COMPLETE NONNEGATIVELY CURVED SPHERES AND PLANES

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To my parents,

PREFACE

This dissertation is submitted for the degree of Doctor of Philosophy at Georgia Institute of Technology. The research described herein was conducted under the supervision of Professor Igor Belegradek in the School of Mathematics, Georgia Institute of Technology, between August 2010 and April 2015.

This work is to the best of my knowledge original, except where acknowledgements and references are made to previous work.

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TABLE OF CONTENTS

SUMMARY

First we study the space of complete Riemannian metrics of nonnegative curvature on the sphere equipped with $C^{k+\alpha}$ topology. In this thesis, unless stated otherwise we assume $\alpha \in [0,1)$. When $\alpha = 0$, it means the compact open C^k topology. We show the space is homogenous for $k \geq 2$ and $\alpha \in (0,1)$. If k is infinite, we show that the space is homeomorphic to the separable Hilbert space. We also prove for finite k and $\alpha \in (0,1)$, the space minus any compact subset is weakly contractible. We also study $\mathrm{Diff}^+_{0,1,\infty}(S^2)$, the group of self-diffeomorphism of the sphere fixing the complex numbers 0, 1, ∞ and isotopic to the identity. For any $k+\alpha$, we show $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ equipped with $C^{k+\alpha}$ topology is not completely metrizable.

Then we study the space of complete Riemannian metrics of nonnegative curvature on the plane equipped with the $C^{k+\alpha}$ topology. If k is infinite, we show that the space is homeomorphic to the separable Hilbert space. For any finite k and $\alpha \in (0,1)$, we prove that the space minus a compact subset is weakly contractible. For any finite $k+\alpha$, we show the space cannot be made disconnected by removing a finite dimensional subset. A similar result holds for the associated moduli space. The proof combines properties of subharmonic functions with results of infinite dimensional topology and dimension theory. A key step is a characterization of the conformal factors that make the standard Euclidean metric on the plane into a complete metric of nonnegative sectional curvature.

CHAPTER I

INTRODUCTION

1.1 Introduction

The spaces of Riemannian metrics have been studied under various geometric assumptions such as positive scalar [34, 35, 37], negative sectional [18, 19, 20, 21], positive Ricci [42], and nonnegative sectional [10] curvatures.

In this thesis we study the spaces of Riemannian metrics under the assumption that the metric is both complete and nonnegativey curved.

Let M be any m-dimensional manifold. Let $\mathcal{R}^{k+\alpha}_{\geq 0}(M)$ denote the set of C^∞ complete Riemannian metrics on M of nonnegative sectional curvature equipped with the $C^{k+\alpha}$ topology, where k is a finite integer or ∞ and $\alpha \in [0, 1)$. When $\alpha = 0$, it is the compact open C^k topology. When $\alpha \in (0, 1)$, it is the Hölder $C^{k+\alpha}$ topology.

For the sphere, Earle-Schwartz [17] implies that ϕ depends in $C^{k+\alpha}$ ($\alpha \in (0, 1)$) on the Beltrami dilatation, i.e. ϕ depends in $C^{k+\alpha}$ ($\alpha \in (0, 1)$) on w in the Beltrami equation $\phi^w_{\bar{z}} = w\phi^w_z$. This easily implies that ϕ depends in $C^{k+\alpha}$ ($\alpha \in (0, 1)$) on the metric g .

For the sphere, we consider a bijection

$$\Pi : \ L_1 \times \mathrm{Diff}^+_{0,1,\infty}(S^2) \to \mathcal{R}_{\geq 0}(S^2),$$

where $L_1 = \{u | u \in C^\infty(S^2), \Delta u_{g_1} \leq 1\}$ and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ is the group of self-diffeomorphism of the sphere fixing the complex numbers 0, 1, ∞ and isotopic to the identity.

For the plane, we consider a bijection

$$\Pi : S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2) \to \mathcal{R}_{\geq 0}(\mathbb{R}^2).$$

In this thesis, we mainly study topology of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$, $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ and how their topological properties vary with k.

The cases of spheres and planes are considered in chapter 2 and chapter 3, respectively. In chapter 2 we study topology of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$. Any metric on S^2 is conformal to the round metric g_1. So up to normalization a metric $g \in \mathcal{R}_{\geq 0}(S^2)$ can be written uniquely as $\phi^* e^{2u} g_1$ where ϕ is a C^∞ self-diffeomorphism of S^2 that fixes 0, 1, ∞, and u is a C^∞ function on S^2. The sectional curvature of g equals $e^{-2u}(1 - \triangle_{g_1} u)$, so nonnegatively curved means $\triangle_{g_1} u \leqslant 1$.

Consider the bijection

$$\Pi: \ L_1 \times \mathrm{Diff}^+_{0,1,\infty}(S^2) \to \mathcal{R}_{\geq 0}(S^2),$$

where $L_1 = \{u | u \in C^\infty(S^2), \Delta u_{g_1} \leq 1\}$ and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ is the group of self-diffeomorphism of the sphere fixing the complex numbers 0, 1, ∞ and isotopic to the identity.

To better understand the topology of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ we prove the following

(1) The map Π is a homeomorphism if L_1 is given the $C^{k+\alpha}$ topology and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ is given the $C^{k+1+\alpha}$ topology, where k is nonnegative integer or ∞ and $\alpha \in (0,1)$ [Theorem 2.4.1].

We prove the following

(2) For $k = \infty$ the spaces $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ and L_1 are both homeomorphic to l^2, so $\mathcal{R}^{\infty}_{\geq 0}(S^2)$ is homeomorphic to l^2 [Theorem 2.2.1].

We also study the metrizability of L_1 with $C^{k+\alpha}$ topology and $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$. We prove the following theorems.

(3) If $k \neq \infty$ and $k \geq 2$, then L_1 equipped with $C^{k+\alpha}$ topology is not completely metrizable [Theorem 2.3.8]. The space $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ is not completely metrizable since $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ equipped with $C^{k+1+\alpha}$ topology is not completely metrizable [Theorem 2.3.9, Theorem 2.3.10].

For finite k we discuss the homeomorphism type of L_1 equipped with $C^{k+\alpha}$ topology. One natural guess is that L_1 equipped with $C^{k+\alpha}$ topology are of the same homeomorphism type for different k's.

A natural question is whether $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ is homogenous. In general, convex subset of Banach spaces need not be homogenous and their homeomorphism classification is wide open. We prove $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ is homogenous for $k \geq 2$ as follows

(4) L_1 equipped with $C^{k+\alpha}$ topology and $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ ($\alpha \in (0,1),\ k \geq 2$) are homogenous [Theorem 2.5.5].

The above theorem hinges on results of T. Banach [12] and the following lemmas:

(5) Let K_l denote the set $\{u \in C^\infty(S^2)|\ \Delta_{g_1} u \leq 1, ||u||_{k+1} \leq l\}$, l is positive integer. Let $\bar{K}_l$ denote the closure of K_l in $C^{k+\alpha}$ topology. Then $L_1 \subset \cup_l(\bar{K}_l \cap L_1)$ and $\bar{K}_l \cap L_1$ is closed in $\mathrm{Aff}(L_1)$.

(6) The set $\bar{K}_l \cap L_1$ is a Z-set in L_1, i.e. every map $f:\ Q \to L_1$ of the Hilbert cube can be uniformly approximated by maps whose range miss the set $\bar{K}_l \cap L_1$ [Theorem 2.5.4].

(7) Zero function is an internal point of L_1, i.e. the set $A = \{a \in L_1 : \exists\, \epsilon > 0\ with\ -\epsilon a \in L_1\}$ is equal to L_1[Theorem 2.5.3].

In chapter 3, we study connectedness properties of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$. The main theme is deciding when two metrics can be deformed to each other through complete nonnegatively curved metrics outside a given subset, and how large the space of such deformation is. The starting point is a result of Blanc-Fiala [8] that any complete nonnegatively curved metric on $\mathbb{R}^2$ is conformally equivalent to the standard Euclidean metric g_0, i.e. isometric to $e^{-2u}g_0$ for some smooth function u, see [23] for generalizations.

The sectional curvature of $e^{-2u}g_0$ equals $e^{2u}\Delta u$, where Δ is the Euclidean Laplacian.

Thus $e^{-2u}g_0$ has nonnegative curvature if and only if u is subharmonic. Characterizing subharmonic functions that correspond to complete metrics is not straightforward, and doing so is the main objective of this chapter. A basic property [25] of a subharmonic function u on $\mathbb{R}^2$ is that the limit

$$\alpha(u) := \lim_{r\to\infty} \frac{M(r,\ u)}{\log r},$$

exists in $[0, \infty]$, where $M(r,u) := \sup\{u(z) : |z| = r\}$. By Liouville's theorem $\alpha(u) = 0$ if and only if u is constant, while any non constant harmonic function u satisfies $\alpha(u) = \infty$ [Theorem 3.1.1].

Appealing to more delicate properties of subharmonic functions [28, 24] we prove the theorem:

(8) The metric $e^{-2u}g_0$ is complete if and only if $\alpha(u) \leq 1$ [Theorem 3.2.3].

(9) $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is metrizable and separable [Theorem 3.3.6].

(10) $\mathcal{R}^{\infty}_{\geq 0}(\mathbb{R}^2)$ is homeomorphic to l^2, the separable Hilbert space.

Any metric conformally equivalent to the standard metric on $\mathbb{R}^2$ can be written uniquely as $\phi^* e^{-2u}g_0$ where g_0 is the standard Euclidean metric, u is a smooth function, and $\phi \in \mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$, the group of self-diffeomorphisms of the plane fixing the complex numbers 0, 1 and isotopic to the identity [Theorem 3.3.4].

Let S_α be the subset of $C^\infty(\mathbb{R}^2)$ consisting of subharmonic functions with $\alpha(u) \leq \alpha$.

The map $(u,\phi) \to \phi^* e^{-2u}g_0$ defines a bijection

$$\Pi : S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2) \to \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2).$$

We prove

(11) Π is a homeomorphism if S_1 is given the $C^{k+\alpha}$ topology and $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ is given the $C^{k+1+\alpha}$ topology, where k is a positive integer or ∞ and $\alpha \in (0, 1)$ [Theorem 3.3.5].

Unless stated otherwise we equip S_α , $C^\infty(\mathbb{R}^2)$ and $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ with the compact-open C^∞ topology. Theorem (11) stated above implies that Π is a homeomorphism for $k = \infty$.

By contrast, if k is finite, then Π is not a homeomorphism, because it factors as the composite of $\Pi : S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2) \to \mathcal{R}^{\infty}_{\geq 0}(\mathbb{R}^2)$ and $\mathbf{id}$: $\mathcal{R}^{\infty}_{\geq 0}(\mathbb{R}^2) \to \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ and the latter map is a continuous bijection that is clearly not a homeomorphism.

The C^∞ topology makes $C^\infty(\mathbb{R}^2)$ into a separable Fréchet space, see [Theorem 3.3.6]. Moreover, we show in [Theorem 3.3.2, Theorem 3.3.3] that the subset S_α of $C^\infty(\mathbb{R}^2)$ is closed, convex, and not locally compact when $\alpha \neq 0$. Since S_0 consists of constants, it is homeomorphic to $\mathbb{R}$.

What makes the continuous bijection Π useful is the fact that the parameter space $S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ is homeomorphic to l^2.

Our first application demonstrates that any two metrics can be deformed to each other in a variety of ways, while by passing a given countable set:

(12) If K is a countable subset of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ and X is a separable metrizable space, then for any distinct points $x_1, x_2 \in X$ and any distinct metrics g_1, g_2 in $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ there is an embedding of X into $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ that takes x_1, x_2 to g_1, g_2, respectively [Theorem 3.4.1].

Some deformations in $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ can be constructed explicitly (e.g. one could slightly change the metric near a point where $K > 0$, or one could join two embedded convex surfaces in $\mathbb{R}^3$ by the path of their convex combination). Yet it it unclear how such methods could yield (12). Instead we use infinite dimensional topology. Then theorem (12) is an easy consequence of the following facts:

- Like any continuous one-to-one map to a Hausdorff space, the map Π restricts to a homeomorphism on every compact subset, e.g. the Hilbert cube.
- Every separable metrizable space embeds into the Hilbert cube [Theorem 3.4.2].
- The complement in l^2 of the countable union of compact sets is homeomorphic to l^2 [3], and hence contains an embedded Hilbert cube.

A topological space is *continuum-connected* if every two points lie in a *continuum* (a compact connected space); thus a continuum-connected space is connected but not necessarily path-connected.

By *dimension* we mean the covering dimension, see [16]. Note that for separable metrizable spaces the covering dimension equals the small and the large inductive dimension, see [16, Theorem 1.1.7]. A topological space is finite dimensional if and only if it embeds into a Euclidean space [16].

By the above theorem (12) any two metrics in $\mathcal{R}^{k+\alpha}_{\geq 0}$ lie in an embedded copy of $\mathbb{R}^n$ for any n. By the fact that $\mathbb{R}^n$ cannot be separated by a subset of codimension ≥ 2, we prove
(13) The complement of every finite dimensional subset of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is continuum-connected. The complement of every closed finite dimensional subset of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is path-connected [Theorem 3.4.5].

Let $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ denote the moduli space of complete nonnegatively curved metrics, i.e. the quotient space of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ by the Diff$(\mathbb{R}^2)$-action via pullback. The moduli space $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is rather pathological, e.g. it is not a T_1 space (in the proof of Π^{-1} is not continuous we exhibit a non-flat metric $g \in \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ whose isometry lies in every neighborhood of the isometry class of g_0). Consider the map $S_1 \to \mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ sending u to the isometry class of $e^{-2u}g_0$. Its fibers lie in the orbits of a Conf(g_0)-action of $C^\infty(\mathbb{R}^2)$, so each fiber is the union of countably many finite dimensional compact sets. By this fact, we prove
(14) The complement of a subset S of $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is path-connected if S is countable, or if S is closed and finite dimensional [Theorem 3.5.2].
Besides these, we also discuss the contractibility of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ minus a compact set K, and $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ minus a compact set K. We prove
(15) $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2) \setminus K$ is weakly contractible for $\alpha \in (0, 1)$[Theorem 2.7.2].

(16) $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ is weakly contractible for $\alpha \in (0, 1)$.

Theorems (11)-(14) may hold for spaces of nonnegatively curved spheres with similar proof. We will investigate them in the future.

CHAPTER II

NON NEGATIVELY CURVED SPHERES

2.1 Interpretation of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$

Definition 2.1.1 *A bilinear form on a vector space V is a bilinear map $V \times V \to K$, where K is the field of scalars. Let Bil(V) denote the vector space of bilinear forms on V.*

Let M be any manifold with dimension m. There is a bundle, associate with TM, over M whose fiber over $p \in M$ is $\mathrm{Bil}(T_pM)$. Let Q denote the total space of the bundle. The base space of the bundle is M. The fiber over $p \in M$ is $\mathrm{Bil}(T_pM)$.
We denote this fiber bundle as $\mathrm{Bil}(\mathbb{R}^m) \to Q \xrightarrow{\pi} M$, where π is the projection of the bundle. Metric on S^2 is a smooth section of $\pi : Q \to M$, a map f in $C^\infty(M, Q)$ with $f(p) \in \mathrm{Bil}(T_pM)$.

Let us think of $\mathcal{R}_{\geq 0}(M)$ as a subset of $C^\infty(M, Q)$. When $\alpha = 0$, the space $C^\infty(M, Q)$ sits in $C^k(M, Q)$, on which we can define the compact open C^k topology with finite k as follows. The set $C^k(M, Q)$ embeds as a closed subset into $C^0(M, J^k(M, Q))$, where $J^k(M, Q)$ is the space of k-jets which is a C^0 manifold. Then by defining compact open topology on $C^0(M, J^k(M, Q))$, we have the compact open C^k topology on $C^\infty(M, Q)$. Based on the compact open C^k topology on $C^\infty(M, Q)$, we can define the C^∞ topology on $C^\infty(M, Q)$.
As a subset of $C^\infty(M, Q)$, the set $\mathcal{R}_{\geq 0}(M)$ can be equipped with C^k topology for any k. Let $\mathcal{R}^k_{\geq 0}(M)$ denote $\mathcal{R}_{\geq 0}(M)$ equipped with C^k topology, where k is nonnegative integer or ∞.

In this chapter, we study the topology of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$, and how they vary with k.

2.2 $\mathcal{R}^{\infty}_{\geq 0}(S^2)$

Theorem 2.2.1 *$\mathcal{R}^{\infty}_{\geq 0}(S^2)$ is homeomorphic to l^2.*

Proof Denote $l^2 := \{(x_n) \in \mathbb{R}^{\infty} : \Sigma_n x_n^2 < \infty\}$, the separable Hilbert space.

Since l^2 is homeomorphic to the countable infinite product of the open interval $(0, 1)$, we know $l^2 \times l^2$ is homeomorphic to l^2. So it is sufficient to show $\mathrm{Diff}^{+}_{0,1,\infty}(S^2)$ and L_1 equipped with C^{∞} topology are both homeomorphic to l^2.

According to [44], the space $\mathrm{Diff}^{+}_{0,1,\infty}(S^2)$ equipped with C^{∞} topology is homeomorphic to l^2. Note that L_1 equipped with C^{∞} topology is closed convex subset of a separable Fréchet space $C^{\infty}(S^2)$, and hence it is also homeomorphic to l^2 [14]. $\square$

2.3 L_1 with $C^{k+\alpha}(k \geq 2)$ topology and $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ are not completely metrizable

In this section, assume k is finite. Let E denote the square $[0,1] \times [0,1]$. Let S denote the set $C^\infty(E)$ equipped with C^k topology and Y denote the set $C^\infty(S^2)$ equipped with C^k topology.

We shall show later that there is a map H from S to Y, such that H is a homeomorphism from S to $H(S)$ and $H(S)$ is a closed subset of Y. To prove Y is not completely metrizable, it is sufficient to show $H(S)$ is not completely metrizable. Since complete metrizability is preserved under homeomorphism, it is equivalent to show S is not completely metrizable.

Before explaining main ideas used in the proof, we introduce some definitions and notations.

Definition 2.3.1 *Given a topological space X, a subset A of X is meagre if it can be expressed as the union of countably many nowhere dense subsets of X. Dually, a comeagre set is one whose complement is meagre, or equivalently, the intersection of countably many sets with dense interiors.*

A subset B of X is nowhere dense if there is no neighborhood on which B is dense.

The useful facts are any subset of a meagre set is meagre and the union of countably many meagre sets is also meagre [36].

Our goal is to show S is not completely metrizable.

Let $D^k = \{f \in C^k(E) :$ for any $i + j = k$, $\frac{\partial^k f}{\partial x^i \partial y^j}$ is differentiable at $(x, y)$for some$(x, y) \in E\}$.

For $k = 0$, let $A^0_{n,m}$=$\{f \in C^0(E)$: there is $(x, y) \in E$ such that

$$|\frac{f(t,y) - f(x,y)}{t - x}| < n \text{ and } |\frac{f(x,s) - f(x,y)}{s - y}| < n$$

if $0 < |x - t| < \frac{1}{m}$ and $0 < |s - y| < \frac{1}{m}$ $\}$

For $k > 0$, let $A^k_{n,m}$=$\{f \in C^k(E)$: there is $(x, y) \in E$ such that

$$|\frac{\frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(t,y) - \frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{t-x}| \leqslant n \text{ and } |\frac{\frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,s) - \frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{s-y}| \leqslant n$$

for any $i + j = k - 1$ if $0 < |x - t| < \frac{1}{m}$ and $0 < |s - y| < \frac{1}{m}$ }.

Let X denote the space ($C^k(E)$, C^k topology). Then our goal is an easy consequence of the following claims proved later:

- From the fact $A^k_{n,m}$ is closed and nowhere dense in X, we know $A^k_{n,m}$ is meagre.
- The key point is S is meagre in X. And this is implied by the fact $S \subset D^k \subset A^k = \cup \cup A^k_{n,m}$
- Under the assumption S is completely metrizable, we show S^c is meagre. Then the space $X = S \cup S^c$ is meagre, which is a contradiction with the fact X is a complete metric space.

Now we show the fact $D^k \subset A^k_{n,m}$.

Lemma 2.3.2 *If $f(x,y) \in D^k$, then $f \in A^k_{n,m}$ for some n and m.*

Proof Step1: for $k = 0$, suppose $f(x,y) \in D^0$, i.e. $f(x,y)$ is differentiable at (x,y), then there exists n such that $|f_x(x,y)| < n$ and $|f_y(x,y)| < n$. So there exists $\delta > 0$ such that

$$|\frac{f(t,y) - f(x,y)}{t-x}| < n \text{ and } |\frac{f(x,s) - f(x,y)}{s-y}| < n$$

if $0 < |t - x| < \delta$ and $0 < |s - y| < \delta$.

Choose m such that $\frac{1}{m} < \delta$, then $f \in A^0_{n,m}$ according to the definition of $A^0_{n,m}$.

Step2: then we can prove the lemma by induction.

Suppose it is true for k, we want to show it is true for $k + 1$. The proof is the same as step 1. □

Then we show $A^k_{n,m}$ is meagre in X by the following two lemmas.

Lemma 2.3.3 *$A^k_{n,m}$ is closed for each k.*

Proof Suppose f_i is a Cauchy sequence in $A^k_{n,m}$ and $f_i \to f$. We shall prove $f \in A^k_{n,m}$. For each i there is $(x_i, y_i) \in K$ such that

$$\left|\frac{\frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(t,y) - \frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{t-x}\right| \leqslant n \text{ and } \left|\frac{\frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,s) - \frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{s-y}\right| \leqslant n$$

for any $i + j = k - 1$ if $0 < |x - t| < \frac{1}{m}$ and $0 < |s - y| < \frac{1}{m}$.

Without loss of generality, we assume that (x_i, y_i) converges.

Suppose (x_i, y_i) converges to (x, y) and suppose $0 < |x - t| < \frac{1}{m}$ and $0 < |s - y| < \frac{1}{m}$, then we have

$$\left|\frac{\frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(t,y) - \frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{t-x}\right| = \lim_{i\to\infty} \left|\frac{\frac{\partial}{\partial x}(\frac{\partial^{k-1} f_i}{\partial x^i \partial y^j})(t,y_i) - \frac{\partial}{\partial x}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x_i,y_i)}{t-x_i}\right| \leqslant n$$

Similarly, we can prove $\left|\frac{\frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,s) - \frac{\partial}{\partial y}(\frac{\partial^{k-1} f}{\partial x^i \partial y^j})(x,y)}{s-y}\right| \leqslant n$. □

We know polynomials are dense in $C^\infty(E)$ equipped with C^k topology. Then we use polynomial to approximate functions in $C^k(E)$.

Lemma 2.3.4 *For any $f(x, y) \in C^k(E)$ and any $\epsilon > 0$, there is an approximation $P(x, y) \in C^k(E)$ s.t. $\|f(x, y) - P(x, y)\|_k < \epsilon$*

The smoothness of $P(x, y)$ implies that $P(x, y) \in A^k_{n,m}$. And this will be used to prove

Lemma 2.3.5 *$A^k_{n,m}$ is nowhere dense.*

Proof Since $A^k_{n,m}$ is closed, it is sufficient to show that $A^k_{n,m}$ does not contain an open ball. Consider the open ball $B_\epsilon(f)$. We must find $g \in B_\epsilon(f)$ with $g \notin A^k_{n,m}$. From the above lemma, we can find $P(x, y) \in C^k(E)$ such that $\|f - P\|_k < \epsilon/2$. Since $P(x, y)$ is smooth and E is compact, there is $M \in \mathbb{N}$ such that $|\frac{\partial^{k+1} P}{\partial^{k+1} y}| \leqslant M$.

There is a function $\phi(x,y)$ such that $\frac{\partial^k\phi}{\partial^k y} \leqslant 1$, $||\phi||_k \leqslant q$ and $\frac{\partial^{k+1}\phi}{\partial^{k+1}y} = \pm t$. Now we construct the function $\phi(x,y)$. Choose $t > \frac{2q(M+n)}{\epsilon}$ and consider the partition of y: $b_i = \frac{i}{t}$ for $i = 0, 1, ..., t$. Let ϕ^k be the piecewise linear function satisfying $\frac{\partial^k\phi}{\partial^k y} = 0$ if i is even and 1 if i is odd. E is compact, so there is q such that $||\phi||_k \leq q$.

Then we construct the goal function $g(x,y)$ as $g(x,y) = P(x,y) + \frac{\epsilon}{2q}\phi(x,y)$.
Since $||f-P||_k < \frac{\epsilon}{2}$ and $||g-P||_k < \frac{\epsilon}{2}$, we know $||f-g||_k < \epsilon$.

We claim that $g \notin A^k_{n,m}$. Let $(x,y) \in K$. If P and ϕ are both kth-differentiable at (x,y), then $|\frac{\partial^{k+1}g}{\partial^{k+1}y}(x,y)| = |\frac{\partial^{k+1}P}{\partial^{k+1}y}(x,y) \pm \frac{\epsilon}{2q}t|$.
Since $|\frac{\partial^{k+1}P}{\partial^{k+1}y}(x,y)| \leqslant M$, we know $|\frac{\partial^{k+1}g}{\partial^{k+1}y}(x,y)| \geqslant n$. So $g \notin A^k_{n,m}$ and $B_\epsilon(f) \subsetneq A^k_{n,m}$. $\square$

From the above two lemmas, $A^k_{n,m}$ is meagre in ($C^k(E)$, C^k-topology). Let $A^k = \cup\cup A^k_{n,m}$, then A^k is meagre as well. Since $D \subseteq A$, D is meagre.
Then our goal is proved as follows.

Lemma 2.3.6 *S is not completely metrizable.*

Proof We consider S as a subspace of X. Suppose S is completely metrizable. We know the fact that S is completely metrizable if and only if S is the intersection of countably many open subsets. Then we consider it as the intersection of countable open sets, i.e. $S = \cap_{i=1}^{\infty} U_i$, where U_i is open set $\cup_{f\in S} B_{\frac{1}{i}}(f)$. Here $B_{\frac{1}{i}}(f) = \{u|\ ||u-f||_k < \frac{1}{i}\}$. Because S is dense in X [26], the set U_i containing S is dense in X .

And we know the fact that a set is comeagre if it can be expressed as the intersection of countably many sets with dense interior. We know U_i has dense interior. So S is comeagre. Thus the complement, i.e. S^c, is meagre.

S is a subset of meagre set D^k, So S is meagre. Then $X = S \cup S^c$ as union set of two meagre sets is meagre. This is a contradiction because a complete metric space is not meagre. □

Theorem 2.3.7 *Y is not completely metrizable.*

Proof We already know S is not completely metrizable. We shall show there is a homeomorphism

$$H : S \to Y$$

such that $H(S)$ is a closed subset of Y.

From [7], for any function $f \in C^\infty(E)$, there is an extension $g \in C^\infty(S^2)$ such that $g|_E = f$. Let $H(f) = g$.

From the fact that any closed subset of a completely metrizable space is completely metrizable, it is sufficient to show $H(S)$ is not completely metrizable, which is a consequence of $H(S)$ is homeomorphic to S and S is not completely metrizable.

Then Y is not completely metrizable as desired. □

Theorem 2.3.8 *L_1 equipped with finite $C^{k+\alpha}$ topology is not completely metrizable.*

Proof For $k \geq 2$, suppose L_1 is completely metrizable. Then as a G_δ set in L_1, the set $\mathring{L}_1 = \{u \in C^\infty : \Delta_{g_1} u \leq 1\}$ is not completely metrizable.

The set C^∞ equipped with $C^{k+\alpha}$ topology has a cover by translates of $\mathring{L}_1$. It is a fact that a metrizable space covered by a family of completely metrizable open sets is completely metrizable. So Y, i.e. $C^\infty(S^2)$ equipped with $C^{k+\alpha}$ topology, is completely metrizable, contradicting with the fact and above theorem [Theorem 2.3.7].

So $\mathring{L}_1$ is not completely metrizable. □

Besides these, we show $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ is not completely metrizable by the following lemma.

Theorem 2.3.9 *Let $l > 0$ and $s = \infty$. If M is a manifold of positive dimension, then* $\mathrm{Diff}^{s,l+\alpha}(M)$ *is not completely metrizable.*

Proof Recall that complete metrizability is inherited by any G_δ (e.g. open or closed) subset. Fix two closed n-disks D, Δ in a coordinate chart of M with $n > 0$ and $D \subset \mathrm{Int}\,\Delta$. Identifying Δ with a hemisphere in S^n, and extending by the identity allows identify $\mathrm{Diff}^{l+\alpha,l+\alpha}(M, \mathrm{rel}\ M \setminus D)$ with a closed subset of $\mathrm{Diff}^{l+\alpha,l+\alpha}(S^n)$. This diffeomorphism group is an open subset of $C^{l+\alpha,l+\alpha}(S^n, S^n)$. The inclusion $S^n \to \mathbb{R}^{n+1}$ identifies $C^{l+\alpha,l+\alpha}(S^n, S^n)$ with a closed subset of the Banach space $C^{l+\alpha,l+\alpha}(S^n, \mathbb{R}^{n+1})$. It follows that $\mathrm{Diff}^{l+\alpha,l+\alpha}(M, \mathrm{rel}\ M \setminus D)$ is completely metrizable.

An application of a Baire category theorem [39, Theorem 2.5.10] is that a subgroup of a completely metrizable group is completely metrizable if and only if it it closed. Since $s > l$, the subgroup $\mathrm{Diff}^{s,l+\alpha}(M, \mathrm{rel}\ M \setminus D)$ is not closed in $\mathrm{Diff}^{l+\alpha,l+\alpha}(M, \mathrm{rel}\ M \setminus D)$, and therefore is not completely metrizable.

Finally, since $\mathrm{Diff}^{s,l+\alpha}(M, \mathrm{rel}\ M \setminus D)$ is closed in $\mathrm{Diff}^{s,l+\alpha}(M)$, we conclude that $\mathrm{Diff}^{s,l+\alpha}(M)$ is not completely metrizable. □

In our setting $M = S^2$, since we can choose D such that $\mathrm{Diff}^{s,l+\alpha}(S^2, S^2 \setminus D)$ is closed in $\mathrm{Diff}^{s,l+\alpha}_{0,1,\infty}(S^2)$, we conclude $\mathrm{Diff}^{s,l+\alpha}_{0,1,\infty}(S^2)$ is not completely metrizable. So $\mathrm{Diff}^{\infty,k+\alpha}_{0,1,\infty}(S^2)$ is not completely metrizable. Since $\mathrm{Diff}^{\infty,k+\alpha}_{0,1,\infty}(S^2)$ is a direct factor and hence a closed subset of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ we conclude:

Theorem 2.3.10 $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$ *is not completely metrizable for* $\alpha \in (0, 1)$.

From the same proof and Theorem 3.3.5 , we have

Theorem 2.3.11 $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ *is not completely metrizable for* $\alpha \in (0, 1)$.

2.4 Beltrami equation on S^2

The Laplace-Beltrami operator is the divergence of the gradient: $\Delta f=$ div grad f.

Suppose M is an oriented Riemannian manifold. In local coordinates,

$$\mathrm{div} X = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}X^i)$$

where the Einstein notation is implied, so that the repeated index i is summed over. The gradient of a scalar function f is the vector field grad f that may be defined through the inner product $<\cdot,\cdot>$ on the manifold, as $<\mathrm{grad} f(x), v_x >= df(x)(v_x)$ for all vectors v_x anchored at point x in the tangent space T_xM of the manifold at point x. In local coordinates, one has $(gradf)^i = \partial^i f = g^{ij}\partial_j f$ where g^{ij} are the components of the inverse of the metric tensor, so that $g^{ij}g_{jk} = \delta^i_k$.

Combining the definitions of the gradient and divergence, the formula for the Laplace-Beltrami operator Δ applied to a scalar function f is , in local coordinates

$$\Delta f = \mathrm{div}\,\mathrm{grad} f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$$

We know any metric on S^2 is conformal to the round metric g_1. So up to normalization a metric $g \in \mathcal{R}^{\infty}_{\geqslant 0}(S^2)$ can be written uniquely as $\phi^* e^{2u} g_1$ where ϕ is a C^∞ self-diffeomorphism of S^2 that fixes 0, 1, ∞, and u is a C^∞ function on S^2. The sectional curvature of g equals $e^{-2u}(1 - \triangle_{g_1} u)$, so nonnegatively curved means $\triangle_{g_1} u \leqslant 1$.

By the result of Earle-Schatz, we get a homeomorphic parametrization of $\mathcal{R}^{k+\alpha}_{\geqslant 0}(S^2)$ by the product of the diffeomorphism group $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ fixing $0, 1, \infty$ and the space $L_1 = \{u \in C^\infty(S^2) | \triangle_{g_1} u \leqslant 1\}$. The $C^{k+\alpha}$ topology on the space of metrics gives rise to the $C^{k+\alpha}$ topology on $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ and L_1.

To state the useful theorem by Earle-Schatz, we first introduce some notations.

Let $\mathcal{U} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$ and $\mathcal{M}^{m+\alpha}(\mathcal{U})$ be the set of functions $u \in C^{m+\alpha}(\mathcal{U})$ such that

$|\mu(z)| < 1$ for all $z \in \mathcal{U}$. And $\mathbb{C}^{m+\alpha}(\mathcal{U})$ consists of those functions on $\mathcal{U}$ having continuous derivatives up to order k and such that the kth partial derivatives are Hölder continuous with exponent α.

The theorem is stated as follows.

"For each $k < 1$, the map $\mu \to f^{\mu}$ is a homeomorphism of the set of $\mu \in \mathcal{M}^{m+\alpha}(\mathcal{U})$ with $\sup\{|\mu(z)| : z \in \mathcal{U}\} \leq k < 1$ onto its image in $C^{m+1+\alpha}(\mathcal{U}, \mathbb{C})$. Here the integer $m \geqslant 0$ and the number $0 < \alpha < 1$ are fixed but arbitrary."

Theorem 2.4.1 *Let k be a nonnegative integer or $k = \infty$ and $\alpha \in (0, 1)$. If L_1 is given the $C^{k+\alpha}$ topology and $Diff^{+}_{0,1,\infty}(S^2)$ is given the $C^{k+1+\alpha}$ topology, then the map Π is a homeomorphism.*

Proof The map $\Pi(u, \phi) = \phi^* e^{2u_1} g_1$ is a bijection. We consider the two cover of S^2. One is S^2 without the north pole. The other one is S^2 without the south pole. In each chart, if $\phi \in \mathrm{Diff}^{+}_{0,1,\infty}(S^2)$ varies in the $C^{k+1+\alpha}$ topology, then its differential varies in the $C^{k+\alpha}$ topology, which implies continuity of Π. It remains to show that Π^{-1} is continuous.

The stereographic projection $E_1 = S^2 \setminus \textit{north pole} \to \mathbb{C}$ is conformal. The pulled back metric $\bar{g}$ on $\mathbb{C}$ is conformally equivalent to the original metric g on the chart. Consider a compact subset K of E_1.

Write $\bar{g} = \psi^* e^f g_0 = e^{f \circ \psi} \psi^* g_0$ with ψ is an orientation-preserving diffeomorphism of $\mathbb{C}$ that fixes 0 and 1. The Jacobian of ψ equals $|\psi_z|^2 - |\psi_{\bar{z}}|^2$, and since ψ is orientation-preserving, we get $|\frac{\psi_{\bar{z}}}{\psi_z}| < 1$. Computing

$$\frac{\psi^* g_0}{|\psi_z|^2} = \frac{|d\psi|^2}{|\psi_z|^2} = |dz + \frac{\psi_{\bar{z}}}{\psi_z} d\bar{z}|^2$$

gives $\bar{g} = e^{f \circ \psi} |\psi_z|^2 |dz + \frac{\psi_{\bar{z}}}{\psi_z} d\bar{z}|^2$. Also we can write $\bar{g} = Edx^2 + 2Fdxdy + Gdy^2$ as $\lambda |dz + \mu d\bar{z}|^2$, where

$$\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}) \text{ and } \mu = \frac{E-G+2iF}{4\lambda}.$$

Positive definiteness of $\bar{g}$ easily implies $|\mu| < 1$ and $\lambda > 0$. We know $E,\ F,\ G$ are of class $C^{k+\alpha}$, then the products EG and F^2 are also of class $C^{k+\alpha}$. So μ and λ depend in $C^{k+\alpha}$ on $\bar{g}$. Comparing the two descriptions of $\bar{g}$ we see that $\psi_{\bar{z}} = \mu\psi_z$, that is, ψ solves the Beltrami equation with dilatation μ. Futhermore, $\lambda = e^{f\circ\psi}|\psi_z|^2$ so that $f = \log(\lambda|\psi_z|^{-2}) \circ \psi^{-1}$.

We are going to show that if a sequence of metrics $\bar{g}_l = \psi_l^* e^{f_l} g_0 = \lambda_l|dz + \mu_l d\bar{z}|^2$, with $\psi_l \in \mathrm{Diff}^+_{0,1}(\mathbb{C})$, converges to g uniformly on compact subsets in the $C^{k+\alpha}$ topology, then ψ_l, f_l converge to ψ, f, respectively, in the same topology. A key ingredient is the smooth dependence of ψ on μ established by Earle-Schatz.

To state their result let U, U' be domains in S^2 whose boundaries are embedded circles, and let a_1, a_2, a_3 and a'_1, a'_2, a'_3 be two triples of distinct points on ∂U and $\partial U'$ respectively. Recall that given a C^∞ function $\beta : U \to \mathbb{C}$ with $|\beta| \le k < 1$ for some constant k, there is a unique homeomorphism $w^\beta : \bar{U} \to \bar{U}'$ that restricts to a diffeomorphism $U \to U'$, maps each a_k to a'_k, and solves the Beltrami equation with dilatation β, see e.g.[33, p.183, 194] for existence and uniqueness and [41, Theorem 2.2 in Section 4 of Chapter 2] for regularity.

The continuity Theorem of Earle-Schatz states that varying β in the $C^{k+\alpha}$ topology results in varying w^β in $C^{k+1+\alpha}$ topology. Strictly speaking, Earle-Schatz assume that U, U' equal the upper half plane, and the two triples of points equal $0, 1, \infty$, but the conformal invariance of the Beltrami dilatation, together with the Riemann mapping theorem give the same conclusion for any U, U' as above.

The Continuity Theorem does not immediately apply in our setting, where $U = \mathbb{C} = U'$ and $|\beta|$ is not bounded way from 1. Instead we use the theorem locally, on an arbitrary disk $B_t = \{z \in \mathbb{C} : |z| < t\}$, but then the difficulty is that the domain $\psi(B_t)$ may change as the diffeomorphism ψ varies with μ. Below we resolve the issue by adjusting $\psi(B_t)$ via an ambient diffeomorphism that is the identity on a given compact set. Exhausting $\mathbb{C}$ by such compact sets yields the smooth dependence of ψ on μ.

Let K be a compact subset of $\mathbb{C}$. Let $\tilde{g} = \tilde{\psi}^* e^{\tilde{f}} g_0 = \tilde{\lambda}|dz + \tilde{\mu} d\bar{z}|^2$ be a metric that is $C^{k+\alpha}$ close to $g = \psi^* e^f g_0 = \lambda|dz + \mu d\bar{z}|^2$ over K, where $\tilde{\psi}, \psi \in \mathrm{Diff}^+_{0,1}(\mathbb{C})$. Choose s with $\tilde{\psi}(K) \subset B_s$. The domains $\psi(B_r), r > 0$ exhaust $\mathbb{C}$, and so do the domains $\tilde{\psi}(B_r)$, which allows us to find r with $\bar{B}_s \subset \psi(B_r) \cap \tilde{\psi}(B_r)$.

It is easy to construct an orientation-preserving self-diffeomorphism h of $\mathbb{C}$ that maps $\tilde{\psi}\bar{B}_r$ onto $\psi(\bar{B}_r)$, equals the identity on B_s, and has the property that $h \circ \tilde{\psi}$ and ψ agree at the points $-r, ir, r$ of ∂B_r. (Indeed, $\tilde{\psi}(\partial B_r)$, $\psi(\partial B_r)$ are homotopic smooth simple closed curves in the open annulus $\mathbb{C} - \bar{B}_s$, and hence they can be moved to each other by a compactly supported ambient isotopy of the annulus. The identity component of $\mathrm{Diff}(S^1)$ acts transitively on the set of triples of distinct points of S^1, e.g. if S^1 is identified to the boundary of the upper half plane, then the map $x \to \frac{(x-a)(c-b)}{(x-b)(c-a)}$ takes a, b, c to $0, \infty, 1$, respectively, and preserves an orientation, and hence is isotopic to the identity of S^1. So given two triples of points in $S^1 \times 0$ there is a compactly supported isotopy of $S^1 \times \mathbb{R}$ that takes one triple to the other one. Here we identity $S^1 \times \mathbb{R}$, $S^1 \times 0$ with $\mathbb{C} - \tilde{B}_s$, $\psi(\partial B_r)$, respectively. Combining the two isotopies, and extending the result by the identity on $\tilde{B}_s$ yields the desired h).

Since $g|_K, \tilde{g}|_K$ are $C^{k+\alpha}$ close, so are the dilations of ψ_K, $\tilde{\psi}|_k = h \circ \tilde{\psi}|_K$. Thus $\psi|_{B_r}, h \circ \tilde{\psi}|_{B_r}$ are diffeomorphisms of B_r onto $\psi(B_r)$ whose dilatations are $C^{k+\alpha}$ close on K. The absolute values of the dilatations are less than 1 (as the diffeomorphisms are orientation-preserving), and hence are bounded away from 1 by compactness of $\bar{B}_r$. Now the Continuity Theorem implies that $\psi|_{B_r}$, $h \circ \tilde{\psi}|_{B_r}$ are $C^{k+1+\alpha}$ close over K. It follows that $\lambda|\psi_z|^{-2}$, $\tilde{\lambda}|\tilde{\psi}_z|^{-2}$ are $C^{k+\alpha}$ close over K.

Thus if a sequence of metrics $\bar{g}_l$ converges to $\bar{g}$ uniformly on compact subsets in the C^k topology, then the corresponding diffeomorphisms ψ_l converge to ψ uniformly on compact subsets in the $C^{k+1+\alpha}$ topology. Since $\mathrm{Diff}^+_{0,1}(\mathbb{C})$ with the $C^{k+1+\alpha}$ topology is a topological group we also have the $C^{k+1+\alpha}$ convergence of ψ_l^{-1} to ψ^{-1}. In lemma 2.2 of [6], it states that " If G is of class $C^{k-1,\alpha}$ and $H \in C^k$, $k \geq 1$, then $G \circ H$ is of class $C^{k-1,\alpha}$. This implies that $\log(\lambda_l|(\psi_l)_z|^{-2}) \circ \psi_l^{-1}$ converges to $\log(\lambda|\psi_z|^{-2}) \circ \psi^{-1}$ in the $C^{k+\alpha}$ topology.

The sphere can be covered by two compact subsets. Then $\lambda|\psi_z|^{-2}$, $\tilde{\lambda}|\tilde{\psi}_z|^{-2}$ are $C^{k+\alpha}$ close over S^2. Also $\log(\lambda_l|(\psi_l)_z|^{-2}) \circ \psi_l^{-1}$ converges to $\log(\lambda|\psi_z|^{-2}) \circ \psi^{-1}$ in the $C^{k+\alpha}$ topology on S^2, which completes the proof. □

2.5 L_1 equipped with $C^{k+\alpha}$ topology is homogenous

Definition 2.5.1 *A point x_0 of a convex set C in a linear topological space is called* ***almost internal*** *if the set $A = \{a \in C : \exists\, \epsilon > 0 \text{ with } x_0 - \epsilon(a - x_0) \in C\}$ is dense in C. If the set A coincides with C, then x_0 is called an* ***internal*** *point of C.*

A subset A of a space X is called a $Z-$ *set* in X if A is closed in X and every map $f : Q \to X$ of the Hilbert cube can be uniformly approximated by maps whose ranges miss the set A.

For a set C in a Fréchet space Aff(C) denotes the affine hull of C and $\bar{C}$ is the closure of C.

A metric space(X, d) is totally bounded if and only if for every real number $\epsilon > 0$, there exists a finite collection of open balls in X of radius ϵ whose union contains X. For a convex set C in a Fréchet space F, let $\mathcal{F}_{ctb}(C)$ be the class of spaces homeomorphic to *totally bounded* (in the natural uniform structure of F) subsets of C that are closed in Aff(C).
Let C be a class of spaces. Denote by σC the class of spaces C that can be expressed as a countable union $C = \cup_{n=1}^{\infty} C_n$, where each C_n is closed in C and $C_n \in C$.

A subset A of a space X is called a $\sigma Z-$ *set* in X if A is a countable union of Z-sets in X. A space X is defined to be a $\sigma Z-$ *space* if X is a $\sigma Z-$set in X.

Based on the above definitions and properties, we recall the theorem [30]:

Theorem 2.5.2 *A convex set C with an almost internal point is topologically homogeneous, provided $C \in \sigma\mathcal{F}_{ctb}(C)$ and C is a σZ-space.*

A space X is *topologically homogeneous* if for every two points $x, y \in X$ there exists an autohomeomorphism h of X with $h(x) = y$.

The round metric on S^2 is $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$. Consider a small neighborhood U of (θ, ϕ). Suppose $\sin\theta > 0$ in U. Then

$$\Delta f = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^j}\left(\sqrt{\det g}\, g^{ij} \frac{\partial f}{\partial x^i}\right) = \cot\theta f_\theta + f_{\theta\theta} + \csc^2\theta f_{\phi\phi},$$

where $g^{11} = 1$, $g^{12} = g^{21} = 0$ and $g^{22} = \csc^2\theta$.

In our setting, $L_1 = \{u \in C^\infty(S^2) : \Delta u \leq 1\}$ with $C^{k+\alpha}$ topology, here k is any non negative integer.

$$C = \{u \in C^\infty(S^2) |\ \Delta u \leq 1\}$$

$$\mathrm{Aff}C = \{u | u \in C^\infty(S^2)\} \text{ equipped with } C^{k+\alpha} \text{ topology.}$$

$$K_l = \{u \in C^\infty(S^2) |\ \Delta u \leq 1, ||u||_{k+1} \leq l\}.$$

Since S^2 is compact, every smooth function on the sphere has bounded C^k norm. So we have $C \subset \cup_l(\bar{K}_l \cap C)$. Here the closure is taken in $C^{k+\alpha}$ norm.

From above we know $C \in \sigma\mathcal{F}_{ctb}(C)$. We shall show C has an almost internal point and C is a $\sigma Z-$space.

Theorem 2.5.3 *Zero function is an internal point of* C.

Proof Let $A = \{a \in C : \exists\, \epsilon > 0\ with - \epsilon a \in C\}$. $\Delta u = g^{jl}\partial_j\partial_l u$ + first-order terms. For any $c \in C$, since the sphere is compact, let $M = \max\{g^{jl}\partial_j\partial_l u$, first-order terms $\}$.

There exists $\epsilon > 0$ such that $\epsilon M \leq 1$. So $c \in A$. c is arbitrary function in C, so $A = C$.

By the definition, zero function is an internal point of the convex set C. □

Theorem 2.5.4 $\bar{K}_l$ *is a Z-set.*

Proof First, we shall show $\bar{K}_l$ is totally bounded by showing $\bar{K}_l$ is sequentially compact.

Recall the Arzela-Ascoli theorem, it can be stated as follows:

Consider a sequence of real-valued continuous functions $f_n, n \in \mathbb{N}$ defined on any compact manifold. If this sequence is uniformly bounded and equicontinuous, then there exists a subsequence f_{n_k} that converges uniformly.

This proves the following corollary:

Let f_n be a uniformly bounded sequence of real-valued differentiable functions such that the derivatives f_n' are uniformly bounded. Then there exists a subsequence f_{n_k} that converges uniformly on the manifold.

In our case, Arzela-Ascoli theorem is applied to S^2.

For any sequence $f_n \in \bar{K}_l$, it is uniformly bounded sequence of real-valued $k+1$-differentiable functions on the sphere such that the $k+1$th derivatives are uniformly bounded. Then there exists a subsequence f_{n_k} that all the kth derivatives converges uniformly on the sphere.

By adapting the above step k times, there exists a subsequence $\hat{f}_k$ that converges uniformly in C^{k+1} topology. And this implies there is a subsequence $\hat{f}_{n_k}$ converges uniformly in $C^{k+\alpha}$ topology. By the above proof, we know ($\bar{K}_l$, $C^{k+\alpha}$-topology) is sequentially compact, so it is totally bounded.

Then we shall show that any continuous map $f : Q \to C$ can be pushed off from $\bar{K}_l$.

Using scaling by constant t, we get function $tf(q)$ satisfying $\Delta(tf(q)) < 1$ for any $q \in Q$. From now on, let f denote tf.

Next we try to approximate f by $g : Q \to C - \bar{K}_l$. Let $g(q) = f(q) + v$, here v is a function satisfying the following conditions:

(1) $\max\{0, \Delta v\}$ is very small such that $\Delta g(q) \leq 1$.

(2) $C^{k+\alpha}$ norm of v is very small, i.e. f can be uniformly approximated by g.

(3) C^{k+1} norm of v is larger than $l + \|f(q)\|_{k+1}$ for any $q \in Q$.

By the third condition, we can show $g(q) \in C - \bar{K}_l$ by contradiction. Suppose there is $q \in Q$ such that $g(q) \in \bar{K}_l \cap C$, i.e. $\|f(q) + v\|_{k+1} \leq l$. Then $\|v\|_{k+1} \leq \|f(q) + v\|_{k+1} + \| - f(q)\|_{k+1} \leq$

$l + \|f(q)\|_{k+1}$, which is a contradiction of the third condition.

Finally, we shall show there exists function v satisfying the above three conditions.

Direct computations show that, in local coordinates x^1, x^2,

$$\nabla u = g^{ki}\partial_k u \partial_i$$
$$\text{Hess}(u)(\partial_i, \partial_j) = \frac{1}{\sqrt{G}} g_{jk}\partial_i(\sqrt{G} g^{kl}\partial_l u)$$
$$\Delta u = \frac{1}{\sqrt{G}}\partial_j(\sqrt{G} g^{jl}\partial_l u),$$

where

$$g_{ij} = \langle \partial_i, \partial_j \rangle$$
$$G = det(g_{ij})$$
$$(g^{ij}) = (g_{ij})^{-1}.$$

In particular

$$\Delta u = g^{jl}\partial_j\partial_l u + \text{first-order terms}$$

is a second order, elliptic operator.

Case $k \geq 2$. For any $0 < \epsilon < 1$, consider any function h in the set $H = \{h \in C, \|h\|_{k+\alpha} < \epsilon\}$. Then for any N, there is $h \in H$ such that $\|h\|_{k+1} > N$. Since $\|h\|_{k+\alpha} < \epsilon$, then $\Delta h = g^{jl}\partial_j\partial_l h$ + first-order terms $\leq M$, where M is a constant.
Choose $\bar{h} = \frac{(1-t)h}{M}$, then $\Delta\bar{h} \leq 1$. And for any N, there is $h \in H$ such that $\|\hat{h}\|_{k+1} > N$. So there is function v as desired.

Case $k = 0, 1$. Consider $k = 0$. There is a nonconstant function u with $\Delta u < 1$ on S^2. We can find a neighborhood U such that $\max\{|\frac{\partial u}{\partial x_1}|, |\frac{\partial u}{\partial x_2}|\} \geq P$, where P is positive. Then we can find a diffeomorphism of S^2 that is supported in U, is C^α close to the identity but not C^1 close to the identity. Such a diffeomorphism can be constructed by hand in a chart on U and then extended by the identity to S^2. Denote the diffeomorphism as h. Since u precomposed h, we have $\Delta u \circ h(x, y) < 1$.

The C^α norm of $u \circ h$ is the same as for u. But its first order partial derivative could be very large by the chain rule. The function $u \circ h$ is as desired.

Similar to case $k = 0$, there is function satisfying the above three conditions when $k = 1$. □

Theorem 2.5.5 *L_1 equipped with $C^{k+\alpha}$ topology is homogenous for $k \geq 2$.*

Proof By the above theorem, a convex set C with an almost internal point is topologically homogeneous, provided $C \in \sigma\mathcal{F}_{ctb}(C)$ and C is a σZ-space.

With the notation $C = \{u \in C^\infty(S^2), \Delta u \leq 1\}$, $K_l = \{u \in C^\infty(S^2) | \Delta u \leq 1, ||u||_{k+1} \leq l\}$, we know $C = \cup_l(\bar{K}_l \cap C)$, where l is positive integer.

Since $\bar{K}_l$ is compact it is closed in the ambient Fréchet space. So $\bar{K}_l \cap C$ is closed in C. Now $\mathrm{Aff}(C) = C^\infty(S^2)$ and C is closed in $C^\infty(S^2)$ for $k \geq 2$. Thus any closed subset of C is closed in $\mathrm{Aff}(C)$, i.e. $\bar{K}_l \cap C$ is closed in $\mathrm{Aff}(C)$. According to theorem 2.5.2, L_1 is homogenous. □

Corollary 2.5.6 *$\mathcal{R}^{k+\alpha}_{\geq 0}$ is homogenous for $k \geq 2$ and $\alpha \in (0, 1)$.*

Proof Since L_1 is homogenous and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ equipped with $C^{k+1+\alpha}$ topology is homogenous as a group, the space $\mathcal{R}^{k+\alpha}_{\geq 0}$, which is homeomorphic to the product of L_1 with C^{k+1} topology and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ with $C^{k+1+\alpha}$ topology, is homogenous for $\alpha \in (0, 1)$. □

2.6 $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ *equipped with the* $C^{k+1+\alpha}$ *topology is contractible*

Theorem 2.6.1 $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ *equipped with the* $C^{k+1+\alpha}$ *topology is contractible.*

Proof By the same proof of Theorem 2.4.1 , we know that $\mathcal{R}^{k+\alpha}(S^2)$ is homeomorphic to the product of $C^\infty(S^2)$ and $\mathrm{Diff}^+_{0,1,\infty}(S^2)$, with $C^{k+\alpha}$ and $k+1+\alpha$ topology, respectively. Since $\mathcal{R}^{k+\alpha}(S^2)$ is convex, it is contractible.

Thus $\mathrm{Diff}^+_{0,1,\infty}(S^2)$ is a retract of a contractible space, and hence it is contractible. □

2.7 $\mathcal{R}^{k+\alpha}_{\geqslant 0}(S^2) \setminus K$ *is weakly contractible for* $k \geq 2$

Actually, the main theorem in this section is already proved later Theorem 3.6.4. The following proof is a different method.

Recall that $\mathcal{R}^{k+\alpha}_{\geqslant 0}(S^2)$ is contractible, we are trying to show that $\mathcal{R}^{k+\alpha}_{\geqslant 0}(S^2)$ minus any compact set K is weakly contractible, i.e. any sphere in the space is null-homotopic to the constant map into the base point. Include $\mathcal{R}^{k+\alpha}_{\geqslant 0}(S^2)$ in the space $\mathcal{R}^{k+\alpha,k+\alpha}_{\geqslant 0}(S^2)$ of $C^{k+\alpha}$ metrics with $C^{k+\alpha}$.

From appendix C, we know $C^\infty(S^2)$ is not dense in $C^{k+\alpha}(S^2)$. In fact the latter is not separable. The convolution with C^∞ kernel is not continuous in $C^{k+\alpha}$ topology when $\alpha \in (0,1)$. But the closure of $C^\infty(S^2)$ in $C^{k+\alpha,k+\alpha}(S^2)$ is separable, and incidentally is called the *little Hölder space* denoted by $c^{k+\alpha}(S^2)$. Note that $c^{k+\alpha,k+\alpha}(S^2)$ is Banach as a closed linear subspace in the Banach space $C^{k+\alpha,k+\alpha}(S^2)$. And the useful space is the space $H = \{u \in c^{k+\alpha,k+\alpha}(S^2) | \, \Delta u \leq 1\}$. The space H is a closed convex subset of a Banach space. So H is weakly contractible.

Now we discuss convolution of functions in H. Through the smooth procedure by convolution, sectional curvature stays positive when $k \geq 2$. So we only consider the case when $k \geq 2$.

First, recall the convolution of functions in $C^k(\mathbb{R}^2, \mathbb{R}^2)$.
Let $\theta : \mathbb{R}^m \to \mathbb{R}$ be a map having compact support. There is a smallest $\sigma \geq 0$ such that Suppθ is contained in the closed ball $B_\sigma(0) \subset \mathbb{R}^m$ of radius σ and center 0. We call σ the *support radius* of θ.
Let $U \subset \mathbb{R}^m$ be open and $f : U \to \mathbb{R}^n$ a map. If $\theta : \mathbb{R}^m \to \mathbb{R}$ has compact support we

define the *convolution* of f by θ to be the map

$$\theta * f : U_\sigma \to \mathbb{R}^n$$

given by

$$\theta * f(x) = \int_{B_\sigma(0)} \theta(y) f(x-y) dy \quad (x \in U_\sigma)$$

where

$$U_\sigma = x \in U : B_\sigma(x) \subset U.$$

The integrand is 0 on the boundary of $B_\sigma(0)$; we extend it to a continuous map $\mathbb{R}^m \to \mathbb{R}$ by defining it to be 0 outside $B_\sigma(0)$. Therefore we have

$$\theta * f(x) = \int_{\mathbb{R}^m} \theta(y) f(x-y) dy \quad (x \in U_\sigma)$$

A map $\theta : \mathbb{R}^m \to \mathbb{R}$ is called a *convolution kernel* if it is nonnegative, has compact support, and $\int_{\mathbb{R}^m} \theta_m = 1$. It is clear that there exist C^∞ convolution kernels of any given support radius.

By theorem 2.3 in Hirsch's book [26]: Let $\theta : \mathbb{R}^m \to \mathbb{R}$ have support radius $\sigma > 0$. Let $U \subset \mathbb{R}^m$ be an open set, and $f : U \to \mathbb{R}^n$ a continuous map. Define $U_\sigma = \{x \in U : B_\sigma(x) \subset U\}$. The convolution $\theta * f$: $U_\sigma \to \mathbb{R}^n$, $\theta * f(x) = \int_{B_\sigma(0)} \theta(y) f(x-y) dy$ $(x \in U_\sigma)$ has the following properties:

(a) If $\theta|_{\text{Int Supp}\theta}$ is C^k, $1 \le k \le \infty$, then so is $\theta * f$; and for each finite k, $D^k(\theta * f)_x(Y_1, ...Y_k) = \int_{\mathbb{R}^m} D^k\theta(x-z)(Y_1, ..., Y_k) f(z) dz$.

(b) If f is C^k then

$$D^k(\theta * f) = \theta * (D^k f).$$

(c) Suppose f is C^r, $0 \le r \le \infty$. Let $K \subset U$ be compact. Given $\epsilon > 0$ there exists $\sigma > 0$ such that $K \subset U_\sigma$, and if θ is a C^r convolution kernel of support radius σ, then $\theta * f$ is C^r and

$$\|\theta * f - f\|_{r,K} < \epsilon.$$

Our goal is to prove the following theorem.

Theorem 2.7.1 *Suppose f is in $c^{k+\alpha}$, $0 \leq k \leq \infty$. Let $K \subset U$ be compact. Given $\epsilon > 0$ there exists $\sigma > 0$ such that $K \subset U_\sigma$, and if θ is a C^∞ convolution kernel of support radius σ, then $\theta * f$ is C^∞ and*

$$||\theta * f - f||_{k+\alpha,K} < \epsilon.$$

Proof Let θ denote the standard convolution C^∞ kernel. Let us try to show that for any f in $C^{k,\alpha}$ the convolution $\theta * f$ is close to f in $C^{k+\alpha}$ norm. Let g be a C^∞ function that approximates f in $C^{k+\alpha}$ norm. Then

$$||f - \theta * f||_{k+\alpha} \leq ||f - g||_{k+\alpha} + ||g - \theta * g||_{k+\alpha} + ||\theta * g - \theta * f||_{k+\alpha}.$$

The first term is small.

The second term is also small because we can choose θ such that g and $\theta * g$ are close in C^{k+1} norm, which ensures closeness of g and $\theta * g$ on $C^{k+\alpha}$ norm.

Write the last term as $||\theta * (g - f)||_{k+\alpha}$. Set $h = g - f$. We need to estimate $||\theta * h||_{k+\alpha}$ in terms of $||h||_{k+\alpha}$, which is small by assumption.

$$\frac{|\theta * h(x) - \theta * h(y)|}{|x-y|^\alpha} = \frac{|\int \theta(t)(h(x-t) - h(y-t))dt|}{|x-y|^\alpha} \leq \frac{\int \theta(t)|h(x-t) - h(y-t)|dt}{|x-y|^\alpha}$$

Since $\frac{|h(x-t)-h(y-t)|}{|x-y|^\alpha} \leq ||h||_\alpha$, we have

$$\frac{\int \theta(t)|h(x-t) - h(y-t)|dt}{|x-y|^\alpha} \leq ||h||_\alpha \int c(t)dt = ||h||_\alpha.$$

So

$$||\theta * g - \theta * f||_{k+\alpha} \leq ||\theta * g - \theta * f||_k + ||h||_\alpha.$$

So the third term is very small. □

Now we discuss the contractibility of $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2)$.

Theorem 2.7.2 *$\mathcal{R}^{k+\alpha}_{\geq 0}(S^2) \setminus K$ is weakly contractible for $k \geq 2$.*

Proof Consider the following diagram:

$$
\begin{tikzcd}
S^n \arrow[r, "f"] \arrow[d, "\cap"'] & (L_1 \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1,\infty}(S^2)) \setminus K \arrow[d, "\cap"] \\
D^{n+1} \arrow[r] & (H \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1,\infty}(S^2)) \setminus K
\end{tikzcd}
$$

Let K_1 and K_2 denote the projection of K onto H and $\mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1,\infty}(S^2)$ respectively. Denote f as (f_1, f_2), where f_i is the restriction of f onto each factor space. Note that $H \setminus K_1$ is weaky contractible. Then f_1 can be extended to continuous map $F_1 : D^{n+1} \to H \setminus K_1$. We scale F_1 by t. The map tF_1 can be considered as the continuous map $\hat{F} : D^{n+1} \times S^2 \to \mathbb{R}$. Then for every x we have $\Delta\hat{F}(x, \cdot) \leq t < 1$. Let us now push $\hat{F}(D^{n+1})$ back to $L_1 \setminus K_1$ by convolution. By Theorem 2.7.1, for any $\epsilon > 0$, there is function $\hat{G}$ such that $||\hat{F} - \hat{G}||_{k+\alpha} < \epsilon$. Since $k \geq 2$, for small enough ϵ we will have $\Delta\hat{G}(x, \cdot) < 1$. Note that the boundary of the disk moves a little by the above procedure. We have to to show that two nearby maps from S^n to an open convex set, i.e. $F_1(S^n)$ and $\hat{G}(S^n)$, are homotopic. This can be accomplished by the straight line homotopy in $L_1 \setminus K_1$.

Therefore, the space $L_1 \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1,\infty}(S^2)$ minus a compact set K is weakly contractible. Since $\mathcal{R}^{k+\alpha}_{\geq 0}(S^2) \setminus K$ is homeomorphic to the product of $L_1 \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(S^2)$ equipped with $C^{k+\alpha}$ topology minus K, we know $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ is weakly contractible. □

CHAPTER III

NON NEGATIVELY CURVED PLANES

3.1 Non negatively curved planes

This section studies connectedness properties of the set of complete nonnegatively curved metrics on $\mathbb{R}^2$ equipped with the compact-open topology. The main theme is deciding when two metrics can be deformed to each other through complete nonnegatively curved metrics outside a given subset, and how large the space of such deformation is.

We consider an open, two-dimensional Riemannian manifold M whose metric is defined by a positive definite quadratic form

$$ds^2 = E(\xi, \eta)d\xi^2 + 2F(\xi, \eta)d\xi d\eta + G(\xi, \eta)d\eta^2,$$

ξ and η denoting local parameters. If E, F and G are sufficiently regular, then it is possible to introduce (local) isothermic parametrics, i.e. there exists a coordinate transformation $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ such that $E = G > 0$, $F = 0$ in the (x, y)-parameter system. Then we can write

$$ds^2 = e^{-2u(x,y)}(dx^2 + dy^2) = e^{-2u(z)}|dz|^2$$

putting $z = x + iy$. Such a transformation always exists.

Any complete nonnegatively curved metric on $\mathbb{R}^2$ is conformally equivalent to the standard Euclidean metric g_0, i.e. isometric to $e^{-2u}g_0$ for some smooth function u.

We consider a two dimensional plane π in the tangent space T_pM, and we consider all geodesics emanating from p that are tangent to the plane π. The union of these geodesics rays forms a two-dimensional surface $\Sigma \subset M$. The surface Σ is defined as $\Sigma = \exp_p(U \cap \pi)$, where $\exp_p : T_pM \to M$ denotes the exponential map and $U \subset T_pM$ denotes a small ball centered at the origin. Then the sectional curvature $K(\pi)$ is defined to be the Gaussian curvature of the two-dimensional surface Σ at the point p.

$$K(\pi) = \frac{R(X,Y,X,Y)}{|X|^2|Y|^2-\langle X,Y\rangle^2}$$

where X, Y is a basis of π. This definition is independent of the choice of the basis X, Y, and if X, Y are chosen to be an orthonormal basis then the denominator is equal to 1.

The sectional curvature of $e^{-2u}g_0$ can be calculated from the E,F,G and their partial derivatives up to the second order. From the above isothermic parameter system we know the sectional curvature has the particular simple expression $e^{2u}\triangle u$, where $\triangle$ is the Euclidean Laplacian, i.e. $\triangle \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$.

The sectional curvature of $e^{-2u}g_0$ equals $e^{2u}\triangle u$. Thus $e^{-2u}g_0$ has nonnegative curvature if and only if u is subharmonic. Characterizing subharmonic functions that correspond to complete metrics is not straightforward, and doing so is the main objective of this section. A basic property of a subharmonic function u on $\mathbb{R}^2$ is that [25, Theorem 2.14]

Theorem 3.1.1 *The limit*

$$\alpha(u) := \lim_{r\to\infty} \frac{M(r,u)}{\log r}$$

exists in $[0, \infty]$, *where* $M(r, u) := \sup\{u(z) : |z| = r\}$.

For example, by Liouville's theorem $\alpha(u) = 0$ if and only if u is constant, while any nonconstant harmonic function u satisfies $\alpha(u) = \infty$ [27, Theorem 2.15].

3.2 Complete non negatively curved planes

Definition 3.2.1 *According to the definition of H.Hopf and W.Rinow the manifold M is called complete if every divergent path on M has infinite length. A path s is said to be divergent (or to tend to the ideal boundary of M) if*
(1) s is the topological image $p = p(t)$ of the half-open interval $0 \leq t < 1$,
(2) given any arbitrary subcompact K of M there always exists a number $t' < 1$ such that $p(t)$ lies outside K for $t > t'$.

A Riemannian manifold is incomplete if and only if it contains a locally rectifiable (or equivalently, a smooth) path that eventually leaves every compact set and has finite length. In the manifold $(\mathbb{R}^2, e^{-2u}g_0)$ the length of a path γ equals $\int_\gamma e^{-u} ds$.

In the separable Fréchet space $C^\infty(\mathbb{R}^2)$ equipped with the compact-open C^∞ topology consider the subset S_α consisting of smooth subharmonic functions with $\alpha(u) \leq \alpha$. Since S_0 consists of constants, it is homeomorphic to $\mathbb{R}$. Later we will prove that the subset S_α is closed convex, and not locally compact when $\alpha \neq 0$.

Let M be a smooth manifold. Let $\mathcal{R}(M)$ be the set of all Riemannian metrics and $C(M) \subset \mathcal{R}(M)$ be the subspace of all complete Riemannian metric on M.

Remark If $g_0, g_1 \in \mathcal{R}(M)$, $g_0 \in C(M)$ and $f : M \to \mathbb{R}$ is a positive C^∞ function which is bounded away from zero then $g = fg_0 + g_1 \in C(M)$.

Proof Let d_0, d be the metric on M induced by g_0, g respectively. It suffices to show any Cauchy sequence in the d metric is Cauchy in the d_0 metric.
Let $L > 0$ be such that $f(x) \geq L$ for all $x \in M$ and p_n a Cauchy sequence in the d metric. Since d_0 is complete, there is a point $p \in M$ such that $p_n \to p$ in the d_0 metric, hence in the manifold topology.

$$\sqrt{L}\epsilon > \inf_\alpha \int \sqrt{fg_0(\dot\alpha, \dot\alpha) + g_1(\dot\alpha, \dot\alpha)} \geq \sqrt{L} \inf_\alpha \int \sqrt{g_0(\dot\alpha, \dot\alpha)} = \sqrt{L} d_0(p_m, p_n)$$

Then $d_0(p_m, p_n) < \epsilon$ if $m, n \geq N$, i.e. p_n is Cauchy in the d_0 metric, which completes the proof. □

This is false if we omit the hypothesis that f is bounded away from zero. Here is a counterexample.
Take $M = \mathbb{R}^1$, g_0 the usual Euclidean metric, $f(x) = e^{-x}$ and g_1 any incomplete metric. It suffices to show $f(x)g_0$ is incomplete. $f(x)g_0(\xi, \eta) = f(x)\xi\eta$. The induced metric is then $d(x, y) = \int_x^y \sqrt{f(t)}dt$.
In this example, $\int_0^\infty \sqrt{f(t)}dt = \int_0^\infty e^{-1/2t}dt < \infty$, then $a_n = n$ is a Cauchy sequence and so $f(x)g_0$ is not complete.
Let $g_s := (s + e^{-2u})g_0$, i.e. $g_s = sg_0 + e^{-2u}g_0$, then g_s is complete for every $s > 0$ by the above statement. Notice $e^{-2u}g_0$ is the endpoint of the curve $g_s = sg_0 + e^{-2u}g_0$. Hence, complete Riemannian metrics on M form a dense subset in the space of all Riemannian metrics.

Lemma 3.2.2 *The metric $g = e^{-2u}g_0$ is complete if* $0 < \alpha(u) < 1$.

Proof Suppose $\alpha(u) \in (0,1)$. Fix $\alpha_1 \in (\alpha(u), 1)$, and any smooth path σ going to infinity. Find r_1 with $M(r) < \alpha_1 \log r$ for all $r > r_1$. Then $u(re^{i\theta}) \leq M(r)$ implies

$$\int_\sigma e^{-u} ds \geq \int_{r_1}^{\infty} r^{-\alpha_1} dr = \infty$$

So g is complete. □

Appealing to more delicate properties of subharmonic functions due to Hub and Hayman we will prove:

Theorem 3.2.3 *The metric $e^{-2u} g_0$ is complete if and only if $\alpha(u) \leq 1$.*

The following two lemmas imply the above theorem.

Lemma 3.2.4 *The metric $g = e^{-2u} g_0$ is complete if $\alpha(u) < 1$ and incomplete if $\alpha(u) > 1$ or $\alpha(u) = \infty$.*

Proof It is sufficient to show g is incomplete if $\alpha(u) > 1$ or $\alpha(u) = \infty$.
If $\alpha(u) = \infty$, then incompleteness of g can be shown from the following theorem in [32]. Let u be subharmonic in $\mathbb{C}$ and suppose that $\lim_{r\to\infty} \frac{M(r)}{\log r} = +\infty$. Then there exists a path Γ tending to ∞ with

$$\int_\Gamma e^{-\lambda\mu} |dz| < +\infty \text{ for each } \lambda > 0$$
$$\frac{u(z)}{\log |z|} \to +\infty \text{ as } z \to \infty \text{ on } \Gamma.$$

If $\alpha(u) = 0$, then u is a constant function. Therefore $g = e^{-2u} g_0$ is complete since g_0 is complete.
So we can assume that $\alpha(u)$ is positive and finite. Recall the definition of ϵ-set from [28] :
We call an ϵ-set any countable set of circles not containing the origin, and subtending angles at the origin whose sum s is finite. The number s will be called the (angular) extent of the ϵ-set.
Suppose that $u(z)$ is subharmonic and not constant in the plane and that

$$B(r,u) = O(\log r), \text{ as } r \to \infty$$

Then $u(re^{i\theta}) = B(r,u) + o(1)$, uniformly as $re^{i\theta} \to \infty$ outside an ϵ-set.

Remark: For almost all fixed θ and $r > r_0(\theta)$, $z = re^{i\theta}$ lies outside the ϵ-set.

In fact this is the case unless the ray $z = re^{i\theta}$, $0 < r < \infty$ meets infinitely many circles of the ϵ-set. We can write $\epsilon = \epsilon' \cup \epsilon$", where ϵ' contains only a finite number of circles and ϵ" has extent less that s. If the ray $z = re^{i\theta}$ meets infinitely many circles of ϵ, then this ray meets ϵ" and the set of such θ has measure at most s, i.e. measure zero.

From the above theorem, there is a constant c and a measure zero subset Z of the unit circle such that $0 \leq M(r) - u(re^{i\theta}) \leq c$ for every $\theta \notin Z$ and all $r > r(\theta)$.

Suppose $\alpha(u) > 1$, and fix $\theta \notin Z$, and the corresponding ray $\gamma(r) = re^{i\theta}$, $r > r(\theta)$ on which $0 \leq M(r) - u(re^{i\theta}) \leq c$. Then $\int_\gamma e^{-u}$ is bounded above and below by positive multiples of $\int_\gamma e^{-M}$. As $\frac{M(r)}{\log r} \to \alpha$, for any $\alpha_0 \in (1, \alpha(u))$ there is r_0 with $M(r) > \alpha_0 \log r$ for all $r > r_0$. Shortening γ to $\gamma > \gamma_0$, we get $\int_\gamma e^{-B} \leq \int_{r_0}^\infty r^{-\alpha_0} < \infty$ proving incompleteness of g. □

We now introduce a new notation based on u and $M(r,u) := \max\{u(z) : |z| = r\}$. Set $\mu(t,u) := M(e^t,u)$; when u is understood we simply write M, μ. Subharmonicity of u implies that μ is a convex function [25].

Hence μ has left and right derivatives everywhere, and they are equal outside a countable subset, and so the same holds for M. By the maximum principle [25], M is strictly increasing (except when u is constant), and hence the same is true for μ.

3.3 *Properties of* $S_1 = \{u \in C^\infty(\mathbb{R}^2) | \, \Delta u \geq 0, \alpha(u) \leq 1\}$

Lemma 3.3.1 *S_1 equals the set of smooth subharmonic functions u such that the metric $e^{-2u}g_0$ is complete.*

Proof If $u \notin S_1$, then $e^{-2u}g_0$ is incomplete by the previous lemma. Suppose $u \in S_1$ while $e^{-2u}g_0$ is incomplete, and we will show the contradiction. By incompleteness of g there is a smooth path γ in $\mathbb{R}^2$ going to infinity such that $\int_\gamma e^{-u} ds < \infty$. Now $u \leq M$ implies $\int_\gamma e^{-M} ds < \infty$. It is convenient to replace M, μ with nearby smooth functions with similar properties which is possible by a result of Azagra [1] as follows:

Let $U \subset \mathbb{R}^d$ be open and convex. For every convex function $f : U \to \mathbb{R}$ and every $\epsilon > 0$ there exists a real-analytic convex function $g : U \to \mathbb{R}$ such that $f - \epsilon \leq g \leq f$.

So there is a smooth convex function ν defined on $\mathbb{R}$ such that $\mu - 1 \leq \nu \leq \mu$. Note that $\frac{\nu(t)}{t} \to 1$ as $t \to \infty$.

For $r > 0$ set $N(r) := \nu(\log r)$; the function $(x, y) \to N(r)$ is subharmonic:

$$\triangle N = \nu''(t_x^2 + t_y^2) + \nu' \triangle t = \nu'' r^{-2}$$

Here t_x, t_y are partial derivatives of $t = \log r$; note that $\triangle t = 0$ while $t_x = \frac{x}{r^2}$ and $t_y = \frac{y}{r^2}$.
Set $d(r) := \frac{N(r)}{\log r} - 1$ so that $e^{-N(r)} = r^{-1-d(r)}$. Since $u \in S_1$, we get $u \notin S_\alpha$ for $\alpha < 1$ so that $d(r) \to 0$ as $r \to \infty$. Also $M - 1 \leq N \leq M$ so that $\int_\gamma r^{-1-d} ds = \int_\gamma e^{-N} ds < \infty$.

In deriving a contradiction it helps consider the following cases.
If $d' \geq 0$ for all large r, then since d converges to zero as $r \to \infty$, we must have $d \leq 0$ for large r, so after shortening γ we get

$$\int_\gamma r^{-1-d} ds \geq \int_\gamma r^{-1} ds = \infty$$

which is a contradiction.

If the sign of d' takes values ± 1 as $r \to \infty$, then there is a point where d' and d'' are both negative, which contradicts subharmonicity of N for $r > 1$ as

$$0 \leq \triangle N = N'' + \tfrac{N'}{r} = d'' \log r + d'(\tfrac{\log r}{r} + \tfrac{2}{r}).$$

It remains to deal with the case when $d' \leq 0$ for large r. Multiply the above expression by $r \log r$ to get

$$0 \leq r \log^2 r (d'' + d'(\tfrac{1}{r} + \tfrac{2}{r \log r})) = (r(\log r)^2 d')'$$

which integrates over $[\rho, r]$ to $d'(\rho)\rho \log^2 \rho \leq d'(r) r \log^2 r$. Since $d' \leq 0$ for all large r, we conclude that $c := d'(\rho)\rho \log^2 \rho$ is a nonnegative constant. Set $f(r) := -\frac{c}{\log c}$ so that $f' = \frac{c}{r \log^2 r} \leq d'$. Integrating $d' - f' \geq 0$ over $[r, R]$ gives $d(R) - f(R) \geq d(r) - f(r)$ and since $d(R)$, $f(R)$ tend to zero as $R \to \infty$, we get $d(r) \leq f(r)$ for all large r. Hence $\int_\gamma r^{-1-d} ds \geq \int_\gamma r^{-1-f} ds = e^c \int_\gamma r^{-1} ds = \infty$ which again is a contradiction. □

Remark If u is harmonic and nonconstant, then $e^{-2u} g_0$ is not complete and $\alpha(u) = \infty$.

Proof If $\alpha(u)$ is finite, then by rescaling we may assume that $\alpha(u) < 1$ so that $e^{-2u} g_0$ is complete by the previous lemma. Since u is harmonic, $e^{-2u} g_0$ has zero curvature, and hence $e^{-2u} g_0 = \phi^* g_0$ for some $\phi \in \mathrm{Diff}(\mathbb{R}^2)$. It follows that ϕ is conformal,and hence ϕ or its composition with the complex conjugation is affine. Therefore $\phi^* g_0$ is a constant multiple of g_0, hence u is constant. □

There is another way to prove the incompleteness of $e^{-2u} g_0$ for u harmonic and nonconstant.

Proof Since u is harmonic, it is the real part of an entire function f. Thus e^u is $|e^f|$ where e^f is an entire function with no zeros, and hence so is e^{-f}. Huber [24] proves that

An entire analytic function $w = f(z)$ is a polynomial if and only if there exists a positive number λ such that

$$\int_{\sigma} |f(z)|^{-\lambda}|dz| = +\infty$$

for every locally rectifiable path σ tending to infinity.

It implies there is a path going to infinity such that the integral of $|e^{-f}|^{-1} = e^{u}$ over the path is finite. (Actually, Huber's result applies to any non-polynomial entire function in place of e^{-f}, and his proof is much simplified when the entire function has finitely many zeros, as happens for e^{-f}; the case of finitely many zeros is explained on [31]. as following

Let $f(z)$ be an entire function, not a polynomial. Let $\lambda > 0$. Then there exists a locally rectifiable path C_{λ} tending to infinity, such that

$$\int_{C_{\lambda}} |f(z)|^{-\lambda}|dz| < \infty.$$

As remarked by Huber, there is no difficulty if $f(z)$ has only a finite number of zeros, so that $f(z) = P(z)\exp[g(z)]$, where P is a polynomial and g is entire. The function

$$w = \Phi(z) = \int_{0}^{z} e^{-\lambda g(z)} dz$$

is then entire and without critical points. If the inverse function $\Phi^{-1}(w)$ has no singular points, then it is also entire, so that $\Phi(z)$ has form $az + b$ and $f(z)$ is a polynomial; hence $\Phi^{-1}(w)$ must have singularities. In particular there must be a functional element of $\Phi^{-1}(w)$ which can be continued from $w = 0$ along a finite segment ending at a singularity w_0. The segment is mapped by $\Phi^{-1}(w)$ on a path C_{λ} in the $z-$plane, on which $z \to \infty$ as $w \to w_0$. Then

$$|w_0| = \int_{C_{\lambda}} |\tfrac{dw}{dz}||dz| = \int_{C_{\lambda}} |e^{g(z)}|^{-\lambda}|dz|.$$

Thus C_{λ} is the desired path if $P(z) \equiv 1$; by removing a finite portion of C_{λ}, one can ensure that $|P(z)| \geq 1$ on the remaining portion C'_{λ}, so that C'_{λ} is the desired path. □

Theorem 3.3.2 *S_{α} is a closed convex subset in the Fréchet space $C^{\infty}(\mathbb{R}^2)$.*

Proof Fist, we shall show S_α is convex.

If $u = su_1 + (1-s)u_0$ with $s \in [0,1]$ and $u_j \in S_\alpha$, then u is subharmonic,and $M(r,u) \leq sM(r,u_1) + (1-s)M(r,u_0)$, so dividing by $\log r$ and taking $r \to \infty$ yields $\alpha(u) \leq s\alpha(u_1) + (1-s)\alpha(u_0) \leq \alpha$. Therefore, $u \in S_\alpha$, i.e. S_α is convex.

Then we shall show S_α is closed. We consider the sequence.

Fix $u_j \in S_\alpha$ and a smooth function u such that $u_j \to u$ in $C^\infty(\mathbb{R}^2)$. Clearly u is subharmonic, and also $M(r,u_j) \to M(r,u)$ for each r, so $\mu(\cdot,u_j) \to \mu(\cdot,u)$ pointwise. Set $\mu_j := \mu(\cdot,u_j)$, $\mu := \mu(\cdot,u)$. Since μ_j, μ are convex, μ'_j, μ' exist outside a countable subset Σ[38], and the convergence $\mu_j \to \mu$ is uniform on compact sets, which easily implies that $\mu'_j \to \mu'$ outside Σ.

By convexity μ'_j, μ' are non-decreasing outside Σ. It follows that $\mu'_j \leq \alpha$ outside Σ for if $\mu'_j(t_1) = \alpha_1 > \alpha$, then $\mu'_j \geq \alpha_1$ for $t \geq t_1$, so integrating we get $\mu_j(t) - \mu_j(t_1) \geq \alpha_1(t-t_1)$ which contradicts

$$\lim_{t\to\infty} \frac{\mu_j(t)}{t} \leq \alpha.$$

Since $\mu'_j \to \mu'$ we get $\mu' \leq \alpha$ outside Σ. Integrating gives $\mu(t) \leq \alpha(t-t_0)+\mu(t_0)$ everywhere, so $u \in S_\alpha$. $\square$

Theorem 3.3.3 *If $\alpha > 0$, then S_α is not locally compact.*

Proof Let us fix and $u \in S_\alpha$ with $0 < \alpha(u) < \infty$, and suppose arguing by contradiction that u has a compact neighborhood V in S_α. Since u is not harmonic, there is a closed disk $D \subset \mathbb{R}^2$ such that $\Delta u > 0$. The Fréchet space is not locally compact because it is an infinite dimensional topological vector space [40]. Homogeneity of $C^\infty(D)$ implies that it contains no compact neighborhood. To get a contradiction we show that the restriction map $\delta : C^\infty(\mathbb{R}^2) \to C^\infty(D)$ takes V to a compact neighborhood of $u|_D$. Compactness of $\delta(V)$ follows from the continuity of δ. If $\delta(V)$ were not a neighborhood of $u|_D$, there would exist a sequence $u_i \to u$ in $C^\infty(\mathbb{R}^2)$ with $u_i|_D \in C^\infty(D) - \delta(V)$.

Let ϕ be a bump function with $\phi|_D \equiv 1$ and such that the support of ϕ lies in a compact neighborhood of D on which $\Delta u > 0$. Then $u + \phi(u_i - u) \in V$ for large i, and $u + \phi(u_i - u)|_D = u_i|_D$, which contradicts the assumption $u_i|_D \notin \delta(V)$.

A closed subset of a locally compact space is locally compact, so since S_α is closed in S_∞, we may assume α is finite. Fix any nonconstant $u \in S_\alpha$ and show that it has no compact neighborhood. Since $a < \alpha < \infty$ we know that u is not harmonic, so there is a disk closed D where $\triangle u > 0$. Now any sufficiently small C^∞ variation $\tilde{u}$ of u is in S_α provided $\tilde{u} - u$ is supported in D, and by a standard argument $\tilde{u}$'s cannot all lie in a compact neighborhood of u. □

Recall that complete Riemannian metrics on any manifold form a dense subset in the space of all Riemannian metrics, e.g. $e^{-2u}g_0$ is the endpoint of the curve $g_s := (s + e^{-2u})g_0$ where the metric g_s is complete for $s > 0$. This led to speculation in [9] that the set of u's for which $e^{-2u}g_0$ is complete and nonnegatively curved is " probably neither closed nor convex".

A Riemannian metric on $\mathbb{R}^2$ can be written as the form

$$ds^2 = E(\xi, \eta)d\xi^2 + 2F(\xi, \eta)d\xi d\eta + G(\xi, \eta)d\eta^2,$$

so it is can be thought of as a smooth map from $\mathbb{R}^2$ to the space of symmetric, positive definite bilinear forms.

Theorem 3.3.4 *If g is conformal to g_0, then there are unique $\phi \in \mathit{Diff}^+_{0,1}(\mathbb{R}^2)$ and $v \in C^\infty(\mathbb{R}^2)$ such that g equals $\phi^* e^v g_0$.*

Proof Any metric g conformal to g_0 can be written as $\phi^* e^f g_0$ where $\phi \in \mathrm{Diff}(\mathbb{R}^2)$ and

$f \in C^\infty(\mathbb{R}^2)$. Note that $\phi^* e^f g_0 = e^{f\circ\phi}\phi^* g_0$. To see uniqueness suppose $\phi_1^* e^{v_1} g_0 = \phi_2^* e^{v_2} g_0$ and rewrite it as

$$(\phi_1 \circ \phi_2^{-1})^* g_0 = e^{(v_2 - v_1)\circ\phi_1\circ\phi_2^{-1}} g_0$$

so that $\phi_1 \circ \phi_2^{-1}$ is a conformal automorphism of $\mathbb{R}^2$ that preserves orientation and fixes 0, 1. It follows that $\phi_1 \circ \phi_2^{-1}$ is the identity, then the above equality implies $v_1 = v_2$.

To prove existence recall that any diffeomorphism of $\mathbb{R}^2$ is isotopic either to the identity or to the reflection $z \to \bar{z}$. The metric $e^{f\circ\phi}\phi^* g_0$ does not change when we compose ϕ with an isometry of g_0, and composing ϕ with an affine map results in rescaling which can be subsumed into $f \circ \phi$ changing it by an additive constant. Thus composing with $z \to \bar{z}$ if needed, and with an affine map we can arrange ϕ to lie in $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$. □

According to the above lemma, any metric conformally equivalent to $\mathbb{R}^2$ can be written uniquely as $\phi^* e^{-2u} g_0$ where g_0 is the standard Euclidean metric, u is a smooth function, and $\phi \in \mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$, the group of self-diffeomorphisms of the plane fixing the complex numbers 0, 1 and isotopic to the identitty. From above theorem, the metric $e^{-2u} g_0$ is complete if and only if $\alpha(u) \leq 1$. So the map $(u, \phi) \to \phi^* e^{-2u} g_0$ defines a bijection

$$\Pi : S_1 \times \mathrm{Diff}^+_{0,1} \to \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2).$$

We equip S_α, $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ with the C^∞ topology. Then Π is a homeomorphism for $k = \infty$ from Theorem 3.3.5. By contrast, if k is finite, the Π is not a homeomorphism, because it factors as the composite of $\Pi :\ S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2) \to \mathcal{R}^\infty_{\geq 0}(\mathbb{R}^2)$ and **id**: $\mathcal{R}^\infty_{\geq 0} \to \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ and the latter map is a continuous bijection that is clearly not a homeomorphism.

Now we prove the following theorem:

Theorem 3.3.5 *Let k be a positive integer or $k = \infty$ and $\alpha \in (0, 1)$. If S_1 is given the $C^{k+\alpha}$ topology and $\mathit{Diff}^+_{0,1,}(\mathbb{R}^2)$ is given the $C^{k+1+\alpha}$ topology, then the map Π is a homeomorphism.*

Proof The map $\Pi(u, \phi) = \phi^* e^{-2u} g_0$ is a bijection.If $\phi \in \mathrm{Diff}^+_{0,1}(\mathbb{C})$ is given the $C^{k+1+\alpha}$ topology, then its differential varies in the $C^{k+\alpha}$ topology, which implies continuity of Π. It remains to show that Π^{-1} is continuous.

Write $g = \phi^* e^f g_0 = e^{f\circ\phi} \phi^* g_0$ with ϕ is an orientation-preserving diffeomorphism of $\mathbb{C}$ that fixes 0, 1. The Jacobian of ϕ equals $|\phi_z|^2 - |\phi_{\bar{z}}|^2$, and since ϕ is orientation-preserving, we get $|\frac{\phi_{\bar{z}}}{\phi_z}| < 1$. Computing

$$\frac{\phi^* g_0}{|\phi_z|^2} = \frac{|d\phi|^2}{|\phi_z|^2} = |dz + \frac{\phi_{\bar{z}}}{\phi_z} d\bar{z}|^2$$

gives $g = e^{f\circ\phi} |\phi_z|^2 |dz + \frac{\phi_{\bar{z}}}{\phi_z} d\bar{z}|^2$. Also we can write $g = Edx^2 + 2Fdxdy + Gdy^2$ as $\lambda |dz + \mu d\bar{z}|^2$, where

$$\lambda = \frac{1}{4}(E + G + 2\sqrt{EG - F^2}) \text{ and } \mu = \frac{E - G + 2iF}{4\lambda}.$$

Positive definiteness of g easily implies $|\mu| < 1$ and $\lambda > 0$. We know $E,\ F,\ G$ are of class $C^{k+\alpha}$, then the products EG and F^2 are also of class $C^{k+\alpha}$. So μ and λ depend in $C^{k+\alpha}$ on $\bar{g}$. Comparing the two descriptions of g we see that $\phi_{\bar{z}} = \mu\phi_z$, that is, ϕ solves the Beltrami equation with dilatation μ. Futhermore, $\lambda = e^{f\circ\phi} |\phi_z|^2$ so that $f = \log(\lambda |\phi_z|^{-2}) \circ \phi^{-1}$.

We are going to show that if a sequencee of metrics $g_l = \phi_l^* e^{f_l} g_0 = \lambda_l |dz + \mu_l d\bar{z}|^2$, with $\phi_l \in \mathrm{Diff}^+_{0,1,\infty}(\mathbb{C})$, converges to g uniformly on compact subsets in the $C^{k+\alpha}$ topology, then ϕ_l, f_l converge to ϕ, f, respectively, in the same topology. A key ingredient is the smooth dependence of ϕ on μ established by Earle-Schatz [17] .

To state their result let U, U' be domains in S^2 whose boundaries are embedded circles, and let a_1, a_2, a_3 and a'_1, a'_2, a'_3 be two triples of distinct points on ∂U and $\partial U'$ respectively. Recall that given a C^∞ function $\beta : U \to \mathbb{C}$ with $|\beta| \le k < 1$ for some constant k, there is a unique homeomorphism $w^\beta : \bar{U} \to \bar{U}'$ that restricts to a diffeomorphism $U \to U'$, maps each a_k to a'_k, and solves the Beltrami equation with dilatation β, see e.g.[33, p.183, 194] for existence and uniqueness and [41, Theorem 2.2 in Section 4 of Chapter 2] for regularity.

The continuity Theorem of Earle-Schatz states that varying β in the $C^{k+\alpha}$ topology results in varying w^{β} in $C^{k+1+\alpha}$ topology. Strictly speaking, Earle-Schatz assume that U, U' equal the upper half plane, and the two triples of points equal $0, 1, \infty$, but the conformal invariance of the Beltrami dilatation, together with the Riemann mapping theorem give the same conclusion for any U, U' as above.

The Continuity Theorem does not immediately apply in our setting, where $U = \mathbb{C} = U'$ and $|\beta|$ is not bounded way from 1. Instead we use the theorem locally, on an arbitrary disk $B_t = \{z \in \mathbb{C} : |z| < t\}$, but then the difficulty is that the domain $\phi(B_t)$ may change as the diffeomorphism ϕ varies with μ. Below we resolve the issue by adjusting $\phi(B_t)$ via an ambient diffeomorphism that is the identity on a given compact set. Exhausting $\mathbb{C}$ by such compact sets yields the smooth dependence of ϕ on μ.

Let K be a compact subset of $\mathbb{C}$. Let $\tilde{g} = \tilde{\phi}^* e^{\tilde{f}} g_0 = \tilde{\lambda}|dz + \tilde{\mu} d\bar{z}|^2$ be a metric that is $C^{k+\alpha}$ close to $g = \phi^* e^f g_0 = \lambda|dz + \mu d\bar{z}|^2$ over K, where $\tilde{\phi}, \phi \in \mathrm{Diff}^+_{0,1}(\mathbb{C})$. Choose s with $\tilde{\phi}(K) \subset B_s$. The domains $\phi(B_r), r > 0$ exhaust $\mathbb{C}$, and so do the domains $\tilde{\phi}(B_r)$, which allows us to find r with $\bar{B}_s \subset \phi(B_r) \cap \tilde{\phi}(B_r)$.

It is easy to construct an orientation-preserving self-diffeomorphism h of $\mathbb{C}$ that maps $\tilde{\phi}\bar{B}_r$ onto $\phi(\bar{B}_r)$, equals the identity on B_s, and has the property that $h \circ \tilde{\phi}$ and ϕ agree at the points $-r, ir, r$ of ∂B_r. (Indeed, $\tilde{\phi}(\partial B_r)$, $\phi(\partial B_r)$ are homotopic smooth simple closed curves in the open annulus $\mathbb{C} - \bar{B}_s$, and hence they can be moved to each other by a compactly supported ambient isotopy of the annulus. The identity component of $\mathrm{Diff}(S^1)$ acts transitively on the set of triples of distinct points of S^1, e.g. if S^1 is identified to the boundary of the upper half plane, then the map $x \to \frac{(x-a)(c-b)}{(x-b)(c-a)}$ takes a, b, c to $0, \infty, 1$, respectively, and preserves an orientation, and hence is isotopic to the identity of S^1. So given two triples of points in $S^1 \times 0$ there is a compactly supported isotopy of $S^1 \times \mathbb{R}$ that takes one triple to the other one. Here we identity $S^1 \times \mathbb{R}$, $S^1 \times 0$ with $\mathbb{C} - \tilde{B}_s$, $\phi(\partial B_r)$, respectively. Combining the two isotopies, and extending the result by the identity on $\tilde{B}_s$ yields the desired h).

Since $g|_K$, $\tilde{g}|_K$ are $C^{k+\alpha}$ close, so are the dilations of ϕ_K, $\tilde{\phi}|_k = h \circ \tilde{\phi}|_K$. Thus $\phi|_{B_r}$, $h \circ \tilde{\phi}|_{B_r}$ are diffeomorphisms of B_r onto $\phi(B_r)$ whose dilatations are $C^{k+\alpha}$ close on K. The absolute values of the dilatations are less than 1 (as the diffeomorphisms are orientation-preserving), and hence are bounded away from 1 by compactness of $\bar{B}_r$. Now the Continuity Theorem implies that $\phi|_{B_r}$, $h \circ \tilde{\phi}|_{B_r}$ are $C^{k+1+\alpha}$ close over K. It follows that $\lambda|\phi_z|^{-2}$, $\tilde{\lambda}|\tilde{\phi}_z|^{-2}$ are $C^{k+\alpha}$ close over K.

Thus if a sequence of metrics g_l converges to g uniformly on compact subsets in the $C^{k+\alpha}$ topology, then the corresponding diffeomorphisms ϕ_l converge to ϕ uniformly on compact subsets in the $C^{k+1+\alpha}$ topology. Since $\mathrm{Diff}^+_{0,1}(\mathbb{C})$ with the $C^{k+1+\alpha}$ topology is a topological group we also have the $C^{k+1+\alpha}$ convergence of ϕ_l^{-1} to ϕ^{-1}. In lemma 2.2 of [6], it states that " If G is of class $C^{k-1,\alpha}$ and $H \in C^k$, $k \geq 1$, then $G \circ H$ is of class $C^{k-1,\alpha}$. This implies that $\log(\lambda_l|(\phi_l)_z|^{-2}) \circ \phi_l^{-1}$ converges to $\log(\lambda|\phi_z|^{-2}) \circ \phi^{-1}$ in the $C^{k+\alpha}$ topology, which completes the proof. □

The existence of the parametrization

$$\Pi : S_1 \times \mathrm{Diff}^+_{0,1} \to \mathcal{R}^k_{\geq 0}(\mathbb{R}^2),$$

has nontrivial consequences because the parameter space $S_1 \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ is homeomorphic to l^2 by the following results of infinite dimensional topology:

1. Any closed convex non-locally-compact subset of a separable Fréchet space is homeomorphic to l^2, the separable Hilbert space. This implies $S_\alpha \subset C^\infty(\mathbb{R}^2)$ with $\alpha \neq 0$ is

homeomorphic to l^2.

2. $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ is homeomorphic to l^2 [44].

3. l^2 is homeomorphic to $(-1,1)^{\mathbb{N}}$, the countably infinite product of open intervals [2].

Here are basic topological properties of $\mathcal{R}^k_{\geq 0}(\mathbb{R}^2)$ (with k finite or infinite).

Theorem 3.3.6 *Let N, M be smooth manifolds and $0 \leq k \leq \infty$. With the $C^{k+\alpha}$ topology the space $C^\infty(M,N)$ is*

(1) separable and metrizable,

(2) completely metrizable if $k = \infty$,

(3) a Fréchet space if N is a Euclidean space and $k = \infty$.

Proof Case $\alpha = 0$. In fact, the space $C^\infty(M,N)$ sits in $C^k(M,N)$ which embeds as a closed subset into $C^0(M, J^k(M,N))$ where $J^k(M,N)$ is the space of k-jets which is a C^0 manifold. Separability is implied by having a countable basis, and since the latter property is inherited by subspaces it suffices to show that $C^0(M, J^k(M,\mathbb{R}))$ has a countable basis, but in general if the spaces X, Y have a countable basis and if X is locally compact, then $C^0(X,Y)$ with the compact-open topology has a countable basis [15]. Similarly, metrizability is inherited by subspaces, and complete metrizability is inherited by closed subspaces, while $C^0(X,Y)$ with the compact-open topology is completely metrizable whenever X is locally compact and Y is completely metrizable [26]. A proof of (3) can be found in [40].

Case $\alpha \in (0,1)$. We only need to show property (1). Consider the identity map:

$$id : C^{\infty,k+1}(M,N) \to C^{\infty,k+\alpha}(M,N).$$

The map id is continuous. The space $C^{\infty,k+1}(M,N)$ is separable, i.e. it has a countable dense subset. Since id is continuous, this subset is also dense in $C^{\infty,k+\alpha}(M,N)$. So $C^{\infty,k+\alpha}(M,N)$ is separable.

Now we want to show the space is metrizable.

One can embed N into some Euclidean space $\mathbb{R}^l$ as a closed submanifold. Then $C^{k+\alpha}(M,N)$

can be identified with a subset of $C^{k+\alpha}(M, \mathbb{R}^l)$ consisting of maps with image in N. The set $C^{k+\alpha}(M, \mathbb{R}^l)$ is metrized as in [29, p. 62] via a countable exhaustion of M by compact domains. Then $C^{k+\alpha}(M, N)$ becomes a subset of a metrizable space and hence it is metrizable.

□

3.4 Connectedness properties of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$

Our first application demonstrates that any two metrics can be deformed to each other in a variety of ways, while bypassing a given countable set:

Theorem 3.4.1 *If K is a countable subset of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ and X is a separable metrizable space, then for any distinct points $x_1, x_2 \in X$ and any distinct metrics g_1, g_2 in $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ there is an embedding of X into $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ that takes x_1, x_2 to g_1, g_2, respectively.*

The above theorem hinges on the following facts:

1. Like any continuous one-to-one map to a Hausdorff space, the map Π restricts to a homeomorphism on every compact subset, e.g. the Hilbert cube.
2. Every separable metrizable space embeds into the Hilbert cube.

Theorem 3.4.2 *Every separable metrizable space X is homeomorphic to a subset of the Hilbert cube $\mathcal{H}$.*

Proof Let d be a metric on X consistent with its topology and let $x_{k k\geq 1}$ be a countable dense subset. For each $k \geq 1$ define the function

$$f_k(x) = \min\{1, d(x, x_k)\}$$

and let $f : X \to \mathcal{H}$ be defined by $f(x) = (f_k(x))_{k\geq 1}$. Each f_k is continuous and so f is continuous. If $f(x) = f(y)$, then we can find a subsequence $x_{k_n n\geq 1}$ of $x_{k k\geq 1}$ such that $x_{k_n} \to x$ and so $d(x_{k_n}, y) \to 0$, from which it follows that $x = y$, i.e. f is one-to-one.

It remains to show that f^{-1} is continuous, i.e. $f(y_n) \to f(y)$ in $\mathcal{H}$, implies that $y_n \to y$ in X. If $f(y_n) \to f(y)$, given $\epsilon > 0$, choose x_k such that $d(y, x_k) < \epsilon$. Since $d(y_n, x_k) \to d(y, x_k)$ as $n \to \infty$, for $n \geq 1$ sufficiently large, we have that $d(y_n, x_k) < \epsilon$. Hence $d(y_n, y) < 2\epsilon$ for $n \geq 1$ large, i.e. $y_n \to y$ in X. □

3. The complement in l^2 of the countable union of compact sets is homeomorphic to l^2, and hence contains an embedded Hilbert cube.

The proof of the theorem is as following:

Proof As was explained, $S_1\times$ $\mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$ is homeomorphic to l^2, and since $\Pi^{-1}(K)$ is a countable union of compact sets, its complement is homeomorphic to l^2, which contains the Hilbert cube and hence an embedded copy of X. Applying an affine self-homeomorphism of l^2 one can ensure that the embedding maps x_1, x_2 to $\Pi^{-1}(g_1), \Pi^{-1}(g_2)$, repectively. Since X sits in an embedded copy of the Hilbert cube (a compact set), and since $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is Hausdorff, the restriction of Π to the embedded copy of X is a homeomorphism onto its image, which has desired properties. $\square$

By dimension we mean the covering dimension.

A space is finite dimensional if and only if it embeds into a Euclidean space from the embedding theorem [16] stated as follows.

Theorem 3.4.3 *Every separable metric space X such that* $0 \leq dimX \leq n$ *is embeddable in Euclidean (2n+1)-space* R^{2n+1}*; if, moreover, the space X is compact, then all homeomorphic embeddings of X in* R^{2n+1} *form a* $G_\delta-$*set dense in the function space* $(\mathbb{R}^{2n+1})^X$.

Definition 3.4.4 *A space is continuum-connected if every two points lie in a continuum (a compact connected space); thus a continuum-connected space is connected but not necessarily path-connected.*

By theorem 3.1.15, Let X be the separable metrizable space $\mathbb{R}^n$, then any two metrics lie in an embedded copy of $\mathbb{R}^2$ for any n. Since $\mathbb{R}^2$ cannot be separated by subspace of codimension ≥ 2 we conclude:

Theorem 3.4.5 *The complement of every finite dimensional subspace of* $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ *is continuum-connected. The complement of every closed finite dimensional subspace of* $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ *is path-connected.*

Proof Let S be a finite dimensional subspace of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$. Fix two points g_1, g_2 in the complement of S. The above theorem 3.1.15 implies g_1, g_2 lie in a subspace X of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ that is homeomorphic to $\mathbb{R}^n$ for any n, especially with $n \geq \dim(S) + 2$.
From theorem 1.1.2 of [16], for every subspace M of a regular space S we have $\operatorname{ind} M \leq \operatorname{ind} X$. So we have $\operatorname{ind}(S \cap X) \leq \operatorname{ind} S$. Since $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is separable and metrizable, its subspace is also separable and metrizable, which means $\operatorname{ind}(S \cap X) = \dim(S \cap X)$ and $\operatorname{ind} S = \dim S$. So $S \cap X$ has $\dim(S \cap X) \leq \dim S$, then its codimension in $X \cong \mathbb{R}^n$ is ≥ 2. Then the points g_1, g_2 lie in a continuum in X that is disjoint from S by theorem 1.8.19 in

[16] stated as follows.

”Mazurkiewicz’s theorem. If a subset M of a region $G \subset \mathbb{R}^n$ satisfies the inequality $\operatorname{ind} M \leq n-2$, then M does not cut G, i.e., for every pair of points $x, y \in G \setminus M$ there exists a continuum $C \subset G \setminus M$ which contains x and y.”

If S is closed one can say more: Suppose that X is homeomorphic to S^n with $n \geq \dim(S)+2$, so $S \cap X$ is a closed subset of X of codimension ≥ 2. By the cohomological characterization of dimension [16], the space $S \cap X$ has trivial Čech cohomology in dimensions $> n-2$, hence by the Alexander duality $X \setminus S$ is path-connected, giving a path in $X \setminus S$ joining g_1, g_2. $\square$

3.5 Moduli space of complete non negatively curved metrics

Let $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ denote the moduli space of complete nonnegatively curved metrics, i.e. the quotient space of $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ by the Diff$(\mathbb{R}^2)$-action via pullback. The moduli space $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is rather pathological, e.g. it is not a T_1 space (in the proof of Π^{-1} is not continuous we exhibit a non-flat metric $g \in \mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ whose isometry lies in every neighborhood of the isometry class of g_0).

Proposition 3.5.1 *There is an isometry class in $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ that lies in every neighborhood of the isometry class of g_0.*

Proof Let $f : [0, \infty) \to [0, \infty)$ be a convex smooth function with $f(x) = 2$ for $x \in [0, 1]$ and $f(x) = x$ if $x \geq 3$. The surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $x \to (x, 0, f(x))$ about the z-axis defines a complete metric g on $\mathbb{R}^2$ of nonnegative curvature. For each r there is a metric r-ball in $(\mathbb{R}^2, g)$ that is isometric to the Euclidean r-ball B_r about the origin in $(\mathbb{R}^2, g_0)$. Extending the isometry to a self-diffeomorphism ϕ_r of $\mathbb{R}^2$ gives $\phi_r^* g|_{B_r} = g_0|_{B_r}$ so $\phi_r^* g$ converge to g_0 uniformly on compact subsets as $r \to \infty$. □

Consider the map $S_1 \to \mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ sending u to the isometry class of $e^{-2u} g_0$. Its fibers lie in the orbits of a Conf$(\mathbb{R}^2)$-action of $C^\infty(\mathbb{R}^2)$, so each fiber is the union of countably many finite dimensional compact sets, which by dimension theory arguments easily implies:

Theorem 3.5.2 *The complement of a subset S of $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is path-connected if S is countable, or if S is closed and finite dimensional.*

Proof Let $q : S_1 \to \mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ denote the continuous surjection sending u to the isometry class of $e^{-2u} g_0$.

If S is countable, it suffices to show that every fiber of q is the union of countably many compact sets because then the complement of a countable subset in $\mathcal{M}^{k+\alpha}_{\geq 0}(\mathbb{R}^2)$ is the image of S_1 with a countable collection of compact subsets removed, which is homeomorphic to l^2 [13], and of course the continuous image of l^2 is path-connected.

A function $v \in S_1$ lies in the fiber over the isometry class of $e^{-2u} g_0$ if and only if $e^{-2v} g_0 = \psi^* e^{-2u} g_0 = e^{-2u\circ\psi} \psi^* g_0$ for some $\psi \in \mathrm{Diff}(\mathbb{R}^2)$. Note that ψ necessarily lies in $\mathrm{Conf}(\mathbb{R}^2)$, the group of conformal automorphism of $\mathbb{R}^2$, i.e. either ψ or $r\psi$ equals $z \to az + b$ for some $a, b \in \mathbb{C}$, where $a \neq 0$ and $r(z) = \bar{z}$. Since $\psi^* g_0 = |a|^2 g_0$, we conclude that $v = u \circ \psi - \log |a|$. In summary, $v, u \in S_1$ lie in the same fiber if and only if $v = u \circ \psi - \log |a|$ where either ψ or $r\psi$ equals $z \to az + b$ with $a \neq 0$. Thus the fiber through u is the intersection of S_1 and the image of the continuous map $o : \mathrm{Conf}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ sending ψ to $u \circ \psi - \log |a|$. Since $\mathrm{Conf}(\mathbb{R}^2)$ is a Lie group, it is the union of countably many compact sets, and hence so is its image under any continuous map. Since S_1 is closed, every fiber is the union of countably many compact sets.

Now suppose that S is closed and finite dimensional. Let $\hat{S}$ be the q-preimage of S. Fix two points g_1, g_2 in the complement of S, which are q-images of $u_1, u_2 \in S_1$, respectively. By the previous theorem we may assume that u_1, u_2 lie in an embedded copy $\hat{Q}$ of the Hilbert cube. It is enough to show that $\hat{Q} \cap \hat{S}$ is finite dimensional in which case $\hat{Q} \setminus \hat{S}$ is path-connected by the proof of the previous theorem; in fact, much more is true: the complement to any finite dimensional closed subset of the Hilbert cube is acyclic from the following lemma proved in [30].

" For $X \cong Q$ or $X \cong s$, where Q is the Hilbert cube and $s = (-1, 1)^\infty$, if $A \subset X$ is an open cube and K is a finite-dimensional closed subset of X, then $A - K$ has the homology of a point."

Since S is closed, $\hat{Q} \cap \hat{S}$ is compact, so the restriction of q to $\hat{Q} \cap \hat{S}$ is a continuous surjection $\hat{q} : \hat{Q} \cap \hat{S} \to q(\hat{Q}) \cap S$ of compact separable metrizable spaces, and in particular a closed map, which is essential for what follows.

Since the target of $\hat{q}$ lies in S, it is finite dimensional. Finite-dimensionality of the domain of $\hat{q}$ would from a uniform upper bound on the dimension of the fiber of $\hat{q}$ [16]. In [16], it says "If $f : X \to Y$ is a closed mapping of a separable metric space X to a separable metric space Y and there exists an integer $k \geq 0$ such that $\operatorname{ind} f^{-1}(y) \leq k$ for every $y \in Y$, then $\operatorname{ind} X \leq \operatorname{ind} Y + k$.

Each fiber lies in the image of the orbit map $o : \mathrm{Conf}(\mathbb{R}^2) \to C^\infty(\mathbb{R}^2)$ desired above. Since $\mathrm{Conf}(\mathbb{R}^2)$ is a Lie group, it is the union of countably many l-dimensional compact domains (here $l = 4$).

Restricting o to each such domain is a continuous map from a compact space to the Hausdorff space, which is therefore a closed map, and hence the dimension of the target is $< 2l$, which is the sum of the dimension of the domain and the largest dimension of the fiber [16]. By the sum theorem for the dimension of a countable union of closed subsets [16] cited as follows, the image of o has dimension $< 2l$.
" If a separable metric space X can be represented as the union of a sequence $F_1,\ F_2, ...$ of subspaces such that $\operatorname{ind} F_i\ \leq n$ and F_i is an $F_\sigma - set$ for $i = 1, 2, ...$, then $\operatorname{ind} X\ \leq n$, where $F_\sigma - sets$ are defined as countable unions of closed sets." □

Remark. The previous three theorems yield a deformation between any two given metric g_1, g_2 that runs in a separable metrizable space, a continuum, or a path, and we now show that this deformation can be arranged to bypass any given set of complete flat metrics $\mathcal{F}$. We do so in the setting of the first theorem; the other two proofs are similar. Set $P_\alpha := S_\alpha \times \mathrm{Diff}^+_{0,1}(\mathbb{R}^2)$, which we identify with l^2. Flat metrics are parametrized by P_0 which is a closed linear subset of infinite infinite codimension in P_1. If $P'_\alpha := P_\alpha\ \Pi^{-1}(g_1), \Pi^{-1}(g_2)$, the P'_0 has property Z in P_1', see [4], so that theorem 3 in [4] implies that $P_1 \setminus P_0'$ is homeomorphic to P_1, after which the proof is finished as in the first theorem.

3.6 $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ is weakly contractible for $\alpha \in (0,1)$

Definition 3.6.1 *A topological space X is said to be locally contractible if it satisfies the following equivalent conditions:*

1. It has a basis of open subsets each of which is a contractible space under the subspace topology,

2. For every $x \in X$ and every open subset $V \ni x$ of X, there exists an open subset $U \ni x$ such that $U \subset V$ and U is a contractible space in the subspace topology from V.

Definition 3.6.2 *A topological space is said to be weakly contractible if all of its homotopy groups are trivial.*

Theorem 3.6.3 $\mathrm{Diff}^{+}_{0,1}(\mathbb{R}^2)$ *equipped with $C^{k+\alpha}$ topology is contractible.*

Proof Consider $\mathbb{R}^2$ as $\mathbb{C}$, so diffeomorpisms in $\mathrm{Diff}^{+}_{0,1}(\mathbb{R}^2)$ are self-maps of $\mathbb{C}$.
Define a family of self-maps $H_t(f)(z)$ of $\mathbb{C}$ as $\frac{f(tz)}{f(t)}$ if $t \in (0,1]$ and as $H_0(f)(z) = I$ for $t = 0$.
Indeed, $H_1(z) = \frac{f(z)}{f(1)} = f(z)$. Also as $t \to 0$, we have

$$\lim_{t\to 0} \frac{f(tz)}{f(t)} = \lim_{t\to 0} \frac{f(tz)}{t} \times \frac{t}{f(t)} = \frac{zf'(0)}{f'(0)} = z.$$

From this, one easily sees that H is continuous simultaneously at t and z.
The map H is continuous even when $\mathrm{Diff}^{+}_{0,1}(\mathbb{C})$ is given the $C^{k+1+\alpha}$ topology. □

Another equivalent definition of weakly contractibility of space X is that any continuous map $f : S^n \to X$ can be extended to a continuous map $\hat{f} : D^{n+1} \to X$ such that $\hat{f}|_{S^n} = f$.

Let $\mathrm{Diff}^{+,\infty,k+\alpha}_{0,1}(\mathbb{R}^2)$ denote the set $\mathrm{Diff}^{+}_{0,1}(\mathbb{R}^2)$ equipped with $C^{k+\alpha}$ topology.
Let $\mathrm{Diff}^{+,k+\alpha}_{0,1}$ be the group of $C^{k+\alpha}$ self-diffeomorphisms of the plane fixing the complex numbers 0, 1 and isotopic to the identity. And let $\mathrm{Diff}^{+,k+\alpha,k+\alpha}_{0,1}$ be $\mathrm{Diff}^{+,k+\alpha}_{0,1}$ equipped with $C^{k+\alpha}$ topology.

Theorem 3.6.4 $\mathcal{R}^{k+\alpha}_{\geq 0}(\mathbb{R}^2) \setminus K$ *is weakly contractible for $\alpha \in (0,1)$.*

Proof It suffices to show any continuous map

$$f : S^n \to (S_1 \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(\mathbb{R}^2)) \setminus K$$

can be extended to a continuous map

$$\hat{f} : D^{n+1} \to (S_1 \times \mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(\mathbb{R}^2)) \setminus K \text{ such that } \hat{f}|_{S^n} = f.$$

Here S_1 is equipped with $C^{k+\alpha}$ topology.

Denote f as (f_1, f_2), where f_1 is the restriction of f onto S_1 and f_2 is the restriction of f onto $\mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(\mathbb{R}^2)$.

Since S_1 is convex, it is weakly contractible. Let K_1 and K_2 denote the projection of K onto S_1 and $\mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(\mathbb{R}^2)$ respectively. Since S_1 is weakly contractible, the map f_1 can be extended to continuous maps $F_1 : D^{n+1} \to S_1$.

The following two lemmas use different methods to prove the theorem. The first lemma shows $S_1 \setminus K_1$ is weakly contractible and the second one shows $\mathrm{Diff}^{+,\infty,k+1+\alpha}_{0,1}(\mathbb{R}^2) \setminus K_2$ is weakly contractible. □

Lemma 3.6.5 *Let C be an infinite-dimensional convex subset in a Fréchet space and let K be a compact subset of C. Then $C \setminus K$ is weakly contractible. In particular, this applies to L_1 and S_1.*

Proof Let $f : S^n \to C \setminus K$ be a continuous map. The closure $\bar{C}$ of C is homeomorphic to l^2, so $l^2 \setminus K$ is homeomorphic to l^2, and hence contractible. Hence f extends to a continuous map $F : D^{n+1} \to \bar{C} \setminus K$. The complement of any compact set is homotopy dense in l^2, so we can assume that $F(D^{n+1})$ and K are disjoint. Finally, C is homotopy dense in $\bar{C}$ [5, exercises 12c and 13 in section 1.2], so we can push $F(D^{n+1})$ into C. Both operations move $F(S^n)$ only slightly, so since $f(S^n)$ and K are disjoint compact, they remain disjoint.

Lemma 3.6.6 *Let G be a non-locally compact contractible metrizable group. Then any compact set K in G has a weakly contractible complement, and any map $F : D^m \to G$ can be uniformly approximated by maps whose range misses K. In particular, this applies to* $\mathrm{Diff}_{0,1,\infty}^{+,\infty,k+1+\alpha}(S^2)$ *and* $\mathrm{Diff}_{0,1}^{+,\infty,k+1+\alpha}(\mathbb{R}^2)$.

Proof Any continuous map $f : S^{m-1} \to G \backslash K$ extends to a continuous map $F : D^m \to G$, so it remains to push the map into $G \backslash K$. Let $K_0 = F(D^m) \cup K$. Let $R = \{g \in G : g(K_0) \cap K_0 \neq \emptyset\}$ where $g(K_0)$ refers to the left G-action on itself. Note that R is compact for if $g_i \in R$ and $g_i(x_i) = y_i$ with $x_i, y_i \in K$, then $g_i = y_i x_i^{-1}$ which subconverges. Since G is not locally compact and homogeneous, no point of G has a compact neighborhood, and in particular R is not a neighborhood of the identity. So there is a sequence $\phi_i \in G \setminus R$ that converges to the identity. Then $\phi_i \circ F$ approximates F and misses K.

We are going to apply the above result to the diffeomorphism group (or rather its subgroup of diffeomorphisms that are isotopic to the identity and fix some finite set). The group is metrizable but not completely metrizable. Hence it is not a Lie group. On the other hand by the solution of the Hilbert-Smith conjecture any locally compact group of C^2 diffeomorphisms acting continuously and effectively on a manifold must be a Lie group. Thus G cannot be locally compact, see [22] and references therein. □

APPENDIX A

DIMENSION

Definition A.0.7 *Let X be a set and $\mathcal{A}$ a family of subsets of X. By the order of the family $\mathcal{A}$ we mean the largest integer n such that the family $\mathcal{A}$ contains $n+1$ sets with a non-empty intersection; if no such integer exists, we say that the family $\mathcal{A}$ has order ∞.*

Thus, if the order of a family $\mathcal{A} = \{A_s\}_{s\in S}$ equals n, then for each $n+2$ distinct indexed $s_1, s_2, ..., s_{n+2} \in S$ we have $A_{s_1} \cap A_{s_2} \cap ... \cap A_{s_{n+2}} = \emptyset$.

Let us recall that a cover $\mathcal{B}$ is a $refinement$ of another cover $\mathcal{A}$ of the same space, in other words $\mathcal{B}$ refines $\mathcal{A}$, if for every $B \in \mathcal{B}$ there exists an $A \in \mathcal{A}$ such that $B \subset A$. Clearly, every subcover $\mathcal{A}_0$ of $\mathcal{A}$ is a refinement of $\mathcal{A}$.

Definition A.0.8 *To every normal space X one assigns the covering dimension of X, denoted by dimX, which is an integer larger than or equal to -1 or the "infinite number ∞"; the definition of the dimension function dim consists in the following conditions:*

(1) $\dim X \leq n$*, where n = -1, 0, 1, ..., if every finite open cover of the space X has a finite open refinement of order $\leq n$;*

(2) $\dim X = n$ *if* $\dim X \leq n$ *and* $\dim X > n-1$*;*

(3) $\dim X = \infty$ *if* $\dim X > n$ *for* $n = -1, 0, 1, ...$

Definition A.0.9 *To every regular space X one assigns the small inductive dimension of X, denoted by indX, which is an integer larger than or equal to -1 or the "infinite number" ∞; the definition of the dimension function ind consists in the following conditions:*

(1) $\operatorname{ind} X = -1$ *if and only if* $X = \emptyset$*;*

(2) $\operatorname{ind} X \leq n$*, where* $n = 0, 1, ...,$ *if for every point $x \in X$ and each neighborhood $V \subset X$ of the point x there exists an open set $U \subset X$ such that*

$$x \in U \subset V \text{ and } \operatorname{ind} \operatorname{Fr} U \leq n-1;$$

(3) $\operatorname{ind} X = n$ *if* $\operatorname{ind} X \leq n$ *and* $\operatorname{ind} X > n-1$*, i.e. the inequality* $\operatorname{ind} X \leq n-1$ *does not hold:*

(4) $\operatorname{ind} X = \infty$ *if* $\operatorname{ind} X > n$ *for* $n = -1, 0, 1, \ldots$

Note that for separable metrizable spaces the covering dimension equals the small inductive dimension.

APPENDIX B

C^K- TOPOLOGY AND HÖLDER SPACE

B.1 C^k-topology

Definition B.1.1 *If M and N are C^r manifolds, $C^r(M,N)$ denotes the sets of C^r maps from M to N. At first we assume r is finite.*
The weak or " compact-open C^r" topology on $C^r(M,N)$ is generated by the sets defined as follows. Let $f \in C^r(M,N)$. Let $(\phi U), (\psi, V)$ be charts on M, N; let $K \subset U$ be a compact set such that $f(K) \subset V$; let $0 < \epsilon \leq \infty$. Define a weak subbasic neighborhood [26]

$$\mathcal{N}^r(f; (\phi, U), (\psi, V), K, \epsilon)$$

to be the set of C^r maps $g : M \to N$ such that $g(K) \subset V$ and

$$\|D^k(\psi f \phi^{-1})(x) - D^k(\psi g \phi^{-1})(x)\| < \epsilon$$

for all $x \in \phi(K)$, $k = 0, ..., r$. This means that the local representations of f and g, together with their first k derivatives, are within ϵ at each point of K. The compact-open C^r topology on $C^r(M,N)$ is generated by these sets.

We now define the spaces $C^\infty_W(M,N)$. The weak topology on $C^\infty(M,N)$ is simply the union of the topologies induced by the inclusion maps $C^\infty(M,N) \to C^r_W(M,N)$ for r finite.

It is convenient to redefine the topologies on $C^r(M,N)$ in a way which avoids coordinate charts. $C^r(M,N)$ will be identified with a subset of $C^0(M, J^r(M,N))$ where $J^r(M,N)$[26] is the manifold of r-jets of maps from M and N.

In this way $C^r(M,N)$ becomes a set of continuous maps. We denote by $C(X,Y)$ the set of continuous maps from a space X to a space Y. The compact open topology on $C(X,Y)$ is generated by the sub base comprising all sets of the form

$$\{f \in C(X,Y) : f(K) \subset V\}$$

where $K \subset X$ is compact and $V \subset Y$ is open.

We briefly use r-jets to explain the notion of C^k-convergence for C^k-maps between manifolds.

First let us fix some notation. By M, N we denote ($C^\infty-$) manifolds of dimension m, n respectively. Charts are denoted by (ϕ, U), where ϕ is the map and U is the domain. By $\mathcal{L}^r(\mathbb{R}^m, \mathbb{R}^n)$ we mean the space of r-linear maps

$$\underbrace{\mathbb{R}^m \times \cdots \times \mathbb{R}^m}_{r\ times} \to \mathbb{R}^n,$$

and $\| \ \|$ denotes the standard norm on the corresponding vector space.

Let $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ be open subsets. For a C^k-map $f: V \to W$, k a non-negative integer, we denote by $f^(k)$ its k-jet, that is

$$f^k : V \to W \times \mathcal{L}^1(\mathbb{R}^m, \mathbb{R}^n) \times \cdots \times \mathcal{L}^k(\mathbb{R}^m, \mathbb{R}^n),$$

$$f^(k)(x) = (f(x), Df(x), ..., D^k f(x)).$$

Definition B.1.2 *Let k be a non-negative integer. A sequence $f_\nu : V \to W$, $\nu \geq 1$, of C^k-maps converges in the C^k-topology to a C^k-map $f : V \to W$ if the sequence $(f_\nu^{(k)})$ of k-jets converges locally uniformly to $f(k)$. Then (f_ν) is also said to converge in C^k. In other word, (f_ν) converges in C^k if all derivatives from order 0 up to order k converge locally uniformly.*

A sequence $f_\nu : V \to W$, $\nu \geq 1$, of C^∞-maps *converges* in C^∞ if it converges in each C^k-topology, $0 \leq k < \infty$.

Lemma B.1.3 *let $U \subset \mathbb{R}^l$, $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$ be open subsets. Assume $g_v : U \to V$ and $f_v : V \to W$, $v \geq 1$, are two sequences of C^k-maps converging in C^k to g and f, respectively. Then $(f_v \circ g_v)$ converges in C^k to $f \circ g$.*

Now we consider the inverse functions of the sequence (f_v).

Lemma B.1.4 *Let $k \geq 1$ be some positive integer and $f_v : U \to V$ be a sequence of diffeomorphisms between open subsets of $\mathbb{R}^m$. If (f_v) converges in C^k to a diffeomorphism f, then the inverse maps (f_v^{-1}) converge in C^k to f^{-1}.*

B.2 Hölder space $C^{k,\alpha}(Q)$

The Hölder space of order $k,\ \alpha$ is the space of function from the space of $k-$ differentiable continuous functions that are "almost" $k+1$ differentiable and continuous. With the word almost, we mean that every $k+1$-differentiable continuous function belongs to the Hölder space, but not all k-differentiable continuous functions do.

Definition B.2.1 *Höoder space $C^{k,\alpha}(Q)$. Let $Q \subset \mathbb{R}^n$ be a region, let $f \in C^k(\bar{Q})$, we say that $f \in C^{k,\alpha}(\bar{Q})$ iff*

$$\max_{n \leq k} \sup_{x,y \in \bar{Q}, x \neq y} \frac{|D^n f(y) - D^n f(x)|}{|y-x|^\alpha} = [f]_{C^{k,\alpha}(\bar{Q}} < \infty.$$

Therefore the Hölder space is a subset of $C^k(\bar{Q})$, such that the functions are *Hölder continuous*, i.e. exists a constant $0 \leq C \in \mathbb{R}$ such that

$$|D^n f(y) - D^n f(x)| \leq C|y-x|^\alpha,\ \forall x, y \in \bar{Q} : x \neq y$$

The Hölder norm is defined as

$$||f||_{C^{k,\alpha}(\bar{Q})} := ||f||_{C^k(\bar{Q})} + [f]_{C^{k,\alpha}(\bar{Q})}$$

There are some useful properties of the Hölder space.

1) $C^{k,\alpha}(\bar{Q})$ is Banach with the norm $||f||_{C^{k,\alpha}(\bar{Q})}$.

2) The set $C^{k+1}[0,1]$ is not dense on $C^{k,\alpha}[0,1]$.

3) The set $C^\infty(\bar{Q})$ is not dense on $C^{k,\alpha}(\bar{Q})$.

4) Let $C^{k,\alpha}(\bar{Q})$ be the Hölder space over $\bar{Q}$, then $C^{k+1} \subset C^{k,\alpha}(\bar{Q}) \forall k \in \mathbb{N}, \forall \alpha \in (0,1)$.

APPENDIX C

BELTRAMI EQUATION

In this appendix, we review basic properties of the Beltrami equation

$$w_{\bar{z}} = \mu w_z. \tag{C.0.1}$$

Here w and μ are complex valued maps on a domain in $\mathbb{C}$. Now we suppose that μ is a function defined in U with $\sup |\mu(z)| < 1$. A function w is called a *generalized solution* of (C.0.1) in U if w is absolutely continuous on lines in G and the derivatives $w_z,\ w_{\bar{z}}$ satisfy (C.0.1) almost everywhere in U.

Now we state the existence theorem

Theorem C.0.2 *If U is an arbitrary domain and μ an arbitrary function in U with*

$$\sup_{z \in U} |\mu(z)| < 1,$$

then there exists a quasiconformal mapping w of U whose complex dilatation coincides with μ almost everywhere in U.

In our case, the uniqueness of the solution can be derived from the following theorem.

Theorem C.0.3 *Let U and U' be conformally equivalent simply connected domains and μ a measurable function in U with* $\sup |\mu(z)| < 1$. *The there exists a quasiconformal mapping $w : U \to U'$ whose complex dilatation coincides with μ almost everywhere. This mapping is uniquely determined up to a conformal mapping of U' onto itself.*

The application of the above two theorems in our proof is as follows.

Let U, U' be domains in S^2 whose boundaries are embedded circles, and let $a_1,\ a_2,\ a_3$ and $a'_1,\ a'_2,\ a'_3$ be two triples of distinct points on ∂U and $\partial U'$ respectively. Given a C^∞ function $\beta : U \to \mathbb{C}$ with $|\beta| \le k < 1$ for some constant k, there is a unique homeomorphism

$w^\beta : \bar{U} \to \bar{U}'$ that restricts to a diffeomorphism $U \to U'$, maps each a_k to a'_k, and solves the Beltrami equation with dilatation β.

To state the useful theorem by Earle-Schatz, we first introduce some notations.

Let $\mathcal{U} = \{z \in \mathbb{C}; \mathrm{Im} z > 0\}$ and $\mathcal{M}^{m+\alpha}(\mathcal{U})$ be the set of functions $u \in \mathbb{C}^{m+\alpha}(\mathcal{U})$ such that $|\mu(z)| < 1$ for all $z \in \mathcal{U}$. And $\mathbb{C}^{m+\alpha}(\mathcal{U})$ consists of those functions on $\mathcal{U}$ having continuous derivatives up to order k and such that the kth partial derivatives are Hölder continuous with exponent α.

A general theorem is stated as follows.

Theorem C.0.4 *For each $k < 1$, the map $\mu \to f^\mu$ is a homeomorphism of the set of $\mu \in \mathcal{M}^{m+\alpha}(\mathcal{U})$ with $\sup\{|\mu(z)| : z \in \mathcal{U}\} \leq k < 1$ onto its image in $C^{m+1+\alpha}(\mathcal{U}, \mathbb{C})$. Here the integer $m \geqslant 0$ and the number $0 < \alpha < 1$ are fixed but arbitrary.*

From the above theorem, we can easily know

Corollary C.0.5 *For each $k < 1$, the map $\mu \to w_\mu$ is a homeomorphism of the set of $\mu \in \mathcal{M}(\mathcal{U})$ with $\sup\{|\mu(z)| : z \in \mathcal{U}\} \leq k < 1$ onto its image in $C^\infty(\mathcal{U}, \mathbb{C})$.*

APPENDIX D

METRIZABILITY

Let $I = [0, 1]$ and let $C^k(I)$ be the set of all functions with continuous k-th derivative, $f : I \to \mathbb{R}$. We consider the metric space $(C^k(I), C^k)$ with C^k norm.

We begin by generalizing some concepts from $\mathbb{R}$ to $C(I)$.

We say that $E \subseteq C(I)$ is nowhere dense if for all open balls $B_\epsilon(f)$, centered at $f \in C(I)$ with radius ϵ, there is an open ball $B_\delta(g) \subseteq B_\epsilon(f)$ with $E \cap B_\delta(g) = \emptyset$.

We say that $E \subseteq C(I)$ is *meager* if

$$E = \cup_{n=1}^{\infty} E_n$$

where each E_n is nowhere dense.

Theorem D.0.6 *$C(I)$ is not meager.*

Proof Suppose $E = \cup E_n$ where each $E_n \subseteq C(I)$ is nowhere dense. We will find $f \in C(I)$ with $f \notin E$ by constructing a sequence (f_n) converging uniformly to f as follows:

Let $f_0 \in C(I)$. Let $\epsilon_0 = 1$. Given f_n and $\epsilon_n > 0$, since E_n is nowhere dense we can find $f_{n+1} \in B_{\epsilon_n}(f_n)$ and $\epsilon_{n+1} > 1$ such that:

1) $\bar{B}_{\epsilon_{n+1}}(f_{n+1}) \subseteq B_{\epsilon_n}(f_n)$;

2) $B_{\epsilon_{n+1}}(f_{n+1}) \cap E_n = \emptyset$.

We claim that the sequence (f_n) is Cauchy. Let $\epsilon > 0$. Choose N such that $\epsilon_N < \epsilon$. If $n > m \geq N$, then $f_n \in \bar{B}_{\epsilon_m}(f_m)$. Hence $\|f_n - f_m\| \leq \epsilon_m < \epsilon$. Thus there is $f \in C(I)$ such that $f_n \to f$. Since $f_n \in \bar{B}_{\epsilon_m}(f_m)$ for all $n > m$, and $\bar{B}_{\epsilon_m}(f_m)$ is closed, we know that $f \in \bar{B}_{\epsilon_m}(f_m)$ for all m.

Since $\bar{B}_{\epsilon_m}(f_m) \cap E_m = \emptyset$, $f \notin E_m$ for any m. Thus $f \in C(I) \setminus E$.

The following lemma is used in our proof.

Lemma D.0.7 *Suppose $F \subseteq C(I)$ is closed. The following are equivalent:*

1) F is nowhere dense;

2) there is no open ball $B_\epsilon(f) \subseteq F$.

Proof 1) $\Rightarrow$ 2) Clear.

2) $\Rightarrow$ 1) Suppose F is not nowhere dense. Then there is an open ball $B_\epsilon(f)$ such that every open ball $F \cap B_\delta(g) \neq \emptyset$ whenever $B_\delta(g) \subseteq B_\epsilon(f)$. We claim that $B_\epsilon(f) \subseteq F$. Let $g \in B_\epsilon(f)$. For each n we can find $f_n \in B_{1/n}(g) \cap E$. Then f_n converges uniformly to g. Hence $g \in E$. Thus $B_\epsilon(f) \subseteq E$.

Now we discuss $C^k(I)$ for any $k \geq 0$.

Lemma D.0.8 *For any $f(x)$ in $C^k(I)$ ($k \in \mathbb{N}$), there is a function $P(x)$ in $C^k(I)$ s.t. $\|f(x) - P(x)\|_k < \epsilon$ and the $k-th$ derivative of $P(x)$, denoted as $P^k(x)$, is a continuous piecewise linear function.*

Proof Use induction on k. For k=0. It is already provenit is.

Suppose for $k = n$, it is true. Consider the case $k = n + 1$.

For a fixed $f(x)$ in $C^{n+1}(I)$, $f'(x)$ is in $C^n(I)$. There is a function $\bar{P}(x)$ in $C^n(I)$ s.t. $\|f'(x) - \bar{P}(x)\|_n < \frac{\epsilon}{2}$. Choose $P(x)$ s.t. $P'(x) = \bar{P}(x)$ and $P(0) = f(0)$. Then $\|f(x) - P(x)\|_{n+1} = \sup_{x\in I} |f(x) - P(x)| + \|f'(x) - \bar{P}(x)\|_n = \sup_{x\in I} | \int_0^x f'(t) - \bar{P}(t)dt| + \|f'(x) - \bar{P}(x)\|_n < \frac{\epsilon}{2} + \frac{\epsilon}{2}$. □

Let $D = \{f \in C^k(I) :\ f^k$ is differentiable at x for some $x \in I\}$. We will prove that D is meagre.

Let $A_{n,m}$ ={$f \in C^k(I)$: there is $x \in I$ such that $|\frac{f^k(t)-f^k(x)}{t-x}| \leqslant n$ if $0 < |x - t| < \frac{1}{m}$ }.

Lemma D.0.9 *If $f(x) \in D$, then $f \in A_{n,m}$ for some n and m.*

Proof Suppose f is differentiable at x, then there exists n such that $|f'(x)| < n$. And there exists $\delta > 0$ such that $|\frac{f(t)-f(x)}{t-x}| < n$ if $0 < |t-x| < \delta$.

Choose m such that $\frac{1}{m} < \delta$, then $f \in A_{n,m}$ according to the definition of $A_{n,m}$.

Lemma D.0.10 *Each $A_{n,m}$ is closed.*

Proof Suppose f_i is a Cauchy sequence in $A_{n,m}$ and $f_i \to f$. We shall prove $f \in A_{n,m}$.

For each i there is $x_i \in K$ such that

$$|\tfrac{f_i(t)-f_i(x_i)}{t-x_i}| \leqslant n \text{ for all } 0 < |x_i - t| < \tfrac{1}{m}.$$

By the Bolzono-Weierstrass Theorem x_i has a convergent subsequence. Without loss of generality, we may assume that x_i converges.

Suppose x_i converges to x and suppose $0 < |x-t| < \frac{1}{m}$, then

$$|\tfrac{f(t)-f(x)}{t-x}| = \lim_{i\to\infty} |\tfrac{f_i(t)-f_i(x_i)}{t-x_i}| \leqslant n.$$

Lemma D.0.11 *For $k = 0$, $A_{n,m}$ is nowhere dense.*

Proof Since $A_{n,m}$ is closed, it suffices to show that $A_{n,m}$ does not contain an open ball. Consider the open ball $B_\epsilon(f)$. We will show there exists $g \in B_\epsilon(f)$ with $g \notin A_{n,m}$. We can find a piecewise linear $p(x)$ such that $\|f - p\| < \epsilon/2$.

Since the graph of p is a finite union of line segments, p is differentiable at all but finitely many points and we can find $M \in \mathbb{N}$ such that $|p'(x)| \leqslant M$ for all x where p is differentiable. Choose $k > \frac{2(M+n)}{\epsilon}$.

There is a continuous piecewise linear function $\phi(x)$ where $|\phi(x)| \leqslant 1$ for all $x \in K$ and $\phi'(x) = \pm k$ for all x where $\phi(x)$ is differentiable. Consider the partition $a_i = i/k$ for $i = 0, ..., k$ and let $\phi(a_i) = 0$ if i is even and 1 if i is odd. Let

$$g(x) = p(x) + \tfrac{\epsilon}{2}\phi(x)$$

Since $\|f-p\| < \epsilon/2$ and $\|g-p\| < \epsilon/2$, $\|f-g\| \leqslant \|f-p\| + \|g-p\| < \epsilon$.

We now show $g \notin A_{n,m}$.

Let $x \in [0,1]$. If p and ϕ are differentiable at x, then $|g'(x)| = |p'(x) \pm \frac{\epsilon}{2}k|$. Then $|g'(x)| > n$ since $|p'(x)| \leqslant M$.

Hence, we can find $l > m$ such that $g|[x, x+\frac{1}{l}]$ and $g|[x-\frac{1}{l}, x]$ are linear and the absolute value of the slope is greater than n. In particular, if $0 < |x-t| < \frac{1}{l} < \frac{1}{m}$, then $|\frac{g(t)-g(x)}{t-x}| > n$ and $g \notin A_{n,m}$. Thus $B_\epsilon(f) \subsetneq A_{n,m}$. □

Lemma D.0.12 *For each k, $A_{n,m}$ is nowhere dense.*

Proof Since $A_{n,m}$ is closed, it suffices to show that $A_{n,m}$ does not contain an open ball. Consider the open ball $B_\epsilon(f)$. We must find $g \in B_\epsilon(f)$ with $g \notin A_{n,m}$. By Previous lemma, we can find a function P with continuous piecewise linear k-th derivative such that $\|f-p\|_k < \frac{\epsilon}{2}$.

Since the graph of p^k is finite union of line segments, p^k is differentiable at all but finitely many points and we can find $M \in \mathbb{N}$ such that $|p^{k+1}(x)| \leqslant M$ for all x where p^k is differentiable. There is a function $\phi(x)$, with a continuous piecewise linear k-derivative $\phi^k(x)$ where $\phi^k(x) \leqslant 1$, $\|\phi\|_k \leqslant k$ and $\phi^{k+1}(x) = \pm t$,[$t > \frac{2k(M+n)}{\epsilon}$, consider the partition $a_i = \frac{i}{t}$ for $i = 0, 1, ..., t$ and let $\phi^k(a_i) = 0$ if i is even and 1 if i is odd.And from this function we constructed, we know there is an original function ϕ satisfying $\|\phi\|_k \leqslant k$],for all x where $\phi^k(x)$ is differentiable.Let

$$g(x) = p(x) + \frac{\epsilon}{2\|\phi(x)\|_k}\phi(x)$$

Since $\|f-p\|_k < \frac{\epsilon}{2}$ and $\|g-p\|_k < \frac{\epsilon}{2}$, $\|f-g\|_k < \epsilon$.

We claim that $g \notin A_{n,m}$. Let $x \in [0,1]$. If p and ϕ are both differentiable at x, then $|g^{k+1}(x)| = |p^{k+1}(x) \pm \frac{\epsilon}{2\|\phi\|_k}t|$. Since $|p^{k+1}(x)| \leqslant M$, $|g^{k+1}(x)| > n$. So $g \notin A_{n,m}$ and $B_\epsilon(f) \subsetneq A_{n,m}$.

□

From above three lemmas, $A_{n,m}$ is meagre. Let $A = \cup\cup A_{n,m}$, then A is meagre. Since $D \subseteq A$, D is meagre.

Theorem D.0.13 *$C^\infty(I)$ is not completely metrizable in C^k-topology.*

Proof Let S be C^∞ in ($C^k(I)$, C^k-topology),which is denoted as X. Suppose S is completely metrizable. S is completely metrizable if and only if S is the intersection of countably many open subsets. Let $S = \cap U_i$, where each U_i is open set containing all the polynomials.

A set is comeagre if it can be expressed as the intersection of countably many sets with dense interiors. So S is comeagre. Thus the complement, i.e. S^c, is meagre.

S is a subset of D. Since D is meagre, S is meagre. The union of countably many meagre sets is meagre, so $X = S \cup S^c$ is meagre. This is a contradiction because a complete metric space is not meagre. □

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