Separators, Brambles, Tree Decompositions and Excluding Clique Minors

JOURNAL OF THE
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A SEPARATOR THEOREM FOR NONPLANAR GRAPHS

NOGA ALON, PAUL SEYMOUR, AND ROBIN THOMAS

1. Introduction

A *separation* of a graph G is a pair (A, B) of subsets of $V(G)$ with $A \cup B = V(G)$, such that no edge of G joins a vertex in $A - B$ to a vertex in $B - A$. Its *order* is $|A \cap B|$. A well-known theorem of Lipton and Tarjan [2] asserts the following. ($\mathbf{R}^+$ denotes the set of nonnegative real numbers. If $w\colon V(G) \to \mathbf{R}^+$ is a function and $X \subseteq V(G)$, we denote $\sum(w(v)\colon v \in X)$ by $w(X)$.)

(1.1) *Let G be a planar graph with n vertices, and let $w\colon V(G) \to \mathbf{R}^+$ be a function. Then there is a separation (A, B) of G of order $\le 2\sqrt{2}\sqrt{n}$, such that $w(A - B), w(B - A) \le \frac{2}{3}w(V(G))$.*

Our object is to prove an extension of (1.1) for nonplanar graphs with a fixed excluded "minor." A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting edges. By an *H-minor* of G we mean a minor of G isomorphic to H. Thus, the Kuratowski–Wagner theorem asserts that planar graphs are those without K_5- or $K_{3,3}$-minors. We prove the following:

(1.2) *Let $h \ge 1$ be an integer, let G be a graph with n vertices and with no K_h-minor, and let $w\colon V(G) \to \mathbf{R}^+$ be a function. Then there is a separation (A, B) of G of order $\le h^{3/2}n^{1/2}$ such that $w(A - B), w(B - A) \le \frac{2}{3}w(V(G))$.*

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1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 05C40.
The first author's research was carried out under a consulting agreement with Bellcore.

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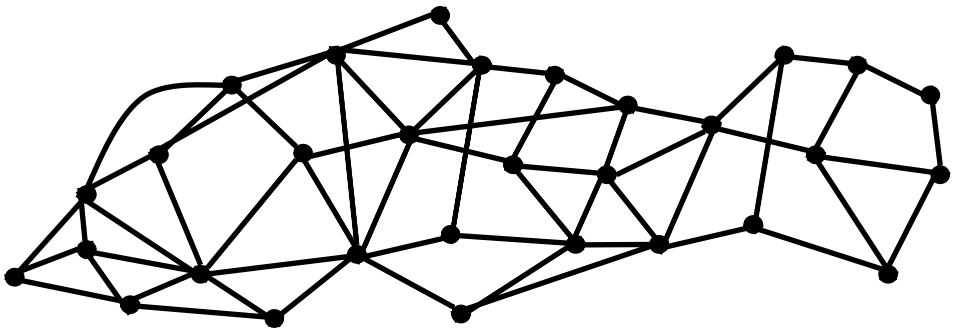
- A *Separator* is a 2/3-Separator

Lipton and Tarjan 1979: Every Planar Graph Has A Separator of Size $2\sqrt{2}\sqrt{|V|}$



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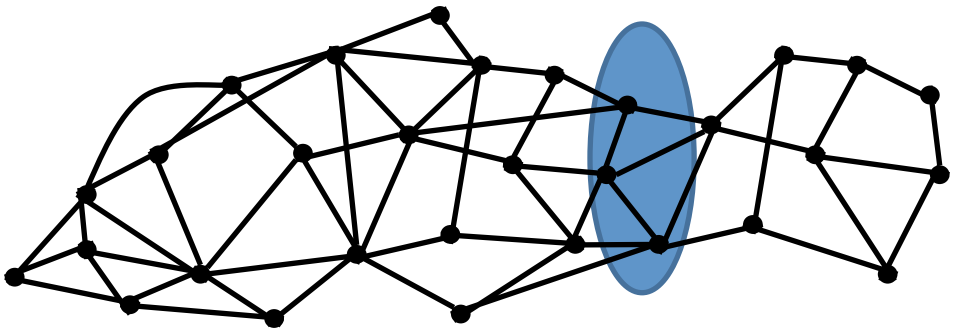

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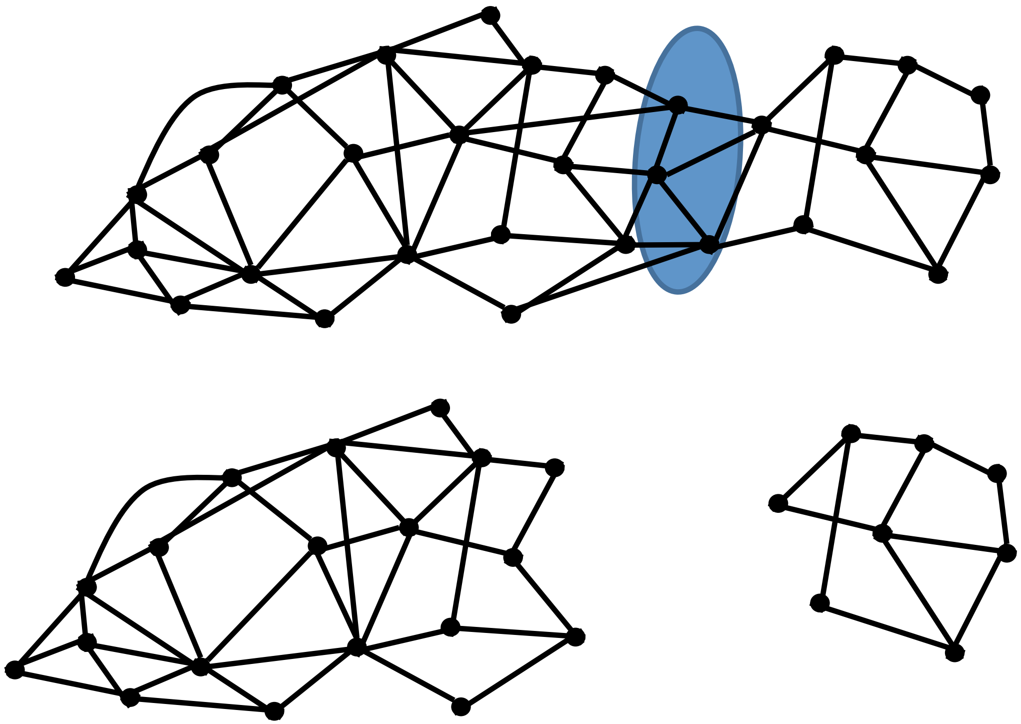
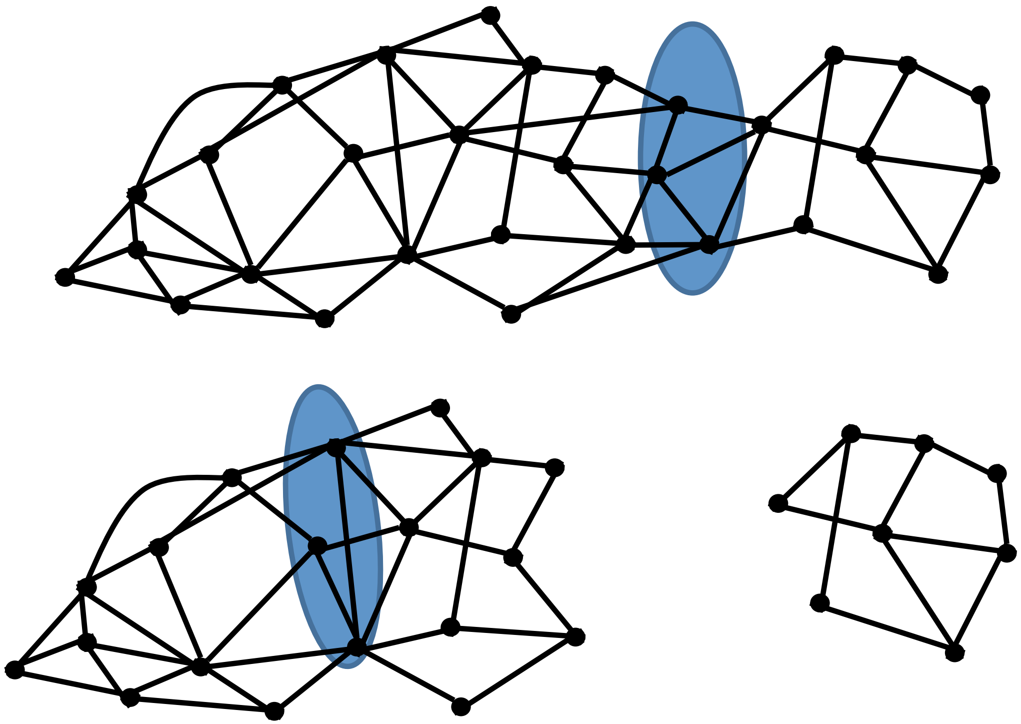


Every Planar Graph Has A ½-Separator of Size $4\sqrt{2}\sqrt{|V|}$



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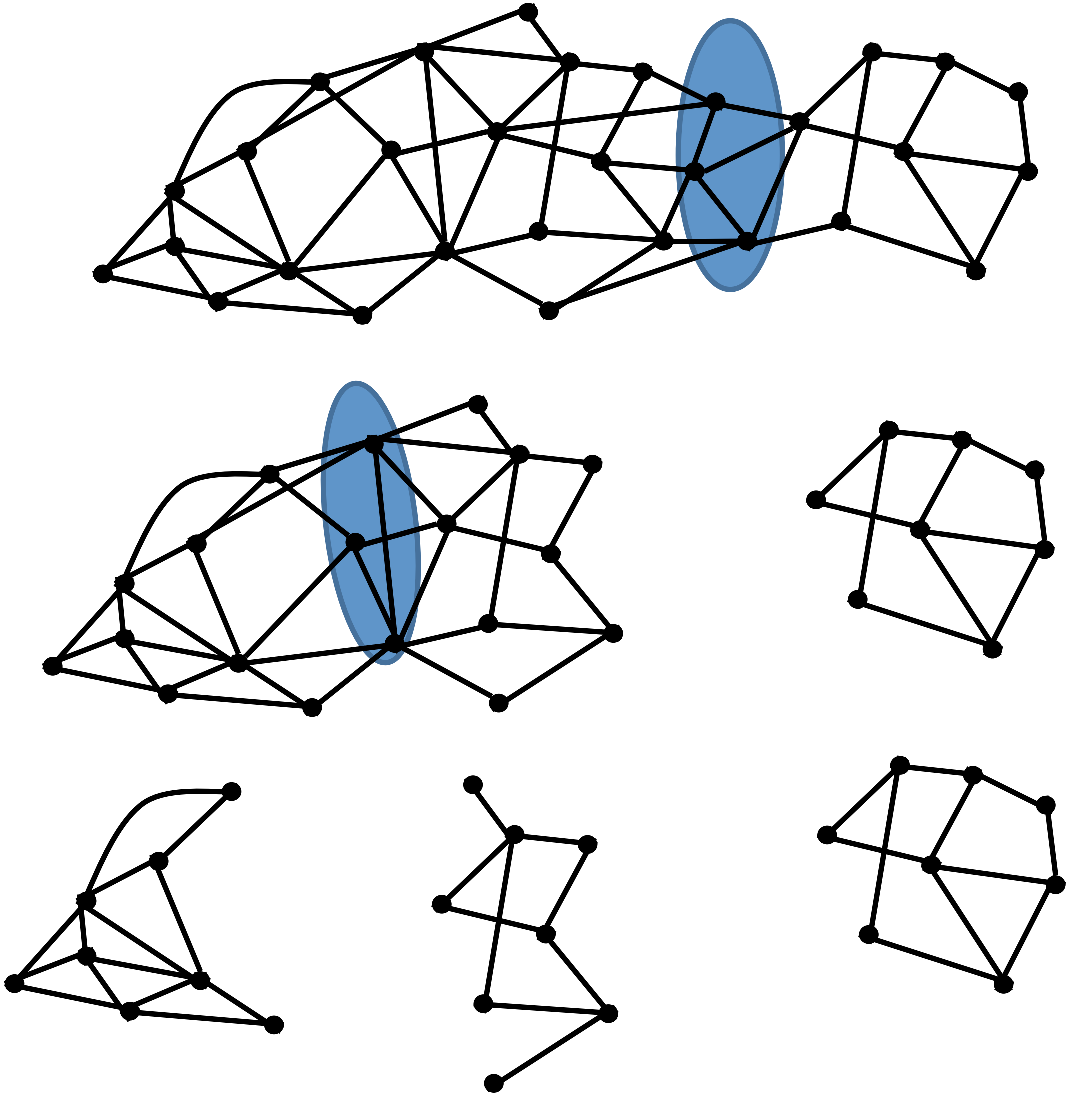
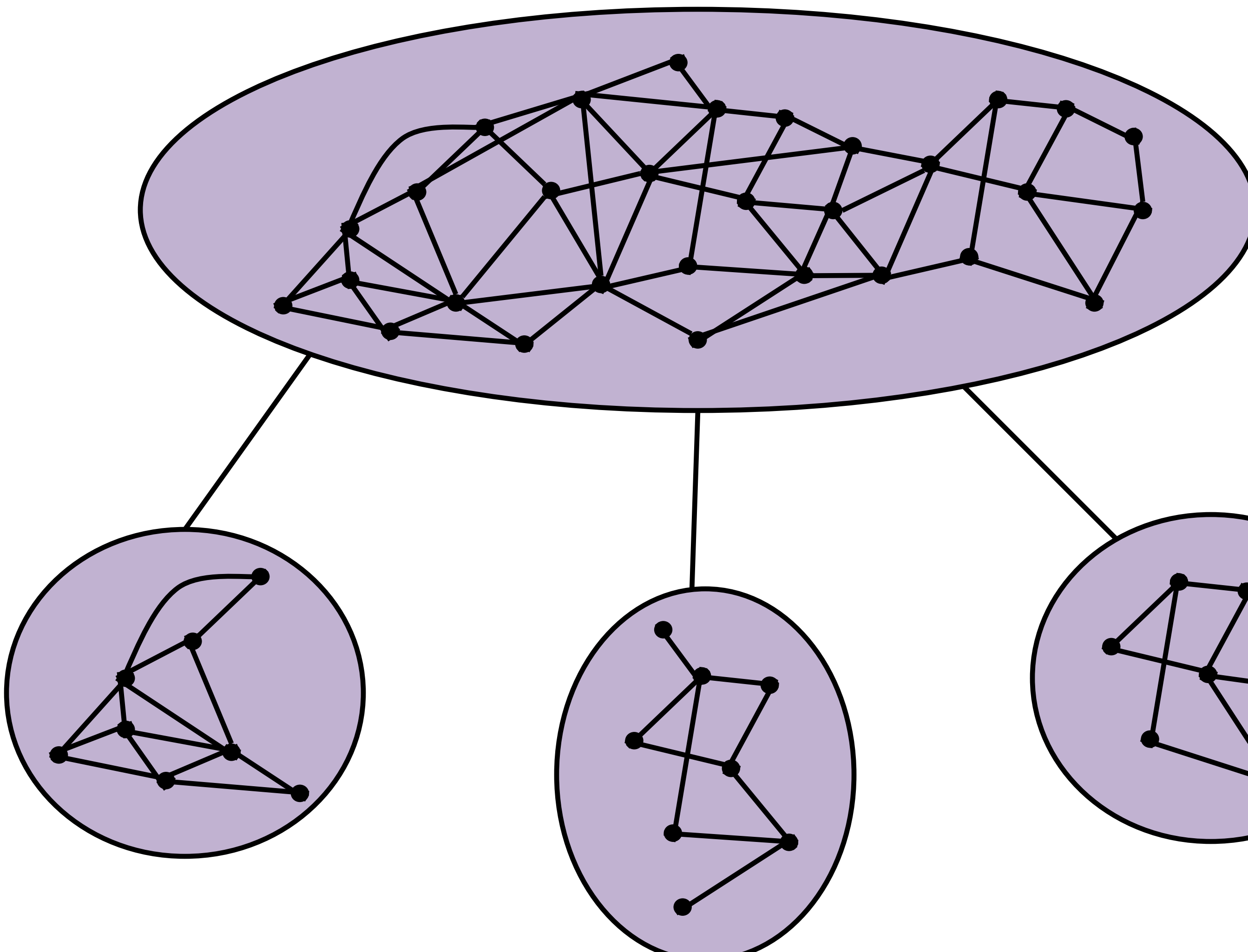
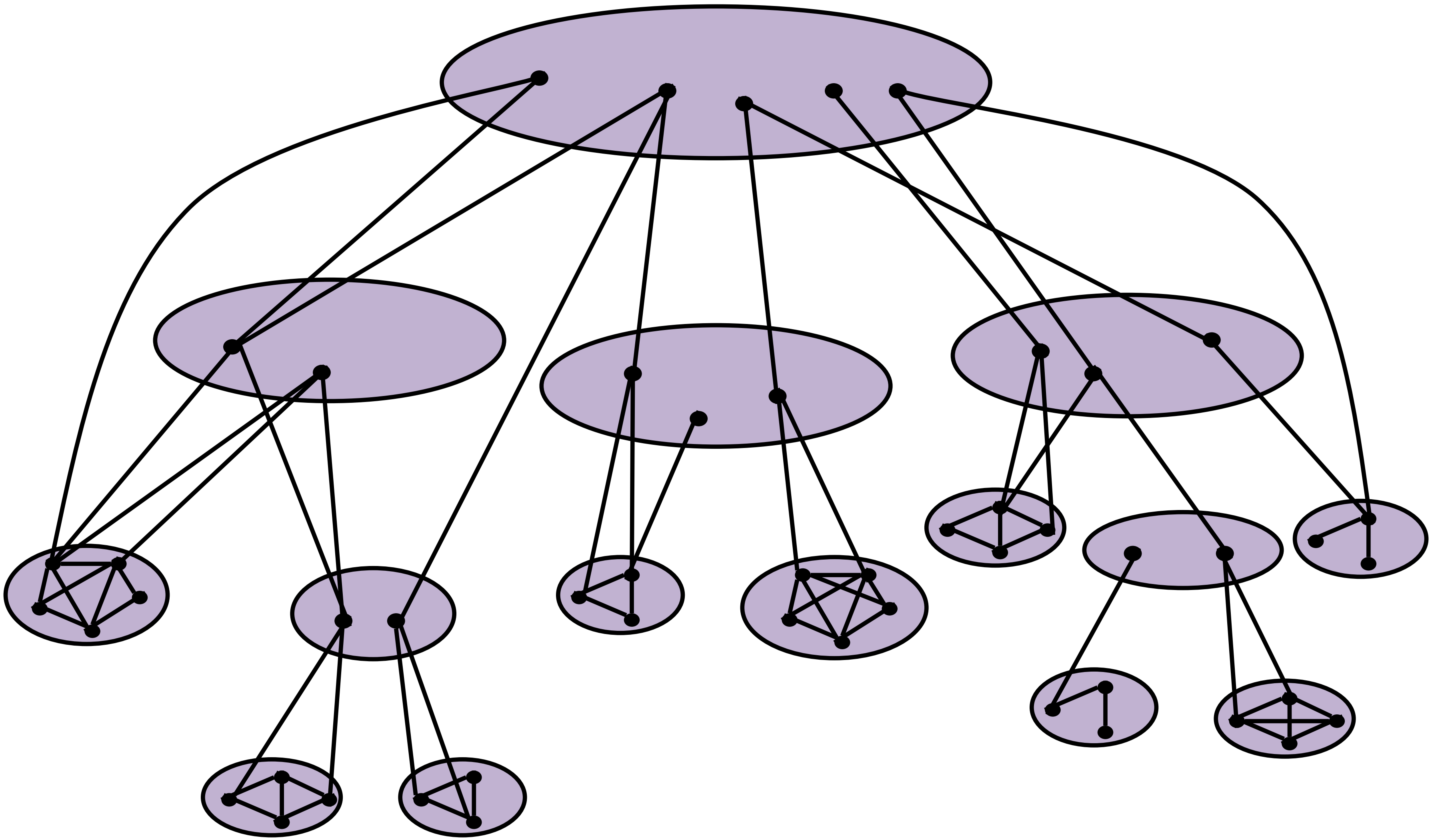


Decomposing On A Separator



All Planar G have a cutset X s.t. |X|=o(|V|) and each component U of G-X has at most log log |V| vertices



Approximation Algorithms

- Can efficiently find a cutset X of o(|V|) vertices s.t. each component of G-X has at most log log |V| vertices in any graph G provided we can find a small separator for each subgraph of G in linear time.

- Can thereby approximately solve many optimization problems in O(|V| log |V|) time on such graphs.

Two Seminal Papers

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Graph Searching and a Min–Max Theorem for Tree-Width

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Received April 19, 1989

The tree-width of a graph G is the minimum k such that G may be decomposed into a "tree-structure" of pieces each with at most $k + 1$ vertices. We prove that this equals the maximum k such that there is a collection of connected subgraphs, pairwise intersecting or adjacent, such that no set of $\leqslant k$ vertices meets all of them. A corollary is an analogue of LaPaugh's "monotone search" theorem for cops trapping a robber they can see (LaPaugh's robber was invisible). © 1993 Academic Press, Inc.

1. INTRODUCTION

Here is a cops-and-robber game, played on a finite, undirected graph G. (All graphs in this paper are undirected, and finite unless we say otherwise.) The robber stands on a vertex of the graph, and can at any time run at great speed to any other vertex along a path of the graph. He is not permitted to run through a cop, however. There are k cops, each of whom at any time either stands on a vertex or is in a helicopter (that is, is temporarily removed from the game). The objective of the player controlling the movement of the cops is to land a cop via helicopter on the vertex occupied by the robber, and the robber's objective is to elude capture. (The point of the helicopters is that cops are not constrained to move along paths of the graph—they move from vertex to vertex

* This research was performed under a consulting agreement with Bellcore and while visiting the DIMACS Center at Rutgers University.

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The Seminal Papers

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  Every subgraph has a small separator.
- Use these to provide an algorithm for finding a small separator in certain nonplanar graphs
- Characterize which graphs have these properties using a slightly different decomposition tree built using separators

S-Separators and Linkedness

- For a set S of vertices, an *S-Separator* is a subset X of V such that every component U of G-X satisfies:

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- *Link(G)* is the maximum k such that there is some S for which there is no S-Separator of size <k.

Brambles and Their Orders

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- *BN(G)* is the maximum order of a bramble in G

BN(G) AND Link(G)

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The order of β_S is the size of a minimum separator for S.

For any X with $|X|<\text{ord}(\beta)$ there is a unique component f(X) of G-X containing an element of β.

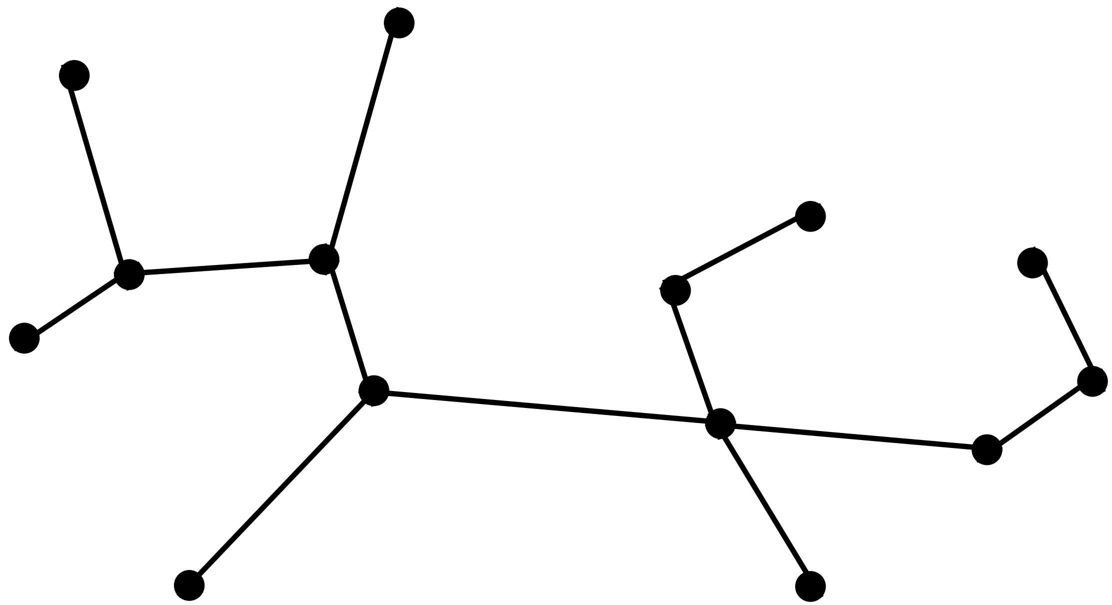
Separators in Planar Graphs

- Easy Proof that a planar graph has no bramble of order exceeding $5\sqrt{|V|}$ vertices.

- Proof that every planar graph has a separator with $2.12\sqrt{|V|}$ vertices

Alon, Seymour, Thomas 1994.

Trees, Single Vertex Separators, And Choosing Sides



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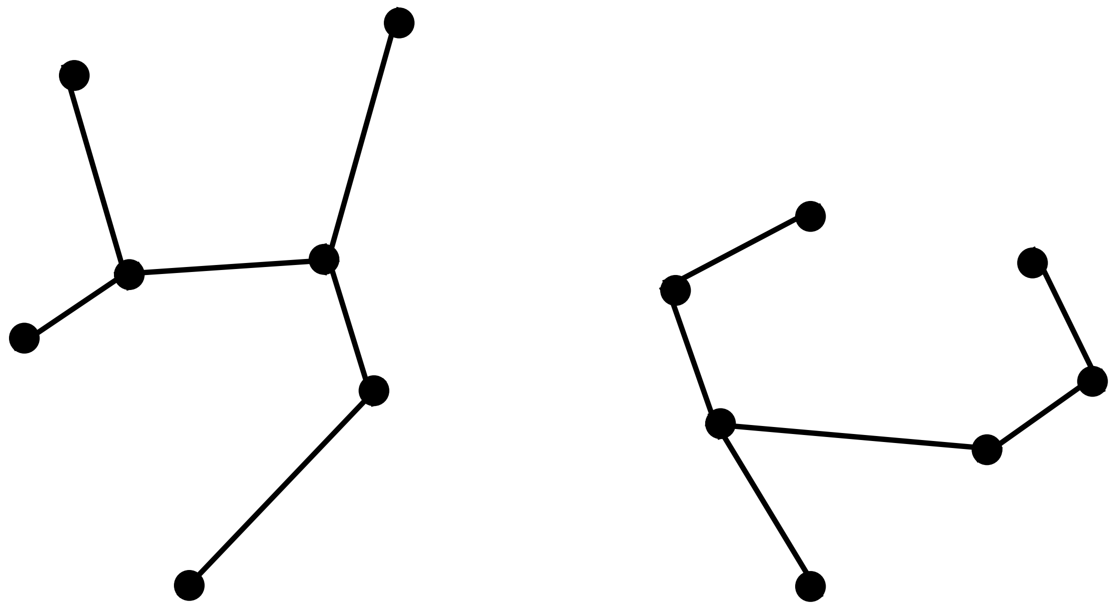

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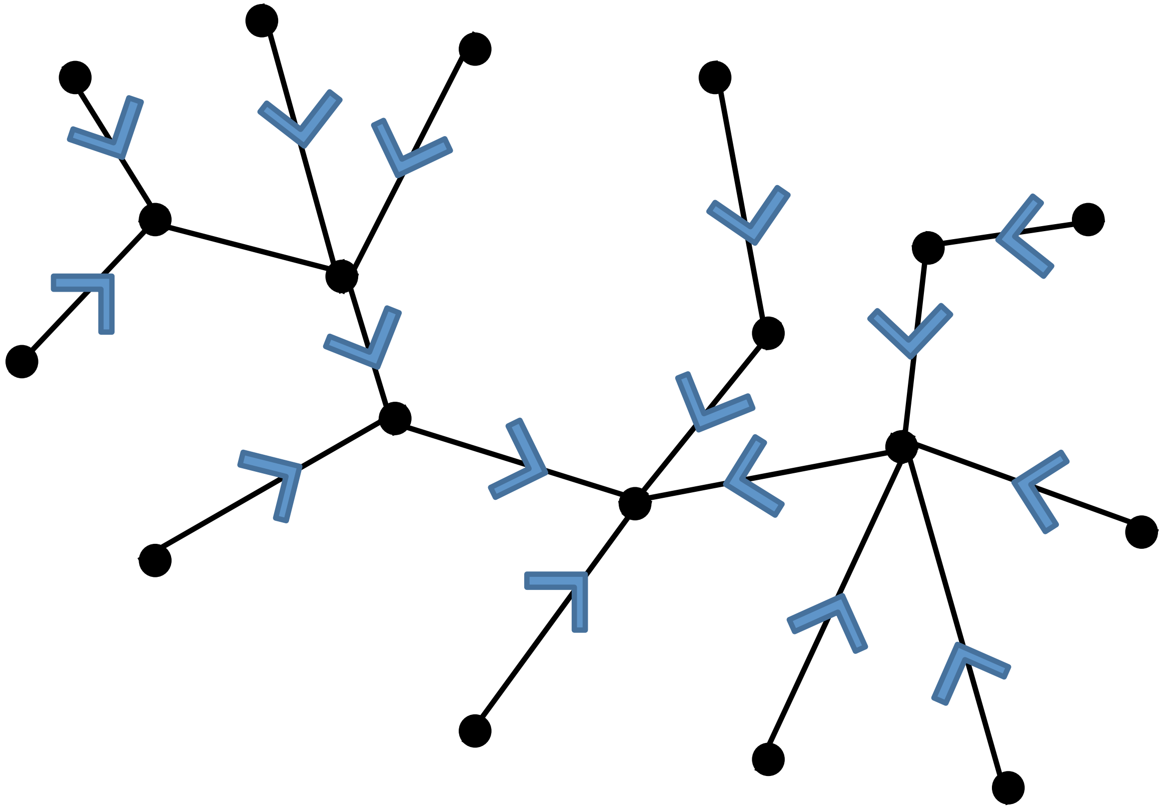

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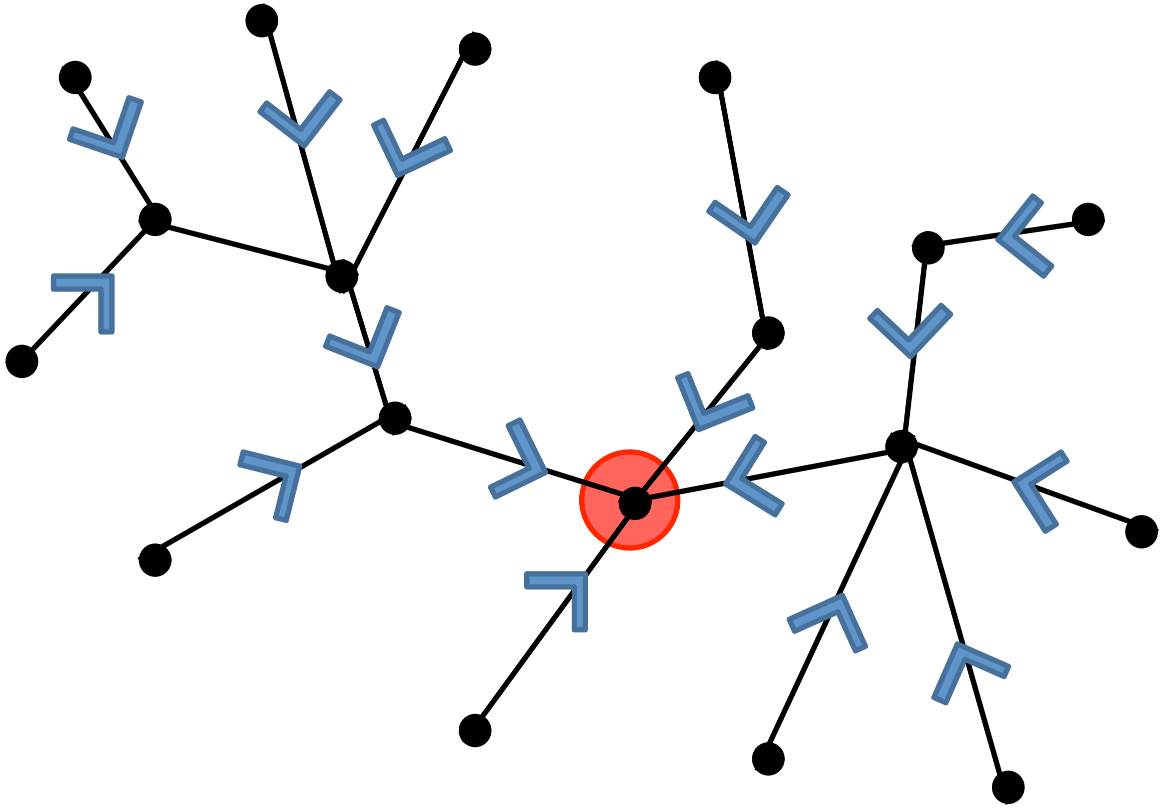
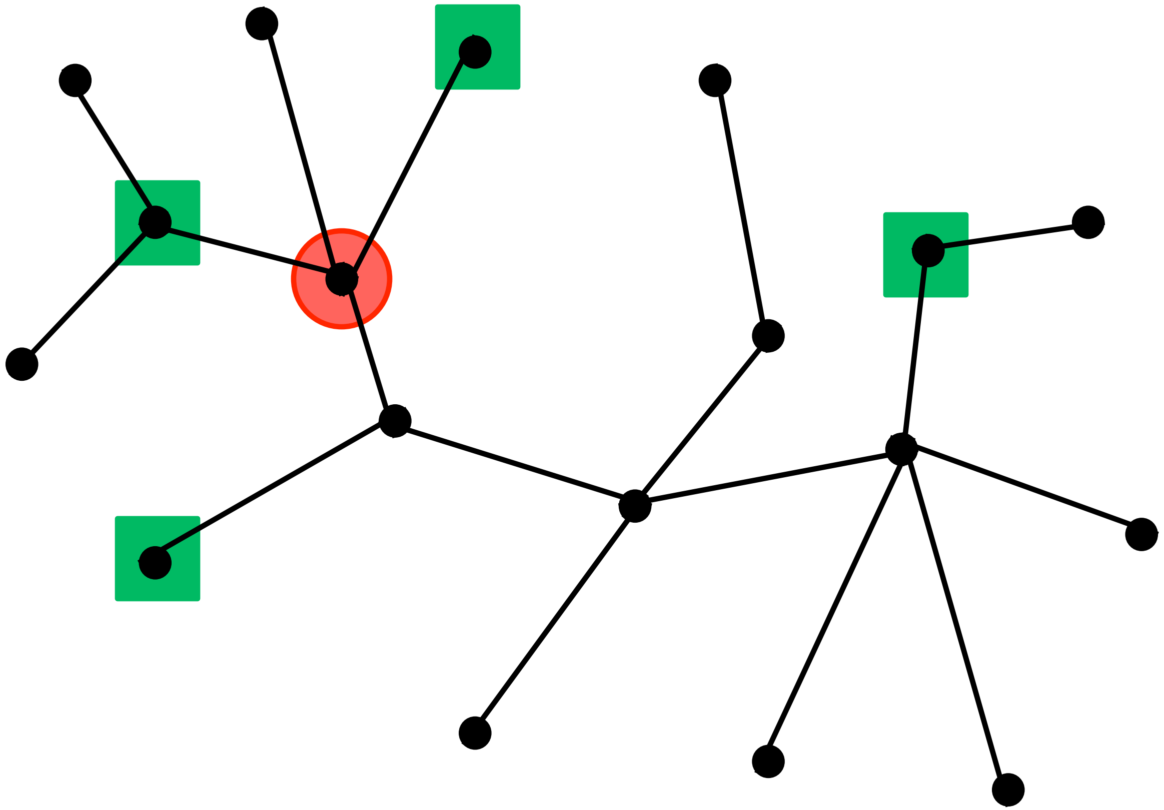


G Has A Single Vertex S-Separator For Every S Precisely If It Is A Forest



Tree Decompositions And Their Widths

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Tree Decompositions And Their Widths

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- The treewidth of G is the minimum of the widths of its tree decompositions.

Tree Width and Link(G)

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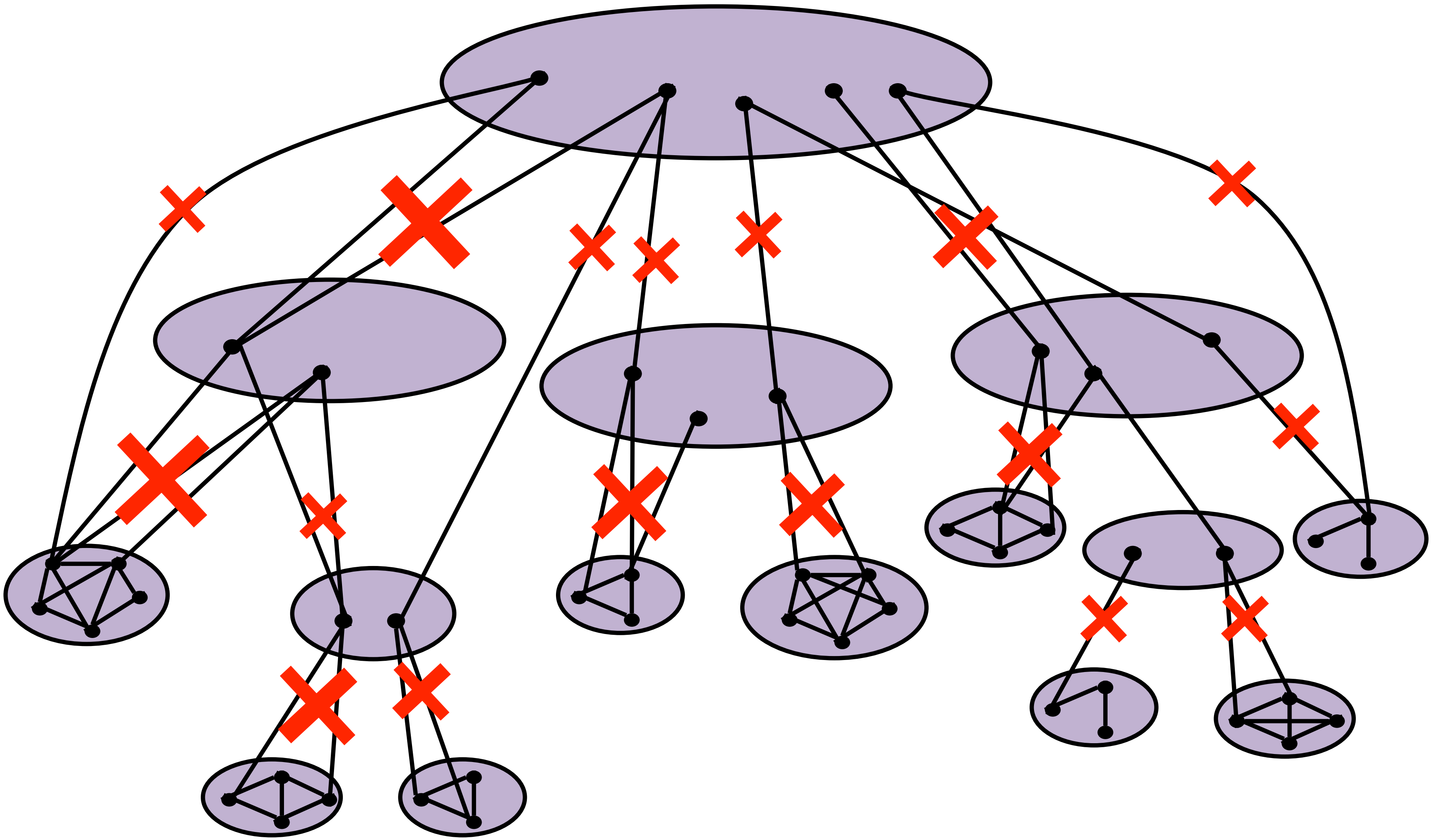
Tree Width and Link(G)

- A Graph of Tree Width k has an S-Separator of size at most k+1 for all $S \subseteq V$

- A graph G which has an S-Separator of size at most k for every $S \subseteq V$ has treewidth at most 3k

Build Tree Decomposition Where Edges Are In The Leaves



Brambles And Tree Decompositions

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- The Tree Width of G is BN(G)-1.

Seymour and Thomas, 1993.

Minors, Models, and Brambles

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- G contains an H-model if and only if H is a minor of G.
- A K_l model is a bramble of order l whose elements do not intersect.

Nice Tree Decompositions of Graphs Without Large Clique Minors

- If G has no K_5 minor then it has a tree decomposition where for each node t of the tree, the graph F_t corresponding to t is planar or a specific 8 vertex graph

Wagner, 1943

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  Wagner, 1943

- If G has no K_l minor then it has a tree decomposition where for each node t, the graph F_t is quasi-embedded in a surface in which K_l cannot be embedded.

  Roberston and Seymour, 2003

Excluding Large Clique Minors

- For any l, can find a separator of size $O(\sqrt{|V|})$ in any graph without a K_l minor in linear time

Kawarabayashi,Li, and Reed 2011

Excluding Large Clique Minors

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  Kawarabayashi,Li, and Reed 2011

- For any l, can find a Robertson-Seymour tree decomposition of a graph without a K_l minor in $O(|V|\log |V|)$ time.

  Kawarabayashi, Li, and Reed 2011

Indistinguishable Brambles

- Two brambles β_1 and β_2 are *indistinguishable* if for every set X such that |X| is less than the minimum of the orders of the two brambles, the two brambles choose the same component of G-X.

Quickly Excluding A Planar Graph

- If G contains a bramble β of order f(k) then it contains a bramble corresponding to a k by k grid which is indistinguishable from it

Robertson, Seymour, Thomas, 1995.

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- The grid model can be found in linear time if β is β_S for some set S which has bouded size or is V.

Finding Small Separator in Graphs Without K_l Models

- For three huge constants k<< c<C, if β_V has order at most c then can find a separator of size C in linear time.

Finding Small Separator in Graphs Without K_l Models

- For three huge constants $k << c < C$, if β_V has order at most c then can find a separator of size C in linear time.
- Otherwise, find a k by k grid model, indistinguishable from $\beta_{V.}$

Finding Small Separator in Graphs Without K_l Models

- For three huge constants k<< c<C, if β_V has order at most c then can find a separator of size C in linear time.
- Otherwise, find a k by k grid model indistinguishable from β_V.
- Find a nice quasi-embedding of the graph with respect to this model.

Finding Small Separator in Graphs Without K_l Models

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- Otherwise, find a k by k grid model indistinguishable from $\beta_{V.}$
- Find a nice quasi-embedding of the graph with respect to this model.
- Find an $O(\sqrt{|V|})$ separator in the part of the graph embedded in a simple surface.

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- Proceed as in bounded tree width case until the root set S has size 2c and no c separator.

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Finding Tree Decompositions of Graphs Without K_l Models

- Proceed as in bounded tree width case until the root set S has size 2c and no c separator.
- Find (i) a k by k grid model indistinguishable from β_S and (ii) the associated quasi-embedding.
- Put the embedded part into the tree node, and have children corresponding to the cutsets cutting off the non-embedded pieces.

One Extra Piece

- For all l and t there is an $\varepsilon > 0$ such that if G has no K_l model then it either contains a matching with $\varepsilon|V|$ edges or a stable set S with $\varepsilon|V|$ vertices such that each vertex of S has the same neighbourhood as t other vertices.

Reed and Wood.

Back To The Basics

- Brambles define a choice of component for each small cutset.
- Can walk towards a bramble to build a simpler bramble which is indistinguishable from it.
- Repeated separation yields a tree decomposition.

Happy Birthday Robin!