Feynman rules for wave turbulence

Vladimir Rosenhaus

Initiative for the Theoretical Sciences
The CUNY Graduate Center

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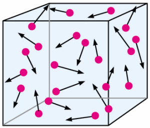
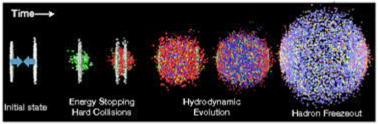
The thermal state

- Statistical physics is built around the study of the thermal state.
- This is for good reason: at late times, generic closed systems with generic initial conditions are, for most purposes, indistinguishable from the thermal state.



Far-from-equilibrium

- There is an increasingly large range of contexts in which one needs to study far-from-equilibrium system.
- This is challenging: the thermal state is now irrelevant, and there is no clear “universal” state to replace it.
- It appears one must study every far-from-equilibrium initial condition on a case-by-case basis.



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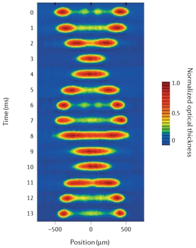

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Forcing and dissipation

- The key is to look at a subsystem, consisting of a range of modes of the underlying field.
- A nonlinear system couples modes of different wavenumber, and there will be a flux of modes passing through the subsystem; the subsystem behaves like an open system.
- As an approximation, we can replace our subsystem with a simple open system: one in which there is a perpetual flux passing through, maintained by external forcing and dissipation acting on an otherwise closed system.

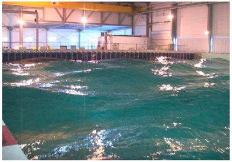
- Of course, systems with forcing and dissipation are by themselves physically relevant, so the motivation in terms of an intermediate stage in the thermalization process is not necessary.
- In the turbulence literature, this is the distinction between freely decaying turbulence versus forced turbulence. In QFT literature, this is called prethermalization.

Open systems

- Once a system is open, it is liberated from the requirement of late time thermalization
- This allows for rich late time behavior.
- Beyond the thermal state, the next simplest possible late time state is a stationary, but non-equilibrium, state.
- We can control the state through the microscopic parameters of the Hamiltonian – the dispersion relation and the nonlinear interaction – as well as the external forcing and dissipation.

Wave turbulence

- Remarkably, even for a weakly interacting nonlinear system, there are cases in which one can find a stationary, non-equilibrium state - the Kolmogorov-Zakharov state.
- This is wave turbulence, or weak turbulence.
- Wave turbulence has been shown to occur in an incredible range of contexts



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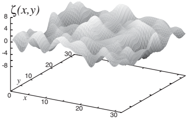


(d)

Figure: (a) Gravity wave turbulence (Falcon et. al) (b) Elastic plate turbulence (During et al.)

Wave turbulence versus turbulence

- Turbulence was originally, and is more commonly, discussed in the context of hydrodynamics – Navier-Stokes equation.
- This is not wave turbulence – there is no weak coupling expansion, and the problem of understanding hydrodynamic turbulence, and the associated range of nonperturbative phenomena, is difficult.
- Turbulence is a state of a nonlinear system in which energy is distributed over many degrees of freedom, in a fashion which strongly deviates from equilibrium, exhibiting chaos in both space and time.

A nonlinear field theory with quartic interaction

- Work in Fourier space, with modes ϕ_k. Do a canonical transformation, replacing the real field and momentum variables, ϕ_k and π_k, with the single complex variable a_k, whose complex conjugate we denote by $a_k^\dagger$,
  $\phi_k = \frac{1}{\sqrt{2\omega_k}}(a_k + a_k^\dagger)$ and $\pi_k = i\frac{\sqrt{\omega_k}}{\sqrt{2}}(a_k^\dagger - a_k)$.
- The Hamiltonian

$$H = \sum_p \omega_p a_p^\dagger a_p + \sum_{p_1, p_2, p_3, p_4} \lambda_{p_1 p_2 p_3 p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \ .$$

- The equations of motion:

$$\dot{a}_k + i\omega_k a_k = -2i \sum_{p_2, p_3, p_4} \lambda_{k p_2 p_3 p_4} a_{p_2}^\dagger a_{p_3} a_{p_4} \ .$$

The kinetic equation

The main player in wave turbulence is the kinetic equation: it governs the occupation number n_k of mode k, $n_k(t) = \langle a_k(t) a_k^\dagger(t) \rangle$.

$$\frac{\partial n_k}{\partial t} = 16\pi \sum_{p_2, p_3, p_4} |\lambda_{k p_2 p_3 p_4}|^2 \left(\frac{1}{n_k} + \frac{1}{n_{p_2}} - \frac{1}{n_{p_3}} - \frac{1}{n_{p_4}} \right) n_k n_{p_2} n_{p_3} n_{p_4}$$
$$\delta(\omega_k + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \delta(k + p_2 - p_3 - p_4)$$

This equation is valid at weak coupling (and a late times).

Boltzmann equation

- The wave kinetic equation for quantum mechanics is similar, but with some of the n_p replaced with $N_p + 1$, where N_p now denotes the occupation number,
- For large occupation number, $N_k \gg 1$, the classical kinetic equation is recovered, while for small occupation numbers, $N_k \ll 1$, waves behave like particles and one recovers the classical Boltzmann equation.

The thermal distribution

What is the stationary solution? i.e. what does one get at late times:

$$\frac{\partial n_k}{\partial t} = 16\pi \sum_{p_2, p_3, p_4} |\lambda_{k p_2 p_3 p_4}|^2 \left(\frac{1}{n_k} + \frac{1}{n_{p_2}} - \frac{1}{n_{p_3}} - \frac{1}{n_{p_4}} \right) n_k n_{p_2} n_{p_3} n_{p_4}$$
$$\delta(\omega_k + \omega_{p_2} - \omega_{p_3} - \omega_{p_4}) \delta(k + p_2 - p_3 - p_4)$$

Can check,

$$n_k = \frac{T}{\omega_k}$$

is a stationary solution. This is the Rayleigh-Jeans distribution, which is the low energy (high temperature; i.e. classical) limit of the Plank distribution (Bose-Einstein).

The Kolmogorov-Zakharov (KZ) solution

There is another stationary solution. For,

$$\omega_p \propto p^{\alpha} \ , \quad \lambda_{p_1 p_2 p_3 p_4} = (p_1 p_2 p_3 p_4)^{\frac{\beta}{4}} U \delta_{p_1,p_2;p_3,p_4} \ ,$$

where U depends only on the ratio of momenta and their mutual angles, one may check that there is a stationary solution to the kinetic equation,

$$n_p \propto p^{-\gamma} \ , \qquad \gamma = \frac{2}{3}\beta + d \ .$$

This is the Kolmogorov-Zakharov (KZ) solution. Since the kinetic equation is only valid at weak nonlinearity, these solutions are called weakly turbulent states.

The KZ solution deviates strongly from equilibrium.

Fluctuations about KZ

- Much of the work on wave turbulence has focused on establishing the existence and properties of the turbulent state.
- A broader question is how to repeat everything we know in statistical mechanics, but based on the wave turbulent state instead of the thermal state. Concretely, how to characterize fluctuations *about* the turbulent state. Our goal is to take a step in that direction.

Plan:

- We take a classical nonlinear field theory, with an arbitrary dispersion relation and arbitrary quartic interaction.
- We add dissipation, as well as external forcing, where the forcing function is drawn from a Gaussian distribution.
- We give a general prescription for computing correlation functions of the field.
- The basic tool that we use is that a classical field theory with stochastic forcing is a quantum field theory, which in turn can be solved perturbatively through Feynman diagrams.

This provides a systematic way of deriving the kinetic equations, going beyond leading order in the coupling.

A note on averaging

- Turbulent cascades are present in interacting, chaotic, many-body systems. To perform calculations, or measurements, some kind of averaging is necessary.
- Assuming statistical spatial homogeneity, one can average over initial conditions, or alternatively one can perform a time average.
- In the context of forced turbulence, one can average over the forcing function.
- For deriving the leading order kinetic equation, which kind of averaging is used is largely irrelevant. At leading order one can simply assume higher-point correlation functions factorize into two-point functions; this is referred to as the random phase approximation.

Current work on Wave Turbulence

Significant active work:

- New physical contexts exhibiting classical wave turbulence, e.g. During, Mordant, Zakharov, Nazarenko, L'vov, Onorato,...
- Wave turbulence in quantum mechanics and the nonlinear Schrödinger equation, e.g. Buckmaster, Nazarenko,...
- Wave turbulence in quantum field theory and related concepts of prethermalization Micha and Tkachev, Berges, Schlichting,...
- Mathematical properties of wave turbulence, e.g. Eyink, Shi, Newell, Tran, Faou,..
- Models such as MMT,...

Relation to other work

- Shavit and Falkovich, '20 stressed the importance of quantities beyond the mode occupation number.
- Gurarie, '95 found the next-to-leading order correction to the kinetic equation governing the mode occupation number. We will reproduce and generalize his result.
- The connection between stochastic classical field theories and quantum field theories, is well-known has appeared before in turbulence, e.g. Wyld, '61, Migdal et al., '95, Martin-Siggia-Rose, '73 .
- However, as far we know, path integral methods have not been applied to classical wave turbulence for the explicit purpose of systematically computing correlation functions perturbatively in the coupling

Outline

1. Show that a classical field theory with Gaussian random forcing is equivalent to a quantum field theory, with a Lagrangian that is the square of the force-free equations of motion.
2. Use this Lagrangian to work out Feynman rules
3. Compute tree-level diagrams (gives leading order kinetic equation)
4. Compute one-loop diagrams (give next-to-leading order kinetic equation)

A nonlinear interacting field with random forcing

$$H = \sum_{p} \omega_p a_p^\dagger a_p + \sum_{p_1,p_2,p_3,p_4} \lambda_{p_1 p_2 p_3 p_4} a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3} a_{p_4} \ .$$

Equations of motion:

$$\dot{a}_k = -i \frac{\partial H}{\partial a_k^\dagger}$$

Add forcing and dissipation,

$$\dot{a}_k = -i \frac{\partial H}{\partial a_k^\dagger} + f_k(t) - \gamma_k a_k \ .$$

Correlation functions

We will want to average over the forcing, drawn from a Gaussian distribution,

$$P\left[f\right] \sim \exp\left(-\int dt \sum_k \frac{|f_k(t)|^2}{F_k}\right)$$

$$\langle f_k(t) f_p^*(t')\rangle = F_k \delta(k-p)\delta(t-t')$$

In the "diagrammatic" approach (Zakharov and L'vov, '75) one solves the equations of motion with some definite forcing, and then averages over the forcing.

This is straightforward, but tedious (Erofeev and Malkin, '89).

Path integral

We would like to streamline the procedure, by integrating out the forcing at the outset.

The expectation value of a general operator $\mathcal{O}(a)$ is found by solving the equations of motion and computing $\mathcal{O}(a)$ for each value of f_k, and then averaging over the f_k as prescribed by the probability distribution $P[f]$

$$\langle \mathcal{O}(a) \rangle = \int \mathcal{D}f \mathcal{D}f^* \, P[f] \, \mathcal{O}(a) \ ,$$

We insert a delta to function, to enforce the equations of motion,

$$\begin{aligned} \langle \mathcal{O}(a) \rangle &= \int \mathcal{D}a \mathcal{D}a^\dagger \, \mathcal{D}f \mathcal{D}f^* \, P[f] \, \mathcal{O}(a) \, \delta\big(\mathrm{Re}(E_f)\big) \ \delta\big(\mathrm{Im}(E_f)\big)\big) \ , \\ \mathrm{E}_f &= \dot{a}_k + i \frac{\delta H}{\delta a_k^\dagger} - f_k(t) + \gamma_k a_k \ , \end{aligned}$$

We write the delta functionals in integral form as,

$$\delta\big(\mathrm{Re}(E_f)\big)\ \delta\big(\mathrm{Im}(E_f)\big)) = \int \mathcal{D}\eta\mathcal{D}\eta^*\ e^{i\int dt \sum_k \left(\eta_k(t)E_f^* + \eta_k^* E_f\right)}\ ,$$

and perform the integral over f,

$$\langle \mathcal{O}(a)\rangle = \int \mathcal{D}\eta\mathcal{D}\eta^*\mathcal{D}a\mathcal{D}a^\dagger\,\mathcal{O}(a)\,e^{-\int dt\, L}$$

$$L = \sum_k \eta_k(t)F_k\eta_k^*(t) - i\eta_k(t)E_{f=0}^* - i\eta_k^*(t)E_{f=0}\ ,$$

Carrying out the Gaussian integral over η_k gives,

$$\langle \mathcal{O}(a)\rangle = \int \mathcal{D}a\mathcal{D}a^\dagger\,\mathcal{O}(a)\,e^{-\int dt\, L}\ , \qquad L = \sum_k \frac{|E_{f=0}|^2}{F_k}\ .$$

Stochastic field theory to quantum field theory

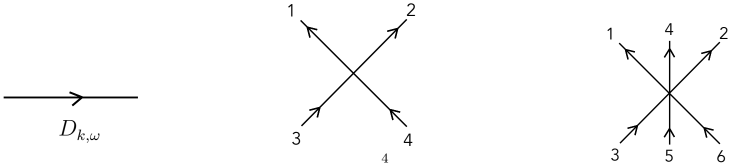
- The result is sensible: We started with equations of motion $E_{f=0}$, added a forcing term f_k, and then averaged over a Gaussian distribution for the f_k.
- The result is an effective Lagrangian which is proportional to the magnitude squared of $E_{f=0}$.
- Notice that initially the only averaging was over the forcing term f_k, with f_k having a Gaussian probability distribution.
- The end result, however, is the Lagrangian in which there is no forcing term, but with a_k as a fluctuating variable; in other words, a quantum field theory.

The Lagrangian

$$L = \sum_k \frac{1}{F_k} \left| \mathscr{D}_k a_k + 2i \sum_{p_2,p_3,p_4} \lambda_{kp_2p_3p_4} a^\dagger_{p_2} a_{p_3} a_{p_4} \right|^2 .$$

$$\mathscr{D}_k a_k \equiv \dot{a}_k + (i\omega_k + \gamma_k) a_k$$

Feynman diagrams, in frequency space:



$D_{k,\omega}$

$-i\lambda_{p_1p_2p_3p_4} \sum_{i=1}^{4} \widehat{G}_i$

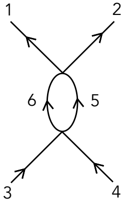
$-4 \sum_{p_7} \frac{1}{F_{p_7}} \lambda_{p_1p_2p_3p_7} \lambda_{p_7p_4p_5p_6}$

(a) (b) (c)

$$D_{k,\omega} \equiv \int dt D_k(t) e^{-i\omega t} = F_k |G_{k,\omega}|^2 \,, \quad \text{where} \quad G_{k,\omega} = \frac{i}{\omega - \omega_k + i\gamma_k} \,.$$

$$\widehat{G}_i = \frac{1}{G^*_{p_i,\omega_i} F_{p_i}} \,, \quad i = 1, 2 \,, \qquad \widehat{G}_i = \frac{-1}{G_{p_i,\omega_i} F_{p_i}} \,, \quad i = 3, 4 \,.$$

One-loop



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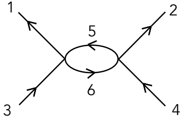

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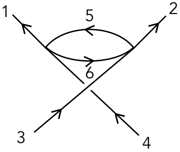

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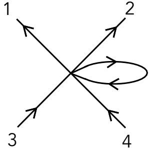

(g)

Kinetic equation

$$
\begin{aligned}
\frac{1}{4}\frac{\partial n_k(t)}{\partial t} &= \sum_{p_i} \delta_{k,p_1} \lambda_{p_3 p_4 p_1 p_2}\, 4\lambda_{p_1 p_2 p_3 p_4} \prod_{i=1}^{4} n_i \left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right) \pi\delta(\omega_{p_1,p_2;p_3,p_4}) \\
&+ \sum_{p_i} \lambda_{p_3 p_4 p_1 p_2} \delta_{k,p_1} \prod_{i=1}^{6} n_i \Big\{ 8\lambda_{p_1 p_2 p_5 p_6} \lambda_{p_5 p_6 p_3 p_4} \Big[\left(\frac{1}{n_5}+\frac{1}{n_6}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right) \frac{\pi\delta(\omega_{p_1,p_2;p_3,p_4})}{\omega_{p_1,p_2;p_5,p_6}} \\
&\qquad + (3,4) \leftrightarrow (5,6) + (1,2) \leftrightarrow (5,6) \Big] \\
&\quad + \Big(16\lambda_{p_1 p_6 p_3 p_5} \lambda_{p_2 p_5 p_4 p_6} \Big[\left(\frac{1}{n_5}-\frac{1}{n_6}\right)\left(\frac{1}{n_1}+\frac{1}{n_2}-\frac{1}{n_3}-\frac{1}{n_4}\right) \frac{\pi\delta(\omega_{p_1,p_2;p_3,p_4})}{\omega_{p_4,p_6;p_2,p_5}} \\
&\qquad + (1,3) \leftrightarrow (5,6) + (4,2) \leftrightarrow (5,6) \Big] + (3 \leftrightarrow 4) \Big) \Big\} \,.
\end{aligned}
$$

$\omega_{p_i,p_j;p_k,p_l} \equiv \omega_{p_i} + \omega_{p_j} - \omega_{p_k} - \omega_{p_l}$.

Summary

- We took a classical field theory with an arbitrary, but small, quartic interaction (a collection of coupled harmonic oscillators) with dissipation and Gaussian-random forcing
- We gave a prescription for computing correlation functions, perturbatively in the coupling.
- We applied this to compute the two-point and four-point correlation functions, to next-to-leading order in the coupling.

- In the limit of vanishing forcing and vanishing dissipation, one might have thought that the properties of the system should be the same as those of a closed system, whose late time state is the thermal state.
- This is not so.
- If the interactions and dispersion relation take the form of homogeneous functions (in the mathematical sense), there is a choice of $n_k \equiv F_k/2\gamma_k$ (kept finite in the limit of vanishing F_k, γ_k) which, at leading order, gives an alternate stationary state – the Kolmogorov-Zakharov state (the turbulent state), which has a flux of energy passing through.
- With this choice of n_k, our equations for the correlation functions at next-to-leading order characterize fluctuations about the Kolmogorov-Zakharov state. This includes the next-to-leading order correction to the Kolmogorov-Zakharov state itself, as characterized by the next-to-leading order kinetic equation.

Goals

There are two broad goals which motivate study of correlation functions in the turbulent state.

1)This is a stationary state which is not the thermal state (and is relevant even in contexts of closed systems with far-from-equilibrium initial conditions) and one would like to develop linear response theory for it. For instance, one would like to find the turbulent-state analog of transport coefficients and the fluctuation-dissipation theorem, concepts familiar for the thermal state.

2) Independently, there has been enormous interest in recent years in the study of many-body chaos. Viewing wave turbulence in light of these new developments may be productive.

Future work

- Higher order in the coupling, higher-point correlation functions, information-theoretic measures (entanglement entropy).
- Invariant measure of a_k (the probability distribution for a_k follows form the correlation functions)
- The next-to-leading order corrections to the correlation functions should, in principle, be measurable quantities. Connect with experiments.
- Generalize to quantum field theory. Goes beyond thermal field theory. A corner of far-from-equilibrium quantum field.