_t$-minor. When $k > t$, clearly the answer is 0. The question is less clear for other values of $k$. This thread dates back to the works of Dirac [7], Mader [33], Jørgensen [22], and Song and Thomas [43] for the cases when $k = 2$ and $t \leq 9$.

For general $t$ and any $k < t$, Wood [47] asked the following question, which was asked again by Fox and Wei [14].

**Question 5.1.2** (Wood [47], Fox and Wei [14])**.** *Let $t, k$ be positive integers such that $k < t$. What is the maximum possible number of cliques of order $k$ in a $K_t$-minor free graph on $n$ vertices?*

For small values of $t$, Wood [47] determined the exact value of ex$(n, K_k, K_t$-minor$)$ for $t \leq 9$ and $k < t$. On the other hand, for larger values of $t$ but for $k = 2$, the asymptotic sharp (in $t$) answer is now known after a series of works by Mader, Kostochka, and Thomason [32, 33, 26, 27, 44, 45]. In particular,

Kostochka [26, 27] and Thomason [44] independently proved that the maximum number of edges in graphs on $n$ vertices and with no $K_t$-minor is $\Theta(t\sqrt{\log_2 t})n$. Thomason [45] later determines the constant $(\alpha + o_t(1))t\sqrt{\ln t} \cdot n$ where $\alpha = 0.319...$ is an explicit constant. This asymptotic extremal configuration can be achieved by random graph $G(n', p')$ with appropriate values of $n'$ and $p'$.

For larger values of $k$, it seems pseudorandom graphs are no longer optimal. As observed by Fox and Wei [14], the average order of the cliques in the complement of a perfect matching of $x$ edges is $2x/3$, and thus a typical random clique in this graph has about this size. Now consider the graph which is a complement of a perfect matching of just less than $2t/3$ edges and is thus $K_t$-free. It has nearly the maximum number of $k$-cliques for $k = 4t/9$, which gives the $4t/9$-clique count $3^{2t/3-o(t)}n$. The complement of a perfect matching is an example of a *Turán graph* that each part has size 2. In general, a candidate for lower bound construction is based on Turán graphs.

Let $T(n, \omega)$ be the *Turán graph*, the complete balanced multipartite graph on $n$ vertices and with $\omega$ parts, where each part has order $\lfloor n/\omega \rfloor$ or $\lceil n/\omega \rceil$. Are disjoint unions of Turán graphs nearly optimal? When $k = t - 1$, Wood [47] shows that the maximum number of $K_{t-1}$ in a $K_t$-minor free graph is exactly $n-t+2$. The construction is called an $(t-2)$-tree (Definition 5.1.1), which is essentially similar to a disjoint union of copies of $K_{t-1}$ where the different copies of $K_{t-1}$ are glued along the same $K_{t-2}$.

The discussion above shows that depending on the range of $k$, the extremal constructions for the exact maximum number of $k$-cliques may have quite different forms. We are interested in the asymptotically sharp bounds for the number of $k$-cliques in graphs on $n$ vertices and without $K_t$-minor, where the asymptotic is up to $o(1)$ in the exponent, similar to what asymptotic means as in Theorem 5.1.1 [14].

Some general upper bounds for this quantity are known. The following simple upper bound is well-known, for example by Wood [47] Lemma 18, the proof of Norine et al. [35], the proof of Lemma 3.1 in Reed and Wood [37]; the proof of Lemma 5 in Fomin et al. [13], or a simplified proof of Theorem 1.1 in Fox and Wei [14].

**Theorem 5.1.3** ([47, 35, 37, 13, 14]). *When $t$ is sufficiently large, for any $k < t$, every graph on $n$ vertices with no $K_t$-minor has at most $\binom{\beta t\sqrt{\ln t}}{k-1}n$ cliques of order $k$. The constant*

$\beta = 0.64$.

Notice that $\beta > 2\alpha$ where constant $\alpha = 0.319...$ is determined by Thomason [45]. This bound is sharp for $k = 2$ up to a multiplicative constant by the aforementioned result of Thomason [45] and by considering a disjoint union of random graphs of appropriate sizes.

Besides this upper bound, Wood [47] made an explicit conjecture on the maximum number of $k$-cliques in $K_t$-minor free graphs on $n$ vertices for large $k$.

**Conjecture 5.1.4** (Wood [47] Conjecture 20)**.** *For some $\lambda \in [1/3, 1)$, for all integers $t > 3$ and $k > \lambda t$ and $n > t - 1$, the number of $k$-cliques in a $K_t$-minor free graph on $n$ vertices is at most*

$$\binom{t-2}{k} + (n-t+2)\binom{t-2}{k-1} = \binom{t-2}{k-1}\left(n - \frac{(k-1)(t-1)}{k}\right).$$

Again, the upper bound is achieved by the $(t-2)$-trees defined below. We will prove an asymptotic version of this conjecture for $\lambda > 2/3$ in Corollary 5.4.2, and show that the claim of this conjecture does not hold for $\lambda < 0.553$.

**Definition 5.1.1** ($(t-2)$-tree)**.** An $(t-2)$-*tree* is a family of graphs defined recursively as follows: We start with the complete graph $K_{t-2}$, which is also an $(t-2)$-tree. For any $(t-2)$-tree $H$, if $C$ is a clique of order $t-2$ in $H$, then by adding another vertex to $H$ that is adjacent only to the vertices in $C$ is also an $(t-2)$-tree. Then the number of cliques of order $k$ in every graph in the $(t-2)$-tree family is $\binom{t-2}{k} + (n-t+2)\binom{t-2}{k-1}$.

For general values of $k < t$ such that $t - k \gg \log_2 t$, we prove asymptotically sharp bounds on the maximum possible number of $k$-cliques in $K_t$-minor free graphs on $n$ vertices in Theorem 5.1.5. Again asymptotic here means up to $o_t(1)$ in the exponent, similar to Theorem 5.1.1 [14].

The main results of this chapter are summarized in the next subsection.

### 5.1.1 Our Results

In the following theorem, we answer Question 5.1.2 (Wood [47], Fox and Wei [14]) up to $o_t(1)$ in the exponent, similar to what asymptotically sharp means as in Theorem 5.1.1 [14]. In other words, we prove a sharp upper bound for the maximum number of cliques of size $k$ in $K_t$-minor free graphs up to $o_t(1)$ in the exponent.

**Definition 5.1.2.** For fixed $k < t$, let $T_t^*(k)$ be the Turán graph $T(2t-\omega-1, \omega)$ maximizing the number of cliques of order $k$ among all $\omega$ such that $k \leq \omega \leq t-1$. Let $C_t^*(k)$ denoted the number of cliques of order $k$ in $T_t^*(k)$.

We will show that $T_t^*(k)$ is $K_t$-minor free for every $t$ and $k < t$ in Lemma 5.3.4. Our main result in this chapter is the following theorem.

**Theorem 5.1.5.** *Assume $t - k \gg \log_2 t$. The number of cliques of order $k$ in a $K_t$-minor free graph on $n$ vertices is at most*

$$n \cdot \left( \frac{C_t^*(k)}{|V(T_t^*(k))|} \right)^{1+o_t(1)}$$

*This bound is sharp up to $o_t(1)$ in the exponent when $n \geq 2t$.*

The matching lower bound construction is by considering $\lfloor n/|V(T_t^*(k))| \rfloor$ disjoint copies of the Turán graph $T_t^*(k)$.

**Remark.** *We have discussed that when $k = 2$, pseudorandom graphs are asymptotically optimal [45]. It turns out that when $k \ll \log\log t$, the random graph construction also matches the bound in Theorem 5.1.5, with a slightly better error bound $o_t(1)$ compared to the Turán graph construction.*

**Remark.** *When $k > 2t/3$, Lemma 5.3.1 will show that $T_t^*(k) = K_t^-$, the complete graph $K_t$ delete a single edge, and thus $C_t^*(k) = \binom{t-1}{k} + \binom{t-2}{k-1}$.*

**Remark.** *To see the quantitative behavior of* $\frac{C_t^*(k)}{|V(T_t^*(k))|}$ *for general values of* $k$*, first notice* $t \leq |T_t^*(k)| \leq 2t$ *by the definition of* $T_t^*(k)$*. We will also show that* $\binom{t-1}{k}\max\left(1, \left(2 - 4\sqrt{2k/t}\right)\right)^k \leq C_t^*(k) \leq \binom{t-1}{k}2^k$ *for* $k \geq 25$ *in Claim 5.3.2 and Lemma 5.3.6. In addition,* $T_t^*(k)$ *has* $\omega$ *parts where* $\sqrt{tk}/4 \leq \omega \leq 10\sqrt{tk}$ *as shown in Proposition 5.1.6.*

We also prove the asymptotic version of Wood's Conjecture 5.1.4 for every $k < t$ such that $t - k \gg \log_2 t$. In Theorem A.0.1 we show that the conjecture is false for $k \leq 0.553t$.

Notice that the known upper bound in Theorem 5.1.3 (Wood [47] and Fox-Wei [14]) is already an asymptotically sharp bound in the sense above when $k < t^{1-\delta}$ for some absolute constant $\delta$. (For more computational detail see the proof of Theorem 5.1.5 in Chapter 5.4.) However, not only have we improved this bound for $k$ in this range, but also showed that the new bounds are asymptotically sharp up to $o(1)$ in the exponent for all $k$ such that $t - k \gg O(\log t)$.

For general values of $k, t$, it is challenging to write down a closed-formula description of $T_t^*(k)$. Later Lemma 5.3.5 tells us that in $T_t^*(k)$, the order of each part is smaller than $\sqrt{\frac{4n-3k}{k}} + 1$; Thus when $k > 4n/7$, the complement of the optimal graph $T_t^*(k)$ is a perfect matching with possibly isolated vertices. It still remains open what the exact description of $T_t^*(k)$ is for general $k$. The order of each part changes as a function of $k$. We could prove an asymptotic result on the size of each part.

**Proposition 5.1.6.** *For every* $t > k \geq 1$*, the optimal* $T_t^*(k)$ *is given by the Turán graph* $T(n, r)$ *with* $n + r = 2t - 1$ *where the number of part* $r$ *satisfies* $\sqrt{tk}/4 \leq r \leq 10\sqrt{tk}$*. When* $k \geq 2t/3$*, the graph* $T_t^*(k)$ *is the Turán graph* $T(t, t-1) = K_t^-$*.*

**Proof Idea** The proof idea started with a peeling process to encode all cliques of order $K$. This peeling process was used in [14], which was highly inspired by the classic paper of Kleitman–Winston [25]. Roughly speaking, the peeling process maps each clique $K$ into a short encoding $I(K)$ and a "dense" graph. The authors in [14] showed that the number of encoding $|\{I(K) : K \subset G\}|$ is small, and it is relatively easier to bound the size of the maximum clique minor in a dense graph. However, as observed in [14], even though the method could provide an almost sharp bound on the total number of cliques in $G$, it fails to provide a satisfactory answer when we fix the clique size $k$. The challenges are two-fold: first, the upper bound on the number of encodings proved in [14] could be too large for some ranges of $k$, and also we need to characterize the optimal dense graphs optimizing the number of $K_k$. In this dissertation, we made three improvements to overcome the difficulties. The first is that, by a careful analysis of the peeling process, we show that, if $I(K)$ is large, either the number of such encoding is small, or we can find a much bigger clique minor in $G$ which would lead to a contradiction. This idea is particularly important when $k$ is in extreme ranges. The second improvement is made by showing a better upper bound for the number of possible representations $I(K)$ when fixing some parameters of $I(K)$. The third is a different method to bound the number of cliques of a given size in the dense graph.

### 5.2 Analysis of the Peeling Process

The development of the hypergraph container's method has been powerful in answering many long-standing questions. It was developed by Balogh-Morris-Samotij [2] and Saxton-Thomason [39]. The idea, which is transferring a general setting into a dense setting, can be traced back to the classical paper of Kleitman–Winston [25] on graphs.

Our Key Lemmas in this section are Lemmas 5.2.2 and 5.2.3, by carefully analyzing the peeling process (container's method [25]) below. The container's method works as follows. Roughly speaking, for each clique $K$ in $G$, we find a way to encode a small number of (ordered) vertices $v_1, \ldots, v_{r(K)}$ in $K$, call it $I(K)$. In other words, we gradually peel out vertices from $G$ with vertices in $I(K)$ be the landmarks. We want the total number of encoding $(v_1, \ldots, v_{r(K)})$ to be small. Different cliques may have the same encoding and we can group all the cliques $K$ by the different encoding. The vertices in $K \setminus I(K)$ are contained in a "dense" subgraph of $G$. And we could bound the number of the cliques (of order $k - r(K)$) ( Lemma 5.3.1 and Proposition 5.3.2).

We now describe the peeling process, which is almost the same procedure as in Fox and the second author [14] which was heavily motivated by [25]. However, the analysis of the peeling process in this dissertation is much more involved, since we would need to bound the number of cliques for a fixed size $k$. We will elaborate on the difficulties and how we overcome them in the next subsection after the description of the peeling process.

### 5.2.1 Description of the Peeling Process

Now we describe how to encode each clique $K$ by some sequence $v_1, \ldots, v_{r(K)}$. To determine the encoding for each clique $K$, we apply the following *peeling process* for $K$.

**Peeling Process.** *Firstly, we preorder vertices of $G$. Let $G_0 = G$. We delete vertices in $G_0$ one by one until some vertex $v_1 \in K$ has the smallest degree. (We break the tie by the predefined ordering on all the vertices in $G$). In this way, we obtain an induced subgraph $G_1$ that contains $K$ in which $v_1$ has the minimum degree. We repeat this process as follows:*

1. *After picking $v_i$ and thus obtaining the associated $G_i$, delete from $G_i$ vertex $v_i$ and its non-neighborhood $D_i$. We called this induced subgraph $G'_i$;*

2. *Delete vertices in $G'_i$ one by one until some vertex in $K$ has the smallest degree. (We break the tie by the predefined ordering). Let this vertex in $K$ be $v_{i+1}$ and the remaining graph be $G_{i+1}$. We call the set of deleted vertices in this deleting process as $Y_i = V(G'_i) \setminus V(G_{i+1})$.*

*Let $n_i$ be the number of vertices in $G_i$, and also let $d_i$ be the missing degree of $v_i$ in $G_i$, i.e. $d_i = |D_i|$. We call the process of finding $v_i$ and $G_i$ from $G_{i-1}$ the $i$-th step.*

*We call step $r$ the stopping step and $G_r$ the terminal graph, and let $r(K) = r$ if $r$ is the least positive integer such that*

1. *$n_r \leq t - r$, or*

*2.* $d_r \leq \frac{1}{2}(n_r + r - t)^{1/2}$*, or*

*3.* $r = |V(K)|$.

For any clique $K$, the peeling process above gives a sequence $v_i$, $G_i$, $D_i$ and $Y_i$. Let the *layer* at step $i$ be denoted as $L_i := D_i \cup Y_i$. Since no more vertices are deleted from the terminal graph $G_r$, for convenience, write $Y_r = \emptyset$.

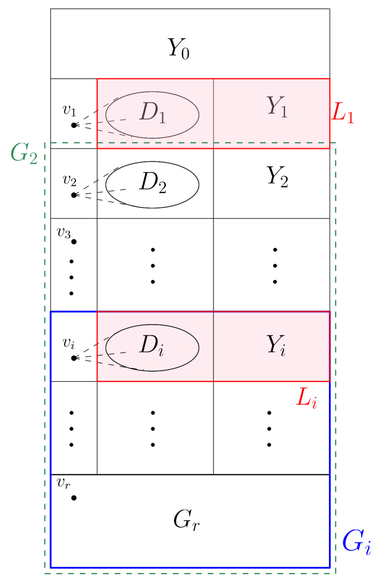


Figure 5.1: Illustration of the notations in the Peeling Process: $v_i$ is the vertex with minimum degree in $G_i$, and $D_i$ are the set of non-neighbors of $v_i$ in $G_i$. The set $Y_i$ are the extra vertices deleted until $v_{i+1}$ is the minimum degree vertex. For instance, the red box at the top indicates the layer $L_1$ which is the union of $D_1$ and $Y_1$. Notice that $Y_i$ can be an empty set.

One reason for applying this peeling process is that, after the first step, we will get a graph $G_2$ whose size is independent from $n$. The result of Thomason [45] implies, as $G$ does not contain a $K_t$-minor, every subgraph of it has a vertex of degree at most $d := \beta t\sqrt{\ln t}$ when $t$ is sufficiently large. In this chapter, without special notice, when we assume $t$ is sufficiently large, we assume $t$ is large enough so that the minimum degree of any $K_t$-minor free graph is at most $\beta t\sqrt{\ln t}$. Since $v_1$ is of minimum degree in $G_1$ and $G_2 \subset N_G(v_1)$, we know

$$d_2 < n_2 = |G_2| \leq d + 1.$$

The stopping condition 2 corresponds to the terminal graph $G_r$ being a "dense" graph, as the maximum missing degree in $G_r$ is small. Recall that the *Hadwiger number* of a graph $G$ is the largest number $t$ such that $K_t$ is a minor of $G$. The idea from [14] is that if a graph is dense, then its Hadwiger number is a simple function in terms of its order and its clique number.

**Lemma 5.2.1** (Lemma 2.1, [14])**.** *Let $G$ be a graph on $n$ vertices with minimum degree $\delta$ and clique number $\omega$ (the order of the largest clique). Let $\bar{\Delta} = n - \delta - 1$, be the maximum missing degree, which is also the maximum degree of the complement of $G$. We say $G$ is dense if $n \geq \omega + 2\bar{\Delta}^2 + 2$ or $\bar{\Delta} \leq 1$. If $G$ is dense, then the Hadwiger number of $G$ is $\lfloor \frac{n+\omega}{2} \rfloor$.*

**Definition 5.2.1.** Given a graph $G$, let $\bar{\Delta}(G)$ be its maximum missing degree, and $\omega(G)$ is the order of the largest clique in $G$. We define the following.

1. Let $\mathcal{D}$ be the family of all dense graphs, i.e., the set of $G$ such that $|V(G)| \geq \omega(G) + 2\bar{\Delta}(G)^2 + 2$ or $\bar{\Delta}(G) \leq 1$.
2. Let $\mathcal{G}_s$ be the family of graphs $G$ such that $\lfloor \frac{|V(G)|+\omega(G)}{2} \rfloor \leq s - 1$.
3. Let $\mathcal{H}_m^s$ be the family of graphs $H$ with at most $m$ vertices and the Hadwiger number of $H$ is at most $s$.

Lemma 5.2.1 guarantees that if $G$ is dense and does not have a $K_s$-minor, then $G$ is in $\mathcal{G}_s$. On the other hand, it also showed that if $G$ is dense and is in $\mathcal{G}_{t+1} - \mathcal{G}_t$, then $G$ must contain a $K_t$ minor. Note that there can be graphs that are not dense but also belong to $\mathcal{G}_t$.

### 5.2.2 Analysis of the Peeling Process

In Fox and Wei [14], the number of all cliques in $K_t$-minor free graphs is bounded by the product of the error term, which is the number of possible encoding with length at most

$r_0 = 4t^{1/2}\log_2{}^{1/4}t$, and the main term, which is the maximum number of all cliques in graphs in $\mathcal{G}_t$.

When the clique size $k$ is fixed, the error term could be too large. Thus, we need to understand how the parameters given by the peeling process are related to the Hadwiger number of the terminal graph. In Lemma 5.2.2 and Lemma 5.2.8, we will show that, in many cases, either the peeling process stops very quickly, and then we have $r(K)$ small and the number of possible encoding for this kind of cliques is small; or the Hadwiger number of the terminal graph $G_r$ is much smaller than $t - r$, resulting in fewer number of such cliques of order $k$. We will also show an improved bound for the number of possible encodings in Lemma 5.2.3.

**Definition 5.2.2.** For a given clique $K$ and its peeling sequences, a vertex subset $A \subseteq V(G)$ such that $A \cap K = \emptyset$ and $A \cap V(G_{r(K)}) = \emptyset$ is called *an extra branch disk* of $K$ if the induced subgraph $G[A]$ is connected and every vertex in $V(K) \cup V(G_r)$ has at least one neighbor in $A$.

To construct a large clique minor, we would like to use the vertices in $K$, together with contracting each branch disk into a single vertex. Thus, we want to find as many disjoint branch disks as possible that are also pairwise adjacent. This motivates us to define the following.

**Definition 5.2.3.** A collection $\mathcal{A} = \{A_1, A_2, \ldots, A_s\}$ of disjoint vertex subsets $A_i$ is called *branch vertex set* of $K$, if each $A_i$ is an extra branch disk of $K$, and for any $1 \le i < j \le s$, the two disks $A_i, A_j$ are adjacent, i.e., there exist $x \in A_i, y \in A_j$ such that $xy \in E(G)$. Let $s(K)$ be the maximum size of branch vertex set $\mathcal{A}$ of $K$.

**Claim 5.2.4.** *Given $K$ and its peeling process which ends in $r = r(K)$ steps. If $G_r$ contains $K_c$ as a minor, then for any branch vertex set $\mathcal{A}$ of $K$, the subgraph induced by $V(G_r) \cup \mathcal{A} \cup \{v_1, v_2, \ldots, v_{r-1}\}$ contains a clique minor of order $r - 1 + c + |\mathcal{A}|$.*

Notice that we could choose $c \geq k-r+1$ as $G_r$ contains $k-r+1$ vertices in the clique $K$.

*Proof.* By the definition of the peeling process, each $v_j$ where $j \leq r-1$ is adjacent to every vertex in $G_r$. Then the claim holds by the connectivity condition in the definitions of extra branch disk and branch vertex set. □

**Definition 5.2.5.** Suppose $M$ is a function in terms of $t$ such that $\log M = o(\log t)$. For a fixed clique $K$, let $R_M(K)$ be the number of $i \in [r(K)-1]$ such that $|Y_i| \geq M$. Let $s_M(r, r_l)$ be the minimum value of $s(K)$ among all cliques $K$ with indexes $r(K) = r$ and $R_M(K) = r_l$. If there is no clique $K$ of order $k$ with $r(K) = r$ and $R_M(K) = r_l$, we set $s_M(r, r_l) = +\infty$.

The main goal of this section is to prove the following two key lemmas. Key Lemma 1 will help us control the number of cliques in the terminal graph $G_r$ by stating that $s_M(r, r_l)$ is relatively large when $r$ or $r_l$ is large.

**Lemma 5.2.2** (Key Lemma 1)**.** *For large enough $t$, for any fixed $r$, and for any $M = M(t) \geq 1$, we have*

$$\min_{r_l < r} s_M(r, r_l) \geq \frac{r}{3} - 7\log_2 t = \frac{r}{3} - O(\log t).$$

*Recall $d = \beta t\sqrt{\ln t}$. Moreover, for any fixed $r$ and $r_l$, and for any $\epsilon \in (0, \frac{1}{6})$ and $M = M(t) \geq 1$,*

$$s_M(r, r_l) \geq r_l - 1 - 7 \cdot \left(\log_{1/(1-\epsilon)} d + \left(8r_l \cdot \log_{1/(2\epsilon)} M\right)/M\right).$$

Key Lemma 2 gives an upper bound on the number of cliques of order $k$ by combining a better error term and the count of $K_{k-r}$ in the terminal graph $G_r$.

**Lemma 5.2.3** (Key Lemma 2)**.** *Let $r_0 = 4t^{1/2}{\log_2}^{1/4} t$ and recall $d = \beta t\sqrt{\ln t}$. When $t$ is sufficiently large, for any function $M = M(t) \geq 0$, the maximum number of cliques of order $k$ in a graph without $K_t$ as a minor is at most*

$$\sum_{r=1}^{\min(r_0,k))} \sum_{\substack{r_l<r: \\ s_M(r,r_l)\leq t-k}} \left(n \cdot \binom{r-1}{r_l} M^{(r-r_l-1)} \binom{r_0}{r_l} \left(\frac{d}{r_0}\right)^{r_l} \mathcal{N}_{k-r}((\mathcal{G}_{t-r-s_M(r,r_l)+1} \cap \mathcal{D}) \cup \mathcal{H}_{t-r}^{t-r-s_M(r,r_l)})\right).$$

### 5.2.3 Proof of Key Lemma 1

In the following part of this chapter, when we say two disjoint vertex sets $A$ and $B$ are adjacent, it means there exist vertices $x \in A$ and $y \in B$ such that $xy \in E(G)$. If $\{v\}$ and $B$ adjacent, we simply say $v$ and $B$ are adjacent. The next claim lists some simple facts about the peeling process.

**Claim 5.2.6.** *The sequence of graphs $G_i, Y_i$ and vertices $v_i \in K$ satisfy the following properties.*

1. *$v_i \in G_i$ and $v_i$ is of minimum degree in $G_i$, and every vertex in $G_i$ has a missing degree at most $d_i$;*

2. *$G_{i+1}$ does not contain $v_i$ and its non-neighbors in $G$;*

3. *$G_{i+1}$ contains $K \setminus \{v_1, \ldots, v_i\}$;*

4. *$G_{i+1}$ is contained in the subgraph of $G$ induced on the vertex set $N_G(v_1) \cap \cdots \cap N_G(v_i)$, where $N_G(u)$ denotes the neighborhood of $u$ in $G$.*

5. *If $A \subseteq V(G_i)$ and $|A| \geq d_i + 1$, then for every $v \in V(G_i)$, either $v \in A$, or $v$ and $A$ are adjacent. Moreover, if $A \subseteq V(G_i)$ and $|A| \geq 2d_i + 1$, then $G[A]$, the subgraph of $G$ induced by $A$, is connected.*

6. *Suppose $Y_i \neq \emptyset$ and let $y_i \in Y_i$ be the last vertex removed in $Y_i$. Then $y_i$ has no less non-neighbors in $G_{i+1}$ than $v_{i+1}$, which means $y_i$ has at least $d_{i+1}$ non-neighbors in*

$G_{i+1}$.

7. *Let $y \in Y_i$ be the last vertex removed in $Y_i$. Then the missing degree of $y$ in $D_i$ is at most $d_i - d_{i+1}$.*

*Proof.* Facts 1-4 are clear from the description of the peeling process.

First, we will prove Fact 5. For every vertex $v \in V(G_i)$, by Fact 1, its missing degree in $G_i$ is at most $d_i$, so $v$ must have at least one neighbor in $A$ as $|A| \geq d_i + 1$. If $A \subseteq V(G_i)$ with $|A| \geq 2d_i + 1$ and $G[A]$ is disconnected, then one connected component of $G[A]$ has at most $d_i$ vertices. Thus any vertex $u$ in this connected component has at least $|A| - d_i \geq d_i + 1$ non-neighbors in $G$. As $A \subseteq V(G_i)$, the missing degree of $u$ in $G_i$ is at least $d_i + 1$, which contradicts with Fact 1.

Suppose Fact 6 is not true. In the subgraph induced by $\{y_i\} \cup V(G_{i+1})$, vertex $y_i$ has less missing degree than $v_{i+1}$. Thus, $v_{i+1}$ should be deleted before $y_i$ which is a contradiction. Suppose Fact 7 is not true. By Fact 1, the missing degree of $y_i$ in $G_i$ is at most $d_i$ and its missing degree in $G_{i+1}$ is less than $d_{i+1}$ which contradicts Fact 6 in Claim 5.2.6. □

To prove Lemma 5.2.2 and Lemma 5.2.3, a main step is to show that the number of encodings is small. The following simple claim states that the length of encoding $r(K)$ in the peeling process cannot be too large.

**Claim 5.2.7.** *The length of encoding for each clique $K$ is small. In other words, when $t$ is sufficiently large, $r(K) < 4t^{1/2}{\log_2}^{1/4}t$.*

*Proof.* This argument is almost the same as in the paper [14] and it is mainly due to the fact that before stopping, the bound on $n_r - n_{r+1}$ deduced from the bound of $d_r$ in the second stop condition guarantees that each time $n_r$ drops a lot. Recall that we set $d = \beta t\sqrt{\ln t}$ where $\beta = 0.64$. Recall that the result of Thomason [45] implies $n_2 = |G_2| \leq d + 1$ when $t$ is sufficiently large. Let $n_i' = n_i + i - t$.

**Fact 5.2.4.** *For every $i < r$, we have $n_i' - n_{i+1}' > \frac{1}{2}(n_i')^{\frac{1}{2}}$, and thus $(n_i')_i$ is strictly decreasing.*

*Proof.* By the definition of $r(K)$, before stopping, $d_i > \frac{1}{2}(n_i')^{\frac{1}{2}}$ for every $i < r$. Thus $n_i' - n_{i+1}' = (n_i + i - t) - (n_{i+1} + i + 1 - t) = n_i - n_{i+1} - 1 = |L_i| \geq |D_i| = d_i > \frac{1}{2}(n_i')^{\frac{1}{2}}$ for every $i < r$.

Because of the first stopping condition, we have $n_i > t - i$ for every $i < r$. Thus, we have $n_i' > 0$ and $d_i \geq 1$ for every $i < r$. Thus, $n_i'$ is strictly decreasing before stopping. □

For each $0 \leq i \leq 2\log_2(d-t)$, let $c_i = (d-t+3)/2^i$. For any $j \leq r$ with $c_i \geq n_j' \geq c_{i+1}$, we have $n_j' - n_{j+1}' > \frac{1}{2}(n_j')^{1/2} \geq \frac{1}{2}c_{i+1}^{1/2}$. Therefore, to drop the $n_j'$ value from $c_i$ to $c_{i+1}$, the number of steps it takes is at most

$$1 + (c_i - c_{i+1})/(\frac{1}{2}c_{i+1}^{1/2}) = 1 + 2c_{i+1}^{1/2} = 1 + 2((d-t+3)/2^{i+1})^{1/2}.$$

Note that, when $t$ is sufficiently large, for each $2 \leq j \leq r$, there is some $i \leq 2\log_2(d-t)$ with $c_i \geq n_j' \geq c_{i+1}$ because $n_j' \leq n_2' \leq d - t + 3$. Thus, $r \leq 1 + \sum_{i=0}^{2\log_2(d-t)}(1 + 2((d - t+3)/2^{i+1})^{1/2}))$
$\leq 1 + 2\log_2 d + 2(d+3)^{1/2}\sum_{i=0}^{\infty} 2^{-(i+1)/2} < 4t^{1/2}\log_2{}^{1/4} t := r_0$. The last inequality holds when $t$ is sufficiently large and plugging in $d = \beta t\sqrt{\ln t}$. □

In the following, we will always let $r_0 = 4t^{1/2}\log_2{}^{1/4} t$. Next, we prove the fact that $s_M(r, r_l)$ is relatively large when $r$ is large, which is the first statement in Lemma 5.2.2.

**Lemma 5.2.5.** *For any $r$ and for any every $k$-clique $K$ with exactly $r$ peeling steps, we have $s(K) \geq \frac{r-2}{3} - 6\log_t d_2$ where $d_2$ is determined by $K$. As a consequence, when $t$ is sufficiently large, for any $r$,*

$$\min_{r_l < r} s_M(r, r_l) \geq \frac{r}{3} - 7\log_2 t.$$

*Proof.* For any $k$-clique $K$ with exactly $r$ peeling steps, we have defined $L_i$ and $D_i$ for every $i \leq r$ by its peeling process. We show that for every three consecutive layers $L_a, L_{a+1}, L_{a+2}$ such that $d_{a+2} \geq \frac{7}{8}d_a$, we can construct an extra branch disk in these three layers. Notice that, by Fact 5 in Claim 5.2.6, every vertex set in $G_a$ with at least $2d_a + 1$ vertices induces a connected subgraph.

Let $A_a = D_a \cup D_{a+1} \cup D_{a+2}$, so $|A_a| \geq d_a + 2 \cdot \frac{7}{8} d_a \geq 2d_a + 1$. By Fact 5 in Claim 5.2.6, $G[A_a]$ is connected and every vertex in $G_a$ is adjacent to a vertex in $A_a$. Thus, we have $A_a$ is adjacent to every vertex in $V(G_r)$ and $V(K) - \{v_1, v_2, \cdots, v_{a-1}\}$. By Fact 2 in Claim 5.2.6, every vertex in $A_a$ is adjacent to $v_a$ for every $i \in [a-1]$. Thus, $A_a$ is an extra branch disk.

Set $i_1 = 2$, recursively define $i_{j+1}$ as the smallest integer such that $d_{i_{j+1}} \leq \frac{7}{8} d_{i_j}$. Then we can partition set of all layers except $L_1$ and $L_r$ into brackets of consecutive layers with brackets $P_j = \{L_{i_j}, L_{i_j+1}, \ldots, L_{i_{j+1}-1}\}$. Thus, there are at most $\log_{\frac{8}{7}} d_2 < 6 \log_2 d_2$ brackets. Assume there are $l$ brackets and let $d_{i_{l+1}} = r$ for convenience. For any three consecutive layers $L_a, L_{a+1}, L_{a+2}$ in the bracket $P_j = \{L_{i_j}, L_{i_j+1}, \ldots, L_{i_{j+1}-1}\}$, we can construct a branch vertex $A_a$, so we can construct $\lfloor \frac{i_{j+1}-i_j}{3} \rfloor$ branch vertices $A_{i_j}, A_{i_j+3}, A_{i_j+6}, \ldots$ in this bracket $P_j$. In total, we construct at least

$$\sum_{j=1}^{l} \lfloor \frac{i_{j+1} - i_j}{3} \rfloor \geq \sum_{j=1}^{l} \left( \frac{i_{j+1} - i_j}{3} - 1 \right) \geq \left( \sum_{j=1}^{l} \frac{i_{j+1} - i_j}{3} \right) - l \geq \frac{r-2}{3} - 6 \log_t d_2$$

disjoint branch vertices. These extra branch disks are pairwise adjacent which means they form a branch vertex set. When $t$ is sufficiently large, we have $d_2 \leq d \leq \beta t \sqrt{\ln t}$ and have $s(K) \geq \frac{r}{3} - 7 \log_2 t$ for every $k$-clique $K$.

□

**Remark.** *With more effort, we can show that, for every $k$-clique $K$ with exactly $r$ peeling steps, $s(K) \geq \frac{r}{2} - O(\log t)$ when $t$ is sufficiently large. However, Lemma 5.2.5 is good enough to prove the main result of this chapter, Theorem 5.1.5, for very large $k$, which is the Theorem 5.4.1.*

When $r_l$ is large, i.e., there are many layers with large $Y_i$, we can expect to find more branch vertices. Now we will prove the second statement of Lemma 5.2.2 in Lemma 5.2.7, which is more technical than the proof of Lemma 5.2.5.

The rough idea is as follows. Suppose we have a sequence of layers whose $D_i$ do not differ by much in sizes, then we will first try to construct the extra branch disk in the topmost layer with non-empty $Y_i$. Recall that $y_i$ is the last removed vertex in $Y_i$. If the subgraph induced by $D_i \cup \{y_i\}$ is connected, by Fact 5 in Claim 5.2.6, it can be contracted as an extra branch disk that is adjacent to every vertex in $G_i$ as $|V(D_i \cup \{y_i\})| = d_i + 1$. If not, we will try to use some vertices in lower layers to connect the different connected components of $D_i \cup \{y_i\}$, and make all these vertices an extra branch disk (Claim 5.2.8). We will try to construct the other extra branch disks greedily. Suppose in layer $j$, the set $U$ is the set of vertices that have not been used. Then Claim 5.2.8 will show either $U$ itself could be an extra branch disk, or we could add to $U$ a small set of vertices from lower layers so that this set together with $U$ is a valid extra branch disk. We will show that by first preprocessing the layers properly, this greedy construction process will work for most of the layers (Claim 5.2.9).

**Claim 5.2.8.** *Fix $\epsilon \in (0, \frac{1}{6})$. Suppose $L_i$ is a layer with nonempty $Y_i$. Let $y = y_i$ be the last removed vertex in $Y_i$ in the peeling process. For every $U \subseteq D_i$ such that $|U| \geq 2\epsilon d_i$ and $d_{i+1} \geq (1-\epsilon)d_i$, and for every $X \subseteq G_{i+1}$ such that $|X| \geq 3.5 d_i$, there exists $W \subseteq X$ that the subgraph induced by $U \cup W \cup \{y\}$ is connected, and $d_i \leq |U \cup W| \leq d_i + 2\log_{\frac{1}{2\epsilon}} d_i$.*

*Proof.* By Fact 7 in Claim 5.2.6, the missing degree of $y$ in $D_i$ is at most $d_i - d_{i+1} \leq \epsilon d_i$, so $|N(y) \cap U| \geq |U| - \epsilon d_i \geq 2\epsilon d_i - \epsilon d_i = \epsilon d_i$ which means $N(y) \cap U$ is non-empty. Let $L$ be the connected component in $L_i$ containing $y$ and $N(y) \cap U$ (clearly $y$ is adjacent to every vertex in $N(y) \cap U$). Let $R = U - L$ and $R_1, R_2, \ldots, R_l$ be the connected components in the graph induced by $R$. Because $R$ and $y$ are not adjacent, $|R|$ is upper bounded by the missing degree of $y$ in $D_i$. In other words,

$$|R| \leq d_i - d_{i+1} \leq d_i - (1-\epsilon)d_i = \epsilon d_i. \tag{5.1}$$

Let $X' = N(y) \cap X$ and $O = D_i - U$. An illustration is shown in Figure 5.2.

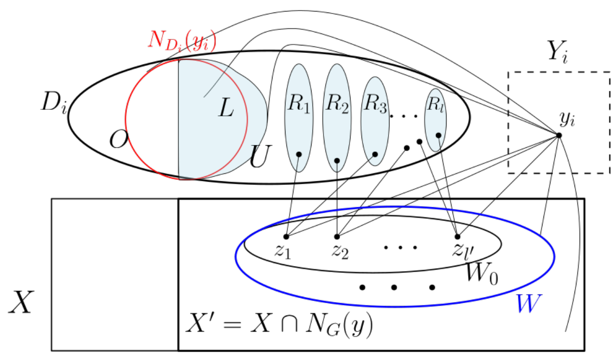


Figure 5.2: Construct an extra branch disk for layer $L_i$

Our goal is to find a vertex set $W$ in $X'$ to connect $L$ and the $l$ connected components of $R$. Since $y$ has at most $d_i$ missing edges in $G_i$, we have $|X'| \geq |X| - d_i \geq 2.5d_i$. To find the proper $W$, we will first try to find a set $W_0 \subseteq X'$ such that $\{y\} \cup U \cup W_0$ is connected and $|W_0| \leq |O| + \log_{\frac{1}{2\epsilon}} d_i$. If $|W_0| \geq |O|$, then we let $W = W_0$; if $|W_0| \leq |O|$, we will add $|O| - |W_0|$ vertices from $X' \setminus W_0$ to $W_0$ to construct $W$. The proof falls into the following two cases:

**Case 1: $|O| \geq l$.** Recall $l$ is the number of connected components in the graph induced by $R$.

We will construct $W_0$ by picking one vertex $z_i$ in $X'$ for each $R_i$ where $i \in [l]$. Here the $z_i$'s do not need to be distinct. For any vertex $v \in R_i \subseteq G_i$, it has at most $d_i$ non-neighbors in $G_i$. Thus, this vertex $v$ has at least one neighbor in $X'$ as $X' \subseteq G_i$ and $|X'| \geq 2.5d_i$. Then we select one of these neighbors arbitrarily as $z_i$. Let $W_0 = \{z_1, z_2, \ldots, z_l\}$.

Because $|W_0| \leq l \leq |O|$, we have $|U \cup W_0| \leq |U| + |O| = |D_i| = d_i$. To construct a proper $W$, we need to add $|O| - |W_0|$ vertices from $X' \setminus W_0$ to $W_0 = \{z_1, z_2, \ldots, z_l\}$ to form the set $W$. We can do this addition because $|X' \setminus W_0| \geq |X'| - |O| \geq 2.5d_i - d_i > d_i \geq |O| - |W_0|$. Thus we have that the subgraph induced by $U \cup W \cup \{y\}$ is connected as every vertex in $W \subset X'$ is adjacent to $y$, and $z_i \in W_0$ is connected to $y$ and the connected component $R_i$. The size satisfies $|U \cup W| = (d_i - |O|) + |W_0| + (|O| - |W_0|) = d_i$.

**Case 2:** $|O| \leq l$**.** Because $l$ is the number of connected components in the graph induced by $R$, we have $|R| \geq l$. Thus, $|O| \leq l \leq |R| \leq 2\epsilon d_i$ by (5.1). For every $j \in [l]$, for any vertex $v \in R_j$, $v$ has at least $|L|$ nonneighbors in $L_i$ because $R_j$ is disconnected from $L$. So $v$ has at most $d_i - |L| = |R| + |O| \leq 4\epsilon d_i$ non-neighbors in $X'$ as we just showed $|O| \leq |R| \leq 2\epsilon d_i$.

Let $l' = \lfloor 2\log_{\frac{1}{2\epsilon}} d_i \rfloor$. We now find a set $W_0 \subset X'$ with size $l'$ such that $\{y\} \cup U \cup W_0$ is connected. To do this, we pick $l'$ vertices independently uniformly at random from $X'$, and let them be the set $W_0$. The event $|W_0| < l'$ happens when some vertex was selected more than once. By a union bound, $\Pr(|W_0| < l') \leq \binom{l'}{2} \cdot \frac{|X'|}{|X'|^2} < \frac{(l')^2}{2|X'|} \leq \frac{(l')^2}{5d_i}$.

When $|W_0| = l'$, for every $j \in [l]$, the probability that there is no edge between $W_0$ and $R_j$ is at most $\left(\frac{4\epsilon d_i}{|X'|}\right)^{l'} \leq \left(\frac{4\epsilon d_i}{2.5d_i}\right)^{l'} = \left(\frac{8\epsilon}{5}\right)^{l'} < (2\epsilon)^{l'}$. By a union bound, the probability that $|W_0| < l'$ or $|W_0| = l'$ but there exists a $j \in [l]$ such that $W_0$ and $R_j$ is not adjacent is at most

$$\frac{(l')^2}{5d_i} + l \cdot (2\epsilon)^{l'} \leq \frac{(l')^2}{5d_i} + 2\epsilon d_i \cdot (2\epsilon)^{l'} \leq \frac{(l')^2}{5d_i} + 2\epsilon d_i \cdot (2\epsilon)^{(2\log_{\frac{1}{2\epsilon}} d_i)-1} = \frac{(2\log_{\frac{1}{2\epsilon}} d_i)^2}{5d_i} + \frac{1}{d_i} < 1$$

The second inequality holds because $l' = \lfloor 2\log_{\frac{1}{2\epsilon}} d_i \rfloor \geq 2\log_{\frac{1}{2\epsilon}} d_i - 1$. The last inequality is true since $1/(2\epsilon) \geq 3$ as $\epsilon \in (0, \frac{1}{6}]$. The union bound showed that there is a subset $W_0 \subseteq X'$ with $l' = \lfloor 2\log_{\frac{1}{2\epsilon}} d_i \rfloor$ vertices such that every $R_j$ in $R$ has at least one neighbor in $W_0$. Thus, together with the fact that every vertex in $X'$ is adjacent to $y$, we have that $\{y\} \cup U \cup W_0$ is connected.

We now construct a set $W$ with the desired size. If $|O| \leq l'$, let $W = W_0$; if $|O| > l'$, then add $|O| - l'$ vertices from $X'$ to $W_0$, and let this new set be $W$. We can do this because $|X' \setminus W_0| \geq |X'| - |O| \geq 2.5d_i - d_i > d_i = |D_i| \geq |O| > |O| - l'$. The first inequality is because $|W_0| = l' \leq |O|$, and the second inequality is because $|X'| \geq 2.5d_i$ and $O \subseteq D_i$. By the definition of $X'$, every vertex in $X'$ is adjacent to $y$. By the fact that $W \subset X'$ and that we have just shown $\{y\} \cup U \cup W_0$ is connected, we have that $U \cup W \cup \{y\}$ is connected. For the size of $U \cup W$, we have $|U \cup W| = d_i - |O| + \max\{|O|, l'\}$. Therefore $d_i \leq |U \cup W| \leq d_i + l' \leq d_i + 2\log_{1/2\epsilon} d_i$. □

In the next claim, we apply Claim 5.2.8 to consecutive layers to construct extra branch disks. To apply Claim 5.2.8, we need to cut all layers into brackets such that $d_i/d_j$ is close to 1 for any two layers $L_i, L_j$ in the same bracket. For a fixed $\epsilon$, we set $p_j = (1-\epsilon)^j d_2$ for every $j \geq 0$, and then let $P_j$ be the set of layers $L_i$ such that $d_i \in (p_j, p_{j-1}]$. Because $d_i$ is non-increasing as $i$ increases, we partition the set of all layers (except $L_1$ and $L_r$) into brackets of consecutive layers $P_j = \{L_{i_j}, L_{i_j+1}, \ldots, L_{i_{j+1}-1}\}$. For any $j$, for any $L_a, L_b \in P_j$, we have

$$d_a > (1-\epsilon)^j d_2 \geq (1-\epsilon)(1-\epsilon)^{j-1} d_2 \geq (1-\epsilon) d_b.$$

Also, there are at most $\log_{\frac{1}{1-\epsilon}} d_2$ brackets. We will try to create branch vertices from vertices in the same bracket.

**Claim 5.2.9.** *For any fixed $\epsilon \in (0, \frac{1}{6}]$, for any $T$ layers $\{L_a, L_{a+1}, \ldots, L_{a+T-1}\}$ in the same bracket where $T \leq \frac{d_a}{4\log_{\frac{1}{2\epsilon}} d_a}$ and $Y_{a+l} \neq \emptyset$ for every $l \in [0, T-8]$, we can create $T-7$ disjoint extra branch disks $\{A_a, A_{a+1}, \ldots, A_{a+T-8}\}$ which are disjoint with $V(K)$, such that $A_i \subseteq V(G_i)$ and $1 + d_i \leq |A_i| \leq 1 + d_i + \log_{\frac{1}{2\epsilon}} d_i$, and $D_i \subseteq A_a \cup A_{a+1} \cup \cdots \cup A_i$ for every $i \in [a, a+T-8]$.*

**Remark.** *The proof does not require these $T$ layers to be consecutive in the original peeling process. For any $T$ layers in the same bracket, if we relabel indices of them by $\{a, a+1, \cdots, a+T-1\}$ based on their original order, then the same result follows.*

*Proof.* For $T \leq 7$, this claim is trivially true. Assume $T \geq 8$. Let $y_i$ be the last removed vertex in $Y_i$ for each $i$ and $Y$ be the set of all $y_i$'s for $a \leq i \leq a+T-1$. We will create $T-7$ branch vertices $\{A_a, A_{a+1}, \ldots, A_{a+T-8}\}$ recursively below:

1. Initial Step: Let $U = D_a$ and $X = L_{a+1} \cup \cdots \cup L_{a+T-1} - Y$. Thus, we have

$|U| = d_a > 2\epsilon d_a$ and $|X| \geq (T-1)|D_{a+T-1}| \geq (T-1)(1-\epsilon)d_a \geq 3.5d_a$, and $d_{a+1} \geq (1-\epsilon)d_a$ because $L_a$ and $L_{a+1}$ are in the same bracket. Then, by Claim 5.2.8, we can find $W \subseteq X$ such that $A_a = U \cup W \cup \{y_a\} \subseteq V(G_a)$ and $d_a \leq |U \cup W| \leq d_a + \log_{\frac{1}{2\epsilon}} d_a$ and $D_a = U \subseteq A_a$.

2. $l$-th Step: Suppose we have found the desired $A_a, A_{a+1}, \ldots, A_{a+l-1}$ for some $l \geq 1$. Let $U$ be the unused vertices in $D_{a+l}$, i.e. $U = D_{a+l} - (A_a \cup \cdots \cup A_{a+l-1})$. Let $X$ be the unused vertices in the lower layers in the same bracket excluding the vertices in $Y$, i.e., $X = (L_{a+l+1} \cup \cdots \cup L_{a+T-1}) - (A_a \cup \cdots \cup A_{a+l-1}) - Y$. By definition, $X \subseteq G_{a+l}$.

   In Fact 5.2.6 below, we will show the conditions of Claim 5.2.8 hold for the $U$ and $X$ defined in the $l$-th step. Then Claim 5.2.8 will guarantee a subset $W \subseteq X$ such that $A_{a+l} = U \cup W \cup \{y_{a+l}\} \subseteq V(G_{a+l})$ and $d_{a+l} \leq |U \cup W| \leq d_{a+l} + \log_{\frac{1}{2\epsilon}} d_{a+l}$ and $D_{a+l} \subseteq U \cup (A_a \cup \cdots \cup A_{a+l-1}) \subseteq A_{a+l} \cup (A_a \cup \cdots \cup A_{a+l-1})$. We call $A_{a+l}$ the branch vertex constructed in layer $L_{a+l}$.

**Fact 5.2.6.** *For every $l \in [0, T-8]$, in the $l$-th step defined in the proof of Claim 5.2.9, we have $d_{a+l+1} \geq (1-\epsilon)d_{a+l}$, and $|U| \geq 2\epsilon d_{a+l}$ and $|X| \geq 3.5d_{a+l}$.*

*Proof.* The case $l = 0$ is already proved in the Initial Step in Claim 5.2.9. Also, the condition $d_{a+l+1} \geq (1-\epsilon)d_{a+l}$ is always true because $L_{a+l}$ and $L_{a+l+1}$ are in the same bracket.

Suppose the Fact 5.2.6 works for the first $l-1$ steps for some $l \geq 2$. We now show the $l$-th step also works. Let $A'_i = A_i - \{y_i\}$ and $L'_i = L_i - \{y_i\}$. By the inductive hypothesis, $|A'_i| \leq d_i + 2\log_{\frac{1}{2\epsilon}} d_i \leq d_i + 2\log_{\frac{1}{2\epsilon}} d_a$ for every $i \in [a, a+l-1]$. Thus we have $|A'_a \cup \cdots \cup A'_{a+l-1}| \leq (\sum_{i=a}^{a+l-1} d_i) + l \cdot 2\log_{\frac{1}{2\epsilon}} d_a$.

By the inductive hypothesis, $D_i \subseteq A_a \cup \cdots \cup A_i$ for every $i \in [a, a+l-1]$. Because $D_i$ is disjoint from any $y_j$ for any $i, j \leq r$, we have $D_i \subseteq A'_a \cup \cdots \cup A'_i$ for every $i \in [a, a+l-1]$.

Thus,

$$D_a \cup D_{a+1} \cup \cdots \cup D_{a+l-1} \subseteq A'_a \cup \cdots \cup A'_{a+l-1}. \tag{5.2}$$

and we know $|D_a \cup D_{a+1} \cup \cdots \cup D_{a+l-1}| = \sum_{i=a}^{a+l-1} d_i$.

We now upper bound the number of used vertex in $G_{a+l}$, which is $|V(G_{a+l}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})|$. Because $D_a \cup D_{a+1} \cup \cdots \cup D_{a+l-1}$ is disjoint from $G_{a+l}$, together with (5.2), we have that

$$|V(G_{a+l}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})| \tag{5.3}$$

$$\leq |A'_a \cup \cdots \cup A'_{a+l-1}| - |D_a \cup D_{a+1} \cup \cdots \cup D_{a+l-1}| \tag{5.4}$$

$$= (\sum_{i=a}^{a+l-1} d_i + l \cdot 2 \log_{\frac{1}{2\epsilon}} d_a) - \sum_{i=a}^{a+l-1} d_i \tag{5.5}$$

$$\leq T \cdot 2 \log_{\frac{1}{2\epsilon}} d_a \leq \frac{d_a}{4 \log_{\frac{1}{2\epsilon}} d_a} \cdot 2 \log_{\frac{1}{2\epsilon}} d_a = d_a/2. \tag{5.6}$$

We can now bound $|U|$. In the $l$-th step, the set of unused vertices in $D_{a+l}$, i.e., the set $U = D_{a+l} \setminus (A_a \cup \cdots \cup A_{a+l-1})$, satisfies $|U| = |D_{a+l}| - |D_{a+l} \cap (A_a \cup \cdots \cup A_{a+l-1})| = |D_{a+l}| - |D_{a+l} \cap (A'_a \cup \cdots \cup A'_{a+l-1})|$ is at least

$$|D_{a+l}| - |V(G_{a+l}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})| \geq d_{a+l} - d_a/2 \geq (1-\epsilon)d_a - d_a/2 \geq 2\epsilon d_a \geq 2\epsilon d_{a+l}.$$

where the first inequality is by (5.6)and the third inequality is true because $\epsilon \leq \frac{1}{6}$.

We now bound $|X|$. Because $L_{a+l+1} \cup \cdots \cup L_{a+T-1} \subseteq V(G_{a+l+1}) \subseteq V(G_{a+l})$, then we have

$$|(L'_{a+l+1} \cup \cdots \cup L'_{a+T-1}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})| \leq |V(G_{a+l}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})| \leq d_a/2. \tag{5.7}$$

We can now show $|X| \geq 3.5d_a$ as

$$\begin{aligned}|X| =&|(L_{a+l+1} \cup \cdots \cup L_{a+T-1}) - (A_a \cup \cdots \cup A_{a+l-1}) - Y| \\ =&|(L'_{a+l+1} \cup \cdots \cup L'_{a+T-1}) - (A'_a \cup \cdots \cup A'_{a+l-1})| \\ =&|L'_{a+l+1} \cup \cdots \cup L'_{a+T-1}| - |(L'_{a+l+1} \cup \cdots \cup L'_{a+T-1}) \cap (A'_a \cup \cdots \cup A'_{a+l-1})| \\ \geq&(T-l)(1-\epsilon)d_a - d_a/2 \geq 4d_a - d_a/2 \geq 3.5d_a \geq 3.5d_{a+l}.\end{aligned}$$

where the first inequality is by (5.7) and the second inequality is by the fact that $l \leq T-8$. □

Next, we show each $A_i$ constructed by this process is an extra branch vertex. The induced subgraph of $A_i$ is connected which is guaranteed by Claim 5.2.8. For $v_1, v_2, \cdots, v_{i-1} \in V(K)$, they are all adjacent to $y_i \in V(G_i)$ by Fact 2 in Claim 5.2.6. Also, $y_i$ and $v_i$ are adjacent by definition of $Y_i$ in the peeling process. The rest of vertices of $K$ and $V(G_r)$ are contained in $V(G_i)$. Because $|A_i| \geq d_i + 1$, by Fact 5 in Claim 5.2.6, we can show the rest of the vertices in $K$ and $V(G_r)$ are all adjacent to $A_i$. Thus, $A_i$ is an extra branch disk. We have completed the proof of Claim 5.2.9 and will show that $A_i$ and $A_j$ are adjacent later. □

**Lemma 5.2.7.** *Let $t$ be sufficiently large. For any fixed $r$ and $r_l$, and for any $\epsilon \in (0, \frac{1}{6}]$ and $M = M(t) \geq 1$, we have*

$$s_M(r, r_l) \geq r_l - 1 - 7 \cdot (\log_{1/(1-\epsilon)} d + 8r_l \cdot \log_{1/(2\epsilon)} M/M).$$

*Proof.* For any $k$-clique $K$ with indices $r(K) = r$ and $R_M(K) = r_l$, we have defined $v_i$, $L_i$, $D_i$, $d_i$ and $Y_i$ for every $i \leq r$ by its peeling process. To prove this lemma, we will try to construct a branch vertex for almost every layer $L_i$ with $|Y_i| > M$ by combining the last removed vertex $y_i$ in $Y_i$ and its neighbors in $D_i$ with a small set of vertices in the lower layers.

We first consider the layers $L_i$ with $|Y_i| \geq \max(M, 2d_i + 1)$. For each such layer $L_i$, we claim $Y_i$ could be a branch disk. This is because by Fact 5 in Claim 5.2.6, the induced subgraph $G[Y_i]$ is connected and adjacent to every vertex in $V(G_i)$. The vertices $v_1, v_2, \cdots, v_{i-1} \in V(K)$ are all adjacent to $Y_i \subset V(G_i)$ by Fact 2 in Claim 5.2.6. Also, $y_i$ and $v_i$ are adjacent by the definition of $Y_i$ in the peeling process. Suppose there are $r_l - r_l'$ layers $L_i$ with $|Y_i| \geq \max(M, 2d_i + 1)$ and thus we have already constructed $r_l - r_l'$ branch vertices only using vertices in the layers $L_i$ where $|Y_i| \geq \max(M, 2d_i + 1)$.

We now only consider the layers with $M \leq |Y_i| < 2d_i + 1$. For convenience, we remove the layers $L_i$ with $|Y_i| < M$ or $|Y_i| \geq \max(M, 2d_i + 1)$. Say there are $r_l' \geq 0$ layers left. To prove the lemma, it suffices to prove that we can construct $r_l' - 1 - 7 \cdot (\log_{\frac{1}{1-\epsilon}} d_2 + (8r_l \cdot \log_{\frac{1}{2\epsilon}} M)/M)$ branch disks in the remaining $r_l'$ layers.

For our convenience, relabel the indices $i$ of the remaining layers $L_i$'s in order, and thus rename the index $i$ in the corresponding $v_i$, $D_i$, $d_i$, and $Y_i$'s. We then have layers $L_1, \ldots, L_{r_l'}$. Because every layers $L_i$ with $|Y_i| < M$ or $|Y_i| \geq 2d_i$ was removed, we have $d_i \geq \frac{1}{2}|Y_i| \geq \frac{M}{2}$ for every $i \leq r_l'$.

Let $T_0 = \frac{d_{r_l'}}{4 \log_{\frac{1}{2\epsilon}} d_{r_l'}}$, so $T_0 \geq \frac{M}{8 \log_{\frac{1}{2\epsilon}} M}$. For each bracket $P_j$, we partition layers in this bracket into intervals of consecutive layers where each interval has $T$ layers except possibly the last interval which may have fewer than $T_0$ layers. We call these intervals the processing intervals in $P_j$.

For any processing interval, suppose $L_a$ is the first layer in this interval. Then $\frac{d_a}{4 \log_{\frac{1}{2\epsilon}} d_a} \geq \frac{d_{r_l'}}{4 \log_{\frac{1}{2\epsilon}} d_{r_l'}} = T_0$. Thus, we can apply Claim 5.2.9 for this interval, and then we can construct an extra branch disk $A_i$ for every layer $L_i$ in this interval except the last 7 layers. Now we are ready to complete the proof of Lemma 5.2.7. Recall that there are at most $\log_{\frac{1}{1-\epsilon}} d_2$ brackets $P_j$, so there are at most $\log_{\frac{1}{1-\epsilon}} d_2$ processing interval with fewer than $T_0$ layers. Furthermore, since there are $r_l'$ layers in total, there are at most $r_l'/T_0$ processing intervals with $T_0$ layers. Because $L_1$ and $L_r$ are not in any brackets and $Y_r = \emptyset$, the number of branch disks we could construct is at least

$$r'_l - 1 - 7 \cdot (\log_{\frac{1}{1-\epsilon}} d_2 + r'_l/T_0) \leq r'_l - 1 - 7 \cdot \left(\log_{\frac{1}{1-\epsilon}} d_2 + \left(8r'_l \cdot \log_{\frac{1}{2\epsilon}} M\right)/M\right).$$

Let $\mathcal{A}$ be the set of all these extra branch disks $A_i$ together with all extra branch disks $Y_j$ for the layer with $|Y_j| \geq \max\{M, 2d_j + 1\}$ (before removing the layers $L_i$ with $|Y_i| < M$ or $|Y_i| \geq \max\{M, 2d_j + 1\}$). We now show that $\mathcal{A}$ is a branch vertex set of $K$. The condition we need to check is that any two extra branch disks $A, B \in \mathcal{A}$ are adjacent. Assume $A$ is the extra branch disk for layer $L_i$ and $B$ is the extra branch disk for layer $L_j$ in the original peeling process (without removing layers). Without loss of generality, assume $i < j$. Because $|A| \geq d_i + 1$ and $B \subseteq V(G_j) \subseteq V(G_i)$. By Claim 5.2.6 Fact 5, every vertex in $B$ is adjacent to some vertices in $A$, so $A$ and $B$ are adjacent.

Thus, by definition of $s_M(r, r_l)$, we have $s_M(r, r_l) \geq |\mathcal{A}|$, and $|\mathcal{A}| \geq (r_l - r'_l) + (r'_l - 1 - 7 \cdot (\log_{\frac{1}{1-\epsilon}} d_2 + \left(8r'_l \cdot \log_{\frac{1}{2\epsilon}} M\right)/M))$. When $t$ is sufficiently large, we have $d_2 \leq d$. Thus, we proved the following lower bound $s_M(r, r_l) \geq |\mathcal{A}| \geq r_l - 1 - 7 \cdot \left(\log_{\frac{1}{1-\epsilon}} d + \left(8r'_l \cdot \log_{\frac{1}{2\epsilon}} M\right)/M\right)$. □

### 5.2.4 Proof of Key Lemma 2

To prove Lemma 5.2.3, a main step is to show that the number of encodings is small. In [14], a crude bound $\binom{\beta t\sqrt{\ln t}}{r_0}$ was sufficient. However, this error bound could be too large if we want to count the cliques of a fixed size $k$. In the next lemma, we provide an improved bound on the number of encoding of $k$-cliques $K$ with indices $r(K) = r$ and $R_M(K) = r_l$. Recall that the length $r(K)$ of encoding of any clique $K$ is at most $r_0 = 4t^{1/2}{\log_2}^{1/4}t$.

**Lemma 5.2.8.** *For fixed $r$, $r_l$ and function $M = M(t)$, the number of possible encoding of $k$-cliques $K$ with $r(K) = r$ and $R_M(K) = r_l$ is at most $n\binom{r-1}{r_l}M^{r-r_l-1}\binom{r_0}{r_l}(\frac{\beta t\sqrt{\ln t}}{r_0})^{r_l}$*

*Proof.* For a given clique $K$, we separate the steps $1 \leq i \leq r(K) = r$ depending on whether $|Y_i|$ is large or not. To be more precise, let $L(K) \cup S(K) = [r]$ be the partition of $[r]$ such that $L(K) = \{i \in [r] | |Y_i| \geq M\}$ and $S(K) = \{i \in [r] | |Y_i| < M\}$. So $|S(K)| = r - r_l$. For any fixed subset $L \subset [r]$, let $C(L)$ be the set of all possible encoding of cliques $K$ such that $L(K) = L, S(K) = [r] \setminus L$.

We first bound the size of $C(L)$ for any given $L \subset [r]$. The first vertex $v_1$ has $n$ choices. Once $v_1$ is fixed, all the rest of the vertices will be picked from $N(v_1)$, which has order at most $d$.

**Claim 5.2.10.** *After picking $v_1$, the vertices $v_2, \ldots, v_i$ are uniquely determined by $n_2, \ldots, n_i$.*

*Proof.* We will prove by induction that both $v_i$ and $G_i$ are uniquely determined by $n_2, \ldots, n_i$ after picking $v_1$. After picking $v_1$, we have the unique $G_1$ where $v_1$ has the minimum degree. This is because $G_1$ is obtained from $G$ by removing vertices one at a time degrees smaller than $v_1$. Then from $G_1$, we remove $v_1$ and the non-neighbors of $v_1$, obtaining $G_1'$. Thus $G_1'$ is also uniquely determined by $v_1$. In $G_1'$, we sequentially remove vertices of degrees smaller than $v_2$ (breaking ties by some predetermined order) until in $G_2$, vertex $v_2$ has the minimum degree. So by knowing how many vertices we delete from $G_1'$ to get $G_2$, we will know $v_2$. The number of vertices we delete in this step is $|G_1'| - |G_2|$. However, $|G_1| - |G_1'|$ is also uniquely determined by $v_1$ as shown before. Thus we know $v_2, G_2$ are uniquely determined by $|G_2| - |G_1|$. The base case holds.

Suppose we have found $v_2, \ldots, v_i$ where $v_j, G_j$ for $j \leq i$ are uniquely determined by $n_2, \ldots, n_j$. We have determined a graph $G_i$ where $v_i$ is of minimum degree. Similarly, the graph $G_{i+1}$ is the induced subgraph of $G_i$ after removing $v_i$ and the non-neighbors of $v_i$, and then we delete from $G_{i+1}$ other vertices till $v_{i+1}$ is the minimum degree (after breaking the tie by some predetermined order). By a similar argument as before, $G_{i+1}$ and $v_{i+1}$ are uniquely determined by knowing how many vertices are deleted from $G_i$ given $G_i$ and $v_i$. Thus the inductive hypothesis holds. □

Recall that we have shown the sequence $n_i'$ is strictly decreasing before stopping in Fact 5.2.4. The claim above has the following corollary.

**Corollary 5.2.9.** *After picking $v_1$, the vertices $v_2, \ldots, v_i$ are uniquely determined by $n'_2, \ldots, n'_i$ where $n'_i = n_i + i - t$ and $n'_i$ are strictly decreasing.*

For any $L$ and $S$, assume we have already selected $v_1, v_2, \cdots, v_{i-1}$, which are the first $i-1$ vertices in some encoding in $C(L)$. Then $G_1, G_2, \cdots, G_{i-1}$ and $n'_1, n'_2, \cdots, n'_{i-1}$ are also determined. To select $v_i$ such that $v_1, v_2, \cdots, v_i$ are the first $i$ vertices in some encoding in $C(L)$, it suffices to select the number $n'_i$ by Corollary 5.2.9. If $i-1 \in S$, then $n'_{i-1} - n'_i$ is not too large by the definition of $S$. Because $n'_i$ are strictly decreasing before stopping, we will see that there are not too many choices for $n'_i$ if $i-1 \in S$. Thus, we define $S' = \{i \in [r] | i-1 \in S\}$ and $L' = \{i \in [r] | i-1 \in L\}$, and we will bound the number of choices of $n_i$ in $L'$ and $S'$ separately. Then we have $L' \cup S' = [r] \setminus \{1\}$ is a partition of $[r] \setminus \{1\}$. In addition, $|L'| = |L|$ and $|S'| = |S| - 1$ because $r \in S$ by the definition of $S$.

Next, we bound the number of possible subsequences of $n'_{i_1}, n'_{i_2}, \cdots, n'_{i_{|L'|}}$ where $i_j \in L'$. Because of the first stopping condition, for every $i \neq r$, we have $n_i > t - r$. Thus, for every $i \neq r$, $n'_i \in [1, d+1-t]$. Also, $n'_r = n_r + r - t \geq (k - r + 1) + r - t = k + 1 - t \geq -t$ because $G_r$ contains $k - r + 1$ vertices in $V(K)$ and $k \geq 2$.

We partition the interval $(0, d+1-t]$ into intervals $I_i = (b_{i+1}, b_i]$, $i \geq 1$, where $b_1 = d + 1 - t$, and for all $i$ where $b_i > 1$, $b_{i+1} = \min(\lceil b_i - c b_i^{1/2} \rceil, b_i - 1)$. In this way, no two values $n'_j, n'_{j'}$ can be in the same interval $I_i$ by the fact that $n'_{j+1} \leq n'_j - c(n'_j)^{1/2}$ and the monotonicity of the function $\min(\lceil x - c\sqrt{x} \rceil, x - 1)$ for integers $x \geq 1$. Assume $[1, d+1-t]$ is partitioned into $l$ intervals, then let $I_{l+1} = [-t+3, 0]$. No two values $n'_j, n'_{j'}$ can be in the interval $I_{l+1}$ because $n'_i$ is positive for every $i \neq r$. Thus the number of choices for $n'_{i_1}, n'_{i_2}, \cdots, n'_{i_{|L'|}}$ is at most

$$\sum_{j_1 < \cdots < j_{|L'|}} |I_{j_1}| \ldots |I_{j_{|L'|}}| \tag{5.8}$$

This is because we first need to pick the $|L'|$ different intervals $I_j$'s'. And once knowing $n'_i$

is in some interval $I_{j_i}$, there are at most $|I_{j_i}|$ ways to choose $n'_i$.

Note that union of the disjoint intervals $I_i$, which is $[-t+3, d+1-t]$, has length $d-2$, and $[-t+3, d+1-t]$ is partitioned into $l+1$ intervals. By convexity, the quantity (5.8) is upper bounded by $\binom{l+1}{|L'|}\left(\frac{d}{l'+1}\right)^{|L'|}$. Furthermore, from the proof of Claim 5.2.7, we know $l+1 < r_0 = 4t^{1/2}\log_2^{1/4} t$. We have (5.8), which is the number of possible subsequences of $n'_{i_1}, n'_{i_2}, \cdots, n'_{i_{|L'|}}$ where $i_j \in L'$, is at most

$$\binom{r_0}{|L'|}\left(\frac{d}{r_0}\right)^{|L'|}. \tag{5.9}$$

Next, we select $n'_i$ for $i \in S' = \{i \in [r] | i-1 \in S\}$ one at a time. Assume $n'_{i-1}$ is already selected. By Corollary 5.2.9, the vertex $v_{i-1}$ is already fixed and $d_{i-1}$ is known. For every $i \in S'$, we have $i-1 \in S$ by the definition of $S$, we have $|Y_{i-1}| < M$. Because $|Y_{i-1}| = |L_{i-1}| - |D_{i-1}| = n'_{i-1} - n'_i - d_{i-1}$, we have $n'_i = n'_{i-1} - d_{i-1} - |Y_{i-1}|$. Thus, $n'_i$ can be selected in the range $[n'_{i-1} - d_{i-1} - (M-1), n'_{i-1} - d_{i-1}]$. Thus, there are at most $M$ choices for each $n'_i$ for $i \in S'$.

Combining the results above, and notice that $v_1$ has $n$ choices, we have shown that, for a fixed $L \cup S$, $|C(L)| \leq n \cdot M^{|S'|}\binom{r_0}{|L'|}\left(\frac{d}{r_0}\right)^{|L'|}$.

For fixed $r, r_l$, we have $|L'| = |L| = r_l$ and $|S'| = |S| - 1 = r - r_l - 1$ by the discussion earlier. There are at most $\binom{r-1}{r_l}$ ways to determine the partition $L \cup S$ of $[r]$ with $r \in S$ and $|L| = r_l$. Thus, the number of possible encoding of cliques $K$ with indices $r, r_l$ is at most $n \cdot \binom{r-1}{r_l} \cdot M^{r-r_l-1}\binom{r_0}{r_l}(\frac{d}{r_0})^{r_l}$, as desired. □

*Proof of Lemma 5.2.3.* We first group the cliques of order $k$ by the encoding $v_1, \ldots, v_{r(K)}$, and then by the values $r(K)$ and $R_M(K)$. Similar to [14], we will bound the number of $k$-cliques $K$ in $G$ with $r(k) = r < k$ and $R_M(K) = r_l$ by the product of the number of possible encoding of $k$-cliques with indices $r$ and $r_l$ proved in Lemma 5.2.8 and the number of $k$-cliques $K$ given $v_1, \ldots, v_{r(K)}$ with $r(K) = r$ and $R_M(K) = r_l$.

Clearly $r(K) \leq k$. If $r(K) = k$, the $k$-clique $K$ is completely determined by the encoding. Next, we will bound the number of $k$-cliques given a fixed encoding given $v_1, \ldots, v_{r(K)}$ with $r(K) = r < k$ and $R_M(K) = r_l$. Recall that the encoding uniquely determines the terminal graph $G_{r(K)}$. We thus bound the number of cliques of order $k - r$ in $G_r$. Recall that $|V(G_r)| = n_r$.

For the cliques with $r(K) = r$ and $R_M(K) = r_l$, we split the cliques into three types: (i) those with $n_r \leq t - r$, (ii) those with $r = k$ and $n_r \geq t - r$, and (iii) those with $r < k$, $n_r \geq t - r$, and $d_r \leq \frac{1}{2}(n_r + r - t)^{1/2}$. By Claim 5.2.4, the Hadwiger number of $G_r$ is a most $t - r - s(K) \leq t - r - s_M(r, r_l)$.

For type (i) where $n_r \leq t - r$, it is not hard to see $G_r \in \mathcal{H}_{t-r}^{t-r-s_M(r,r_l)}$ where recall $\mathcal{H}_m^s$ is the family of graphs $H$ with at most $m$ vertices and its clique minor has size at most $s$. Thus, the number of cliques of order $k - r$ in the terminal graph $G_r$ which are of type (i) is at most $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s_M(r,r_l)})$. For the cliques of type (ii), i.e., cliques of order $k$ with $r = k$ and with the same encoding, the encoding of length $r = k$ uniquely determines the clique $K$ of order $k$.

Finally, we bound the number of cliques of type (iii), i.e., cliques with $n_r > t - r$, $r < k$, and $d_r \leq \frac{1}{2}(n_r + r - t)^{1/2}$.

**Claim 5.2.11.** *If we stop at step $r$ with graph $G_r$ such that $r < k$ and $n_r \geq t - r$, then $G_r \in \mathcal{G}_{t-r-s_M(r,r_l)+1} \cap \mathcal{D}$.*

*Proof.* In this case, recall in the graph $G_r$ the vertex $v_r$ has the minimum degree. Thus, the complement of $G_r$ has maximum degree $\Delta = |D_r| = d_r \leq \frac{1}{2}(n_r + r - t)^{1/2}$ by the definition of $r(K)$. This means in $G_r$

$$\Delta \leq \frac{1}{2}(n_r + r - t)^{1/2}. \tag{5.10}$$

Let $\omega$ be the clique number of $G_r$. Since $G$ has no $K_t$-minor and $G_r$ is in the common neighborhood of $v_1, \ldots, v_r$, we need $\omega < t-r$, so $n_r - \omega > n_r - (t-r) \geq (2\Delta)^2 \geq 2\Delta^2 + 2$ where the second inequality holds by (5.10) and the last inequality holds when $\Delta \geq 1$.

Hence, we have $G_r \in \mathcal{D}$ and the condition of Lemma 5.2.1 is satisfied. By Lemma 5.2.1, the Hadwiger number of $G_r$ is $\lfloor \frac{n_r+\omega}{2} \rfloor \leq t - r - s_M(r, r_l)$, and so $G_r \in \mathcal{G}_{t-r-s_M(r,r_l)+1}$ where $\mathcal{G}_s$ is the family of graphs $G$ such that $\lfloor \frac{|V(G)|+\omega(G)}{2} \rfloor \leq s - 1$ where $\omega(G)$ is the order of the largest clique in $G$. □

Therefore, the number of $k$-cliques $K$ of type (iii) (with indices $r$ and $r_l$) after fixing $v_1, \ldots, v_r$ is bounded above by $\mathcal{N}_{k-r}(\mathcal{G}_{t-r-s_M(r,r_l)+1} \cap \mathcal{D})$. The next claim will give a range of $(r, r_l)$ where it is possible to have a $k$-clique $K$ with $r(K) = r$ and $R_M(K) = r_l$.

**Claim 5.2.12.** *Let $\lambda = t - k$. Let $K$ be a clique of order $k$ with pair of indexes $(r, r_l)$ and let $G_r$ be its terminal graph, it must holds that $s_M(r, r_l) \leq \lambda$.*

*Proof.* We prove by contradiction. Suppose $s_M(r, r_l) > \lambda$. From the Claim 5.2.4, the Hadwiger number of $G_r$ is a most $t - r - s(K) \leq t - r - s_M(r, r_l) < k - r$, which implies we cannot find any clique of order $k - r$ in $G_r$. However, the subgraph induced by $K - \{v_1, v_2, \ldots, v_{r-1}\}$ in $G_r$ is a clique of order $k - r$, a contradiction. □

Thus, combining the results on the cliques of types (i), (ii), (iii), the number of $k$-cliques with indices $r$ and $r_l$ after fixing $v_1, \ldots, v_r$ is bounded above by $\mathcal{N}_{k-r}((\mathcal{G}_{t-r-s_M(r,r_l)+1} \cap \mathcal{D}) \cup \mathcal{H}_{t-r}^{t-r-s_M(r,r_l)})$. Combining with the bound on the number of possible encoding $v_1, \ldots, v_r$ by Lemma 5.2.8, the desired quantity is proved, and here the summation over $r_l$ where $s_M(r, r_l) \leq \lambda$ is by Claim 5.2.12. □

**Remark.** *The reason we need to bound $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s_M(r,r_l)})$ separately is because the optimizer candidate $T(t-r, t-r-s)$ is not in the family $\mathcal{G}_{t-r-s_M(r,r_l)+1}$ unless $s = 0$ which will be discussed in the proof of the next Corollary 5.2.10.*

Sometimes we do not need this elaborated upper bound in the Lemma 5.2.3. If we group cliques only by $r(K)$, the length of their encoding, then we can get the following cruder upper bound.

**Corollary 5.2.10.** *Let $r_0 = 4t^{1/2}{\log_2}^{1/4}t$. When $t$ is sufficiently large, the maximum number of cliques of order $k$ in a $K_t$-minor free graph on $n$ vertices is at most*

$$n \cdot \sum_{r=1}^{\min(r_0,k))} \left( \binom{r_0}{r-1} \left( \frac{\beta t \sqrt{\ln t}}{r_0} \right)^{r-1} \mathcal{N}_{k-r}(\mathcal{G}_{t-r+1} \cap \mathcal{D}) \right).$$

*Proof.* We could assume $t \geq r_0 + 1$. For any fixed $r \leq r_0$, by the definition of $\mathcal{G}_t$ and $\mathcal{H}_m^s$, we have $\mathcal{G}_{t-r-s_M(r,r_l)+1} \subseteq \mathcal{G}_{t-r+1}$ and $\mathcal{H}_{t-r}^{t-r-s_M(r,r_l)} \subseteq \mathcal{H}_{t-r}^{t-r}$. Because $K_{t-r} \in \mathcal{H}_{t-r}^{t-r}$ and every $H \in \mathcal{H}_{t-r}^{t-r}$ is a subgraph of $K_{t-r}$, we have $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r}) = \mathcal{N}_{k-r}(\{K_{t-r}\})$. It is not hard to check that $K_{t-r}$ is in $\mathcal{G}_{t-r+1} \cap \mathcal{D}$ by the definitions of $\mathcal{G}_t$ and dense graph. Thus, we have $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s_M(r,r_l)}) \leq \mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r}) \leq \mathcal{N}_{k-r}(\mathcal{G}_{t-r+1} \cap \mathcal{D})$ for any fixed $r \leq r_0$.

Now we bound the number of all possible encoding of cliques with index $r$ for any fixed $r \leq r_0$. Set $M = 0$ in the expression in Lemma 5.2.3 and then, among all $r_l \leq r-1$, the only possibly non-zero summand is when $r_l = r-1$. Thus the second sum only has one term where $r_l = r-1$. Plugging in $r_l = r-1$, the second sum equals to $n\binom{r_0}{r-1}\left(\frac{\beta t\sqrt{\ln t}}{r_0}\right)^{r-1} \mathcal{N}_{k-r}((\mathcal{G}_{t-r-s_M(r,r-1)+1} \cap \mathcal{D}) \cup \mathcal{H}_{t-r}^{t-r-s_M(r,r-1)})$. By the argument above, we have this quantity is at most $n \cdot \binom{r_0}{r-1}\left(\frac{d}{r_0}\right)^{r-1} \mathcal{N}_{k-r}(\mathcal{G}_{t-r+1} \cap \mathcal{D})$. We can finish the proof by adding up the quantities among all possible values of $r$. $\square$

## 5.3 Structure of the optimal graph $T_t^*(k)$

For fixed $t$ and $k$, recall that $T_t^*(k)$ is the Turán graph $T(2t-\omega-1, \omega)$ maximizing the number of cliques of order $k$ among all $\omega$ such that $k \leq \omega \leq t-1$, and $C_t^*(k)$ denoted the number of cliques of order $k$ in $T_t^*(k)$. In this chapter, we will illustrate the asymptotic structure of graph $T_t^*(k)$ for any pairs of $t$ and $k$ when $t$ is sufficiently large.

### 5.3.1 Structure of the optimal graph $T_t^*(k)$ for $k > 2(t+1)/3$

It is a simple computation to show that for any $k > 2(t+1)/3$ , $T_t^*(k)$ is the graph $K_t^-$, the complete graph $K_t$ minus an edge. We will show this result in the following lemma.

**Lemma 5.3.1.** *Fix $t$ and $k > 2(t+1)/3$. The maximum possible number of cliques of order $k$ in $G$ among all $G \in \mathcal{G}_{t+1}$ is at most $\binom{t}{k} + \binom{t-1}{k-1}$. This bound is sharp as the graph $K_{t+1}^-$ has $\binom{t}{k} + \binom{t-1}{k-1}$ cliques of order $k$. Moreover, we have $T_{t+1}^*(k) = K_{t+1}^-$.*

*Proof.* Let $f(t,k)$ be the quantity we want. We want to show $f(t,k) \leq \binom{t}{k} + \binom{t-1}{k-1}$ for $k > 2(t+1)/3$.

For any graph $G$, for simplicity let $n(G)$ be the number of vertices in $G$, and $\omega(G)$ be the order of the largest clique in $G$. We also define $x(G) = \lfloor (n(G)+\omega(G))/2 \rfloor$. Therefore a graph $G \in \mathcal{G}_{t+1}$ is equivalent to $x(G) \leq t$. We prove the desired result by induction on $x(G) = t$.

The base case is when $t \leq 3$. When $t = 1$, then $\lfloor (n+\omega)/2 \rfloor \leq 1$ and $k > 1$. We have $n = 2, \omega = 1$. This is a graph of two isolated vertices. The result also holds.

When $t = 2$, then $\lfloor (n+\omega)/2 \rfloor \leq 2$ and $k \geq 3$. Then $n + \omega = 4$ or 5. Then the graph is either an edge or a path of two edges. The result clearly holds.

When $t = 3$, then $\lfloor (n+\omega)/2 \rfloor \leq 3$ and $k \geq 3$. Then $n + \omega = 6$ or 7. Similarly, we can assume $\omega \geq 3$. So we have two options: $n = \omega = 3$, or $n = 4, \omega = 3$. The result clearly holds in the first case. For the latter case, the graph is a subgraph of $K_4^-$. In this case, the number of cliques of order 3 is at most 2. The result clearly holds.

Now we assume $t \geq 4$. Assuming the result holds for $x(G) = 1, 2, \ldots, t-1$, we want to show it holds for $x(G) = t$.

We first show if we are in the next two cases then we are done. In the graph $G$, let $d_v$ be the missing degree of $v$ in $G$ for any $v \in V(G)$.

**Case 1:** Suppose there are two non-adjacent vertices $u, v \in V(G)$ with $d_u, d_v \geq 2$.

There are three types of cliques of order $k$ in $G$:

Type 1: cliques not containing $u, v$. Then we count cliques of order $k$ in $G \setminus \{u, v\}$. Since $x(G \setminus \{u, v\}) \leq x(G) - 1$, there are at most $f(t-1, k)$ of them. Since $k > 2(t+1)/3$, then $k > 2(t-1+1)/3$. Thus by inductive hypothesis, $f(t-1,k) \leq \binom{t-1}{k} + \binom{t-2}{k-1}$.

Type 2: cliques containing $v$. Thus it does not contain the vertices not adjacent to $v$. Thus we count cliques of order $k-1$ in $G$ removing $v$ and the non-neighbors of $v$. Call this graph $G'$. Then $n(G') = n(G) - d_v - 1$, and $\omega(G') \le \omega(G) - 1$. Because $\lfloor (a-b)/2 \rfloor \le \lfloor a \rfloor - \lfloor b \rfloor$ for any integers $a \ge b$. Therefore $x(G') \le x(G) - \lfloor (d_v+2)/2 \rfloor$. Thus the number of cliques of type 2 is at most $f(t - \lfloor (d_v+2)/2 \rfloor \rfloor, k-1)$. In order to apply the inductive hypothesis, we need to check the condition $k-1 > 2(t - \lfloor (d_v+2)/2 \rfloor + 1)/3$. It is true because $k \ge 2(t+1)/3$ and $2\lfloor (d_v+2)/2 \rfloor/3 \ge 1$ when $d_v \ge 2$. Thus we can apply the inductive hypothesis for $f(t - \lfloor (d_v+2)/2 \rfloor \rfloor, k-1)$.

Type 3 are the cliques containing $u$. Similarly, when $d_u \ge 2$, the number of cliques of type 3 is at most $f(t - \lfloor (d_v+2)/2 \rfloor, k-1)$ and this can be bounded by the inductive hypothesis.

Combining the three types we just need to check

$$f(t-1,k) + f(t - \lfloor (d_v+2)/2 \rfloor, k-1) + f(t - \lfloor (d_u+2)/2 \rfloor, k-1) \le \binom{t}{k} + \binom{t-1}{k-1}.$$

Since $d_u, d_v \ge 2$, by the inductive hypothesis, it suffices to show

$$\binom{t-1}{k} + \binom{t-2}{k-1} + 2\left(\binom{t-2}{k-1} + \binom{t-3}{k-2}\right) \le \binom{t}{k} + \binom{t-1}{k-1}.$$

By using $\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$, it is equivalent to show

$$2\left(\binom{t-2}{k-1} + \binom{t-3}{k-2}\right) \le \binom{t-1}{k-1} + \binom{t-2}{k-2}.$$

It suffices to show the following two inequalities hold simultaneously:

$$2\binom{t-2}{k-1} \le \binom{t-1}{k-1} \iff t-1 \ge 2(t-k);$$

and

$$2\binom{t-3}{k-2} \leq \binom{t-2}{k-2} \iff t-2 \geq 2(t-k);$$

This holds when $t \geq 2$ since we have assumed $k > 2(t+1)/3$.

**Case 2**

Now we suppose we have two adjacent vertices $u, v$ and there are $d_u$ vertices not adjacent to $u$; and $d_v$ vertices not adjacent to $v$ in $G$. Let $d$ be the number of vertices not adjacent to either $u$ or $v$. Suppose $d_u, d_v \geq 1, d \geq 2$.

Similar to before, we have four types of cliques to consider.

Type 1: cliques not containing $u, v$. Then we count cliques of order $k$ in $G \setminus \{u, v\}$. There are at most $f(t-1, k)$ of them. Since $k > 2(t+1)/3$, then $k > 2(t-1+1)/3$. Thus by the inductive hypothesis, $f(t-1, k) \leq \binom{t-1}{k} + \binom{t-2}{k-1}$.

Type 2: cliques containing $u$ but not $v$. Thus it does not contain the vertices not adjacent to $u$. Thus we count cliques of order $k-1$ in $G$ removing the non-neighbors of $v$ and vertices $u, v$. Call this graph $G'$. Then $n(G') = n(G) - d_v - 2$, and $\omega(G') \leq \omega(G) - 1$. Thus the number of cliques of this type is at most $f(t - \lfloor (d_v+3)/2 \rfloor, k-1)$. In order to apply the inductive hypothesis, we need to check $k - 1 > 2(t - \lfloor (d_v+3)/2 \rfloor + 1)/3$. It is true because $k \geq 2(t+1)/3$ and $2\lfloor (d_v+3)/2 \rfloor/3 \geq 1$ when $d_v \geq 1$. Thus we can apply the inductive hypothesis for $f(t - \lfloor (d_v+3)/2 \rfloor, k-1)$.

Type 3: Similarly, when $d_u \geq 1$, the number of cliques containing $u$ but not $v$ is at most $f(t - \lfloor (d_u+3)/2 \rfloor, k-1)$ and to which we can apply the inductive hypothesis.

Type 4: Cliques containing both $u, v$. We only need to count the number of cliques of order $k-2$ in $G'$ which is $G$ removing $\{u, v\}$ and the $d$ vertices not adjacent to either $u$ or $v$. $n(G') = n(G) - d - 2$, and $\omega(G') \leq \omega(G) - 2$. The number of cliques of this type is bounded above by $f(t - \lfloor (4+d)/2 \rfloor, k-2)$. Again, it is not hard to check $4k - 2 > 2(t - \lfloor (4+d)/2 \rfloor + 1)/3$ when $d \geq 2$ and $k \geq 2(t+1)/3$.

We want to check

$$f(t-1,k)+f(t-\lfloor(d_v+3)/2\rfloor,k-1)+f(t-\lfloor(d_u+3)/2\rfloor,k-1)+f(t-\lfloor(4+d)/2\rfloor,k-2) \le \binom{t}{k}+\binom{t-1}{k-1}$$

Since we assumed $d_u, d_v \ge 1, d \ge \max(d_u, d_v)$ and $d \ge 2$, it suffices to prove

$$f(t-1,k) + 2f(t-2,k-1) + f(t-3,k-2) \le \binom{t}{k} + \binom{t-1}{k-1}.$$

By plugging in the inductive hypothesis and the fact $\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$, it suffices to prove

$$2\left(\binom{t-2}{k-1} + \binom{t-3}{k-2}\right) + \binom{t-3}{k-2} + \binom{t-4}{k-3} \le \binom{t-1}{k-1} + \binom{t-2}{k-2}$$

By subtracting $\left(\binom{t-2}{k-1} + \binom{t-3}{k-2}\right)$ from both sides and utilizing $\binom{a}{b} - \binom{a-1}{b} = \binom{a-1}{b-1}$, it is equivalent to check $\left(\binom{t-2}{k-1} + \binom{t-3}{k-2}\right) + \binom{t-3}{k-2} + \binom{t-4}{k-3} \le \binom{t-2}{k-2} + \binom{t-3}{k-3}$. By a similar reasoning, it is equivalent to check $\binom{t-2}{k-1} + \binom{t-3}{k-2} \le \binom{t-3}{k-3} + \binom{t-4}{k-4}$. It is easy to check both holds when $k > 2(t+1)/3$ and $t \ge 4$.

Therefore if in $G$, there are two non-adjacent vertices both having missing degrees at least 2 then we are done by Case 1. Hence if there are at least two vertices with missing degrees at least 2, they are adjacent. But then we are done by Case 2. As a consequence at most one vertex in $G$ has a missing degree of at least 2, call it $v$ if it exists.

Notice that all the vertices adjacent to $v$ should have zero missing degrees since otherwise we are done again by Case 2. Thus if $v$ with missing degree at least 2 exists, the complement of $G$ is a star with center $v$ with some isolated vertices. Suppose this star has $z$ edges. Then in $G$ with $n$ vertices, $\omega(G) = n - 1$. By assumption $\lfloor(n + (n-1))/2\rfloor \le t$ which means $n \le t + 1$. The number of cliques of order $k$ is $\binom{n-1}{k} + \binom{n-z-1}{k-1}$ where the first term is when the center vertex $v$ of the missing star is not picked for the clique; the second term is when the center vertex $v$ of the missing star is picked for the clique, and thus all the $z$ non-neighbors of the vertex cannot be part of the clique. This is maximized when $n = t + 1$ and $z = 2$, and this is strictly smaller than the bound desired.

Therefore there is no vertex of missing degree at least 2. Thus the complement of $G$ is a matching. However, if the matching has at least two edges, then we are in Case 2 again. Therefore the matching has exactly one edge. Thus again $G$ has $n \leq t + 1$ vertices. The number of cliques of order $k$ in $G$ is at most $\binom{n-1}{k} + \binom{n-2}{k-1} \leq \binom{t}{k} + \binom{t-1}{k-1}$. Thus the desired upper bound is proved; and it can be achieved if and only if in this case where $n(G) = t+1$ and $G$ has only one missing edge, i.e., $G \cong K_{t+1}^-$.

Moreover, by the definition of $\mathcal{G}_t$, every Turán graph $T(2t - \omega + 1, \omega)$ with $\omega \leq t$ is a graph in $\mathcal{G}_{t+1}$. The optimal graph $K_{t+1}^-$ in $G_{t+1}$ also maxized the number of cliques of order $k$ among Turán graphs $T(2t - \omega + 1, \omega)$ for every $\omega \leq t$. Thus, by definition of $T_t^*(k)$, we have $T_{t+1}^*(k) = K_{t+1}^-$. □

### 5.3.2 Asymptotic structure of $T_t^*(k)$ for sufficiently large $t$

To prove Theorem 5.1.5 for $k \leq 2t/3$, we need to get a better understanding of the bound $\mathcal{N}_k(\mathcal{G}_t \cap \mathcal{D})$, we will prove the following proposition which shows the graph in $\mathcal{G}_t$ that achieved the maximum number of $k$-cliques is a $K_t$-minor free Turán graph $T_t^*(k)$. This directly implies Proposition 5.1.6.

**Proposition 5.3.2.** *Among all the graphs $G \in \mathcal{G}_t$, the one that maximizes the number of cliques of order $k$ is the Turán graph $T_t^*(k)$. Thus, $\mathcal{N}_k(\mathcal{G}_t \cap \mathcal{D}) \leq C_t^*(k)$. In addition, $T_t^*(k)$ is $K_t$-minor free. Quantitatively, we have the following bound of $C_t^*(k)$:*

$$\binom{t-1}{k} \max\left(1, \left(2 - 4\sqrt{2k/t}\right)\right)^k \leq C_t^*(k) \leq \binom{t-1}{k} 2^k.$$

*Moreover, the number of parts $\omega$ in $T_t^*(k)$ is bounded by $\sqrt{tk}/4 \leq \omega \leq 10\sqrt{tk}$.*

Recall that "dense" in our application means the condition as in Lemma 5.2.1 is satisfied, i.e., when the maximum missing degree $\bar{\Delta}$ in a graph $G$ is such that

$$|V(G)| \geq \omega(G) + 2\bar{\Delta}^2 + 2 \text{ or } \bar{\Delta} \leq 1. \tag{5.11}$$

Recall that $\mathcal{G}_t$ is the family of graphs $G$ such that $\lfloor \frac{|V(G)|+\omega(G)}{2} \rfloor \leq t-1$. Note that there can be graphs that are not dense but also belong to $\mathcal{G}_t$. We will call $G$ an *optimizer in* $\mathcal{G}_t$ if it maximizes the number of $k$-cliques among all graphs in $\mathcal{G}_t$. Let a *balanced complete multipartite graph* be a complete multipartite graph where the orders of each part differ by at most 1. Clearly all Turán graphs are balanced complete multipartite graphs.

The proof goes as follows. In the whole subsection, we always assume $t$ is large.

1. We first show that the optimizer $G$ in $\mathcal{G}_t$ is given by Turán graphs, which are complete multipartite graphs where the orders of different parts differ by at most 1. (Lemma 5.3.3). Furthermore, we show $|V(G)| + \omega(G) = 2t - 1$ and $G = T_t^*(k)$. (Claim 5.3.1).

2. Next, we will show that every balanced complete multipartite graph satisfying $|V(G)| + \omega(G) = 2t - 1$ is $K_t$-minor free. (Lemma 5.3.4).

3. To illustrate the structure of the optimizer Turán graph $G$, we prove an upper bound of the maximum missing degree of $G$. (Lemma 5.3.5).

4. If $T_t^*(k)$ is the optimizer, we obtain asymptotically the value for $C_t^*(k)$ by proving a simple upper bound (Claim 5.3.2) and constructing a lower bound (Lemma 5.3.6).

5. By further comparing the number of $k$-cliques in $T(n, w)$ to the upper and lower bounds for $C_t^*(k)$ as mentioned above, we are able to determine asymptotically the number of parts in $T_t^*(k)$ (Lemma 5.3.7).

Fix $n, \omega$ such that $\lfloor (n + \omega)/2 \rfloor \leq t - 1$. Zykov's theorem [48] states that the graph on $n$ vertices without $K_{\omega+1}$-subgraph and with the most number of $k$-cliques is achieved by the Turán graph $T(n, \omega)$, which is a balanced complete multipartite graph on $n$ vertices and with $\omega$ parts. Therefore we have the following simple lemma.

**Lemma 5.3.3.** *For all graphs $G \in \mathcal{G}_t$, the ones with the maximum number of cliques of order $k$ are balanced complete multipartite graphs.*

By Lemma 5.3.3, to determine the optimizer in $\mathcal{G}_t$, we only need to consider balanced complete multipartite graphs. Suppose the optimizer $G$ is a graph with $l$ parts and the parts $A_1, \ldots, A_l$ have orders $a_1, \ldots, a_l$ respectively. Thus $\omega(G) = l$. Also, because $K_t^- \in \mathcal{G}_t$ contains some $k$-clique, we need $l = \omega(G) \geq k$. Otherwise, $G$ does not contain any $k$-cliques which contradicts the definition of the optimizer.

**Claim 5.3.1.** *The graph $G \in \mathcal{G}_t$ which maximizes the number of $k$-cliques satisfies $\sum_{i=1}^{l} a_i + l = 2t - 1$.*

*Proof.* For $G \in \mathcal{G}_t$, we need $\lfloor (n + \omega)/2 \rfloor \leq t - 1$. In this case, $n = \sum a_i$ and $\omega(G) = l$. Thus the condition is equivalent to $\lfloor (\sum_{i=1}^{l} a_i + l)/2 \rfloor \leq t - 1$. Therefore $\sum_{i=1}^{l} a_i + l \leq 2(t-1) + 1$. Suppose the claim does not hold, i.e., $\sum_{i=1}^{l} a_i + l < 2(t-1) + 1$. We will produce a new graph $G'$ in $\mathcal{G}_t$ but with more $k$-cliques. Let $G'$ be given by $a'_1 = a_1 + 1, a'_i = a_i$ for $1 < i \leq l$. In this case, $(\sum_{i=1}^{l} a'_i + l)/2 < (2(t-1) + 1 + 1)/2 = t$. Since $t$ is an integer, $\lfloor (\sum_{i=1}^{l} a'_i + l)/2 \rfloor \leq t - 1$. This means $G' \in \mathcal{G}_t$. While on the other hand, because $l \geq k$, the number of cliques of order $k$ in $G'$ is strictly larger than the value for $G$, a contradiction. $\square$

This claim shows that the optimizer $G$ in $\mathcal{G}_t$ is a Turán graph $T(2t - l - 1, l)$ for some $l \leq t - 1$. Thus, we have $G = T_t^*(K)$. Next, we show $G$ is $K_t$-minor free by the following lemma.

**Lemma 5.3.4.** *For any $t \geq 2$ and $l \leq t - 1$, the Turán graph $T(2t - l - 1, l)$ does not contains a $K_t$-minor.*

*Proof.* We will prove this statement by contradiction. Suppose the statement is not true, then there exists $t \geq 2$ and $l \leq t-1$ such that $G = T(2t-1-l, l)$ contains a $K_t$-minor. Let the vertex set of this $K_t$-minor be $\{v_1, v_2, \cdots, v_s\} \cup B_1 \cup B_2 \cup \cdots \cup B_{t-s}$ where $v_i \in V(G)$ and $|B_i| \geq 2$ and $B_i$ is contracted to be a vertex in this $K_t$-minor. For every $i, j \in [s]$ with $i \neq j$, we have $v_i v_j \in E(G)$. Thus, $v_i$ and $v_j$ can not belong to the same part of $G = T(2t-l-1, l)$ as each part of a Turán graph is an independent set. Thus, we have $s \leq l$ and the $K_t$-minor has at least $s + 2(t-s) = 2t - s \geq 2t - l$ vertices which contradicts with $|V(G)| \leq 2t - l - 1$. $\square$

To find the optimizer in $\mathcal{G}_t$, we are doing the following integer optimization problem to find solutions $\{a_i\}_{i \leq l}, l$:

$$\max \sum_{1 \leq i_1 < \cdots < i_k \leq l} a_{i_1} \dots a_{i_k}. \tag{5.12}$$

$$\text{s. t. } a_i \geq 1, l \geq 1, \sum_{i=1}^{l} a_i + l = 2t - 1. \tag{5.13}$$

The objective function (5.12) is the number of $k$-cliques in the (balanced) complete multipartite graph with part $i$ has order $a_i$ and in total $l$ parts. The constraint (5.13) is from Claim 5.3.1.

Next, we prove the following lemma which gives an upper bound of the maximum missing degree $\max a_i - 1$ of $G$.

**Lemma 5.3.5.** *In the optimal complete multipartite $G$ satisfying the constraint (5.13) and optimizing (5.12), the order of each part $a_i$ satisfies $(a_i - 1)^2 < \frac{4n-3k+7-4a_i}{k-1}$ or $a_i < 3$, where $n$ is the number of vertices in $G$.*

**Remark.** *When $k \geq 25$, by this lemma, we can show $G$ is a dense graph. This means $\mathcal{N}_k(\mathcal{G}_t \cap \mathcal{D}) = C_t^*(k) = \mathcal{N}_k(\mathcal{G}_t)$ when $k \geq 25$.*

*Proof.* Suppose $a_1 \geq 3$. Let $G'$ be a graph with parts $B_1, B_2, A_2, \dots, A_l$ where $A_2, \dots, A_l$ are the same as in $G$, and the part $A_1$ in $G$ splits into $B_1, B_2$, where $|B_1| + |B_2| = |A_1| - 1 =$

$a_1 - 1$, and $|B_1|, |B_2| \geq 1$. This is possible since $a_1 \geq 3$. Clearly $G'$ also satisfies (5.13). Let $|B_1| = b_1, |B_2| = b_2$.

Then the objective function for $G'$, i.e., number of cliques of order $k$ in $G'$ can be written as

$$\sum_{2\leq i_1<\cdots<i_k\leq l} a_{i_1}\dots a_{i_k} + \sum_{2\leq i_2<\cdots<i_k\leq l} (b_1+b_2)a_{i_2}\dots a_{i_k} + \sum_{2\leq i_3<\cdots<i_k\leq l} (b_1b_2)a_{i_3}\dots a_{i_k}$$

where the first term counts the number of cliques of order $k$ not containing a vertex in $B_1 \cup B_2$, while the second term counts the ones containing exactly one vertex from $B_1 \cup B_2$, and the third term counts the ones containing one vertex in $B_1$ and one vertex in $B_2$.

Recall the number of cliques of order $k$ in $G$ is exactly

$$\sum_{2\leq i_1<\cdots<i_k\leq l} a_{i_1}\dots a_{i_k} + \sum_{2\leq i_2<\cdots<i_k\leq l} a_1 a_{i_2}\dots a_{i_k}$$

where the first term is the number of cliques of order $k$ in $G$ with no vertex in $A_1$, and the second term is the ones with a vertex in $A_1$.

As $G$ has more $K_k$ than $G'$, the number of $K_k$ in $G$ minus the one in $G'$ satisfies

$$\begin{aligned}
&\sum_{2\leq i_2<\cdots<i_k\leq l} a_1 a_{i_2}\dots a_{i_k} - \sum_{2\leq i_2<\cdots<i_k\leq l} (b_1+b_2)a_{i_2}\dots a_{i_k} - \sum_{2\leq i_3<\cdots<i_k\leq l} (b_1b_2)a_{i_3}\dots a_{i_k} \\
= &\sum_{2\leq i_2<\cdots<i_k\leq l} (a_1-b_1-b_2)a_{i_2}\dots a_{i_k} - \sum_{2\leq i_3<\cdots<i_k\leq l} (b_1b_2)a_{i_3}\dots a_{i_k} \\
= &\sum_{2\leq i_2<\cdots<i_k\leq l} a_{i_2}\dots a_{i_k} - (b_1b_2)\sum_{2\leq i_3<\cdots<i_k\leq l} a_{i_3}\dots a_{i_k} \geq 0,
\end{aligned} \tag{5.14}$$

the last equality holds because $a_1 = b_1 + b_2 + 1$. Notice that $\sum_{2\leq i_2<\cdots<i_k\leq l} a_{i_2}\dots a_{i_k}$ is the number of cliques of order $k-1$ in $G[A_2 \cup \dots A_l]$, and $\sum_{2\leq i_3<\cdots<i_k\leq l} a_{i_3}\dots a_{i_k}$ is the number of cliques of order $k-2$ in the same graph.

Next, we will bound the ratio between these two numbers by a double-counting argument. We will count the number of pairs $(H_1, H_2)$ such that $H_1 \subset H_2 \subset G[A_2 \cup \dots A_l]$

and $H_1$ is a $(k-2)$-clique and $H_2$ is a $(k-1)$-clique. In the graph $G[A_2 \cup \ldots A_l]$, each clique of order $k-1$ has $k-1$ cliques of order $k-2$; and each clique of order $k-2$ is in at most $|A_2 \cup \cdots \cup A_l| - (k-2) = n - a_1 - (k-2)$ cliques of order $k-1$. Thus

$$(n - a_1 - (k-2) - a_1) \sum_{2 \leq i_3 < \cdots < i_k \leq l} a_{i_3} \ldots a_{i_k} \geq (k-1) \sum_{2 \leq i_2 < \cdots < i_k \leq l} a_{i_2} \ldots a_{i_k}.$$

Thus inequality in (5.14) implies

$$\sum_{2 \leq i_2 < \cdots < i_k \leq l} a_{i_2} \ldots a_{i_k} \geq (b_1 b_2) \cdot (k-1) \sum_{2 \leq i_2 < \cdots < i_k \leq l} a_{i_2} \ldots a_{i_k} / (n - (k-2) - a_1).$$

This is equivalent to say $n - (k-2) - a_1 > (k-1)(b_1 b_2)$, for any $b_1 + b_2 = a_1 - 1$ and $b_1, b_2 > 0$. By choosing $b_1$ and $b_2$ with difference at most 1, we have $b_1 b_2 \geq \min\{(\frac{a_1-1}{2})^2, (\frac{a}{2})(\frac{a-2}{2})\} = \frac{a_1^2 - 2a_1}{4}$. Thus, we have $n - (k-2) - a_1 > (k-1)\frac{(a_1-1)^2-1}{4}$. Rearranging, we have $(a_1 - 1)^2 < \frac{4n-3k+7-4a_1}{k-1}$. Proof of this bound for other $a_i$ is same as the proof above for $a_1$. □

Proofs of the last two parts of Proposition 5.3.2 are simple computations. We first give an upper bound on $C_t^*(k)$ which is the number of cliques of order $k$ in $T_t^*(k)$, and also the optimal objective function value for (5.12).

**Claim 5.3.2.**

$$C_t^*(k) \leq \binom{t-1}{k} 2^k.$$

*Proof.* Suppose the optimal graph, which is a balanced complete multipartite graph by Lemma 5.3.3, has $x$ parts of order $a \geq 1$ and $y$ parts of order $a+1$. Thus Claim 5.3.1 implies $|V(G)| + \omega(G) = (a+1)x + (a+2)y = 2t - 1$.This implies

$$x + y \leq \lfloor (2t-1)/(a+1) \rfloor. \tag{5.15}$$

The number of cliques of order $k$ in this graph is

$$\sum_{i=0}^{k} \binom{x}{i} \binom{y}{k-i} a^i (a+1)^{k-i}. \tag{5.16}$$

We can upper bound the above quantity by

$$\sum_{i} \binom{x}{i} \binom{y}{k-i} (a+1)^i (a+1)^{k-i} = \binom{x+y}{k} (a+1)^k. \tag{5.17}$$

By (5.15), the number of cliques of order $k$ is at most

$$\binom{x+y}{k}(a+1)^k \leq \binom{\lfloor \frac{2t-1}{a+1} \rfloor}{k} (a+1)^k := f(a+1).$$

It can be checked that the function $f(a+1)$ is monotone decreasing in $a$. Thus the largest value is chosen when $a = 1$, which is $\binom{t-1}{k}2^k$. $\square$

In fact, when $k \ll t$, the above upper bound is essentially correct. We construct a lower bound for $c_k(T_t^*(k))$ which almost matches the upper bound in Claim 5.3.2.

**Lemma 5.3.6.**

$$C_t^*(k) \geq \binom{t-1}{k} \max\left(1, \left(2 - 4\sqrt{2k/t}\right)\right)^k.$$

*This bound can be achieved by considering $T(n, w)$ where $w = \sqrt{kt/2}$ and $n = 2t-1-w$.*

*Proof.* Notice by considering a clique on $t-1$ vertices, we have $C_t^*(k) \geq \binom{t-1}{k}$. Given $n, w$, each part of $T(n, w)$ has size between $n/w - 1$ and $n/w + 1$. Thus the number of cliques of size $k$ in $T(n, w)$ is at least

$$\binom{w}{k}(n/w - 1)^k \geq \binom{w}{k}(n/w)^k \left(\frac{n/w - 1}{n/w}\right)^k = \frac{\prod_{i=0}^{k-1}(w-i)}{k!}(n/w)^k \left(1 - w/n\right)^k. \tag{5.18}$$

Plugging in $n = 2t - w - 1$, the right hand side of (5.18) is

$$\frac{\prod_{i=0}^{k-1}(w-i)}{k!}((2t-w-1)/w)^k\,(1-w/n)^k = (2t)^k\frac{\prod_{i=0}^{k-1}(w-i)}{k!}\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)^k(1-w/n)^k$$

We know $i \leq k-1$ and $k \leq w < t \leq n$, thus

$$\begin{aligned}&(w-i)\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)(1-w/n) > (w-k)\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)(1-w/n)\\ =&1-\frac{w}{2t}-\frac{1}{2t}-\frac{k}{w}+\frac{k}{2t}+\frac{k}{2tw}-\frac{w}{n}+\frac{w^2}{2nt}+\frac{w}{2tn}+\frac{k}{n}-\frac{kw}{2tn}-\frac{k}{2tn}\geq 1-\frac{2w}{t}-\frac{k}{w}.\end{aligned}$$

To maximize this lower bound, we choose $w = \sqrt{tk/2}$. Then we have $1-\frac{2w}{t}-\frac{k}{w} = 1-2\sqrt{\frac{2k}{t}}$. Therefore the right hand side of (5.18) is at least $\frac{(2t)^k}{k!}\left(1-2\sqrt{2k/t}\right)^k \geq 2^k\binom{t}{k}\left(1-2\sqrt{2k/t}\right)^k$. □

We can now prove asymptotically the number of parts in the optimal graph $T_t^*(k)$ by comparing the number of $k$-cliques in $T(n,w)$ to the upper and lower bounds above. It turns out that the construction in Lemma 5.3.6 is of the correct order. To be more specific, the Turán graph $T(n,w)$ which is $T_t^*(k)$ is such that $w = \Theta(\sqrt{tk})$.

**Lemma 5.3.7** (Restatement of the first part in Proposition 5.1.6)**.** *For any $k \leq t$, the optimal graph $T_t^*(k)$ has $\omega$ parts where $\sqrt{tk}/4 \leq \omega \leq 10\sqrt{tk}$.*

*Proof.* Again assume the optimal graph has $x$ parts of order $a \geq 1$ and $y$ parts of order $a+1$.

Given $n, w$, each part of $T(n,w)$ has size between $n/w-1$ and $n/w+1$. Thus by the AM-GM inequality, the number of cliques of size $k$ in $T(n,w)$, which is $\binom{n}{k}_w$, satisfies

$$\binom{n}{k}_w \leq \binom{w}{k}(n/w)^k = \frac{\prod_{i=0}^{k-1}(w-i)}{k!}(n/w)^k. \tag{5.19}$$

Plugging in $n = 2t - w - 1$, the right hand side of equation (5.19) is

$$\frac{\prod_{i=0}^{k-1}(w-i)}{k!}((2t-w-1)/w)^k = (2t)^k \frac{\prod_{i=0}^{k-1}(w-i)}{k!}\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)^k \tag{5.20}$$

$$\leq (2t)^k \frac{(w-(k-1)/2)^k}{k!}\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)^k. \tag{5.21}$$

The last inequality is by the fact that $(w-i)(w-(k-1-i)) \leq (w-(k-1)/2)^2$ for all $0 \leq i \leq k-1$. We know $i \leq k-1$ and $k \leq w < t \leq n$, thus

$$\begin{aligned}
&(w-(k-1)/2)\left(\frac{1}{w}-\frac{1}{2t}-\frac{1}{2tw}\right)\\
=&1-\frac{w}{2t}-\frac{1}{2t}-\frac{(k-1)/2}{w}+\frac{(k-1)/2}{2t}+\frac{(k-1)/2}{2tw}\\
<&1-\frac{w}{2t}-\left(\frac{1}{2t}-\frac{(k-1)/2}{2tw}\right)+\left(-\frac{(k-1)/2}{w}+\frac{(k-1)/2}{2w}\right) \qquad \left(\text{since } \frac{(k-1)/2}{2t}<\frac{(k-1)/2}{2w}\right)\\
\leq&1-\frac{w}{2t}-\frac{(k-1)/2}{2w} \qquad \left(\text{since } \frac{(k-1)/2}{w}\leq 1\right).
\end{aligned}$$

It can be seen that the maximum of the right-hand side is achieved when $w = \sqrt{(k-1)t/2}$. On the other hand, if $w \geq 10\sqrt{kt} > 10\sqrt{(k-1)t}$ and $k \geq 2$, the right-hand side is at most

$$1-\frac{10\sqrt{(k-1)t}}{2t}-\frac{(k-1)/2}{20\sqrt{(k-1)t}} < 1-5\sqrt{(k-1)/t} < 1-2\sqrt{2k/t}.$$

This means that if $w \geq 10\sqrt{kt} > 10\sqrt{(k-1)t}$, then the objective function (5.19) is at most

$$\frac{(2t)^k}{k!}\left(1-5\sqrt{(k-1)/t}\right)^k < \frac{(2t)^k}{k!}\left(1-2\sqrt{2k/t}\right)^k$$

where the right-hand side is a lower bound for the optimal objective function as has been proved in the Lemma 5.3.6. This means in the optimal graph,

$$w \leq 10\sqrt{kt}. \tag{5.22}$$

On the other hand, by Lemma 5.3.5, we know the size of each part $a_i$ in $T_t^*(k)$ satisfies

$(a_i-1)^2 < \frac{4n-3k+7-4a_i}{k-1}$ or $a_i < 3$. Let $a_1$ be the size of largest part of $T_t^*(k)$, then we have $a_1 = \lceil \frac{n}{w} \rceil \geq \frac{n}{w}$. This means the number of parts $w$ in $T_t^*(k)$ satisfies $n/w \leq a_1 < 3$ or

$$(n/w-1)^2 \leq (a_1-1)^2 \leq \frac{4n-3k+7-4a_1}{k-1} < \frac{4n-3k}{k}.$$

Thus $w > n/3$ or $\frac{n}{w} \leq \sqrt{\frac{4n-3k}{k}} + 1$. Since $t \leq n \leq 2t$ and $k \leq n$, we have $w > t/3$ or $\frac{n}{w} \leq 2\sqrt{\frac{4n-3k}{k}}$. When the second case happens, we have

$$w \geq \frac{1}{2}\sqrt{\frac{n^2k}{4n-3k}} \geq \frac{1}{2}\sqrt{\frac{n^2k}{4n}} \geq \frac{1}{4}\sqrt{nk} \geq \frac{1}{4}\sqrt{tk}$$

Thus, we have $w \geq \min\{t/3, \sqrt{tk}/4\} = \sqrt{tk}/4$. Therefore combining with (5.22), we proved that in the optimal graph $T_t^*(k)$, the number of parts is of the order $\Theta(\sqrt{tk})$. □

## 5.4 Asyptotic number of $k$-cliques in $K_t$-minor-free graphs

In this chapter, we will apply Lemma 5.2.3, Corollary 5.2.10, Lemma 5.3.1 and Proposition 5.3.2 to prove Theorem 5.1.5 for all $k$ such that $t-k \ll \log_2 t$.

### 5.4.1 Asympototic number of $k$-cliques for large $k$

In this subsection, we prove will Theorem 5.4.1 which shows that when $k \geq 2t/3+\tilde{O}(t^{1/2})$, the asymptotically maximum number of cliques in a graph on $n$ vertices with no $K_t$ minor is given by a graph which is a disjoint union of $T(t, t-1)$.

**Theorem 5.4.1.** *Suppose $k \geq 2t/3 + 2\sqrt{t}\log_2{}^{1/4}t$. When t is sufficiently large, the number of cliques of order $k$ in graphs on $n$ vertices and with no $K_t$-minor is at most*

$$n \cdot \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right) \cdot t^{10\log_2 t} \cdot 2^{\min\{4r_0 \log_2 t, 160(t-k)\ln\ln t\}}$$

For any $k$ in the range above, we proved in Lemma 5.3.1 that $T_t^*(k)$ is the graph $K_t^-$, the complete graph $K_t$ minus an edge. The number of $k$-cliques in $K_t^-$ is $\binom{t-1}{k} + \binom{t-2}{k-1}$.

Clearly $K_t^-$ is $K_t$-minor free. By considering $n/t$ disjoint copies of $K_t^-$, we thus have the following corollary which implies the Theorem 5.1.5 when $k \geq (t/3+r_0+2)/3$ and when $t-k \gg \log t$.

**Corollary 5.4.2** (Corollary of Theorem 5.4.1)**.** *Let $t$ be sufficiently large. Suppose $k \geq 2t/3 + 2\sqrt{t}{\log_2}^{1/4}t$ and $t-k \gg \log_2 t$. Then the number of cliques of order $k$ in graphs on $n$ vertices and with no $K_t$-minor is at most*

$$n \cdot \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right)^{1+o_t(1)} = n \cdot \left( \frac{C_t^*(k)}{|V(T_t^*(k))|} \right)^{1+o_t(1)}.$$

*Proof of Corolloary 5.4.2 form Theorem 5.4.1 and Lemma 5.3.1.* Let $\lambda = t-k$. Because $k \geq 2t/3 + 2\sqrt{t}{\log_2}^{1/4}t > 2t/3$, by Lemma 5.3.1, we have $T_t^*(k) = K_t^-$ and $C_t^*(k) = \binom{t-1}{k-1} + \binom{t-2}{k-2}$. In addition, $|T_t^*(k)| = t$.

By assumption, we have $\lambda \leq t/3$. Thus when $\lambda \gg \log_2 t$, we have $t^{10\log_2 t} = \binom{t}{\lambda}^{o_t(1)}$. Similarly, when $\lambda \geq t^{1/2}{\log_2}^{3/2}t$, we have $2^{4r_0 \log_2 t} \leq 3^{\lambda o_t(1)} \leq \binom{t}{\lambda}^{o_t(1)}$ as $\lambda \leq t/3$. When $\lambda \leq t^{1/2}{\log_2}^{3/2}t$, we have $\lambda \leq t^{2/3}$ and then $2^{160\lambda \ln\ln t} = (\frac{t}{\lambda})^{\lambda o_t(1)} \leq \binom{t}{\lambda}^{o_t(1)}$. Thus, for every $\lambda \gg \log_2 t$, we have $2^{\min\{4r_0 \log_2 t, 160\lambda \ln\ln t\}} = \binom{t}{\lambda}^{o_t(1)}$.

Because $t/3 \geq \lambda \gg \log_2 t$, we have $\binom{t-1}{\lambda-1} \geq \binom{t}{\lambda}/t > \binom{t}{\log_2 t}/t \geq \sqrt{t}^{\log_2 t}$. Then we have $\binom{t}{\lambda} \leq t \cdot \binom{t-1}{\lambda-1} = t^2 \cdot \frac{\binom{t-1}{\lambda-1}}{t} \leq (\frac{\binom{t-1}{\lambda}}{t})^{o_t(1)} \cdot \frac{\binom{t-1}{\lambda-1}}{t} \leq \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right)^{1+o_t(1)}$. Combine this result with two results above, we have

$$n \cdot \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right) \cdot t^{10\log_2 t} \cdot 2^{\min\{4r_0 \log_2 t, 160\lambda \ln\ln t\}} = n \cdot \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right)^{1+o_t(1)}.$$

The corollary holds. □

The rest of this subsection dedicates to the proof of Theorem 5.4.1. To prove this theorem, we will apply Lemma 5.2.3. To apply Lemma 5.2.3, we want to bound $\mathcal{N}_{k-r}(\mathcal{G}_{t-r-s(r,r_l)+1} \cup \mathcal{H}_{t-r}^{t-r-s(r,r_l)})$, which is the maximum possible number of cliques of order $k-r$ in a dense graph $G_r \in \mathcal{G}_{t-r-s(r,r_l)+1} \cup \mathcal{H}_{t-r}^{t-r-s(r,r_l)}$. Because $k \geq 2t/3 + 2\sqrt{t}{\log_2}^{1/4}t$, we will see that

$\mathcal{N}_{k-r}(\mathcal{G}_{t-r-s(r,r_l)+1})$ can be bounded tightly by Lemma 5.3.1. To prove Theorem 5.4.1, we also need to bound $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s})$ by the following Lemma 5.4.3.

**Lemma 5.4.3.** *When $t \geq 6r_0$ (or $t \geq 2000$) and $\lambda = t - k \leq t/3$, for every $r \leq r_0$, we have $\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s}) \leq \binom{t-r-s}{k-r} \cdot 2^s$.*

*Proof.* When $s > \lambda$, there is no clique of order $k - r$ in $\mathcal{H}_{t-r}^{t-r-s}$, so this statement is trivially true. Now assume $s \leq \lambda = o(t)$. Zykov's theorem [48] states that the graph on $n$ vertices without $K_{\omega+1}$-subgraph and with the most number of $(k-r)$-cliques is achieved by the Turán graph $T(n, \omega)$. Thus, we only need to bound the number of $k$-cliques in $T(t-r, t-r-s)$. Because $t - r - s \geq t - r_0 - \lambda \geq t/2$, then each part of the Turán graph $T(t-r, t-r-s)$ has size 1 or 2. Also, there are $s$ parts that have sizes of 2, and $t - r - 2s$ parts have sizes of 1.

Any two vertices in a clique $K_{k-r}$ in $T(t-r, t-r-s)$ can not belong to the same part of $T(t-r, t-r-s)$ as each part of a Turán graph is an independent set. For any given $k-r$ distinct parts of $T(t-r, t-r-s)$, there are at most $2^s$ distinct copies of $K_{k-r}$ using these parts since each part has at most 2 vertices and there are $s$ parts of size 2 in the Turán graph. Thus, the number of cliques of order $k-r$ in $T(t-r, t-r-s)$ is at most $\binom{t-r-s}{k-r} \cdot 2^s$. □

**Claim 5.4.1.** *For any fixed $t$ and $k$ such that $\lambda = t - k \leq t/3$, let $f_{t,k}(s) = \binom{t-s}{k} \cdot 2^s$. Then $f_{t,k}(s)$ is strictly decreasing for $s \in [0, \lambda]$.*

*Proof.* For any $s \in [0, \lambda - 1]$, we have $\dfrac{f_{t,k}(s+1)}{f_{t,k}(s)} = 2 \cdot \dfrac{t-(s+1)-k+1}{t-s} = 2 \cdot \dfrac{\lambda - s}{t-s} \leq \dfrac{2\lambda}{t} < 1$. □

Now we will proceed to prove Theorem 5.4.1:

*Proof of Theorem 5.4.1.* When $t$ is sufficiently large, we will use the bound in Lemma 5.2.3 to bound the maximum number of cliques of order $k$ in a graph without a $K_t$-minor. It is easy to see that

$$\binom{r_0}{r-1}\left(\frac{\beta t\sqrt{\ln t}}{r_0}\right)^{r-1} \leq \binom{\beta t\sqrt{\ln t}}{r-1}. \tag{5.23}$$

Applying (5.23) to Lemma 5.2.3, our goal is to bound the quantity

$$n \cdot r_0^2 \cdot \left( \max_{(r,r_l):s_M(r,r_l)\leq\lambda} \binom{r}{r-r_l} M^{r-r_l} \binom{\beta t\sqrt{\ln t}}{r_l} \mathcal{N}_{k-r}(\mathcal{G}_{t-r-s(r,r_l)+1} \cup \mathcal{H}_{t-r}^{t-r-s(r,r_l)}) \right).$$

Let $s_M(r_l) = \lfloor r_l - 1 - 7 \cdot \left(\log_{\frac{1}{1-\epsilon}} d + (8r_l \cdot \log_{\frac{1}{2\epsilon}} M)/M\right) \rfloor$. When $t$ is sufficiently large, by Lemma 5.2.7, we have $s_M(r, r_l) \geq s_M(r_l)$. When $\lambda \gg \log t$, if $r \geq 4\lambda$, we have $s_M(r, r_l) > \lambda$ by Lemma 5.2.5 which contradicts with Claim 5.2.12. Thus, we only consider the term with $r_l < r \leq \min\{r_0, 4\lambda\}$, and in this range, we have

$$r_l - s_M(r_l) \leq 8\left(\log_{\frac{1}{1-\epsilon}} d + (32\lambda \log_{\frac{1}{2\epsilon}} M)/M\right).$$

Because $k > (2t+r_0+2)/3$, we have $k > (2t+r+2)/3$ and then $(k-r) > 2(t-r+1)/3$ for all $r \leq r_0 - 1$. By Lemma 5.3.1, we have:

$$\mathcal{N}_{k-r}(\mathcal{G}_{t-r-s(r,r_l)+1}) \leq \binom{t-r-s_M(r,r_l)}{k-r} + \binom{t-r-s_M(r,r_l)-1}{k-r-1} \leq 2\binom{t-r-s_M(r_l)}{k-r}.$$

Because $(k-r) > 2(t-r+1)/3$ and $r_0 \leq t/6$ when $t$ is sufficiently large, and $s_M(r, r_l) \geq s_M(r_l)$, by Lemma 5.4.3 and Claim 5.4.1, we have

$$\mathcal{N}_{k-r}(\mathcal{H}_{t-r}^{t-r-s(r,r_l)})) \leq \binom{t-r-s_M(r,r_l)}{k-r} \cdot 2^{s_M(r,r_l)} \leq \binom{t-r-s_M(r_l)}{k-r} \cdot 2^{s_M(r_l)}.$$

For our convenience, let $r_\lambda = \min\{r_0, 4\lambda\}$. With the fact that $\binom{cn}{k} \leq c^k\binom{n}{k}$ for any integers $n \geq k \geq 0$ and $c \geq 1$, we have the maximum number of cliques of order $k$ in a graph without $K_t$ as a minor is at most

$$
\begin{aligned}
& n \cdot r_0^2 \cdot \max_{r_l < r \leq r_\lambda} \left( \binom{r}{r_l} M^{r-r_l} \binom{t}{r_l} \cdot (\beta\sqrt{\ln t})^{r_l} \cdot \binom{t-r-s_M(r_l)}{k-r} \cdot 2^{(1+s_M(r_l))} \right) \\
\leq & n \cdot r_0^2 \cdot \max_{r_l < r \leq r_\lambda} \left( \binom{r_\lambda}{r_l} \cdot M^{r_\lambda} \binom{t}{s_M(r_l) + (r_l - s_M(r_l))} \cdot (\beta\sqrt{\ln t})^{r_\lambda} \cdot \binom{t-r-s_M(r_l)}{(t-r-s_M(r_l)) - (k-r)} \cdot 2 \right. \\
\leq & n \cdot r_0^2 \cdot M^{r_\lambda} \cdot (\ln t)^{r_\lambda} \cdot \max_{r_l < r \leq r_\lambda} \left( 2^{r_\lambda} \cdot \binom{t}{s_M(r_l) + (r_l - s_M(r_l))} \right) \cdot \binom{t-r-s_M(r_l)}{\lambda - s_M(r_l)} \cdot 2^{r_\lambda} \Bigg) \qquad \text{(A)} \\
\leq & n \cdot r_0^2 \cdot (4M)^{r_\lambda} \cdot (\ln t)^{r_\lambda} \max_{r_l < r_0 : s_M(r_l) \leq \lambda} \left( \binom{t}{s_M(r_l)} \cdot t^{(r_l - s_M(r_l))} \cdot \binom{t}{\lambda - s_M(r_l)} \right) \qquad \text{(B)} \\
\leq & n \cdot r_0^2 \cdot (4M)^{r_\lambda} \cdot (\ln t)^{r_\lambda} \cdot \max_{r_l < r_0 : s_M(r_l) \leq \lambda} \left( \binom{t}{s_M(r_l)} \cdot \binom{t}{\lambda - s_M(r_l)} \right) \cdot t^{8\left(\log_{\frac{1}{1-\epsilon}} d + (32\lambda \log_{\frac{1}{2\epsilon}} M)/M\right)} \\
\leq & n \cdot r_0^2 \cdot (4M)^{r_\lambda} \cdot (\ln t)^{r_\lambda} \cdot 2^{\min\{2r_0 \log_2 t, 4\lambda\}} \cdot \binom{t}{\lambda} \cdot t^{8\left(\log_{\frac{1}{1-\epsilon}} d + (32\lambda \log_{\frac{1}{2\epsilon}} M)/M\right)}. \qquad \text{(C)}
\end{aligned}
$$

Inequality (A) holds because $\beta < 1$ and $\binom{n}{k} \leq 2^n$ for every $n \geq k \geq 0$, and $(t - r - s_M(r_l)) - (k - r) = t - k - s_M(r_l) = \lambda - s_M(r_l)$. Inequality (B) holds because $\binom{t-r-s_M(r_l)}{\lambda - s_M(r_l)} = 0$ if $s_M(r_l) > \lambda$. Therefore the maximum happens at a value of $r_l$ where $s_M(r_l) \leq \lambda$. It also used the inequality $\binom{n}{a+b} \leq \binom{n}{a} \cdot n^b$.

For inequality (C), we use the following inequality: for any $1 \leq a \leq \lambda$, $\binom{t}{\lambda-a}\binom{t}{a} \leq (\frac{et}{\lambda-a})^{(\lambda-a)}(\frac{et}{a})^a \leq (\frac{e\lambda}{\lambda-a})^{(\lambda-a)}(\frac{e\lambda}{a})^a(\frac{t}{\lambda})^\lambda \leq e^\lambda e^\lambda \binom{t}{\lambda}$. Here the last inequality holds because $f(x) = (\frac{en}{x})^x$ is increasing in the range $x \in [1, n]$ and is decreasing when $x \geq n$, and we have $f(x) \leq e^n$. Thus, we have $\binom{t}{s_M(r_l)} \cdot \binom{t}{\lambda - s_M(r_l)} \leq e^{2\lambda}\binom{t}{\lambda}$. Because $s_M(r_l) \leq r_l < r_0$ and because $\binom{n}{k} \leq t\binom{n}{k+1}$, we can also show that $\binom{t}{s_M(r_l)} \cdot \binom{t}{\lambda - s_M(r_l)} \leq t^{s_M(r_l)} \cdot t^{s_M(r_l)} \binom{t}{\lambda} \leq t^{2r_0}\binom{t}{\lambda}$. Thus, we have $\binom{t}{s_M(r_l)} \cdot \binom{t}{\lambda - s_M(r_l)} \leq 2^{\min\{2r_0 \log_2 t, 4\lambda\}}\binom{t}{\lambda}$.

In the rest of this proof, we assume $t$ is sufficiently large. As $\epsilon < \frac{1}{2}$, we have $r_0^2 \cdot t^{8 \log_{\frac{1}{1-\epsilon}} d} \leq t^{9 \log_2 t}$. Set $M = (\ln t)^2$, and then we have $t^{256\left((\lambda \log_{\frac{1}{2\epsilon}} M)/M\right)} = 2^{o(\lambda)}$. After setting $M = (\ln t)^2$, we have $(4M)^{r_\lambda} \cdot (\ln t)^{r_\lambda} \leq (\ln t)^{10 r_\lambda} = 2^{20 r_\lambda (\log_2 \ln t)}$.

When $\lambda \geq \frac{1}{2} r_0 \log_2 t$, we have $\lambda \geq r_0/4$ and $20 r_\lambda (\log_2 \ln t) = 20 r_0 (\log_2 \ln t) < r_0 \log_2 t$. Thus, the bound above is $n \cdot \binom{t}{\lambda} \cdot 2^{4 r_0 \log_2 t}$. When $\lambda \leq \frac{1}{2} r_0 \log_2 t$, we have

$\min\{2r_0 \log_2 t, 4\lambda\} = 4\lambda \leq \lambda \ln\ln t$. We also have $20 r_\lambda \log_2 \ln t \leq 80 \log_2 e \cdot \lambda \ln\ln t$. Thus, the bound above is $n \cdot \binom{t}{\lambda} \cdot 2^{160\lambda \ln\ln t} \cdot t^{9\log_2 t}$. Because $\binom{t}{\lambda} = \binom{t}{k} \leq t \cdot \binom{t-1}{k}$, the bound above is

$$n \cdot \left( \frac{\binom{t-1}{k} + \binom{t-2}{k-1}}{t} \right) \cdot t^2 \cdot t^{9\log_2 t} \cdot 2^{\min\{4r_0 \log_2 t, 160\lambda \ln\ln t\}}.$$

Then the Theorem 5.4.1 follows as $t^2 < t^{\log_2 t}$ when $t$ is sufficiently large. $\square$

### 5.4.2 Asympototic number of $k$-cliques for $k$ in the middle range

In this subsection, we will give a bound in the following theorem which can prove Theorem 5.1.5 for $k$ such that $\min(k, t-k) \gg O(t^{1/2}{\log_2}^{5/4} t)$. For fixed $t$ and $k$, recall that $T_t^*(k)$ is the Turán graph $T(2t - \omega - 1, \omega)$ maximizing the number of cliques of order $k$ among all $\omega$ such that $k \leq \omega \leq t-1$, and $C_t^*(k)$ denoted the number of cliques of order $k$ in $T_t^*(k)$.

**Theorem 5.4.4.** *When $t$ is sufficiently large and $\min(k, t-k) \gg O(t^{1/2}{\log_2}^{5/4} t)$, the number of cliques of order $k$ in a $K_t$-minor free graph on $n$ vertices is at most*

$$n \cdot \frac{C_t^*(k)}{|T_t^*(k)|} \cdot 2^{8t^{1/2}{\log_2}^{5/4} t}.$$

**Remark.** *We will check in the next corollary that this bound is asymptotically sharp up to multiplicative error $2^{O(t^{1/2}{\log_2}^{5/4} t)}$ for every $k$ in this range when considering disjoint copies of the graph $T_t^*(k)$ which was proved to be $K_t$-minor free in the next the Proposition 5.3.2.*

**Remark.** *This bound is true for $k$ in any range. However, when $k$ is too large or too small, the error term in this bound will be much larger than the main term.*

The following corollary will imply Theorem 5.1.5 when $\min(k, t-k) \gg O(t^{1/2}{\log_2}^{5/4} t)$.

**Corollary 5.4.5** (Corollary of Theorem 5.4.4). *Suppose $\min(k, t-k) \gg O(t^{1/2}{\log_2}^{5/4} t)$. The number of cliques of order $k$ in a $K_t$-minor free graph on $n$ vertices is at most $n \cdot \left( \frac{C_t^*(k)}{|T_t^*(k)|} \right)^{(1+o_t(1))}$.*

*Proof of Corollary assuming Theorem 5.4.4 and Proposition 5.3.2.* By the definition of $T_t^*(k)$, we have $|T_t^*(k)| \le 2t$. Furthermore, $C_t^*(k) \ge \binom{t-1}{k}$ by Proposition 5.3.2. Thus, $C_t^*(k)/|T_t^*(k)| \ge \binom{t-1}{k}/2t$. Because $\min(t, t-k) \gg \log_2 t$, we have $\binom{t-1}{k} \ge \binom{t}{k}/t > \binom{t}{\log_2 t}/t \ge \sqrt{t}^{\log_2 t}$. Then we have $\binom{t}{k} \le 2t^2 \cdot (\frac{\binom{t-1}{k}}{2t}) \le (\frac{\binom{t-1}{k}}{2t})^{1+o_t(1)}$. Let $m = \min(k, t-k)$. As $m \le t/2$, we have $2^{8t^{1/2}{\log_2}^{5/4}t} = 2^{m o_t(1)} \le 2^{m(\log_2 t - \log_2 m) o_t(1)} = \binom{t}{m}^{o_t(1)} = \binom{t}{\lambda}^{o_t(1)}$. Thus, we have $2^{8t^{1/2}{\log_2}^{5/4}t} = \left(\frac{\binom{t-1}{k}}{2t}\right)^{o_t(1)} \le \left(\frac{C_t^*(k)}{|T_t^*(k)|}\right)^{o_t(1)}$. We can finish the proof by applying this inequality to the Theorem 5.4.4. □

The rest of this subsection dedicates to prove Theorem 5.4.4 from Corollary 5.2.10, and we need following lemmas:

**Lemma 5.4.6.** *Let $h(a, b) = \mathcal{N}_b(\mathcal{G}_a \cap \mathcal{D})$ where $a \ge b$. Then, for every non-negative integer $i$, we have $h(a-i, b-i) \le h(a, b)$.*

*Proof.* By definition, $h(a-i, b-i)$ is the maximum number of cliques of order $b-i$ in a dense graph $G$ with $|V(G)| + \omega(G) \le 2(a-i) - 1$. Suppose $G^*$ is the optimizer. Consider $G'$ to be $G^*$ with $i$ extra vertices which form a clique and these $i$ vertices are complete to all the vertices in $G^*$. Then clearly in $G'$, we have $\omega(G') = \omega(G) + i$, and thus $|V(G')| + \omega(G') = |V(G)| + i + \omega(G) + i \le 2(a-i) - 1 + 2i = 2a - 1$. Moreover, $G'$ is also dense because $|V(G')| - \omega(G') = |V(G)| - \omega(G)$ and $\Delta(G') = \Delta(G)$. Each clique of order $b-i$ in $G^*$ can be extended to a unique clique of order $b$ in $G'$ by extending this clique to the $i$ added vertices. This means the number of cliques of order $b$ in $G'$ is at least the number of cliques of order $b-i$ in $G^*$, which is $h(a-i, b-i)$. On the other hand, the number of cliques of order $b$ in $G'$ is at most $h(t, k)$. Thus we know $h(t, k) \ge h(t-i, k-i)$. □

**Lemma 5.4.7.** *For any $t > k \ge 1$, we have $C_t^*(k-1) \le 4t^2 \cdot C_t^*(k)$. In fact, we will prove $C_t^*(k-1) \le \max\{\lceil \frac{2t-k}{k-1} \rceil^2, (2t-2k)\} \cdot C_t^*(k)$.*

*Proof.* By definition, suppose $T_t^*(k-1) = T(2t - \omega^* - 1, \omega^*)$ for some $k-1 \le \omega^* \le t-1$.

Suppose $\omega^* = k-1$. We may assume that in $T(2t-k, k-1)$, there are $a$ parts of size $\lceil \frac{2t-k}{k-1} \rceil$ and $b$ parts of size $\lfloor \frac{2t-k}{k-1} \rfloor$ with $a + b = k - 1$. Thus the number of $(k-1)$-

cliques in $T(2t-k,k-1)$ is at most $\lceil\frac{2t-k}{k-1}\rceil^a\lfloor\frac{2t-k}{k-1}\rfloor^b$. To bound $C_t^*(k)/C_t^*(k-1)$, we construct a $k$-partite graph $H$ with $a-2$ parts of size $\lceil\frac{2t-k}{k-1}\rceil$, $b$ parts of size $\lfloor\frac{2t-k}{k-1}\rfloor$, and three parts of size 1. Then we have $\omega(H)=(a-2)+b+3=k$. Because $\lceil\frac{2t-k}{k-1}\rceil\geq 2$, we have $|V(H)|\leq|V(T(2t-k,k-1))|-4+3=2t-k-1$. By Zykov's theorem [48], Turán graph $T(2t-k-1,k)$ maximized the number of the number of $k$-cliques among all the $K_{k+1}$-free graph with at most $2t-k-1$ vertices. Thus, the number of $K_k$ in $T(2t-k-1,k)$ is at least the number of $K_k$ in $H$, which is $\lceil\frac{2t-k}{k-1}\rceil^{a-2}\lfloor\frac{2t-k}{k-1}\rfloor^b$. This means $C_t^*(k-1)\leq\lceil\frac{2t-k}{k-1}\rceil^2\cdot C_t^*(k)$.

Suppose $k\leq\omega^*\leq t-1$, because of the structure of Turán graphs, every $(k-1)$-clique $K'$ in $T(2t-\omega^*-1,\omega^*)$ is contained in a $k$-clique $K$; and every $k$-clique in $T(2t-\omega^*-1,\omega^*)$ contains at most $2t-\omega^*-k\leq 2t-2k$ cliques of order $k-1$. This means there are at most $(2t-2k)C_t^*(k)$ cliques of order $k-1$ in $T(2t-\omega^*-1,\omega^*)$ for any $k\leq\omega^*\leq t-1$.

The two cases above imply $C_t^*(k-1)\leq\max\{\lceil\frac{2t-k}{k-1}\rceil^2,(2t-2k)\}\cdot C_t^*(k)\leq 4t^2\cdot C_t^*(k)$. □

*Proof of Theorem 5.4.4.* Recall $r_0=4t^{1/2}{\log_2}^{1/4}t\ll k<t$. Assume $t$ is sufficiently large. By Corollary 5.2.10 and inequality (5.23), the maximum number of $k$-cliques in a graph on $n$ vertices without a $K_t$-minor is at most

$$
\begin{aligned}
&n\min(r_0,k)\cdot\left(\max_{r\leq\min(r_0,k)}\binom{\beta t\sqrt{\ln t}}{r-1}\cdot\mathcal{N}_{k-r}(\mathcal{G}_{t-r+1}\cap\mathcal{D})\right).\\
\leq& n\cdot r_0\cdot\binom{\beta t\sqrt{\ln t}}{r_0-1}\cdot\max_{1\leq r\leq r_0}h(t-r+1,k-r)\leq n\cdot r_0\cdot\left(\frac{e\beta t\sqrt{\ln t}}{r_0-1}\right)^{r_0-1}\cdot h(t,k-1)\\
\leq& n\cdot r_0\cdot t^{r_0-1}\cdot h(t,k-1)\leq n\cdot 2^{(r_0\log_2 t)}\cdot C_t^*(k-1)\leq n\cdot 2^{(r_0\log_2 t)}\cdot 4t^2\cdot C_t^*(k).
\end{aligned}
$$

The third inequality is true because $r_0>\sqrt{t}>e\beta\sqrt{\ln t}$ when $t$ is large. Because $|T_t^*(k)|\leq 2t$, the bound above is at most $n\cdot 2^{(r_0\log_2 t)}\cdot 8t^3\cdot\frac{C_t^*(k)}{|T_t^*(k)|}\leq n\cdot 2^{2(r_0\log_2 t)}\cdot\frac{C_t^*(k)}{|T_t^*(k)|}$. □

**Remark.** *When $r_0\leq k\ll t$, we can improve the bound in Theorem 5.4.4 by approximating the maximum point of $\binom{r_0}{r-1}\left(\frac{\beta t\sqrt{\ln t}}{r_0}\right)^{r-1}C_{t-r+1}^*(k-r)$ among $r\leq\min(r_0,k)$. More* *precisely, for any $r_0 \leq k < t^{2/3}$, the number of cliques of order $k$ in every graph on $n$ vertices with no $K_t$-minor is at most*

$$nr_0 \cdot \left(\frac{\beta t\sqrt{\ln t}}{r_0}\right)^{r_0} \binom{t-r_0}{k-r_0} 2^{k-r_0} 2^{O(\sqrt{\log t})r_0}.$$

*It is not hard to show this bound is better than the bounds in Theorem 5.4.4.*

### 5.4.3 The proof of Theorem 5.1.5

In this subsection, we will complete the proof of the main theorem of this chapter.

*Proof of Theorem 5.1.5.* We may assume $t$ is sufficiently large. When $k = 2$, we recall that Thomason [45] proved that the number of edges in graphs on $n$ vertices and with no $K_t$-minor is at most $0.32t\sqrt{\ln t})n$. By Proposition 5.3.2, $\frac{C_t^*(k)}{|V(T_t^*(k))|} \geq \frac{(t(t-1)/2)-1}{2t} > t/5$ as $t > 40$. Thus $t\sqrt{\ln t} \leq (\frac{t}{5})^{1+o_t(1)} \leq \left(\frac{C_t^*(k)}{|V(T_t^*(k))|}\right)^{1+o_t(1)}$ which proves the case when $k = 2$.

When $k \geq 5t/6$, we have $k > 2t/3 + 2\sqrt{t}{\log_2}^{1/4}t$. Then we can apply Corollary 5.4.2 to prove this theorem for $k$ in this range. When $5t/6 > k \geq t^{2/3}$, we have $k \gg t^{1/2}{\log_2}^{5/4}t$ and $t - k \geq t/6 \gg t^{1/2}{\log_2}^{5/4}t$. Then we can apply Corollary 5.4.5 to prove this theorem for $k$ in this range.

We will finish the proof by showing this theorem is true when $3 \leq k < t^{2/3}$.

By Theorem 5.1.3, the number of $k$-cliques in a $K_t$-minor free graph with $n$ vertices is at most $\binom{\beta t\sqrt{\ln t}}{k-1}n$, which is at most $n(\frac{t}{k})^{k-1+o_t(1)}$. Because $k < t^{2/3}$, by Proposition 5.3.2, $C_t^*(k) \geq \binom{t}{k}2^k(1-4t^{-1/3})^k \geq \binom{t}{k}1.82^k$ when $t \geq 45^3$. Then $\frac{C_t^*(k)}{|V(T_t^*(k))|} \geq \frac{(t/k)^k 1.82^k}{2t} \geq (\frac{t}{k})^{k-1}$ because $1.82^k > 2k$ for every $k \geq 3$. Therefore, when $3 \leq k < t^{2/3}$, the number of $k$-cliques in a $K_t$-minor free graph with $n$ vertices is at most $n\binom{\beta t\sqrt{\ln t}}{k-1} \leq n(\frac{t}{k})^{(k-1)(1+o_t(1))} \leq n \cdot \left(\frac{C_t^*(k)}{|V(T_t^*(k))|}\right)^{1+o_t(1)}$.

□

## 5.5 Concluding Remarks

In this chapter, we studied the problem ex$(n, K_k, K_t$-minor$)$ and proved an essentially sharp bound, up to $o_t(1)$ in the exponent, for all $k < t$ such that $t - k \gg \log_2 t$. In other words, we showed ex$(n, K_k, K_t$-minor$) = C(k,t)^{1+o(1)}n$ where we have a matching lower bound construction which contain $C(k,t)n$ cliques of size $k$ but with no $K_t$-minor. The exact bound in the conjecture of Wood 5.1.4 remains open.

An analog question is to study the number of $K_k$ in a graph forbidding $K_t$-subdivision instead of $K_t$-minor is also mentioned in this chapter. In the case of forbidding $K_t$-subdivision, we even do not know ex$(n, K_2, K_t$-subdivision$)$.

**Question 5.5.1.** *What are the exact values of ex$(n, K_k, K_t$-subdivision$)$?*

# Appendices

# APPENDIX A

## DISPROOF OF CONJECTURE 5.1.4

We now give a construction and show Wood's Conjecture 5.1.4 does not hold for $\lambda \leq 0.553$.

**Theorem A.0.1.** *Let $k = \lambda t$ where $\lambda \leq 0.553$. Then when $t$ is large, there exists a graph on $n$ vertices without $K_t$ as a minor, and the number of cliques of order $k$ in this graph is strictly larger than $\binom{t-2}{k-1}n$.*

*Proof.* Consider a graph $G$ on $n$ vertices which is a union of the complement of a perfect matching on $2(t-1)/3$ edges. We can assume $t \equiv 1 \mod 3$ and $n$ is divisible by $4(t-1)/3$. Thus by Lemma 5.2.1, the Hadwiger number of $G$ is $t-1$. On the other hand, the number of cliques of order $k$ in $G$ is

$$\binom{2(t-1)/3}{k} 2^k \cdot \frac{n}{4(t-1)/3}.$$

The last term $\frac{n}{4(t-1)/3}$ is the number of copies of the graph which is the complement of a perfect matching. Each copy has exactly $\binom{2(t-1)/3}{k}2^k$ cliques of order $k$; this is because each edge in the matching can contribute to at most one vertex in the clique.

We want to show that when $t$ is large,

$$\binom{2(t-1)/3}{k} 2^k \cdot \frac{n}{4(t-1)/3} > \binom{t-2}{k-1} n. \tag{A.1}$$

Assume $k = \lambda t$ where $1/3 < \lambda < 0.553$. Then by Stirling's formula applied to the binomial coefficient, letting $h(x) = x \log_2 x$, the left-hand side of (A.1) is at least

$$nc_1 \sqrt{\frac{2t/3}{k(2t/3-k)}} 2^{h(2(t-1)/3) - h(k) - h(2(t-1)/3-k) + k},$$

where $c_1$ is some absolute constant. Similarly, the right hand side of (A.1) is at most

$$nc_2\sqrt{\frac{t}{k(t-k)}}2^{h(t-2)-h(k-1)-h(t-k-1)},$$

where again $c_2$ is some absolute constant. It suffices to show that for each $\lambda$, there is some constant $\epsilon$ such that

$$h(2(t-1)/3)-h(k)-h(2(t-1)/3-k)+k > \epsilon t+h(t-2)-h(k-1)-h(t-k-1). \quad \text{(A.2)}$$

If this is the case, then to prove (A.1), it suffices to show

$$nc_1\sqrt{\frac{2t/3}{k(2t/3-k)}}2^{\epsilon t} > nc_2\sqrt{\frac{t}{k(t-k)}}.$$

This clearly holds when $k = \lambda t$ where $\lambda$ is fixed and $t$ sufficiently large. Thus it suffices to prove (A.2) for some $\epsilon > 0$.

As $h'(x) = \log x + 1/\ln(2)$, for $a > b > 1$, $0 \leq h(a)-h(b) \leq (b-a)(\log a + 1/\ln(2))$. If $b-a \ll a$, we will have when $a$ sufficiently large, $h(b) = h(a)+O(\log a)$. Thus to prove (A.2) for some $\epsilon > 0$, it suffices to prove there exists a constant $\epsilon' > 0$ such that when $t$ is sufficiently large,

$$h(2t/3) - h(k) - h(2t/3 - k) + k > \epsilon' t + h(t) - h(k) - h(t-k).$$

Removing $h(k)$ from both sides, it suffices to prove $h(2t/3) - h(2t/3 - k) + k > \epsilon' t + h(t) - h(t-k)$. Using $k = \lambda t$, notice

$$\begin{aligned} h(t) - h(t-k) &= t\log t - (t-k)\log(t-k) = t\log t - (t-k)\log(t(1-k/t)) \\ &= t\log t - (t-k)\log t - (t-k)\log(1-\lambda) = \lambda t\log t - (1-\lambda)t\log(1-\lambda). \end{aligned}$$

Similarly, for the left hand side,

$$h(2t/3) - h(2t/3 - k) = \lambda t \log(2t/3) - (2/3 - \lambda)t \log(1 - 3\lambda/2)$$

$$= \lambda t \log(t) + \lambda t \log(2/3) - (2/3 - \lambda)t \log(3/2) - (2/3 - \lambda)t \log(2/3 - \lambda)$$

$$= \lambda t \log(t) + 2t/3 \log(2/3) - (2/3 - \lambda)t \log(2/3 - \lambda)$$

Therefore we want to prove

$$-(1-\lambda)t \log(1-\lambda) + \lambda t \log t + \epsilon' t < k + \lambda t \log t + 2t/3 \log(2/3) - (2/3-\lambda)t \log(2/3-\lambda).$$

Removing $\lambda t \log t$ from both ends, and dividing both sides by $t$, it is equivalent to show $\epsilon' - h(1-\lambda) < \lambda + h(2/3) - h(2/3-\lambda)$. The function $f(\lambda) = \lambda + h(2/3) - h(2/3-\lambda) + h(1-\lambda)$ is strictly positive for $\lambda \leq 0.553$, which means the existence of positive $\epsilon'$. $\square$

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