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COMPACT AND RELATED MAPPINGS

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CHAPTER 0

INTRODUCTION

A continuous function from one topological space to another is *compact* if the inverse image of every compact subset of the range is compact. Such functions are also sometimes called *proper* (German: *eigentlich.* French: *propré.*). This concept appeared first around 1947 and has since been the subject of much study.

The class of compact mappings (continuous functions) is of interest largely because they have, in a more general setting, so much of the character of a mapping on a compact domain space. Many results originally obtained for mappings on a compact space have been established later for compact mappings; the celebrated Whyburn-Eilenberg Factorization Theorem is a notable example. The compact mapping is also a generalization of the notion in complex function theory of having a pole at infinity. Compact mappings thus have, in a more general setting, many of the topological properties of polynomial mappings.

In this work, we study the relationships between compactness of a mapping and certain other common mapping properties; e.g., closed, open, quasi-open. In addition, we study certain properties of convergent nets of mappings and their limits.

Chapter 0 is an introductory chapter; it contains general results to be used in the remainder of the work. There is in this chapter a discussion of the so-called Whyburn unified space of a mapping, including some new extensions of G. T. Whyburn's original work in this area.

Chapter II is concerned with certain quasi-open mappings and their relation to compact mappings on separable metric spaces. Central to this investigation is the concept of component degree, or multiplicity, for a mapping. Several sufficient conditions for compactness of quasi-open mappings are obtained in terms of this multiplicity concept.

Chapter III deals with mappings defined on subsets of n-dimensional Euclidean space into n-dimensional Euclidean space. The results in this chapter are similar to those in Chapter II but are somewhat sharper because of the availability of the notion of topological, or Brouwer, degree. Also obtained in this chapter are some polynomial-like properties of compact mappings.

In Chapter IV, we investigate the relationship between closed mappings and compact mappings.

Chapter V is concerned with convergent nets of mappings from one space into another. In particular, we investigate conditions which insure that the limit of a net will have certain common mapping properties such as compactness, openness, monotoneity. Some of the results obtained are extensions to more general spaces of results known for metric spaces; most of the results are, to the best of my knowledge, new even in the metric case.

Throughout this work we shall use notation and standard results that may be found in most textbooks of general topology, such as Kelley [5] or Hocking and Young [2]. We assume that all spaces are Hausdorff.

CHAPTER I

PRELIMINARY RESULTS

This chapter contains basic definitions and results which are used throughout the remainder of this work. There is a discussion of the unified space of a mapping, first introduced by G. T. Whyburn [10], including a summary of Whyburn's major results together with extensions. These extensions are interesting *per se*, but are primarily obtained to be applied in the sequel.

1. Definitions and General Comments

(1.1) Definition: A continuous function from one topological space into another is a *mapping*.

(1.2) Definition: A mapping $f:X \to f(X) = Y$ is *compact* if for any compact set $K \subset Y$, $f^{-1}(K)$ is compact.

(1.3) Definition: Given a mapping $f:X \to f(X) = Y$, then $p \in Y$ is a *singular point* of f (with respect to compactness) if every neighborhood of p contains a compact set K for which $f^{-1}(K)$ is not compact.

We shall consistently use the letter S to denote the subset of the range of f consisting of all singular points of f.

The following two theorems are well known but are included to show the genesis of the concept of a compact mapping. The first of the two theorems shows that a compact mapping is indeed a generalization of a mapping with compact domain, while the second demonstrates that a compact mapping generalizes the notion of an entire function's having a pole at infinity.

(1.4) <u>Theorem:</u> Let $f:X \to f(X)$ be a mapping with X a compact space. Then f is a compact mapping.

<u>Proof:</u> Let $K \subset f(X)$ be compact. Then K is closed (all spaces are assumed Hausdorff), which means that $f^{-1}(K)$ is closed. Being a closed subset of the compact space X, $f^{-1}(K)$ is compact.

(1.5) <u>Theorem:</u> Let $f:X \to f(X)=Y$ be a mapping. Let X^* and Y^* be the one point compactifications of X and Y, respectively. Define the function $g:X^* \to Y^*$ to be the extension of f obtained by setting $g(w_x) = w_y$, where w_x and w_y are the ideal points of X^* and Y^*, respectively. Then g is a continuous function if and only if f is a compact mapping.

<u>Proof:</u> First, suppose g is continuous, and let K be a compact subset of Y. Then $Y^* - K$ is an open neighborhood of w_y, from which follows that $g^{-1}(Y^* - K)$ is an open neighborhood of $w_x = g^{-1}(w_y)$.

But $g^{-1}(Y^* - K) = X^* = g^{-1}(K) = X^* - f^{-1}(K)$, so that $X^* - f^{-1}(K)$ is an open neighborhood of w_x. This means, of course, that $f^{-1}(K)$ is compact.

Now assume that f is compact and let $U^* \subset Y^*$ be an open neighborhood of w_y. Then $U = U^* - w_y$ is the complement of a compact subset of Y. Thus $f^{-1}(Y - U) = X - f^{-1}(U)$ is compact subset of X. We have, therefore, that $w_x + f^{-1}(U)$ is an open neighborhood of w_x, and the continuity of g follows from the fact that $g^{-1}(U^*) = w_x + f^{-1}(U)$.

Among entire complex functions, polynomials are the only compact mappings.

(1.6) Example: Let X be the subset of the plane defined by
$X = \{(x,y) \mid 0 \le x < 1,\ 1 \le y \le 2\} + \{(x,y) \mid 1 \le x \le 2,\ 1 < y \le 2\} + \{(x,y) \mid 2 < x \le 3,\ 1 \le y \le 2\}$.

Let $Y = \{(x,y) \mid 0 \le x \le 3,\ y = 0\}$, and define $f\colon X \to Y$ by $f\colon (x,y) \to (x,0)$.

Notice that f is not compact; the inverse of the point $(3/2,0)$, is, for instance, not compact. In this example, the set of singular points S is all points in Y whose x coordinate lies on the closed interval $1 \le x \le 2$.

(1.7) Example: This example shows that even though a mapping is such that $f^{-1}(y)$ is compact for every point $y \in Y$, it still is not necessarily compact.

Once again X and Y are subsets of the complex plane. Let $X = \{z \mid |z| < 1\} - \{z = -1/2\}$ and $Y = \{z \mid |z| < 1\}$. Define f by $f(z) = z^2$.

Clearly $f^{-1}(y)$ is compact for all $y \in Y$. If, however, K is any compact subset of Y which contains the point $z = 1/4$ in its interior, then $f^{-1}(K)$ is not compact.

For this particular mapping, the set of singular points consists of the one point $z = 1/4$.

2. The Whyburn Unified Space

In 1953, G. T. Whyburn [10] introduced the concept of a so-called unified space for mappings. This unified space has proven to be a useful tool in this study of mappings and we make extensive use of this important notion.

Given a mapping $f:X \to Y$ from one Hausdorff space onto another, we define a new space $Z = X' + Y'$ consisting of a point $x' = h(x)$ for each x in X and a point $y' = k(y)$ for each y in Y so that $h(X) = X'$, $k(Y) = Y'$, h and k are one to one, and $X' \cdot Y' = \emptyset$.

A subset $Q \subset Z$ is defined to be open provided

(i) $h^{-1}(Q \cdot X')$ is open in X,

(ii) $k^{-1}(Q \cdot Y')$ is open in Y,

(iii) for any compact set $K \subset k^{-1}(Q \cdot Y')$, $f^{-1}(K) \cdot [X - h^{-1}(Q \cdot X')]$ is compact.

In reference [10], Whyburn establishes that the collection of all such sets Q is indeed a topology for Z. Items (2.1), (2.2), and (2.3) below are results of Whyburn's that will be used in the sequel.

(2.1) h is a strongly open mapping of X into Z and k is a strongly closed mapping of Y into Z. Hence h and k are homeomorphisms.

(2.2) The function $r:Z \to r(Z) = Y'$ of Z onto Y' defined by $r(z) = z$ if z is in Y', $r(z) = kfh^{-1}(z)$ if z is in X' is a compact mapping.

(2.3) Z is T_1, and if X and Y are locally compact separable metric spaces, then so also is Z.

Item (2.2) is the major result of Whyburn's paper [10]; it shows that an arbitrary mapping from one Hausdorff space onto another is the restriction of a compact mapping.

(2.4) Theorem: A point p in Y is a singular point of f if and only if $k(p)$ is an accumulation point of X'.

Proof: (i) Suppose there is an open neighborhood U of $p' = k(p)$ such that $U \cdot X' = \emptyset$. Then $X - h^{-1}(U \cdot X') = X$, so that for any compact $K \subset k^{-1}(U)$, the property that $f^{-1}(K) \cdot [X - h^{-1}(U \cdot X')]$ is compact implies that $f^{-1}(K)$ is compact. Thus p is not a singular point.

(ii) Suppose p is not a singular point and let V be an open neighborhood of p such that $f^{-1}(K)$ is compact for any compact $K \subset V$.

$k(V)$ is an open subset of Z, for $k(V) \subset Y'$ and $f^{-1}(K) \cdot [X - h^{-1}(X - V)] = f^{-1}(K)$ is compact. Thus p is not an accumulation point of X'.

(2.5) Corollary: $S' = k(S) = Fr(Y')$, where $Fr(Y')$ denotes the boundary of Y'.

(2.6) Corollary: S is a closed subset of Y.

(2.7) Corollary: f is a compact mapping if and only if X' and Y' are separated.

(2.8) Corollary: f is a compact mapping if and only if S is empty.

(2.9) Theorem: If Y is locally compact, then for any p in Y - S and any open set U containing $f^{-1}(p)$ there is a neighborhood V of p such that for any q in V, $f^{-1}(q) \subset U$.

Proof: Let W be a neighborhood of p such that for any compact $K \subset W$, $f^{-1}(K)$ is compact. Let V be an open neighborhood of p such that the closure of V, $\bar{V}$, is contained in W, and is compact. The existence of such a V follows from the fact that Y is locally compact. (Recall that a locally compact Hausdorff space is regular.)

Suppose the theorem is false. Then there is a net $\{x_n, D\}$ in $f^{-1}(\bar{V}) - U$ such that $\{f(x_n), D\}$ is a net in Y with $f(x_n) \to p$. $f^{-1}(\bar{V})$ is compact, so $\{x_n, D\}$ has a convergent subnet, say $\{x_j\}$, $x_j \to x \in f^{-1}(\bar{V}) - U$ and $f(x) = p$, a contradiction.

(2.10) Corollary: A compact, one to one mapping is a homeomorphism.

Proof: In this case, $f^{-1}(p)$ is one point, so that the conclusion of the theorem is exactly continuity of f^{-1}.

This corollary is well known.

(2.11) Remark: Theorem (2.9) and the corollary are valid if Y is second countable instead of locally compact. The proof is similar, using sequences rather than nets.

CHAPTER II

QUASI-OPEN MAPPINGS

In this chapter, we study the relationships between certain quasi-open mappings and compact mappings. Central to this investigation is the concept of degree, or multiplicity, for a mapping. We use the so-called *crude* multiplicity function, and study mappings on separable metric spaces.

3. Definitions and General Comments

(3.1) Definition: A mapping $f:X \to f(X) \subset Y$ is *quasi-open* if for any $y \in Y$ and any open set U containing a compact component of $f^{-1}(y)$, then $y \in \operatorname{int} f(U)$, where $\operatorname{int} f(U)$ denotes the interior of $f(U)$, relative to Y.

(3.2) Definition: A mapping $f:X \to f(X) \subset Y$ is *open* provided that for any open set $U \subset X$, $f(U)$ is open in $f(X)$. f is *strongly open* if $f(U)$ is open in Y.

(3.3) Definition: A mapping $f:X \to f(X) = Y$ is *monotone* provided $f^{-1}(y)$ is compact and connected for every y in Y.

(3.4) Definition: A mapping $f:X \to f(X) = Y$ is *light* provided $f^{-1}(y)$ is totally disconnected for every y in Y.

Obviously a strongly open mapping is quasi-open, and a light, quasi-open mapping is strongly open.

The following theorem was proved by Whyburn [8] for separable metric spaces. We need the more general result for subsequent use, particularly in Chapter V.

(3.5) Theorem: Let f be a quasi-open mapping from a locally connected Hausdorff space X into a Hausdorff space Y. If R is an open connected subset of Y and Q is a conditionally compact component of $f^{-1}(R)$, then $f(Q) = R$.

Proof: We show first that $f(Q)$ is an open subset of R.

Let $y \in f(Q) \subset R$ and define $F = f^{-1}(y) \cdot \bar{Q}$. F is compact since $\bar{Q}$ is compact by hypothesis.

Q is a component of $f^{-1}(R)$, so $f^{-1}(y) \cdot Fr(Q) = \emptyset$. Thus $F = f^{-1}(y) \cdot \bar{Q} = f^{-1}(y) \cdot Q$. (Q is open because X is locally connected.) Q is then an open set containg a compact component of $f^{-1}(y)$, so $y \in \operatorname{int} f(Q)$.

Suppose $R - f(Q) \neq \emptyset$. R is open connected set, so there must be a $z \in R - f(Q)$ which is an accumulation point of $f(Q)$, otherwise $R - f(Q)$ would be open in R, and there would be a separation of R.

Let (y_n, D) be a net in $f(Q)$ with limit $z \in R - f(Q)$. Choose $x_n \in f^{-1}(y_n) \cdot Q$. Q is conditionally compact implies the existence of a subnet of (x_n, D) with limit x. By continuity, $f(x) = z$, so $z \in f(Q)$ since Q is a component of $f^{-1}(R)$. This is a contradiction.

(3.6) Theorem: A monotone quasi-open map on a locally compact space is compact.

Proof: Suppose not and let K be a compact subset of Y such that $f^{-1}(K)$ is not compact.

Let (x_n, D) be a net in $f^{-1}(K)$ that has no convergent subnet. Then $(f(x_n), D)$ is a net in K, which is compact. There is, then, a convergent subnet, call it $(f(x_n), D)$, also.

Suppose $f(x_n) \to y$ in K and consider $f^{-1}(y)$. There is a conditionally compact open set U containing $f^{-1}(y)$ since $f^{-1}(y)$ is compact. We know also, by the quasi-openness of f, that $y \varepsilon \operatorname{int} f(U)$.

Thus $(f(x_n), D)$ is frequently in $f(U)$, so that $f^{-1}f(x_n)$ intersects U frequently. The net (x_n, D) is eventually not in U; otherwise there would be a convergent subnet of (x_n, D). We know that all the $f^{-1}f(x_n)$ are connected, so there is a net (z_n, D), with $z_n \varepsilon$ $f^{-1}f(x_n) \cdot Fr(U)$ and $z_n \to z \varepsilon Fr(U)$, since $Fr(U)$ is compact. From the continuity of f, $f(z) = y$, a contradiction.

(3.7) <u>Theorem</u>: A monotone mapping from one locally compact space onto another is compact if and only if it is quasi-open.

<u>Proof</u>: Theorem (3.6) is the "if" part. Let us assume f is compact mapping from locally compact space X onto locally compact space Y.

Let $y \varepsilon Y$ and let K be the compact component of $f^{-1}(y)$. Suppose R is an open set containing K. Let G be a conditionally compact open set such that $K \subset G \subset R$. Such a G exists since X is locally compact.

Suppose $y \not\varepsilon \operatorname{int} f(G)$. Let V be compact neighborhood of y. It follows that $f^{-1}(V)$ is compact neighborhood of K. Let (y_n, D) be a net in $V \cdot (Y - f(G))$ such that $y_n \to y$.

Take $x_n \varepsilon f^{-1}(y_n)$. (x_n, D) is a net in $f^{-1}(V) \cdot (X - G)$, a compact set. There is a convergent subnet, call it (x_n, D) also, so that $x_n \to x \varepsilon f^{-1}(V) \cdot (X - G)$.

Continuity of f implies that $f(x) = y$, which means $x \varepsilon f^{-1}(y) \subset G$, clearly a contradiction.

(3.8) Corollary: If f is a compact mapping onto a locally compact Hausdorff space, y any point in the range space, G an open set containing $f^{-1}(y)$, then $y \varepsilon \text{int } f(G)$.

(3.9) Remark: Theorem (3.7) and the corollary are also valid for range space second countable instead of locally compact. (*cf.* Remark (2.11).)

The separable metric version of Theorem (3.7) is wellknown and first proved by Whyburn [9].

4. The Crude Multiplicity Function

In this section, we assume $f:X \to f(X) = Y$ to be a quasi-open mapping from one locally compact separable metric space onto another, and that $f^{-1}(y)$ is compact with a finite number of components.

(4.1) Definition: Given a mapping $f:X \to f(X) = Y$, the integer valued function $m(y)$ defined on Y by $m(y)$ = number of components of $f^{-1}(y)$ is called the (component) *crude multiplicity function*.

(4.2) Lemma: Suppose $f^{-1}(y)$ is compact and has n components, K_1, K_2, ..., K_n. Then there exist conditionally compact open sets U_1, U_2, ..., $U_n \subset X$, such that $K_i \subset U_i$, $i = 1, 2, \ldots, n$, $U_i \cdot U_j = \emptyset$ for $i \neq j$; and there is an open neighborhood of y, V, such that $V \subset f(U_i)$ and $f^{-1}(V) \cdot Fr(U_i) = \emptyset$ for each $i = 1, 2, \ldots, n$.

Proof: Let U_1, U_2, ..., U_n be pairwise disjoint conditionally compact open sets with $K_1 \subset U_1$, $K_2 \subset U_2$, ..., $K_n \subset U_n$.

The $Fr(U_i)$ are all compact, so $f(Fr(U_i))$ are compact. The quasi-openness of f implies that $y \in \text{int } f(U_i)$. Hence we need only take V to be a neighborhood of y for which $V \subset \Pi \text{ int } f(U_i)$ and $f^{-1}(V) \cdot \sum_i Fr(U_i) = \emptyset$.

(4.3) <u>Theorem</u>: The multiplicity function m(y) is lower semicontinuous.

<u>Proof</u>: If m(y) = n, then by previous lemma, there is a neighborhood of y, V, so that for any z in V, $f^{-1}(z)$ has at least n components. This is because $V \subset f(U_i)$ and $f^{-1}(V) \cdot Fr(U_i) = \emptyset$ for each i = 1, 2, ..., n.

(4.4) <u>Theorem</u>: If m(y) is continuous at a point y = p, then $p \in Y - S$, where S denotes the set of singular points of f. In other words, p is not a singular point of f.

<u>Proof</u>: Suppose m(p) = k. Let $U_1, U_2, \ldots, U_k \subset X$ and $V \subset Y$ be as in Lemma (4.2).

Choose $W \subset V$ a neighborhood of p such that m(y) = k for all $y \in W$. The possibility of this choice is a consequence of the continuity of m(y) at p and the fact that m(y) is integer valued.

Suppose $f^{-1}(W) \cdot (X - \sum U_i) \neq \emptyset$. Then there is a point $y \in W$ such that $f^{-1}(y) \cdot (X - \sum U_i) \neq \emptyset$. We know, however, from Lemma (4.2) that $f^{-1}(y) \cdot U_i \neq \emptyset$ and $f^{-1}(y) \cdot Fr(U_i) = \emptyset$ for i = 1, ..., k. This means that $f^{-1}(y)$ has more than k components, a contradiction.

Thus $f^{-1}(W) \subset \sum_i U_i$ and for any compact $K \subset W$, $f^{-1}(K)$ is closed subset of the conditionally compact set $\sum_i U_i$ and is therefore compact.

(4.5) <u>Corollary</u>: If m(y) is constant on every component of Y, and Y is locally connected, then f is a compact mapping.

Proof: This would mean that every point of Y is a point of continuity, so by (2.8) f is compact.

(4.6) Corollary: Suppose there is an integer n so that $m(y) \leq n$ for all $y \in Y$. If p is such that $m(p) = n$, then $p \in Y - S$.

Proof: From lower semicontinuity of $m(y)$, there is a neighborhood of p on which $m(y) \geq m(p) = n$. Therefore $m(y) = n$ on this neighborhood and $m(y)$ is continuous at p.

(4.7) Theorem: The interior of S is empty.

Proof: Let p be in int S, and let U be an open conditionally compact neighborhood of p with $U \subset$ int S and $m(y) \geq m(p)$ for all y in U. ($m(y)$ is lower semicontinuous.)

$p \in S$ implies that $m(y)$ is not continuous at p, so there is a point $y_1 \in U$ for which $m(y_1) \neq m(p)$. Hence we must have $m(y_1) > m(p)$, or $m(y_1) \geq m(p) + 1$.

Let U_2 be an open neighborhood of y_1 on which $m(y) \geq m(y_1)$. Since $y_1 \in S$, there is a $y_2 \in U_2$ for which $m(y_2) \geq m(y_1) + 1 \geq m(p) + 2$. For any integer k, we thus obtain an open set $U_k \subset U_{k-1}$ on which $m(y) \geq m(p) + k - 1$. (We can choose $\bar{U}_k \subset U_{k-1}$.)

All the U_k are subsets of the conditionally compact set U, so there is a point z in U_k for $k = 2, 3, \ldots$. This is, however, a contradiction; z cannot possibly be in U_i for $i > m(z)$.

Theorem (4.4) shows that the set of singular points of f is a subset of the set of discontinuities of the crude multiplicity function m. Example (1.7) shows that S can be a proper subset of the discontinuities of m. In this example, m is discontinuous at $z = 0$ and $z = 1/4$, but 0 is not a singular point.

We now characterize the set S by considering the number of components of inverses of arbitrarily small neighborhoods.

(4.8) Theorem: Suppose X and Y are locally connected. Then p is contained in S if and only if every neighborhood of p contains a neighborhood of p, V, such that $f^{-1}(V)$ has more than $m(p)$ components.

Proof: Suppose $m(p) = k$. Let $K_1, K_2, \ldots, K_k$ be the components of $f^{-1}(p)$. Let $U_1, U_2, \ldots U_k \subset X$ and V, a neighborhood of p, be as in Lemma (4.2).

(i) Assume $p \notin S$. Take $W^* \subset V$ to be a neighborhood of p such that the inverse of every compact subset of W^* is compact. Let $W \subset W^*$ be connected, conditionally compact open neighborhood of p.

Clearly $f^{-1}(W)$ has at least k components, since $f^{-1}(p) \subset f^{-1}(W)$ and $f^{-1}(W) \cdot Fr(U_i) = \emptyset$ for $i = 1, \ldots, k$, so there is a component of $f^{-1}(W)$ containing each component of $f^{-1}(p)$. Suppose there is another component of $f^{-1}(W)$, call it G. Now $f(\bar{G}) \subset \overline{f(G)} \subset \bar{W}$, so $\bar{G} \subset f^{-1}(\bar{W})$, which means that $\bar{G}$ is compact.

G is therefore a conditionally compact component of $f^{-1}(W)$, and according to Theorem (3.5), $f(G) = W$. This, however, is not possible because $f^{-1}(p) \cdot G = \emptyset$. There is, then, no such G so $f^{-1}(W)$ has exactly k components.

(ii) Let $Q \subset V$ be a neighborhood of p such that $f^{-1}(Q)$ has exactly k components. Then $f^{-1}(Q) \subset \sum_i U_i$, and so for any compact $K \subset Q$, $f^{-1}(K)$ is a closed subset of $\sum_i U_i$, which is conditionally compact. In other words, $p \notin S$.

(4.9) Theorem: A quasi-open mapping f from one locally compact, locally connected, separable metric space onto another such that $f^{-1}(y)$ is compact and has only a finite number of components for each y is compact if and only if every point y in the range has arbitrarily small neighborhoods whose inverse has the same number of components as $f^{-1}(y)$.

Looking at Example (1.7) again, we see that the inverses of $z = 0$ and $z = 1/4$ both have only one component. There are, however, arbitrarily small neighborhoods of $z = 0$ whose inverse has only one component, while this is not true for $z = 1/4$. Thus $z = 1/4$ is a singular point and $z = 0$ is not.

We end this section by showing that for any mapping f satisfying conditions stated at the beginning of the section and with m(y) bounded, there is a finite decomposition of the domain $X = D_1 + D_2 + \dots + D_n$ such that the restriction of f to each element of the decomposition D_i is a compact mapping.

(4.10) Lemma: An open or closed subspace of a locally compact space is locally compact.

Proof: (i) If $A \subset X$ is an open subset of the locally compact space X, then about any $p \in A$, there is a compact X neighborhood of p, V, such that $V \subset A$; so V is also an A neighborhood of p.

(ii) Let V be compact neighborhood of a point p in A. If A is closed, then $A \cdot V$ is closed subset of the compact set V and is therefore compact.

(4.11) Lemma: If $A \subset X$ is an inverse set (that is, $A = f^{-1}f(A)$), then the restriction of f to A, $f|A$, is quasi-open.

Proof: Let y be in $f(A)$. Then $f^{-1}(y) \subset A$. Let W be open in A and K be a compact component of $f^{-1}(y)$, $K \subset W$. Then $W = U \cdot A$ for some open set U. Thus $K \subset U$ implies that $y \in \operatorname{int} f(U)$.

Let V be a Y neighborhood of y, with $V \subset \operatorname{int} f(U)$. Now $V \cdot f(A) \subset [\operatorname{int} f(U)] \cdot f(A) \subset f(U) \cdot f(A)$, or $y \in V \cdot f(A) \subset f(U \cdot A)$, since A is inverse set.

(4.12) Theorem: Suppose there is an integer n such that $m(y) \leq n$ for all y in Y, and let $D_k = f^{-1}m^{-1}(k)$. Then $f|D_k$ is compact for every integer k.

Proof: $m^{-1}(n) = m^{-1}([p|p \geq n])$, therefore by lower semicontinuity of m, $m^{-1}(n)$ is an open subset of Y. (We are assuming that n is the smallest integer for which $m^{-1}(n)$ is not empty.) From Lemma (4.10), we know D_n and $f(D_n) = m^{-1}(n)$ are locally compact separable metric spaces, and Lemma (4.11) tells us that $f|D_n$ is quasi-open. The compactness of $f|D_n$ now follows from Theorem (4.4).

The spaces $X_1 = X - D_n$ and $Y_1 = f(X_1) = Y - m^{-1}(n)$ are closed and hence, by Lemma (4.10), locally compact. X_1 is inverse set so $f|X_1$ is quasi-open. Repeat the argument of the previous paragraph for

$f:X_1 \to Y_1$ to obtain $f|D_{n-1}$ compact mapping. Then consider $X_2 = X_1 - D_{n-1}$, and so on.

CHAPTER III

MAPPINGS IN EUCLIDEAN SPACES

In this chapter, we continue the study of the relationships between certain mappings and compact mappings. We investigate mappings defined on subsets of n-dimensional Euclidean space into n-dimensional Euclidean space.

In section 5, we obtain results similar to those of the previous chapter, but sharpened somewhat due to the availability of the notion of topological, or Brouwer, degree. We define the degree of a compact mapping and develop some of its properties in section 6.

5. The Algebraic Multiplicity Function

Let E_n denote n-dimensional Euclidean space. In this section, $f:X \to f(X) = Y \subset E_n$ is a mapping with X an open set of E_n and $f^{-1}(y)$ compact for all y in Y.

Given G, with $\bar{G} \subset X$, and G conditionally compact open subset of E_n, a point $p \in E_n - f(Fr(G))$, then the *degree of f with respect to G at the point p* is an integer denoted deg(f, G, p). For the definition and complete development of this notion, we refer the reader to Bers [1, Ch. XIV]. Listed below are the properties of deg(f, G, p) used in the sequel.

(5.1) If $f(x) \neq p$ for any x in G, then deg(f, G, p) = 0.

(5.2) If p and q are in the same component of $E_n - f(Fr(G))$, then

$\deg(f, G, p) = \deg(f, G, q)$.

(5.3) If $G_1, G_2, \ldots, G_k$ is a sequence of disjoint open subsets of G, and $f(x) \neq p$ for x in $G - \sum_{i=1}^{k} G_i$, then $\deg(f, G, p) = \sum_{i=1}^{k} \deg(f, G_i, p)$.

(5.4) If for x in $Fr(G)$, $d(f(x), p) > e$ and g is a mapping for which $d(f(x), g(x)) < e$, then $\deg(f, G, p) = \deg(g, G, p)$. ($d(r,s)$ denotes the distance between r and s.)

We now develop and study an algebraic multiplicity function analogous to the crude multiplicity function of the previous chapter.

(5.5) Lemma: For the mapping $f:X \to f(X) = Y$, and y in Y, let G and H be conditionally compact open sets for which $f^{-1}(y) \subset G \subset \bar{G} \subset X$, and $f^{-1}(y) \subset H \subset \bar{H} \subset X$. Then $\deg(f, G, y) = \deg(f, H, y)$.

Proof: Property (5.3) yields at once that

$$\deg(f, G, y) = \deg(f, G \cdot H, y) \quad \text{and}$$
$$\deg(f, H, y) = \deg(f, G \cdot H, y),$$

since $f^{-1}(y) \subset G \cdot H$.

(5.6) Definition: Given the mapping $f:X \to f(X) = Y$, let $y \in Y$ and let G be a conditionally compact open neighborhood of $f^{-1}(y)$ such that $\bar{G} \subset X$. Define the *algebraic multiplicity function* from Y into the integers by $\mu(y) = \deg(f, G, y)$.

The function $\mu(y)$ is well defined; since $f^{-1}(y)$ is compact, there is such a set G, and according to Lemma (5.5) $\deg(f, G, y)$ is independent of which such G is used. Such a G will hereafter be referred to as *admissible for the calculation of* $\mu(y)$.

(5.7) Theorem: If $p \in Y - S$, then $\mu(y)$ is continuous at $y = p$.

Proof: Let G be admissible for the calculation of $\mu(y)$. Let V be a neighborhood of p that is completely contained in the same component of $E_n - f(Fr(G))$ as p. (E_n is locally connected, so components of open sets are open.)

Let W be a neighborhood of p such that $f^{-1}(W) \subset G$. Theorem (2.9) insures the existence of W. Now $W \cdot V$ is a neighborhood of p, and for any y in $W \cdot V$, G is admissible for calculation of $\mu(y)$. By (5.2), deg (f, G, y) = deg(f, G, p). Thus $\mu(p) = \mu(y)$ for all y in a neighborhood of p.

(5.8) Corollary: If f is a compact mapping, then $\mu(y)$ is continuous.

(5.9) Corollary: If f is a compact mapping and Y is connected, then $\mu(y)$ is constant.

(5.10) Definition: A mapping $f:X \to f(X) = Y$ is *locally sense preserving at* $x \in X$ if there is an open set U which contains the component K of $f^{-1} f(x)$ containing x, so that for any conditionally compact open set G such that $K \subset G \subset U$, $Fr(G) \cdot f^{-1}f(x) = \emptyset$, then $\deg(f, G, f(x)) > 0$.

If f is locally sense preserving at each $x \in X$, we say simply that f is *locally sense preserving*. (*cf.* Titus and Young [6].)

(5.11) Theorem: A locally sense preserving mapping is quasi-open.

Proof: Let $f:X \to f(X) = Y$ be a locally sense preserving mapping, and let K be a component of $f^{-1}(y)$. If U is open set about K, there is conditionally compact open set G so that $K \subset G \subset U$ and $\deg(f, G, y) > 0$. (Lemma (5.12).)

If V is a neighborhood of y completely contained in the same component of $E_n - f(Fr(G))$ as y, then $\deg(f, G, z) > 0$ for all z in V. Property (5.1) then implies that $V \subset f(G) \subset f(U)$ and we have $y \varepsilon \operatorname{int} f(U)$.

(5.12) Lemma: Let F be a closed subset of a locally compact Hausdorff space X. If K is a compact component of F, and G is a neighborhood of K, there is an open set U such that $K \subset U \subset G$ and $Fr(U) \cdot F = \emptyset$.

Proof: Let $V \subset G$ be an open neighborhood of K such that $\bar{V}$ is compact. Consider the set

$$F^* = F \cdot \bar{V} + Fr(V).$$

F^* is a closed subset of X. By Theorem 2-14 of Hocking and Young [2, p. 47], we have

$$F^* = A + B, \text{ a separation of } F^*,$$

with $K \subset A$ and $Fr(V) \subset B$.

A and B are closed in F^*, therefore they are closed in $\bar{V}$. Now $\bar{V}$ is a compact Hausdorff space and hence normal, so there exist sets U, W, open in $\bar{V}$ such that $U \cdot W = \emptyset$, and $A \subset U$, $B \subset W$.

$Fr(V) \subset B \subset W$ implies that $U \subset V$. This means that U is open in X. Thus $K \subset A \subset U$, and $Fr(U) \cdot F = \emptyset$.

(5.13) Theorem: If f is locally sense preserving, then the inverse of any y in Y, $f^{-1}(y)$ has at most a finite number of components, and the number of components, $m(y)$, is not greater than $\mu(y)$.

<u>Proof</u>: Let U be admissible for the calculation of $\mu(y)$. Denote the components of $f^{-1}(y)$ by $\{K_\alpha\}$. Since f is locally sense preserving, there is for each K_α, an open set $U_\alpha \subset U$ such that $\bar{U}_\alpha$ is compact, $K_\alpha \subset U_\alpha$, and $\deg(f, U_\alpha, y) > 0$.

$f^{-1}(y)$ is compact, so it is covered by a finite number of the U_α's, call them $U_1, U_2, \ldots, U_k$. Define open sets $W_1, W_2, \ldots, W_k$ as follows:

$$W_1 = U_1$$

$$W_2 = U_2 - \bar{U}_1$$

$$\cdot$$
$$\cdot$$
$$\cdot$$

$$W_k = U_k - (\bar{U}_1 + \bar{U}_2 + \ldots + \bar{U}_{k-1})$$

Clearly $f^{-1}(y) \subset \sum_{j=1}^{k} w_j$, and $W_i \cdot W_j = \emptyset$ for $i \neq j$. Property (5.3) then tells us that

$$(*)\ \mu(y) = \deg(f, U, y) = \sum_{k=1}^{k} \deg(f, W_i, y).$$

Now suppose that W_j contains more than $\deg(f, W_j, y) = r$ components of $f^{-1}(y)$. Let $K_1, K_2, \ldots, K_{r+1}$ be components of $f^{-1}(y)$ in W_j. (Not necessarily all such components.) Let $V_1, \ldots, V_{r+1}$ be disjoint open sets for which $V_i \subset W_j$ and $K_i \subset V_i$. By Lemma (5.12) we can now find open sets $R_1, \ldots, R_{r+1}$ such that $K_i \subset R_i \subset V_i$ and $Fr(R_i) \cdot f^{-1}(y) = \emptyset$. Define $R_{r+2} = W_j - (\bar{R}_1 + \bar{R}_2 + \ldots + \bar{R}_{r+1})$.

By locally sense preserving property, we have $\deg(f, R_i, y) > 0$ for $i = 1, 2, \ldots, r+1$ and $\deg(f, R_{r+2}, y) \geq 0$. We have also that

$$\deg(f, W_j, y) = \sum_{i=1}^{r+2} \deg(f, R_i, y)$$

$$\geq r + 1,$$

which is a contradiction.

Each W_j thus contains at most $\deg(f, W_j, y)$ components of $f^{-1}(y)$. The equation (*) shows that $f^{-1}(y)$ has at most $\mu(y)$ components.

(5.14) Corollary: If f is a compact, locally sense preserving mapping defined on a region X, then there is an integer n such that each $f^{-1}(y)$ contains at most n components; that is, $m(y) \leq n$ for all y in Y.

Proof: f compact implies that $\mu(y)$ = constant for all y in Y.

(5.15) Theorem: If f is locally sense preserving, then $\mu(y)$ is lower semicontinuous.

Proof: Let $p \in Y$ and let G be admissible for the calculation of $\mu(p)$. Let V be a neighborhood of p such that V is contained in the same component of $E_n - f(Fr(G))$ as p. Then $\deg(f, G, p) = \deg(f, G, y)$ for all y in V.

Let z be a point of V and let $U \subset G$ be admissible for the calculation of $\mu(z)$. Then

$$\mu(z) = \deg(f, U, z) = \deg(f, U - \bar{G}, z) + \deg(f, G, z)$$

$$= \deg(f, U - \bar{G}, z) + \mu(p).$$

From the fact that f(z) has finite number of components, it follows that $\deg(f, U - \bar{G}, z) \geq 0$. We have, therefore, $\mu(z) \geq \mu(p)$.

The following theorem characterizes the singular points of a locally sense preserving mapping as the discontinuities of the algebraic multiplicity function.

(5.16) Theorem: If f is locally sense preserving, then $p \varepsilon Y - S$ if and only if $\mu(y)$ is continuous at p.

Proof: The "only if" part is Theorem (5.7).

Suppose $\mu(y)$ is continuous at $y = p$ and let G be admissible for the calculation of $\mu(p)$. Let V be an open neighborhood of p such that V is completely contained in the same component of $E_n - f(Fr(G))$ as p and such that $\mu(y) = \mu(p)$ for all y in V. The existence of V follows from the local connectedness of E_n and the continuity of $\mu(y)$ at p.

Suppose there is a y in V for which $f^{-1}(y) \cdot (X - \bar{G}) \neq \emptyset$. Let $H \subset G$ be admissible for the calculation of $\mu(y)$. Then

$$\mu(y) = \deg(f, H, y) = \deg(f, H - \bar{G}, y) + \deg(f, G, y)$$

$$= \deg(f, H - \bar{G}, y) + \deg(f, G, p)$$

$$= \deg(f, H - \bar{G}, y) + \mu(p)$$

Hence $\deg(f, H - \bar{G}, y) = 0$.

Now $f^{-1}(y) \cdot (H - \bar{G}) \neq \emptyset$, so let $K_1, \ldots, K_k$ be the components of $f^{-1}(y)$ contained in $H - \bar{G}$. Theorem (5.13) insures that there only a finite number of these components. By the locally sense preserving property, there are disjoint open sets $U_1, \ldots, U_k$ such that $K_i \subset U_i \subset H - \bar{G}$ and $\deg(f, U_i, y) > 0$ for $i = 1, \ldots, k$. But

$$\deg(f, H - \bar{G}, y) = \sum_{i=1}^{h} \deg(f, U_i, y)$$

$$> 0,$$

which is a contradiction.

Therefore $f^{-1}(V) \subset \bar{G}$. For any compact set $K \subset V$, we then have $f^{-1}(K)$ a closed subset of the compact set $\bar{G}$.

(5.17) Corollary: A locally sense preserving mapping whose domain is a region is a compact mapping if and only if $\mu(y)$ is constant.

6. The Degree of a Compact Mapping

In this section, we extend some of the results obtained by R. H. Kasriel [4] for compact mappings from a region in E_2 into E_2. We assume throughout this section that f is a compact mapping from a region in E_n

into E_n.

(6.1) Definition: The number $\mu(y)$ for some y in Y is the *degree* of f, denoted by $d(f)$.

Corollary (5.9) assures us that $d(f)$ is well defined in that $\mu(y)$ is independent of y.

We shall see that the degree of a compact mapping has many properties analagous to those of the degree of a polynomial.

(6.2) Remark: If f is a compact locally sense preserving mapping with $d(f) = k$, then the inverse of any point y contains at most k components. This, of course, is merely a restatement of Theorem (5.13).

(6.3) Lemma: Let f be a compact mapping and let $y \varepsilon Y$. If G is admissible for the calculation of $\mu(y,f)$, then there is an $e > 0$ such that if g is any mapping defined on X for which $d(f, g) < e$, then G is admissible for the calculation of $\mu(y, g)$. ($\mu(y, f)$ and μ (y, g) denote algebraic multiplicity functions for f and g, respectively.)

Proof: Let V be a neighborhood of y such that $f^{-1}(V) \subset G$. (Theorem (2.9).) Let $e > 0$ be such that the spherical neighborhood of y of radius e, $N(e)$ is contained in V.

Let g be a mapping defined on X such that $d(f, g) < e$, and suppose G is not admissible for the calculation of $\mu(y, g)$. There must be then an x in $X - G$ for which $g(x) = y$. But $d(f(x), g(x)) = d(f(x), y) < e$, which means that $f(x)$ is contained in $N(e) \subset V$, or $x \varepsilon f^{-1}(V) \subset G$, a contradiction.

(6.4) Theorem: Let f be a compact mapping with degree not zero. There is, for any y in $f(X)$, an $e > 0$ such that if g is any mapping defined on X with $d(f, g) < e$, then y is in $g(X)$.

Proof: Let G be admissible for the calculation of $\mu(y) = d(f) \neq 0$. Let $e = d(f(Fr(G)), y)$. ($Fr(G)$ is a compact set.) Property (5.4) says that $\deg(g, G, y) \neq 0$, so, according to property (5.1), there is an x in G such that $g(x) = y$.

(6.5) Corollary: If f is compact and has nonzero degree, then $f(X)$ is open subset of E_n.

Proof: This follows from VI B of Hurewicz and Wallman [3, p. 78].

(6.6) Theorem: If f is a compact mapping with degree not zero, there is an $e > 0$ such that any compact mapping g defined on X for which $d(f, g) < e$ has the same degree as f.

Proof: Let $y \in f(X)$ and let G be admissible for the calculation of $\mu(y, f)$.

According to Theorem (6.4), there is an e_1 so that if $d(f, g) < e_1$, then $y \in g(X)$. Lemma (6.3) then implies that there is e_2 so that G is admissible for the calculation of $\mu(y, g)$ if $d(f, g) < e_2 < e_1$.

Now choose $e < e_2$ and less than $d(f(Fr(G), y)$. For g, $d(f, g) < e$, we then have

$$d(f) = \mu(y, f) = \deg(f, G, y) = \deg(g, G, y) = \mu(y, g) = d(g)$$

The next theorem is an exception to our assumption in this section that all domain spaces are open subsets of E_n and that all range spaces are subsets of E_n. Also, f in the following theorem is not necessarily compact.

(6.7) <u>Theorem</u>: Let $f:X \to f(X) \subset Y$ be a mapping from one locally compact, locally connected, separable metric space into another. Let y in f(X) be such that $f^{-1}(y)$ is compact and $m(y, f) = k$. Then there is an $e > 0$ such that if the mapping $g:X \to g(X) \subset Y$ is quasi-open and $d(f, g) < e$, then $m(y, g) \geq m(y, f)$. ($m(y, f)$ and $m(y, g)$ denote the crude multiplicity functions for f and g, respectively.)

<u>Proof</u>: Let $K_1, \ldots, K_k$ be the components of $f^{-1}(y)$, and let $U_1, \ldots, U_k$ be disjoint conditionally compact open sets such that $K_i \subset U_i$ for $i = 1, \ldots, k$. Now let

$$e^* = d(y, \sum_{i=1}^{k} f(Fr(U_i)) \quad / 2.$$

Let N(y) be the e^*-neighborhood of y and denote by R the component of N(y) which contains y. R is a region since Y is locally connected.

Let $e = d(y, Y - R)$ and suppose g is quasi-open and $d(f, g) < e$. Note that $e \leq e^*$. Assume there is a point z in $g(Fr(U_i)) \cdot R$. Then there is an $x \in Fr(U_i)$ for which $g(x) \in R$; but $f(x) \in f(Fr(U_i)$ and

$$d(y, f(x)) \leq d(y, g(x)) + d(g(x), f(x))$$

Thus $$d(y, f(x)) < 2e^* \leq d(y, \sum_{i=1}^{k} f(Fr(U_i)),$$

which is a contradiction. There is, therefore, no such z and $g(Fr(U_i)) \cdot R = \emptyset$ for $i = 1, \ldots, k$.

We have also that $d(y, g(K_i)) < e$ so that $g^{-1}(R) \cdot U_i \neq \emptyset$ for $i = 1, \ldots, k$. There are, therefore, at least k conditionally compact components of $g^{-1}(R)$, $Q_1, \ldots, Q_k$. According to Theorem (3.5), $g(Q_i) = R$ for each i; therefore, $g^{-1}(y)$ has at least k components.

We return now to our previous setting--$f:X \to f(X)$ is a compact mapping and X is a region in E_n, with f(X) contained in E_n.

(6.8) Theorem: Let f be a locally sense preserving compact mapping. If y in f(X) is such that $m(y, f) = d(f)$, the degree of f, then there is an $e > 0$ such that if g is a compact quasi-open mapping defined on X and $d(f, g) < e$, then $f^{-1}(y)$ and $g^{-1}(y)$ have the same number of components.

Proof: f is locally sense preserving so d(f) is not zero. Choose e_1 according to Theorem (6.6) so that f and g have the same degree. Let $e < e_1$ be so that $m(y, g) \geq m(y, f)$. We have also that $d(f) = m(y, f)$ and $d(g) \geq m(y, g)$.

Thus $m(y, f) = d(f) = d(g) \geq m(y, g)$, and the conclusion of the theorem follows from $m(y, f) \geq m(y, g)$ and $m(y, g) \geq m(y, f)$.

CHAPTER IV

CLOSED MAPPINGS

In this chapter, we investigate the relationships between closed mappings and compact mappings. We shall see that these two classes of mappings are very closely related. In particular, every compact mapping into a locally compact Hausdorff space is a closed mapping; also, closed mappings from one locally compact separable metric space into another are "almost" compact in that the set S of singular points of these mappings is "small."

7. Definitions and General Results

(7.1) Definition: A mapping $f:X \to f(X) \subset Z$ is *closed* if $f(A)$ is closed in $f(X)$ for any closed set $A \subset X$. If $f(A)$ is closed in Z, then f is *strongly closed.*

(7.2) Definition: A mapping $f:X \to f(X) = Y$ is a *local homeomorphism* if for every x in X, there is an open neighborhood U of x and an open neighborhood V of f(x) such that $f|U$ is a homeomorphism of U onto V.

Obviously, every local homeomorphism is an open mapping.

(7.3) Theorem: A compact mapping f from a Hausdorff space X onto a locally compact Hausdorff space Y is a closed mapping.

Proof: Suppose there is a closed set $A \subset X$ such that $f(A)$ is not closed. Let $y \notin f(A)$ be an accumulation point of $f(A)$.

Let K be a compact neighborhood of y. Then $f^{-1}(K)$ is compact, and so also is $A \cdot f^{-1}(K)$. Now $f(A \cdot f^{-1}(K)) = f(A) \cdot K$ since $f^{-1}(K)$ is an inverse set. Thus $f(A) \cdot K$ is compact and hence closed. This, however, is a contradiction, because y is an accumulation point of $f(A) \cdot K$.

(7.4) <u>Remark</u>: This theorem is valid if, instead of locally compact, Y is second countable.

(7.5) <u>Theorem</u>: Suppose f is a closed mapping and $f^{-1}(y)$ is compact for each y in Y. Then f is compact.

<u>Proof</u>: Let K be a compact subset of Y and let $\{U_\alpha\}$ be an open covering of $f^{-1}(K)$.

Denote by $\{U_{\alpha y}\}$ the subcollection of $\{U_\alpha\}$ with the property that $U_{\alpha y} \cdot f^{-1}(y) \neq \emptyset$. $\{U_{\alpha y}\}$ is an open covering of $f^{-1}(y)$, so there is a finite subcover. Let W_y denote the union of all the $U_{\alpha y}$ in this finite subcover.

Now $f(X - W_y)$ is a closed set, and $y \notin f(X - W_y)$. If we set $V_y = Y - f(X - W_y)$, then the collection $\{V_y\}$, $y \in K$ is an open covering of K. Select a finite subcover, say $V_{y1}, V_{y2}, \ldots, V_{yp}$. Then $f^{-1}(V_{y1}), \ldots, f^{-1}(V_{yp})$ is a cover of $f^{-1}(K)$.

Suppose $x \in f^{-1}(V_y)$. Then $f(x)$ is not in $f(X - W_y)$, which means that x is not in $X - W_y$. In other words, $x \in W_y$. Thus $f^{-1}(V_{yi}) \subset W_{yi}$ for $i = 1, \ldots, p$, and we have $W_{y1}, \ldots, W_{yp}$, a covering of $f^{-1}(K)$. But each W_{yi} is the union of a finite number of members of the original open covering $\{U_\alpha\}$.

The separable metric versions of the preceding two theorems are due to Whyburn [9]. In separable metric spaces we see that a mapping is compact if and only if it is closed and has compact point inverses.

8. Closed Mappings on Separable Metric Spaces

Throughout this section, the mapping $f:X \to f(X) = Y$ is assumed to be a closed mapping and X and Y are assumed to be locally compact separable metric spaces.

Our first theorem is an extension of Theorem (7.5).

(8.1) Theorem: If y in Y is such that $f^{-1}(y)$ is either compact or nondense, then $y \in Y - S$.

Proof: Consider the unified space $Z = X' + Y'$. Suppose there is a point y in S, the set of singular points of f. Then $y' = k(y) \in Fr(Y')$, according to Corollary (2.5).

Let $K = h(f^{-1}(y))$. There are arbitrarily small neighborhoods U of y' such that $(U - K) \cdot X' \neq \emptyset$. If $f^{-1}(y)$ is compact, the existence of U follows from the fact that K is compact. Otherwise $f^{-1}(y)$, and hence K, is nondense so that $(U - K) \cdot X'$ cannot possibly be empty for any open set U.

There is a sequence $\{x_i'\}$ in $(U - K) \cdot X'$ with $x_i' \to y'$ since $y' \in Fr(X')$. Thus $\{x_i\}$, $x_i = h^{-1}(x_i')$, is a sequence in X, and $f(x_i) \to y = k^{-1}(y')$. Moreover, $f(x_i) \neq y$ since $x_i' \notin K$.

$\sum_{i=1}^{\infty} x_i$ is a closed subset; consequently $f(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{\infty} f(x_i)$ is a closed subset of Y. This is a contradiction; y is an accumulation point of $\sum_{i=1}^{\infty} f(x_i)$.

(8.2) Corollary: If f is not compact, then $f^{-1}(S)$ has a nonempty interior.

Proof: If int $f^{-1}(S) = \emptyset$, then $f^{-1}(p)$ is nondense for all p in S. For all p in $Y - S$, $f^{-1}(p)$ is compact.

(8.3) Corollary: If f is not compact, then $f^{-1}(S)$ is not compact.

(8.4) Corollary: If for every y in Y, $f^{-1}(y)$ is either compact or nondense, then f is a compact mapping.

(8.5) Remark: Corollary (8.4) can be established directly, without the locally compact hypothesis on the spaces.

(8.6) Lemma: In the unified space, $y' \in k(S) = S'$ if and only if y' is an accumulation point of $h(f^{-1}k^{-1}(y'))$.

Proof: If y' is an accumulation point of $h(f^{-1}(y))$, then y' is an accumulation point of X' and is, according to Corollary (2.5), contained in S'. ($y = k^{-1}(y')$.)

Next assume y' is not an accumulation point of $h(f^{-1}(y))$. Let U be an open neighborhood of y' such that $U \cdot h(f^{-1}(y)) = \emptyset$. Then $f^{-1}(y) \cdot [X - h^{-1}(U \cdot X')] = f^{-1}(y)$ is compact. Thus by Theorem (8.1), y is not in S; or, y' is not in S'.

(8.7) Theorem: If S is compact, it is finite.

Proof: Once again we use the unified space. By Corollary (2.5), $k(S) = S' = Fr(Y')$.

Suppose S is compact and infinite. Then, of course, S' is also compact and infinite. Let $\{y_i'\}$ be a sequence of distinct points in S' with $y_i' \to y'$ in S'.

Every neighborhood of y' contains points y_i', each of which is an accumulation point of $h(f^{-1}(y_i))$ by Lemma (8.6). ($y_i = k^{-1}(y_i')$). We can, therefore, pick a sequence $\{x_i'\}$, x_i' in X', $x_i' \to y'$, and $x_i' \in h(f^{-1}(y_i))$. Set $x_i = h^{-1}(x_i')$. Then $f(x_i) \to y$. The set $\sum_{i=1}^{\infty} x_i$ has no accumulation points and is therefore closed. Thus the set $f(\sum_{i=1}^{\infty} x_i) = \sum_{i=1}^{\infty} f(x_i)$ is closed. This, however, is clearly a contradiction since the $f(x_i) = y_i$ are all distinct and not equal to y.

(8.8) Corollary: If the range space Y is compact, then S is finite.

Proof: S is, according to Corollary (2.6), a closed set.

(8.9) Theorem: The set S is completely scattered; that is, for any p in S, there is a neighborhood of p that intersects S in only p.

Proof: Let y be a point in Y, and let V be a compact neighborhood of y. Set $A = f^{-1}(V)$. Then $g = f|A$ is a closed mapping, with compact range, V. Also, $f^{-1}(V)$ is locally compact. Corollary (8.8) insures there are only a finite number of singular points of g, y_i in V.

Let $W \subset V$ be a compact neighborhood of y such that $W \cdot (y_1 + \ldots + y_m)$ is empty except for possibly y itself. Let N be a compact neighborhood of a point in W. Then $f^{-1}(N) = g^{-1}(N)$, and so f and g have the same singular points in W.

(8.10) Corollary: The set S is at most countable.

(8.11) Theorem: If X is connected and f is open and nonconstant, then f is compact.

Proof: Suppose f is not compact. By Corollary (8.4), there is a point y in f(X) for which $f^{-1}(y)$ contains an open set, U. Then y = f(U) is both open and closed subset of the connected set f(X). This means f(X) = y, contradicting the fact that f is not constant.

(8.12) Theorem: If X is connected and f is a local homeomorphism, then f is a closed mapping if and only if $f^{-1}(y)$ consists of exactly k points for some integer k, all y in Y.

Proof: If f is closed, it is compact by Theorem (8.11); the conclusion then follows from the proof of Theorem (6.2) of Whyburn [7, p. 200].

If the number of points in $f^{-1}(y)$ is constant for all y in Y, the conclusion follows from Corollary (4.5).

(8.13) Example: This simple example of a closed mapping that is not compact is given merely to illustrate some of the results of this section.

Let X be the open unit disc in the plane with the center removed. Let Y be the open unit disc. Let the mapping f take all x in X whose distance from the center of the disc is $\leq 1/4$ onto the center of Y and take the rest of X homeomorphically onto the rest of Y.

Now f is a closed mapping. S consists of the center of Y, and $f^{-1}(S)$ is the closed disc of radius 1/4, without the center. Note that S is compact and finite, and that $f^{-1}(S)$ is not compact and has a non-empty interior.

CHAPTER V

NETS OF MAPPINGS

This chapter is concerned with convergent nets of mappings from one space into another. The natural setting for this investigation is the uniform space. For all definitions, notation, and related results on uniform spaces, the reader should see Kelley [5, Ch. 6].

9. Compactness of Limit Mappings

(9.1) Definition: A mapping $f:X \to f(X) = Y \subset Z$ is *strongly compact* if, for any compact set $K \subset Z$, $f^{-1}(K)$ is compact.

(9.2) Lemma: A mapping $f:X \to f(X) = Y \subset Z$ is strongly compact if and only if f is a compact mapping and for any compact set $K \subset Z$, $K \cdot Y$ is compact.

Proof: If f is compact and $K \cdot Y$ is compact for any compact K, then f is obviously strongly compact.

Suppose f is strongly compact, and let K be a compact subset of Z. $f^{-1}(K)$ is compact, so $ff^{-1}(K) = K \cdot Y$ is compact.

(9.3) Remark: If f is strongly compact and Z is a k-space (Kelley [5, p. 230]), then f(X) is a closed subset of Z. Every locally compact Hausdorff space is a k-space, so Theorem (7.3) implies that a strongly compact mapping is strongly closed.

(9.4) Lemma: If $(Z, \mathcal{U})$ is a locally compact uniform space and K is a compact set in Z, there is a V in U such that $\overline{V[K]}$ is compact.

Proof: For each y in K, let U_y be a compact neighborhood of y. Choose $V_y \in U$ so that $V_y \circ V_y [y] \subset U_y$. There is a finite subcollection of the V_y, say $V_{y1}, \ldots, V_{yn}$ such that $\sum_{i=1}^{n} V_{yi} [y_i]$ contains K.

Let $V \in U$ be so that $V \subset \prod_{i=1}^{n} V_{yi}$. Let z be a point in V [K]. There is a y in K such that $(y, z) \in V$. Also, $(y_i, y) \in V_{yi}$ for some yi. Now $V \subset V_{yi}$, so $(y_i, z) \in V_{yi} \circ V_{yi}$, and $z \in U_{yi}$. We have then that V [K] is contained in the set $\sum_{i=1}^{n} U_{yi}$, which is compact.

(9.5) Theorem: Let $\{f_n, D\}$ be a uniformly convergent net of mappings from a space X into a locally compact uniform space (Z, U). The limit mapping f is strongly compact if and only if for every compact set $K \subset Z$, there is an $N \in D$ such that $f_n^{-1}(K)$ is compact for all $n \geq N$.

Proof: First suppose f is strongly compact, and let $K \subset Z$ be a compact set. Let $V \in U$ be such that $\overline{V[K]}$ is compact. The existence of V is insured by Lemma (9.4).

Choose $N \in D$ so that $(f_n(x), f(x)) \in V$ for all x in X and all n $\geq N$. Let p be a point in $f_n^{-1}(K)$. Then $f(p) \in V[K]$; consequently, $p \in f^{-1}(\overline{V[K]})$, which is compact. Hence $f_n^{-1}(K) \subset f^{-1}(\overline{V[K]})$ and is compact.

We now prove the converse. Let F be a compact subset of Z, and let $V \in U$ be such that $\overline{V[F]}$ is compact. Choose $N \in D$ so that $(f_n(x), f(x)) \in V$ and $f_n^{-1}(\overline{V[F]})$ is compact for all $n \geq N$.

We claim that $f^{-1}(F) \subset f_n^{-1}(\overline{V[F]})$, so that $f^{-1}(F)$ is closed subset of a compact subset, and hence compact. Let p be a point of $f^{-1}(F)$. Then $f(p) \in F$, and $f_n(p) \in V[F]$. This establishes our claim.

(9.6) Corollary: If each f_n is a compact mapping, and for every compact set $A \subset Z$ there is an $N \in D$ such that $A \cdot f_n(x)$ is compact for all $n \geq N$, then f is a compact mapping.

(9.7) Corollary: If f_n is a compact mapping *onto* Z for every n in D, then the limit map f is a compact mapping.

10. Quasi-Open and Related Mappings

(10.1) Definition: A net of mappings $\{f_n, D\}$ from a space X into a uniform space $(Y, \mathcal{U})$ is *almost uniformly open* if for any x in X and any neighborhood V of x, there is an $N \in D$ and a $U \in \mathcal{U}$ such that $f_n(V) \supset U[f_n(x)]$ for all $n \geq N$.

(10.2) Theorem: If $\{f_n, D\}$ is an almost uniformly open net of mappings from a locally compact space X into a uniform space $(Y, \mathcal{U})$ that converges uniformly on compact sets to the mapping f, then f is strongly open.

Proof: Let G be an open subset of X, and let x be a point in G. X is a locally compact Hausdorff space, so it is regular. Choose V to be an open neighborhood of x so that $\bar{V} \subset G$ and $\bar{V}$ is compact.

Let U in $\mathcal{U}$ and N_1 in D be so that

$$\oplus \quad f_n(V) \supset U[f_n(x)] \quad \text{for all } n \geq N_1.$$

Choose W in $\mathcal{U}$ so that $W \circ W \subset U$ and so that W is symmetric. Let $N_2 \in D$

be so that $(f(x), f_n(x)) \in W$ for all $n \geq N_2$.

For p in $W[f(x)]$, we have $(f(x), p) \in W$, so that $(f_n(x), p) \in W \circ W \subset U$, and $p \in U[f_n(x)]$. In other words,

$$\boxed{x} \quad U[f_n(x)] \supset W[f(x)] \text{ for } n \geq N_2 .$$

Suppose there is a z in $W[f(x)]$ such that z is not in $f(\bar{V})$. $f(\bar{V})$ is compact, so there is a symmetric M in U for which $M[z] \cdot f(\bar{V}) = \emptyset$.

Let N_3 in D be such that $(f_n(y), f(y)) \in M$ for $n \geq N_3$ and y in $\bar{V}$. Let n be a fixed member of D that is larger than N_1, N_2, and N_3. Then $z \in U[f_n(x)]$ by $\boxed{x}$. The relation $\oplus$ then implies that z is in $f_n(V)$, so there is a w in V for which $f_n(w) = z$. Now $(f_n(w), f(w)) \in M$; or $f(w) \in M[z]$, which is a contradiction. There is, therefore, no such z, and we have $f(G) \supset f(\bar{V}) \supset W[f(x)]$, so f is strongly open.

(10.3) <u>Example:</u> This example shows that even if all the f_n are strongly open, the limit map may fail to be open.

Let X be the subset of the plane obtained by adding to the intervals I_0 from $(0, 0)$ to $(0, 1)$ and J from $(0, 0)$ to $(1, 0)$ the sequence of intervals $I_1, I_2, \ldots$, where I_n joins $(0, 0)$ to $(1, 1/n)$.

For each positive integer n, we define an autohomeomorphism h_n on X so that:

$$h_n(I_0) = I_n, \; h_n(I_k) = I_{k-1} \quad \text{for } 1 \leq k \leq n$$

$$h_n(I_k) = I_k \quad \text{for } k > n \quad \text{and } h_n(J) = J .$$

where each partial mapping is a homeomorphism and keeps the origin fixed. It is easily seen that each h_n is a homeomorphism of X onto itself; the sequence $\{h_n\}$ converges uniformly to the mapping h which maps both I_0 and J topologically onto J and maps I_k topologically onto I_{k-1} for all $k > 0$.

Notice that the image under h of any open subset of I_0 is not open in X. By considering a neighborhood of a point of I_0, we see that the sequence $\{h_n\}$ is not almost uniformly open.

By restricting somewhat the class of spaces, we obtain the following result, the separable metric version of which is well known and due to Whyburn [8].

(10.4) Theorem: Let $\{f_n, D\}$ be a net of quasi-open mappings from a locally connected, locally compact, space X into a locally connected, locally compact, uniform space $(Y, \mathcal{U})$. If $\{f_n, D\}$ converges uniformly on compact sets to the mapping f, then f is quasi-open.

Proof: Let $y \in f(X)$, and let K be a compact component of $f^{-1}(y)$. If G is an open set containing K, there is, according to Lemma (5.12), an open set R such that $K \subset R \subset G$ and $\bar{R}$ compact, $Fr(R) \cdot f^{-1}(y) = \emptyset$. We shall show that $y \in \text{int } f(\bar{R})$.

Let C be the component of $U[y]$ containing y, where U is a symmetric member of $\mathcal{U}$ such that $U \circ U[y]$ and $f(Fr(R))$ are disjoint. Y is locally connected, so there is a symmetric V in $\mathcal{U}$ such that $V \subset U$ and $V[y] \subset C$.

Let $N \in D$ be so that $(f_n(x), f(x)) \in V$ for all x in $\bar{R}$, and all

$n > N$. Then for such n, we have $f_n(K) \subset V[y]$.

Suppose there is a point z in $C \cdot f_n(Fr(R))$. Then there is a w in $Fr(R)$ such that $f_n(w) = z$. Now $(z, y) \in U$ and $(f(w), f_n(w)) = (f(w), z) \in U$, so $(f(w), y) \in U \circ U$, which contradicts $U \circ U[y] \cdot f(Fr(R)) = \emptyset$. There is, therefore, no such z, and we have $C \cdot f_n(Fr(R)) = \emptyset$ for all $n > N$.

Let C_n be the component of $Y - f_n(Fr(R))$ containing y. Then C C_n; otherwise, $C \cdot f_n(Fr(R)) \neq \emptyset$. Let R_n be the component of $f^{-1}(C_n)$ containing K. Then $R_n \subset R$. By Theorem (3.5), $f_n(R_n) = C_n \supset C \supset V[y]$. Thus $f_n(\bar{R}) \supset f_n(R_n) \supset V[y]$ for all $n > N$.

Suppose there is a point p in $V[y]$, such that p is not in $f(\bar{R})$. There is a $W \in \mathcal{U}$ so $W[p] \cdot f(\bar{R}) = \emptyset$, since $f(\bar{R})$ is compact. Let $n \in D$ be $> N$ and large enough to insure that $(f_n(x), f(x)) \in W$ for all x in $\bar{R}$. Choose $w_n \in f_n^{-1}(p) \subset R$. Then $(f_n(x_n), f(x_n)) \in W$. This means that $f(x_n) \in W[p]$, clearly a contradiction.

Example (10.3) shows that without the local connectedness assumptions, this theorem is not necessarily valid.

(10.5) Theorem: Let $\{f_n, D\}$ be a net of quasi-open mappings from a locally connected, locally compact space into a locally connected uniform space $(Y, \mathcal{U})$. If $\{f_n\}$ converges uniformly on compact sets to the light mapping f, then $\{f_n, D\}$ is almost uniformly open.

Proof: For x in X, let V be conditionally compact open neighborhood of x such that $Fr(V) \cdot f^{-1}f(x) = \emptyset$. (Lemma (5.12).) Let $C = Fr(V)$. C and $f(C)$ are compact.

$y = f(x) \notin f(C)$, therefore there is a symmetric $U \in \mathcal{U}$ such that

$$\oplus \quad U \circ U\,[y] \cdot f(C) = \emptyset.$$

Let Q denote the component of U [y] which contains y. Let W in $\mathcal{U}$ be such that $W\,[y] \subset Q$. Choose n large enough to have $(f_n(z), f(z)) \in W$ for all z in $C + \{x\}$. Let R_n be the component of $f_n^{-1}(Q)$ which contains x. (x is in $f_n^{-1}(Q)$ since $f_n(x) \in W\,[y] \subset Q$.)

Suppose there is a point p in C such that $f_n(p) \in U\,[y]$. Then $(f_n(p), y) \in U$, $(f_n(p), f(p)) \in U$, so we have $(y, f(p)) \in U \circ U$, which contradicts $\oplus$. There is then no such point p and $f_n^{-1}(U[y]) \cdot C = \emptyset$ for all n sufficiently large. In other words, $f_n^{-1}(U[y]) \subset X - C$. Also, $f_n^{-1}(Q) \subset f_n^{-1}(U[y]) \subset X - C$.

Now $X - C = V + (X - \bar{V})$ is a separation of $X - C$, and $V \cdot R_n \neq \emptyset$, so $R_n \subset V$.

By Theorem (3.5), $f_n(R_n) = Q \supset W[y]$. Let S in $\mathcal{U}$ be such that $S \circ S \subset W$ and take n large enough to have $f_n(x) \in S\,[y] \subset W\,[y]$. If p is a point in $S\,[f_n(x)]$, then $(p, f_n(x)) \in S$ and $(f_n(x), y) \in S$, so that $(p, y) \in S \circ S \subset W$. Thus p is in W [y]. That is, $S\,[f_n(x)]$ $W\,[y]$. We then have $f_n(V) \supset f_n(R_n) = Q \supset W\,[y] \supset S\,[f_n(x)]$ for all sufficiently large n.

11. Monotone and Related Mappings

(11.1) Definition: Let $\{f_n, D\}$ be a net of mappings from a uniform space $(X, \mathcal{U})$ into a space Y is *almost uniformly one to one* if given any y in Y and any U in $\mathcal{U}$, there is an N in D such that for any $n > N$, $f_n^{-1}(y) \subset U[x]$ for any $x \in f_n^{-1}(y)$.

(11.2) Definition: A mapping $f:X \to f(X) \subset Y$ from a uniform space $(X, \mathcal{U})$ into a space Y is a $\mathcal{U}$-*map* if for U in $\mathcal{U}$ it is true that $f^{-1}f(x) \subset U[x]$ for all x in X.

(11.3) Definition: A subset A of a uniform space $(X, \mathcal{U})$ is $\mathcal{U}$-*chained* if for any two points x and y in A, there is a finite sequence of points $x_1, x_2, \ldots, x_n$ in A such that $U[x] \cdot U[x_1] \neq \emptyset$, $U[x_i] \cdot U[x_{i+1}] \neq \emptyset$, for $i = 1, \ldots, n-1$, and $U[x_n] \cdot U[y] \neq \emptyset$, for U in $\mathcal{U}$.

(11.4) Definition: A mapping $f:X \to f(X) \subset Y$ from a uniform space $(X, \mathcal{U})$ is $\mathcal{U}$-*approximately monotone* if for every y in f(X), $f^{-1}(y)$ is $\mathcal{U}$-chained for U in $\mathcal{U}$.

(11.5) Theorem: Let $\{f_n, D\}$ be a net of mappings from a uniform space $(X, \mathcal{U})$ into a uniform space (Y, γ). Suppose $\{f_n\}$ converges uniformly on compact sets to the mapping f. If $\{f_n\}$ is almost uniformly open and almost uniformly one to one, then the limit f is one to one.

Proof: Suppose there are distinct points x_1 and x_2 in X such that $f(x_1) = f(x_2)$.

Let U in $\mathcal{U}$ be symmetric and so that $U[x_1] \cdot U[x_2] = \emptyset$. Let N_1 in D and V in γ be such that

$$f_n(W[x_1]) \supset V[f_n(x_1)] \quad \text{and}$$

⊕

$$f_n(W[x_2]) \supset V[f_n(x_2)] \quad \text{for all } n > N_1,$$

where $W \varepsilon \mathcal{U}$ is symmetric and such that $W \circ W \subset U$.

Let $y = f(x_1) = f(x_2)$, and choose N_2 in D so that $f_n^{-1}(y) \subset W[x]$ for any x in $f_n^{-1}(y)$, $n > N_2$.

By uniform convergence, let n be $> N_1$ and N_2 and also large enough to insure that $(y, f_n(x_1)) \varepsilon V$ and $(y, f_n(x_2)) \varepsilon V$. Then $y \varepsilon V[f_n(x_1)] \cdot V[f_n(x_2)]$, so according to ⊕, we have

$$f_n^{-1}(y) \cdot U[x_1] \neq \emptyset \quad \text{and} \quad f_n^{-1}(y) \cdot U[x_2] \neq \emptyset.$$

Take $x \varepsilon f_n^{-1}(y) \cdot U[x_1]$. Suppose $W[x] \cdot U[x_2] \neq \emptyset$. Then there is a point p so that $(p, x) \varepsilon W$ and $(p, x_2) \varepsilon W$; in other words, $(x, x_2) \varepsilon W \circ W \subset U$, and $x \varepsilon U[x_2]$, contradicting the disjointness of $U[x_1]$ and $U[x_2]$. Thus $W[x] \cdot W[x_2] = \emptyset$. This is, however, not possible since $f_n^{-1}(y) \subset W[x]$, and $f_n^{-1}(y) \cdot U[x_2] \neq \emptyset$.

There are no such points x_1 and x_2, so f is one to one.

(11.6) <u>Corollary</u>: If X is locally compact, then f is a homeomorphism.

<u>Proof</u>: This corollary follows from Theorem (10.2).

Example (10.3) shows that it is not sufficient in Theorem (11.5) to have $\{f_n\}$ almost uniformly one to one; the almost uniformly open hypothesis is not superfluous.

(11.7) <u>Theorem</u>: Let $\{f_n, D\}$ be a uniformly convergent net of compact mappings from a uniform space $(X, \mathcal{U})$ *onto* a locally connected, locally compact uniform space (Y, γ). Let $f: X \to f(X) = Y$ denote the limit map. If for any U in $\mathcal{U}$ there is an N in D such that f_n is U-approximately monotone for $n > N$, then f is a compact monotone (and hence quasi-open) mapping.

<u>Proof:</u> f is compact by Corollary (9.7). Suppose f is not monotone. Then there is a y in Y for which

$$f^{-1}(y) = K_1 + K_2 \quad \text{separate.}$$

K_1 and K_2 are compact, so there is a W in $\mathcal{U}$ such that $W[K_1] \cdot W[K_2] = \emptyset$. Let U be such that $U \circ U \circ U \circ U \subset W$.

Choose n large enough to have f_n U-approximately monotone. If $G = X - (U \circ U[K_1] + U \circ U[K_2])$, then there is an open neighborhood H of y so that for n sufficiently large, $f_n(\bar{G}) \cdot H = \emptyset$. To show this, let V in γ be so that $V \circ V[y] \subset Y - f(\bar{G})$. ($f(\bar{G})$ is closed since f is a compact mapping.) Let n be large enough to have $(f(z), f_n(z)) \in V$, for all z in X. Suppose there is a point p in $f_n(\bar{G}) \cdot V[y]$. Let q in G be so that $f_n(q) \in V[y]$. We have then $(f_n(q), y) \in V$ and also $(f_n(q), f(q)) \in V$. (We may assume V is symmetric.) Thus $(f(q), y) \in V \circ V$, or $f(q) \in V \circ V[y]$, which is a contradiction.

All the f_n are onto maps, so $f_n(\bar{G}) \cdot H = \emptyset$ implies that

$$H \subset f_n(X - \bar{G}) \subset f_n(\overline{U \circ U[K_1]} + \overline{U \circ U[K_2]}).$$

Let Q be component of H which contains y; then $Q \cdot f_n(\overline{U \circ U[K_i]})$, $i = 1, 2$, are closed in Q. There must, therefore, be at least one point z in $f_n(\overline{U \circ U[K_1]}) \cdot f_n(\overline{U \circ U[K_2]})$. Then

$$f_n^{-1}(q) \cdot \overline{U \circ U [K_1]} \neq \emptyset \quad \text{and}$$

$$f_n^{-1}(z) \cdot \overline{U \circ U [K_2]} \neq \emptyset.$$

$f_n^{-1}(z)$ is U-chained, so there are points x_k in $U \circ U [K_1]$ x_{k+1} in $U \circ U [K_2]$ such that $(x_k, x_{k+1}) \varepsilon U$. But this implies x_k in $U \circ U \circ U [K_2]$, which is a contradiction.

(11.8) <u>Corollary:</u> If all f_n are compact monotone maps, then the limit f is compact monotone.

Example (10.3) shows that the locally connected hypothesis is not superfluous. This example and the separable metric version of the previous corollary are due to Whyburn [11].

(11.9) <u>Definition:</u> A net $\{f_n, D\}$ of mappings from a uniform space $(X, \mathcal{U})$ into a space Y is *uniformly almost uniformly one to one* if given a U in $\mathcal{U}$, there is an N in D such that $f_n^{-1}(y) \subset U [x]$ for any $x \varepsilon f_n^{-1}(y)$, for all y in Y.

(11.10) <u>Theorem:</u> If $\{f_n, D\}$ is a net of compact monotone mappings from a uniform space $(X, \mathcal{U})$ onto a locally connected, locally compact, uniform space (Y, γ) that converges uniformly to the mapping f, and if $\{f_n, D\}$ is uniformly almost uniformly one to one, then f is a compact monotone mapping.

<u>Proof:</u> $\{f_n, D\}$ being uniformly almost uniformly one to one clearly implies the existence of an N in D so that for any U in $\mathcal{U}$, f_n is U-

approximately monotone for $n \geq N$.

(11.11) Corollary: If $\{f_n, D\}$ is uniformly convergent net of homeomorphisms from a uniform space X onto a locally connected, locally compact uniform space Y, then the limit f is a compact monotone mapping.

(11.12) Theorem: Let $\{f_n, D\}$ be a net of mappings from one compact uniform space $(X, \mathcal{U})$ onto another (Y, γ). If $\{f_n\}$ converges uniformly to a U-mapping f, then there is an N in D so that for all $n \geq N$, f_n is a U-mapping.

Proof: Define $F: X \times X \to Y \times Y$ by $F(x_1, x_2) = (f(x_1), f(x_2))$. F is continuous and compact. (Theorem (1.4).)

$F(X \times X - U)$ does not intersect the diagonal of $Y \times Y$; otherwise there would be $(x_1, x_2) \notin U$ for which $f(x_1) = f(x_2)$. This cannot happen because f is a U-map. F is a compact mapping, so $W = Y \times Y - F(X \times X - U)$ is a neighborhood of the diagonal; thus W is in γ.

Let V in γ be such that $V \circ V \subset W$. (Take V symmetric.) Let N be so that $(f_n(x), f(x)) \in V$ for all x in X, $n \geq N$. Let z_1 and z_2 be points in X for which $f_n(z_1) = f_n(z_2)$ for some $n > N$. Then $(f(z_1), f_n(z_1)) \in V$ and $(f(z_2), f_n(z_2)) \in V$. Therefore, $(f(z_1), f(z_2)) \in V \circ V \subset W = Y \times Y - F(X \times X - U)$, so that $(z_1, z_2) \in U$.

(11.13) Corollary: If the limit mapping f is one to one, then $\{f_n, D\}$ is almost uniformly one to one.

(11.14) Theorem: Let $\{f_n, D\}$ be an almost uniformly open net of U-mappings from one uniform space $(X, \mathcal{U})$ into another (Y, γ). If $\{f_n\}$

converges to the mapping f, then f is a $U \circ U \circ U$ - mapping.

Proof: Suppose there is an x in X for which

$$f^{-1}f(x) \cdot (X - U \circ U \circ U\,[x]) \neq \emptyset.$$

Let z be in $f^{-1}f(x) \cdot (X - U \circ U \circ U\,[x])$. Then $U\,[z] \cdot U \circ U\,[x]$ is empty; otherwise $(x, z) \in U \circ U \circ U$.

There is an N in D and a V in γ such that

$$f_n(U\,[x]) \supset V\,[f_n(x)] \text{ and}$$

$$\oplus$$

$$f_n(U\,[z]) \supset V\,[f_n(z)] \text{ for } n > N.$$

Choose n large enough to have $(f_n(x), f(x))$ and $(f_n(z), f(z))$ in V. (We assume V symmetric.) Thus we have

$$f(x) \in V\,[f_n(x)] \text{ and } f(z) \in V\,[f_n(z)].$$

Hence

$$f_n^{-1}(y) \cdot U\,[x] \neq \emptyset \text{ and } f_n^{-1}(y) \cdot U\,[z] \neq \emptyset,$$

where $y = f(x) = f(z)$. This follows from $\oplus$.

Take $q \in f_n^{-1}(y) \cdot U\,[x]$. Then $f_n^{-1}(y) \subset U\,[q]$ since f_n is a U-map. Therefore, there is a point p contained in $U\,[q] \cdot U\,[z] \neq \emptyset$. Now (p, z) and (p, q) in U implies that $(z, q) \in U \circ U$. But $(q, x) \in U$, so that we have $(x, z) \in U \circ U \circ U$, a contradiction.

(11.15) <u>Theorem:</u> Let $\{f_n, D\}$ be an almost uniformly open net of mappings from one compact uniform space X onto another, Y. Suppose $\{f_n\}$ converges uniformly to the mapping f. Then f is a homeomorphism if and only if $\{f_n, D\}$ is almost uniformly one to one.

<u>Proof:</u> Corollary (11.13) and Corollary (11.6).

(11.16) <u>Theorem:</u> Let $\{f_n, D\}$ be an almost uniformly one to one net of quasi-open mappings from a locally connected, locally compact uniform space X into a locally connected uniform space Y. Suppose $\{f_n\}$ converges uniformly to the mapping f. Then f is a homeomorphism if and only if $\{f_n, D\}$ is almost uniformly open.

<u>Proof:</u> Corollary (11.6) and Theorem (10.5).

(11.17) <u>Corollary:</u> A uniformly convergent net of homeomorphisms $\{h_n, D\}$ from one locally compact locally connected uniform space onto another has its limit a homeomorphism if and only if it is almost uniformly open.

REFERENCES

1. Bers, L., *Topology*. New York, New York: New York University Institute of Mathematical Sciences, 1956.

2. Hocking, J. G., and Young, G. S., *Topology*. Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1961.

3. Hurewicz, W., and Wallman, H., *Dimension Theory*. Princeton University Press, 1948.

4. Kasriel, R. H., "Stability of Certain Quasi-Open Mappings," *Duke Mathematical Journal*, Vol. 28, 1961, pp. 595-606.

5. Kelley, J. L., *General Topology*. Princeton, New Jersey: D. Van Nostrand Co., Inc., 1955.

6. Titus, C. J., and Young, G. S., "The Extension of Interiority, with some Applications," *Transactions of the American Mathematical Society*, Vol. 103, 1962, pp. 329-340.

7. Whyburn, G. T., *Analytic Topology*, American Mathematical Society Colloquium Publications, Vol. XXVIII. Providence, Rhode Island: American Mathematical Society, 1942.

8. ________, *Topological Analysis*. Princeton, New Jersey: Princeton University Press, 1958.

9. ________, "Open Mappings on Locally Compact Spaces," *Memoirs of the American Mathematical Society*, no. 1, 1950.

10. ________, "A Unified Space for Mappings," *Transactions of the American Mathematical Society*, Vol. 74, 1953, pp. 344-350.

11. ________, "Monotoneity of Limit Mappings," *Duke Mathematical Journal*, Vol. 29, 1962, pp. 465-470.

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