A STOCHASTIC FLOW FOR FEATURE EXTRACTION

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ABSTRACT

In recent years evolution of level sets of two-dimensional functions or images in time through a partial differential equation has emerged as an important tool in image processing. Curve evolutions, which may be viewed as an evolution of a single level curve, has been applied to a wide variety of problems such as smoothing of shapes, shape analysis and shape recovery. We give a stochastic interpretation of the basic curve smoothing equation, the so called geometric heat equation, and show that this evolution amounts to a rotational diffusion movement of the particles along the contour. Moreover, assuming that a priori information about the orientation of objects to be preserved is known, we present new flows which amount to weighting the geometric heat equation nonlinearly as a function of the angle of the normal to the curve at each point.

1. INTRODUCTION

In recent years evolution of level sets of two-dimensional functions or images in time/scale via a Partial Differential Equation (PDE) has emerged as an important tool in image processing. Curve evolutions, which may be viewed as an evolution of a single level curve, have been applied to a wide variety of problems such as smoothing of shapes, shape analysis and shape recovery. The connection between the evolution of two-dimensional functions and the evolution of curve coordinates has been well-established [1, 2]. The usefulness of these evolutions comes from the fact that they provide a scale-space analysis where fine to coarse features and properties of a signal can be fully traced and observed.

The goal of feature and shape extraction in recognition and classification problems has long been hampered by noise and processing artifacts. The idea of feature-driven progressive smoothing and scale tracking is widely viewed as a promising new avenue of research and hence has been of increasing interest to researchers in the field. To the best of our knowledge, previous literature about the nonlinear diffusion topic has adopted a deterministic approach (aside from [3, 4] which address a different problem). Our primary interest in this paper is to provide a stochastic solution to a specific evolution equation, namely the geometric heat equation. We subsequently use this insight to propose a class of nonlinear diffusions specifically aimed at extracting desired features in a possibly noisy environment.

In the remainder of the paper, we first briefly review in Section 2, some theoretical concepts on scale space analysis and describe a natural, well-known geometric technique based upon the curve shortening flow. In Section 3, we provide a stochastic equivalent equation which in turn unveils a new shape/feature-driven flow described in detail in Section 4, and which we believe offers a variety of possible applications outside the recognition and classification problems. We finally present some illustrating and substantiating examples.

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2. BACKGROUND AND FORMULATION

The well-known low-pass filtering in signal processing by Gaussian smoothing, can be obtained by evolving an image $u_0(x, y)$ by a diffusion equation [5],

$$\begin{aligned} u(0,x,y) &= u_0(x,y), \\ u_t(t,x,y) &= k\Delta u(t,x,y), \quad t>0 \end{aligned} \tag{1}$$

where k is a constant, Δ is the Laplacian operator, $k\Delta u = k(u_{xx}+u_{yy})$,equivalently expressed as $k\Delta u = \nabla \cdot k\nabla u$, (here "$\nabla\cdot$" is the divergence operator, and "∇" is the gradient operator). The solution to this equation is a parameterized collection of functions $u(t, x, y), t > 0$, and is equivalent to filtering $u_0(x, y)$ with a Gaussian filter of variance $2t, t > 0$.

An image $u(x, y)$ can also be thought of as a collection of iso-intensity contours, or level curves. On an iso-intensity contour, let η be the direction normal to the contour (the gradient direction), and let ξ be the direction tangent to the contour (level set direction)

$$\eta = \frac{(u_x, u_y)}{\sqrt{{u_x}^2 + {u_y}^2}}, \quad \xi = \frac{(-u_y, u_x)}{\sqrt{{u_x}^2 + {u_y}^2}},$$

which are depicted in Fig. 1. The η direction is usually

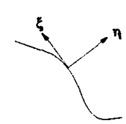

Figure 1: Normal and tangent directions to a level curve.

considered as the direction across the image features since it points to the gradient direction of a level curve which is

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given by $\nabla u = (u_x, u_y)$, whereas the ξ direction is along the image features [6]. Thus, the idea is to smooth less across the image features in order not to damage the boundaries/edges between different regions, and to allow more smoothing along the features to eliminate noise. Denoting second-order directional derivatives in the directions of η and ξ respectively as $u_{\eta\eta}$ and $u_{\xi\xi}$,

$$u_{\eta\eta} = \frac{u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy}}{u_x^2 + u_y^2}, \quad (2)$$

$$u_{\xi\xi} = \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2}, \quad (3)$$

then the diffusion operator can also be written as $\Delta u = u_{\xi\xi} + u_{\eta\eta}$ which diffuses isotropically.

Perona and Malik introduced anisotropic diffusion equation [7], which includes a nonlinear coefficient in the divergence term and given by $u_t = \nabla \cdot (g(\mid \nabla u \mid)\nabla u)$, where $g(\cdot)$ is typically a monotonically decreasing function which suppresses diffusion when the gradient is high (encounter of an edge). One possible choice is $g(|\nabla u|) = \frac{1}{\sqrt{1+|\nabla u|^2}}$. A corresponding selective evolution which unequally weights the diffusion terms $u_{\eta\eta}$ and $u_{\xi\xi}$ was proposed in [6]. The attempt there is to precisely guide the diffusion along the features (i.e. along the ξ direction) and away from the boundaries as just discussed above. Such a diffusion amounts to smoothing one iso-intensity curve without affecting the others.

For the sake of clarity and conciseness, we will focus in this paper on an evolution of a single level set, and use as a starting point the level set method first proposed in [1] for propagating interfaces. A parameterized curve $\mathcal{C}(t,p) = (\mathcal{X}(t,p), \mathcal{Y}(t,p))$, is first embedded into a surface called a level set function $\Phi(t,x,y) : [0,T] \times \mathbb{R}^2 \mapsto \mathbb{R}$, and where t is an evolutionary step, and p is a parameter (e.g. arc length) along the curve. The curve $\mathcal{C}$ is the zero-level set of $\Phi(t,x,y)$, i.e., $\mathcal{C} = \{(x,y) : \Phi(t,x,y) = 0\}$. The evolution equation for $\Phi(t,x,y)$ is derived from the constraint that at any time t, we should have $\Phi(t, \mathcal{C}(t)) = \Phi(t, \mathcal{X}(t), \mathcal{Y}(t)) = 0$, and by differentiating this constraint with respect to t (see [8] for details). A general evolution of any planar curve $\mathcal{C}(t,p)$ is given by

$$\mathcal{C}_t(t,p) = \beta(t,p)\vec{N}, \quad (4)$$

$$\mathcal{C}(0,p) = \mathcal{C}_0(p). \quad (5)$$

where $\vec{N}$ is the inward unit normal to the curve, $\beta(\cdot,\cdot)$ is the evolution velocity, and $\mathcal{C}_0(p)$ is the initial curve [2]. Considering curve evolutions which only depend on a function of the curvature κ, $\beta(\cdot,\cdot)$ can be written as $F(\kappa)$, which leads to the following level set equation

$$\Phi_t + \nabla\Phi \cdot F(\kappa)\vec{N} = \Phi_t - F(\kappa)|\nabla\Phi| = 0, \quad (6)$$

using the fact that the inward unit normal vector $\vec{N} = -\nabla\Phi/|\nabla\Phi|$. A solution to Eq. (4) is then obtained by merely evolving Eq. (6). Various selections of the speed function $F(\kappa)$ are presented in [8], with the simplest form $F(\kappa) = \kappa$ resulting in

$$\Phi_t = \kappa\ |\nabla\Phi|, \quad (7)$$

(referred to as geometric heat equation). Its effect on a particular level curve $\mathcal{C}$ is given by the curve shortening flow, $\mathcal{C}_t = \kappa\vec{N}$, under which curves embedded in $\mathbb{R}^2$ evolve regularly toward circles, eventually collapsing into points as shown by Grayson [9]. The curvature κ can also be expressed in terms of the level set function Φ as

$$\kappa = \nabla \cdot \left(\frac{\nabla\Phi}{|\nabla\Phi|}\right) = \frac{\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy}}{(\Phi_x^2 + \Phi_y^2)^{3/2}}. \quad (8)$$

Substituting this form of κ into Eq. (7), leads to the following form,

$$\Phi_t = \frac{\Phi_y^2\Phi_{xx} - 2\Phi_x\Phi_y\Phi_{xy} + \Phi_x^2\Phi_{yy}}{\Phi_x^2 + \Phi_y^2}. \quad (9)$$

3. STOCHASTIC FORMULATION OF A GEOMETRIC HEAT EQUATION

Observe that Eq. (9) is $\Phi_t = \Phi_{\xi\xi}$, i.e. the contour is smoothed maximally along the tangential direction to the contour. It is clear that the convergence to a point of every shape/contour subjected to a geometric heat flow [9] will not preserve features of a level curve. Our goal is to then investigate this problem and propose a solution as a result of the following development.

Let us call the angle between the outward normal to the curve and the x-axis θ. The outward unit normal $\vec{N}$ can then be expressed as $\vec{N} = (\cos\theta, \sin\theta)$, which is re-written in terms of $\Phi(\cdot)$ as $\vec{N} = (\Phi_x, \Phi_y)/\sqrt{{\Phi_x}^2 + {\Phi_y}^2}$. It follows, $\theta(\Phi_x, \Phi_y) = \tan^{-1}(\frac{\Phi_y}{\Phi_x})$. Using these equations, and defining an operator $\mathbf{A}_h$ of the form

$$\mathbf{A}_h\Phi = \sin^2\theta\ \Phi_{xx} - 2\sin\theta\ \cos\theta\ \Phi_{xy} + \cos^2\theta\ \Phi_{yy}, \quad (10)$$

the geometric heat equation (9) can be re-written as

$$\Phi_t(t,x,y) = A_h\Phi(t,x,y), \quad (11)$$

$$\Phi(0,x,y) = f(x,y), \quad (12)$$

where $f(x,y)$ is the initial level set function. We next show that an evolution equation in fact corresponds to an infinitesimal generator of a Stochastic Differential Equation (SDE), by using Ito diffusions and the Kolmogorov backward diffusion theorem [10].

3.1. Ito Diffusion

A diffusion of a particle is usually well modeled by an SDE which, in turn, represents the underlying process of an evolution. The dynamics of this evolution are captured by a PDE henceforth also referred to as a generator (infinitesimal) of the diffusion.

Definition 1. *Suppose we want to describe the motion of a small particle suspended in a moving liquid, subject to random molecular bombardments. If $b(t,x) \in \mathbb{R}^3$ is the velocity of fluid at the point x at time t, then a widely used mathematical model for the position X_t of the particle at time t is an SDE of the form*

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t, \quad (13)$$

where B_t is a 3-dimensional Brownian motion. Generally, in such an SDE where $X_t \in \mathbb{R}^n, b(t,x) \in \mathbb{R}^n, \sigma(t,x) \in \mathbb{R}^{n\times m}$, and B_t is m-dimensional Brownian motion, $b(\cdot,\cdot)$ is called the drift coefficient, and $\sigma(\cdot,\cdot)$ is called the diffusion coefficient.

The first term in this equation corresponds to a non-random motion, whereas the second term models randomness or noise in the motion.

The solution of such an SDE may be thought of as a mathematical description of the motion of a small particle in a moving fluid, and such stochastic processes are called (Ito) diffusions [10]. For many applications, a second order partial differential operator $\mathbf{A}$ can be associated to an Ito diffusion X_t given by Eq. (13). The basic connection between $\mathbf{A}$ and X_t is that $\mathbf{A}$ is the generator of the process X_t. If $g \in C^2(\mathbb{R}^n)$, i.e. continuous functions with continuous derivatives up to order 2, and g has a compact support, $(g \in C_0^2(\mathbb{R}^n))$, then $\mathbf{A}$ is given in the form

$$\mathbf{A}\ g(x) = \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 g}{\partial x_i\,\partial x_j} + \sum_i b_i(x)\frac{\partial g}{\partial x_i}. \quad (14)$$

Theorem 1. *(Kolmogorov's backward equation)*
Define $u(t,x,y) = E^{x,y}[f(X_t)]$, where $X_t = (X_t^{(1)}, X_t^{(2)})$, and $E^{x,y}[\cdot]$ is the expectation operator with respect to the probability law of X_t starting at the point (x,y), then there exists an operator $\mathbf{A}$ such that,

$$\frac{\partial u}{\partial t} = \mathbf{A}u, \quad t>0,\ (x,y)\in\mathbb{R}^2, \quad (15)$$

$$u(0,x,y) = f(x,y), \quad (x,y)\in\mathbb{R}^2. \quad (16)$$

3.2. Stochastic Formulation

In light of the foregoing development, a natural question which arises is: *given a PDE which governs a curve shortening flow, can we obtain a corresponding SDE of the underlying diffusion?*
Towards that end, we have the following,

Proposition 1. *The evolution equation given in Eqs. (11), (12) with its operator defined in Eq. (10), generates a diffusion of individual pixels whose equation is given by,*

$$dX_t = \sqrt{2}\begin{pmatrix} -sin\theta \\ cos\theta \end{pmatrix} dB_t.$$

Proof: The operator $\mathbf{A}_h$ in Eq. (10) is first rewritten as,

$$\mathbf{A}_h = \frac{1}{2}\sigma\sigma^{\mathbf{T}} = \begin{pmatrix} \sin^2\theta & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \cos^2\theta \end{pmatrix} \odot \mathbf{H},$$

where $\mathbf{H}$ is a Hessian operator and $\odot$ is a Hadamard product. The factorization of $\mathbf{A}_h$ leads to $\sigma = \sqrt{2}\,(-\sin\theta\ \cos\theta)^T$ and by identification, $b=0$. Given $\Phi(t,x,y)$ as a solution to Eqs. (11),(12), we define drift coefficient and diffusion coefficient of an Ito diffusion X_t as follows,

$$b(t,x,y) = 0, \quad (17)$$

$$\sigma(t,x,y) = \sqrt{2}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}, \quad (18)$$

where
$\sin(\theta(t,\Phi_x,\Phi_y)) = \Phi_y(t,x,y)/\sqrt{{\Phi_x}^2(t,x,y)+{\Phi_y}^2(t,x,y)}$,
and
$\cos(\theta(t,\Phi_x,\Phi_y)) = \Phi_x(t,x,y)/\sqrt{{\Phi_x}^2(t,x,y)+{\Phi_y}^2(t,x,y)}$.
Define

$$u(t,x,y) = E^{x,y}[f(X_t)] \quad (19)$$

where X_t satisfies the SDE . in Eq. (13), i.e.,

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \sqrt{2}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} dB_t, \quad (20)$$

with the generator in Eq. (10). By Kolmogorov's backward theorem then u satisfies $\frac{\partial u}{\partial t} = \mathbf{A}_h u$. $\Phi(t,x,y)$ also satisfies the same set of equations, and by uniqueness of the solution of Kolmogorov's equation, we can write the level set function $\Phi(t,x,y)$ as illustrated in Fig. (2) as

$$\Phi(t,x,y) = E^{x,y}[f(X_t)], \quad (21)$$

to complete the proof. ■

A similar development can be carried out for the normal motion (i.e. along η). Due to space limitations, we defer the details to [11], where existence of strong solutions is shown as well as other properties [10]. One can nevertheless infer that the nonlinear diffusion $dX_t = \sqrt{2}\,\vec{T}\,dB_t$ is tangential on the unit circle as illustrated in Fig. (3). This in turn leads us to propose in the next section a more general and feature/shape adapted flow that obey the maximum principle and for which strong solutions exist.

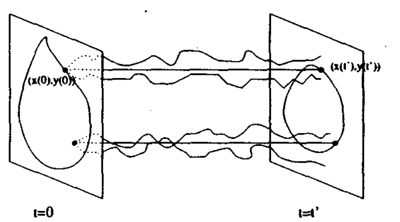

Figure 2: Points of the zero-level set, i.e. initial contour $(\mathcal{X}(0),\mathcal{Y}(0))$, at time t=0, is shown on the left. Those points whose sample realizations result in an average value of zero at time $t=t'$ $(\Phi(t',x,y) = E^{x,y}[f(X_{t'})] = 0)$ form the new contour$(\mathcal{X}(t'),\mathcal{Y}(t'))$ (on the right).

Figure 3: (a) Tangential direction motion, (b) Normal direction motion.

4. A NEW CLASS OF FLOWS

The geometric heat equation produces a self-similar solution which is a circle, so it evolves all shapes into circles. A natural generalization of the proposed approach is to construct an SDE with an arbitrary but carefully chosen functional

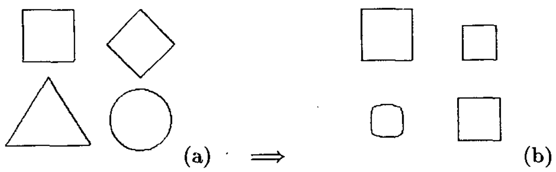

Figure 4: (a) Original set of shapes (b) Result of evolution under $\Phi_t = \cos^2(2\theta)\ \Phi_{\xi\xi}$.

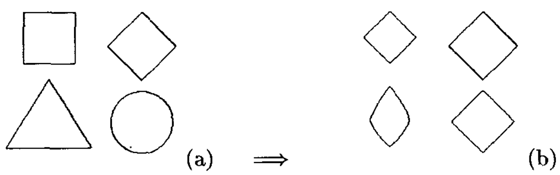

Figure 5: (a) Original set of shapes (b) Result of evolution under $\Phi_t = \sin^2(2\theta)\ \Phi_{\xi\xi}$.

$h(\theta)$ which reflects specific desired goals and also leads to a solution (i.e. Lipschitz properties[10]),

$$\begin{pmatrix} dX_t^{(1)} \\ dX_t^{(2)} \end{pmatrix} = \sqrt{2}\, h(\theta) \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix} dB_t.$$

One such class of functional that we can show the flow of to lead to $2-n$-gones [11] is given by simple trigonometric functions[1]

$$h^2(\theta) = \begin{cases} \cos^2(n\theta) \\ \sin^2(n\theta) \end{cases},$$

where $n \in \mathbb{N}$. For instance, we can obtain flows where we can keep a square shape as the invariant shape among all other shapes in a given initial collection by selecting $h^2(\theta) = \cos^2(2\theta)$. We therefore find a flow

$$\Phi_t = \cos^2(2\theta)\ \Phi_{\xi\xi} \tag{22}$$

which produces square-like shapes. For example, the initial collection of shapes in Fig. 4(a) will be converted to squarish shapes as shown in Fig. 4 (b), so this flow keeps square shapes invariant. With the same reasoning, the PDE $\Phi_t = \sin^2(2\theta)\ \Phi_{\xi\xi}$ does maximal smoothing at those points with orientations $\pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, and their neighborhoods, and no smoothing at points with orientations $0, \pi/2, \pi, 3\pi/2$. Therefore, this flow keeps diamond-like shapes invariant as shown in Fig. 5. Similarly, one can find various flows in the form of $\Phi_t = \cos^2(n\theta)\Phi_{\xi\xi}$, which partition θ space between $[0, 2\pi]$ into n axes, and thus tend to produce 2n-sided polygon-like shapes (oriented vertically and horizontally). Other example flow results are given in Fig. 6(a), and (b), where the initial set of shapes is the one given in Fig. 4(a). Heuristically, the flow in Eq. (22) consists of stopping the diffusion at four diagonal orientations, and allow maximal diffusion at four horizontal and vertical orientations. The generator in the form $A\Phi = \cos^2(2\theta)\ \Phi_{\xi\xi}$ corresponds to a diffusion X_t satisfying the SDE $\left(dX_t^{(1)}\ dX_t^{(2)}\right)^T = \sqrt{2}\ \cos(2\theta)\,(-\sin(\theta)\ \cos(\theta))\ dB_t$. These particular flows can be useful in various shape analysis applications, particularly in recognition of man-made objects like jeeps, cars, tanks and many others, where straight edges exist.

[1] Note added in proof: Recently discovered that Sethian qualitatively mentions a similar idea in his very recent book, with a different approach.

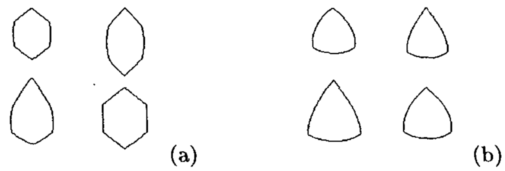

Figure 6: Result of (a)Flow $\Phi_t = \cos^2(3\theta)\ \Phi_{\xi\xi}$, which tends to produce hexagons, (b)Flow $\Phi_t = \sin^2(1.5(\theta - \pi/2))\ \Phi_{\xi\xi}$, which tends to produce triangle-like shapes.

5. CONCLUSIONS

We have presented an alternative view of geometric heat equation in terms of stochastic flows. We also proposed a new class of flows which produce polygonal-invariant shapes that can be useful in various shape recognition tasks.

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