ERDŐS-PÓSA THEOREMS FOR UNDIRECTED GROUP-LABELLED GRAPHS

A Dissertation
Presented to
The Academic Faculty

By

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In Partial Fulfillment
of the Requirements for the Degree
Doctor of Philosophy in
Algorithms, Combinatorics, and Optimization

School of Mathematics
Georgia Institute of Technology

August 2022

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Date approved: June 10, 2022

To my parents

ACKNOWLEDGMENTS

I would like to thank my advisor Xingxing Yu for his support, patience, and for always being honest and attentive to my ideas. He has given me many fresh perspectives and has constantly encouraged me to see the bigger picture.

I am grateful to have been guided by Robin Thomas in the first few years of my graduate studies. He was an exemplary leader and researcher with utmost resilience. My sincere respect for him continues to guide me in many ways.

I would like to thank Anton Bernshteyn, Greg Blekherman, Chun-Hung Liu, and Mohit Singh for serving on my thesis committee. I am also thankful to the ACO program and the School of Mathematics for providing a stimulating and supportive environment. I would also like to thank my friends and colleagues at Georgia Tech and elsewhere.

Lastly, I thank my parents for their dedication and unwavering support to pursue my interests throughout my life.

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SUMMARY

Erdős and Pósa proved in 1965 that cycles satisfy an approximate packing-covering duality. Finding analogous approximate dualities for other families of graphs has since become a highly active area of research due in part to its algorithmic applications. In this thesis we investigate the Erdős-Pósa property of various families of constrained cycles and paths by developing new structural tools for undirected group-labelled graphs.

Our first result is a refinement of the flat wall theorem of Robertson and Seymour to undirected group-labelled graphs. This structure theorem is then used to prove the Erdős-Pósa property of A-paths of length 0 modulo p for a fixed odd prime p, answering a question of Bruhn and Ulmer. Further, we obtain a characterization of the abelian groups Γ and elements $\ell \in \Gamma$ for which A-paths of weight ℓ satisfy the Erdős-Pósa property. These results are from joint work with Robin Thomas.

We extend our structural tools to graphs labelled by multiple abelian groups and consider the Erdős-Pósa property of cycles whose weights avoid a fixed finite subset in each group. We find three types of topological obstructions and show that they are the only obstructions to the Erdős-Pósa property of such cycles. This is a far-reaching generalization of a theorem of Reed that Escher walls are the only obstructions to the Erdős-Pósa property of odd cycles. Consequently, we obtain a characterization of the sets of allowable weights in this setting for which the Erdős-Pósa property holds for such cycles, unifying a large number of results in this area into a general framework. As a special case, we characterize the integer pairs (ℓ, z) for which cycles of length ℓ mod z satisfy the Erdős-Pósa property. This resolves a question of Dejter and Neumann-Lara from 1987. Further, our description of the obstructions allows us to obtain an analogous characterization of the Erdős-Pósa property of cycles in graphs embeddable on a fixed compact orientable surface. This is joint work with Pascal Gollin, Kevin Hendrey, O-joung Kwon, and Sang-il Oum.

CHAPTER 1

INTRODUCTION

1.1 Erdős-Pósa property

Erdős and Pósa [16] showed that cycles satisfy an approximate packing-covering duality; that is, there exists a function $f(k) = O(k \log k)$ such that in every graph, either there are k vertex-disjoint cycles or there is a set of at most $f(k)$ vertices intersecting every cycle. This result has generated extensive activity on whether various families of graphs satisfy a similar approximate duality, often referred to as the *Erdős-Pósa property*.

Let $\mathcal{F}$ be a family of graphs. An *$\mathcal{F}$-packing of size k* is a set of k vertex-disjoint graphs in $\mathcal{F}$, and a *half-integral $\mathcal{F}$-packing of size k* is a multiset of $2k$ graphs in $\mathcal{F}$ such that every vertex occurs in at most two graphs in the multiset. In a graph G, a vertex set $Z \subseteq V(G)$ is an *$\mathcal{F}$-hitting set* if $G - Z$ does not contain a subgraph in $\mathcal{F}$.

We say that $\mathcal{F}$ satisfies the *(half-integral) Erdős-Pósa property* if there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that in every graph G, either there is a (half-integral) $\mathcal{F}$-packing of size k or there is an $\mathcal{F}$-hitting set of size at most $f(k)$.

A beautiful result of Robertson and Seymour [31], obtained as a corollary of the grid minor theorem (see Theorem 2.6.1), says that for a fixed graph H, the family of *H-expansions* (graphs which contain H as a minor) satisfies the Erdős-Pósa property if and only if H is planar. Since cycles are exactly the minimal graphs which contain K_3 as a minor, this implies Erdős and Pósa's original result as a special case.

In many cases, obstructions to the Erdős-Pósa property are of topological nature as we will see throughout. Besides their mathematical interest, the Erdős-Pósa property has many algorithmic applications, for example in approximation algorithms and fixed-parameter tractability of the corresponding maximum packing or minimum cover problems (e.g. [25]).

In this thesis we investigate the Erdős-Pósa property of various families of constrained cycles and paths, unifying many known results in the literature and answering some open problems.

1.2 Cycles

1.2.1 Background

One type of constraint often studied is a modularity condition on the cycle lengths. For example, Thomassen [37] showed that for every positive integer z, the Erdős-Pósa property holds for cycles of length 0 modulo z. On the other hand, Lovász and Schrijver (see [37]) found a class of graphs, so called *Escher walls* (see Figure 1.1(a)), which demonstrate that such a duality does not hold for odd cycles. Escher walls are certain types of projective planar grids in which a cycle has odd length if and only if it is a one-sided closed curve in the projective-planar embedding. It is well-known that no two one-sided closed curves in the projective plane are disjoint, hence Escher walls do not contain two vertex-disjoint odd cycles, but they can require arbitrarily many vertices to intersect every odd cycle.

Reed [30] showed that large Escher walls are contained in every graph that contains neither many vertex-disjoint odd cycles nor a small odd cycle hitting set, yielding a structural characterization of the graphs failing to satisfy this approximate duality for odd cycles. But since Escher walls contain large half-integral packings of odd cycles, Reed concluded that the half-integral Erdős-Pósa property holds for odd cycles.

Escher walls can be modified to give infinitely many pairs (ℓ, z) for which the Erdős-Pósa property fails for cycles of length ℓ modulo z. This was essentially shown by Dejter and Neumann-Lara [13], who then asked to find all pairs (ℓ, z) of integers for which an analogue of the Erdős-Pósa theorem does hold for cycles of length ℓ modulo z.

Question 1 (Question 4 in [13]). *For which pairs of positive integers (ℓ, z) does the family of cycles of length $\ell \mod z$ satisfy the Erdős-Pósa property?*

Note that the half-integral Erdős-Pósa property does hold for all pairs (ℓ, z) (see [20]).

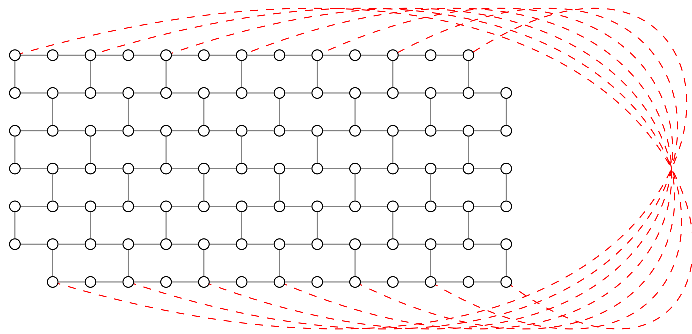
Other types of constraints for cycles have been considered. Given a vertex set S, an S-*cycle* is a cycle containing a vertex in S. Kakimura, Kawarabayashi, and Marx [26] showed that the Erdős-Pósa property holds for S-cycles. Birmelé, Bondy, and Reed [2] showed that it also holds for *long* cycles, defined as cycles of length at least L for some fixed positive integer L. Bruhn, Joos, and Schaudt [5] combined these results to show the Erdős-Pósa property of long S-cycles. Note that there cannot be an extension of their result to odd S-cycles due to Escher walls, but they are not the only obstructions, see Figure 1.1(b). Again, the half-integral Erdős-Pósa property holds for odd S-cycles, as shown by Kakimura and Kawarabayashi [24].

Given a family $\mathcal{S}$ of sets, an $\mathcal{S}$-*cycle* is a cycle containing at least one vertex from each member of $\mathcal{S}$. Huynh, Joos, and Wollan [23] proved an analogue of the Erdős-Pósa theorem for (S_1, S_2)-cycles. An extension of their result to (S_1, S_2, S_3)-cycles fails, and a third type of obstruction appears in this setting, see Figure 1.1(c). Again, the half-integral Erdős-Pósa property holds for all finite families $\mathcal{S}$ (see [20]).

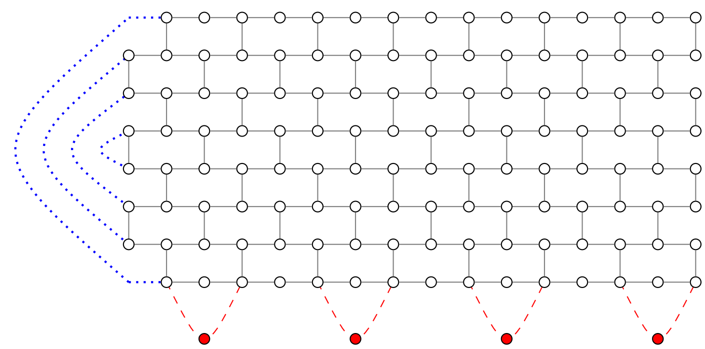
1.2.2 Some of our results

We consider a unified approach to deal with a large number of such constraints in a common setting. For an abelian group Γ, a Γ-*labelled graph* is a pair (G, γ) of a graph G and a Γ-*labelling* $\gamma\colon E(G) \to \Gamma$. The *weight* of (G, γ) (or the γ-*value* of G) is defined to be the sum of $\gamma(e)$ over all edges e of G. We say that (G, γ) is Γ-*nonzero* (or that G is γ-*nonzero*) if its weight (or its γ-value) is a non-identity element of Γ.

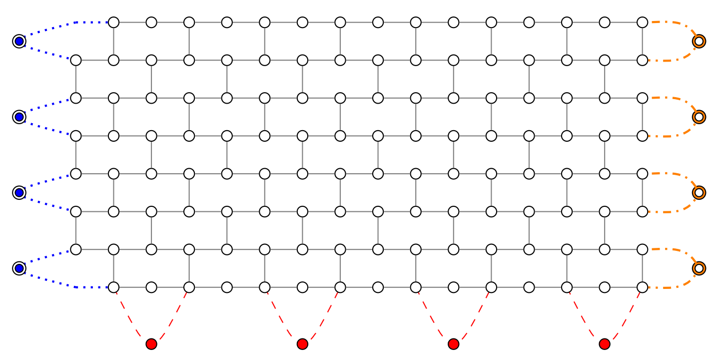
Modularity constraints on cycles (say modulo z) can be naturally encoded in the setting of $\mathbb{Z}/z\mathbb{Z}$-labelled graphs, where each edge is labelled (the congruence class) $1 + z\mathbb{Z}$ and the target cycles have values exactly $\ell + z\mathbb{Z}$. Moreover, S-cycles can be encoded as $\mathbb{Z}$-nonzero cycles with respect to the $\mathbb{Z}$-labelling which assigns the label 1 to edges incident with vertices in S and 0 to all other edges. Using multiple abelian groups, we may encode



(a) An *Escher Wall*, the obstruction for odd cycles. We refer to the arrangement of the red paths around the wall as ‘crossing’.



(b) An obstruction for odd S-cycles, where vertices in S are shown in red. We refer to the arrangement of the blue dotted paths around the wall as ‘nested’, and of the red dashed paths as ‘in series’.



(c) An obstruction for (S_1, S_2, S_3)-cycles, where vertices in S_1, S_2, and S_3 are shown in red, blue, and orange colors respectively.

Figure 1.1: Obstructions for Erdős-Pósa type results for constrained cycles. Dashed or dotted lines represent paths of odd length and solid lines represent paths of even length.

cycles satisfying multiple properties simultaneously (e.g. odd S-cycles).

In joint work with Thomas [36], we proved a structure theorem (Theorem 3.1.1) which refines the flat wall theorem (Theorem 2.6.2) of Robertson and Seymour [33] to Γ-labelled graphs. This is analogous to a result of Huynh, Joos, and Wollan [23] who proved a flat wall theorem for a *directed* model of group-labelled graphs. An easy consequence of our structure theorem is the following Erdős-Pósa result for Γ-nonzero cycles:

Theorem 1.2.1. *Let Γ be an abelian group. Then the family of Γ-nonzero cycles satisfies the half-integral Erdős-Pósa property. Moreover, if Γ has no element of order two, then the family of Γ-nonzero cycles satisfies the Erdős-Pósa property.*

This strengthens a result of Wollan [39] that Γ-nonzero cycles satisfy the Erdős-Pósa property if and only if Γ has no element of order two. Furthermore, our structure theorem implies that if Γ has an element of order two, then the only obstruction to the Erdős-Pósa property of Γ-nonzero cycles is an analogue of the Escher wall labelled by a fixed element of order two. In this sense, Theorem 3.1.1 generalizes Reed's result for odd cycles.

In joint work with Gollin, Hendrey, Kwon, and Oum [19], we extended Theorem 1.2.1 by considering graphs labelled by multiple abelian groups $\Gamma_1, \dots, \Gamma_m$ and considering more general forms of the set $A \subseteq \Gamma := \prod_{j=1}^{m} \Gamma_j$ of 'allowable' values for the cycles, beyond a 'Γ-nonzero' constraint. Namely, we give a characterization of the sets $A \subseteq \Gamma$ for which the *allowable* cycles (cycles with weights in A) satisfy the Erdős-Pósa property, under the assumption that A is the set of all elements of Γ avoiding a fixed finite set of elements of each Γ_i (see Theorems 1.2.3 and 1.2.4). This is derived as a corollary of our main result (Theorem 3.3.2) which characterize the obstructions to the Erdős-Pósa property of allowable cycles in this setting. This is a far-reaching generalization of Reed's result that Escher walls are the only obstructions to the Erdős-Pósa property of odd cycles.

As one consequence, we obtain a characterization of the integer pairs (ℓ, z) for which the family of cycles of length ℓ mod z satisfies the Erdős-Pósa property, completely resolving Question 1.

Theorem 1.2.2. *Let ℓ and z be integers with $z \geq 2$, and let $p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorization of z with $p_i < p_{i+1}$ for all $i \in [n-1]$. The following statements are equivalent.*

- *Cycles of length ℓ mod z satisfy the Erdős-Pósa property.*
- *Both of the following conditions are satisfied.*
  1. *If $p_1 = 2$, then $\ell \equiv 0 \pmod{p_1^{a_1}}$.*
  2. *There do not exist distinct $i_1, i_2, i_3 \in [n]$ such that $\ell \not\equiv 0 \pmod{p_{i_j}^{a_{i_j}}}$ for each $j \in [3]$.*

Furthermore, our results allow us to combine different types of constraints to obtain corresponding characterizations, for example on when S-cycles of length ℓ mod z satisfy the Erdős-Pósa property (see Corollary 3.4.2).

Let us now give a loose description of the obstructions. Each obstruction consists of a wall, in which every cycle has value zero in every group, together with a collection of sets of paths arranged around the boundary of the wall, so that each set of paths is 'nested', 'crossing', or 'in series' (see Figure 1.1 for examples). Moreover, this collection of sets is minimally sufficient to find allowable cycles, in that every allowable cycle contains at least one path from each of these sets, and every cycle which contains exactly one path from each set is allowable. Additionally, every allowable cycle must contain an odd number of paths from each set that is not in series. Finally, one of the following conditions must be satisfied:

- the number of crossing sets of paths is odd (see for example Figure 1.1(a)),
- at least one but not all sets are arranged in series (see for example Figure 1.1(b)), or
- at least three sets of these paths are arranged in series (see for example Figure 1.1(c)).

As we show in Subsection 3.3.3, these obstructions do not contain a packing of more than two allowable cycles.

The conditions on the set A of allowable values which avoid these obstructions can be characterized as follows. Consider a product of m abelian groups $\Gamma = \prod_{i \in [m]} \Gamma_i$. If $g = (g_i : i \in [m]) \in \Gamma$, then we write $\pi_i(g)$ to denote $g_i \in \Gamma_i$.

Theorem 1.2.3. *For every pair of positive integers m and ω, there is a function $f_{m,\omega} \colon \mathbb{N} \to \mathbb{N}$ satisfying the following property. Let $\Gamma = \prod_{i \in [m]} \Gamma_i$ be a product of m abelian groups, and for each $i \in [m]$, let Ω_i be a subset of Γ_i with $|\Omega_i| \leq \omega$. Let A be the set of all elements $g \in \Gamma$ such that $\pi_i(g) \in \Gamma_i \setminus \Omega_i$ for all $i \in [m]$, and suppose that*

(1) for all $a \in A$, we have $\langle 2a \rangle \cap A \neq \emptyset$,

(2) for all $a, b, c \in \Gamma$ with $\langle a, b, c \rangle \cap A \neq \emptyset$, we have $(\langle a, b \rangle \cup \langle b, c \rangle \cup \langle a, c \rangle) \cap A \neq \emptyset$.

Then the family of Γ-labelled cycles with weights in A satisfies the Erdős-Pósa property.

Conversely, we have the following negative result.

Theorem 1.2.4. *Let A be a nonempty subset of an abelian group Γ such that A does not satisfy at least one of the following properties:*

(1) for all $a \in A$, we have $\langle 2a \rangle \cap A \neq \emptyset$,

(2) for all $a, b, c \in \Gamma$ with $\langle a, b, c \rangle \cap A \neq \emptyset$, we have $(\langle a, b \rangle \cup \langle b, c \rangle \cup \langle a, c \rangle) \cap A \neq \emptyset$.

Then the family of Γ-labelled cycles with weights in A does not satisfy the Erdős-Pósa property.

Note that for fixed m and ω, Theorem 1.2.3 produces a single function $f_{m,\omega}$ which does not depend on the specific abelian groups considered. These theorems completely characterize when such a duality holds in the setting where the set of allowable cycles are those whose values avoid a fixed finite subset of each abelian group. Considering additional restrictions on the structure of the graphs or group labellings, we further strengthen Theorem 1.2.3 by observing that, when checking conditions (1) and (2), we may ignore any group Γ_i for which every large subwall of G contains a γ_i-nonzero cycle (see Theorem 3.4.1). This strengthening allows us to encode a wider variety of properties of cycles. For example, for fixed integers p, ℓ and given a subgraph H of tree-width at most p in a graph G, consider the cycles containing at least ℓ edges not contained in H. Such cycles can be represented with the $\mathbb{Z}$-labelling which assigns value 1 to edges not in H and 0 to all edges in H. If H has no edges, then these are exactly the cycles of length at least ℓ.

Observe that for finite abelian groups Γ, Theorems 1.2.3 and 1.2.4 give a complete characterization of the sets $A \subseteq \Gamma$ for which the allowable cycles satisfy the Erdős-Pósa property without any additional assumptions (take $\Gamma = \Gamma_1$ and $\Omega_1 = \Gamma \setminus A$). On the other hand, if Γ is infinite, then A must also be infinite, as we prove in subsection 3.4.4.

Theorem 1.2.5. *Let A be a nonempty finite subset of an infinite abelian group Γ. For integers $s \geq 2$ and $t \geq 1$, there is a graph G with a Γ-labelling γ such that*

- *for every set of s cycles of G whose γ-values are in A, there is a vertex that belongs to all of the s cycles and*
- *there is no hitting set of size at most t for the set of all cycles of G whose γ-values are in A.*

Another consequence of our characterization of the obstructions is an application to graphs of bounded orientable genus (for example, planar graphs). Since large Escher Walls are not embeddable in any fixed compact orientable surface, if we restrict to graphs embeddable on a fixed compact orientable surface, then we obtain a different characterization of the Erdős-Pósa property. For example, odd cycles do satisfy the Erdős-Pósa property when restricted to planar graphs (see [17, 27] for related work). In Subsection 3.4.3, we give a characterization analogous to Theorems 1.2.3 and 1.2.4 for graphs that are embeddable in a fixed compact orientable surface. Additionally, we obtain the following analogue of Theorem 1.2.2 for planar graphs.

Theorem 1.2.6. *Let ℓ and z be integers with $z \geq 2$, let $p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorization of z with $p_i < p_{i+1}$ for all $i \in [n-1]$, and let $\mathbb{S}$ be a compact orientable surface. The following statements are equivalent.*

- *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer k, every graph embeddable in $\mathbb{S}$ contains k vertex-disjoint cycles of length ℓ modulo z or a set of at most $f(k)$ vertices hitting all such cycles.*

- *Both of the following conditions are satisfied.*

  1. *If $p_1 = 2$, then either $\ell \equiv 0 \pmod{p_1^{a_1}}$ or $\ell \equiv 0 \pmod{z/p_1^{a_1}}$.*
  2. *There do not exist distinct $i_1, i_2, i_3 \in [n]$ such that $\ell \not\equiv 0 \pmod{p_{i_j}^{a_{i_j}}}$ for each $j \in [3]$.*

For graphs embedded in a compact orientable surface, our results allow us to derive an Erdős-Pósa type theorem for the cycles whose $\mathbb{Z}_2$-homology class is in a fixed set of allowable values. This result complements an analogous half-integral Erdős-Pósa type theorem for graphs embedded in an arbitrary compact surface (see [20, Corollary 8.10]). We discuss this in more detail in Subsection 3.4.3.

Let us take a moment to highlight the differences between our results and the work of Huynh, Joos, and Wollan [23], who considered group labellings of orientations of edges in a graph, where the two orientations of each edge are assigned labels that are inverse to each other. For a graph imbued with two such labellings, they considered cycles with nonzero value in each coordinate and obtained a refinement of the flat wall theorem which proves the half-integral Erdős-Pósa property of such cycles. They also provide additional conditions which are sufficient to derive the (integral) Erdős-Pósa property of such cycles.

There is no general translation between the labellings of edges which we use and the labellings of orientations of edges which they considered, but many interesting properties (except modularity constraints on the cycle lengths with modulus greater than 2) can be encoded in either setting. As an example, they apply their result to obtain canonical obstructions to an Erdős-Pósa type result for odd cycles intersecting a prescribed set $\mathcal{S}$, and our structural theorem gives the same result. Whereas their result applies to arbitrary groups, dealing with nonabelian groups is more complicated in our setting, and it is unclear how to extend our result to nonabelian groups. However, we do not only consider the cycles that are nonzero in each coordinate, and we are able to consider any finite number of group labellings, rather than just two.

1.3 A-paths

1.3.1 Background

Let A be a vertex set. An *A-path* is a nontrivial path whose intersection with A is exactly its endpoints. A classical result of Gallai [18], which generalizes the Tutte-Berge formula for matchings to A-paths, shows that for every graph G and every positive integer k, either G contains k vertex-disjoint A-paths or there is a set of at most $2k-2$ vertices intersecting every A-path. Mader [28] showed that the same conclusion holds more generally for $\mathcal{S}$-paths, where $\mathcal{S}$ is a partition of A and an $\mathcal{S}$-path is an A-path whose endpoints are in distinct parts of $\mathcal{S}$.

This was further generalized by Chudnovsky et al. [9] to *directed* group-labelled graphs. Let Γ be a group with additive operation and identity 0, where Γ may be nonabelian. A *directed Γ-labelled graph* is a pair $(\vec{G},\gamma)$ where $\vec{G}$ is an orientation of an undirected graph G and $\gamma : E(G) \to \Gamma$ is a *Γ-labelling* of G. The *weight* of a walk $W = v_0e_1v_1 \dots v_{m-1}e_mv_m$ in G is defined to be $\gamma(W) = \gamma(e_1, v_1) + \cdots + \gamma(e_m, v_m)$, where for an edge $e = uv$ oriented from u to v, $\gamma(e, v) = \gamma(e)$ and $\gamma(e, u) = -\gamma(e)$. We say that W is *Γ-nonzero* if $\gamma(W) \neq 0$.

Theorem 1.3.1 (Theorem 1.1 in [9])**.** *Let Γ be a group and let $(\vec{G},\gamma)$ be a directed Γ-labelled graph with $A \subseteq V(G)$. Then for all positive integers k, either $(\vec{G},\gamma)$ contains k vertex-disjoint Γ-nonzero A-paths or there is a set of at most $2k-2$ vertices intersecting every Γ-nonzero A-path.* $\square$

With suitable choices of Γ and γ, one immediately obtains the results of Gallai and Mader, and many more. In the setting of undirected group-labelled graphs, Wollan [40] showed that Γ-nonzero A-paths in undirected Γ-labelled graphs also satisfy the Erdős-Pósa property, albeit with a worse bound $f(k) = O(k^4)$ (Theorem 1.1 in [40]). In particular, for all positive integers m, A-paths of length $\neq 0 \mod m$ satisfy the Erdős-Pósa property.

1.3.2 Our results

Here we address the opposite problem of packing A-paths of weight 0, which we call *Γ-zero A-paths*. This was first investigated by Bruhn, Heinlein, and Joos who showed that even A-paths satisfy the Erdős-Pósa property (Theorem 7 in [4]), whereas A-paths of length $0 \mod m$ do *not* satisfy the Erdős-Pósa property for all composite $m > 4$ (Proposition 8 in [4]). Interestingly, the composite number 4 does not adhere to this trend, as shown by Bruhn and Ulmer:

Theorem 1.3.2 (Theorem 1 in [6])**.** *A-paths of length 0 modulo 4 satisfy the Erdős-Pósa property.* $\square$

In the same paper, they asked whether the Erdős-Pósa property holds for A-paths of length $0 \mod p$ when p is an odd prime (Problem 22 in [6]). We provide an affirmative answer to their question:

Theorem 1.3.3. *Let p be an odd prime. Then A-paths of length 0 modulo p satisfy the Erdős-Pósa property.*

Using Theorem 1.3.3, we characterize the abelian groups Γ and elements $\ell \in \Gamma$ for which A-paths of weight ℓ in undirected Γ-labelled graphs satisfy the Erdős-Pósa property:

Theorem 1.3.4. *Let Γ be an abelian group and let $\ell \in \Gamma$. Then, in undirected Γ-labelled graphs, A-paths of weight ℓ satisfy the Erdős-Pósa property if and only if*

- *$\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^k$ where $k \in \mathbb{N}$ and $\ell = 0$,*

- $\Gamma \cong \mathbb{Z}/4\mathbb{Z}$ *and* $\ell \in \{0, 2\}$*, or*

- $\Gamma \cong \mathbb{Z}/p\mathbb{Z}$ *where* p *is prime (and* $\ell \in \Gamma$ *is arbitrary).*

We also prove the following characterization for Γ-zero A-paths in *directed* group-labelled graphs:

Theorem 1.3.5. *Let* Γ *be a group. Then, in directed* Γ*-labelled graphs,* Γ*-zero* A*-paths satisfy the Erdős-Pósa property if and only if* Γ *is finite.*

We remark that the "if" part of Theorem 1.3.5 was proved independently by Böltz [3]. We nevertheless provide the proof since it is short and Theorem 1.3.5 is used in our proof of Theorem 1.3.4. Besides, at the time of this writing, [3] is only available in German.

1.4 Organization

Our proofs follow a well-established approach in the area, developing several new tools along the way. This approach involves the use of *tangles* (see subsection 2.3) introduced by Robertson and Seymour in their graph minors project. If $\mathcal{F}$ is any family of connected graphs, then a graph which does not have a large $\mathcal{F}$-packing nor a small $\mathcal{F}$-hitting set admits a large tangle oriented towards the members of $\mathcal{F}$ in the graph (see Lemmas 2.3.1, 5.1.1, 6.1.1). Using the grid minor theorem (Theorem 2.6.1), we obtain a large wall and we look for paths attaching to the boundary of the wall as in Figure 1.1 to produce certain desired configurations.

The preliminary definitions and lemmas are given in Chapter 2. In Chapter 3, we give the statements of our main results in full technical detail, sketch their proofs, and discuss their applications. In Chapter 4, we prove our first structural result, Theorem 3.1.1, which refines the flat wall theorem to undirected group-labelled graphs. In Chapter 5, we use Theorem 3.1.1 to prove Theorem 1.3.3, that the Erdős-Pósa property holds for A-paths of length 0 mod p for every odd prime p. In Chapter 6, we prove our main structural result, Theorem 3.3.2, characterizing the obstructions to the Erdős-Pósa property of allowable cycles.

CHAPTER 2
PRELIMINARIES

2.1 Basic notation and terminology

All graphs and group-labelled graphs are assumed to be undirected and may have parallel edges but no loops. A graph is *simple* if it has no parallel edges. We denote set differences with the notation $S - T = \{s \in S : s \notin T\}$. For a graph G, let $V_{\neq 2}(G)$ denote the set of vertices of G whose degree is not equal to 2. Unless explicitly stated otherwise, we say disjoint to mean vertex-disjoint whenever applicable. For a set $\mathcal{G}$ of graphs, we denote by $\bigcup \mathcal{G}$ the union of the graphs in $\mathcal{G}$. By slight abuse of notation, we say two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs are *disjoint* if the graphs $\bigcup \mathcal{G}_1$ and $\bigcup \mathcal{G}_2$ are (vertex-)disjoint.

Let G be a graph and let $A, B \subseteq V(G)$. The subgraph induced by A in G is denoted $G[A]$. We write $G - A$ to denote $G[V(G) - A]$ and if H is a graph, then we write $G - H$ to denote $G - V(H)$. For a positive integer k, we say that G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected for all $X \subseteq V(G)$ with $|X| < k$.

An *A-path* is a nontrivial path in G such that both endpoints are in A and no internal vertex is in A. An *A-B-path* is a (possibly trivial) path in G such that one endpoint is in A, the other endpoint is in B, and the path is internally disjoint from $A \cup B$. If A or B are singletons, say $A = \{a\}$, $B = \{b\}$, or both, then we also refer to such a path as an a-B-path, A-b-path, or a-b-path respectively. If H_1, H_2 are subgraphs of G, we also write H_1-H_2-path to mean a $V(H_1)$-$V(H_2)$-path. If T is a tree and $u, v \in V(T)$, then the unique u-v-path in T is denoted uTv. Given a sequence of such paths $v_0T_1v_1, v_1T_2v_2, \dots, v_{n-1}T_nv_n$, the concatenation of these paths in their given order is denoted $v_0T_1v_1T_2v_2 \dots v_{n-1}T_nv_n$. The last vertex v_n may be omitted in this notation (e.g. $v_0T_1v_1T_2$) if T_n is a path, v_n is an endpoint of T_n, and the direction of traversal is obvious from context.

An *A-bridge* of G is either a subgraph consisting of an edge in G with both endpoints in A, or a connected component H of $G - A$ together with the vertices of A adjacent to H and the edges of G with one endpoint in A and the other in $V(H)$. The *attachments* of an A-bridge are the vertices of the A-bridge that are also in A.

2.2 Group-labelled graphs

Let (G, γ) be a Γ-labelled graph. The *weight* of (G, γ) and the *γ-value* of G are both defined as $\sum_{e \in E(H)} \gamma(e)$. A *$\Gamma$-labelled subgraph of* (G, γ) is a Γ-labelled graph $(H, \gamma|_H)$ where H is a subgraph of G and $\gamma|_H$ is the Γ-labelling of H obtained by restricting γ to $E(H)$. When it is understood that H is a subgraph of G, we simply write (H, γ) to denote $(H, \gamma|_H)$. If (G, γ) does not contain a Γ-nonzero cycle as a (Γ-labelled) subgraph, then we say that (G, γ) is *Γ-bipartite*, or that G is *γ-bipartite*.

Let $g \in \Gamma$ be an element such that $2g = 0$ (that is, either $g = 0$ or g has order two). Given a vertex $v \in V(G)$, define a new Γ-labelling γ' of G where

$$\gamma'(e) = \begin{cases} \gamma(e) + g & \text{if } e \text{ is incident with } v \\ \gamma(e) & \text{if } e \text{ is not incident with } v \end{cases}$$

We call this operation *shifting at v by g*. Since $2g = 0$, this preserves the weights of cycles and also of paths which do not contain v as an endpoint. We say that (G, γ_1) and (G, γ_2) are *shift-equivalent* if one can be obtained from the other by a sequence of shifting operations.

Let $\mathbf{0}$ denote the Γ-labelling that labels all edges 0. Clearly, if (G, γ) is shift-equivalent to $(G, \mathbf{0})$, then (G, γ) is Γ-bipartite. If G is 3-connected, then the converse also holds as we now show. First we need the following lemma.

Lemma 2.2.1. *Let Γ be an abelian group and let (G, γ) be a Γ-labelled graph such that $2\gamma(e) = 0$ for all $e \in E(G)$. If (G, γ) is Γ-bipartite, then (G, γ) is shift-equivalent to $(G, \mathbf{0})$.*

Proof. We proceed by induction on $|E(G)|$. If $|E(G)| = 0$ then there is nothing to prove. Otherwise let $e = uv \in E(G)$. Then $(G - e, \gamma)$ is also Γ-bipartite so there is a sequence of shift operations in (G, γ) resulting in a Γ-labelling γ' such that $\gamma'(f) = 0$ for all $f \in E(G) - e$. If e is a bridge in G, then we obtain $(G, \mathbf{0})$ by possibly shifting by $\gamma'(e)$ at each vertex in one side of the bridge e (here we use the assumption that $2\gamma(e) = 0$). Otherwise, e belongs to a cycle. But since shift operations preserve weights of cycles and (G, γ) is Γ-bipartite, it follows that $\gamma' = \mathbf{0}$. $\square$

Recall that graphs are assumed to have no loops; a cycle that is not simple consists of two parallel edges.

Lemma 2.2.2. *Let Γ be an abelian group and let (G, γ) be a Γ-labelled graph such that G is 3-connected and (G, γ) has no simple Γ-nonzero cycle. Then (G, γ) is Γ-bipartite and shift-equivalent to $(G, \mathbf{0})$.*

Proof. Let $e = uv$ be an edge of G. Since G is 3-connected, G contains two internally disjoint u-v-paths P_1 and P_2, each with at least 3 vertices. Since the three simple cycles in $P_1 \cup P_2 \cup \{e\}$ are Γ-zero, we have $\gamma(e) = -\gamma(P_1) = -\gamma(P_2)$ and, hence, $2\gamma(e) = 0$. If there is an edge e' parallel to e, then $\gamma(e') = \gamma(e)$ since otherwise either $P_1 \cup \{e\}$ or $P_1 \cup \{e'\}$ would be a simple Γ-nonzero cycle. Thus (G, γ) is Γ-bipartite, and since $2\gamma(e) = 0$ for all $e \in E(G)$, it follows from Lemma 2.2.1 that (G, γ) is shift-equivalent to $(G, \mathbf{0})$. $\square$

We reiterate that shifting in (undirected) group-labelled graphs can only be done by elements $g \in \Gamma$ such that $2g = 0$. In particular, in Lemma 2.2.2, if Γ has no element of order two, then the conclusion is that in fact $\gamma = \mathbf{0}$.

The next two lemmas show how 3-connectivity can be used to find a Γ-nonzero path.

Lemma 2.2.3. *Let Γ be an abelian group, (G, γ) a Γ-labelled graph, and let C be a cycle in G. Let w_1, w_2, w_3 be three distinct vertices on C and, for each $i \in [3]$, let Q_i denote the w_j-w_k-path in C that is disjoint from w_i, where $\{j, k\} = [3] - \{i\}$.*

(a) *If* $2\gamma(Q_1) \neq 0$*, then for some* $j \in \{2,3\}$*, the two* w_1*-*w_j*-paths in* C *have different weights.*

(b) *If* C *is* Γ*-nonzero, then for some distinct pair* $i, j \in [3]$*, the two* w_i*-*w_j*-paths in* C *have different weights.*

Proof. If the two w_1-w_2-paths in C have the same weight, then $\gamma(Q_3) = \gamma(Q_1) + \gamma(Q_2)$. If the two w_1-w_3-paths in C also have the same weight, then $\gamma(Q_2) = \gamma(Q_1) + \gamma(Q_3)$. Adding the two equalities gives $2\gamma(Q_1) = 0$, proving (a). If, in addition, the two w_2-w_3-paths in C have the same weight, then $\gamma(Q_1) = \gamma(Q_2) + \gamma(Q_3)$. Adding the three equalities gives $\gamma(Q_1) + \gamma(Q_2) + \gamma(Q_3) = 0$, proving (b). $\square$

Lemma 2.2.4. *Let* Γ *be an abelian group and let* (G, γ) *be a* Γ*-labelled graph. Let* C *be a* Γ*-nonzero cycle, let* $A \subseteq V(G)$*, and let* P_1, P_2, P_3 *be three disjoint* A*-*$V(C)$*-paths in* (G, γ)*. Then* $C \cup P_1 \cup P_2 \cup P_3$ *contains a* Γ*-nonzero* A*-path.*

Proof. Let w_i denote the endpoint of P_i in C for each $i \in [3]$, and define Q_i as in Lemma 2.2.3. If $|A \cap V(C)| \geq 2$, then at least one of the A-paths in C is Γ-nonzero. If $A \cap V(C) = \emptyset$, then the conclusion follows immediately from Lemma 2.2.3(b). So we may assume $|A \cap V(C)| = 1$ and, without loss of generality, that $w_3 \in A \cap V(C)$ (i.e. P_3 is a trivial path). Suppose that every A-path in $C \cup P_1 \cup P_2$ is Γ-zero. Considering the three A-paths in $C \cup P_1 \cup P_2$ containing Q_3, we deduce that $\gamma(P_2) = \gamma(Q_1)$ and $\gamma(P_1) = \gamma(Q_2)$. But this implies $0 = \gamma(P_1) + \gamma(Q_3) + \gamma(P_2) = \gamma(Q_2) + \gamma(Q_3) + \gamma(Q_1) = \gamma(C) \neq 0$, a contradiction. $\square$

The definitions of Erdős-Pósa property extend in the obvious way to families of group-labelled graphs. The family of Γ-nonzero A-paths satisfy the Erdős-Pósa property, as shown by Wollan [40]:

Theorem 2.2.5 (Wollan [40] Theorem 1.1)**.** *Let* k *be a positive integer, let* Γ *be an abelian group, let* (G, γ) *be a* Γ*-labelled graph, and let* $A \subseteq V(G)$*. Then* G *contains* k *disjoint* γ*-*

nonzero A-paths or a vertex set of size at most $f_{2.2.5}(k) := 50k^4 - 4$ hitting all γ-nonzero A-paths.

We remark that the theorem stated in [40] gives the bound $|X| \leq 50k^4$ rather than $|X| \leq 50k^4 - 4$, but this difference is clearly negligible in the proof in [40]. The modified bound will be convenient in some of our calculations.

If Γ is the product $\prod_{i\in[m]} \Gamma_i$ of m abelian groups for a positive integer m, then we denote by γ_i the composition of γ with the projection to Γ_i. For a subset $I \subseteq [m]$ we denote by Γ_I be the subgroup of Γ of all $g \in \Gamma$ with $\pi_i(g) = 0$ for all $i \in [m] \setminus I$. For a Γ-labelled graph (G, γ) and a subgroup Λ of Γ, $\Gamma/\Lambda = \{g + \Lambda : g \in \Gamma\}$ denotes the quotient group and the *induced* (Γ/Λ)*-labelling* of (G, γ) is the Γ/Λ-labelling λ defined by $\lambda(e) := \gamma(e) + \Lambda$ for all edges $e \in E(G)$.

2.3 Tangles

A *separation* in a graph G is an ordered pair of subgraphs (C, D) such that C and D are edge-disjoint and $C \cup D = G$. The *order* of a separation (C, D) is $|V(C) \cap V(D)|$. A separation of order at most k is a k*-separation*. A *tangle* $\mathcal{T}$ *of order* k is a set of $(k-1)$-separations of G such that

(T1) for every $(k-1)$-separation (C, D), either $(C, D) \in \mathcal{T}$ or $(D, C) \in \mathcal{T}$,

(T2) $V(C) \neq V(G)$ for all $(C, D) \in \mathcal{T}$, and

(T3) $C_1 \cup C_2 \cup C_3 \neq G$ for all $(C_1, D_1), (C_2, D_2), (C_3, D_3) \in \mathcal{T}$.

Given $(C, D) \in \mathcal{T}$, we say that C and D are the two *sides* of (C, D); C is the $\mathcal{T}$*-small side* and D is the $\mathcal{T}$*-large side* of (C, D).

Tangles can be thought of as an orientation of all small order separations so that they point to some "highly-connected" part of the graph in a consistent manner. For example, it is well-known that a connected graph on at least 2 vertices has a tree-decomposition into *blocks* (maximal subgraphs that are either 2-connected or isomorphic to K_2). For each block B, there is a tangle of order 2 consisting of all 1-separations (C, D) such that $B \subseteq D$. Examples of higher order tangles associated with large K_t-models and with large walls are given in sections 2.5 and 2.6 respectively.

Here, we describe another class of high order tangles which arise from counterexamples to the Erdős-Pósa property. Suppose $f : \mathbb{N} \to \mathbb{N}$ is *not* a (half-integral) Erdős-Pósa function for a family $\mathcal{F}$ of Γ-labelled graphs. Let us say that $((G, \gamma), k)$ is a *minimal counterexample to f being a (half-integral) Erdős-Pósa function for $\mathcal{F}$* if (G, γ) does not contain a (half-integral) $\mathcal{F}$-packing of size k nor an $\mathcal{F}$-hitting set of size at most $f(k)$, and moreover k is minimum among all such $((G, \gamma), k)$.

A standard argument appearing in various forms [6, 30, 37, 39] shows that, if $\mathcal{F}$ is a family of *connected* Γ-labelled graphs that does not satisfy the Erdős-Pósa property, then a minimal counterexample admits a tangle $\mathcal{T}$ of large order such that no $\mathcal{T}$-small side of a separation in $\mathcal{T}$ contains a Γ-labelled subgraph in $\mathcal{F}$. Recall that if (G, γ) is a Γ-labelled graph and H is a subgraph of G, then (H, γ) denotes the Γ-labelled subgraph $(H, \gamma|_H)$ of (G, γ).

Lemma 2.3.1. *Let Γ be an abelian group and let $\mathcal{F}$ be a family of connected Γ-labelled graphs. Let $f : \mathbb{N} \to \mathbb{N}$ be a function, and suppose t is a positive integer such that $t \leq f(k) - 2f(k-1)$ and $t \leq f(k)/3$. If $((G, \gamma), k)$ is a minimal counterexample to f being a (half-integral) Erdős-Pósa function for $\mathcal{F}$, then G admits a tangle $\mathcal{T}$ of order $t + 1$ such that for each $(C, D) \in \mathcal{T}$, (C, γ) does not contain a Γ-labelled subgraph in $\mathcal{F}$ and $(D - C, \gamma)$ contains a Γ-labelled subgraph in $\mathcal{F}$.*

Proof. Let (C, D) be a t-separation in G. We first show that exactly one of (C, γ) and (D, γ) contains a Γ-labelled subgraph in $\mathcal{F}$. If neither side contains a Γ-labelled subgraph in $\mathcal{F}$, then $V(C \cap D)$ is an $\mathcal{F}$-hitting set of size at most $t \leq f(k)$, a contradiction. Next suppose that both sides contain a Γ-labelled subgraph in $\mathcal{F}$. Then neither $(C - D, \gamma)$ nor $(D - C, \gamma)$ contains a (half-integral) $\mathcal{F}$-packing of size $k - 1$. By minimality of k, $(C - D, \gamma)$ and $(D - C, \gamma)$ contain $\mathcal{F}$-hitting sets X and Y respectively, each of size at most $f(k-1)$. Since every Γ-labelled graph in $\mathcal{F}$ is connected, every Γ-labelled subgraph of $(G - X - Y, \gamma)$ in $\mathcal{F}$ intersects $C \cap D$. Thus $Z := X \cup Y \cup V(C \cap D)$ is an $\mathcal{F}$-hitting set with $|Z| \leq 2f(k-1) + t \leq f(k)$, a contradiction.

Let $\mathcal{T}$ be the set of t-separations (C, D) of G such that (C, γ) does not contain a Γ-labelled subgraph in $\mathcal{F}$. Note that $(D - C, \gamma)$ contains a Γ-labelled subgraph in $\mathcal{F}$ since otherwise $V(C \cap D)$ would again be a small hitting set.

It remains to show that $\mathcal{T}$ is a tangle. Clearly, $\mathcal{T}$ satisfies **(T1)** and **(T2)**. To see **(T3)**, suppose there exist (C_1, D_1), (C_2, D_2), $(C_3, D_3) \in \mathcal{T}$ such that $C_1 \cup C_2 \cup C_3 = G$. Since no (C_i, γ) contains a Γ-labelled subgraph in $\mathcal{F}$ and every Γ-labelled graph in $\mathcal{F}$ is connected, every Γ-labelled subgraph of (G, γ) in $\mathcal{F}$ intersects $V(C_j \cap D_j)$ for some $j \in [3]$. But this implies that $Z := V(C_1 \cap D_1) \cup V(C_2 \cap D_2) \cup V(C_3 \cap D_3)$ is an $\mathcal{F}$-hitting set with $|Z| \leq 3t \leq f(k)$, a contradiction. $\square$

Let $\mathcal{T}$ be a tangle of order k in a graph G. Given a positive integer $k' \leq k$, the set $\mathcal{T}'$ of $(k'-1)$-separations (C, D) in G such that $(C, D) \in \mathcal{T}$ is a tangle of order k', called the *truncation* of $\mathcal{T}$ to order k'. If $X \subseteq V(G)$ is a set of fewer than k vertices, then there is a tangle of order $k - |X|$ in $G - X$, consisting of the $(k - |X| - 1)$-separations of $G - X$ which can be written as $(C - X, D - X)$ for some $(C, D) \in \mathcal{T}$. We denote this tangle by $\mathcal{T} - X$.

2.4 3-blocks

Due to the 3-connectivity condition that arises naturally in undirected group-labelled graphs (e.g. Lemmas 2.2.2–2.2.4), we will need to work with *3-blocks* of graphs. The decomposition of 2-connected graphs into a tree structure of 3-connected components was first given by Tutte [38]. Here, we use a definition of 3-blocks adapted from the terminology of k-*blocks* studied in [29, 7, 8].

Let G be a graph. A separation (C, D) of G *properly separates* two vertices u and v if $V(C - D)$ and $V(D - C)$ each contain one of $\{u, v\}$. A vertex set $U \subseteq V(G)$ is *2-*

inseparable in G if no two vertices of U are properly separated by a 2-separation in G (that is, for every 2-separation (C, D) of G, we have either $U \subseteq V(C)$ or $U \subseteq V(D)$). A 2-inseparable set U is *maximal* if there does not exist a 2-inseparable set properly containing U. Observe that if U is a maximal 2-inseparable set, then every U-bridge of G has at most 2 attachments.

Let (G, γ) be a Γ-labelled graph. A *3-block of* (G, γ) is a Γ-labelled graph (B, γ_B) obtained from a maximal 2-inseparable set $U = V(B)$ of G as follows: For each $u, v \in U$ and $\alpha \in \Gamma$, if there exists a U-path in (G, γ) with endpoints u, v and weight α, then add a new (possibly parallel) edge uv with label α. Note that B may not be a subgraph of G. For example, if G is a subdivision of a simple 3-connected graph H, then $V(H)$ is a maximal 2-inseparable set in G, and the corresponding 3-block is (H, γ_H) where for each $e \in E(H)$, $\gamma_H(e)$ is the weight of the path in (G, γ) corresponding to e. Also observe that if $|U| \geq 4$, then B is a 3-connected graph.

The following proposition is immediate from the definition of 3-blocks of Γ-labelled graphs.

Proposition 2.4.1. *Let Γ be an abelian group and let (B, γ_B) be a 3-block of a Γ-labelled graph (G, γ). For each subgraph P_B of (B, γ_B) that is either a simple cycle or a path, there exists a cycle or a path P respectively in (G, γ) with weight equal to the weight of P_B such that $V(P) \cap V(B) = V(P_B)$, and the order of the vertices in $V(P_B)$ appearing in P is the same as the order appearing in P_B.* $\square$

We will primarily be concerned with a particular 3-block associated with a given tangle.

Lemma 2.4.2. *Let $\mathcal{T}$ be a tangle of order 3 in a graph G. Then there is a unique maximal 2-inseparable set U that is contained in every $\mathcal{T}$-large side (that is, we have $U \subseteq V(D)$ for all $(C, D) \in \mathcal{T}$). Moreover, we have $|U| \geq 4$.*

Proof. If U_1 and U_2 are distinct maximal 2-inseparable sets of G, then there is a 2-separation of G properly separating a vertex in U_1 and a vertex in U_2. It follows from **(T1)** that there is at most one maximal 2-inseparable set of G contained in every $\mathcal{T}$-large side.

We now show that such a maximal 2-inseparable set exists. Let us say that a 2-separation (C, D) of G is *good* if $(C, D) \in \mathcal{T}$ and, if $|V(C \cap D)| = 2$, then there is a path in C connecting the two vertices of $V(C \cap D)$. Let us also say that a 2-separation (C, D) is *tight* if it is good and there does not exist a good separation $(C', D') \in \mathcal{T}$ such that $C \subseteq C'$ and $D' \subsetneq D$. Let $U \subseteq V(G)$ be the set of vertices which belong to the intersection $V(C \cap D)$ of some tight separation $(C, D) \in \mathcal{T}$. We claim that U is a maximal 2-inseparable set contained in every $\mathcal{T}$-large side, and that $|U| \geq 4$.

Let us first show that $|U| \geq 3$. Suppose $|U| \leq 2$. Let $(C, D) \in \mathcal{T}$ be a separation such that $V(C \cap D) \subseteq U$ and, subject to this condition, D is minimal. Note that the first condition is satisfied by the separation $(G_U, G) \in \mathcal{T}$ where G_U denotes the subgraph of G with vertex set U and no edges. Now by **(T2)**, $V(D - C)$ is nonempty, and it follows from **(T3)** and the minimality of D that $D - C$ is connected. Let $v \in V(D - C)$. The separation $(G[\{v\}], G)$ is good but not tight (since $v \notin U$), so there is a tight separation $(C_v, D_v) \in \mathcal{T}$ such that $V(C_v \cap D_v) \subseteq U$ and $v \in V(C_v - D_v)$. Since $D - C$ is connected and D is minimal, this implies that $D \subseteq C_v$, hence $G = C \cup D = C \cup C_v$, contradicting **(T3)**. We thus have $|U| \geq 3$.

Next we show that U is contained in every $\mathcal{T}$-large side. Suppose to the contrary that there exists $(C, D) \in \mathcal{T}$ such that $u \in V(C - D)$ for some $u \in U$. By the definition of U, there is a tight separation $(C_u, D_u) \in \mathcal{T}$ such that $u \in V(C_u \cap D_u)$. Consider the separation $(C \cup C_u, D \cap D_u)$. Since the orders of (C, D) and (C_u, D_u) are each at most 2, and since $u \in V(C - D)$, the order of $(C \cup C_u, D \cap D_u)$ is at most 3. If its order is equal to 3, then the orders of (C, D) and (C_u, D_u) are both 2, $V(C \cap D) \subseteq V(D_u - C_u)$, and $V(C_u \cap D_u) - \{u\} \subseteq V(D - C)$ (see Figure 2.1a). Since $V(C \cap D) \cap V(C_u)$ is empty, this contradicts the assumption that (C_u, D_u) is good.

We may thus assume that $(C \cup C_u, D \cap D_u)$ is a 2-separation. It follows from **(T1)** and

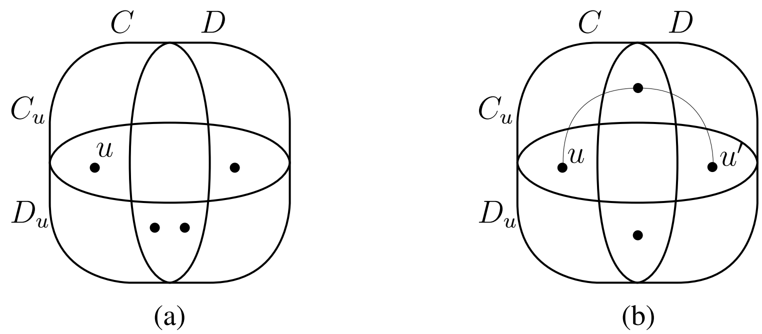


Figure 2.1: Since (C_u, D_u) is good, if $|V(C_u \cap D_u)| = 2$, then there is a path in C_u connecting the two vertices of $V(C_u \cap D_u)$. In (b), $(C \cup C_u, D \cap D_u)$ is not good, so there is a 1-separation (C', D') with $V(C' \cap D') = \{u'\}$ which violates the tightness of (C_u, D_u).

(T3) that $(C \cup C_u, D \cap D_u) \in \mathcal{T}$. Note that $C_u \subseteq C \cup C_u$ and, since $u \in V(D_u - D)$, we have $D \cap D_u \subsetneq D_u$. By the assumption that (C_u, D_u) is tight, we have that $(C \cup C_u, D \cap D_u)$ is not good; that is, $|V((C \cup C_u) \cap (D \cap D_u))| = 2$ and every $V((C \cup C_u) \cap (D \cap D_u))$-bridge of $C \cup C_u$ has at most one attachment (see Figure (2.1b) for one possible configuration). But this implies that there is a 1-separation $(C', D') \in \mathcal{T}$ such that $C_u \subseteq C'$, $V(C' \cap D') \subseteq V((C \cup C_u) \cap (D \cap D_u))$, and $D' \subseteq D_u - \{u\} \subsetneq D_u$, contradicting the tightness of (C_u, D_u).

Hence, U is contained in every $\mathcal{T}$-large side. Note that this also implies that U is 2-inseparable. To see that U is a maximal 2-inseparable set, let $u' \in V(G) - U$. Then the separation $(G[\{u'\}], G)$ is good but not tight, so there exists a tight separation $(C, D) \in \mathcal{T}$ such that $u' \subseteq V(C - D)$, $D \subsetneq G$, and $V(C \cap D) \subseteq U$. Since $|U| \geq 3$ and U is contained in $V(D)$, this implies that (C, D) properly separates u' from a vertex in U. We conclude that U is the unique maximal 2-inseparable set that is contained in every $\mathcal{T}$-large side.

It remains to show that $|U| \geq 4$. We have already shown $|U| \geq 3$, so suppose $|U| = 3$ and write $U = \{u_1, u_2, u_3\}$. For $i \in [3]$, consider the separation $(C_i, D_i) \in \mathcal{T}$ such that $V(C_i \cap D_i) = U - \{u_i\}$ and, subject to this condition, C_i is maximal. Since $U \subseteq D_i$ for all $i \in [3]$, every U-bridge of G (which has at most two attachments in U) is contained in C_i for some $i \in [3]$. This implies that $C_1 \cup C_2 \cup C_3 = G$, contradicting **(T3)**. $\square$

Lemma 2.4.3. *Let $\mathcal{T}$ be a tangle of order $k \geq 3$ in a graph G, and let $X \subseteq V(G)$ with $|X| \leq k-3$. Then there is a unique maximal 2-inseparable set U of $G - X$ such that $U \cup X$ is not contained in any $\mathcal{T}$-small side. Moreover, we have $|U| \geq 4$.*

Proof. Let $\mathcal{T}_X$ denote the tangle of order 3 in $G - X$ that is a truncation of $\mathcal{T} - X$ (which has order $k - |X| \geq 3$). Let U be the unique maximal 2-inseparable set of $G - X$ contained in every $\mathcal{T}_X$-large side, given by Lemma 2.4.2. If U' is a maximal 2-inseparable set $G - X$ distinct from U, then there is a 2-separation $(C_X, D_X) \in \mathcal{T}_X$ such that $U \subseteq V(C_X)$, so $U' \cup X$ is contained in $G[V(C_X) \cup X]$, which forms the $\mathcal{T}$-small side of a $(k-1)$-separation in G.

Now suppose that $U \cup X$ is contained in a $\mathcal{T}$-small side. Let $(C, D) \in \mathcal{T}$ such that $U \cup X \subseteq V(C)$ and, subject to this condition, D is minimal. Then $D - X$ is connected by **(T3)**. Since U is a maximal 2-inseparable set of $G - X$ and $U \cup X \subseteq V(C)$, it follows that $D - X$ is contained in a U-bridge H of $G - X$. Note that H has at most two attachments in U. Hence, H forms the $\mathcal{T}_X$-small side of a 2-separation in $G - X$, which implies that $G[V(H) \cup X]$ forms the $\mathcal{T}$-small side of a $(k-1)$-separation in $\mathcal{T}$. Since $D \subseteq G[V(H) \cup X]$, we have $G = C \cup D = C \cup G[V(H) \cup X]$, contradicting **(T3)**. Therefore, U is the unique maximal 2-inseparable set of $G - X$ such that $U \cup X$ is not contained in any $\mathcal{T}$-small side. Finally, note that $|U| \geq 4$ by Lemma 2.4.2. $\square$

Let (G, γ) be a Γ-labelled graph. Let $\mathcal{T}$ be a tangle of order $k \geq 3$ in G and let $X \subseteq V(G)$ with $|X| \leq k - 3$. The *$\mathcal{T}$-large 3-block of* $(G - X, \gamma)$ is the 3-block (B, γ_B) of $(G - X, \gamma)$ obtained from the unique maximal 2-inseparable set U of $G - X$ such that $U \cup X$ is not contained in any $\mathcal{T}$-small side, as given by Lemma 2.4.3. Note that B is 3-connected, since $|U| \geq 4$ by Lemma 2.4.3.

2.5 K_m-models

Let $v_1, \ldots, v_m$ denote the vertices of the complete graph K_m. A *K_m-model* μ consists of a collection of disjoint trees $\mu(v_i)$ for $i \in [m]$ and edges $\mu(v_iv_j)$ for distinct $i, j \in [m]$ such that $\mu(v_iv_j)$ has one endpoint in $\mu(v_i)$ and the other in $\mu(v_j)$. It is easy to see that a graph G contains a K_m-model if and only if K_m can be obtained from G by a sequence of vertex deletions, edge deletions, and edge contractions; the trees $\mu(v_i)$ correspond to the subgraphs of G that were contracted to form the vertex v_i of K_m.

Let $U \subseteq \{v_1, \ldots, v_m\}$. Then $\mu[U]$ denotes the graph defined by

$$\mu[U] = \bigcup_{v_i \in U} \mu(v_i) \cup \bigcup_{v_i, v_j \in U} \mu(v_iv_j).$$

If we are given U explicitly, say $U = \{v_{i_1}, \ldots, v_{i_k}\}$, then we simply write $\mu[v_{i_1}, \ldots, v_{i_k}]$. When there is no room for ambiguity, we also write μ to refer to the subgraph $\mu[V(K_m)] = \mu[\{v_1, \ldots, v_m\}]$. The *$K_{|U|}$-submodel π of μ restricted to U* is the $K_{|U|}$-model of the complete graph on the vertex set U, given by $\pi(v_i) = \mu(v_i)$ and $\pi(v_iv_j) = \mu(v_iv_j)$ for all $v_i, v_j \in U$. If $n \leq m$ and η is a K_n-model such that each tree of η contains some tree of μ, then we say that η is an *enlargement* of μ. Note that a submodel of μ is also an enlargement of μ.

We now describe a tangle associated with a K_m-model. Consider an $(m-1)$-separation (C, D) in a graph G containing a K_m-model μ. Then there is exactly one side of (C, D) that intersects every tree $\mu(v_i)$ of μ. Indeed, since $\mu(v_1), \ldots, \mu(v_m)$ are disjoint trees and $|V(C \cap D)| \leq m - 1$, there is a tree $\mu(v_i)$ disjoint from $C \cap D$; assume without loss of generality that $\mu(v_i) \subseteq D - C$. Since every other tree $\mu(v_j)$ is adjacent to $\mu(v_i)$ by the edge $\mu(v_iv_j) \in E(G)$, D is the unique side of (C, D) intersecting every tree of μ. The set of all such $(m-1)$-separations satisfies properties **(T1)** and **(T2)**. Note however that it may violate **(T3)** if $m \geq 3$ (for example, consider K_3).

Define $k = \lceil \frac{2m}{3} \rceil$ and let $\mathcal{T}_\mu$ be the set of $(k-1)$-separations (C, D) in G such that D

intersects every tree of μ. It is straightforward to verify that $\mathcal{T}_\mu$ satisfies **(T3)** (see [33]), hence $\mathcal{T}_\mu$ is a tangle of order k. We call $\mathcal{T}_\mu$ the *tangle induced by* μ. If η is a submodel or an enlargement of μ, then $\mathcal{T}_\eta$ is a truncation of $\mathcal{T}_\mu$.

Let (G, γ) be a Γ-labelled graph. We say that a K_m-model μ in G is Γ-*bipartite* if for every choice of four distinct indices $i, j, k, l \in [m]$, we have that $(\mu[v_i, v_j, v_k, v_l], \gamma)$ is a Γ-bipartite Γ-labelled graph. Oppositely, if for every choice of four distinct indices $i, j, k, l \in [m]$, $(\mu[v_i, v_j, v_k, v_l], \gamma)$ contains a Γ-nonzero cycle, then we say that μ is Γ-*odd*.

Remark 2.5.1. For *directed* group-labelled graphs, the definition of Γ-bipartite (resp. Γ-odd) K_m-models in [23] only require that for every *three* distinct indices $i, j, k \in [m]$, $(\mu[v_i, v_j, v_k], \gamma)$ is Γ-bipartite (resp. not Γ-bipartite). However, the property we actually want of a Γ-bipartite K_m-model μ is for $(\mu[V(K_m)], \gamma)$, as a Γ-labelled graph, to be Γ-bipartite. And in contrast to the directed setting, the above condition does not suffice for undirected group-labelled graphs. For example, let $\Gamma = \mathbb{Z}/3\mathbb{Z}$, $G = K_m$, $\gamma(e) = 1$ for all $e \in E(G)$, and let μ be a K_m-model in G. Then every triangle in G is Γ-zero, so $(\mu[v_i, v_j, v_k], \gamma)$ is Γ-bipartite for all distinct $i, j, k \in [m]$, but $(\mu(V(K_m)), \gamma)$ is clearly not Γ-bipartite for $m \geq 4$. We will show in Lemma 4.1.3 that our definition of Γ-bipartite K_m-models does give this desired property.

2.6 Walls

Let $r, s \geq 2$ be integers. An $r \times s$-*grid* is a graph with vertex set $[r] \times [s]$ and edge set

$$\{(i, j)(i', j') : |i - i'| + |j - j'| = 1\}.$$

An *elementary* $r \times s$-*wall* is the subgraph of an $(r+1) \times (2s+2)$-grid obtained by deleting the edges

$$\left\{(2i-1, 2j)(2i, 2j) : i \in \left[\left\lceil \tfrac{r}{2} \right\rceil\right], j \in [s+1]\right\} \cup \left\{(2i, 2j-1)(2i+1, 2j-1) : i \in \left[\left\lceil \tfrac{r-1}{2} \right\rceil\right], j \in [s+1]\right\},$$

then deleting the two vertices of degree 1. An *elementary r-wall* W is an elementary $r \times r$-wall. We say that an $r \times s$-wall is a *wall of order* $\min\{r+1, s+1\}$. Figure 2.2 shows an elementary 6×5-wall.

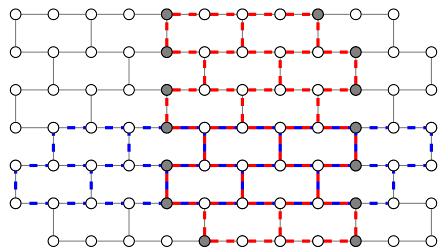


Figure 2.2: A 6×5-wall with a 3-column-slice highlighted in red and a 3-row-slice highlighted in blue. The column-boundary of the red 3-column-slice is indicated by the solid vertices.

Let W be an elementary $r \times s$-wall. There is a set of $r+1$ disjoint paths called the *rows* of W such that each path has vertex set $\{(i', j') \in V(W) : i' = i\}$ for some $i \in [r+1]$. Let R_i^W, $i \in [r+1]$, denote the i-th row from top to bottom of some fixed planar embedding of W. Then there is a unique set of $s+1$ disjoint R_1^W-R_{r+1}^W-paths in W called the *columns* of W. The i-th column from left to right is denoted by C_i^W. For $(i, j) \in [r] \times [s]$, the (i, j)*-th brick* is the facial cycle of length 6 contained in the union $R_i^W \cup R_{i+1}^W \cup C_j^W \cup C_{j+1}^W$. Note that the order of the i-th vertical/rows may be reversed depending on the orientation of the embedding.

The *perimeter* of W is the cycle in the union $R_1^W \cup R_{r+1}^W \cup C_1^W \cup C_{s+1}^W$. The *corners* of W are the four vertices that are endpoints of R_1^W or R_{r+1}^W (equivalently, the endpoints of C_1^W or C_{s+1}^W).

An $r \times s$*-wall* or r*-wall* is a subdivision of an elementary $r \times s$-wall or r-wall respectively. The *nails* or the *branch vertices* of an r-wall W are the vertices corresponding to those of the elementary r-wall before subdivision, and the set of nails of W is denoted by either N^W or $b(W)$. The *top nails* are the nails on the first row that are not corners. All other terminology on elementary walls in the preceding paragraphs extend to walls in the obvious way. Note that, given a wall W, there may be many possible choices for its nails of degree 2. If this choice is not explicitly described, it is assumed implicitly that a choice of nails accompanies each wall. Other definitions which depend on the choice of nails are assumed to be with respect to this implicit choice.

Let $r' \leq r$ and $s' \leq s$. An $r' \times s'$-*subwall* of an $r \times s$-wall W is an $r' \times s'$-wall W' that is a subgraph of W such that each horizontal and column of W' is a subpath of some horizontal and column of W respectively. If, in addition, the set of indices i such that R_i^W contains a row of W' and the set of indices j such that C_j^W contains a column of W' both form contiguous subsets of $[r+1]$ and $[s+1]$ respectively, then we say that W' is a *compact* subwall of W. A subwall W' is k-*contained* in W if R_i^W and C_j^W are disjoint from W' for all $i, j \leq k$ and for all $i > r - k + 1$ and $j > s - k + 1$. If W' is 1-contained in W, then there is a unique choice of nails of W' such that they have degree 3 in W. We call these the *nails of* W' *with respect to* W.

For an integer $c \geq 3$, we call a subwall W' of a wall W a c-*column-slice of* W if

- the set of nails of W' is exactly $N^W \cap V(W')$,
- there is a column of W' that is a column of W, and
- W' has exactly c columns, see Figure 2.2 for an example.

Similarly, for an integer $r \geq 3$, we call a subwall W' of a wall W an r-*row-slice of* W if

- the set of nails of W' is exactly $N^W \cap V(W')$,
- there is a row of W' that is a row of W, and
- W' has exactly r rows.

Note that in an r-row-slice W' of W, depending on the location, the first column of W' may be in the last column of W by the definition of a wall.

Let W be a wall in a graph G. The *column-boundary of* W is the set of all endvertices of rows of W. A W-*handle* is a nontrivial W-path in G whose endvertices are in the column-boundary of W.

Let W be a (c, r)-wall and let W' be a c'-column-slice of W for some $3 \leq c' \leq c$. For a path P whose endvertices are nails of W, the *row-extension of P to W' in W* is a W'-handle containing P that is contained in the union of P and the rows of W. We can easily observe that if such a W'-handle exists, then it is unique. Note that the row-extension of a W-handle to W' always exists. For a set $\mathcal{P}$ of disjoint W-handles, we define the *row-extension of $\mathcal{P}$ to W' in W* to be the set of row-extensions of the paths in $\mathcal{P}$ to W' in W. Note that these W'-handles are disjoint.

Let W be an r-wall contained in a graph G. If (C, D) is an r-separation in G, then exactly one side of (C, D) contains a row of W, and the set of r-separations (C, D) such that D contains a row forms a tangle $\mathcal{T}_W$ of order $r + 1$ [33], called the *tangle induced by* W. A tangle $\mathcal{T}$ in G *dominates* the wall W if $\mathcal{T}_W$ is a truncation of $\mathcal{T}$. If W' is a subwall of W, then $\mathcal{T}_{W'}$ is a truncation of $\mathcal{T}_W$ (that is, $\mathcal{T}_W$ dominates W').

Let W be an r-wall contained in a graph G and let P be a $V_{\neq 2}(W)$-path in W with endpoints $x, y \in N^W$. Suppose there is an x-y-path R in G such that R is internally disjoint from $W - P$, and let W' be the wall obtained from W by replacing P with R. Then we say that W' is a *local rerouting* of W. Note that if W' is a local rerouting of W, then $\mathcal{T}_{W'} = \mathcal{T}_W$.

The following fundamental result of Robertson and Seymour is also known as the grid minor theorem:

Theorem 2.6.1 (Robertson, Seymour, and Thomas [34]). *There exists a function $f_{2.6.1} : \mathbb{N} \to \mathbb{N}$ such that if $g \geq 3$ is an integer and $\mathcal{T}$ is a tangle in a graph G of order at least $f_{2.6.1}(g)$, then $\mathcal{T}$ dominates a g-wall W in G.* $\square$

A Γ-labelled wall (W, γ) is *facially Γ-odd* if every brick is a Γ-nonzero cycle. We say that (W, γ) is *strongly Γ-bipartite* if it is shift-equivalent to (W, γ') such that every $b(W)$-path in (W, γ') is Γ-zero. Note that this definition depends on the choice of the nails of W. This is a slightly stronger condition than just requiring (W, γ) as a Γ-labelled graph to be Γ-bipartite, but the difference is superficial; if (W, γ) is a Γ-bipartite r-wall with $r \geq 3$, then by Lemma 2.2.2, the $(r-2)$-wall 1-contained in (W, γ) with the choice of nails with respect to W is strongly Γ-bipartite.

2.6.1 Flat walls

Let (C, D) be a separation of order $k \leq 3$ such that there is a vertex $v \in V(D-C)$ and k paths from v to $C \cap D$ pairwise disjoint except at v. Let H be the graph obtained from C by adding an edge between each nonadjacent pair of vertices in $C \cap D$. Then we say that H is an *elementary reduction of* G *with respect to* (C, D). Let $X \subseteq V(G)$. If a graph H can be obtained from G by a sequence of elementary reductions with respect to separations (C, D) for which $X \subseteq V(C)$, then we say that H is an X*-reduction* of G.

Let W be a wall in a graph G and let O denote the perimeter of W. Suppose there is a separation (C, D) of G such that $V(C \cap D) \subseteq V(O)$, $V(W) \subseteq V(D)$, and the nails of W are in C. If there is a $V(C \cap D)$-reduction of D that can be embedded on a closed disk Δ so that $V(C \cap D)$ lies on the boundary of Δ and the order of $V(C \cap D)$ along the boundary of Δ agrees with the order along O, then we say that the wall W is *flat in* G and that the separation (C, D) *certifies* that W is flat. Note that a subwall of a flat wall is flat, and a local rerouting of a flat wall is also flat.

We can now state the flat wall theorem [32, 10].

Theorem 2.6.2 (Theorem 2.2 in [10])**.** *Then there exists a function* $f_{2.6.2} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that if a graph* G *contains an* $f_{2.6.2}(r, t)$*-wall* W*, then one of the following outcomes hold:*

1. G *contains a* K_t*-model* μ *such that* $\mathcal{T}_\mu$ *is a truncation of* $\mathcal{T}_W$*.*

2. *There exists* $Z \subseteq V(G)$ *with* $|Z| \leq t-5$ *and an* r*-subwall* W' *of* W *that is disjoint from* Z *and flat in* $G - Z$*.* $\square$

2.7 Linkages and handlebars

Let G be a graph and let $A, B \subseteq V(G)$. A *linkage* is a set of disjoint paths. An A-*linkage* is a linkage of A-paths and an A-B-*linkage* is a linkage of A-B-paths.

For a set $\mathcal{G}$ of graphs, we denote by $\bigcup \mathcal{G}$ the union of the graphs in $\mathcal{G}$. By slight abuse of notation, we say two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs are *disjoint* if the graphs $\bigcup \mathcal{G}_1$ and $\bigcup \mathcal{G}_2$ are disjoint.

Let $(X, \prec)$ be a linearly ordered set. We say two disjoint subsets $\{x_1, x_2\}$ and $\{y_1, y_2\}$ of X of size 2 with $x_1 \prec x_2$ and $y_1 \prec y_2$ are

- *in series* if either $x_2 \prec y_1$ or $y_2 \prec x_1$;
- *nested* if either $x_1 \prec y_1 \prec y_2 \prec x_2$ or $y_1 \prec x_1 \prec x_2 \prec y_2$; and
- *crossing* otherwise.

A set $S \subseteq \binom{X}{2}$ of pairwise disjoint sets is *in series*, *nested*, or *crossing*, respectively, if its elements are pairwise in series, nested, or crossing, respectively, and S is called *pure* if it is in series, nested, or crossing. If $\mathcal{P}$ is an X-linkage, then it is *in series, nested, crossing,* and *pure* if the set

$$\{\{x, y\} : \{x, y\} \text{ is the set of endpoints of a path in } \mathcal{P}\}$$

is in series, nested, crossing, and pure respectively.

A straightforward argument shows the following lemma (see also [23, Lemma 25]).

Lemma 2.7.1. *Let t be a positive integer, let $(X, \prec)$ be a linearly ordered set, and let $S \subseteq \binom{X}{2}$ be a set of pairwise disjoint sets. If S has size greater than t^3, then S contains a pure subset of size greater than t.*

Proof. First consider the partial order $\prec_1$ on S such that for $\{a, b\}, \{c, d\} \in S$ with $a \prec b$ and $c \prec d$ we have $\{a, b\} \prec_1 \{c, d\}$ if $b \prec c$. By Dilworth's Theorem [15], S contains either a chain of size greater than t with respect to $\prec_1$ or an antichain of size greater than t^2 with respect to $\prec_1$. In the first case we have a subset that is in series, so suppose instead that there is some $S' \subseteq S$ of size greater than t^2 that is an antichain with respect to $\prec_1$. Let $\prec_2$ be the partial order on S' such that for $\{a,b\}, \{c,d\} \in S$ with $a \prec b$ and $c \prec d$ we have $\{a,b\} \prec_2 \{c,d\}$ if $a \prec c$ and $b \prec d$. Again by Dilworth's Theorem, there is some $S'' \subseteq S'$ of size greater than t such that S'' is either a chain or an antichain with respect to $\prec_2$, and hence either crossing or nested, respectively. $\square$

Let (W,γ) be a Γ-labelled wall in (G,γ) with the set N of top nails. A *linkage of* (W,γ) is an N-linkage in $G-(W-N)$. Note that this definition depends on the choice of the top nails of W, which may be implicit. A linkage of (W,γ) is *pure* if it is pure with respect to a linear ordering of N given by the top row of (W,γ). A linkage $\mathcal{P}$ of (W,γ) is Γ-*odd* if $(W \cup P,\gamma)$ contains a Γ-nonzero cycle containing P for all $P \in \mathcal{P}$. If (W,γ) is strongly Γ-bipartite and $\mathcal{P}$ is a Γ-odd linkage of (W,γ), then (G,γ) is shift-equivalent to (G,γ') such that every $b(W)$-path Q in W satisfies $\gamma'(Q)=0$ and every path P in $\mathcal{P}$ satisfies $\gamma'(P) \neq 0$.

Let W be an $r \times c$-wall. Let $\prec_W$ be the linear order on the column-boundary of W such that $v \prec_W w$ if at least one of the following holds.

- v is in the first column and w is in the last column.
- Both v and w are in the first column and the index of the row containing v is lower than the index of the row containing w.
- Both v and w are in the last column and the index of the row containing v is higher than the index of the row containing w.

A set $\mathcal{P}$ of W-handles is *pure*, *nested*, *in series*, or *crossing*, respectively, if the set of sets of endvertices of all the paths in $\mathcal{P}$ is pure, nested, in series, or crossing, respectively, with respect to $\prec_W$. We call a set $\mathcal{P}$ of disjoint W-handles a W-*handlebar* if $\mathcal{P}$ is pure and there are two paths A and B in $C_1^W \cup C_c^W$ such that each W-handle in $\mathcal{P}$ is a $V(A)$-$V(B)$-path. Observe that if $\mathcal{P}$ is a W-handlebar in series having at least two W-handles, then all the endvertices of W-handles in $\mathcal{P}$ are in C_i^W for some $i \in \{1, c\}$.

Two W-handlebars $\mathcal{P}_1$ and $\mathcal{P}_2$ are *non-mixing* if for each $i \in [2]$ there are (not necessarily disjoint) paths A_i and B_i in $C_1^W \cup C_c^W$ such that $\mathcal{P}_i$ is a set of $V(A_i)$-$V(B_i)$-paths and $A_1 \cup B_1$ and $A_2 \cup B_2$ are disjoint.

CHAPTER 3

STATEMENTS OF MAIN RESULTS AND APPLICATIONS

3.1 Flat wall theorem for undirected group-labelled graphs

Recall the flat wall theorem (Theorem 2.6.2) which says that, given a large wall, either there is a large piece of the wall that is almost flat, or there is a large K_t-model that is highly connected to the wall. Our structure theorem imposes additional conditions on the weights of cycles in the K_t-model or the flat wall.

Theorem 3.1.1. *Let Γ be an abelian group and let $r, t \geq 1$ be integers. Then there exist integers $g(r,t)$ and $h(r,t)$, where $h(r,t) \leq g(r,t) - 3$, such that if a Γ-labelled graph (G, γ) contains a wall (W, γ) of size at least $g(r,t)$, then one of the following outcomes hold:*

(1) There is a Γ-odd K_t-model μ in G such that $\mathcal{T}_\mu$ is a truncation of $\mathcal{T}_W$.

(2) There exists $Z \subseteq V(G)$ with $|Z| \leq h(r,t)$ and a flat $50r^{12}$-wall (W_0, γ) in $(G-Z, \gamma)$ such that $\mathcal{T}_{W_0}$ is a truncation of $\mathcal{T}_W$ and either

(a) (W_0, γ) is facially Γ-odd, or

(b) (W_0, γ) is strongly Γ-bipartite and there is a pure Γ-odd linkage of (W_0, γ) of size r.

(3) There exists $Z \subseteq V(G)$ with $|Z| \leq h(r,t)$ such that the $\mathcal{T}_W$-large 3-block of $(G - Z, \gamma)$ is Γ-bipartite.

3.1.1 Proof outline of Theorem 3.1.1

The proof of Theorem 3.1.1 follows the proof outline of Huynh, Joos, and Wollan [23] for directed group-labelled graphs. We proceed first by applying the flat wall theorem

(Theorem 2.6.2) to obtain one of its two outcomes. If there is a large K_m-model π in G such that $\mathcal{T}_\pi$ is a truncation of $\mathcal{T}_W$, then by Ramsey's theorem for 4-uniform hypergraphs, we obtain a large submodel μ of π that is either Γ-odd or Γ-bipartite. The first case satisfies outcome (1) of Theorem 3.1.1. In the second case, we show that the Γ-labelled subgraph (μ, γ) is Γ-bipartite (see also Remark 2.5.1). We then look to enlarge μ to a Γ-odd K_t-model by choosing a vertex s_i in each tree $\mu(v_i)$ and finding many disjoint Γ-nonzero S-paths, where $S = \{s_1, \ldots, s_t\}$. With an appropriate choice of such paths, we obtain a Γ-odd enlargement of μ whose trees contain the union of a pair of trees $\mu(v_i)$ and the Γ-nonzero S-path connecting them (see Figure 4.3 in section 4.1).

Finding the appropriate S-paths, however, seems to be considerably more difficult in undirected group-labelled graphs compared to the directed setting. The main obstacle is that Lemma 2.2.2 requires 3-connectivity for a Γ-bipartite graph to be shift-equivalent to the labelling $\mathbf{0}$, whereas the directed analog of Lemma 2.2.2 (which is essentially equivalent to Lemma 2.2.1) does not require any connectivity assumptions. This means that, in the Γ-bipartite graph (μ, γ), we can only guarantee that paths between *branching* vertices of μ (defined in section 4.1) are Γ-zero. This requires us to choose the vertices s_i more carefully and keep track of how each s_i branches to the other trees of μ throughout the proof.

In the second outcome of the flat wall theorem, we also employ a Ramsey-type argument to the given wall (W, γ) to obtain a smaller wall (W_0, γ) that is either facially Γ-odd or Γ-bipartite. The first case satisfies outcome (2)-(a) of Theorem 3.1.1. In the second case, we find many disjoint Γ-nonzero N_0-paths outside of the wall W_0, where N_0 is the set of top nails of W_0, and apply Lemma 2.7.1 to obtain a pure Γ-odd linkage of (W_0, γ). In this part, the 3-connectivity condition adds only minor obstacles.

In both outcomes, if the desired Γ-nonzero S-paths or N_0-paths do not exist, then we show that outcome (3) of Theorem 3.1.1 is satisfied using Theorem 2.2.5.

3.1.2 Proof outline of Theorem 1.2.1

Theorem 1.2.1 follows readily from Theorem 3.1.1 and the tools presented in chapter 2. Although Theorem 1.2.1 is implied by Theorem 3.3.2, we nevertheless sketch the proof here as a simple demonstration of our proof techniques.

Let Γ be an abelian group and consider a minimal counterexample (G, γ) to the family of Γ-nonzero cycles satisfying the Erdős-Pósa property. Then (G, γ) admits a large tangle $\mathcal{T}$ such that no Γ-nonzero cycle is contained in the small side of a separation in $\mathcal{T}$, as we saw in Lemma 2.3.1. By Theorem 2.6.1, there is a large wall (W, γ) such that $\mathcal{T}_W$ is a truncation of $\mathcal{T}$. We then apply Theorem 3.1.1 to (W, γ) to obtain one of its outcomes.

It is not difficult to see that there is a large packing of Γ-nonzero cycles in outcomes (1) and (2)(a), and in outcome (2)(b) in the cases where the pure Γ-odd linkage is either in series or nested. Now suppose the linkage is crossing. If Γ does not have an element of order two, then again it is not difficult to see that there is a large packing of Γ-nonzero cycles. If Γ has an element g of order two, then every path in the linkage could have weight g (this is essentially an Escher wall) and we may not have a large packing, but we do have a large half-integral packing of Γ-nonzero cycles. Finally, outcome (3) implies that there is a small hitting set for the Γ-nonzero cycles, contradicting the definition of a minimal counterexample.

This proof sketch essentially shows that a large Escher wall is the only obstruction to the Erdős-Pósa property of Γ-nonzero cycles. In Chapter 6 we will proceed in a similar manner to characterize the obstructions in a more general setting.

3.2 A-paths

Our main result on A-paths is Theorem 1.3.3, that for every odd prime p, the family of A-paths of length 0 mod p satisfies the Erdős-Pósa property. This is proved in Chapter 5 using Theorem 3.1.1. Here, we derive a characterization of the abelian groups Γ and elements $\ell \in$

Γ for which the family of A-paths of weight ℓ satisfies the Erdős-Pósa property, assuming Theorem 1.3.3.

We begin with a straightforward corollary of Theorem 1.3.1. Given two vertex set A, B of a graph G, an *A-B-A-path* is either an A-path containing a vertex in B, or a trivial path $\{a\}$ where $a \in A \cap B$.

Corollary 3.2.1. *Let G be a graph and let $A, B \subseteq V(G)$. Then for all positive integers k, either G contains k disjoint A-B-A-paths or there is a set of at most $2k-2$ vertices intersecting every A-B-A-path.*

Proof. Since each vertex in $A \cap B$ forms a trivial A-B-A-path, we may assume without loss of generality that $A \cap B = \emptyset$. Let Γ be the free group generated by $|E(G)|$ elements. Let $\vec{G}$ be an arbitrary orientation of G. Label each edge e incident to B with a distinct generator $\gamma(e)$ of Γ, and label all other edges 0. Then A-B-A-paths in G correspond exactly to Γ-nonzero A-paths in $(\vec{G}, \gamma)$. $\square$

3.2.1 A-paths of a fixed weight in infinite groups

We first take care of the infinite case by showing that if Γ is infinite, then for all $\ell \in \Gamma$, the Erdős-Pósa property does not hold for A-paths of weight ℓ in either model of group-labelling. The following construction also implies that the Erdős-Pósa function for Γ-zero A-paths necessarily grows with the order of the group Γ.

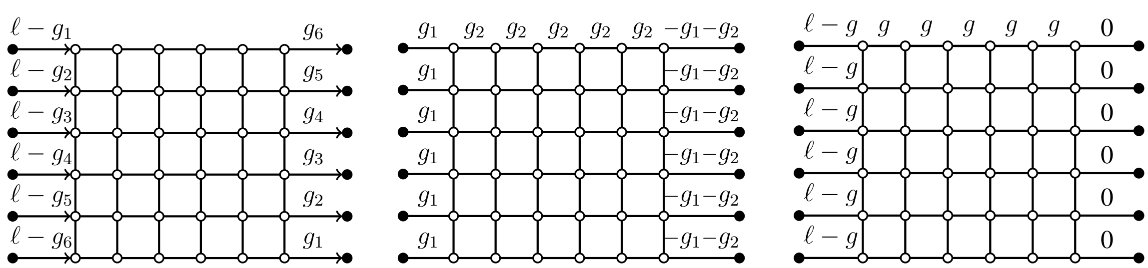
Lemma 3.2.2. *Let Γ be an infinite group and let $\ell \in \Gamma$. Then A-paths of weight ℓ do not satisfy the Erdős-Pósa property in both models of group-labelling.*

Proof. Let n be a positive integer and let H_n denote the $n \times n$-grid with vertex set $\{v_{i,j} : i, j \in [n]\}$ and edge set $\{v_{i,j}v_{i',j'} : |i-i'|+|j-j'| = 1\}$. Let G_n denote the graph obtained from H_n by adding $2n$ new vertices $u_1, \ldots, u_n, w_1, \ldots, w_n$ and adding the edges $u_i v_{1,i}$ and $v_{n,i} w_i$ for each $i \in [n]$.

Since Γ is infinite, there exists a sequence of elements $g_1, g_2, \dots \in \Gamma$ such that $g_k \notin \{g_j, \ell - g_j, g_j \pm \ell, g_j \pm 2\ell\}$ for all $j < k$. For the directed model, orient the edges $u_i v_{1,i}$ from u_i to $v_{1,i}$, orient the edges $v_{n,i} w_i$ from $v_{n,i}$ to w_i, and orient the remaining edges arbitrarily to obtain an orientation $\vec{G}$ of G. Define the Γ-labelling

$$\gamma_n(e) = \begin{cases} \ell - g_i & \text{if } e = u_i v_{1,i} \text{ for } i \in [n] \\ g_{n+1-i} & \text{if } e = v_{n,i} w_i \text{ for } i \in [n] \\ 0 & \text{otherwise} \end{cases}$$

and define $A = \{u_1, \dots, u_n, w_1, \dots, w_n\}$ (see Figure 3.1 (a)). In both (G_n, γ_n) and $(\vec{G}_n, \gamma_n)$, it follows from our choice of $g_1, g_2, \dots$ that if an A-path has both endpoints in $\{u_i\}$, both endpoints in $\{w_i\}$, or endpoints u_i and w_{n+1-j} where $i \neq j$, then it cannot have weight ℓ. So A-paths of weight ℓ are exactly the A-paths with endpoints u_i and w_{n+1-i} for some $i \in [n]$, and clearly no two such paths are disjoint. On the other hand, no vertex set of size less than n intersects all such paths. Therefore, A-paths of weight ℓ do not satisfy the Erdős-Pósa property. $\square$



(a) $(\vec{G}_6, \gamma_6)$ (b) (G_6, γ'_6). (c) (G_6, γ''_6).

Figure 3.1: The black vertices constitute A and all unlabelled edges have weight 0.

3.2.2 Γ-zero A-paths in directed Γ-labelled graphs

An *A-tree* is a tree whose intersection with A is exactly its set of leaves. Let $\ell(T)$ denote the number of leaves of a tree T. Our proof of Theorem 1.3.5 applies the so-called *frame* argument expounded in [4].

Lemma 3.2.3. *Let Γ be a finite group and let k be a positive integer. If $(\vec{T}, \gamma)$ is a directed Γ-labelled graph where T is a subcubic A-tree with $\ell(T) \geq (2k-1)|\Gamma| + 1$, then $(\vec{T}, \gamma)$ contains k disjoint zero A-paths.*

Proof. We proceed by induction on k. Let $k = 1$. Choose an interal vertex v of T and let $P_1, \ldots, P_{|\Gamma|+1}$ be distinct $\{v\}$-A-paths in T. Then $\gamma(P_i) = \gamma(P_j)$ for some $i \neq j$, and the symmetric difference of P_i and P_j is a zero A-path. This proves the base case.

Let $k > 1$ and assume that the statement holds for all $k' < k$. Fix a leaf a of T. For a vertex v of degree 3, let T_1' denote the connected component of $T - v$ containing a, and let T_1 denote the maximal A-tree contained in T_1'. Let $T_2 = T - T_1'$, which is also an A-tree, and note that $\ell(T_1) + \ell(T_2) = \ell(T)$.

Now choose v to be a vertex of degree 3 that is farthest from a subject to the condition that $\ell(T_2) \geq |\Gamma| + 1$. Then $\ell(T_2) \leq 2|\Gamma|$ by our choice of v, so $\ell(T_1) \geq 2((k-1)-1)|\Gamma| + 1$. By the inductive hypothesis, $(\vec{T_1}, \gamma)$ contains $k-1$ disjoint zero A-paths and $(\vec{T_2}, \gamma)$ contains a zero A-path, yielding k disjoint zero A-paths in $(\vec{T}, \gamma)$. $\square$

Theorem 3.2.4. *Let Γ be a finite group and let k be a positive integer. Then every directed Γ-labelled graph $(\vec{G}, \gamma)$ has either k disjoint zero A-paths or a vertex set $X \subseteq V(G)$ with $|X| < 6(k-1)|\Gamma|$ such that $(\vec{G} - X, \gamma)$ has no zero A-path.*

Proof. Let F be an inclusion-wise maximal forest in G such that each connected component of F is a subcubic A-tree that contains a zero A-path. Then we may assume that F has at most $k-1$ connected components. Let X denote the set of vertices of degree 1 or 3 in F.

Suppose $(\vec{G} - X, \gamma)$ contains a zero A-path P. Then P intersects $F - X$ since otherwise $F \cup P$ violates the maximality of F. Let P' be a subpath of P such that $|V(P') \cap A| = 1 = |V(P') \cap V(F - X)|$. Then the vertex in $V(P') \cap V(F - X)$ has degree 2 in F by the definition of X, so $F \cup P'$ again violates the maximality of F.

Therefore, $(\vec{G} - X, \gamma)$ does not contain a zero A-path. To show the upper bound on $|X|$, let $T_1, \ldots, T_c$ denote the connected components of F and let k_i be the largest integer such that $\ell(T_i) \geq (2k_i - 1)|\Gamma| + 1$. Then $\ell(T_i) \leq (2k_i + 1)|\Gamma|$ and T_i contains k_i disjoint zero A-paths by Lemma 3.2.3, so we may assume that $\sum_{i=1}^{c} k_i \leq k - 1$. Since a nontrivial subcubic tree T has exactly $\ell(T) - 2$ vertices of degree 3, we have

$$|X| = \sum_{i=1}^{c}(2\ell(T_i) - 2) \leq \sum_{i=1}^{c}(2(2k_i + 1)|\Gamma| - 2) < 4(k-1)|\Gamma| + 2c|\Gamma| \leq 6(k-1)|\Gamma|.$$

□

Theorem 1.3.5. *Let Γ be a group. Then, in directed Γ-labelled graphs, Γ-zero A-paths satisfy the Erdős-Pósa property if and only if Γ is finite.*

Proof. The proof follows immediately from Lemma 3.2.2 and Theorem 3.2.4 □

3.2.3 Γ-zero A-paths in undirected Γ-labelled graphs

Here we give the undirected analog of Theorem 1.3.5, using Theorem 1.3.3.

Theorem 3.2.5. *Let Γ be an abelian group. Then, in undirected Γ-labelled graphs, zero A-paths satisfy the Erdős-Pósa property if and only if $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some positive integer k or $\Gamma \cong \mathbb{Z}/m\mathbb{Z}$ where m is either equal to 4 or a prime.*

Proof. We may assume that Γ is finite by Lemma 3.2.2. If $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^k$ for some positive integer k, then every element of Γ has order two, so the two models of Γ-labelled graphs are equivalent and Γ-zero A-paths satisfy the Erdős-Pósa property by Theorem 3.2.4. If $\Gamma \cong \mathbb{Z}/4\mathbb{Z}$ or $\Gamma \cong \mathbb{Z}/p\mathbb{Z}$ for an odd prime p, then Γ-zero A-paths satisfy the Erdős-Pósa property by Theorem 1.3.2 and Theorem 1.3.3 respectively.

Now suppose Γ is a finite abelian group not isomorphic to any of the above groups.

Claim 3.2.5.1. *There exist nonzero elements $g_1, g_2 \in \Gamma$ such that the order of $\langle g_2 \rangle + g_1$ in the quotient group $\Gamma/\langle g_2 \rangle$ is greater than two.*

Proof. First suppose $|\Gamma|$ is not a power of 2. Then there is a prime $q_1 > 2$ dividing $|\Gamma|$. If Γ is cyclic, then since $\Gamma \ncong \mathbb{Z}/q_1\mathbb{Z}$, we may choose a generator g_1 of Γ and let $g_2 = q_1 g_1 \neq 0$. If Γ is not cyclic, we may choose g_1, g_2 such that g_1 has order q_1, g_2 has prime order, and the two subgroups $\langle g_1 \rangle$ and $\langle g_2 \rangle$ are distinct. It is easy to see that these choices of g_1, g_2 satisfy the claim.

Now suppose $|\Gamma|$ is a power of 2. If Γ is cyclic, then it has an element g_1 of order 8 (since $\Gamma \ncong \mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/4\mathbb{Z}$) and we may choose $g_2 = 4g_1$. If Γ is not cyclic, then since $\Gamma \ncong (\mathbb{Z}/2\mathbb{Z})^k$, it contains a subgroup H isomorphic to $(\mathbb{Z}/4\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ and we may choose g_1 and g_2 so that $H = \langle g_1, g_2 \rangle$ and g_1 has order 4. These choices again satisfy the claim. $\square$

Let q_2 be the order of g_2 and let $q_1 > 2$ be the order of $\langle g_2 \rangle + g_1$ in $\Gamma/\langle g_2 \rangle$. Let G_n be the graph obtained from the $n \times n$-grid as in Lemma 3.2.2. Define the Γ-labelling

$$\gamma'_n(e) = \begin{cases} g_1 & \text{if } e \text{ is incident to } u_i \text{ for some } i \in [n] \\ -g_1 - g_2 & \text{if } e \text{ is incident to } w_i \text{ for some } i \in [n] \\ g_2 & \text{if } e = v_{1,i}v_{1,i+1} \text{ for some } i \in [n-1] \\ 0 & \text{otherwise} \end{cases}$$

and let $A = \{u_1, \ldots, u_n, w_1, \ldots, w_n\}$ (see Figure 3.1 (b)). Then every Γ-zero A-path in (G_n, γ'_n) has one endpoint in $\{u_i : i \in [n]\}$ and one endpoint in $\{v_i : i \in [n]\}$, since every other A-path P has its weight in the coset $\langle g_2 \rangle + 2g_1$ or $\langle g_2 \rangle - 2g_1$, neither of which are zero in $\Gamma/\langle g_2 \rangle$ (since $q_1 > 2$). Moreover, such a path contains an edge of the form $v_{1,i}v_{1,i+1}$ for some $i \in [n-1]$ since otherwise its weight would be equal to $-g_2 \neq 0$.

Clearly, any two such paths intersect, so there does not exist two disjoint zero A-paths. On the other hand, the smallest size of a vertex set intersecting every zero A-path can be made arbitrarily large for with large enough n. Therefore, zero A-paths in Γ-labelled graphs do not satisfy the Erdős-Pósa property. $\square$

3.2.4 A-paths of a fixed weight in undirected group-labelled graphs

Let p be a prime. A *p-group* is a group in which the order of every element is a power of p. We will use the following well-known fact about p-groups.

Theorem 3.2.6 (Theorem 12.5.2 in [21]). *A finite p-group which contains only one subgroup of order p is either cyclic or a generalized quaternion group.* □

Note that generalized quaternion groups are nonabelian. We now prove Theorem 1.3.4.

Theorem 1.3.4. *Let Γ be an abelian group and let $\ell \in \Gamma$. Then, in undirected Γ-labelled graphs, A-paths of weight ℓ satisfy the Erdős-Pósa property if and only if*

- $\Gamma \cong (\mathbb{Z}/2\mathbb{Z})^k$ *where $k \in \mathbb{N}$ and $\ell = 0$,*
- $\Gamma \cong \mathbb{Z}/4\mathbb{Z}$ *and $\ell \in \{0, 2\}$, or*
- $\Gamma \cong \mathbb{Z}/p\mathbb{Z}$ *where p is prime (and $\ell \in \Gamma$ is arbitrary).*

Proof. We may assume that Γ is finite by Lemma 3.2.2. If $\ell = 0$, we can apply Theorem 3.2.5, so we may also assume $\ell \neq 0$.

Suppose there exists a nonzero element $g \in \Gamma$ such that $\ell \notin \langle g \rangle$. Let G_n be the graph obtained from the $n \times n$-grid as before. Define the Γ-labelling

$$\gamma''_n(e) = \begin{cases} \ell - g & \text{if } e \text{ is incident to } u_i \text{ for some } i \in [n] \\ g & \text{if } e = v_{1,i}v_{1,i+1} \text{ for some } i \in [n-1] \\ 0 & \text{otherwise} \end{cases}$$

and let $A = \{u_1, \dots, u_n, w_1, \dots, w_n\}$ (see Figure 3.1 (c)). It follows from our choice of g that every A-path of weight ℓ has one endpoint in each $\{u_i : i \in [n]\}$ and $\{w_i : i \in [n]\}$ and uses edge of the form $v_{1,i}v_{1,i+1}$. Therefore, A-paths of weight ℓ do not satisfy the Erdős-Pósa property.

So we may assume that $\ell \in \langle g \rangle$ for all nonzero $g \in \Gamma$. This implies that the order of ℓ in Γ is prime; if its order is equal to mn where $m, n > 1$ are integers, then $\ell \notin \langle m\ell \rangle$. Let p denote the order of ℓ in Γ. Then Γ is a p-group; if there is a distinct prime q dividing $|\Gamma|$, then for every element $g \in \Gamma$ of order q we have $\ell \notin \langle g \rangle$. Similarly, if Γ' is a subgroup of Γ with $|\Gamma'| = p$ and $\Gamma' \neq \langle \ell \rangle$, then for any nonzero $g \in \Gamma'$ we have $\ell \notin \langle g \rangle$.

Thus, Γ is a p-group and $\langle \ell \rangle$ is the unique subgroup of order p. By Theorem 3.2.6 (and since Γ is abelian), $\Gamma \cong \mathbb{Z}/p^a\mathbb{Z}$ for some $a \in \mathbb{N}$. First, if $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$ and $\ell = 1$, then A-paths of weight ℓ are exactly the Γ-nonzero A-paths. Since the two models of group-labelling are equivalent for $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$, these A-paths satisfy the Erdős-Pósa property by Theorem 1.3.1. Otherwise, we have $a \geq 2$ in which case $\ell \in \langle p^{a-1} \rangle - \{0\}$, or $p > 2$ (or both). In all cases (except $\Gamma \cong \mathbb{Z}/2\mathbb{Z}$), there exists $g \in \Gamma$ such that $\ell = 2g$.

Now we claim that the Erdős-Pósa property holds for A-paths of weight $\ell = 2g$ if and only if it holds for Γ-zero A-paths. Indeed, given a Γ-labelled graph (G, γ) with $A \subseteq V(G)$, define a new Γ-labelling $\gamma' : E(G) \to \Gamma$ where $\gamma'(e) = \gamma(e) - g$ if e is incident with A and $\gamma'(e) = \gamma(e)$ otherwise. Then A-paths of weight ℓ in (G, γ) correspond exactly to the Γ-zero A-paths in (G, γ').

It then follows from Theorem 3.2.5 that, if $\Gamma \not\cong \mathbb{Z}/2\mathbb{Z}$ and $\ell \neq 0$, then A-paths of weight $\ell \neq 0$ satisfy the Erdős-Pósa property if and only if $\Gamma \cong \mathbb{Z}/4\mathbb{Z}$ and $\ell = 2$, or $\Gamma \cong \mathbb{Z}/p\mathbb{Z}$ for some odd prime p. This completes the proof of Theorem 1.3.4. $\square$

We pose the following problem of determining the directed analog of Theorem 1.3.4. Note that, in directed group-labelled graphs, reversing the direction of traversal of a walk W inverts its weight $\gamma(W)$.

Problem 1. *Characterize the groups Γ and elements $\ell \in \Gamma$ such that, in* directed *Γ-labelled graphs, A-paths of weight in $\{\ell, -\ell\}$ satisfy the Erdős-Pósa property.*

It suffices to consider finite groups by Lemma 3.2.2 and nonzero ℓ by Theorem 1.3.5. If $\Gamma \cong \mathbb{Z}/3\mathbb{Z}$ and $\ell \neq 0$, then the problem is equivalent to nonzero A-paths since $\Gamma =$

$\{0, 1, -1\}$. The counterexample in Figure 3.1 (c) can also be adapted to the directed setting in the natural way to show that the Erdős-Pósa property does not hold if there exists $g \in \Gamma$ such that $\ell \notin \langle g \rangle$. It would therefore suffice to deal with the two outcomes of Theorem 3.2.6.

3.3 Cycles

3.3.1 Obstructions to the Erdős-Pósa property of cycles

Let Γ be an abelian group and let $A \subseteq \Gamma$. We now describe a class of obstructions for the Erdős-Pósa property of allowable cycles (cycles with weights in A) in Γ-labelled graphs.

Definition 3.3.1. For positive integers κ and θ, an abelian group Γ, and $A \subseteq \Gamma$, let $\mathcal{C}(\kappa, \theta, \Gamma, A)$ be the class of all Γ-labelled graphs (G, γ) having a wall W of order at least θ and a nonempty family $(\mathcal{P}_i \colon i \in [t])$ of disjoint non-mixing W-handlebars each of size at least κ such that

(1) G is the union of W and $\bigcup\{\bigcup \mathcal{P}_i \colon i \in [t]\}$,

(2) every N^W-path in W is γ-zero,

(3) $\sum_{i\in[t]} \gamma(P_i) \in A$ for any family $(P_i \colon i \in [t])$ such that $P_i \in \mathcal{P}_i$ for all $i \in [t]$,

(4) for each $i \in [t]$, we have $\langle \gamma(P) \colon P \in \bigcup_{j\in[t]\setminus\{i\}} \mathcal{P}_j \rangle \cap A = \emptyset$,

(5) if $f \colon \bigcup_{j\in[t]} \mathcal{P}_j \to \mathbb{Z}$ is a function satisfying $\sum_{j\in[t]} \sum_{P\in\mathcal{P}_j} f(P)\gamma(P) \in A$, then for each $i \in [t]$, either $\mathcal{P}_i$ is in series or $\sum_{P\in\mathcal{P}_i} f(P)$ is odd, and

(6) at least one of the following properties holds.

 (a) The number of crossing W-handlebars in $(\mathcal{P}_i \colon i \in [t])$ is odd.

 (b) At least one but not all W-handlebars in $(\mathcal{P}_i \colon i \in [t])$ are in series.

 (c) At least three W-handlebars in $(\mathcal{P}_i \colon i \in [t])$ are in series.

The Γ-labelled graphs in $\mathcal{C}(\kappa, \theta, \Gamma, A)$ are obstructions to the Erdős-Pósa property of allowable cycles in the following sense. They admit a large half-integral packing of allowable cycles (see Theorem 3.3.2 (iii)), which certifies that there is no small hitting set for the allowable cycles. On the other hand, they do not admit a packing of three disjoint allowable cycles, as we will show in Subsection 3.3.3. Hence, these graphs form counterexamples to the Erdős-Pósa property for the allowable cycles.

Our main theorem is that they are the only obstructions, under the additional assumption that $\Gamma = \prod_{j \in [m]} \Gamma_j$ is a product of a finite number of abelian groups and that A is the set of all elements of Γ avoiding a fixed finite set of elements from each Γ_j.

Recall that for a product $\Gamma = \prod_{j \in [m]} \Gamma_j$ of m abelian groups and for a subset $J \subseteq [m]$, we denote by Γ_J the subgroup consisting of all $g \in \Gamma$ with $\pi_j(g) = 0$ for all $j \in [m] \setminus J$.

Theorem 3.3.2. *For every two positive integers m and ω, there is a function $f_{m,\omega} \colon \mathbb{N}^3 \to \mathbb{Z}$ satisfying the following property. Let $\Gamma = \prod_{j \in [m]} \Gamma_j$ be a product of m abelian groups, and for every $j \in [m]$, let Ω_j be a subset of Γ_j with $|\Omega_j| \leq \omega$. For each $j \in [m]$, let $A_j := \pi_j^{-1}(\Gamma_j \setminus \Omega_j) \subseteq \Gamma$ and $A := \bigcap_{j \in [m]} A_j$. Let G be a graph with a Γ-labelling γ and let $\mathcal{O}$ be the set of all cycles of G whose γ-value is in A. Then for every three positive integers k, κ, and θ, there exists a Γ-labelling γ' of G that is shifting equivalent to γ such that at least one of the following statements is true.*

(i) There are k disjoint cycles in $\mathcal{O}$.

(ii) There is a hitting set for $\mathcal{O}$ of size at most $f_{m,\omega}(k, \kappa, \theta)$.

(iii) There is a subgraph H of G such that for some $J \subseteq [m]$ and for the (Γ/Γ_J)-labelling γ'' induced by the restriction of γ' to H, we have $(H, \gamma'') \in \mathcal{C}(\kappa, \theta, \Gamma/\Gamma_J, A+\Gamma_J)$, and H contains a half-integral packing of κ cycles in $\mathcal{O}$.

3.3.2 Proof outline

Applying similar arguments as in Subsection 3.1.2, we will obtain a large wall W which, after possibly shifting the labelling, satisfies the following properties for some set $Z \subseteq [m]$ of coordinates:

- every N^W-path is γ_j-zero for all $j \in Z$,
- every large subwall contains a γ_j-nonzero cycle for all $j \in [m] \setminus Z$.

We repeatedly apply Theorem 2.2.5 to obtain a collection of W-handlebars that is sufficient to generate a value that is allowable with respect to the coordinates in Z. These handlebars will be restricted so that for each coordinate $j \in Z$, the γ_j-values of the handles within a handlebar are either all equal or all distinct. We then restrict these handlebars so that they become pairwise non-mixing, and further throw away any handlebar that is unnecessary to generate a value that is allowable with respect to the coordinates in Z. Using the outer columns of the wall, we combine handles within each handlebar to form a set of handlebars for a subwall of W which now satisfies property (3) of Definition 3.3.1. Since we already threw away all unnecessary handlebars, property (4) is also satisfied. Each new handlebar whose handles contain an even number of handles in $\mathcal{P}$ will now be in series, which allows us to do this in such a way that we additionally satisfy property (5).

Following the approach from [20], we find a half-integral packing of cycles in $\mathcal{O}$ of size κ, and since we assumed that statement (iii) fails, we can conclude that property (6) fails. This means either that each handlebar is in series and there are at most two of them, or that no handlebar is in series and the number of crossing handlebars is even. In the first case, it is not hard to find a packing of k cycles whose values are allowable with respect to the coordinates in Z, and techniques from [20] enable us to deal with the coordinates in $[m] \setminus Z$ easily and obtain a packing of k cycles in $\mathcal{O}$ (see Subsection 6.1.4). In the second case, we iteratively combine pairs of 'adjacent' handlebars to obtain one handlebar for a subwall of W, where each new handle contains exactly one handle of each constituent handlebar. This will form a nested handlebar, enabling us once again to find a packing of k cycles in $\mathcal{O}$. Thus, we have the desired contradiction in each case.

3.3.3 The obstructions have no packing of three allowable cycles

We now demonstrate that the graphs described in Definition 3.3.1 do not contain three disjoint allowable cycles.

Proposition 3.3.3. *Let κ and θ be positive integers, let Γ be an abelian group, and let $A \subseteq \Gamma$. If $(G, \gamma) \in \mathcal{C}(\kappa, \theta, \Gamma, A)$, then G has no three disjoint cycles whose γ-values are in A, and if G has two disjoint cycles whose γ-values are in A, then (G, γ) satisfies property (6)(b).*

Proof. Let W be the wall and let $\mathfrak{P} = (\mathcal{P}_i \colon i \in [t])$ be the family of W-handlebars described in Definition 3.3.1, and let $o := 3$ if property (6)(b) holds and let $o := 2$ otherwise. Suppose that G has a set $\mathcal{O} = \{O_i \colon i \in [o]\}$ of o disjoint cycles whose γ-values are in A.

Suppose first that property (6)(a) or (b) holds. Let n be the number of nested W-handlebars in $\mathfrak{P}$, let x be the number of crossing W-handlebars in $\mathfrak{P}$, and let s be the number of W-handlebars in $\mathfrak{P}$ that are in series. Note that $n + x \geq 1$ because property (6)(a) or (b) holds. By rearranging indicies, we may assume that $\mathcal{P}_i$ is nested for all $i \in [n]$, crossing for all $i \in [n + x] \setminus [n]$, and in series for all $i \in [n + x + s] \setminus [n + x]$.

Consider the complex closed unit disc $D := \{z \in \mathbb{C} \colon |z| \leq 1\}$ and let S be the complex unit circle $\{z \in \mathbb{C} \colon |z| = 1\}$. Let $\xi := e^{i\pi/(x+n)}$ and for $\alpha, \beta \in [0, 2(x + n)]$, let $\operatorname{arc}(\alpha, \beta)$ be the open arc $\{\xi^\gamma \colon \alpha < \gamma < \beta\}$ in S. We now form a surface in which G embeds by carefully selecting a pair of closed arcs in S to paste together for each nested and crossing handlebar in $\mathfrak{P}$. For each $j \in [n+x]$, let P_j be a W-handle in $\mathcal{P}_j$ and let $\{v_\ell \colon \ell \in [2n+2x]\}$ be the set of endvertices of paths in $\{P_j \colon j \in [n+x]\}$, where $v_\ell \prec_W v_k$ if and only if $k < \ell$. Let f and g be injective maps from $[n + x]$ to $[2n + 2x]$ such that for all $j \in [n + x]$, the endvertices of P_j are $v_{f(j)}$ and $v_{g(j)}$, and $f(j) < g(j)$.

Let $\sim$ be the equivalence relation on S obtained by taking the transitive closure with respect to the following properties;

- $\xi^{f(j)+\alpha} \sim \xi^{g(j)+1-\alpha}$ for each $j \in [n]$ and each $\alpha \in [0,1]$,

- $\xi^{f(j)+\alpha} \sim \xi^{g(j)+\alpha}$ for each $j \in [n+x] \setminus [n]$ and each $\alpha \in [0,1]$.

Finally, let $\mathbb{S}$ be the surface $D/\sim$. If $|z| < 1$, then in $D/\sim$, z is not identified with any other points of D and therefore we write z to denote the equivalence class $\{z\}$ in $\mathbb{S}$ when $|z| < 1$ for convenience. Let $D^* := \{z \in \mathbb{S} \colon z \in \mathbb{C}, |z| < 1\}$, and let S^* be the complement of D^* in $\mathbb{S}$.

There is an embedding ϕ of G in $\mathbb{S}$ such that

(i) $W \cup \bigcup\{\bigcup \mathcal{P}_j \colon j \in [n+x+s] \setminus [n+x]\}$ is embedded in D^*,

(ii) for each $j \in [n+x]$ and each $P \in \mathcal{P}_j$, the subset of D corresponding to $\phi(P)$ is the union of two curves of positive length, each of which intersects S exactly once, at equivalent points in the arcs $\operatorname{arc}(f(j), f(j)+1)$ and $\operatorname{arc}(g(j), g(j)+1)$, and

(iii) for each $j \in [n+x+s] \setminus [n+x]$, there is a component of $\mathbb{S} \setminus \phi(G)$ whose boundary in $\mathbb{S}$ contains $\phi(\bigcup \mathcal{P}_j)$.

For each $j \in [o]$ and $k \in [2n+2x]$, let $X_{j,k}$ be the set of points in $\operatorname{arc}(k, k+1)$ corresponding to points in $\phi(O_j) \cap S^*$ and let $X_j := \bigcup_{k \in [2n+2x]} X_{j,k}$. By (ii), $X_{j,k}$ is a finite set. Note that the elements of $\{X_i \colon i \in [o]\}$ are pairwise disjoint since $\mathcal{O}$ is a set of disjoint cycles. Also, property (5) implies that for each $j \in [o]$ and $k \in [2n+2x]$, $|X_{j,k}|$ is odd. This implies that for all $k \in [2n+2x]$, $|X_{1,k} \cup X_{2,k}|$ (and hence $|X_1 \cup X_2|$) is even.

Let $\{z_j \colon j \in [|X_1 \cup X_2|]\}$ be the enumeration of $X_1 \cup X_2$ such that if $z_j = \xi^\alpha$ and $z_k = \xi^\beta$ for some $j, k \in [|X_1 \cup X_2|]$ and $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta < 2n+2x$, then $j < k$. For $j \in [2]$, let M_j be the subset of D corresponding to $\phi(O_j)$. Each component of $M_1 \cup M_2$ is a curve C which separates D and contains exactly two points in $X_1 \cup X_2$. It follows that each component of $D \setminus C$ contains an even number of points in $X_1 \cup X_2$, and hence C contains exactly one point in $Z_1 := \{z_{2j-1} \colon j \in [\frac{1}{2}|X_1 \cup X_2|]\}$ and exactly one point in $Z_2 := \{z_{2j} \colon j \in [\frac{1}{2}|X_1 \cup X_2|]\}$. We therefore have $|X_1 \cap Z_1| = |X_1 \cap Z_2|$.

Let $j \in [n+x]$ and $k, \ell \in [|X_1 \cup X_2|]$ be such that z_k and z_ℓ are equivalent and are in $\operatorname{arc}(f(j), f(j)+1)$ and $\operatorname{arc}(g(j), g(j)+1)$ respectively. For each $a \in [g(j)-1] \setminus [f(j)]$ we have that $|X_{1,a} \cup X_{2,a}|$ is even. For each $a, b \in [|X_1 \cup X_2|]$ such that $z_a \in X_{1,f(j)} \cup X_{2,f(j)} \setminus \{z_k\}$ and z_b is the point in $X_{1,g(j)} \cup X_{2,g(j)}$ equivalent to z_a, we have that $|\{z_a, z_b\} \cap \operatorname{arc}(z_k, z_\ell)|$ is even if and only if $j \in [n]$. Therefore, if $j \in [n]$, then $\ell - k$ is odd and $|(X_{1,f(j)} \cup X_{1,g(j)}) \cap Z_1| = |(X_{1,f(j)} \cup X_{1,g(j)}) \cap Z_2|$, and if $j \in [n+x] \setminus [n]$, then $\ell - k$ is even and $|(X_{1,f(j)} \cup X_{1,g(j)}) \cap Z_1| - |(X_{1,f(j)} \cup X_{1,g(j)}) \cap Z_2|$ is congruent to 2 modulo 4. Now,

$$\begin{aligned}
0 &= |X_1 \cap Z_1| - |X_1 \cap Z_2| \\
&= \sum_{j=1}^{n+x} \Big(|X_{1,f(j)} \cup X_{1,g(j)} \cap Z_1| - |X_{1,f(j)} \cup X_{1,g(j)} \cap Z_2| \Big) \\
&= \sum_{j=n+1}^{n+x} \Big(|X_{1,f(j)} \cup X_{1,g(j)} \cap Z_1| - |X_{1,f(j)} \cup X_{1,g(j)} \cap Z_2| \Big)
\end{aligned}$$

and therefore x is even. Hence, we may assume that property (6)(b) holds, and so $o = 3$.

For a set $\mathcal{O}'$ of disjoint cycles in G, we define an auxiliary multigraph $H(\mathcal{O}')$ whose vertex set is the set of all components of $\mathbb{S} \setminus \bigcup\{\phi(O) \colon O \in \mathcal{O}'\}$ where for each $O \in \mathcal{O}'$, there is an edge e_O between the components that contain O in their boundary. We remark that if there is only one component whose boundary contains O, then e_O is a loop.

Claim. *If $\mathcal{O}'$ is a subset of $\mathcal{O}$ of size at least 2, then the graph $H(\mathcal{O}')$ has no loop.*

Proof. Without loss of generality, we may assume that $\{O_1, O_2\} \subseteq \mathcal{O}'$, and it is sufficient to prove the claim when $\{O_1, O_2\} = \mathcal{O}'$. Recall that $|X_{j,k}|$ is odd for each $j \in [2]$ and $k \in [2n+2x]$. Hence, $|X_1 \cup X_2|$ is even. For all $j \in [n]$ and $\alpha \in [0,1]$, let $A_{j,\alpha} := \operatorname{arc}(f(j)+\alpha, g(j)+1-\alpha)$ and note that $A_{j,\alpha} \setminus \{\xi^k \colon k \in [2(n+x)]\}$ is the disjoint union

$$\operatorname{arc}(f(j)+\alpha, f(j)+1) \cup \operatorname{arc}(g(j), g(j)+1-\alpha) \cup \bigcup\{\operatorname{arc}(k, k+1) \colon k \in [g(j)-1] \setminus [f(j)]\}.$$

Observe that $|(X_1 \cup X_2) \cap A_{j,\alpha}|$ is even, because

$$|(X_1 \cup X_2) \cap \mathrm{arc}(f(j) + \alpha, f(j) + 1)| = |(X_1 \cup X_2) \cap \mathrm{arc}(g(j), g(j) + 1 - \alpha)|.$$

Similarly, for all $j \in [n + x] \setminus [n]$ and $\alpha \in [0, 1]$, if $\xi^{f(j)+\alpha} \notin X_1 \cup X_2$, then

$$|(X_1 \cup X_2) \cap \mathrm{arc}(f(j) + \alpha, g(j) + \alpha)| \equiv 0 \pmod{2},$$

because

$$|X_k \cap (\mathrm{arc}(f(j) + \alpha, f(j) + 1) \cup \mathrm{arc}(g(j), g(j) + \alpha))| = |X_{k,f(j)}|$$

for each $k \in [2]$. It follows that we can 2-colour the components of $S^* \setminus \phi(O_1 \cup O_2)$ such that every point in $\phi(O_1 \cup O_2) \cap S^*$ is on the boundary of two components of different colours. Now consider a curve C in $\mathbb{S} \setminus \phi(O_1 \cup O_2)$ between two points in S^* whose interior is entirely in D^*. Let X be the set of points in D corresponding to points in $\phi(O_1 \cup O_2)$ and let Y be the set of points in D corresponding to points in C. Now each component of X contains exactly two points in $X_1 \cup X_2$, so each component of $S \setminus Y$ contains an even number of points in $X_1 \cup X_2$. It follows that the endpoints of C are in components of $S^* \setminus \phi(O_1 \cup O_2)$ with the same colour. Now for every component Z of $\mathbb{S} \setminus \phi(O_1 \cup O_2)$, the components of $S^* \setminus \phi(O_1 \cup O_2)$ contained in Z all have the same colour. Therefore, $H(\mathcal{O}')$ is 2-colourable, and hence contains no loop. □

By property (6)(b), we have that $s \geq 1$. By property (4) and (iii), there is a component of $\mathbb{S} \setminus \phi(\bigcup \mathcal{O})$ whose boundary intersects each of $\phi(O_1)$, $\phi(O_2)$, and $\phi(O_3)$, which means that some vertex of $H(\mathcal{O})$ is incident with all three edges. If two edges of $H(\mathcal{O})$ are parallel, say e_{O_1} and e_{O_2}, then $H(\{O_1, O_3\})$ has a loop, contradicting the claim. It follows that $H(\mathcal{O})$ is isomorphic to the star $K_{1,3}$.

Without loss of generality, there are points $z_1 \in X_{1,1}$ and $z_2 \in X_{2,1}$ such that $X_{3,1} \subseteq$

$\operatorname{arc}(z_1, z_2)$. Since $|X_{3,1}|$ is odd, there are points $z_a, z_b \in X_{1,1} \cup X_{2,1}$ such that $\operatorname{arc}(z_a, z_b) \cap (X_1 \cup X_2)$ is empty and $\operatorname{arc}(z_a, z_b) \cap X_3$ is odd. It follows that each of the two components of $\mathbb{S} \setminus \phi(\bigcup \mathcal{O})$ whose boundary contains $\phi(O_3)$ also contains either $\phi(O_1)$ or $\phi(O_2)$ in its boundary, contradicting that $H(\mathcal{O})$ is isomorphic to $K_{1,3}$.

We conclude that neither property (6)(a) nor property (6)(b) holds. Hence, property (6)(c) holds and $n = x = 0$. Let G' be a graph obtained from G by adding for each $i \in [3]$ a vertex v_i with neighbourhood $V(\bigcup \mathcal{P}_i) \cup \{v_j \colon j \in [3] \setminus \{i\}\}$. Note that G' is a planar graph. Since W is connected, there is a $V(O_1)$-$V(O_2)$-path P in W that contains some edge e. Note also that for each $i \in [2]$ and $j \in [3]$, the cycle O_i contains a path in $\mathcal{P}_j$ by property (4). Now

$$G'[\{v_1, v_2, v_3\} \cup V(O_1 \cup O_2 \cup P)]/(E(O_1 \cup O_2 \cup P) \setminus \{e\})$$

is isomorphic to K_5, a contradiction. $\square$

3.4 Applications and discussion

3.4.1 Characterization of the Erdős-Pósa property

In this subsection we derive Theorems 1.2.3 and 1.2.4. The following result directly implies Theorem 1.2.3.

Theorem 3.4.1. *For every three positive integers m, ω, and θ, there is a function $f_{m,\omega,\theta} \colon \mathbb{N} \to \mathbb{N}$ satisfying the following property. Let $\Gamma = \prod_{i \in [m]} \Gamma_i$ be a product of m abelian groups and let $m' \in \{0\} \cup [m]$. For every $i \in [m]$, let Ω_i be a subset of Γ_i with $|\Omega_i| \leq \omega$ and let A_i be the set of all elements $g \in \Gamma$ such that $\pi_i(g) \in \Gamma_i \setminus \Omega_i$. Let $A := \bigcap_{i \in [m]} A_i$ and let $A' := \bigcap_{i \in [m']} A_i$. Suppose that*

(1) $\langle 2a \rangle \cap A' \neq \emptyset$ for all $a \in A'$ and

(2) for all $a, b, c \in \Gamma$, if $\langle a, b, c \rangle \cap A' \neq \emptyset$, then $(\langle a, b \rangle \cup \langle b, c \rangle \cup \langle a, c \rangle) \cap A' \neq \emptyset$.

Let G be a graph with a Γ-labelling γ such that for each $i \in [m] \setminus [m']$, every wall in G of order at least θ contains a cycle whose γ_i-value is nonzero.[1] *Let $\mathcal{O}$ be the set of all cycles of G whose γ-value is in A. Then for all $k \in \mathbb{N}$ there exists either a set of k disjoint cycles in $\mathcal{O}$, or a hitting set for $\mathcal{O}$ of size at most $f_{m,\omega,\theta}(k)$.*

Proof. We set $f_{m,\omega,\theta}(k) := f_{m,\omega}(k, k, \theta)$ for the function $f_{m,\omega}$ as in Theorem 3.3.2.

Assume for a contradiction that there exists neither a set of k disjoint cycles in $\mathcal{O}$, nor a hitting set for $\mathcal{O}$ of size at most $f_{m,\omega,\theta}(k)$. Then by Theorem 3.3.2 there exists a Γ-labelling γ' of G that is shifting equivalent to γ and a subgraph H of G such that for some $J \subseteq [m]$ and for the (Γ/Γ_J)-labelling γ'' induced by the restriction of γ' to H, we have $(H, \gamma'') \in \mathcal{C}(\kappa, \theta, \Gamma/\Gamma_J, A + \Gamma_J)$. Let W be a wall of order θ, t be a positive integer, and $(\mathcal{P}_i \colon i \in [t])$ be a family of pairwise disjoint non-mixing W-handlebars with $H = W \cup \bigcup\{\bigcup \mathcal{P}_i \colon i \in [t]\}$ as in Definition 3.3.1.

Since by Definition 3.3.1(2) every cycle in W is γ''-zero, we conclude that $[m] \setminus [m'] \subseteq J$. By property (1) and Definition 3.3.1(5), we obtain that $\mathcal{P}_i$ is in series for each $i \in [t]$. In particular, Definition 3.3.1(6)(b) and (6)(a) do not hold. By property (2), Definition 3.3.1(3) and (4), we obtain that Definition 3.3.1(6)(c) does not hold as well, contradicting that $(H, \gamma'') \in \mathcal{C}(\kappa, \theta, \Gamma/\Gamma_J, A + \Gamma_J)$. $\square$

As a consequence of Proposition 3.3.3, we now straightforwardly obtain Theorem 1.2.4.

Theorem 1.2.4. *Let A be a nonempty subset of an abelian group Γ such that A does not satisfy at least one of the following properties:*

(1) for all $a \in A$, we have $\langle 2a \rangle \cap A \neq \emptyset$,

(2) for all $a, b, c \in \Gamma$ with $\langle a, b, c \rangle \cap A \neq \emptyset$, we have $(\langle a, b \rangle \cup \langle b, c \rangle \cup \langle a, c \rangle) \cap A \neq \emptyset$.

Then the family of Γ-labelled cycles with weights in A does not satisfy the Erdős-Pósa property.

[1] Instead, we could restrict to Γ-labelled graphs such that for each $i \in [m] \setminus [m']$, every subgraph of G of tree-width at least θ contains a cycle whose γ_i-value is nonzero.

Proof. First, suppose that for some $a \in A$ we have that $\langle 2a \rangle \cap A = \emptyset$. Let G be a graph consisting of a wall W of order at least $t+1$ and a crossing W-handlebar $\mathcal{P}$ of size $t+1$ such that each row of W contains at most one vertex of $\bigcup \mathcal{P}$. Defining γ such that $\gamma(e) = 0$ for all $e \in E(W)$ and $\gamma(P) = a$ for all $P \in \mathcal{P}$ yields that $(G, \gamma) \in \mathcal{C}(t+1, t+1, \Gamma, A)$. It now follows from Proposition 3.3.3 that there are no two disjoint cycles in $\mathcal{O}$. Now consider a set $T \subseteq V(G)$ of size at most t. Note that there is some W-handle $P \in \mathcal{P}$ such that for the two rows R_i^W and R_j^W which intersect P and for some column C_k^W we have that $P \cup R_i^W \cup R_j^W \cup C_k^W$ is disjoint from T and contains a cycle in $\mathcal{O}$. Hence, T is not a hitting set for $\mathcal{O}$ as desired.

Now suppose that there are $a_1, a_2, a_3 \in \Gamma$ forming a counterexample to property (2). By possibly replacing a_i with another element of $\langle a_i \rangle$ for each $i \in [3]$, we may assume that $(a_1 + a_2 + a_3) \in A$. Let G be a graph consisting of a wall W of order at least $t+2$ and a set $\mathfrak{P} = \{\mathcal{P}_i \colon i \in [3]\}$ of three pairwise disjoint non-mixing W-handlebars each of size $t+1$ and each in series. Defining γ such that $\gamma(e) = 0$ for all $e \in E(W)$ and $\gamma(P) = a_i$ for all $i \in [3]$ and $P \in \mathcal{P}_i$ yields that $(G, \gamma) \in \mathcal{C}(t+1, t+2, \Gamma, A)$. It now follows from Proposition 3.3.3 that there are no two disjoint cycles in $\mathcal{O}$. Now consider a set $T \subseteq V(G)$ of size at most t. Note that there are two columns $C_{\ell_1}^W$, $C_{\ell_2}^W$ of W and for each $i \in [3]$, there is a W-handle $P_i \in \mathcal{P}_i$ such that for the two rows $R_{j_i}^W$ and $R_{k_i}^W$ that intersect P_i, we have that $\bigcup \{P_i \cup R_{j_i}^W \cup R_{k_i}^W : i \in [3]\} \cup C_{\ell_1}^W \cup C_{\ell_2}^W$ is disjoint from T and contains a cycle in $\mathcal{O}$. Hence, T is not a hitting set for $\mathcal{O}$ as desired. $\square$

3.4.2 $\mathcal{S}$-cycles of length ℓ modulo z

We now prove a generalization of Theorem 1.2.2 which additionally allows us to recover many known Erdős-Pósa type results, as discussed in the introduction.

Corollary 3.4.2. *Let ℓ, z, t, and L be integers with $z \geq 1$ and $t \geq 0$, and let $p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorization of z with $p_i < p_{i+1}$ for all $i \in [n-1]$. The following statements are equivalent.*

- *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every graph G with a family $\mathcal{S}$ of t subsets of $V(G)$ and every positive integer k, either G contains k disjoint $\mathcal{S}$-cycles of length ℓ modulo z and of length at least L or a set of at most $f(k)$ vertices hitting all such cycles.*

- *All of the following conditions are satisfied.*

  (1) $t \leq 2$.

  (2) If $p_1 = 2$, *then* $\ell \equiv 0 \pmod{p_1^{a_1}}$.

  (3) There do not exist $3 - t$ *distinct* $i \in [n]$ *for which* $\ell \not\equiv 0 \pmod{p_i^{a_i}}$.

Proof. Let $m := t + 2, \omega := \max\{L, z\}, \theta := 3$, and $m' := t + 1$. For all $i \in [t+2] \setminus \{t+1\}$, let $\Gamma_i := \mathbb{Z}$ and $\Gamma_{t+1} := \mathbb{Z}_z$. Let $\Gamma := \prod_{i \in [m]} \Gamma_i$. For each $i \in [t]$, let $\Omega_i := \{0\}$. Let $\Omega_{t+1} = \mathbb{Z}_z \setminus \{\ell\}$ and $\Omega_{t+2} := [L-1]$. For each $i \in [m]$, let A_i be the set of all $g \in \Gamma$ such that $\pi_i(g) \in \Gamma_i \setminus \Omega_i$. Let $A := \bigcap_{i \in [m]} A_i$ and $A' := \bigcap_{i \in [m']} A_i$. For any graph G together with a family $\mathcal{S} = (S_i \colon i \in [t])$ of subsets of $V(G)$, we define a Γ-labelling $\gamma_{G,\mathcal{S}}$ as follows. For each $i \in [t]$, let $\gamma_i(e) = 1$ if $e \in E(H)$ is incident with S_i and $\gamma_i(e) = 0$ otherwise, and let $\gamma_i(e) := 1$ for all $e \in E(H)$ and each $i \in [m] \setminus [t]$. Let $\gamma_{G,\mathcal{S}}$ be the Γ-labelling of G for which $\pi_i \circ \gamma_{G,\mathcal{S}} = \gamma_i$ for all $i \in [m]$. Let $\mathcal{O}_{G,\mathcal{S}}$ be the set of all $\mathcal{S}$-cycles in G of length ℓ modulo z and of length at least L. Then the $\gamma_{G,\mathcal{S}}$-value of a cycle of G is in A if and only if it is in $\mathcal{O}_{G,\mathcal{S}}$.

Suppose that conditions (1), (2), and (3) are satisfied and let $f := f_{m,\omega,\theta}$. To apply Theorem 3.4.1, we verify that the two properties in Theorem 3.4.1 are satisfied for the subset A' of Γ.

Let $g \in A'$. By condition (2), $\gcd(2\ell, z) = \gcd(\ell, z)$, which implies that $\langle 2\ell \rangle = \langle \ell \rangle$ in $\mathbb{Z}_z$. Let x be a nonzero integer such that $\ell \equiv x(2\ell) \pmod{z}$. Now for all $i \in [t]$, we have that $x\pi_i(g) \neq 0$ since $\pi_i(g) \neq 0$. We conclude that $\langle 2g \rangle \cap A' \neq \emptyset$.

Let $g_1, g_2, g_3 \in \Gamma$ be such that $\langle g_1, g_2, g_3 \rangle \cap A' \neq \emptyset$. Then there exist integers x_1, x_2, x_3 such that

- $x_1\pi_i(g_1) + x_2\pi_i(g_2) + x_3\pi_i(g_3) \neq 0$ for all $i \in [t]$ and

- $x_1\pi_{t+1}(g_1) + x_2\pi_{t+1}(g_2) + x_3\pi_{t+1}(g_3) = \ell$.

Let $I \subseteq [n]$ denote the set of indices $i \in [n]$ such that $\ell \not\equiv 0 \pmod{p_i^{a_i}}$. Note that for each $i \in I$, we have that $\gcd(\ell, p_i^{a_i}) \geq \gcd(\pi_{t+1}(g_1), \pi_{t+1}(g_2), \pi_{t+1}(g_3), p_i^{a_i})$, and so there exists $d_i \in \{g_1, g_2, g_3\}$ such that $\gcd(\ell, p_i^{a_i}) \geq \gcd(\pi_{t+1}(d_i), p_i^{a_i})$. For each $i \in I$, let $q_i := \prod_{j \in [n]\setminus\{i\}} p_j^{a_j}$ and let y_i be an integer such that $(y_i q_i)\pi_{t+1}(d_i) \equiv \ell \pmod{p_i^{a_i}}$. Let

$$\hat{g} := \sum_{i \in I} (y_i q_i) d_i.$$

Note that for every $g \in \Gamma$, we have that $\pi_{t+1}(\hat{g} + zg) = \ell$. Let K be an integer that is greater than $|\pi_i(g)|$ for all $i \in [t]$ and $g \in \{g_1, g_2, g_3, \hat{g}\}$. For $i \in [t]$, let $c_i \in \{g_1, g_2, g_3\}$ such that $\pi_i(c_i) \neq 0$. Such an element exists because $x_1\pi_i(g_1) + x_2\pi_i(g_2) + x_3\pi_i(g_3) \neq 0$. By conditions (1) and (3), we have that $\{c_j \colon j \in [t]\} \cup \{d_j \colon j \in I\}$ is a subset of $\{g_1, g_2, g_3\}$ of size at most 2, and by construction we have that

$$\hat{g} + \sum_{i \in [t]} (K^i z) c_i \in \langle \{c_i \colon i \in [t]\} \cup \{d_i \colon i \in I\} \rangle \cap A'.$$

Therefore, both properties of Theorem 3.4.1 are satisfied. Let G be a graph and let $\mathcal{S} = (S_i \colon i \in [t])$ be a family of subsets of $V(G)$ and let $\gamma := \gamma_{G,\mathcal{S}}$ and $\mathcal{O} := \mathcal{O}_{G,\mathcal{S}}$ as defined above. Note that since the γ_m-value of every edge of G is positive, every wall in G contains a cycle whose γ_m-value is nonzero. Now applying Theorem 3.4.1, we conclude that either G contains k disjoint cycles in $\mathcal{O}$ or a hitting set for $\mathcal{O}$ of size at most $f(k)$.

Now suppose that f is a function as in the first statement. Let $I \subseteq [n]$ denote the set of indices $i \in [n]$ such that $\ell \not\equiv 0 \pmod{p_i^{a_i}}$. Let G be a graph consisting of a wall W of order at least $f(2) + 2$ together with a set $\mathfrak{P} = \{\mathcal{P}_i \colon i \in I\} \cup \{\mathcal{Q}_i \colon i \in [t]\}$ of size $|I| + t$ of pairwise disjoint non-mixing W-handlebars, each of size at least $f(2) + 1$ such that

1. every N^W-path in W has length $(|L| + 1)z$

2. for all $i \in [t]$, every W-handle in $\mathcal{Q}_i$ has length $2z$,

3. for all $i \in I$, every W-handle in $\mathcal{P}_i$ has length congruent to ℓ modulo $p_i^{a_i}$ and congruent to 0 modulo $p_j^{a_j}$ for all $j \in [n] \setminus I$,

4. for every $\mathcal{P} \in \mathfrak{P}$, if $1 \in I$, $p_1 = 2$ and $\mathcal{P} = \mathcal{P}_1$, then $\mathcal{P}$ is crossing and disjoint from the first column of W, and $\mathcal{P}$ is in series and disjoint from the last column of W, otherwise.

For each $i \in [t]$, let $S_i := V(\bigcup \mathcal{Q}_i) \setminus V(W)$, let $\mathcal{S} := (S_i \colon i \in [t])$. Let γ be the $(\Gamma/\Gamma_{\{m\}})$-labelling induced by $\gamma_{G,S}$ and let $\mathcal{O} := \mathcal{O}_{G,S}$ as defined above. Note that a cycle of G is in $\mathcal{O}$ if and only if its γ-value is in $A + \Gamma/\Gamma_{\{m\}}$. Note that for $\mathcal{C}(f(3)+1, f(3)+2, \Gamma/\Gamma_{\{m\}}, A + \Gamma_{\{m\}})$, we have that (G, γ) satisfies properties (1)–(5) of Definition 3.3.1. In particular, every cycle of G which contains exactly one W-handle from each $\mathcal{P} \in \mathfrak{P}$ is in $\mathcal{O}$. Consider a set $T \subseteq V(G)$ of size at most $f(3)$. Now there are two columns C and C' of W and for each $\mathcal{P} \in \mathfrak{P}$ there is a W-handle $P_{\mathcal{P}} \in \mathcal{P}$ such that for the two rows $R_{\mathcal{P}}$ and $R'_{\mathcal{P}}$ which intersect P we have that $\bigcup\{P_{\mathcal{P}} \cup R_{\mathcal{P}} \cup R'_{\mathcal{P}} \colon \mathcal{P} \in \mathfrak{P}\} \cup C \cup C'$ is disjoint from T and contains a cycle in $\mathcal{O}$. By the definition of f, we have that G contains three disjoint cycles in $\mathcal{O}$. Hence, for the $(\Gamma/\Gamma_{\{m\}})$-labelling γ' induced by γ, we have that $(G, \gamma') \notin \mathcal{C}(f(2)+1, f(3)+2, \Gamma/\Gamma_{\{m\}}, A + \Gamma_{\{m\}})$ by Proposition 3.3.3. Now if $p_1 = 2$, then since property (6)(a) is not satisfied, we have that $1 \notin I$. Therefore, condition (2) holds. Now every W-handlebar in $\mathfrak{P}$ is in series, and so given that property (6)(c) is not satisfied, we have that conditions (1) and (3) hold, as desired. $\square$

3.4.3 Restriction to graphs embeddable in orientable surfaces

In this subsection, we study the implications of Theorem 3.3.2 when restricting to the class of graphs embeddable in a fixed orientable surface.

Proposition 3.4.3. *Let Γ be an abelian group, let $A \subseteq \Gamma$ and let $\kappa \geq 2$ and θ be positive integers. The following statements are equivalent.*

(i) *Every graph in* $\mathcal{C}(\kappa, \theta, \Gamma, A)$ *satisfies property (6)(a).*

(ii) *Every graph in* $\mathcal{C}(\kappa, \theta, \Gamma, A)$ *is non-planar.*

(iii) *For every* $X \subseteq \Gamma$ *such that* $\sum_{g \in X} g \in A$ *and* $\langle Y \rangle \cap A = \emptyset$ *for every* $Y \subsetneq X$*, we have* $|X| \leq 2$ *and if* $|X| = 2$*, then* $\langle 2X \rangle \cap A \neq \emptyset$*.*

Proof. It is easy to see that a wall together with a crossing handlebar of size at least 2 forms a non-planar graph, and hence that statement (i) implies statement (ii).

Now suppose that statement (iii) holds, and consider some $(G, \gamma) \in \mathcal{C}(\kappa, \theta, \Gamma, A)$. Let W be the wall and let $\mathfrak{P} = (\mathcal{P}_i \colon i \in [t])$ be the family of W-handlebars described in Definition 3.3.1. Let $(P_i \colon i \in [t])$ be a family such that $P_i \in \mathcal{P}_i$ for all $i \in [t]$, and let $X = \{\gamma(P_i) \colon i \in [t]\}$. By property (4), for every $Y \subsetneq X$ we have that $\langle Y \rangle \cap A = \emptyset$, so $|X| \leq 2$ and $\langle 2X \rangle \cap A \neq \emptyset$. Now G does not satisfy property (6)(c) since this would require $|X| \geq 3$, so in particular at least one W-handlebar in $\mathfrak{P}$ is not in series. Hence, $\langle 2X \rangle \cap A = \emptyset$ by property (5) and so $|X| = 1$. We conclude that G satisfies property (6)(a), and therefore statement (i) holds.

Now suppose that $X \subseteq \Gamma$ is a counterexample to the statement (iii). If $|X| \geq 3$, then it is easy to see that $\mathcal{C}(\kappa, \theta, \Gamma, A)$ contains some (G, γ) where $(\mathcal{P}_x \colon x \in X)$ is the family of W-handlebars, each of which is in series, with $\gamma(P) = x$ for all $P \in \mathcal{P}_x$, and G is planar. Otherwise, let $X = \{x_1, x_2\}$, where $\langle 2x_1, 2x_2 \rangle \cap A = \emptyset$. If $\langle 2x_1, x_2 \rangle \cap A = \emptyset$, then set $(g_1, g_2) := (x_1, x_2)$ and otherwise let a and b be integers such that $ax_2 + 2b(x_1 + x_2) \in A$ and set $(g_1, g_2) := (ax_2, 2b(x_1 + x_2))$. Observe that $\{g_1, g_2\}$ is a counterexample to the statement (iii) with $\langle 2g_1, g_2 \rangle \cap A = \emptyset$. Now it is easy to see that $\mathcal{C}(\kappa, \theta, \Gamma, A)$ contains some (G, γ) where $(\mathcal{P}_1, \mathcal{P}_2)$ is the family of W-handlebars with $\mathcal{P}_1$ nested, $\mathcal{P}_2$ in series, and $\gamma(P) = g_i$ for all $P \in \mathcal{P}_i$. As before, G is planar. Therefore, statement (ii) implies statement (iii). $\square$

If every graph in $\mathcal{C}(\kappa, \theta, \Gamma, A)$ is non-planar, then no Γ-labelled planar graph contains any of them, so restricting the class of graphs in Theorem 3.3.2 to planar graphs yields an

Erdős-Pósa type result. In fact, it is not hard to see that for sufficiently large κ and θ every graph in $\mathcal{C}(\kappa, \theta, \Gamma, A)$ which satisfies property (6)(a) contains a large Escher wall. It is known that large Escher walls are not embeddable on any fixed compact orientable surface (see for example [1]). Thus we obtain the following corollary.

Corollary 3.4.4. *For every pair of positive integers m and ω and every compact orientable surface $\mathbb{S}$, there is a function $f\colon \mathbb{N} \to \mathbb{N}$ satisfying the following property. Let $\Gamma = \prod_{i\in[m]} \Gamma_i$ be a product of m abelian groups, and for each $i \in [m]$, let Ω_i be a subset of Γ_i with $|\Omega_i| \leq \omega$. Let A be the set of all elements $g \in \Gamma$ such that $\pi_i(g) \in \Gamma_i \setminus \Omega_i$ for all $i \in [m]$. Suppose that*

> *for every $X \subseteq \Gamma$, if $\sum_{g\in X} g \in A$, then there are some integer $y \leq 2$ and a subset Y of X of size y such that $\langle yY \rangle \cap A \neq \emptyset$.*

Let G be a Γ-labelled graph embeddable in $\mathbb{S}$ with Γ-labelling γ and let $\mathcal{O}$ be the set of all cycles of G whose γ-value is in A. Then for all $k \in \mathbb{N}$ there exists either a set of k disjoint cycles in $\mathcal{O}$, or a hitting set for $\mathcal{O}$ of size at most $f(k)$. □

Corollary 3.4.5. *Let A be a nonempty subset of an abelian group Γ such that A does not satisfy the following property:*

(1) for every $X \subseteq \Gamma$, if $\sum_{g\in X} g \in A$, then there are some integer $y \leq 2$ and a subset Y of X of size y such that $\langle yY \rangle \cap A \neq \emptyset$.

For every positive integer t, there is a planar graph G with a Γ-labelling γ such that for the set $\mathcal{O}$ of cycles of G with values in A, there are no three disjoint cycles in $\mathcal{O}$ and there is no hitting set for $\mathcal{O}$ of size at most t.

Proof. Let $X \subseteq \Gamma$ be a counterexample to (1) of minimum size. If $|X| \geq 3$, then A does not satisfy the second property in Theorem 1.2.4, and the graph constructed in the proof of Theorem 1.2.4 is planar. Hence we may assume that $X = \{x_1, x_2\}$ for some pair of distinct elements of Γ (since property (1) is trivially satisfied by subsets of Γ of size at most 1). As in the proof of Proposition 3.4.3, we may assume that $\langle 2x_1, x_2 \rangle \cap A = \emptyset$.

Let G be a graph consisting of a wall W of order at least $t+2$ and a pair $\mathfrak{P} = \{\mathcal{P}_i \colon i \in [2]\}$ of disjoint non-mixing W-handlebars each of size $t+1$, such that $\mathcal{P}_1$ is nested and $\mathcal{P}_2$ is in series. Defining γ such that $\gamma(e) = 0$ for all $e \in E(W)$ and $\gamma(P) = x_i$ for all $i \in [2]$ and $P \in \mathcal{P}_i$ yields that $(G, \gamma) \in \mathcal{C}(t+1, t+2, \Gamma, A)$. It now follows from Proposition 3.3.3 that there are no three disjoint cycles in $\mathcal{O}$. Now consider a set $T \subseteq V(G)$ of size at most t. Note that there are two columns $C^W_{\ell_1}$, $C^W_{\ell_2}$ of W and for each $i \in [2]$, there is a W-handle $P_i \in \mathcal{P}_i$ such that for the two rows $R^W_{j_i}$ and $R^W_{k_i}$ that intersect P_i, we have that $\bigcup\{P_i \cup R^W_{j_i} \cup R^W_{k_i} \colon i \in [3]\} \cup C^W_{\ell_1} \cup C^W_{\ell_2}$ is disjoint from T and contains a cycle in $\mathcal{O}$. Hence, T is not a hitting set for $\mathcal{O}$ as desired. $\square$

For example, cycles that are either odd or of length 16 modulo 30 satisfy an Erdős-Pósa type theorem when restricted to planar graphs, whereas the cycles that are either odd or of length 106 modulo 210 do not, and neither do cycles of length 1 modulo 6.

We will now derive the exact characterization of when cycles of length ℓ modulo z satisfy an Erdős-Pósa type result in planar graphs.

Theorem 1.2.6. *Let ℓ and z be integers with $z \geq 2$, let $p_1^{a_1} \cdots p_n^{a_n}$ be the prime factorization of z with $p_i < p_{i+1}$ for all $i \in [n-1]$, and let $\mathbb{S}$ be a compact orientable surface. The following statements are equivalent.*

- *There is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every integer k, every graph embeddable in $\mathbb{S}$ contains k vertex-disjoint cycles of length ℓ modulo z or a set of at most $f(k)$ vertices hitting all such cycles.*
- *Both of the following conditions are satisfied.*
  1. *If $p_1 = 2$, then either $\ell \equiv 0 \pmod{p_1^{a_1}}$ or $\ell \equiv 0 \pmod{z/p_1^{a_1}}$.*
  2. *There do not exist distinct $i_1, i_2, i_3 \in [n]$ such that $\ell \not\equiv 0 \pmod{p_{i_j}^{a_{i_j}}}$ for each $j \in [3]$.*

Proof. First suppose that the second statement holds. By Theorem 1.2.2, we may assume that $p_1 = 2$, $\ell \not\equiv 0 \pmod{p_1^{a_1}}$, and $\ell \equiv 0 \pmod{z/p_1^{a_1}}$. Then $\ell = q2^{t-1}(z/p_1^{a_1})$ for some $t \in [a_1]$ and some odd integer q. Let $\Gamma := \mathbb{Z}_z$ and let $A := \{\ell\}$. Let X be a subset of Γ such that $\sum_{g \in X} g = \ell$. Let Γ' be the subgroup of Γ generated by 2^t together with $\{z/p_i^{a_i} \colon i \in [n] \setminus \{1\}\}$, and note that $\ell \notin \Gamma'$. In particular there is some $x \in X \setminus \Gamma'$. Note that $x \cdot (z/2^{a_1}) = q'2^{t'-1}$ for some $t' \in [t]$ and some odd integer q'. In particular, $Y := \{x\}$ is a subset of X of size 1 such that $\ell \in \langle Y \rangle$. The first statement now follows from Corollary 3.4.4.

Suppose instead that the second statement does not hold, and let $\Gamma := \mathbb{Z}_z$ and $A := \{\ell\} \subseteq \Gamma$. Let $J \subseteq [n]$ be the set of indices such that $\ell \not\equiv 0 \pmod{p_j^{a_j}}$, and let $(b_j \colon j \in J)$ be a family of integers such that for the set $X := \{b_j(z/p_j^{a_j}) \colon j \in J\}$ we have that $\sum_{g \in X} g = \ell$. Now if $|J| \geq 3$, then X has no subset Y of size at most 2 such that $\ell \in \langle Y \rangle$. Otherwise we have that $\{2\} \subsetneq \{p_j \colon j \in J\}$, from which it follows that $|X| = 2$ and $\ell \notin \langle 2X \rangle$. Now Corollary 3.4.5 implies that the first statement does not hold. $\square$

When considering surface embeddings of graphs, it is also natural to consider the homology classes of cycles. For graphs embedded in a fixed compact surface, Huynh, Joos, and Wollan obtained a half-integral Erdős-Pósa result for the non-null-homologous cycles of the embedding [23, Theorem 6], and an integral Erdős-Pósa result for these cycles when the surface is orientable [23, Corollary 41]. They did this by considering a different type of group labelling, where the two orientations of each edge are assigned labels that are inverse to each other.

Since in our setting we do not distinguish between the two orientations of an edge, we are unable to directly apply our results to homology classes in the first homology group with coefficients in $\mathbb{Z}$. However orientations can be ignored when considering the first homology group with coefficients in $\mathbb{Z}_2$, and so our results are applicable. Note that for a closed orientable surface, the set of simple closed curves homologous to zero for the $\mathbb{Z}_2$-homology is exactly the same as for the $\mathbb{Z}$-homology. This follows the universal coefficient theorem (see [22]), which allows us to relate the $\mathbb{Z}$-homology with the $\mathbb{Z}_2$-homology by taking all coefficients modulo 2. We then apply a classical result which states that no simple closed curve has $\mathbb{Z}$-homology class kh for any integer $k \geq 2$ and any nonzero element h of the $\mathbb{Z}$-homology (see for example [35]).

The following elementary observation allows us to encode the $\mathbb{Z}_2$-homology classes of cycles in our group labelling setting (see [20, Proposition 3.5] for a proof). A graph H is called *even* if every vertex of H has even degree. For a graph G, let $\mathcal{C}(G)$ denote the *cycle space* of G over $\mathbb{Z}_2$, which is the vector space of all even subgraphs H of G with the symmetric difference as the operation.

Observation 3.4.6. *Let G be a graph, let Γ be an abelian group, and let $\phi\colon \mathcal{C}(G) \to \Gamma$ be a group homomorphism. Then there is a Γ-labelling γ of G such that $\gamma(H) = \phi(H)$ for every even subgraph H of G.*

We also need the following lemma.

Lemma 3.4.7 ([14, Lemma B.6])**.** *Let $\mathbb{S}$ be a compact surface, and let $\mathcal{C}$ be a finite set of disjoint circles in $\mathbb{S}$. Assume that*

- *$\mathbb{S} \setminus \bigcup \mathcal{C}$ has a component D_0 whose closure in $\mathbb{S}$ meets every circle in $\mathcal{C}$, and*
- *no circle in $\mathcal{C}$ bounds a disk in $\mathbb{S}$ that is disjoint from D_0.*

Then $|\mathcal{C}|$ is at most the Euler characteristic of $\mathbb{S}$.

We now obtain a strengthening of the integral Erdős-Pósa result of Huynh, Joos, and Wollan for graphs embedded in a fixed orientable surface [23, Corollary 41].

Corollary 3.4.8. *Let $\mathbb{S}$ be a compact orientable surface with $\mathbb{Z}_2$-homology group Γ and let A be a set of $\mathbb{Z}_2$-homology classes of $\mathbb{S}$. There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for all $k \in \mathbb{N}$ and every graph G embedded in $\mathbb{S}$ there exists either k disjoint cycles whose $\mathbb{Z}_2$-homology classes are in A, or a hitting set for these cycles of size at most $f(k)$.*

Proof. We will apply Theorem 3.3.2 with $m := 1, \omega := |\Gamma \setminus A|, \Gamma_1 := \Gamma$, and $\Omega_1 := \Gamma \setminus A$. Let κ and θ be integers such that κ is greater than the Euler characteristic of $\mathbb{S}$ and no graph containing a wall W of order θ and a crossing W-handlebar of size κ is embeddable in $\mathbb{S}$, and let $f(k) := f_{1,\omega}(k, \kappa, \theta)$. Let G be a graph embedded in $\mathbb{S}$, and let γ be a Γ-labelling of G such that $\gamma(H)$ is the $\mathbb{Z}_2$-homology class of H for every even subgraph H of G (see Observation 3.4.6). Assume for a contradiction that there are neither k disjoint cycles whose $\mathbb{Z}_2$-homology classes are in A, nor a hitting set for these cycles of size at most $f(k)$. By Theorem 3.3.2, for some γ' that is shifting-equivalent to γ there is a subgraph H of G such that for some $J \subseteq [1]$ and for the (Γ/Γ_J)-labelling γ'' induced by the restriction of γ' to H, we have $(H, \gamma'') \in \mathcal{C}(\kappa, \theta, \Gamma/\Gamma_J, A + \Gamma_J)$. Note that by properties (3) and (4) of Definition 3.3.1, we have that Γ/Γ_J is not the trivial group, and hence $J = \emptyset$. Let W be the wall in H and let $\mathfrak{P}$ be the family W-handlebars in H described in Definition 3.3.1. By our choice of κ and θ, we have that $\mathfrak{P}$ contains no crossing W-handlebar, so by property (6) some W-handlebar $\mathcal{P}$ in $\mathfrak{P}$ is in series. Consider the set S of cycles in the union of the first and last column of W together with $\bigcup \mathcal{P}$, and note that $|S| = \kappa$. Hence by Lemma 3.4.7 there is a cycle O in S whose image in $\mathbb{S}$ bounds a disc, and hence $\gamma(O) = \gamma'(O) = 0$. But now by property (2) for the $P \in \mathcal{P}$ contained in O we have $\gamma'(P) = 0$, contradicting properties (3) and (4). $\square$

3.4.4 A negative result for finite allowable subsets of infinite groups

In this subsection, we show that when the set of allowable values of cycles is a nonempty finite subset of an infinite abelian group, then the Erdős-Pósa property fails for the allowable cycles. In fact, we show that a $(1/n)$-integral analogue of the Erdős-Pósa theorem fails for every positive integer n.

Theorem 3.4.9. *Let A be a nonempty finite subset of an infinite abelian group Γ. For integers $s \geq 2$ and $t \geq 1$, there is a graph G with a Γ-labelling γ such that*

- *for every set of s cycles of G whose γ-values are in A, there is a vertex that belongs* *to all of the s cycles and*

- *there is no hitting set of size at most t for the set $\mathcal{O}$ of all cycles of G whose γ-values are in A.*

Proof. Let $\alpha \in A$. We claim that there is an infinite set $\{g_i : i \in \mathbb{N}\}$ of elements of Γ such that for all integers k' with $0 \leq k' \leq s(t+1)$ and for all distinct finite subsets $S_1, S_2 \subseteq \mathbb{N}$, we have

$$k'\alpha + \sum_{i \in S_1} g_i - \sum_{j \in S_2} g_j \notin A. \tag{3.1}$$

Indeed, if Γ has an element g' of infinite order, then we may choose a sufficiently large multiple g of g' so that no nonzero element of $\langle g \rangle$ is in the finite set $\{\alpha' - k'\alpha : \alpha' \in A,\ 0 \leq k' \leq s(t+1)\}$. Then $\{2^i g : i \in \mathbb{N}\}$ satisfies (3.1). Otherwise, if every element of Γ has finite order, then we may sequentially choose an arbitrary element $g_i \notin \langle A \cup \{g_j : 1 \leq j \leq i-1\} \rangle$ for all $i \in \mathbb{N}$. This proves the claim.

We will construct a graph by constructing $s(t+1)$ edge-disjoint cycles with the property that any set of s of them share a common vertex but no vertex is contained in more than s of them.

Let V be the set of subsets of $[s(t+1)]$ of size s, let $W := [s(t+1)] \times [\binom{s(t+1)-1}{s-1}]$ and let G be the complete bipartite graph with bipartition (V, W). For each $i \in [s(t+1)]$, let O_i be a cycle of G whose vertex set is exactly the union of $\{i\} \times [\binom{s(t+1)-1}{s-1}]$ and the set of vertices in V which contain i. Let e_i be an arbitrary edge of O_i. Observe that $E(O_i) \cap E(O_j) = \emptyset$ for distinct i, j. Let γ be a Γ-labelling of G assigning each edge in $E(G) \setminus \{e_i : i \in [s(t+1)]\}$ a distinct value in $\{g_i : i \in \mathbb{N}\}$, and for each $i \in [s(t+1)]$ assigning e_i the value $\alpha - \gamma(O_i - e_i)$. Each vertex of G is contained in at most s cycles in $\{O_i : i \in [s(t+1)]\}$, so every hitting set for $\{O_i : i \in [s(t+1)]\}$ has size at least $t+1$, and by construction for every set of s cycles in $\{O_i : i \in [s(t+1)]\}$, there is a vertex that belongs to all of the s cycles. We will finish the proof by showing that $\mathcal{O} = \{O_i : i \in [s(t+1)]\}$.

By definition, we have $\gamma(O_i) = \alpha \in A$ for each $i \in [s(t+1)]$. Now suppose that $O \in \mathcal{O}$. Let $I := \{i \in [s(t+1)] \colon e_i \in E(O)\}$, let $F := E(O) \setminus \{e_i \colon i \in I\}$, and let $F' := \bigcup_{i \in I}(E(O_i) \setminus \{e_i\})$. Then

$$\gamma(O) = \sum_{i \in I}(\alpha - \gamma(O_i - e_i)) + \sum_{e \in F} \gamma(e) = |I|\alpha + \sum_{e \in F} \gamma(e) - \sum_{e \in F'} \gamma(e),$$

so by (3.1), we have $F = F'$. Since O is a cycle and cycles in $\{O_i \colon i \in [s(t+1)]\}$ are edge-disjoint, we deduce that $|I| = 1$ and $O = O_i$ for some $i \in [s(t+1)]$. Hence, $\mathcal{O} = \{O_i \colon i \in [s(t+1)]\}$. $\square$

3.4.5 Open problems

We now discuss some interesting directions for future research in this area.

Problem 2. *Let Γ be an abelian group. Characterize the subsets A of Γ for which there is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that for every positive integer k, every Γ-labelled graph (G, γ) either contains k disjoint cycles whose γ-value is in A or a hitting set for those cycles of size at most $f(k)$.*

As seen in Theorem 3.4.9, for an infinite group such a set A needs to be infinite as well. This problem is already interesting for the group $\mathbb{Z}$.

As a surprising negative result, for every positive integer t, there exists a $\mathbb{Z}$-labelled graph (G, γ) with no two disjoint cycles with γ-value at least 0 and no hitting set for these cycles of size at most t. Let G be the graph with vertex set $\{v_i \colon i \in [4t+4]\}$, where each vertex with an even index $2i$ is adjacent to all vertices with odd indices j for which $j \leq 2i+1$. Let γ be the $\mathbb{Z}$-labelling of G which assigns value $t+3$ to the edge $v_{2i}v_{2i+1}$ for all $i \in [2t+1]$ and value $-1-t$ to all other edges. In every cycle of this graph, both edges incident to the vertex of highest index in the cycle and both edges incident to the vertex of lowest index in the cycle have value $-1-t$, so there are at least two more edges of value $-1-t$ than of value $t+3$. From this it is easy to verify that the construction satisfies the desired properties, assuming $t \geq 2$. This construction can easily be adapted to apply to cycles of γ-value at least L for any integer L. This is in contrast to the case of cycles of length at least L, where Thomassen [37] showed that an Erdős-Posa result holds. Thus we also present the following variant of Problem 2.

Problem 3. *Characterize the subsets A of $\mathbb{N}$ for which there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer k, every graph G either contains k disjoint cycles whose lengths are in A or a hitting set for these cycles of size at most $f(k)$.*

The construction presented above can also be adapted to show that a $(1/n)$-integral analogue of the Erdős-Pósa theorem fails for cycles of non-negative value in $\mathbb{Z}$-labelled graphs for every positive integer n. Interestingly, we know of no natural example where a half-integral analogue of the Erdős-Pósa theorem fails but some fractional analogue of the Erdős-Pósa theorem holds. In fact, we conjecture the following.

Conjecture 3.4.10. *Let Γ be an abelian group, let $A \subseteq \Gamma$, let $s \geq 3$ be an integer and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for every Γ-labelled graph (G, γ) and every positive integer k, either G contains k cycles whose γ-values are in A such that no subset of s of these cycles share a common vertex, or there is a hitting set for these cycles of size at most $f(k)$. Then there is a function $f' : \mathbb{N} \to \mathbb{N}$ such that for every Γ-labelled graph (G, γ) and every positive integer k, either G contains k cycles whose γ-values are in A such that three of these cycles share a common vertex, or there is a hitting set for these cycles of size at most $f'(k)$.*

If we allow ourselves to restrict the class of Γ-labelled graphs considered, then there are examples for which this conjecture fails. However, we know of no counterexample if we only allow ourselves to restrict the class of graphs considered, but do not restrict the Γ-labellings of these graphs.

CHAPTER 4

FLAT WALL THEOREM FOR UNDIRECTED GROUP-LABELLED GRAPHS

This chapter is dedicated to proving Theorem 3.1.1. We will separately deal with the two outcomes of the flat wall theorem (Theorem 2.6.1).

4.1 Large K_t-model

In this section we consider the case where G contains a large K_t-model. As discussed in section 3.1, we need some additional definitions and lemmas related to the trees of a K_t-model.

Let T be a tree and let $U \subseteq V(T)$. Then the smallest subtree of T containing all vertices in U is the *U-subtree of T*. For example, if $|U| = 2$, say $U = \{u, v\}$, then U-subtree of T is the path uTv. Suppose T is a tree in a graph G and $F \subseteq E(G)$ is such that each edge in F has exactly one endpoint in T and no two edges of F share an endpoint outside of T (so that $T \cup F$ is a tree). Then the *F-extension of T* is the $V(F)$-subtree of $T \cup F$, where $V(F)$ is the set of endpoints of the edges in F. For $n \in \mathbb{N}$, an *n-star* is a graph isomorphic to a subdivision of $K_{1,n}$. If $n \geq 3$, the *center* of an n-star is the unique vertex of degree greater than 2 and a *leg* is a path from its center to a leaf.

Let μ be a K_m-model in a Γ-labelled graph (G, γ) where $V(K_m) = \{v_1, \ldots, v_m\}$. For distinct $i, j \in [m]$, let us denote the two endpoints of the edge $\mu(v_iv_j)$ in the trees $\mu(v_i)$ and $\mu(v_j)$ by $\mu(v_iv_j)_i$ and $\mu(v_iv_j)_j$ respectively. Fix $i \in [m]$ and let $d \in \mathbb{N}$. We say that a vertex $s \in V(\mu(v_i))$ is *d-central* if no connected component of $\mu(v_i) - s$ contains $\mu(v_iv_j)_i$ for at least $m - 1 - d$ distinct indices $j \in [m] - \{i\}$. In other words, if $e \in E(\mu(v_i))$ is incident to s, then the connected component of $\mu(v_i) - e$ containing s contains $\mu(v_iv_j)_i$ for more than d indices $j \in [m] - \{i\}$.

It is easy to see that a d-central vertex always exists if $d < \frac{m-1}{2}$: start from an arbitrary vertex s in $\mu(v_i)$ and, as long as the current vertex is not d-central, move towards the (unique) component T of $\mu(v_i) - s$ for which there are at least $m - 1 - d > \frac{m-1}{2}$ indices j such that $\mu(v_i v_j)_i \in V(T)$. Since $d < \frac{m-1}{2}$, this process cannot backtrack and must end eventually.

Let $\mathcal{C}_i^d$ denote the set of d-central vertices in $\mu(v_i)$. The vertices of $\mathcal{C}_i^d$ form a subtree in $\mu(v_i)$: if s', s'' are distinct vertices in $\mathcal{C}_i^d$ and s is in the interior of the path $s'\mu(v_i)s''$, then each connected component T of $\mu(v_i) - s$ is contained in a connected component of either $\mu(v_i) - s'$ or $\mu(v_i) - s''$, so there are less than $m - 1 - d$ indices j such that $\mu(v_i v_j)_i \in V(T)$.

Let $u \in V(\mu(v_i))$ and let $j_1, j_2, j_3 \in [m] - \{i\}$ be distinct. If the $\{\mu(v_i v_{j_1}), \mu(v_i v_{j_2}), \mu(v_i v_{j_3})\}$-extension of $\mu(v_i)$ is a 3-star centered at u, then we say that u *branches to* $\{\mu(v_{j_1}), \mu(v_{j_2}), \mu(v_{j_3})\}$, or simply to $\{v_{j_1}, v_{j_2}, v_{j_3}\}$ if there is no ambiguity of the model. If $Y \subseteq [m] - \{i\}$, we say that u *branches avoiding* Y if there exist $j_1, j_2, j_3 \in [m] - \{i\} - Y$ such that u branches to $\{v_{j_1}, v_{j_2}, v_{j_3}\}$. We say that a vertex $u \in V(\mu(v_i))$ is d-*branching* if u branches avoiding Y for all $Y \subseteq [m] - \{i\}$ with $|Y| \leq d$. If u is 0-branching, we simply say that u is *branching*. The set of branching vertices of a K_m-model μ is denoted $b(\mu)$. The following proposition is immediate.

Proposition 4.1.1. *Let μ be a K_m-model with $m \geq 4$. Then the union of all $b(\mu)$-paths in μ is a subdivision of a 3-connected graph H where $V(H) = b(\mu)$ and the edges of H correspond to the $b(\mu)$-paths in μ.* □

If $m \geq 4$, then clearly each tree $\mu(v_i)$ contains a branching vertex. If $d \geq 1$ however, then a d-branching vertex need not always exist. For example, $\mu(v_i)$ could be a path with each vertex incident to only a few edges of the form $\mu(v_i v_j)$. The following lemma shows that this is essentially the only obstruction:

Lemma 4.1.2. *Let $0 < d < \frac{m-1}{2}$ be an integer and suppose $\mu(v_i)$ does not contain a d-branching vertex. Then*

1. *the $\mathcal{C}_i^d$-subtree of $\mu(v_i)$ is a path R, and*

2. *for each $s \in \mathcal{C}_i^d$, there are at most $3d$ indices $j \in [m] - \{i\}$ such that the (possibly trivial) $\mu(v_iv_j)_i$-$\mathcal{C}_i^d$-path in $\mu(v_i)$ ends at s. In particular, at least $m-1-6d$ of these paths end at an internal vertex of R.*

Proof. Suppose that the $\mathcal{C}_i^d$-subtree of $\mu(v_i)$ is not a path. Then there exists a vertex $u \in \mathcal{C}_i^d$ that is adjacent in $\mu(v_i)$ to at least three vertices in $\mathcal{C}_i^d$, say s_1, s_2, and s_3. For each $k \in [3]$, since s_k is d-central, there are more than d indices $j \in [m]-\{i\}$ such that $\mu(v_iv_j)_i \in V(T_k)$, where T_k is the connected component of $\mu(v_i)-u$ containing s_k. Thus, for all $Y \subseteq [m]-\{i\}$ with $|Y| \leq d$, there is an index $j_k \notin Y$ such that $\mu(v_iv_{j_k})_i \in V(T_k)$. This implies that u is d-branching and proves the first statement of the lemma.

Now let $s \in \mathcal{C}_i^d$. For each connected component T of $\mu(v_i) - s$ not containing a vertex in $\mathcal{C}_i^d$, there are at most d indices $j \in [m] - \{i\}$ such that $\mu(v_iv_j)_i \in V(T)$, since otherwise the neighbour of s in T would be d-central. Let J denote the set of indices $j \in [m] - \{i\}$ such that the $\mu(v_iv_j)_i$-$\mathcal{C}_i^d$-path in $\mu(v_i)$ ends at s. The second statement of the lemma asserts that $|J| \leq 3d$.

Suppose $|J| \geq 3d+1$. We show that s must then be d-branching. Let $Y \subseteq [m] - \{i\}$ with $|Y| \leq d$ and let c denote the number of indices $j \in J - Y$ such that $s = \mu(v_iv_j)_i$. Note that $|J - Y| \geq 2d+1$.

If $c \geq 3$, then clearly s branches avoiding Y, so we may assume $c \in \{0,1,2\}$. Then there are at least $2d-c+1$ indices $j \in J-Y$ such that $\mu(v_iv_j)_i$ is contained in a connected component of $\mu(v_i) - s$ not containing a vertex in $\mathcal{C}_i^d$. But each such component contains $\mu(v_iv_j)_i$ for at most d indices $j \in J - Y$, so there are at least $3-c$ distinct connected components of $\mu(v_i) - s$ containing $\mu(v_iv_j)_i$ for some $j \in J - Y$. These $3-c$ components together with the c edges on s imply that s branches avoiding Y. Hence s is d-branching and this proves the second statement of the lemma. □

We now show, as discussed in Remark 2.5.1, that if μ is a Γ-bipartite K_m-model, $m \geq 4$, then (μ, γ) is a Γ-bipartite Γ-labelled graph.

Lemma 4.1.3. *Let $m \geq 4$ and let μ be a Γ-bipartite K_m-model in a Γ-labelled graph (G, γ). Then $(\mu[V(K_m)], \gamma)$ is a Γ-bipartite Γ-labelled graph.*

Proof. Write $V(K_m) = \{v_1, \ldots, v_m\}$. Let us first prove the following claim.

Claim 4.1.3.1. *If P is a $b(\mu)$-path in μ, then $2\gamma(P) = 0$.*

Proof. If $m = 4$, then $\mu[V(K_m)]$ is Γ-bipartite by definition and the claim follows by Proposition 4.1.1 and Lemma 2.2.2.

So suppose $m \geq 5$ and let P be a $b(\mu)$-path in μ with endpoints u and w. First suppose that u and w are in different trees of μ, say in $\mu(v_1)$ and $\mu(v_2)$ respectively. Since u is branching, there are two indices in $[m] - \{1, 2\}$, say 3 and 4, such that u branches to $\{v_2, v_3, v_4\}$. Thus u is also a branching vertex in the K_4-submodel η of μ restricted to $\{v_1, v_2, v_3, v_4\}$. Since the claim holds for $m = 4$, if w is also branching in η, then $2\gamma(P) = 0$ as desired. Otherwise, since w is branching in μ but not in η, there exists another index in $[m] - \{1, 2, 3, 4\}$, say 5, such that w branches to both $\{v_1, v_3, v_5\}$ and $\{v_1, v_4, v_5\}$. Furthermore, u must branch to at least one of $\{v_2, v_3, v_5\}$ or $\{v_2, v_4, v_5\}$. Without loss of generality, assume that u branches to $\{v_2, v_3, v_5\}$. Then u and w are both branching in the K_4-submodel of μ restricted to $\{v_1, v_2, v_3, v_5\}$, so $2\gamma(P) = 0$. See Figure 4.1a.

So we may assume that u and w are in the same tree, say $\mu(v_1)$. Since u and w are both branching in μ, there are four indices in $[m] - \{1\}$, say 2, 3, 4, and 5, such that u branches to both $\{v_2, v_3, v_4\}$ and $\{v_2, v_3, v_5\}$ and w branches to both $\{v_2, v_4, v_5\}$ and $\{v_3, v_4, v_5\}$.

For the rest of the proof of this claim, we only consider the K_5-submodel of μ restricted to $\{v_1, \ldots, v_5\}$. For notational convenience we simply assume that μ is a K_5-model.

Let η be the K_4-submodel of μ restricted to $\{v_2, v_3, v_4, v_5\}$ and, for $i \in \{2, 3, 4, 5\}$, let x_i be the unique vertex in $\mu(v_i)$ that is branching in η. Let Q_{ij} denote the unique x_i-x_j-path in $\mu[v_i, v_j]$ for $i, j \in \{2, 3, 4, 5\}$. Note that $2\gamma(Q_{ij}) = 0$ by the $m = 4$ case of the claim.

Let P_i denote the u-x_i-path in $\mu[v_1, v_i]$ for $i \in \{2, 3\}$ and let y_i be the closest vertex to u on P_i such that $y_i \in V(\mu(v_i))$ and y_i is branching in μ. Similarly, for $j \in \{4, 5\}$, let P_j

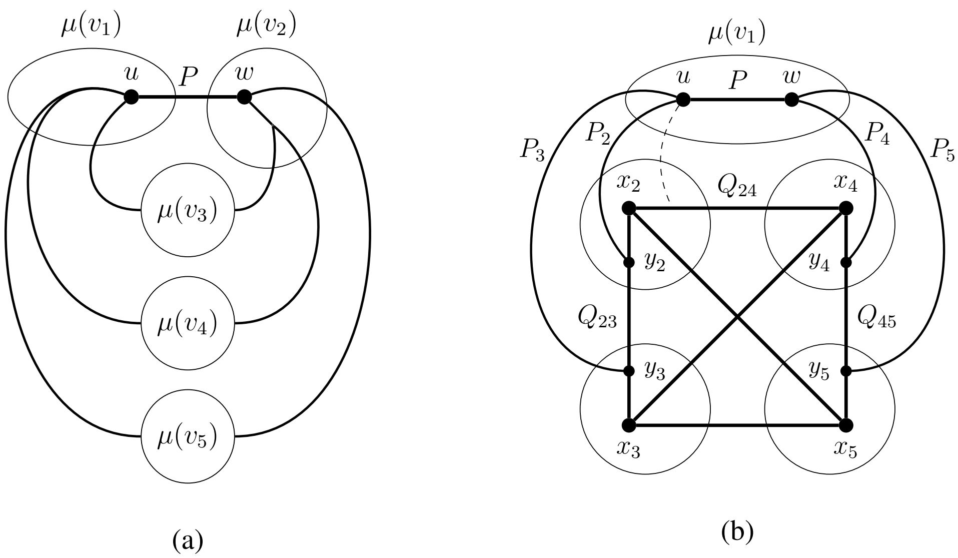


Figure 4.1: A $b(\mu)$-path P in a K_5-model μ. Each ellipse indicates a tree $\mu(v_i)$.

denote the w-x_j-path in $\mu[v_1, v_j]$ and let y_j be the closest vertex to w on P_j such that $y_j \in V(\mu(v_j))$ and y_j is branching in μ. Note that $2\gamma(uP_2y_2) = 2\gamma(uP_3y_3) = 2\gamma(wP_4y_4) = 2\gamma(wP_5y_5) = 0$ since these are $b(\mu)$-paths with endpoints in different trees.

Suppose $y_2 \in V(Q_{24} \cup Q_{25})$. Then y_2 branches to $\{v_1, v_4, v_5\}$, so it is a branching vertex in the K_4-submodel π of μ restricted to $\{v_1, v_2, v_4, v_5\}$. Since w is also branching in π, we have $2\gamma(y_2P_2uPw) = 0$ by the $m = 4$ case. But as previously noted, we also have $2\gamma(uP_2y_2) = 0$, which implies $2\gamma(P) = 0$. Therefore we may assume that $y_2 \in V(Q_{23}) - x_2$ and, by symmetry, $y_3 \in V(Q_{23}) - x_3$. Similarly, we may assume that $y_j \in V(Q_{45}) - x_j$ for $j \in \{4, 5\}$. See Figure 4.1b.

Since the two cycles $P \cup P_4 \cup Q_{24} \cup P_2$ and $P \cup P_4 \cup Q_{34} \cup P_3$ are in $\mu[v_1, v_2, v_3, v_4]$, they are both Γ-zero, and so $\gamma(Q_{24}) + \gamma(P_2) = \gamma(Q_{34}) + \gamma(P_3)$. But $P_2 \cup Q_{24} \cup Q_{34} \cup P_3$ is also a cycle in $\mu[v_1, v_2, v_3, v_4]$, so $\gamma(P_2) + \gamma(Q_{24}) + \gamma(Q_{34}) + \gamma(P_3) = 0$, which gives $0 = 2\gamma(P_2) + 2\gamma(Q_{24}) = 2\gamma(P_2)$ since $2\gamma(Q_{24}) = 0$. By symmetry, we have $2\gamma(P_2) = 2\gamma(P_4) = 0$. Now since $P \cup P_4 \cup Q_{24} \cup P_2$ is a Γ-zero cycle, it follows that $2\gamma(P) = 2\gamma(P) + 2\gamma(P_2) + 2\gamma(Q_{24}) + 2\gamma(P_4) = 0$. This completes the proof of the claim. ■

Now suppose $\mu[V(K_m)]$ contains a Γ-nonzero cycle and choose such a cycle C minimizing the number ℓ of edges of the form $\mu(v_iv_j)$. Then $\ell \geq 5$ by the definition of a Γ-bipartite K_m-model. Let $\mu(v_{i_1}), \mu(v_{i_2})$, and $\mu(v_{i_3})$ be three consecutive trees visited by C in that order. Then $\mu[v_{i_1}, v_{i_3}]$ contains a $V(C)$-path Q that is not an edge of C, and the two cycles C_1, C_2 in $C \cup Q$ distinct from C both contain less than ℓ edges of the form $\mu(v_iv_j)$. Hence $\gamma(C_1) = \gamma(C_2) = 0$ by minimality of ℓ. Moreover, the two vertices in $V(C) \cap V(Q)$ are clearly branching in μ, so Q is a concatenation of $b(\mu)$-paths in μ, and hence $2\gamma(Q) = 0$ by the Claim. We thus have $0 = \gamma(C_1)+\gamma(C_2) = \gamma(C)+2\gamma(Q) = \gamma(C)$, a contradiction. This completes the proof of the lemma. $\square$

Next we prove two technical lemmas which will be used in the proof of Lemma 4.1.6.

Lemma 4.1.4. *Let Γ be an abelian group and let (G, γ) be a Γ-labelled graph. Let P be a Γ-nonzero path in (G, γ) with endpoints s_1, s_2. Let $s'_1, s'_2 \in V(G) - V(P)$ and for each $i \in [2]$, let Q_i be a path from s'_i to $(V(P) - \{s_1, s_2\})$ such that Q_1 is disjoint from Q_2. Then there is a Γ-nonzero $\{s_1, s_2, s'_1, s'_2\}$-path in $P \cup Q_1 \cup Q_2$ with different endpoints than P.*

Proof. Note that $P \cup Q_1 \cup Q_2$ is a tree with four leaves $\{s_1, s_2, s'_1, s'_2\}$ and two vertices of degree 3, say x_1 and x_2 where x_i is the endpoint of Q_i in $V(P)$ for each $i \in [2]$. Let us assume without loss of generality that s_1, x_1, x_2, s_2 occur in this order on P. If the four $\{s_1, s_2\}$-$\{s'_1, s'_2\}$-paths in $P \cup Q_1 \cup Q_2$ are all Γ-zero, then for each $i \in [2]$ we have $\gamma(s_1Px_i) = \gamma(x_iPs_2) = -\gamma(Q_i)$, which gives $2\gamma(Q_i) = -\gamma(P) \neq 0$. Then $\gamma(x_1Px_2) = \gamma(x_1Ps_2) - \gamma(x_2Ps_2) = -\gamma(Q_1) + \gamma(Q_2)$, but this implies that the s'_1-s'_2-path $s'_1Q_1x_1Px_2Q_2s'_2$ has weight $\gamma(Q_1) + (-\gamma(Q_1) + \gamma(Q_2)) + \gamma(Q_2) = 2\gamma(Q_2) \neq 0$. $\square$

In the following lemma, we use the notation arising in the proof of Lemma 4.1.6.

Lemma 4.1.5. *Let Γ be an abelian group. Let (G', γ') be a Γ-labelled graph and let $s_1, s_2, s'_{j^1_1}, s'_{j^1_2}, s'_{j^2_1}, s'_{j^2_2}$ be six distinct vertices in $V(G')$. Let $(Q^1_1, Q^1_2, Q^2_1, Q^2_2, Q, P_1)$ be a tuple of six paths in (G', γ') satisfying the following properties (see Figure 4.2):*

1. *For each $i, k \in [2]$, Q^i_k is an s_i-$s'_{j^i_k}$-path.*

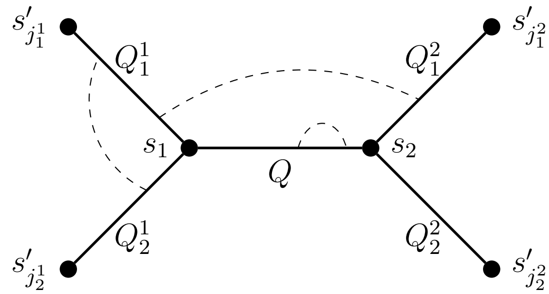


Figure 4.2: The tree $T = Q_1^1 \cup Q_2^1 \cup Q_1^2 \cup Q_2^2 \cup Q$. The dashed lines indicate some possible $V(T)$-subpaths P of P_1.

2. *Q is a Γ-zero s_1-s_2-path.*
3. *The five paths $Q_1^1, Q_2^1, Q_1^2, Q_2^2, Q$ are disjoint except that for each $i \in [2]$, the vertex s_i belongs to exactly three paths Q_1^i, Q_2^i, Q.*
4. *P_1 is a Γ-nonzero s_1-s_2-path.*

Then the union of the six paths $Q_1^1, Q_2^1, Q_1^2, Q_2^2, Q, P_1$ contains a Γ-nonzero $\{s'_{j_1^1}, s'_{j_2^1}, s'_{j_1^2}, s'_{j_2^2}\}$-path.

Proof. Let $S'' = \{s'_{j_1^1}, s'_{j_2^1}, s'_{j_1^2}, s'_{j_2^2}\}$ and let $T = Q_1^1 \cup Q_2^1 \cup Q_1^2 \cup Q_2^2 \cup Q$. Suppose there does not exist a Γ-nonzero S''-path in $T \cup P_1$. Then the edges of P_1 can be partitioned into a sequence of maximal paths contained in T and $V(T)$-paths not contained in T. Let P be a subpath of P_1 that is a $V(T)$-path not contained in T. Let u and v denote the two endpoints of P.

Suppose that u and v both lie in Q. If $\gamma'(uQv) \neq \gamma'(P)$, then we get a Γ-nonzero S''-path say from $s'_{j_1^1}$ to $s'_{j_1^2}$ by rerouting the path $s'_{j_1^1} T s'_{j_1^2}$ along P, a contradiction. On the other hand, if $\gamma'(uQv) = \gamma'(P)$, then we can reroute Q through P and still satisfy the five conditions of the lemma, while only reducing the union of the six paths. Therefore, we may assume that u and v do not both lie in Q. Similarly, we may assume that u and v do not both lie in any one path Q_k^i.

Next suppose that u and v are in $Q_1^i \cup Q_2^i$ for some $i \in [2]$. Assume without loss of generality that $u \in V(Q_1^i) - s_i$ and $v \in V(Q_2^i) - s_i$. Since there is no Γ-nonzero S''-path in

$T \cup P$, by Lemma 2.2.3(a), we have $2\gamma'(P) = 2\gamma'(uQ_1^i s_i) = 2\gamma'(vQ_2^i s_i) = 0$. Moreover, we have $\gamma'(P) = \gamma'(uTv)$, since otherwise there is a Γ-nonzero $s'_{j_1^i}$-$s'_{j_2^i}$-path in $T \cup P$. Similarly, if one endpoint of P is in Q_k^i and the other is in Q, or if one endpoint is in Q_k^1 and the other is in Q_ℓ^2, then $2\gamma'(P) = 2\gamma'(uTs_i) = 2\gamma'(vTs_i) = 0$ for each $i \in [2]$ (since $\gamma'(Q) = 0$), and $\gamma'(P) = \gamma'(uTv)$. Thus, if U denotes the set consisting of every vertex that is an endpoint of a subpath of P_1 that is a $V(T)$-path not contained in T, then for all $u \in U$, we have $2\gamma'(uTs_i) = 0$ for each $i \in [2]$. This then implies that every path R in T with both endpoints in $U \cup \{s_1, s_2\}$ satisfies $2\gamma'(R) = 0$.

Now let $(s_1 = u_1, u_2, \ldots, u_n = s_2)$ denote the sequence of vertices in $U \cup \{s_1, s_2\}$ that occur on P_1 in this order. Let $W = u_1Tu_2Tu_3 \ldots u_{n-1}Tu_n$, which is a walk from s_1 to s_2 contained in T. We have shown above that $\gamma(u_iTu_{i+1}) = \gamma(u_iP_1u_{i+1})$ and $2\gamma(u_iTu_{i+1}) = 0$ for all $i \in [n-1]$. The first equality implies that $\gamma(W) := \sum_{i=1}^{n-1} \gamma(u_iTu_{i+1}) = \gamma(P_1) \neq 0$, whereas the second equality implies that $\gamma(W) = \gamma(s_1Ts_2) = \gamma(Q) = 0$, a contradiction. $\square$

Let $R_4(n, m)$ denote the Ramsey numbers for 4-uniform hypergraphs. In other words, if $r \geq R_4(n, m)$, then for every red-blue colouring of the hyperedges of the complete 4-uniform hypergraph on r vertices, there is either a red complete hypergraph on n vertices or a blue complete hypergraph on m vertices.

We now prove the main lemma of this section.

Lemma 4.1.6. *Let Γ be an abelian group and let $t \geq 2$ be an integer. Let (G, γ) be a Γ-labelled graph containing a $K_{R_4(t,m(t))}$-model π where $m(t) = 50(150t^4)^4 + 1 + 300t^4$. Then either there is a Γ-odd K_t model in (G, γ) that is an enlargement of π, or there exists $X \subseteq V(G)$ with $|X| < 50(150t^4)^4$ such that the $\mathcal{T}_\pi$-large 3-block of $(G - X, \gamma)$ is Γ-bipartite.*

Proof. Let $\mathcal{H}$ be the complete 4-uniform hypergraph on the vertex set of $K_{R_4(t,m(t))}$. Colour a hyperedge $\{w, x, y, z\}$ of $\mathcal{H}$ red if $\pi[w, x, y, z]$ contains a Γ-nonzero cycle and blue if

$\pi[w,x,y,z]$ is Γ-bipartite. Then there is either a Γ-odd K_t-submodel or a Γ-bipartite $K_{m(t)}$-submodel of π. In the first case we are done (since submodels of π are also enlargements of π), so we assume the existence of a Γ-bipartite $K_{m(t)}$-model μ that is an enlargement of π. Write $V(K_{m(t)}) = \{v_1, \dots, v_{m(t)}\}$. By Lemma 4.1.3, (μ, γ) is a Γ-bipartite Γ-labelled graph and we may assume by Lemma 2.2.2 and Proposition 4.1.1 that, after possibly shifting,

$$\text{every } b(\mu)\text{-path in } (\mu, \gamma) \text{ is } \Gamma\text{-zero.} \tag{$*$}$$

For each $i = 1, 2, \dots, 50(150t^4)^4$, sequentially in this order, pick a vertex $s_i \in \mu(v_i)$ as follows. If $\mu(v_i)$ has a $50t^4$-branching vertex, then define s_i to be such a vertex. Otherwise, by Lemma 4.1.2, the $\mathcal{C}_i^{50t^4}$-subtree of $\mu(v_i)$ is a path R and there are at least $50(150t^4)^4$ indices $j \in [m(t)] - \{i\}$ such that the $\mu(v_i v_j)_i$-R-path in $\mu(v_i)$ ends at an internal vertex of R. Pick one such index j such that

$$j \notin \{\kappa(i') : i' < i \text{ and } \kappa(i') \text{ was previously defined}\},$$

and define $\kappa(i) = j$. Such an index j always exists as long as $i \leq 50(150t^4)^4$. Define s_i to be the (internal) vertex of R such that the $\mu(v_i v_{\kappa(i)})_i$-R-path in $\mu(v_i)$ ends at s_i. Note that s_i is branching in μ as it is an internal vertex of R.

Define $S = \{s_i : i \in [50(150t^4)^4]\}$. Since each s_i is branching in μ, we have $S \subseteq b(\mu)$. By Theorem 2.2.5, either there exist $150t^4$ disjoint Γ-nonzero S-paths in (G, γ) or there exists $X \subseteq V(G)$ with $|X| \leq 50(150t^4)^4 - 3$ such that $(G - X, \gamma)$ does not contain a Γ-nonzero S-path.

Claim 4.1.6.1. *If $X \subseteq V(G)$, $|X| \leq 50(150t^4)^4 - 3$, and $(G - X, \gamma)$ does not contain a Γ-nonzero S-path, then the $\mathcal{T}_\pi$-large 3-block (B, γ_B) of $(G - X, \gamma)$ is Γ-bipartite.*

Proof. Since $|S| = 50(150t^4)^4$, there are three vertices of S, say s_1, s_2, s_3 without loss of generality, such that X is disjoint from $\mu(v_1) \cup \mu(v_2) \cup \mu(v_3)$. Note that for each $i \in [3]$,

$\mu(v_i)$ intersects $V(B)$ by the definition of $\mathcal{T}_\pi$. For $i \in [3]$, define $x_i \in V(\mu(v_i))$ to be s_i if $s_i \in V(B)$ and, otherwise, a closest vertex to s_i in $\mu(v_i)$ that is in $V(B)$.

Now suppose that (B, γ_B) contains a simple Γ-nonzero cycle C_B. Since B is 3-connected, there exist three disjoint $V(C_B)$-$\{x_1, x_2, x_3\}$-paths P_1', P_2', P_3' in B. Applying Proposition 2.4.1, we obtain a Γ-nonzero cycle C of $(G - X, \gamma)$ corresponding to C_B and three disjoint $V(C_B)$-$\{x_1, x_2, x_3\}$-paths P_1, P_2, P_3 in $(G - X, \gamma)$ corresponding to P_1', P_2', P_3' respectively. Since X is disjoint from $\mu(v_i)$ for each $i \in [3]$, the three paths $x_i\mu(v_i)s_i$ together with $C \cup P_1 \cup P_2 \cup P_3$ gives three disjoint $V(C)$-$\{s_1, s_2, s_3\}$-paths in $(G - X, \gamma)$. By Lemma 2.2.4, there exists a Γ-nonzero S-path, a contradiction. Thus (B, γ_B) has no simple Γ-nonzero cycle and by Lemma 2.2.2, (B, γ_B) is Γ-bipartite. ■

So we may assume that there exist $150t^4$ disjoint Γ-nonzero S-paths in (G, γ). Let $\mathcal{P} = \{P_1, \ldots, P_{150t^4}\}$ be a set of $150t^4$ disjoint Γ-nonzero S-paths minimizing the number of edges in $\cup\mathcal{P}$ not in a tree $\mu(v_i)$ of μ; that is, we minimize

$$\left|\left(\bigcup_{j=1}^{150t^4} E(P_j)\right) - \left(\bigcup_{j=1}^{m(t)} E(\mu(v_j))\right)\right|.$$

By relabelling indices in $[50(150t^4)^4]$ and updating κ accordingly, we may assume that P_i has endpoints s_{2i-1} and s_{2i} for all $i \in [150t^4]$. Note that $|S| = 50(150t^4)^4$ and $|\mathcal{P}| = 150t^4$.

Claim 4.1.6.2. *There are at most $|\mathcal{P}|$ indices j such that $2|\mathcal{P}| < j \leq |S|$ (i.e. s_j is not an endpoint of a path in $\mathcal{P}$) and $\mu(v_j)$ intersects a path in $\mathcal{P}$.*

Proof. Suppose there are more than $|\mathcal{P}|$ indices j such that $j > 2|\mathcal{P}|$ and $\mu(v_j)$ intersects a path in $\mathcal{P}$. For each such index j, pick a closest vertex x_j to s_j in $\mu(v_j)$ such that x_j is in a path in $\mathcal{P}$. Then there exist two such indices j_1 and j_2 such that x_{j_1} and x_{j_2} belong to one path P of $\mathcal{P}$. For each $i \in [2]$, let $Q_i = s_{j_i}\mu(v_{j_i})x_{j_i}$. Then each Q_i is disjoint from every path in $\mathcal{P} - \{P\}$, and every S-path contained in $P \cup Q_1 \cup Q_2$ distinct from P has fewer edges not in a tree of μ than P. But by Lemma 4.1.4, $P \cup Q_1 \cup Q_2$ contains a Γ-nonzero S-path P' distinct from P, so $(\mathcal{P} - \{P\}) \cup \{P'\}$ contradicts our choice of $\mathcal{P}$. ■

By relabelling among indices j with $2|\mathcal{P}| < j \leq |S|$ and updating κ accordingly, we may assume that no path in $\mathcal{P}$ contains a vertex in $\mu(v_j)$ for all j with $3|\mathcal{P}| < j \leq |S|$. We now construct a minor G' of G as follows. Let $J' = \{j : 3|\mathcal{P}| < j \leq |S|\}$ and for each $j \in J'$, contract the tree $\mu(v_j)$ into a single vertex s'_j and delete all loops. Let $S' = \{s'_j : j \in J'\}$.

Define a Γ-labelling γ' of G' as follows. Let $e = x'y' \in E(G')$. If e is not incident to S', then define $\gamma'(e) = \gamma(e)$. If $x' = s'_j$ and $y' \notin S'$, then let x be the endpoint of e in G in $\mu(v_j)$ and define $\gamma'(e) = \gamma(e) + \gamma(x\mu(v_j)s_j)$. Similarly, if $x' = s'_j$ and $y' = s'_k$, then let x and y be the corresponding endpoints of e in G and define $\gamma'(e) = \gamma(e) + \gamma(x\mu(v_j)s_j) + \gamma(y\mu(v_k)s_k)$. Then each S'-path P' in (G', γ') with endpoints s'_j and s'_k corresponds to an s_j-s_k-path P in (G, γ) of the same weight, obtained by extending the endpoints of P' in G along $\mu(v_j)$ and $\mu(v_k)$ to s_j and s_k respectively.

Let μ' be the resulting $K_{m(t)}$-model in G' obtained from μ. In other words, $\mu'(v_i) = \mu(v_i)$ for $i \notin J'$, $\mu'(v_j) = \{s'_j\}$ for $j \in J'$, and $\mu'(v_iv_j) = \mu(v_iv_j)$ for all $i, j \in [m(t)]$. Note that for $i \notin J'$, the d-central and d-branching vertices in $\mu(v_i)$ and $\mu'(v_i)$ are the same.

Claim 4.1.6.3. *There exist t disjoint Γ-nonzero S'-paths in (G', γ').*

Proof. Suppose not. By Theorem 2.2.5, there exists $Y \subseteq V(G')$ with $|Y| \leq 50t^4 - 4$ such that $(G' - Y, \gamma')$ does not contain a Γ-nonzero S'-path. Recall that no path in $\mathcal{P}$ intersects $\mu(v_j)$ for $j \in J'$, so the paths in $\mathcal{P}$ are unaffected by the minor operations used to obtain G'. We may thus consider $\mathcal{P}$ also as a linkage in (G', γ') whose paths are disjoint from S'.

Since $|Y| < 50t^4$ and $|\mathcal{P}| = 150t^4$, there are more than $100t^4$ paths P_i that are disjoint from $Y - S'$. Among these paths there are more than $50t^4$ paths P_i such that Y is also disjoint from $\mu'(v_{2i-1}) \cup \mu'(v_{2i})$. Among these, there is a path P_i such that Y is also disjoint from $\mu'(v_{\kappa(2i-1)}) \cup \mu'(v_{\kappa(2i)})$, where we define $\mu'(v_{\kappa(i)}) = \emptyset$ if κ is not defined on i. By relabelling the paths in $\mathcal{P}$ and updating the indices in $[2|\mathcal{P}|]$ and κ accordingly, we may assume that Y is disjoint from $P_1 \cup \mu'(v_1) \cup \mu'(v_2) \cup \mu'(v_{\kappa(1)}) \cup \mu'(v_{\kappa(2)})$.

Let $Y' \subseteq [m(t)] - \{1\}$ be the set of indices j such that either $\mu'(v_j)$ contains a vertex in Y or j is equal to the index $\kappa(2)$, if defined. Then $|Y'| \leq |Y| + 1 \leq 50t^4 - 3$.

Define j_1^1, j_2^1, j_3^1 as follows.

Case 1: s_1 is $50t^4$-branching in μ'.

By the definition of a $50t^4$-branching vertex, there exist $j_1^1, j_2^1, j_3^1 \in [m(t)] - \{1\} - Y'$ such that s_1 branches to $\{\mu'(v_{j_1^1}), \mu'(v_{j_2^1}), \mu'(v_{j_3^1})\}$ in μ'.

Case 2: s_1 is not $50t^4$-branching in μ'.

By Lemma 4.1.2, the $\mathcal{C}_1^{50t^4}$-subtree of $\mu'(v_1)$ is a path R. By definition, s_1 is an internal vertex of R and the $\mu'(v_1 v_{\kappa(1)})_1$-$R$-path in $\mu'(v_1)$ ends at s_1. Since s_1 is an internal vertex of R, there are exactly two connected components R_1 and R_2 of $\mu'(v_1) - s_1$ containing a vertex in $\mathcal{C}_1^{50t^4}$, and for each $k \in [2]$, there are more than $50t^4$ indices j such that $\mu'(v_1 v_j)_1 \in V(R_k)$ (by the definition of a $50t^4$-central vertex). Choose one such index $j_k^1 \notin Y'$ for each $k \in [2]$ and define $j_3^1 = \kappa(1)$. Then s_1 branches to $\{\mu'(v_{j_1^1}), \mu'(v_{j_2^1}), \mu'(v_{j_3^1})\}$.

We choose j_1^2, j_2^2, j_3^2 in a similar manner for $\mu'(v_2)$ with the additional condition that $j_k^2 \notin \{j_1^1, j_2^1, j_3^1\}$.

Case 1: s_2 is $50t^4$-branching in μ'.

Since $|Y'| \leq 50t^4 - 3$, we may choose $j_1^2, j_2^2, j_3^2 \in [m(t)] - \{2\} - (Y' \cup \{j_1^1, j_2^1, j_3^1\})$ such that s_2 branches to $\{\mu'(v_{j_1^2}), \mu'(v_{j_2^2}), \mu'(v_{j_3^2})\}$ in μ'.

Case 2: s_2 is not $50t^4$-branching in μ'.

The $\mathcal{C}_2^{50t^4}$-subtree of $\mu'(v_2)$ is a path R and s_2 is an internal vertex of R. Let R_1 and R_2 denote the two connected components of $\mu'(v_2) - s_2$ containing a vertex in $\mathcal{C}_2^{50t^4}$. Then for each $k \in [2]$, there are more than $50t^4$ indices j such that $\mu'(v_2 v_j)_2 \in V(R_k)$, and since $|Y'| \leq 50t^4 - 3$, we may choose one such index $j_k^2 \notin Y' \cup \{j_1^1, j_2^1, j_3^1\}$. Define $j_3^2 = \kappa(2)$.

Note that $\mu'(v_{j_k^i})$ is disjoint from Y for all $i \in [2]$ and $k \in [3]$. For $i \in [2]$, let T_i denote the $\{\mu'(v_i v_{j_1^i}), \mu'(v_i v_{j_2^i}), \mu'(v_i v_{j_3^i})\}$-extension of $\mu'(v_i)$. Then T_i is a 3-star centered at s_i. We modify T_i by extending its legs if necessary so that its leaves are in S' by the following procedure. For each $i \in [2]$ and $k \in [3]$, if $j_k^i \notin J'$, then choose a new $\ell_k^i \in J' - \{j_1^1, j_2^1, j_3^1, j_1^2, j_2^2, j_3^2\}$ so that the ℓ_k^i are distinct and Y is disjoint from each $\mu'(v_{\ell_k^i})$. Extend the leg in T_i ending with $\mu'(v_i v_{j_k^i})$ through $\mu'(v_{j_k^i})$ and through the edge $\mu'(v_{j_k^i} v_{\ell_k^i})$. Since $\mu'(v_{j_k^i})$ is disjoint from Y, T_i is still disjoint from Y. Redefine j_k^i to be ℓ_k^i.

After this procedure, $\{s'_{j_1^1}, s'_{j_2^1}, s'_{j_3^1}\}$ and $\{s'_{j_1^2}, s'_{j_2^2}, s'_{j_3^2}\}$ are the leaves of T_1 and T_2 respectively, the six leaves are in $S' - Y'$ and distinct, and T_1 is disjoint from T_2. For each $i \in [2]$ and $k \in [3]$, let Q_k^i denote the path from s_i to $s'_{j_k^i}$ in T_i. Now consider the unique path Q from s_1 to s_2 in $\mu'[v_1, v_2]$. Then for each $i \in [2]$, at least two of the paths Q_1^i, Q_2^i, Q_3^i intersect Q only at s_i. Without loss of generality, assume that Q_1^i, Q_2^i intersect Q only at s_i. Note that $\gamma'(Q) = \gamma(Q) = 0$ by $(*)$. Hence, the five paths $(Q_1^1, Q_2^1, Q_1^2, Q_2^2, Q)$ satisfy the first three conditions of Lemma 4.1.5.

Recall that P_1 (considered as a path in (G', γ')) is a Γ-nonzero path from s_1 to s_2 in $(G' - Y, \gamma')$ disjoint from S'. We thus have a tuple of paths $(Q_1^1, Q_2^1, Q_1^2, Q_2^2, Q, P_1)$ in $(G' - Y, \gamma')$ satisfying the four conditions of Lemma 4.1.5, from which it follows that there is a Γ-nonzero $\{s'_{j_1^1}, s'_{j_2^1}, s'_{j_1^2}, s'_{j_2^2}\}$-path (hence a Γ-nonzero S'-path) in $(G' - Y, \gamma')$, a contradiction. This completes the proof of the claim. ■

Let $\mathcal{Q}' = \{Q'_1, \ldots, Q'_t\}$ be a set of t disjoint Γ-nonzero S'-paths in (G', γ') minimizing the number of edges in $\cup\mathcal{Q}'$ not contained in a tree $\mu'(v_k)$ of μ'; that is, we minimize

$$\left|\left(\bigcup_{i=1}^{t} E(Q'_i)\right) - \left(\bigcup_{k=1}^{[m(t)]} E(\mu'(v_k))\right)\right|.$$

Let $j_1, \ldots, j_{2t} \in J'$ be the indices such that Q'_i has endpoints $s'_{j_{2i-1}}$ and $s'_{j_{2i}}$. Since each Q'_i is an S'-path, no path in $\mathcal{Q}'$ contains a vertex s'_j with $j \in J' - \{j_1, \ldots, j_{2t}\}$.

Claim 4.1.6.4. *There are at most $3t$ indices $k \notin J'$ such that $\mu'(v_k)$ intersects a path in $\mathcal{Q}'$.*

Proof. Suppose there are more than $3t$ indices $k \notin J'$ such that $\mu'(v_k)$ intersects a path in $\mathcal{Q}'$. Choose $3t+1$ such indices $k_1, \ldots, k_{3t+1}$. Let $\ell_1, \ldots, \ell_{3t+1} \in J' - \{j_1, \ldots, j_{2t}\}$ be distinct. For each $i \in [3t+1]$, let $x_{k_i} \in V(\mu'(v_{k_i}))$ be a closest vertex to s'_{ℓ_i} in $\mu'[v_{k_i}, v_{\ell_i}]$ such that x_{k_i} is in a path in $\mathcal{Q}'$. Then there is a path Q' in $\mathcal{Q}'$ containing at least four of the vertices x_{k_i}, say $x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}$. Let us assume without loss of generality that the vertices $x_{k_1}, x_{k_2}, x_{k_3}, x_{k_4}$ occur in this order on Q'. Define $R_2 = x_{k_2}\mu'[v_{k_2}, v_{\ell_2}]s'_{\ell_2}$ and $R_3 = x_{k_3}\mu'[v_{k_3}, v_{\ell_3}]s'_{\ell_3}$. Note that R_2 and R_3 are disjoint from every path in $\mathcal{Q}' - \{Q'\}$, and every S'-path in $Q' \cup R_2 \cup R_3$ distinct from Q' contains fewer edges not contained in a tree $\mu'(v_k)$ than Q'. But by Lemma 4.1.4, $Q' \cup R_2 \cup R_3$ contains a Γ-nonzero S'-path Q'' distinct from Q', so $(\mathcal{Q} - \{Q'\}) \cup \{Q''\}$ contradicts our choice of $\mathcal{Q}'$. ■

We now use $\mathcal{Q}'$ to construct a Γ-odd K_t-model in (G, γ) that is an enlargement of μ. Recall that each S'-path $Q'_i \in \mathcal{Q}'$ corresponds to an S-path Q_i in (G, γ) of the same weight, obtained by extending the endpoints of Q'_i in G along $\mu(v_{j_{2i-1}})$ and $\mu(v_{j_{2i}})$ to $s_{j_{2i-1}}$ and $s_{j_{2i}}$ respectively. Note in particular that Q_i is disjoint from $\mu(v_{j_k})$ for all $k \in [2t] - \{2i-1, 2i\}$, and that $Q_i \cap \mu(v_{j_{2i-1}})$ and $Q_i \cap \mu(v_{j_{2i}})$ are (possibly trivial) paths.

Let $\mathcal{Q} = \{Q_1, \ldots, Q_t\}$. For simplicity of notation, we relabel the indices in $[50(150t^4)^4]$ so that Q_i has endpoints s_{2i-1} and s_{2i} (we will not need $\mathcal{P}$ for the remainder of the proof). After this relabelling, we have that for each $i \in [t]$, Q_i is disjoint from $\mu(v_j)$ for $j \in [2t] - \{2i-1, 2i\}$, and that $Q_i \cap \mu(v_{2i-1})$ and $Q_i \cap \mu(v_{2i})$ are paths. Furthermore, by Claim 4.1.6.4, we may assume that for all $j > 5t$, $\mu(v_j)$ does not intersect a path in $\mathcal{Q}$.

To construct the Γ-odd K_t-model, first define $L = \emptyset$. For each $j = 1, \ldots, 2t$, sequentially, choose an index $\ell_j \notin [5t]$ as follows.

Case 1: s_j is $50t^4$-branching in μ.

Since $Q_{\lceil j/2 \rceil} \cap \mu(v_j)$ is a path, we may choose $\ell_j \notin L \cup [5t]$ such that the s_j-$\mu(v_{\ell_j})$-path in $\mu[v_j, v_{\ell_j}]$ is internally disjoint from $Q_{\lceil j/2 \rceil}$. Add ℓ_j to L.

Case 2: s_j is not $50t^4$-branching in μ.

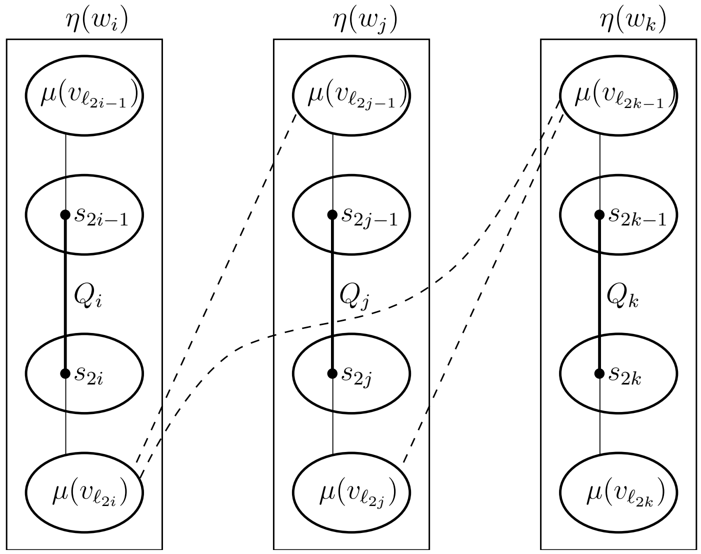


Figure 4.3: Three trees of the Γ-odd K_t-model η. Each ellipse represents a tree of μ and each rectangle represents a tree of η. The three dashed lines indicate the edges $\eta(w_iw_j)$, $\eta(w_iw_k)$, and $\eta(w_jw_k)$.

By Lemma 4.1.2, the $\mathcal{C}_j^{50t^4}$-subtree of $\mu(v_j)$ is a path and s_j is an internal vertex of this path. Then there is a connected component T of $\mu(v_j) - s_j$ containing a vertex in $\mathcal{C}_j^{50t^4}$ such that T is disjoint from $Q_{\lceil j/2\rceil}$ (since $Q_{\lceil j/2\rceil} \cap \mu(v_j)$ is a path). Moreover, there are more than $50t^4$ indices k such that $\mu(v_jv_k)_j \in V(T)$. Choose one such index ℓ_j such that $\ell_j \notin L \cup [5t]$, and add ℓ_j to L.

Define a K_t-model η as follows. Let $\{w_1, \ldots, w_t\}$ denote the vertices of K_t. For each $i \in [t]$, define the tree

$$\eta(w_i) = \mu(v_{\ell_{2i-1}}) \cup \mu(v_{\ell_{2i-1}}v_{2i-1}) \cup \mu(v_{2i-1}) \cup Q_i \cup \mu(v_{2i}) \cup \mu(v_{2i}v_{\ell_{2i}}) \cup \mu(v_{\ell_{2i}}).$$

Note that the path from $\mu(v_{\ell_{2i-1}})$ to $\mu(v_{\ell_{2i}})$ in $\eta(w_i)$ contains Q_i as a subpath. For $i < j$, define the edges of η as $\eta(w_iw_j) = \mu(v_{\ell_{2i}}v_{\ell_{2j-1}})$. Clearly, η is a K_t-model that is an enlargement of μ and hence of π.

To show that η is Γ-odd, we show in fact that the cycle contained in any three trees of η is Γ-nonzero. See Figure 4.3. Let $1 \le i < j < k \le t$ and let C be the unique cycle in $\eta[w_i, w_j, w_k]$. Note that C contains the edges $\{\eta(w_i w_j), \eta(w_i w_k), \eta(w_j w_k)\} = \{\mu(v_{\ell_{2i}} v_{\ell_{2j-1}}), \mu(v_{\ell_{2i}} v_{\ell_{2k-1}}), \mu(v_{\ell_{2j}} v_{\ell_{2k-1}})\}$, so we have $C \cap \eta(w_i) \subseteq \mu(v_{\ell_{2i}})$ and $C \cap \eta(w_k) \subseteq \mu(v_{\ell_{2k-1}})$. Moreover, C is the internally disjoint union of Q_j (which is Γ-nonzero) and the s_{2j-1}-s_{2j}-path in μ through

$$\mu(v_{2j-1}),\ \mu(v_{\ell_{2j-1}}),\ \mu(v_{\ell_{2i}}),\ \mu(v_{\ell_{2k-1}}),\ \mu(v_{\ell_{2j}}),\ \mu(v_{2j})$$

in this order. By $(*)$, we have $\gamma(E(C) - E(Q_j)) = 0$. Hence, $\gamma(C) = \gamma(Q_j) \neq 0$ and η is a Γ-odd K_t-model that is an enlargement of π. $\square$

4.2 Large flat wall

In this section we deal with the second outcome of the flat wall theorem.

Lemma 4.2.1. *Let $t \geq 4$ be an integer and let Γ be an abelian group. Let (G, γ) be a Γ-labelled graph containing a flat $(t+2)^2$-wall (W, γ). Then there is a flat t-wall (W_1, γ) with certifying separation (C_1, D_1) such that $\mathcal{T}_{W_1}$ is a truncation of $\mathcal{T}_W$ and either*

(i) (W_1, γ) is facially Γ-odd, or

(ii) the 3-block of (D_1, γ) containing the degree 3 vertices of (W_1, γ) is Γ-bipartite.

Proof. Let (W_2, γ) denote the t-subwall of (W, γ) contained in the union of the $((t+2)i - t)$-th rows and columns of (W, γ), $i \in [t+1]$. Note that (W_2, γ) is flat and 1-contained in (W, γ) since $(t+2)(t+1) - t < (t+2)^2$. For $i, j \in [t+1]$, let $R_i^{W_2}$ and $C_j^{W_2}$ denote the i-th row and the j-th column of W_2 respectively, and for $i, j \in [t]$ let $B_{i,j}$ denote the (i,j)-th brick of W_2. Then the union of the $t+3$ rows and columns of W intersecting $B_{i,j}$ contains a $(t+2)$-subwall of W, which 1-contains a flat t-subwall $W_{i,j}$ of W. Let $(C_{i,j}, D_{i,j})$ be a certifying separation for $W_{i,j}$ minimizing $|V(D_{i,j})|$. Then $D_{i,j}$ is disjoint from $D_{i',j'}$ for $(i,j) \neq (i',j')$.

If for some $i, j \in [t]$, the 3-block of $(D_{i,j}, \gamma)$ containing the degree 3 vertices of $(W_{i,j}, \gamma)$ is Γ-bipartite, then $(W_{i,j}, \gamma)$ and $(C_{i,j}, D_{i,j})$ satisfies outcome (ii). So we may assume that the 3-block of $(D_{i,j}, \gamma)$ containing the degree 3 vertices of $(W_{i,j}, \gamma)$ contains a Γ-nonzero cycle $O_{i,j}$ for all $i, j \in [t]$.

For $i, j \in [t]$, let $P_{i,j}$ denote the subpath of $C_{j+1}^{W_2}$ that is a $R_i^{W_2}$-$R_{i+1}^{W_2}$-path. Let $H_{i,j}$ be the union of $D_{i,j}$, $P_{i,j}$, and the subpaths of rows of W that are $W_{i,j}$-$P_{i,j}$-paths. Then there are three disjoint paths from the interior of $P_{i,j}$ to $O_{i,j}$ in $H_{i,j}$, so by Lemma 2.2.3(b), there is a path $R_{i,j}$ in $H_{i,j}$ having the same endpoints as $P_{i,j}$ such that $\gamma(P_{i,j}) \neq \gamma(R_{i,j})$. Note that replacing $P_{i,j}$ with $R_{i,j}$ yields a local rerouting of (W_2, γ).

We then obtain a facially Γ-odd t-wall (W_1, γ) from (W_2, γ) by a sequence of local reroutings where, for each $(i, j) \in [t]^2$ in lexicographic order, we replace $P_{i,j}$ with $R_{i,j}$ if necessary to make the (i, j)-th brick Γ-nonzero. $\square$

Lemma 4.2.2. *Let $r \geq 4$ be an integer and let Γ be an abelian group. Let (G, γ) be a Γ-labelled graph containing a flat $(150r^{12} + 2)^2$-wall (W, γ). Then one of the following outcomes hold:*

(1) There is a flat $50r^{12}$-wall (W_1, γ) such that $\mathcal{T}_{W_1}$ is a truncation of $\mathcal{T}_W$ and either

(i) (W_1, γ) is facially Γ-odd, or

(ii) (W_1, γ) is strongly Γ-bipartite and there is a pure Γ-odd linkage of (W_1, γ) of size r.

(2) There exists $Z \subseteq V(G)$ with $|Z| < 50r^{12}$ such that the $\mathcal{T}_W$-large 3-block of $(G - Z, \gamma)$ is Γ-bipartite.

Proof. Applying Lemma 4.2.1 with $t = 150r^{12}$, we obtain a flat $150r^{12}$-wall (W_0, γ) with top nails N_0 and certifying separation (C_0, D_0) satisfying the conclusion of Lemma 4.2.1. If (W_0, γ) is facially Γ-odd, then outcome (1)-(i) is satisfied by taking a compact $50r^{12}$-subwall. So we may assume that the 3-block (B_0, γ_0) of (D_0, γ) containing the vertices of degree 3 in (W_0, γ) is Γ-bipartite. Since W_0 is a $150r^{12}$-wall and $r \geq 4$, we have $|V(B_0)| \geq 4$, hence B_0 is 3-connected. By Lemma 2.2.2, we may assume by possibly shifting in (G, γ) that

$$\text{every } V(B_0)\text{-path in } (D_0, \gamma) \text{ is } \Gamma\text{-zero.} \tag{†}$$

Also note that given any 1-contained subwall W' of W_0 with the choice of nails with respect to W_0, its branch vertices all have degree 3 in W_0, so $b(W') \subseteq V(B_0)$ and every $b(W')$-path in (D_0, γ) is Γ-zero by (†).

Since (W_0, γ) is a flat $150r^{12}$-wall, it $50r^{12}$-contains a flat $50r^{12}$-subwall (W_1, γ). Let N_1 denote its top nails with respect to (W_0, γ) and let (C_1, D_1) be a certifying separation for (W_1, γ) such that $|V(D_1)|$ is minimized. Note that $D_1 \subseteq D_0$ and hence, by (†), every path in (D_1, γ) between branch vertices of (W_1, γ) is Γ-zero. In particular, (W_1, γ) is strongly Γ-bipartite.

If there exist r^3 disjoint Γ-nonzero N_1-paths in $(G-(V(D_1)-N_1), \gamma)$, then by Lemma 2.7.1, there is a pure linkage $\mathcal{P}$ of (W_1, γ) whose paths are Γ-nonzero. Since every N_1-path in (D_1, γ) is Γ-zero, $\mathcal{P}$ is Γ-odd and outcome (1)-(ii) is satisfied. So by Theorem 2.2.5, we may assume that there exists $Z \subseteq V(G - (V(D_1) - N_1))$ with $|Z| \leq 50r^{12} - 3$ such that $(G - (V(D_1) - N_1) - Z, \gamma)$ does not contain a Γ-nonzero N_1-path.

Since $|Z| \leq 50r^{12} - 3$ and W_1 is $50r^{12}$-contained in W_0, there are two columns of W_0, one to the left of W_1 and one to the right, and two rows of W_0, one above W_1 and one below, such that the four paths are all disjoint from Z and not contained in the perimeter of W_0. Let O denote the unique cycle in the union of these four paths. See Figure 4.4. Since W_1 is a $50r^{12}$-wall, there are two disjoint $V(O)$-N_1-paths P_1 and P_2 that are subpaths of columns of W_0 disjoint from Z. Note that $\gamma(P_1) = \gamma(P_2) = 0$ since the endpoints of each P_i have degree 3 in W_0.

Let W° denote the compact subwall of W_0 whose perimeter is O, and let (C°, D°)

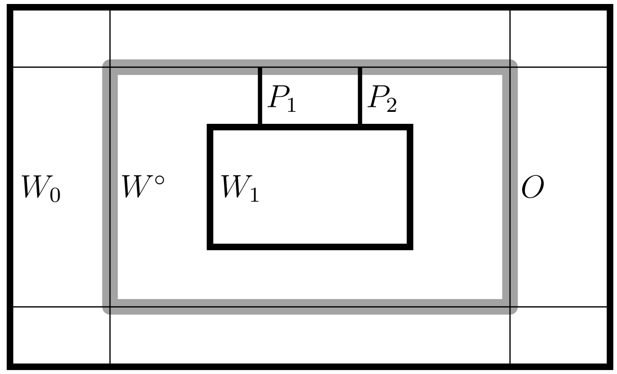


Figure 4.4: W_0 is a flat $150r^{12}$-wall which $50r^{12}$-contains a compact $50r^{12}$-subwall W_1. The cycle O (highlighted in grey) around W_1 is disjoint from Z and forms the perimeter of a compact subwall W° of W_0. The two $V(O)$-N_1-paths P_1 and P_2 are also disjoint from Z.

be a certifying separation for W° in G minimizing $|V(C^\circ \cap D^\circ) - V(B_0)|$. Note that $W_1 \subseteq W^\circ \subseteq D^\circ \subseteq D_0$ and $V(C^\circ \cap D^\circ) \subseteq V(O)$. We claim that $V(C^\circ \cap D^\circ) \subseteq V(B_0)$. Suppose otherwise and let $v \in V(C^\circ \cap D^\circ) - V(B_0)$. Then v is contained in the interior of a $V(B_0)$-bridge $\mathcal{B}$ of D_0 with two attachments, say $a_1, a_2 \in V(B_0) \cap V(O)$. But then $(C^\circ - (\mathcal{B} - \{a_1, a_2\}), D^\circ \cup \mathcal{B})$ is also a certifying separation for W° with fewer vertices in $V(C^\circ \cap D^\circ) - V(B_0)$, contradicting our choice of (C°, D°). It follows from (†) that every $V(C^\circ \cap D^\circ)$-path in (D°, γ) is Γ-zero.

We now show that outcome (2) is satisfied. Let (B, γ_B) be the $\mathcal{T}_{W_1}$-large 3-block of $(G-Z, \gamma)$. Note that B is 3-connected and contains all vertices of degree 3 in W_1. Suppose contrary to outcome (2) that (B, γ_B) contains a simple Γ-nonzero cycle S' (using Lemma 2.2.2), and let S be a Γ-nonzero cycle in $(G - Z, \gamma)$ corresponding to S' as given by Proposition 2.4.1.

We claim that $S \not\subseteq D^\circ$. Otherwise, there exist three disjoint $V(C^\circ \cap D^\circ)$-$V(S)$-paths in D°, since S is 3-connected to $b(W_1)$ which is in turn highly connected to $V(C^\circ \cap D^\circ)$. By Lemma 2.2.4, we obtain a Γ-nonzero $V(C^\circ \cap D^\circ)$-path in (D°, γ), a contradiction.

We also claim that $S \not\subseteq C^\circ$. Otherwise, there exist three disjoint $V(C^\circ \cap D^\circ)$-$V(S)$-paths in $(C^\circ - Z, \gamma)$ (since there are three disjoint paths from S to $b(W_1)$ which must go through $V(C^\circ \cap D^\circ)$). By Lemma 2.2.4, there exists a Γ-nonzero $V(C^\circ \cap D^\circ)$-path in C°. Extending the endpoints of this path along $O \cup P_1 \cup P_2$, we obtain a Γ-nonzero N_1-path in $(G - (V(D_1) - N_1) - Z, \gamma)$, a contradiction.

Therefore, S intersects both $C^\circ - D^\circ$ and $D^\circ - C^\circ$, so the edges of S can be partitioned into $V(C^\circ \cap D^\circ)$-paths, each contained in either C° or D°. Those in D° are all Γ-zero by (†), so C° contains a Γ-nonzero $V(C^\circ \cap D^\circ)$-path. This similarly gives a contradictory Γ-nonzero N_1-path in $(G - (V(D_1) - N_1) - Z, \gamma)$, and therefore outcome (2) holds. $\square$

4.3 Proof of Theorem 3.1.1

The proof of Theorem 3.1.1, restated below, now follows readily from Lemma 4.1.6 and Lemma 4.2.2.

Theorem 3.1.1. *Let Γ be an abelian group and let $r, t \geq 1$ be integers. Then there exist integers $g(r,t)$ and $h(r,t)$, where $h(r,t) \leq g(r,t) - 3$, such that if a Γ-labelled graph (G,γ) contains a wall (W,γ) of size at least $g(r,t)$, then one of the following outcomes hold:*

(1) There is a Γ-odd K_t-model μ in G such that $\mathcal{T}_\mu$ is a truncation of $\mathcal{T}_W$.

(2) There exists $Z \subseteq V(G)$ with $|Z| \leq h(r,t)$ and a flat $50r^{12}$-wall (W_0,γ) in $(G-Z,\gamma)$ such that $\mathcal{T}_{W_0}$ is a truncation of $\mathcal{T}_W$ and either

(a) (W_0,γ) is facially Γ-odd, or

(b) (W_0,γ) is strongly Γ-bipartite and there is a pure Γ-odd linkage of (W_0,γ) of size r.

(3) There exists $Z \subseteq V(G)$ with $|Z| \leq h(r,t)$ such that the $\mathcal{T}_W$-large 3-block of $(G - Z,\gamma)$ is Γ-bipartite.

Proof. Let $r' = (150r^{12}+2)^2$ and $t' = R_4(t, m(t))$ where $m(t) = 50(150t^4)^4 + 1 + 300t^4$. Let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that $g(r,t) \geq f_{2.6.2}(r',t')$ where $f_{2.6.2}$ is the function from Theorem 2.6.2, and let $h(r,t) = t' + 50(150t^4)^4 + 50r^{12}$. Note that we may assume $g(r,t) \geq h(r,t) + 3$.

Suppose (G,γ) contains an $g(r,t)$-wall W. By Theorem 2.6.2, either (G,γ) contains a $K_{t'}$-model π such that $\mathcal{T}_\pi$ is a truncation of $\mathcal{T}_W$ or there exists $X \subseteq V(G)$ with $|X| \leq t' - 5$ and an r'-subwall W' of W that is disjoint from X and flat in $G - X$.

Suppose we are in the first case, that there is a $K_{R_4(t,m(t))}$-model π such that $\mathcal{T}_\pi$ is a truncation of $\mathcal{T}_W$. By Lemma 4.1.6, either there is a Γ-odd K_t-model μ in G such that $\mathcal{T}_\mu$ is a truncation of $\mathcal{T}_\pi$ (hence of $\mathcal{T}_W$), or there exists $Y \subseteq V(G)$ with $|Y| < 50(150t^4)^4$ such that the $\mathcal{T}_\pi$-large 3-block of $(G - Y, \gamma)$ is Γ-bipartite. Since $\mathcal{T}_\pi$ is a truncation of $\mathcal{T}_W$, this 3-block is also $\mathcal{T}_W$-large. The first outcome satisfies (1). The second outcome satisfies (3) with $Z = Y$.

Now suppose we are in the second case, that there exists $X \subseteq V(G)$ with $|X| \leq t' - 5$ and a flat $(150r^{12} + 2)^2$-wall (W', γ) in $(G - X, \gamma)$ such that $\mathcal{T}_{W'}$ is a truncation of $\mathcal{T}_W$. If there exists $Y \subseteq V(G - X)$ with $|Y| < 50r^{12}$ such that the $\mathcal{T}_{W'}$-large 3-block of $(G - X - Y, \gamma)$ is Γ-bipartite, then this 3-block is also $\mathcal{T}_W$-large, so (3) is satisfied with $Z = X \cup Y$. Otherwise, by Lemma 4.2.2, there is a flat $50r^{12}$-wall (W_1, γ) such that $\mathcal{T}_{W_1}$ is a truncation of $\mathcal{T}_W$ and either (W_1, γ) is facially Γ-odd or (W_1, γ) is strongly Γ-bipartite and there is pure Γ-odd linkage of (W_1, γ) of size r. These two outcomes satisfy (2)-(a) and (2)-(b) respectively. $\square$

CHAPTER 5

A-PATHS OF LENGTH ZERO MODULO A PRIME

In this chapter we prove Theorem 1.3.3, that for every odd prime p, the family of A-paths of length 0 mod p satisfies the Erdős-Pósa property.

5.1 Preliminary results

We will need the following variant of Lemma 2.3.1 for A-paths.

Lemma 5.1.1 (Lemma 8 in [6])**.** *Let t be a positive integer and let $f : \mathbb{N} \to \mathbb{N}$ be function such that $f(k) \geq 2f(k-1) + 3t + 10$ and let $((G, \gamma), k)$ be a minimal counterexample to f being an Erdős-Pósa function for the family of Γ-zero A-paths. Then $G - A$ admits a tangle $\mathcal{T}$ of order t such that for each $(C, D) \in \mathcal{T}$, $G[A \cup C]$ does not contain a Γ-zero A-path and $G[A \cup (D - C)]$ contains a Γ-zero A-path.* □

Let μ be a K_t-model in a graph G. We say that a linkage $\mathcal{P}$ *nicely links to* μ if each path in $\mathcal{P}$ has exactly one endpoint in μ, has no internal vertex in $\mu[V(K_t)]$, and each tree of μ intersects at most one path of $\mathcal{P}$. The following lemma allows us to find a large linkage that nicely links to a large submodel of a given K_t-model. We use the formulation in [6], but we remark that the lemma also follows from the proof of (5.3) in [32].

Lemma 5.1.2 (Lemma 10 in [6]; see also (5.3) in [32])**.** *Let $\ell, t \in \mathbb{N}$ with $t \geq 3\ell$. Let G be a graph with $A \subseteq V(G)$, and let μ be a K_t-model in G disjoint from A. Then there is a $K_{t-2\ell}$-submodel η of μ such that either there is an A-η-linkage of size ℓ that nicely links to η, or there exists $X \subseteq V(G)$ with $|X| < 2\ell$ separating A from η.* □

Let W be a wall with top nails N in a graph G. We say that a linkage $\mathcal{P}$ *nicely links to* W if each path in $\mathcal{P}$ is contained in $G - (W - N)$, has exactly one endpoint in N, and has no internal vertex in N. The next lemma allows us to find a large linkage that nicely links to a large subwall of a given wall.

Lemma 5.1.3 (Lemma 12 in [6]). *Let $r, t \in \mathbb{N}$ with $r \geq t$. Let G be a graph with $A \subseteq V(G)$, and let W be a wall of size at least $4tr$ in G disjoint from A. Then W has an r-subwall W_1 such that either there are t disjoint A-W_1-paths that nicely link to W_1 or there exists $X \subseteq V(G)$ with $|X| < 3t^2$ separating A from W_1.* $\square$

5.2 Γ-nonzero A-cycle-chains

In this section we deal with outcomes (1) and (2) of Theorem 3.1.1.

Let l be a positive integer. A *cycle-chain of length l* is a tuple of paths $(P, Q_1, \ldots, Q_l)$ consisting of a *core path* P and l disjoint $V(P)$-paths Q_i such that the $V(Q_i)$-subpaths P_i of P are disjoint from each other. A cycle-chain in a Γ-labelled graph is Γ-*nonzero* if $\gamma(P_i) \neq \gamma(Q_i)$ for all $i \in [l]$. An A-*cycle-chain* is a cycle-chain $(P, Q_1, \ldots, Q_l)$ such that P is an A-path and Q_i is disjoint from A for all $i \in [l]$.

Let $\mathcal{C} = (P, Q_1, \ldots, Q_l)$ be a Γ-nonzero A-cycle-chain in a Γ-labelled graph where $\Gamma = \mathbb{Z}/p\mathbb{Z}$ and p is prime. If l is large enough, then $\mathcal{C}$ contains A-paths of all possible weights since every nonzero element of Γ is a generator. The optimal bound can be obtained from the Cauchy-Davenport Theorem [12] which states that if $X, Y \subseteq \mathbb{Z}/p\mathbb{Z}$ and p is prime, then $|X + Y| \geq \min(|X| + |Y| - 1, p)$.

Proposition 5.2.1. *Let $\Gamma = \mathbb{Z}/p\mathbb{Z}$ where p is prime. Then a Γ-nonzero A-cycle-chain of length $p - 1$ contains a Γ-zero A-path.*

Proof. Let $(P, Q_1, \ldots, Q_{p-1})$ be a Γ-nonzero A-cycle-chain. Let P_i denote the $V(Q_i)$-subpath of P and let $\alpha_i = \gamma(Q_i) - \gamma(P_i) \neq 0$. By the Cauchy-Davenport Theorem, $\{0, \alpha_1\} + \{0, \alpha_2\} + \cdots + \{0, \alpha_{p-1}\} = \mathbb{Z}/p\mathbb{Z}$, hence there is a rerouting of P through some of the paths Q_i to obtain an A-path of any desired weight. $\square$

Note that Proposition 5.2.1 does not hold for cycle-chains of length $p-2$ (consider the case $\gamma(P) = 1 = \alpha_i$ for all $i \in [p-2]$). The condition that p is prime is also necessary; if Γ has a nontrivial proper subgroup Γ', $\gamma(P) \notin \Gamma'$, and $\gamma(Q_i) - \gamma(P_i) \in \Gamma' - \{0\}$ for all i, then the weight of every A-path in $P \cup Q_1 \cup \cdots \cup Q_k$ is in the coset $\Gamma' + \gamma(P)$, hence nonzero.

Let us now show how to find a large packing of Γ-nonzero A-cycle-chains (hence of Γ-zero A-paths).

Throughout this subsection, we fix an odd prime p, fix $\Gamma = \mathbb{Z}/p\mathbb{Z}$ (hence Γ has no element of order two), and assume the existence of a Γ-labelled graph (G, γ) and a large tangle $\mathcal{T}$ of $(G - A, \gamma)$ such that

(a) there does not exist $X \subseteq V(G)$ with $|X| < 108k^2$ intersecting every Γ-zero A-path, and

(b) for all $(C, D) \in \mathcal{T}$, $(G[A \cup C], \gamma)$ does not contain a Γ-zero A-path and $(G[A \cup (D - C)], \gamma)$ contains a Γ-zero A-path.

Lemma 5.2.2. *Let $l \in \mathbb{N}$ and let μ be a Γ-odd K_{5l+1}-model. Then there is a Γ-nonzero cycle-chain $\mathcal{C}$ of length l contained in μ whose core path is a $\mu(v_1)$-$\mu(v_{5l+1})$-path. Moreover, the cycles in $\mathcal{C}$ are disjoint from $\mu(v_1) \cup \mu(v_{5l+1})$.*

Proof. We first prove the lemma for $l = 1$. Since μ is Γ-odd, there is a Γ-nonzero cycle C in $\mu[v_2, v_3, v_4, v_5]$, and since C intersects at least three of the trees in $\{\mu(v_2), \mu(v_3), \mu(v_4), \mu(v_5)\}$, we may assume without loss of generality that C intersects $\mu(v_2), \mu(v_3)$, and $\mu(v_4)$. For each $i \in \{2, 3, 4\}$, let P_i denote the unique $\mu(v_6)$-C-path in $\mu[v_i, v_6]$ and let w_i denote the endpoint of P_i in C. Since C is a Γ-nonzero cycle, without loss of generality, we may assume that the w_3-w_4-path in C that is disjoint from w_2 is Γ-nonzero. Let R denote the unique $\mu(v_1)$-C-path in $\mu[v_1, v_2]$. Then, by Lemma 2.2.3(a) and by the assumption that Γ has no element of order two, there is a $j \in \{3, 4\}$ such that the two $\mu(v_1)$-$\mu(v_6)$-paths in $R \cup C \cup P_j$ have different weights. This gives the desired Γ-nonzero cycle-chain of length

1 whose core path is a $\mu(v_1)$-$\mu(v_6)$-path and whose cycle C is disjoint from $\mu(v_1) \cup \mu(v_6)$ (as C is contained in $\mu[v_2, v_3, v_4, v_5]$).

Now for $l > 1$, we apply the $l = 1$ case to the K_6-submodel

$$\mu_i := \mu[v_{5(i-1)+1}, v_{5(i-1)+2}, \ldots, v_{5(i-1)+6}]$$

for each $i \in [l]$ to obtain a Γ-nonzero cycle-chain $\mathcal{C}_i$ of length 1 contained in μ_i whose core path is a $\mu(v_{5(i-1)+1})$-$\mu(v_{5i+1})$-path and whose cycle is disjoint from $\mu(v_{5(i-1)+1}) \cup \mu(v_{5i+1})$. Note that the core path is internally disjoint from $\mu(v_{5(i-1)+1}) \cup \mu(v_{5i+1})$. Hence, for each $i \in [l-1]$, the unique path in $\mu(v_{5i+1})$ between the endpoints of the core paths of $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$ in $\mu(v_{5i+1})$ is internally disjoint from every $\mathcal{C}_j$, $j \in [l]$. By connecting consecutive cycle-chains in the trees $\mu(v_{5i+1})$, $i \in [l-1]$, we obtain a Γ-nonzero cycle-chain of length l contained in μ whose core path is a $\mu(v_1)$-$\mu(v_{5l+1})$-path and whose cycles are disjoint from $\mu(v_1) \cup \mu(v_{5l+1})$. $\square$

Lemma 5.2.3. *Let μ be a Γ-odd K_{5kp}-model in (G, γ) disjoint from A such that $\mathcal{T}_\mu$ is a truncation of $\mathcal{T}$. Then there exist k disjoint Γ-zero A-paths.*

Proof. We apply Lemma 5.1.2 with $\ell = 2k$ and $t = 5kp$ to obtain a $K_{k(5p-4)}$-submodel η of μ such that either there is an A-η-linkage of size $2k$ that nicely links to η or there exists $X \subseteq V(G)$ with $|X| < 4k$ separating A from η. Note that the order of $\mathcal{T}_\eta$ is greater than $4k$.

Suppose the latter outcome holds. Then there exists a separation (C, D) of G with $V(C \cap D) = X$ such that $A \subseteq V(C)$ and $V(\eta) \subseteq V(D)$. Then $(C - A, D)$ is a $4k$-separation in $G - A$ and, since $V(\eta) \subseteq V(D)$, we have $(C - A, D) \in \mathcal{T}_\eta \subseteq \mathcal{T}$. By (b), every Γ-zero A-path intersects $D - C$ and, since $A \subseteq V(C)$, it follows that X intersects all Γ-zero A-paths, contradicting (a).

So there exists an A-η-linkage $\mathcal{P} = \{P_1, \ldots, P_{2k}\}$ of size $2k$ that nicely links to η. Assume without loss of generality that P_i has an endpoint in $\eta(v_i)$. Then there exist k

disjoint K_{5p-4}-submodels η_i of η such that η_i contains $\eta(v_{2i-1})$ and $\eta(v_{2i})$. By Lemma 5.2.2, there is a Γ-nonzero cycle-chain of length $p-1$ in η_i whose core path is a $\eta(v_{2i-1})$-$\eta(v_{2i})$-path. Extending the core path through $\eta(v_{2i-1}) \cup \eta(v_{2i}) \cup P_{2i-1} \cup P_{2i}$, we obtain k disjoint Γ-nonzero A-cycle-chains, each of length $p-1$. The lemma now follows from Proposition 5.2.1. $\square$

Lemma 5.2.4. *Let (W, γ) be a facially Γ-odd $3l \times 2$-wall and let R_i^W denote the i-th row of W for each $i \in [3l+1]$. Then there is a Γ-nonzero cycle-chain $\mathcal{C}$ of length l in (W, γ) whose core path is a R_1^W-R_{3l+1}^W-path. Moreover, the cycles in $\mathcal{C}$ are disjoint from $R_1^W \cup R_{3l+1}^W$.*

Proof. We first prove the lemma for $l = 1$. Let (W, γ) be a facially Γ-odd 3×2-wall. Let $B_{i,j}$ denote the (i, j)-th brick of W for $i \in [3]$ and $j \in [2]$. We assume without loss of generality that the wall is oriented in such a way that $B_{2,1}$ shares an edge with $B_{1,1}$ and $B_{1,2}$. Let w_1, w_2, and w_3 denote the three vertices on $B_{2,1}$ that have degree 3 in $B_{1,2} \cup B_{2,1} \cup B_{2,2}$. Since W is facially Γ-odd, by Lemma 2.2.3(b), there is a distinct pair $i, j \in [3]$ such that the two w_i-w_j-paths in $B_{2,1}$ have different lengths. In each of the three possible cases, it is easy to see that there is a Γ-nonzero cycle-chain of length 1 whose core path is a R_1^W-R_4^W-path and whose cycle is $B_{2,1}$ (see Figure 5.1).

Now for $l > 1$, let W_i be the 3×2-subwall of W whose first and last row is R_{3i-2}^W and R_{3i+1}^W respectively. We apply the $l = 1$ case of the lemma to each W_i to obtain a Γ-nonzero cycle-chain of length 1 whose core path is a R_{3i-2}^W-R_{3i+1}^W-path and whose cycle is disjoint from $R_{3i-2}^W \cup R_{3i+1}^W$. Connecting consecutive cycle-chains in R_{3i+1}^W, $i \in [l-1]$, we obtain the desired Γ-nonzero cycle-chain of length l. $\square$

Lemma 5.2.5. *Let (W, γ) be a facially Γ-odd wall of size at least $2592k^3p$ in (G, γ) disjoint from $A \subseteq V(G)$ such that $\mathcal{T}_W$ is a truncation of $\mathcal{T}$. Then there exist k disjoint Γ-zero A-paths.*

Proof. Let $t = 6k$ and $r = 108k^2p$. Note that $r \geq \max\{3(p-1), 3t^2\}$ and $2592k^3p = 4tr$. By Lemma 5.1.3, there exists an r-subwall W_1 of W such that either there exist t disjoint

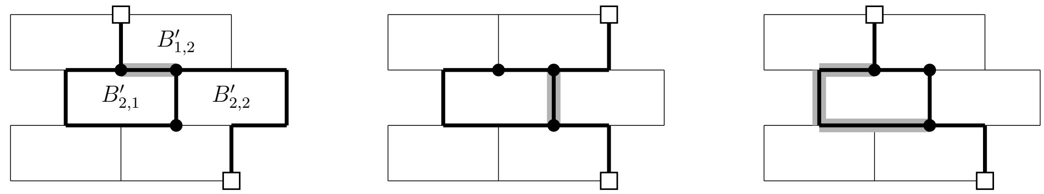


Figure 5.1: The three black vertices are w_1, w_2, and w_3. Since $B_{2,1}$ is a Γ-nonzero cycle, at least one of these three cycle-chains is Γ-nonzero.

A-W_1-paths that nicely link to W_1 or there exists $X \subseteq V(G)$ with $|X| < 3t^2$ separating A from W_1.

Suppose the latter case holds. Then there exists a separation (C, D) of G with $X = V(C \cap D)$ such that $A \subseteq V(C)$ and $V(W_1) \subseteq V(D)$. Since $|X| < 3t^2$ and W_1 has size at least $3t^2$, it follows that $(C - A, D) \in \mathcal{T}_{W_1} \subseteq \mathcal{T}$, which implies that every Γ-zero A-path intersects $D - C$ by (b). But since $A \subseteq C$, X intersects every Γ-zero A-path, violating (a).

So there exists an A-W_1-linkage $\mathcal{P}$ of size $6k$ that nicely links to W_1. Since $|\mathcal{P}| = 6k$, there exist $2k$ paths $\mathcal{P}' = \{P_1, \dots, P_{2k}\} \subseteq \mathcal{P}$ and $2k$ disjoint compact $r \times 2$-subwalls $U_1, \dots, U_{2k}$ of W_1 such that each U_i contains the endpoint of exactly one path in $\mathcal{P}'$, say P_i. Assume without loss of generality that $U_1, \dots, U_{2k}$ are positioned from left to right. Let R_1 and R_2 denote the first and $(\frac{3}{2}(p-1)+1)$-th row of W_1 respectively.

Recalling that $r > \frac{3}{2}(p-1)$, we apply Lemma 5.2.4 to each U_i to obtain a Γ-nonzero cycle-chain of length $\frac{1}{2}(p-1)$ whose core path is a R_1-R_2-path that is internally disjoint from $R_1 \cup R_2$ and whose cycles are disjoint from $R_1 \cup R_2$. It follows that $P_{2i-1} \cup P_{2i} \cup U_{2i-1} \cup U_{2i} \cup R_2$ contains a Γ-nonzero A-cycle-chain of length $p-1$ for each $i \in [k]$. Therefore, there exist k disjoint Γ-zero A-paths by Proposition 5.2.1. □

Lemma 5.2.6. *Let (W, γ) be a strongly Γ-bipartite wall and let $\mathcal{L}$ be a pure Γ-odd linkage of (W, γ) with $|\mathcal{L}| = 3l$. Then there is a Γ-nonzero cycle-chain of length l contained in $R_1^W \cup (\cup \mathcal{L})$ whose core path is a subpath of R_1^W. Moreover, if $\mathcal{L}$ is nested or crossing, then the core path intersects exactly one endpoint of each path in $\mathcal{L}$.*

Proof. Since Γ has no element of order two and (W, γ) is strongly Γ-bipartite, every $b(W)$-

path in (W, γ) is Γ-zero and every path in $\mathcal{L}$ is Γ-nonzero.

If $\mathcal{L}$ is in series, then the conclusion is trivial as $R_1^W \cup (\cup\mathcal{L})$ itself is a Γ-nonzero cycle-chain with core path R_1^W. So we assume that $\mathcal{L}$ is nested or crossing. Let $L_1, \ldots, L_{3l}$ denote the paths of $\mathcal{L}$ and let x_i and y_i denote the left and right endpoint of L_i, $i \in [3l]$. Then there exist disjoint subpaths R_x and R_y of R_1^W such that $\{x_1, \ldots, x_{3l}\} \subseteq V(R_x)$ and $\{y_1, \ldots, y_{3l}\} \subseteq V(R_y)$. We may assume that x_i is positioned to the left of x_j for $i < j$.

First consider the case $l = 1$. We claim that there exist $i, j \in [3]$, $i < j$, such that $\gamma(L_i) \neq -\gamma(L_j)$. Indeed, otherwise we have $\gamma(L_1) = -\gamma(L_2) = \gamma(L_3) = -\gamma(L_1)$, which gives $2\gamma(L_1) = 0$, a contradiction as Γ has no element of order two. Now choose such $1 \leq i < j \leq 3$ with $\gamma(L_i) \neq -\gamma(L_j)$. Then $(x_i R_x x_j, x_i L_i y_i R_y y_j L_j x_j)$ is a Γ-nonzero cycle-chain of length 1 whose core path $x_i R_x x_j$ is a subpath of R_1^W intersecting exactly one endpoint of each path in $\mathcal{L}$.

Now for $l > 1$, we apply the $l = 1$ case above to obtain a Γ-nonzero cycle-chain of length 1 contained in

$$x_{3i-2} R_x x_{3i} \cup L_{3i-2} \cup L_{3i-1} \cup L_{3i} \cup y_{3i-2} R_y y_{3i}$$

for each $i \in [l]$. Connecting consecutive cycle-chains along R_x, we obtain the desired Γ-nonzero cycle-chain of length l. □

Lemma 5.2.7. *Let (W, γ) be a Γ-bipartite wall of size at least $2664k^3p$ with a pure Γ-odd linkage $\mathcal{L}$ with $|\mathcal{L}| \geq 18kp$ such that W and the paths in $\mathcal{L}$ are disjoint from $A \subseteq V(G)$ and such that $\mathcal{T}_W$ is a truncation of $\mathcal{T}$. Then there exist k disjoint Γ-zero A-paths.*

Proof. Let W' be a $36kp$-contained compact $2592k^3p$-subwall of W and define $t = 6k$ and $r = 108k^2p$. Note that $r \geq \max\{36kp, 3t^2\}$ and $2592k^3p = 4tr$. By Lemma 5.1.3, there exists a compact r-subwall W_1 of W' such that either there exist t disjoint A-W_1-paths that nicely link to W_1 or there exists $X \subseteq V(G)$ with $|X| < 3t^2$ separating A from W_1. The latter case is impossible as before, so we may assume that there exists an A-W_1-linkage $\mathcal{P}^1$

with $|\mathcal{P}^1| = 6k$ that nicely links to W_1.

Since W_1 is $36kp$-contained in W, we may extend the endpoints of the paths of $\mathcal{L}$ through W to obtain a pure linkage $\mathcal{L}^1$ of W_1 with $|\mathcal{L}^1| = 18kp$. Note that $\mathcal{L}^1$ is also Γ-odd.

We first modify the paths in $\mathcal{P}^1$ and $\mathcal{L}^1$ so that they become disjoint from each other, at the cost of losing a few paths in $\mathcal{L}^1$. Let H denote the union of all paths in $\mathcal{P}^1$ and in $\mathcal{L}^1$.

Claim 5.2.7.1. *There is an A-W_1-linkage $\mathcal{P}^2$ in H with $|\mathcal{P}^2| = 6k$ that nicely links to W_1 and a subset $\mathcal{L}^2 \subseteq \mathcal{L}^1$ with $|\mathcal{L}^2| = 18k(p-1)$ such that each path in $\mathcal{P}^2$ is disjoint from each path in $\mathcal{L}^2$.*

Proof. Let $\mathcal{P}^2$ be an A-W_1-linkage in H of size $6k$ that nicely links to W_1 minimizing the number of edges not in a path in $\mathcal{L}^1$. Suppose $L \in \mathcal{L}^1$ intersects a path in $\mathcal{P}^2$. Let x be an endpoint of L and let y be the closest vertex to x on L such that y is in some path in $\mathcal{P}^2$, say P'. If P' does not have an endpoint in $V(L)$, then rerouting P' through L to x, we obtain another A-W_1-linkage in H of size $6k$ that nicely links to W_1 using strictly fewer edges not in a path in $\mathcal{L}^1$, a contradiction. Therefore, every path $L \in \mathcal{L}^1$ intersecting a path in $\mathcal{P}^2$ contains an endpoint of a path in $\mathcal{P}^2$, and the number of such paths in $\mathcal{L}^1$ is at most $|\mathcal{P}^2| = 6k$. We may then take a subset $\mathcal{L}^2 \subseteq \mathcal{L}^1$ with $|\mathcal{L}^2| = 18k(p-1) \leq 18kp - 6k$ excluding the paths that intersect $\mathcal{P}^2$. ■

Let R denote the top row of W_1 and let $v_1, \ldots, v_{6k}$ denote the top nails, from left to right, that are endpoints of a path in $\mathcal{P}^2$. Let $R_1 = v_1 R v_{2k}, R_2 = v_{2k+1} R v_{4k}$, and $R_3 = v_{4k+1} R v_{6k}$. Then each path in $\mathcal{L}^2$ is disjoint from at least one of R_1, R_2, or R_3, so there exists $m \in \{1, 2, 3\}$ such that there are $6k(p-1)$ paths in $\mathcal{L}^2$ that are disjoint from R_m. We fix such $m \in \{1, 2, 3\}$.

Let $\mathcal{P}^3 = \{P_1, \ldots, P_{2k}\}$ be the set of $2k$ paths in $\mathcal{P}^2$ containing an endpoint in R_m. We relabel the vertices $v_1, \ldots, v_{6k}$ so that v_i is an endpoint of P_i for $i \in [2k]$ and v_i is to the left of v_j for $i < j$.

Let $\mathcal{L}^3 = \{L_1, \ldots, L_{6k(p-1)}\}$ be a set of $6k(p-1)$ paths in $\mathcal{L}^2$ disjoint from R_m. For $i \in [6k(p-1)]$, let x_i and y_i denote the left and right endpoint of L_i on R respectively. Assume without loss of generality that x_i is to the left of x_j for $i < j$. Then, up to reorientation of the wall, we may assume that the following hold:

1. If $\mathcal{L}^3$ is in series, then there is a subpath R' of R containing $\{x_1, y_1, \ldots, x_{k(p-1)}, y_{k(p-1)}\}$ such that R' is disjoint from R_m.

2. If $\mathcal{L}^3$ is crossing or nested, then the two (disjoint) subpaths $R_x = x_1 R x_{3k(p-1)}$ and $R_y = y_1 R y_{3k(p-1)}$ are disjoint from R_m.

First suppose $\mathcal{L}^3$ is in series. Then $R' \cup \{L_1, \ldots, L_{k(p-1)}\}$ is a Γ-nonzero cycle-chain of length $k(p-1)$ which can be partitioned into k disjoint Γ-nonzero cycle-chains, each of length $p-1$, and each of whose core path is a subpath of R'. By linking the endpoints of the k cycle-chains to the endpoints of $\mathcal{P}^3$ through the W_1, we obtain k disjoint Γ-nonzero A-cycle-chains each of length $p-1$ and hence k disjoint Γ-zero A-paths by Proposition 5.2.1.

Let us assume now that $\mathcal{L}^3$ is crossing or nested. By Lemma 5.2.6, there is a Γ-nonzero cycle-chain of length $k(p-1)$ contained in $R \cup \{L_1, \ldots, L_{3k(p-1)}\}$ whose core path is R_x or R_y. Say R_x. Again we partition into k disjoint Γ-nonzero cycle-chains, each of length $p-1$, and each of whose core path is a subpath of R_x. Linking the endpoints to $\mathcal{P}^3$ through W_1, we obtain k disjoint Γ-nonzero A-cycle-chains each of length $p-1$ and hence k disjoint Γ-zero A-paths by Proposition 5.2.1. □

5.3 Γ-bipartite 3-block

In this section we deal with outcome (3) of Theorem 3.1.1. Let us first sketch the proof, as it is more involved than the previous section.

5.3.1 Proof outline

Let (B, γ) be a Γ-bipartite 3-block of $(G - A, \gamma)$. Since (B, γ) is Γ-bipartite and 3-connected, we may assume that every $V(B)$-path in $(G - A, \gamma)$ has weight 0 by Lemma 2.2.2. Thus, the weight of an A-path containing vertices in $V(B)$ is determined only by its A-$V(B)$-subpaths.

The goal is to find two large A-$V(B)$-linkages $\mathcal{P}$ and $\mathcal{Q}$ such that every path in $\mathcal{P}$ has weight ℓ and every path in $\mathcal{Q}$ has weight $-\ell$ for some $\ell \in \Gamma$, and such that they can be linked in (B, γ) to obtain many disjoint Γ-zero A-paths. The main obstacle is that the edges of the 3-block (B, γ) are not necessarily edges in the original graph (G, γ); rather, they are "virtual" edges representing $V(B)$-bridges of (G, γ) with two attachments. Since the vertices in A may be adjacent to vertices in $V(B)$-bridges not necessarily in $V(B)$, it is not immediately clear how the ends of such A-$V(B)$-paths can be linked to yield the desired Γ-zero A-paths (see Figure 5.2). To aid with this step, we use a large wall W in (B, γ) as an intermediary structure and find two linkages of weight ℓ and $-\ell$ respectively that nicely link to a subwall of W.

This then raises the natural question of whether an approximate version of Menger's theorem holds for paths of weight ℓ. In other words, given disjoint vertex sets A, U of a Γ-labelled graph, can we find either many disjoint A-U-paths of weight ℓ or a small vertex set hitting all such paths? This is false in general as easily seen from the constructions in Figure 3.1 (b) and (c). Nonetheless, we show in Lemma 5.3.1 that this is true under the additional assumption that U is contained in a Γ-bipartite 3-block of $(G - A, \gamma)$. This result is used in Lemma 5.3.2 to find paths of weight ℓ that nicely link to some large subwall of a given wall in a Γ-bipartite 3-block of $(G - A, \gamma)$. In Lemma 5.3.3, we apply the strategy discussed above using Lemma 5.3.2.

5.3.2 Menger type theorems for paths of weight ℓ

We remark that the results in this subsection apply to all abelian groups Γ, and that Lemmas 5.3.1 and 5.3.2 do not assume the existence of a large tangle.

Given $U \subseteq V(B)$, we define the *initial segment* of an A-U-path to be its A-$V(B)$-subpath. The *end segments* of an A-path going through $V(B)$ are its two A-$V(B)$-subpaths.

Lemma 5.3.1. *Let Γ be an abelian group with $\ell \in \Gamma$, let t be a positive integer, let (G, γ) be a Γ-labelled graph with $A \subseteq V(G)$, and let (B, γ) be a 3-block of $(G - A, \gamma)$ such that (B, γ) is Γ-bipartite. Let $U \subseteq V(B)$. If there does not exist $X \subseteq V(G)$ with $|X| < 12t$ intersecting all A-U-paths of weight ℓ, then there exist t disjoint A-U-paths of weight ℓ in (G, γ).*

Proof. Since (B, γ) is a Γ-bipartite 3-block of $(G - A, \gamma)$, Lemma 2.2.2 gives a sequence of shifting operations of (G, γ), only shifting at vertices in $V(G - A)$, resulting in a Γ-labelling γ' of G such that every $V(B)$-path in $(G - A, \gamma')$ has weight 0. Note that the weights of A-paths are unchanged by such shifting operations. So we may assume without loss of generality that, after possibly shifting, every $V(B)$-path in $(G - A, \gamma)$ has weight 0.

Let B_ℓ be the unlabelled graph obtained from (the graph) B by adding the vertex set A and, for each $a \in A$ and $b \in V(B)$, adding an edge ab if there is an A-$V(B)$-path of weight ℓ with endpoints a and b in (G, γ). Then for each A-U-path of weight ℓ in (G, γ), there is a corresponding A-U-path in B_ℓ with the same endpoints and same sequence of vertices in $V(B)$.

The converse does not necessarily hold: Let $b_1 b_2 \in E(B)$, $a \in A$, and suppose P is an A-U-path in B_ℓ with a, b_1, b_2 as its first three vertices in this order. Let R denote the union of all $V(B)$-bridges of $G - A$ whose sets of attachments in B are equal to $\{b_1, b_2\}$. If

- there does not exist an a-b_2-path of weight ℓ going through b_1 in $(G[R + a], \gamma)$, and
- there does not exist an a-b_1-path with weight ℓ in (G, γ) that is internally disjoint from $A \cup V(B) \cup V(R)$,

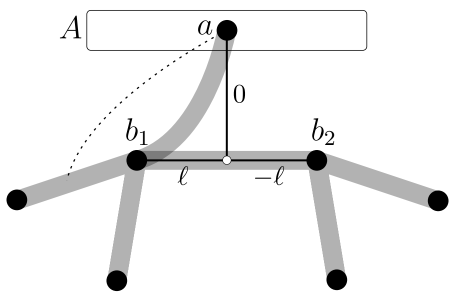


Figure 5.2: The black filled vertices are in B_ℓ and the highlighted curves represent edges of B_ℓ. If a is not adjacent to another $V(B)$-bridge of $G - A$ attaching to b_1, then an A-U-path in B_ℓ starting with the vertices a, b_1, b_2 does not have a corresponding path of weight ℓ in (G, γ), and is improper.

then there does not exist a corresponding A-U-path of weight ℓ in (G, γ) with the same endpoints and same sequence of vertices in $V(B)$ as P. See Figure 5.2. In this case, let us call the A-U-path P in B_ℓ *improper*. Otherwise, there clearly exists a corresponding A-U-path of weight ℓ in (G, γ) and we call P *proper*. If $a \in A, b_1 \in U$, and $ab_1 \in E(B_\ell)$, then we also call the path ab_1 proper and it (by the definition of B_ℓ) also has a corresponding A-U-path of weight ℓ in (G, γ). In all cases, we call b_1 the *first attachment* and b_2, if it exists, the *second attachment* of P.

Note that the definition of proper and improper A-U-paths in B_ℓ depend only on their first and second attachments. Moreover, if two A-U-paths in B_ℓ have the same endpoint in A and same first attachment but distinct second attachments, then at least one of the two paths is proper.

Given a linkage $\mathcal{P}$ of proper A-U-paths in B_ℓ, let $A(\mathcal{P})$ and $U(\mathcal{P})$ denote the sets of vertices in A and U respectively that are endpoints of a path in $\mathcal{P}$, and let $F_\mathcal{P}$ denote the set of vertices in $V(B)$ that are first attachments of A-U-paths in $B_\ell - A(\mathcal{P})$. In other words, $F_\mathcal{P}$ is the set of vertices in $V(B)$ that are adjacent to $A - A(\mathcal{P})$ in B_ℓ.

Let $\mathcal{P}$ be a (cardinality-wise) maximum linkage of proper A-U-paths in B_ℓ. If $|\mathcal{P}| \geq 2t$, then the corresponding $2t$ A-U-paths of weight ℓ in (G, γ) are disjoint except possibly in their initial segments. But since the initial segment of such a path can intersect at most one other, there is a subset of $\mathcal{P}$ of t disjoint A-U-paths of weight ℓ in (G, γ), as desired. So we may assume that $|\mathcal{P}| < 2t$.

By Corollary 3.2.1, either there are $4t$ disjoint U-$F_{\mathcal{P}}$-U-paths in $B_\ell - A$ or there exists $Y \subseteq V(B_\ell) - A$ with $|Y| < 8t$ such that $B_\ell - A - Y$ does not contain a U-$F_{\mathcal{P}}$-U-path.

Case 1: There exist $4t$ disjoint U-$F_{\mathcal{P}}$-U-paths in $B_\ell - A$.

Let us choose linkages $\mathcal{P}$ and $\mathcal{Q}$ such that

(i) $\mathcal{P}$ is a maximum linkage of proper A-U-paths in B_ℓ,

(ii) $\mathcal{Q}$ is a linkage of $4t$ U-$F_{\mathcal{P}}$-U-paths in $B_\ell - A$, and

(iii) subject to (i) and (ii), the number of edges in $(\cup\mathcal{P}) \cup (\cup\mathcal{Q})$ is minimum.

First suppose there exists $Q \in \mathcal{Q}$ that is disjoint from $\cup\mathcal{P}$. Let $b \in V(Q) \cap F_{\mathcal{P}}$ and let $a \in A - A(\mathcal{P})$ be adjacent to b in B_ℓ. Then $Q + a + ab$ contains a proper A-U-path by the definition of improper paths, and moreover this path is disjoint from $\cup\mathcal{P}$ since Q is disjoint from $\cup\mathcal{P}$. This contradicts the maximality of $\mathcal{P}$ and we may thus assume that every path in $\mathcal{Q}$ intersects a path in $\mathcal{P}$.

Since $|\mathcal{P}| < 2t$ and $|\mathcal{Q}| = 4t$, we can choose a subset $\mathcal{Q}' = \{Q'_1, \ldots, Q'_{2t}\}$ of $\mathcal{Q}$ with $|\mathcal{Q}'| = 2t$ such that no path in $\mathcal{Q}'$ contains a vertex in $U(\mathcal{P})$. For each $i \in [2t]$, let u_i be an endpoint of Q'_i and let z_i be the closest vertex to u_i on Q'_i that is contained in a path in $\mathcal{P}$.

Since $|\mathcal{P}| < 2t$ and $|\mathcal{Q}'| = 2t$, there exist distinct $i, j \in [2t]$ such that $z_i, z_j \in V(P)$ for some $P \in \mathcal{P}$. Let a and u denote the two endpoints of P in A and U respectively, and assume without loss of generality that a, z_i, z_j, u occur in this order on P. Then $P' := aPz_jQ'_ju_j$ is a proper A-U-path since P is proper and P and P' have the same first and second attachments. See Figure 5.3. Moreover, P' is disjoint from each path in $\mathcal{P} - P$ by our choice of z_j. But since Q'_j does not contain a vertex in $U(\mathcal{P})$, P' uses strictly fewer edges that are not in $\cup\mathcal{Q}$ than P.

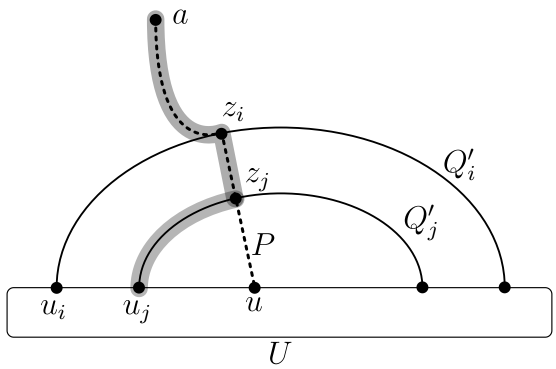


Figure 5.3: Lemma 5.3.1, Case 1. The highlighted path is a proper A-U-path using fewer edges not in $\cup\mathcal{Q}$, contradicting the choice of $\mathcal{P}$ and $\mathcal{Q}$.

Since $\mathcal{P}$ is a maximum linkage of proper A-U-paths in B_ℓ, so is $\mathcal{P}' := \mathcal{P} - P + P'$. Since $A(\mathcal{P}) = A(\mathcal{P}')$ (hence $F_{\mathcal{P}} = F_{\mathcal{P}'}$), $\mathcal{P}'$ and $\mathcal{Q}$ also satisfy (i) and (ii). But $(\cup\mathcal{P}') \cup (\cup\mathcal{Q})$ has fewer edges than $(\cup\mathcal{P}) \cup (\cup\mathcal{Q})$, contradicting (iii).

Case 2: There exists $Y \subseteq V(B_\ell) - A$ with $|Y| < 8t$ such that $B_\ell - A - Y$ does not contain a U-$F_{\mathcal{P}}$-U-path.

Let H denote the graph $B_\ell - A - Y$. Since H does not contain a U-$F_{\mathcal{P}}$-U-path, for all $b_1 \in F_{\mathcal{P}} - Y$, there exists a 1-separation (C_{b_1}, D_{b_1}) in H such that $b_1 \in V(C_{b_1} - D_{b_1})$ and $U - Y \subseteq V(D_{b_1})$.

Claim 5.3.1.1. *Let $a \in A - A(\mathcal{P})$, $b_1 \in F_{\mathcal{P}} - Y$, and suppose that there exists an improper A-U-path Q contained in $H + a + ab_1$. Let b_2 denote the second attachment of Q. Then there exists a proper A-U-path in $H + a + ab_1$ if and only if b_1b_2 is not a cut-edge in H separating b_1 from $U - Y$.*

Proof. If b_1b_2 is a cut-edge in H separating b_1 from $U - Y$, then any a-U-path in $H + a + ab_1$ must start with the vertices a, b_1, b_2 in this order, and any such path is improper since it has the same first and second attachments as Q. Conversely, if b_1b_2 is not such a cut-edge, then there exists a b_1-U-path in H avoiding the edge b_1b_2. Combining this path with the edge ab_1 gives a proper A-U-path in $H + a + ab_1$. See

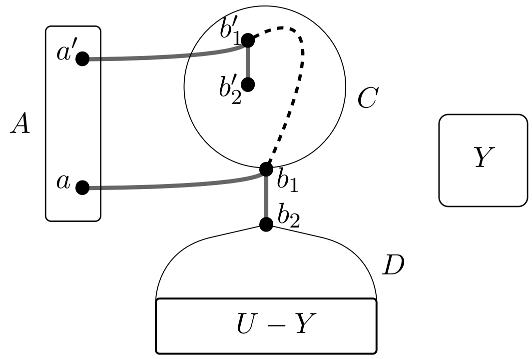


Figure 5.4: Lemma 5.3.1, Case 2. If there is an improper A-U-path using a, b_1, b_2 such that b_1b_2 is a cut-edge, then the edge ab_1 is deleted in H'. Otherwise, as with a', b'_1, b'_2 in the figure, there is a proper A-U-path obtained by rerouting within C, and the edge $a'b'_1$ remains in H'.

Figure 5.4. ■

Let H' be the graph obtained from $B_\ell - A(\mathcal{P}) - Y$ by deleting the edge ab_1 for each $a \in A - A(\mathcal{P})$ and $b_1 \in F_\mathcal{P}$ such that:

> There is an improper A-U-path in $H + a + ab_1$ with second attachment b_2 such that b_1b_2 is a cut-edge in H separating b_1 from $U - Y$.

We now show that the problem reduces to Menger's theorem on H'.

Claim 5.3.1.2. *Let $\mathcal{Q}' = \{Q'_1, \dots, Q'_k\}$ be a linkage of A-U-paths in H'. Then there is a linkage $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ of proper A-U-paths in $B_\ell - A(\mathcal{P}) - Y$ such that Q'_i and Q_i have the same endpoints for all $i \in [k]$.*

Proof. Let $a^1, \dots, a^k$ denote the endpoints of $Q'_1, \dots, Q'_k$ respectively in $A - A(\mathcal{P})$. For each $i \in [k]$, define an A-U-path Q_i in $B_\ell - A(\mathcal{P}) - Y$ as follows. If Q'_i is proper, then set $Q_i = Q'_i$. If Q'_i is improper, let b^i_1 and b^i_2 denote the first and second attachments of Q'_i respectively and let z^i denote the unique vertex in $V(C_{b^i_1} \cap D_{b^i_1})$. Then z^i is a cut-vertex separating b^i_1 from $U - Y$ in H, so $z^i \in V(Q'_i)$. Also, $b^i_1b^i_2$ is not a cut-edge in H separating b^i_1 from $U - Y$ since otherwise the edge $a^ib^i_1$ would have been deleted in H'.

It follows that there exists a b_1^i-z^i-path R_i in $C_{b_1^i}$ avoiding the edge $b_1^i b_2^i$. Define Q_i to be the A-U-path $a^i b_1^i R_i z^i Q_i'$. Then Q_i is proper, contained in $B_\ell - A(\mathcal{P}) - Y$, and Q_i has the same endpoints as Q_i'. Moreover, since Q_i was obtained by modifying Q_i' only inside $C_{b_1^i}$ (which is separated by the cut-vertex z^i from U in H), it follows that $\mathcal{Q} := \{Q_1, \ldots, Q_k\}$ is a linkage of proper A-U-paths in $B_\ell - A(\mathcal{P}) - Y$. ■

If H' contains $2t$ disjoint A-U-paths, then we obtain $2t$ disjoint proper A-U-paths in $B_\ell - A(\mathcal{P}) - Y$ (hence in B_ℓ), contradicting the assumption that $\mathcal{P}$ is a maximum such linkage. Thus, by Menger's theorem, there exists $Z \subseteq V(H')$ with $|Z| < 2t$ such that $H' - Z$ does not contain an A-U-path.

Claim 5.3.1.3. *$(G - A(\mathcal{P}) - Y - Z, \gamma)$ does not contain an A-U-path of weight ℓ.*

Proof. Suppose Q is an A-U-path of weight ℓ in $(G - A(\mathcal{P}) - Y - Z, \gamma)$, and let a denote the endpoint of Q in $A - A(\mathcal{P})$. Then Q corresponds to a proper A-U-path Q' in $B_\ell - A(\mathcal{P}) - Y - Z$ with the same endpoints and same sequence of vertices in $V(B)$. Let b_1 denote the vertex succeeding a in Q'.

Since $H' - Z$ does not contain an A-U-path, the edge ab_1 must have been deleted in H'. By the definition of H', there exists an improper A-U-path in $H + a + ab_1$ with second attachment say b_2 such that $b_1 b_2$ is a cut-edge in H separating b_1 from $U - Y$. But by Claim 5.3.1.1, there does not exist a proper A-U-path in $B_\ell - A(\mathcal{P}) - Y$ starting with the edge ab_1, contradicting the existence of Q'. ■

Thus $X := A(\mathcal{P}) \cup Y \cup Z \subseteq V(G)$ is a hitting set for A-U-paths of weight ℓ in (G, γ) with $|X| \leq |\mathcal{P}| + |Y| + |Z| < 2t + 8t + 2t = 12t$. This completes the proof of the lemma.

□

We next prove a generalization of Lemma 5.1.3 for paths of weight ℓ that nicely link to a wall in a Γ-bipartite 3-block.

Lemma 5.3.2. *Let Γ be an abelian group with $\ell \in \Gamma$, let r, t be positive integers with $r \geq 12t$, and let $T = 3(36t)^2$. Let (G, γ) be a Γ-labelled graph with $A \subseteq V(G)$ and let (B, γ) be a Γ-bipartite 3-block of $(G - A, \gamma)$. Let W be an s-wall in $G - A$ where $s \geq (2r + 1)(2T + 1)$ such that W is 1-contained in a wall W' and $b(W) \subseteq V(B)$ where $b(W)$ is the set of branch vertices of W with respect to W'. Suppose in addition that there does not exist $X \subseteq V(G)$ with $|X| < 12T$ intersecting all A-$b(W)$-paths of weight ℓ in (G, γ). Then W contains a compact r-subwall W_1 such that there are t disjoint A-W_1-paths of weight ℓ that nicely link to W_1.*

Proof. By Lemma 5.3.1, there exist T disjoint A-$b(W)$-paths of weight ℓ in (G, γ). Let $\mathcal{P}$ be a linkage of T such paths minimizing the number of edges in $\cup\mathcal{P} - E(W)$.

Claim 5.3.2.1. *There are at most T $b(W)$-paths Q in W such that Q intersects $\cup\mathcal{P}$ and neither endpoint of Q is in $\cup\mathcal{P}$.*

Proof. Let Q be a $b(W)$-path in W with endpoints $w_1, w_2 \notin V(\cup\mathcal{P})$ and let $P = x_0x_1 \ldots x_m$ be a path in $\mathcal{P}$ with $x_0 \in A$ and $x_m \in b(W)$ such that $Q \cap P \neq \emptyset$. We may choose P and $x_i \in V(Q \cap P)$ such that $w_1Qx_i - x_i$ does not intersect $\cup\mathcal{P}$.

Suppose x_0Px_i intersects $W - Q$. Then $x_i \in V(B)$; otherwise, there is a 2-separation (C, D) of $G - A$ such that $x_i \in V(C - D)$, $b(W) \subseteq V(D)$, and $V(C \cap D) = \{u, v\}$ where $u, v \in V(Q)$ and $x_i \in uQv$. But both x_0Px_i and x_iPx_m intersect $W - Q$ and, therefore, they both contain one of $\{u, v\}$. Since one of u or v is in $w_1Qx_i - x_i$, this contradicts the assumption that $w_1Qx_i - x_i$ does not intersect $\cup\mathcal{P}$. Now $x_i \in V(B)$ implies that $\gamma(w_1Qx_i) = \gamma(x_iPx_m) = 0$ since (B, γ) is Γ-bipartite. Thus $P' := x_0Px_iQw_1$ is an A-$b(W)$-path of weight ℓ disjoint from $\cup\mathcal{P} - P$ with fewer edges not in W, contradicting the minimality of $\mathcal{P}$.

Therefore, we may assume that x_0Px_i does not intersect $W - Q$. In other words, Q is the first $b(W)$-path in W that P intersects. Since $|\mathcal{P}| = T$, there are at most T such paths Q. ■

Let $W(\mathcal{P})$ denote the vertices of $b(W)$ that are endpoints of a path in $\mathcal{P}$. Let $S \subseteq b(W)$ be the vertex set obtained from $W(\mathcal{P})$ by adding, for each $b(W)$-path Q in W whose interior intersects $\cup\mathcal{P}$, one endpoint of Q. We have $|S| \leq 2T$ by Claim 5.3.2.1.

Since W is a wall of size at least $(2r+1)(2T+1)$, there are $2r+1$ consecutive rows and $2r+1$ consecutive columns of W that are all disjoint from S and hence from $\cup\mathcal{P}$. Let W_0 denote the compact $2r$-subwall of W contained in the union of these $2r+1$ rows and columns of W. Let W_1 denote the compact r-subwall of W_0 disjoint from the first r rows and columns of W_0. Let N_1 denote the set of top nails of W_1 and let H denote the graph $G - (V(W_1) - N_1)$.

Claim 5.3.2.2. *There does not exist $Y \subseteq V(H)$ with $|Y| < 12t$ intersecting all A-N_1-paths of weight ℓ in (H, γ).*

Proof. Suppose to the contrary that Y is such a hitting set. Since $r \geq 12t$, there exists a row R^W and a column C^W of W intersecting W_0 and disjoint from W_1 and Y. There also exists a column C_*^W of W containing a vertex in N that is disjoint from Y.

Since $|\mathcal{P}| = T = 3(36t)^2 > 2(36t)^2 + 12t$, there exists $\mathcal{P}' \subseteq \mathcal{P}$ with $|\mathcal{P}'| = 2(36t)^2$ such that each path in $\mathcal{P}'$ is disjoint from Y. Then there exists $\mathcal{P}'' \subseteq \mathcal{P}'$ with $|\mathcal{P}''| = 36t$ such that the vertices of $W(\mathcal{P}'')$ either lie in distinct rows or in distinct columns of W.

Suppose the vertices of $W(\mathcal{P}'')$ lie in distinct rows (resp. columns) of W. Since $|Y| < 12t$, there are $24t$ distinct rows (resp. columns) $Q_1, \ldots, Q_{24t}$ of W, in this order in W and disjoint from Y, such that Q_i contains a vertex w_i in $W(\mathcal{P}'')$. Let P_i denote the path in $\mathcal{P}''$ containing w_i as an endpoint and let a_i denote the endpoint of P_i in A.

Let y_i denote the vertex in $P_i \cap Q_i$ that is closest to C^W (resp. R^W) on the w_i-C^W-subpath (resp. the w_i-R^W-subpath) of Q_i. If $y_i \in V(B)$, then $\gamma(a_i P_i y_i) = \ell$ and we obtain an A-N_1-path of weight ℓ in $(H - Y, \gamma)$ in $a_i P_i y_i \cup Q_i \cup C^W \cup R^W \cup C_*^W$, a contradiction.

So we may assume that $y_i \notin V(B)$ for all $i \in [24t]$. Then there is a 2-separation (C_i, D_i) of $G - A$ with $y_i \in V(C_i - D_i)$, $V(B) \subseteq V(D_i)$, and $V(C_i \cap D_i) = \{x_i, z_i\}$ where $x_i, z_i \in V(Q_i) \cap V(B)$. Assume without loss of generality that w_i, x_i, y_i, z_i occur in

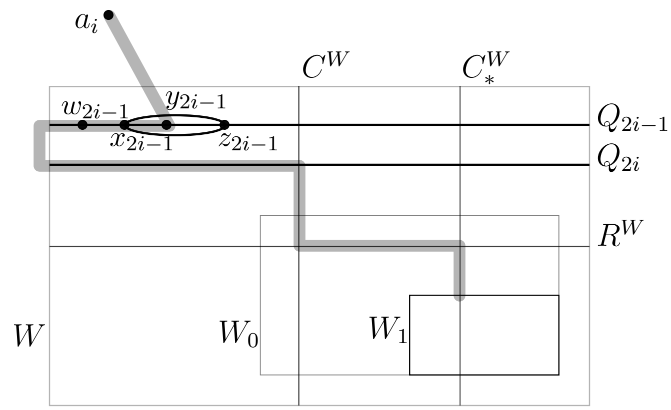


Figure 5.5: The highlighted path is an A-N_1-path of weight ℓ in $(H - Y, \gamma)$ as described in the proof of Claim 5.3.2.2. At least one such path is disjoint from Y.

this order on Q_i (where possibly $w_i = x_i$ and $z_i \in C^W$). Then the A-$V(B)$-subpath of P_i is contained in $G[C_i + a_i]$ and ends at x_i. Let $P_i' = a_i P_i x_i Q_i w_i$.

Recall that W is 1-contained in a wall W'. For $i \in [24t]$, let Q_i' denote the rows (resp. columns) of W' containing Q_i. For $i \in [12t]$, let R_i be the A-C^W-path (resp. A-R^W-path) obtained from P_{2i-1}' by continuing along Q_{2i-1}' to the first or last column (resp. row) of W', using it to reach Q_{2i}', then going back along Q_{2i}' to C^W (resp. R^W). See Figure 5.5.

Then $\gamma(R_i) = \ell$ and $R_1, \ldots, R_{12t}$ are disjoint. Since $|Y| < 12t$, some R_i is disjoint from Y and we thus obtain a contradictory A-N_1-path of weight ℓ in $(H - Y, \gamma)$ in $R_i \cup C^W \cup R^W \cup C_*^W$. This completes the proof of the claim. ∎

Let (H', γ') be the Γ-labelled graph obtained from (H, γ) by adding an edge between each pair of vertices in N_1 with label 0. Let (B', γ') be the 3-block of (H', γ') containing N_1. Then (B', γ') is Γ-bipartite: if C is a simple Γ-nonzero cycle in (B', γ'), then there are three disjoint $V(C)$-N_1-paths which give a Γ-nonzero N_1-path in H by Lemma 2.2.4. But since $N_1 \subseteq V(B)$, this contradicts the assumption that (B, γ) is Γ-bipartite.

Applying Lemma 5.3.1 to (H', γ') and N_1, we obtain t disjoint A-W_1-paths that nicely link to W_1, completing the proof of the lemma. □

For each positive integer k, let $BR(k)$ denote the smallest integer b such that any red-blue coloring of the edges of $K_{b,b}$ contains either a red $K_{k,k}$ or a blue $K_{k,k}$. These are called the *Bipartite Ramsey numbers* and it is known that $BR(k) \leq (1 + o(1))2^{k+1} \log k$ (see [11]). We further define the following parameters:

$$\begin{aligned}
t &= t(k) = 16BR(k) \\
T &= T(k) = 3(36t)^2 \\
r_0 &= r_0(k) = 12t \\
r_i &= r_i(k) = (2r_{i-1} + 1)(2T + 1) + 2t \qquad \text{for } i \geq 1.
\end{aligned}$$

Lemma 5.3.3. *Let Γ be a finite abelian group and let k be a positive integer. Then there exists an integer $\beta(k, |\Gamma|)$ such that the following holds: If (G, γ) is a Γ-labelled graph with $A \subseteq V(G)$ such that*

(I) there does not exist $Y \subseteq V(G)$ with $|Y| < 12T(k)|\Gamma|$ intersecting every Γ-zero A-path.

(II) $(G - A, \gamma)$ contains an $\beta(k, |\Gamma|)$-wall W' inducing a tangle $\mathcal{T} = \mathcal{T}_{W'}$ in $G - A$ such that the $\mathcal{T}$-large 3-block (B, γ) of $(G - A, \gamma)$ is Γ-bipartite, and

(III) if $(C, D) \in \mathcal{T}$, then $(G[A \cup C], \gamma)$ does not contain a Γ-zero A-path and $(G[A \cup (D - C)], \gamma)$ contains a Γ-zero A-path,

then (G, γ) contains k disjoint Γ-zero A-paths.

Proof. Define $s = s(k, |\Gamma|) = r_{|\Gamma|+1}$. We show that $\beta(k, |\Gamma|) = s + 2$ suffices. Suppose $G - A$ contains an $(s + 2)$-wall W' with $\mathcal{T} = \mathcal{T}_{W'}$ satisfying (II) and (III). Since (B, γ) is the $\mathcal{T}$-large 3-block of $(G - A, \gamma)$, every vertex of degree 3 in W' is in $V(B)$. Let W be the s-wall that is 1-contained in W' with the natural choice of corners and nails, so that $b(W) \subseteq V(B)$.

Since (B, γ) is Γ-bipartite, we may assume by Lemma 2.2.2 that, after possibly shifting, every $V(B)$-path in $(G - A, \gamma)$ has weight zero. Let P be a Γ-zero A-path in (G, γ). Then

P intersects $V(B)$ in at least two vertices, since otherwise there would be a 3-separation $(C, D) \in \mathcal{T}$ such that $P \subseteq G[A \cup C]$, violating (III). In particular, P contains two disjoint end segments whose weights are ℓ and $-\ell$ for some $\ell \in \Gamma$.

Claim 5.3.3.1. *Let $\ell \in \Gamma$ and let W^* be a compact r-subwall of W such that $r \geq 12T$. Let $X \subseteq V(G)$ with $|X| < 12T$. If $(G - X, \gamma)$ does not contain an A-$b(W^*)$-path of weight ℓ, then $(G - X, \gamma)$ does not contain a Γ-zero A-path whose end segments have weights $\pm\ell$.*

Proof. Suppose P is a Γ-zero A-path in $(G - X, \gamma)$ whose end segments have weights $\pm\ell$, and let $B(P) = V(P) \cap V(B)$. If there exist two disjoint $B(P)$-$b(W^*)$-paths in $G - A - X$, then the union of these two paths and P contains an A-$b(W^*)$-path of weight ℓ in $(G - X, \gamma)$ and we are done. Otherwise, there exists a 1-separation (C, D) in $G - A - X$ with $B(P) \subseteq V(C)$ and $b(W^*) - X \subseteq V(D)$.

Consider the $12T$-separation $(G[C \cup (X - A)], G[D \cup (X - A)])$ in $G - A$. Since W^* has size $r \geq 12T$ and $b(W^*) \subseteq V(G[D \cup (X - A)])$, we have $(G[C \cup (X - A)], G[D \cup (X - A)]) \in \mathcal{T}_{W^*} \subseteq \mathcal{T}$. But $P \subseteq G[A \cup C \cup X]$, violating (III). ■

Claim 5.3.3.2. *Let W° be a compact r-subwall of W such that $r \geq r_i$ for some $i \geq 1$. Then there exists $\ell \in \Gamma$ such that, for any choice of $\ell^\circ \in \{\ell, -\ell\}$, there is a compact t-contained r_{i-1}-subwall W_1° of W° such that there are t disjoint A-W_1°-paths of weight ℓ° that nicely link to W_1°.*

Proof. Let W_0° be a t-contained $(2r_{i-1} + 1)(2T + 1)$-subwall of W°. The size of W_0° is clearly greater than $12T$. Suppose that for every $\ell \in \Gamma$, there exists $X_\ell \subseteq V(G)$ with $|X_\ell| < 12T$ such that either X_ℓ intersects every A-$b(W_0^\circ)$-path of weight ℓ or X_ℓ intersects every A-$b(W_0^\circ)$-path of weight $-\ell$. Then by Claim 5.3.3.1 (applied to W_0°, ℓ, and X_ℓ for every $\ell \in \Gamma$), $Y := \cup_{\ell \in \Gamma} X_\ell$ intersects every Γ-zero A-path and $|Y| < 12T|\Gamma|$, violating (I). So there exists $\ell \in \Gamma$ for which such a set X_ℓ does not exist.

Let $\ell^\circ \in \{\ell, -\ell\}$. We have shown above that there does not exist $X \subseteq V(G)$ with $|X| < 12T$ intersecting every A-$b(W_0^\circ)$-path of weight ℓ°. By Lemma 5.3.2 (applied to

W_0°, ℓ°, r, and t), we obtain a compact r_{i-1}-subwall W_1° of W_0° such that there are t disjoint A-W_1°-paths of weight ℓ° that nicely link to W_1°. ■

We apply Claim 5.3.3.2 repeatedly starting with W to obtain a sequence of elements $\ell_1, \dots, \ell_{|\Gamma|+1}$, subwalls $W \supseteq W_1 \supseteq \cdots \supseteq W_{|\Gamma|+1}$, and linkages $\mathcal{P}_1, \dots, \mathcal{P}_{|\Gamma|+1}$ such that $\mathcal{P}_i$ is a set of t disjoint A-W_i-paths of weight ℓ_i that nicely links to W_i. We have $\ell_i = \ell_j$ for some $i < j$ and, since we are free to choose either ℓ_j or $-\ell_j$ at the j-th iteration of Claim 5.3.3.2, we may assume that $\ell_j = -\ell_i$ for some $i < j$. We can then extend the linkages $\mathcal{P}_i$ and $\mathcal{P}_j$ through W_i and W_j respectively so that they link nicely to $W_{|\Gamma|+1}$ (since $W_{|\Gamma|+1}$ is t-contained in each of the previous walls). Note that $W_{|\Gamma|+1}$ has size $r_0 = 12t$.

Renaming, we have thus obtained a $12t$-wall W_* and two linkages $\mathcal{P}$ and $\mathcal{Q}$ of A-W_*-paths of weight ℓ and $-\ell$ respectively that nicely link to W_*, with $|\mathcal{P}| = |\mathcal{Q}| = t$. For an A-W_*-linkage $\mathcal{R}$, let $A(\mathcal{R})$ and $W_*(\mathcal{R})$ denote the set of endpoints of $\mathcal{R}$ in A and W_* respectively.

Recall that $t = 16BR(k)$ where $BR(k)$ is the bipartite Ramsey number. The $16BR(k)$ paths of $\mathcal{P}$ (resp. $\mathcal{Q}$) contain a set $\mathcal{P}^1 \subseteq \mathcal{P}$ (resp. $\mathcal{Q}^1 \subseteq \mathcal{Q}$) with $|\mathcal{P}^1| = |\mathcal{Q}^1| = 8BR(k)$ such that no B-bridge of $(G - A, \gamma)$ contains the initial segments of two paths in $\mathcal{P}^1$ (resp. $\mathcal{Q}^1$).

Now take an arbitrary subset $\mathcal{P}^2 \subseteq \mathcal{P}^1$ with $|\mathcal{P}^2| = 4BR(k)$. Then the interior of the initial segment of each path in $\mathcal{P}^2$ intersects at most one path in $\mathcal{Q}^1$, so there is a subset $\mathcal{Q}^2 \subseteq \mathcal{Q}^1$ with $|\mathcal{Q}^2| = 4BR(k)$ such that no path in $\mathcal{Q}^2$ intersects the interiors of initial segments of paths in $\mathcal{P}^2$. Similarly, take a subset $\mathcal{Q}^3 \subseteq \mathcal{Q}^2$ with $|\mathcal{Q}^3| = 2BR(k)$ and choose a subset $\mathcal{P}^3 \subseteq \mathcal{P}^2$ with $|\mathcal{P}^3| = 2BR(k)$ such that no path in $\mathcal{P}^3$ intersects the interiors of initial segments of paths in $\mathcal{Q}^3$.

We may then choose $\mathcal{P}^4 \subseteq \mathcal{P}^3$ and $\mathcal{Q}^4 \subseteq \mathcal{Q}^3$ with $|\mathcal{P}^4| = |\mathcal{Q}^4| = BR(k)$ such that each vertex in A belongs to at most one path in $\mathcal{P}^4 \cup \mathcal{Q}^4$. Note that every $A(\mathcal{P}^4)$-$A(\mathcal{Q}^4)$-path in $(\cup\mathcal{P}^4) \cup (\cup\mathcal{Q}^4) \cup W_*$ is the union of an initial segment of a path in $\mathcal{P}^4$ (which has weight ℓ), an initial segment of a path in $\mathcal{Q}^4$ (which has weight $-\ell$), and a $V(B)$-path (which has weight 0). Hence every $A(\mathcal{P}^4)$-$A(\mathcal{Q}^4)$-path in $(\cup\mathcal{P}^4) \cup (\cup\mathcal{Q}^4) \cup W_*$ is a Γ-zero A-path.

If there exist linkages $\mathcal{P}^5 \subseteq \mathcal{P}^4$ and $\mathcal{Q}^5 \subseteq \mathcal{Q}^4$ with $|\mathcal{P}^5| = |\mathcal{Q}^5| = k$ such that the paths in $\mathcal{P}^5 \cup \mathcal{Q}^5$ are disjoint, then we obtain k disjoint Γ-zero A-paths by linking $W_*(\mathcal{P}^5)$ to $W_*(\mathcal{Q}^5)$ through W_*. Otherwise, by the definition of $BR(k)$, there exist linkages $\mathcal{P}^5 \subseteq \mathcal{P}^4$ and $\mathcal{Q}^5 \subseteq \mathcal{Q}^4$ with $|\mathcal{P}^5| = |\mathcal{Q}^5| = k$ such that every path in $\mathcal{P}^5$ intersects every path of $\mathcal{Q}^5$. Let $H = (\cup\mathcal{P}^5) \cup (\cup\mathcal{Q}^5)$.

Since $H \subseteq (\cup\mathcal{P}^4) \cup (\cup\mathcal{Q}^4) \cup W_*$, every $A(\mathcal{P}^5)$-$A(\mathcal{Q}^5)$-path in H is an $A(\mathcal{P}^4)$-$A(\mathcal{Q}^4)$-path in $(\cup\mathcal{P}^4) \cup (\cup\mathcal{Q}^4) \cup W_*$, hence a Γ-zero A-path. If there does not exist k disjoint $A(\mathcal{P}^5)$-$A(\mathcal{Q}^5)$-paths in H, then by Menger's theorem there exists $Z \subseteq V(H)$ with $|Z| < k$ separating $A(\mathcal{P}^5)$ from $A(\mathcal{Q}^5)$ in H. But this is a contradiction since such a set Z is disjoint from at least one path in $\mathcal{P}^5$ and at least one path in $\mathcal{Q}^5$, and since these two paths intersect, their union contains an $A(\mathcal{P}^5)$-$A(\mathcal{Q}^5)$-path, hence a Γ-zero A-path. $\square$

5.4 Proof of Theorem 1.3.3

We are now ready to prove Theorem 1.3.3.

Theorem 5.4.1 (Theorem 1.3.3 restated)**.** *Let p be an odd prime and let $\Gamma = \mathbb{Z}/p\mathbb{Z}$. Then Γ-zero A-paths satisfy the Erdős-Pósa property.*

Proof. For each positive integer k define $r_* = r_*(k) = 18kp$ and $t_* = t_*(k) = 5kp$. Let $f_{2.6.1}, g, h, \beta$ be the functions given by Theorem 2.6.1, Theorem 3.1.1, and Lemma 5.3.3. Define $\varphi(k) = g(r_*, t_*) + h(r_*, t_*) + \beta(k, p)$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that $f(k) \geq 2f(k-1) + 3f_{2.6.1}(\varphi(k)) + 10$ and $f(k) \geq h(r_*, t_*) + 12T(k)p + 108k^2$, where $T(k)$ is the function appearing in condition (I) of Lemma 5.3.3.

Let $((G, \gamma), k)$ with $A \subseteq V(G)$ be a minimal counterexample to f being an Erdős-Pósa function for Γ-zero A-paths. That is, (G, γ) does not contain k disjoint Γ-zero A-paths, there does not exist $X \subseteq V(G)$ with $|X| \leq f(k)$ intersecting every Γ-zero A-path, and subject to these two conditions, k is minimum.

By Lemma 5.1.1, $G - A$ admits a tangle $\mathcal{T}$ of order $f_{2.6.1}(\varphi(k))$ such that for each $(C, D) \in \mathcal{T}$, $G[A \cup C]$ does not contain a Γ-zero A-path and $G[A \cup (D - C)]$ contains a Γ-zero A-path. By Theorem 2.6.1, $G - A$ contains a $\varphi(k)$-wall W such that $\mathcal{T}_W$ is a truncation of $\mathcal{T}$. We apply Theorem 3.1.1 to (W, γ), r_*, and t_* and obtain one of its outcomes.

In outcomes (1) and (2), note that conditions (a) and (b) of section 5.2 are satisfied. In outcome (1), we have a Γ-odd K_{5kp}-model μ in $(G - A, \gamma)$ such that $\mathcal{T}_\mu$ is a truncation of $\mathcal{T}_W$. Lemma 5.2.3 implies that (G, γ) contains k disjoint Γ-zero A-paths. In outcome (2), we have a $50r_*^{12}$-wall (W_0, γ) in $(G - A, \gamma)$ such that $\mathcal{T}_{W_0}$ is a truncation of $\mathcal{T}_W$. Note that $50r_*^{12} \geq 2664k^3p$. In outcome (2)-(a), (W_0, γ) is facially Γ-odd and by Lemma 5.2.5, (G, γ) contains k disjoint Γ-zero A-paths. In outcome (2)-(b), (W_0, γ) is a Γ-bipartite wall with a pure Γ-odd linkage of (W_0, γ) of size $r_* = 18kp$, and by Lemma 5.2.7, (G, γ) again contains k disjoint Γ-zero A-paths. In all cases, we obtain k disjoint Γ-zero A-paths, contradicting the assumption that $((G, \gamma), k)$ is a minimal counterexample.

Therefore outcome (3) holds and there exists $Z \subseteq V(G - A)$ with $|Z| \leq h(r_*, t_*)$ such that the $\mathcal{T}$-large 3-block of $(G - A - Z, \gamma)$ is Γ-bipartite. Since W has size $\varphi(k) = g(r_*, t_*) + h(r_*, t_*) + \beta(k, p)$, there is a $\beta(k, p)$-subwall W_1 of W in $(G - A - Z, \gamma)$.

Then $(G - Z, \gamma)$ and W_1 satisfy the three hypotheses of Lemma 5.3.3. Indeed, since $\mathcal{T}_{W_1}$ is a truncation of $\mathcal{T}$, the $\mathcal{T}$-large 3-block of $(G - A - Z, \gamma)$ is also $\mathcal{T}_{W_1}$-large, so hypothesis (II) holds. Similarly, every separation in $\mathcal{T}_{W_1}$ is also in $\mathcal{T}$, so (III) holds as well. Furthermore, since $f(k) \geq h(r_*, t_*) + 12T(k)p$ and (G, γ) does not contain a hitting set of size less than $f(k)$, $(G - Z, \gamma)$ does not contain a hitting set Y with $|Y| < 12T(k)p$, satisfying hypothesis (I). By Lemma 5.3.3, $(G - Z, \gamma)$ contains k disjoint Γ-zero A-paths, a contradiction. $\square$

CHAPTER 6
OBSTRUCTIONS TO THE ERDŐS-PÓSA PROPERTY OF ALLOWABLE CYCLES

6.1 Preliminaries

First we collect several tools from Gollin et al. [20].

6.1.1 Packing functions

Let G be a graph and let ν be a function from the set of subgraphs of G to the set of non-negative integers. For subgraphs $H, H' \subseteq G$, we say

- ν is *monotone* if $\nu(H) \leq \nu(H')$ whenever H is a subgraph of H',
- ν is *additive* if $\nu(H \cup H') = \nu(H) + \nu(H')$ whenever H and H' are disjoint, and
- ν is a *packing function for* G if it is monotone and additive.

Now let ν be a packing function for a graph G. For a subgraph $H \subseteq G$, we say a set $T \subseteq V(H)$ is a ν*-hitting set for* H if $\nu(H - T) = 0$. We define $\tau_\nu(H)$ as the size of a smallest ν-hitting set of H. Note that in the traditional sense of the word, a ν-hitting set of G is a hitting set for the minimal subgraphs $H \subseteq G$ for which $\nu(H) \geq 1$.

Lemma 6.1.1 (Gollin et al. [20, Lemma 4.1])**.** *Let ν be a packing function for a graph G and let $T \subseteq V(G)$ be a minimum ν-hitting set for G of size t. Let $\mathcal{T}_T$ be the set of all separations (A, B) of G of order less than $t/6$ such that $|B \cap T| > 5t/6$. If $\tau_\nu(H) \leq t/12$ whenever H is a subgraph of G with $\nu(H) < \nu(G)$, then $\mathcal{T}_T$ is a tangle of order $\lceil t/6 \rceil$.*

6.1.2 Cleaning the wall

We will need the following notation. Let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups and let (G,γ) be a Γ-labelled graph. Given a subset $Z \subseteq [m]$ and an integer ℓ, we say that a wall W in G is (γ, Z, ℓ)-*clean* if

(1) every N^W-path in W is γ_j-zero for all $j \in Z$ and

(2) W has no ℓ-subwall that is γ_j-bipartite for all $j \in [m] \setminus Z$.

We write $\mathbb{N}_{\geq 3}$ to denote the set of integers greater than or equal to 3.

Lemma 6.1.2 (Gollin et al. [20, Lemma 5.1]). *Let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups, let (G,γ) be a Γ-labelled graph, let $\psi\colon \{0\} \cup [m+1] \to \mathbb{N}_{\geq 3}$ be a function, and let W be a wall of order $\psi(0)+2$ in G. Then there exist a Γ-labelling γ' of G shifting-equivalent to γ, a subset Z of $[m]$, and a $(\gamma', Z, \psi(|Z|+1)+2)$-clean $\psi(|Z|)$-wall that is dominated by $\mathcal{T}_W$.*

6.1.3 Collecting handles

Lemma 6.1.3 (Gollin et al. [20, Lemma 4.3]). *Let u, k be positive integers such that $f_{2.2.5}(k) < u-2$. Let Γ be an abelian group, let (G,γ) be a Γ-labelled graph, and let ν be a packing function for G such that*

- *every minimal subgraph H of G with $\nu(H) \geq 1$ is a γ-nonzero cycle,*
- *$\tau_\nu(H) \leq 3u$ for every subgraph H of G with $\nu(H) < \nu(G)$, and*
- *$\tau_\nu(G) \geq u$.*

Let $T \subseteq V(G)$ be a minimum ν-hitting set for G and let $N \subseteq V(G)$ such that for every $S \subseteq V(G)$ of size less than u, there is a component of $G - S$ containing a vertex of N and at least $4u$ vertices of T. Then G contains k disjoint γ-nonzero N-paths.

A *corridor* of a graph G is a $V_{\neq 2}(G)$-path of length at least 1.

Lemma 6.1.4 (Gollin et al. [20, Lemma 6.1])**.** *There exist functions $w_{6.1.4}\colon \mathbb{N}^2 \to \mathbb{N}$ and $f_{6.1.4}\colon \mathbb{N} \to \mathbb{N}$ satisfying the following. Let k, t, and c be positive integers with $c \geq 3$, let Γ be an abelian group, and let (G, γ) be a Γ-labelled graph. Let W be a wall in G of order at least $w_{6.1.4}(k, c)$ such that all corridors of W are γ-zero. For each $i \in [t-1]$, let $\mathcal{P}_i$ be a set of $4k$ W-handles in G such that the paths in $\bigcup_{i\in[t-1]} \mathcal{P}_i$ are disjoint. If G contains at least $f_{6.1.4}(k)$ disjoint γ-nonzero $V_{\neq 2}(W)$-paths, then there exist a c-column-slice W' of W and a set $\mathcal{Q}_i$ of k disjoint W'-handles for each $i \in [t]$ such that*

(i) for each $i \in [t-1]$, the set $\mathcal{Q}_i$ is a subset of the row-extension of $\mathcal{P}_i$ to W' in W,

(ii) the paths in $\bigcup_{i\in[t]} \mathcal{Q}_i$ are disjoint, and

(iii) the paths in $\mathcal{Q}_t$ are γ-nonzero.

6.1.4 Finding allowable cycles

A clean wall can help us to build cycles whose values will be allowable, as the following lemma demonstrates.

Lemma 6.1.5 (Gollin et al. [20, Lemma 8.1])**.** *There exist functions $c_{6.1.5}, r_{6.1.5}\colon \mathbb{N}^4 \to \mathbb{N}$ satisfying the following. Let t, ℓ, m, and ω be positive integers with $\ell \geq 3$, let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups, for each $j \in [m]$ let Ω_j be a subset of Γ_j of size at most ω, and let (G, γ) be a Γ-labelled graph. Let Z be a subset of $[m]$ and let W be a (γ, Z, ℓ)-clean $r \times c$-wall with $c \geq c_{6.1.5}(t, \ell, m, \omega)$ and $r \geq r_{6.1.5}(t, \ell, m, \omega)$. Then for every set $\mathcal{P}$ of at most t disjoint W-handles such that $\gamma_j\left(\bigcup \mathcal{P}\right) \notin \Omega_j$ for all $j \in Z$, there is a cycle O in $W \cup \bigcup \mathcal{P}$ such that $\gamma_j(O) \notin \Omega_j$ for all $j \in [m]$.*

6.2 Handling handlebars

Recall that two sets $\mathcal{G}_1$ and $\mathcal{G}_2$ of graphs are said to be disjoint if $\bigcup \mathcal{G}_1$ and $\bigcup \mathcal{G}_2$ are disjoint. First we show that given a family of pairwise disjoint sets of W-handles, we can throw away some W-handles from each set to obtain a family of pairwise disjoint non-mixing W-handlebars.

Lemma 6.2.1. *There is a function $f_{6.2.1}\colon \mathbb{N}^2 \to \mathbb{N}$ satisfying the following property. Let t, θ, c, and r be positive integers with $r \geq 3$ and $c \geq 3$, let W be a $r \times c$-wall, and let $(\mathcal{P}_i \colon i \in [t])$ be a family of pairwise disjoint sets of $f_{6.2.1}(t, \theta)$ W-handles. If the W-handles in $\bigcup_{i=1}^{t} \mathcal{P}_i$ are disjoint, then there exists a family $(\mathcal{P}_i^* : i \in [t])$ of pairwise non-mixing W-handlebars such that $\mathcal{P}_i^* \subseteq \mathcal{P}_i$ and $|\mathcal{P}_i^*| \geq \theta$ for all $i \in [t]$.*

Proof. Let

$$f_{6.2.1}(t, \theta) := \begin{cases} \max\{3((2t-1)\theta - 1)^3 + 1, 30 f_{6.2.1}(t-1, \theta)\} & \text{if } t > 1, \\ 3(\theta - 1)^3 + 1 & \text{if } t = 1. \end{cases}$$

We proceed by induction on t. If $t = 1$, then there is a subset $\mathcal{P}' \subseteq \mathcal{P}_1$ of size $(\theta - 1)^3 + 1$ whose paths all have the same number of endvertices in C_1^W. The result then follows from Lemma 2.7.1.

Suppose $t \geq 2$. By the above argument, there is a W-handlebar $\mathcal{P}' = \{P_j' \colon j \in [(2t-1)\theta]\} \subseteq \mathcal{P}_t$ of size $(2t-1)\theta$. For each $j \in [(2t-1)\theta]$, let v_j and w_j be the endvertices of P_j' with $v_j \prec_W w_j$. Without loss of generality, we may assume that for all $j, j' \in [(2t-1)\theta]$ with $j < j'$, we have $v_j \prec_W v_{j'}$. For each $x \in [2t-1]$, let A_x be the subpath of $C_1^W \cup C_c^W$ from $v_{1+(x-1)\theta}$ to $v_{x\theta}$, and let B_x be the subpath of $C_1^W \cup C_c^W$ from $w_{1+(x-1)\theta}$ to $w_{x\theta}$. Note that for distinct x and y in $[2t-1]$, we have that $A_x \cup B_x$ and $A_y \cup B_y$ are disjoint. Hence, for each $i \in [t-1]$, there are at most two integers $x \in [2t-1]$ such that $A_x \cup B_x$ contains more than a third of the endvertices of paths in $\mathcal{P}_i$. Hence, there exists $x \in [2t-1]$ such that $A_x \cup B_x$ contains at most a third of the endvertices of paths in $\mathcal{P}_i$ for all $i \in [t-1]$. For every $i \in [t-1]$, let $\mathcal{P}_i' \subseteq \mathcal{P}_i$ of size at least $10 f_{6.2.1}(t-1, \theta)$ such that $A_x \cup B_x$ and $\bigcup \mathcal{P}_i'$ are disjoint.

Since the graph $H := C_1^W \cup C_c^W - V(A_x \cup B_x)$ has at most four components and $10 =$

$\binom{4}{1} + \binom{4}{2}$, for each $i \in [t-1]$, there is a subset $\mathcal{P}_i'' \subseteq \mathcal{P}_i'$ of size $f_{6.2.1}(t-1, \theta)$ such that for every pair of W-handles $P, P' \in \mathcal{P}_i''$, each component of H contains the same number of endvertices of P and P'. By the induction hypothesis, there is a family $(\mathcal{P}_i^* : i \in [t-1])$ of pairwise non-mixing W-handlebars such that $\mathcal{P}_i^* \subseteq \mathcal{P}_i''$ and $|\mathcal{P}_i^*| \geq \theta$ for each $i \in [t-1]$. Together with $\mathcal{P}_t^* := \{P_j' \colon j \in [x\theta] \setminus [(x-1)\theta]\}$, these W-handlebars satisfy the lemma. □

The paths of a W-handlebar $\mathcal{P}$ can be pieced together through the outer columns of W to form a W^*-handlebar $\mathcal{P}^*$ for some column-slice W^* of W such that each path in $\mathcal{P}^*$ contains exactly d paths of $\mathcal{P}$ for any desired d, provided that $\mathcal{P}$ and W are large enough. The following lemma shows that this can be done simultaneously to a family of pairwise disjoint non-mixing W-handlebars so that the resulting family of W^*-handlebars is also pairwise disjoint and non-mixing.

Lemma 6.2.2. *Let t, c, r, and θ be positive integers with $c \geq 5$ and $r \geq 3$, and let d_i be a positive integer for each $i \in [t]$. Let W be a $r \times c$-wall in a graph G and let W^* be a $(c-2)$-column-slice of W containing C_2^W and C_{c-1}^W. Let $(\mathcal{P}_i \colon i \in [t])$ be a family of pairwise disjoint non-mixing W-handlebars with $|\mathcal{P}_i| \geq \theta d_i$ for all $i \in [t]$. Then there is a family $(\mathcal{P}_i^* \colon i \in [t])$ of pairwise disjoint non-mixing W^*-handlebars each of size θ such that for each $i \in [t]$ and $Q \in \mathcal{P}_i^*$, there is a set $\{P_{j,Q} \in \mathcal{P}_i \colon j \in [d_i]\}$ of size d_i such that*

$$\bigcup_{j=1}^{d_i} P_{j,Q} \subseteq Q \subseteq W \cup \bigcup_{j=1}^{d_i} P_{j,Q}.$$

Moreover, for each $i \in [t]$, if d_i is even, then $\mathcal{P}_i^*$ is in series and if d_i is odd, then $\mathcal{P}_i^*$ is of the same type as $\mathcal{P}_i$.

Proof. For each $i \in [t]$, let $\mathcal{P}_i =\colon \{P_{i,x} \colon x \in [\theta d_i]\}$ such that if $x, y \in [\theta d_i]$ with $x < y$, then some endvertex of $P_{i,x}$ is $\prec_W$-smaller than both endvertices of $P_{i,y}$. For each $i \in [t]$ and $y \in [\theta]$, it is easy to verify that there is a unique path in $C_1^W \cup C_c^W \cup \bigcup_{x=(y-1)d_i+1}^{yd_i} P_{i,x}$

that contains $\bigcup_{x=(y-1)d_i+1}^{yd_i} P_{i,x}$ whose set of endvertices contains the $\prec_W$-smallest endvertex of $P_{i,(y-1)d_i+1}$ and some endvertex of P_{i,yd_i}. Let $Q_{i,y}$ denote the row-extension of this path to W^*. Now with $\mathcal{P}_i^* := \{Q_{i,y} \colon y \in [\theta]\}$, we easily observe that $(\mathcal{P}_i^* \colon i \in [t])$ is as desired. $\square$

Next we show that if $(\mathcal{P}_i : i \in [t])$ is a family of disjoint non-mixing W-handlebars none of which are in series, then we can construct a W'-handlebar for some subwall W' of W such that each W'-handle contains exactly one path from each $\mathcal{P}_i$.

Lemma 6.2.3. *Let t, k, c, and r be positive integers with $k \geq 2$ and $c, r \geq 3$. Let W be a $r' \times c'$-wall in a graph G with $c' \geq c_{6.2.3}(t,k,c) := c + kt$ and $r' \geq r_{6.2.3}(k,r) := r + k$. Let $(\mathcal{P}_i \colon i \in [t])$ be a family of pairwise disjoint non-mixing W-handlebars in G, each of size k, such that no $\mathcal{P}_i$ is in series for all $i \in [t]$. Then there exist a subwall W' of W having at least c columns and at least r rows and a W'-handlebar $\mathcal{Q}$ in G of size k such that for each $Q \in \mathcal{Q}$, there is a set $\{P_{i,Q} \in \mathcal{P}_i \colon i \in [t]\}$ such that*

$$\bigcup_{i=1}^{t} P_{i,Q} \subseteq Q \subseteq W \cup \bigcup_{i=1}^{t} P_{i,Q}.$$

Moreover, $\mathcal{Q}$ is crossing if and only if the number of crossing W-handlebars in $(\mathcal{P}_i \colon i \in [t])$ is odd.

Proof. We proceed by induction on t. This lemma is trivial if $t = 1$ and therefore we may assume that $t > 1$. First, suppose that for some distinct $j', j'' \in [t]$, there is a path Q in $C_1^W \cup C_{c'}^W$ that contains exactly one endvertex of each path in $\mathcal{P}_{j'} \cup \mathcal{P}_{j''}$ and no endvertex of any path in $\bigcup\{\mathcal{P}_x \colon x \in [t] \setminus \{j', j''\}\}$. Without loss of generality, we may assume that $j' = t-1$, $j'' = t$, and $Q \subseteq C_1^W$. Let $(a_j \colon j \in [2k])$ be a strictly increasing sequence of integers in $[r']$ such that $R_{a_j}^W \cap Q$ contains an endvertex of a path in $\mathcal{P}_{t-1} \cup \mathcal{P}_t$ for all $j \in [2k]$. For $j \in [k]$, let Q_j be a subpath of C_{k+1-j}^W from a vertex in $R_{a_j}^W$ to a vertex in $R_{a_{2k+1-j}}^W$.

Then it is easy to observe that for each $j \in [k]$, there is a unique path in

$$\bigcup \mathcal{P}_{t-1} \cup R^W_{a_j} \cup Q_j \cup R^W_{a_{2k+1-j}} \cup \bigcup \mathcal{P}_t$$

that contains exactly one path in $\mathcal{P}_{t-1}$ and exactly one path in $\mathcal{P}_t$. Let W^* be a $(c'-k)$-column-slice of W containing C^W_{k+1} and $C^W_{c'}$. Then the row-extensions of all of these paths to W^* yield a W^*-handlebar $\mathcal{P}'_{t-1}$ that is disjoint and non-mixing with the row-extension of $\mathcal{P}_i$ to W^* for each $i \in [t-2]$. Note that $\mathcal{P}'_{t-1}$ is crossing if and only if exactly one of $\mathcal{P}_{t-1}$ and $\mathcal{P}_t$ is crossing. By applying the induction hypothesis to $\mathcal{P}'_{t-1}$ and row-extensions of $\mathcal{P}_i$ to W^* for all $i \in [t-2]$, we deduce the lemma in this case.

Now suppose that there is no path Q as defined above for any pair of W-handlebars in $(\mathcal{P}_i \colon i \in [t])$. Since no W-handlebar in $(\mathcal{P}_i \colon i \in [t])$ is in series, it follows that each of C^W_1 and $C^W_{c'}$ meets at most one W-handlebar in $(\mathcal{P}_i : i \in [t])$ and so $t = 2$. Let W'' be a $(c'-2k)$-column-slice of W containing C^W_{k+1} and $C^W_{c'-k}$ and let W' be a $(r'-k)$-row-slice of W'' containing $R^{W''}_{k+1}$ and $R^{W''}_{r'}$. Without loss of generality, we may assume that the endvertices of $\mathcal{P}_1$ are contained in C^W_1 and the endvertices of $\mathcal{P}_2$ are contained in $C^W_{c'}$. Let $(a_j \colon j \in [k])$ be a strictly increasing sequence of integers in $[r']$ such that $R^W_{a_j}$ contains the endvertex of a path in $\mathcal{P}_1$ that is $\prec_W$-smaller than its other endvertex for all $j \in [k]$, and let $(b_j \colon j \in [k])$ be a strictly increasing sequence of integers in $[r']$ such that $R^W_{b_j}$ contains the endvertex of a path in $\mathcal{P}_2$ that is $\prec_W$-larger than its other endvertex for all $j \in [k]$. Let W^0 be the k-column-slice of W containing C^W_1 and let W^1 be the k-column-slice of W containing $C^W_{c'}$. For $j \in [k]$, let P_j be a subpath of C^W_j from a vertex in $R^W_{a_j}$ to a vertex in R^W_j and let P'_j be a subpath of $C^W_{c'+1-j}$ from a vertex in R^W_j to a vertex in $R^W_{b_j}$. Again, it is easy to observe that for each $j \in [k]$, there is a unique path in

$$\bigcup \mathcal{P}_1 \cup R^{W^0}_{a_j} \cup P_j \cup R^W_j \cup P'_j \cup R^{W^1}_{b_j} \cup \bigcup \mathcal{P}_2$$

that contains exactly one path in $\mathcal{P}_1$ and exactly one path in $\mathcal{P}_2$. Now the row-extensions of all of these paths to W' yield a W'-handlebar $\mathcal{Q}$ as desired. As before, note that $\mathcal{Q}$ is crossing if and only if exactly one of $\mathcal{P}_1$ and $\mathcal{P}_2$ is crossing. This completes the proof. $\square$

The final lemma of this section shows that if $(\mathcal{P}_i : i \in [q])$ is a family of pairwise disjoint non-mixing W-handlebars that does not satisfy any of the three properties of Definition 3.3.1(6), then it is possible to find many disjoint cycles each containing exactly one path from each $\mathcal{P}_i$ (with the help of Lemma 6.1.5).

Lemma 6.2.4. *Let k, c, and r be positive integers with $k \geq 2$ and $c, r \geq 3$. Let $q \in \{0,1,2\}$. Let W be a $r' \times c'$-wall in a graph G with $c' \geq c_{6.2.4}(k,c) := c + 6k$ and $r' \geq r_{6.2.4}(k,r) := k(r+2)$. Let $(\mathcal{P}_i \colon i \in [q])$ be a family of pairwise disjoint non-mixing W-handlebars in G, each of size k, such that one of the following holds.*

(i) $q = 0$.

(ii) $q = 1$ *and $\mathcal{P}_1$ is either nested or in series.*

(iii) $q = 2$ *and $\mathcal{P}_1$ and $\mathcal{P}_2$ are both in series.*

Then for each $x \in [k]$, there exist an $r \times c$-subwall W_x, a set $\mathcal{H}_x = \{H_{x,i} \colon i \in [q]\}$ of q disjoint W_x-handles, and a set $\{P_{x,i} \in \mathcal{P}_i \colon i \in [q]\}$ such that

1. *for distinct $x, x' \in [k]$ the graphs $W_x \cup \bigcup \mathcal{H}_x$ and $W_{x'} \cup \bigcup \mathcal{H}_{x'}$ are disjoint and*
2. *$P_{x,i} \subseteq H_{x,i} \subseteq W \cup P_{x,i}$ for each $x \in [k]$ and each $i \in [q]$.*

Proof. Without loss of generality, we may assume that if $q > 0$, then the paths in $\mathcal{P}_1$ have at least one endvertex in C_1^W and if $q = 2$, then each endvertex of each path in $\mathcal{P}_1$ is $\prec_W$-smaller than each endvertex of each path in $\mathcal{P}_2$ (since $\mathcal{P}_1$ and $\mathcal{P}_2$ are both in series). If $q \geq 1$, then let $\{P_{1,x} \colon x \in [k]\}$ be an enumeration of $\mathcal{P}_1$ such that for all $x \in [k-1]$, the $\prec_W$-smallest endvertex of $P_{1,x}$ is $\prec_W$-smaller than both endvertices of $P_{1,x+1}$, and additionally if $q = 2$, then let $\{P_{2,x} \colon x \in [k]\}$ be an enumeration of $\mathcal{P}_2$ such that for all $x \in [k-1]$, the $\prec_W$-smallest endvertex of $P_{2,x}$ is $\prec_W$-larger than both endvertices of $P_{2,x+1}$.

Let W^0 be a $4k$-column-slice of W containing C_1^W and let W^1 be a $(c'-c-4k)$-column-slice of W containing $C_{c'}^W$. Let W^* be a c-column-slice of W disjoint from $W^0 \cup W^1$. Let $\{W_x : x \in [k]\}$ be a set of k disjoint $r \times c$-subwalls of W^* such that W_x intersects both $R_{(x-1)r+1}^W$ and R_{xr}^W for each $x \in [k]$. For each $x \in [2k]$ and $z \in \{0,1\}$,

- let v_x^z be the unique nail in the column-boundary of W^z that is contained in both $R_{\lceil xr/2 \rceil}^W$ and $C_{z(c+1)+4k}^W$ and
- let w_x^z be the unique nail in the column-boundary of W^z that is contained in both $R_{(r'+1-x)}^W$ and $C_{z(c+1)+4k}^W$.

Note that for each $x \in [k]$, the nails v_{2x-1}^0, v_{2x}^0, v_{2x-1}^1, and v_{2x}^1 are each contained in a row of W that intersects W_x. For each $x \in [2k]$, let T_x be the unique path in $R_{r'+1-x}^W \cup C_{c'+1-x}^W \cup R_{\lceil xr/2 \rceil}^W$ from w_x^0 to v_x^1. Note that $\mathcal{T} := \{T_x : x \in [2k]\}$ is a set of $2k$ disjoint paths that are internally disjoint from $W^0 \cup \bigcup\{W_j : j \in [k]\}$.

If $q = 0$, then W_x with $\mathcal{H}_x = \emptyset$ for each $x \in [k]$ satisfies the condition and therefore we may assume $q > 0$.

Suppose that $q = 1$ and $\mathcal{P}_1$ is in series. As $k \geq 2$, each path in $\mathcal{P}_1$ has both of its endvertices in W^0. Since W^0 has at least $2k$ columns, there is a set $\mathcal{Q}$ of $2k$ disjoint paths from the endvertices of the paths in $\mathcal{P}_1$ to the set $\{v_x^0 : x \in [2k]\}$ in W^0. By the planarity of W, we conclude that for $x \in [k]$, the endvertices of $P_{1,x}$ are linked by two paths Q_x^* and Q_x^{**} in $\mathcal{Q}$ to $\{v_{2x-1}^0, v_{2x}^0\}$. Moreover, for each $x \in [k]$, the path $Q_x^* \cup Q_x^{**} \cup P_{1,x}$ can be easily extended to a W_x-handle $H_{x,1}$ such that all desired properties are satisfied.

Now suppose that $q = 1$ and $\mathcal{P}_1$ is nested. If each path in $\mathcal{P}_1$ has one endvertex in W^0 and one endvertex in W^1, then there is a set $\mathcal{Q}$ of $2k$ disjoint paths containing for each $z \in \{0,1\}$ a subset of k paths from the endvertices in W^z of the paths in $\mathcal{P}_1$ to $\{v_{2x}^z : x \in [k]\}$ in W^z. If each path in $\mathcal{P}_1$ has both of its endvertices in W^0, then there are $2k$ disjoint paths from the endvertices of the paths in $\mathcal{P}_1$ to $\{v_{2x}^0 : x \in [k]\} \cup \{w_{2x}^0 : x \in [k]\}$ in W^0, which together with the paths in $\{T_{2x} : x \in [k]\}$ yield a set $\mathcal{Q}$ of $2k$ disjoint paths from the endvertices of $\mathcal{P}_1$ to $\{v^z_{2x} \colon x \in [k], z \in \{0,1\}\}$. Hence, in both of these cases, the set $\mathcal{Q}$ avoids $\bigcup\{W_j \colon j \in [k]\}$. By the planarity of W, we conclude that for $x \in [k]$, the endvertices of the path $P_{1,x} \in \mathcal{P}_1$ are linked by two paths Q^*_x and Q^{**}_x in $\mathcal{Q}$ to $\{v^0_{2x}, v^1_{2x}\}$. As before, for each $x \in [k]$, the path $Q^*_x \cup Q^{**}_x \cup P_{1,x}$ can be easily extended to a W_x-handle $H_{x,1}$ such that all desired properties are satisfied.

Therefore, we may assume that $q = 2$. Recall that the paths in $\mathcal{P}_1$ have both their endvertices in W^0. If each path in $\mathcal{P}_2$ has both of its endvertices in W^1, then since each of W^0 and W^1 has at least $2k$ columns, there exist a set $\mathcal{Q}_1$ of $2k$ disjoint paths from the endvertices of the paths in $\mathcal{P}_1$ to $\{v^0_x \colon x \in [2k]\}$ in W^0 and a set $\mathcal{Q}_2$ of $2k$ disjoint paths from the endvertices of the paths in $\mathcal{P}_2$ to $\{v^1_x \colon x \in [2k]\}$ in W^1. If each path in $\mathcal{P}_2$ has both of its endvertices in W^0 as well, then since W^0 has $4k$ columns, there are $4k$ disjoint paths from the set of endvertices of $\mathcal{P}_1 \cup \mathcal{P}_2$ to the set $\{v^0_x \colon x \in [2k]\} \cup \{w^0_x \colon x \in [2k]\}$ in W^0. In this case, let $\mathcal{Q}_1$ be the subset of these paths with endvertices in $\{v^0_x \colon x \in [2k]\}$ and let $\mathcal{Q}_2$ be the concatenation of the subset of these paths with endvertices in $\{w^0_x \colon x \in [2k]\}$ together with the paths in $\{T_x \colon x \in [2k]\}$. Hence, in both of these cases, by the planarity of W, for each $i \in [2]$, the endvertices of $P_{i,x}$ are linked by two paths $Q^*_{i,x}$ and $Q^{**}_{i,x}$ in $\mathcal{Q}_i$ to $\{v^{i-1}_{2x-1}, v^{i-1}_{2x}\}$ and these paths avoid $\bigcup\{W_j \colon j \in [k]\}$. As before, for each $i \in [2]$ and $x \in [k]$, the path $Q^*_{i,x} \cup Q^{**}_{i,x} \cup P_{i,x}$ can be easily extended to a W_x-handle $H_{x,i}$ such that all desired properties are satisfied. $\square$

6.3 Lemmas for products of abelian groups

In this section we present some additional lemmas from [20] and prove useful extensions on finding allowable values. The first lemma says that if a set of elements of Γ generates an allowable value, then it does so using each element a bounded number of times.

Lemma 6.3.1 (Gollin et al. [20, Corollary 7.2]). *Let m, t, and ω be positive integers, let $\Gamma = \prod_{j \in [m]} \Gamma_j$ be a product of m abelian groups and for all $j \in [m]$, let Ω_j be a subset of Γ_j of size at most ω. For all $i \in [t]$ and $j \in [m]$, let $g_{i,j}$ be an element of Γ_j. If there* *exist integers $c_1, \ldots, c_t$ such that $\sum_{i=1}^{t} c_i g_{i,j} \notin \Omega_j$ for all $j \in [m]$, then there exist integers $d_1, \ldots, d_t$ with $d_i \in [2^{m\omega}]$ for each $i \in [t]$ such that $\sum_{i=1}^{t} d_i g_{i,j} \notin \Omega_j$ for all $j \in [m]$.*

The next lemma allows us to find large sets of elements of Γ such that for each $j \in [m]$, their γ_j-values are either all equal or all distinct.

Lemma 6.3.2 (Gollin et al. [20, Lemma 7.6])**.** *There exists a function $f_{6.3.2} \colon \mathbb{N}^2 \to \mathbb{N}$ satisfying the following. Let m, t, and N be positive integers with $N \geq f_{6.3.2}(t, m)$ and let $\Gamma = \prod_{j \in [m]} \Gamma_j$ be a product of m abelian groups. Then for every sequence $(g_i \colon i \in [N])$ over Γ, there exists a subset I of $[N]$ with $|I| = t$ such that for each $j \in [m]$, either*

- *$\pi_j(g_i) = \pi_j(g_{i'})$ for all $i, i' \in I$ or*
- *$\pi_j(g_i) \neq \pi_j(g_{i'})$ for all distinct $i, i' \in I$.*

Furthermore, if Z is a subset of $[m]$ such that for all distinct i and i' in $[N]$ there exists $j \in Z$ such that $\pi_x(g_i) \neq \pi_x(g_{i'})$, then the second condition holds for some $j \in Z$.

For the coordinates j for which the γ_j-values are all distinct, we have the following extension of Lemma 6.3.2.

Lemma 6.3.3. *Let t, m, and n be positive integers, let $\Gamma = \prod_{j \in [m]} \Gamma_j$ be a product of m abelian groups, and let $(g_i \colon i \in [n])$ be a family of elements of Γ such that*

$$\pi_j(g_i) \neq \pi_j(g_{i'})$$

for all $j \in [m]$ and distinct i and i' in $[n]$. If $n \geq f_{6.3.3}(t, m) := m3^{t-1} + t$, then there is a subset $I \subseteq [n]$ of size t such that

$$\pi_j\big(\sum_{i \in S} g_i\big) \neq \pi_j\big(\sum_{i \in T} g_i\big)$$

for every $j \in [m]$ and any pair of distinct subsets S and T of I.

Proof. Let I be a maximal subset of $[n]$ such that $\pi_j(\sum_{i\in S} g_i) \neq \pi_j(\sum_{i\in T} g_i)$ for every $j \in [m]$ and every pair of distinct subsets S and T of I. Suppose that $|I| < t$. By the maximality of I, for each $a \in [n] \setminus I$, there are disjoint subsets S' and T' of I such that $\pi_j(g_a) = \pi_j(\sum_{i\in S'} g_i - \sum_{i\in T'} g_i)$ for some $j \in [m]$. Note that there are $3^{|I|}$ ways to choose the disjoint subsets S' and T' of I. Since $\pi_j(g_a) \neq \pi_j(g_{a'})$ for every $j \in [m]$ and every pair of distinct elements a and a' in $[n] \setminus I$, we have that $n - (t-1) \leq n - |I| \leq m3^{|I|} \leq m3^{t-1}$, contradicting the assumption on n. $\square$

We will apply Lemma 6.3.2 multiple times to obtain a family $(S_i : i \in [t])$ of large subsets of Γ each satisfying the conclusion of Lemma 6.3.2. The following lemma says that there is a choice of an element from each S_i so that the sum of the chosen elements is allowable in each coordinate j for which at least one S_i has all distinct γ_j-values.

Lemma 6.3.4 (Gollin et al. [20, Lemma 7.4])**.** *Let m, t, and ω be positive integers, let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups, and for all $j \in [m]$, let Ω_j be a subset of Γ_j of size at most ω. Let $(S_i : i \in [t])$ be a family of subsets of Γ such that for each $j \in [m]$, there exists $i \in [t]$ such that $\pi_j(g) \neq \pi_j(g')$ for all distinct g, g' in S_i. If $|S_i| > m\omega$ for all $i \in [t]$, then for every $h \in \Gamma$, there is a sequence $(g_i \colon i \in [t])$ of elements of Γ such that*

(i) $g_i \in S_i$ *for each* $i \in [t]$ *and*

(ii) $\pi_j\left(h + \sum_{i\in[t]} g_i\right) \notin \Omega_j$ *for all* $j \in [m]$.

The final lemma is an extension of Lemma 6.3.4 that given a family $(S_i : i \in [t])$ of large subsets of Γ satisfying the conclusion of Lemma 6.3.2, there are large subsets S_i' of S_i so that for *every* choice of an element from each S_i', the sum of the chosen elements is allowable in each coordinate j for which at least one S_i has all distinct γ_j-values.

Lemma 6.3.5. *Let m, ω, κ, t, and s be positive integers with $s \geq f_{6.3.5}(m, \omega, \kappa, t) := \kappa + m\omega\kappa^{t-1}$, let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups, and for each $j \in [m]$, let Ω_j be a subset of Γ_j of size at most ω. Let $(g_{i,x} : i \in [t], x \in [s])$ be a family of elements of Γ such that for each $j \in [m]$, we have*

(a) $|\{\pi_j(g_{i,x})\colon x \in [s]\}| \in \{1, s\}$ *for each* $i \in [t]$ *and*

(b) $\pi_j(\sum_{i\in[t]} g_{i,1}) \notin \Omega_j$.

Then there are subsets $I_i \subseteq [s]$ *for* $i \in [t]$*, each of size at least* κ*, such that*

$$\pi_j(\sum_{i\in[t]} g_{i,a_i}) \notin \Omega_j$$

for every $j \in [m]$ *and every* $(a_i \in I_i\colon i \in [t])$.

Proof. Let $(I_i \subseteq [s]\colon i \in [t])$ be a family satisfying

(1) $1 \in I_i$ for all $i \in [t]$,

(2) $|I_i| \leq \kappa$ for all $i \in [t]$,

(3) for every $j \in [m]$ and every $(a_i \in I_i\colon i \in [t])$, we have $\pi_j(\sum_{i\in[t]} g_{i,a_i}) \notin \Omega_j$, and

(4) subject to the previous conditions, $\sum_{i\in[t]}|I_i|$ is maximized.

By (b), such a family $(I_i : i \in [t])$ exists.

Suppose for contradiction that $|I_x| < \kappa$ for some $x \in [t]$. Without loss of generality, we assume that $x = t$. By properties (3) and (4), for each $y \in [s] \setminus I_t$, there exist $j \in [m]$ and $(a_i \in I_i\colon i \in [t-1])$ such that $\pi_j(g_{t,y} + \sum_{i\in[t-1]} g_{i,a_i}) \in \Omega_j$. Since $s \geq \kappa + m\omega\kappa^{t-1}$, we have

$$\frac{|[s] \setminus I_t|}{m\prod_{i\in[t-1]}|I_i|} \geq \frac{|[s] \setminus I_t|}{m\kappa^{t-1}} > \omega \geq \max_{j\in[m]}|\Omega_j|,$$

so by the pigeonhole principle, there exist $j \in [m]$, $(a_i \in I_i : i \in [t-1])$, and distinct $y, y' \in [s] \setminus I_t$ such that $\pi_j(g_{t,y} + \sum_{i\in[t-1]} g_{i,a_i}) = \pi_j(g_{t,y'} + \sum_{i\in[t-1]} g_{i,a_i}) \in \Omega_j$. This implies that $\pi_j(g_{t,y}) = \pi_j(g_{t,y'})$ and by (a), we have $|\{\pi_j(g_{t,x})\colon x \in [s]\}| = 1$. It follows that $\pi_j(g_{t,y}) = \pi_j(g_{t,1})$ and so $\pi_j(g_{t,1} + \sum_{i\in[t-1]} g_{i,a_i}) \in \Omega_j$, contradicting properties (1) and (3). $\square$

6.4 Proof of Theorem 3.3.2

Theorem 3.3.2. *For every two positive integers m and ω, there is a function $f_{m,\omega}\colon \mathbb{N}^3 \to \mathbb{Z}$ satisfying the following property. Let $\Gamma = \prod_{j\in[m]} \Gamma_j$ be a product of m abelian groups, and for every $j \in [m]$, let Ω_j be a subset of Γ_j with $|\Omega_j| \leq \omega$. For each $j \in [m]$, let $A_j := \pi_j^{-1}(\Gamma_j \setminus \Omega_j) \subseteq \Gamma$ and $A := \bigcap_{j\in[m]} A_j$. Let G be a graph with a Γ-labelling γ and let $\mathcal{O}$ be the set of all cycles of G whose γ-value is in A. Then for every three positive integers k, κ, and θ, there exists a Γ-labelling γ' of G that is shifting equivalent to γ such that at least one of the following statements is true.*

(i) There are k disjoint cycles in $\mathcal{O}$.

(ii) There is a hitting set for $\mathcal{O}$ of size at most $f_{m,\omega}(k,\kappa,\theta)$.

(iii) There is a subgraph H of G such that for some $J \subseteq [m]$ and for the (Γ/Γ_J)-labelling γ'' induced by the restriction of γ' to H, we have $(H,\gamma'') \in \mathcal{C}(\kappa,\theta,\Gamma/\Gamma_J, A{+}\Gamma_J)$, and H contains a half-integral packing of κ cycles in $\mathcal{O}$.

Proof. For fixed positive integers m, ω, κ, and θ, we will define $f_{m,\omega}(k,\kappa,\theta)$ by recursion on k. First, we set $f_{m,\omega}(1,\kappa,\theta) := 0$. Assume that $k > 1$ and $f_{m,\omega}(k-1,\kappa,\theta)$ is already defined. We define $k^\star := \max\{k,\kappa\}$.

For integers p and z_0 with $p > 0$ and $0 \leq z_0 \leq m$, let $\alpha(p,z_0)$ and $\rho(z_0)$ be recursively defined as follows. For every positive integer p, we define

$$\rho(0) := m,$$
$$\alpha(p,0) := f_{6.2.1}\Big(\rho(0), f_{6.3.3}\big(2^{m\omega+1} f_{6.3.5}(m,\omega,k^\star,\rho(0)), m\big)\Big),$$

and for $z_0 > 0$, we recursively define

$$\rho(z_0) := m + f_{6.3.2}(\alpha(1, z_0 - 1), m),$$

$$\alpha(p, z_0) := \begin{cases} \alpha(1, z_0 - 1) & \text{if } p \geq \rho(z_0), \\ \max\Big\{ 4f_{6.3.2}(\alpha(p+1, z_0), m), \\ \qquad f_{6.2.1}\Big(p, f_{6.3.3}\big(2^{m\omega+1} f_{6.3.5}(m, \omega, k^\star, p), m\big)\Big)\Big\} & \text{otherwise.} \end{cases}$$

Let $\hat{p} := \rho(m)$. Note that $\alpha(x, z_0) \geq \alpha(\rho(z_0), z_0) = \alpha(1, z_0 - 1) \geq \alpha(x, z_0 - 1)$ for $x > 0$ and $z_0 > 0$. Thus, α is increasing in the second argument. We may also assume that $f_{6.3.2}$ is increasing in its first argument. These two properties imply that $\rho(z_0) \leq \hat{p}$ for all $z_0 \leq m$. Let

$$u := \max\{\lceil f_{m,\omega}(k-1, \kappa, \theta)/3 \rceil, f_{2.2.5}(f_{6.1.4}(f_{6.3.2}(\alpha(1, m), m))) + 3\}.$$

We recursively define $\beta(p, z_0, z)$ for integers p, z_0, and z with $0 \leq z_0 \leq z \leq m$ and $0 \leq p \leq \hat{p}$, as well as $\psi(z)$ for an integer z with $0 \leq z \leq m+1$ and $c_x(z)$, $r_x(z)$ for $x \in \{0, 1, 2\}$ and a non-negative integer $z \leq m$ as follows. We define

$$\psi(m+1) := 3,$$

and for $z \leq m$ we define

$$\begin{aligned}
c_0(z) &:= c_{6.1.5}(2, \psi(z+1)+2, m, \omega),\\
r_0(z) &:= r_{6.1.5}(2, \psi(z+1)+2, m, \omega),\\
c_1(z) &:= c_{6.2.4}(k, c_0(z)),\\
r_1(z) &:= r_{6.2.4}(k, r_0(z)),\\
c_2(z) &:= \max\big\{\theta, c_{6.2.3}(\hat{p}, k, c_1(z)),\ \max(\kappa, k)\cdot c_{6.1.5}(\hat{p}, \psi(z+1)+2, m, \omega)\big\},\\
r_2(z) &:= \max\big\{\theta, r_{6.2.3}(k, r_1(z)),\ r_{6.1.5}(\hat{p}, \psi(z+1)+2, m, \omega)\big\},\\
\beta(p, z_0, z) &:= \begin{cases} \max\big\{u,\ c_2(z)+2\big\} & \text{if } z_0 = 0,\\ \beta(1, z_0-1, z) & \text{if } z_0 > 0 \text{ and } p = \hat{p},\\ w_{6.1.4}(f_{6.3.2}(\alpha(p+1, z_0), m),\ \beta(p+1, z_0, z)) & \text{if } z_0 > 0 \text{ and } p < \hat{p}; \end{cases}\\
\psi(z) &:= \max\big\{\psi(z+1),\ \beta(0, z, z),\ r_2(z)\big\}.
\end{aligned}$$

Observe that $\beta(p, z_0, z) \geq u$. Lastly, we define

$$f_{m,\omega}(k, \kappa, \theta) := \max\big\{6 f_{2.6.1}(\psi(0)+2),\ 6u,\ 12 f_{m,\omega}(k-1, \kappa, \theta)\big\}.$$

We proceed by induction on k. The case $k = 1$ is clear. Suppose that $k > 1$. For every subgraph H of G, let $\nu(H)$ denote the maximum number of disjoint cycles O in H with $\gamma(O) \in A$. Observe that ν is a packing function for G.

Suppose for contradiction that $\nu(G) < k$, $\tau_\nu(G) > f_{m,\omega}(k, \kappa, \theta)$, and there is no Γ-labelling γ' of G that is shifting equivalent to γ such that the statement (iii) holds. Let T be a minimum ν-hitting set of size $t := \tau_\nu(G)$. By assumption, $t > f_{m,\omega}(k, \kappa, \theta) > f_{m,\omega}(k-1, \kappa, \theta)$. By the induction hypothesis, G contains $k-1$ disjoint cycles in $\mathcal{O}$ and therefore $\nu(G) = k-1$.

For each subgraph H of G, if $\nu(H) < \nu(G)$, then by the induction hypothesis,

$$\tau_\nu(H) \le f_{m,\omega}(k-1, \kappa, \theta) \le f_{m,\omega}(k, \kappa, \theta)/12 < t/12.$$

Let $\mathcal{T}_T$ be the set of all separations (A, B) of G of order less than $t/6$ with $|B \cap T| > 5t/6$. By Lemma 6.1.1, $\mathcal{T}_T$ is a tangle of order $\lceil t/6 \rceil > f_{2.6.1}(\psi(0)+2)$. By Theorem 2.6.1, G has a wall of order $\psi(0)+2$ dominated by $\mathcal{T}_T$. By Lemma 6.1.2, this wall has a $\psi(|Z|)$-subwall W that is $(\gamma', Z, \psi(|Z|+1)+2)$-clean for some subset $Z \subseteq [m]$ and some Γ-labelling γ' of G shifting-equivalent to γ and dominated by $\mathcal{T}_T$. Since $\gamma(O) = \gamma'(O)$ for every cycle O in G, we may assume without loss of generality that $\gamma = \gamma'$.

Claim 6.4.0.1. *There exist an integer $c \ge \beta(1, 0, |Z|)$, a set $I \subseteq [\hat{p}]$, a c-column-slice W' of W, a family $(\mathcal{P}_i \colon i \in I)$ of pairwise disjoint non-mixing W'-handlebars, a family $(Z_i \colon i \in I)$ of subsets of Z, and a family $(g_i \colon i \in I)$ of elements of Γ such that*

(a) *if $I \neq \emptyset$, then $|\mathcal{P}_i| \ge 2^{m\omega+1} f_{6.3.5}(m, \omega, k^\star, |I|)$ for each $i \in I$,*

(b) *$|\pi_j(\gamma(\mathcal{P}_i))| = |\mathcal{P}_i|$ for all $i \in I$ and $j \in Z_i$,*

(c) *$\pi_j(\gamma(\mathcal{P}_i)) = \{\pi_j(g_i)\}$ for all $i \in I$ and $j \in Z \setminus Z_i$,*

(d) *there is some $g \in \langle g_i \colon i \in I \rangle$ such that $\pi_j(g) \notin \Omega_j$ for all $j \in Z \setminus \bigcup_{i \in I} Z_i$,*

(e) *for every $i \in I$ and every $g \in \langle g_{i'} \colon i' \in I \setminus \{i\} \rangle$, there is some $j \in Z \setminus \bigcup_{i' \in I \setminus \{i\}} Z_y$ such that $\pi_j(g) \in \Omega_j$, and*

(f) *for each $i \in I$ and $j \in Z_i$, and every pair of distinct subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{P}_i$, we have*

$$\pi_j\left(\sum_{P \in \mathcal{S}} \gamma(P)\right) \neq \pi_j\left(\sum_{P \in \mathcal{T}} \gamma(P)\right).$$

Proof. For non-negative integers c, q, and p with $q \le p$, we say that a triple $(W', \mathfrak{P}, \mathcal{Z})$ consisting of a wall W', a family $\mathfrak{P} := (\mathcal{P}_i \colon i \in [p])$ of disjoint sets of W'-handles, and a family $\mathcal{Z} := (Z_i \colon i \in \{0\} \cup [p])$ of subsets of Z is a (c, q, p)-***McGuffin*** if W' is a c-column-slice of W such that

(1) if $p \neq 0$, then $|\mathcal{P}_i| \geq \alpha(p, |Z_0|)$ for all $i \in [p]$,

(2) $|\pi_j(\gamma(\mathcal{P}_i))| = |\mathcal{P}_i|$ for all $i \in [p]$ and $j \in Z_i$,

(3) $|\pi_j(\gamma(\mathcal{P}_i))| = 1$ for all $i \in [p]$ and $j \in Z \setminus Z_i$,

(4) $Z_0 = Z \setminus \bigcup_{i \in [q]} Z_i$,

(5) $Z_i \setminus \bigcup_{i' \in [i-1]} Z_{i'} \neq \emptyset$ for all $i \in [q]$, and

(6) for all distinct $i, i' \in [p] \setminus [q]$ there is $j \in Z_0$ such that $\pi_j(\gamma(\mathcal{P}_i)) \cap \pi_j(\gamma(\mathcal{P}_{i'})) = \emptyset$.

Note that $(W, \emptyset, (Z))$ is a $(\psi(|Z|), 0, 0)$-McGuffin and by the definition, $\psi(|Z|) \geq \beta(0, |Z|, |Z|)$. Furthermore, if $(W', \mathfrak{P}, \mathcal{Z})$ is a (c, q, p)-McGuffin, then $q \leq |Z|$ by (5) and $|Z_0| \leq m$, which implies that $\rho(|Z_0|) \leq \hat{p}$. Let (q, p) be a lexicographically maximal pair of non-negative integers with $q \leq p \leq \hat{p}$ for which there is a (c, q, p)-McGuffin $(W', \mathfrak{P}, \mathcal{Z})$ for some $c \geq \beta(p, |Z_0|, |Z|)$.

First, we claim that $p < \rho(|Z_0|)$. Suppose that $p \geq \rho(|Z_0|)$. Then

$$p - q \geq \rho(|Z_0|) - m \geq f_{6.3.2}(\alpha(q+1, |Z_0| - 1), m)$$

since $q \leq m$ by (5) and α is decreasing in its first argument. Let $\mathcal{P}''$ be a set of $p - q$ disjoint W'-handles containing exactly one element of $\mathcal{P}_i$ for each $i \in [p] \setminus [q]$. For $i \in [q]$, let $\mathcal{P}'_i := \mathcal{P}_i$ and $Z'_i := Z_i$. Note that $|\mathcal{P}_i| \geq \alpha(p, |Z_0|) \geq 4 f_{6.3.2}(\alpha(p+1, |Z_0|), m) \geq 4\alpha(p+1, |Z_0|)$ for each $i \in [p]$. Thus, by Lemma 6.3.2, there is a subset $\mathcal{P}'_{q+1}$ of $\mathcal{P}''$ with $|\mathcal{P}'_{q+1}| = \alpha(q+1, |Z_0| - 1)$ such that for each $j \in [m]$, either

- $\pi_j(\gamma(P)) = \pi_j(\gamma(Q))$ for all $P, Q \in \mathcal{P}'_{q+1}$ or
- $\pi_j(\gamma(P)) \neq \pi_j(\gamma(Q))$ for all distinct $P, Q \in \mathcal{P}'_{q+1}$,

and the second condition holds for some $j \in Z_0$ since by (6), for all distinct paths P and Q in $\mathcal{P}''$, there exists $j \in Z_0$ such that $\pi_j(\gamma(P)) \neq \pi_j(\gamma(Q))$. Let

$$Z'_{q+1} := \{j \in Z_0 \colon \pi_j(\gamma(P)) \neq \pi_j(\gamma(Q)) \text{ for all distinct } P, Q \in \mathcal{P}'_{q+1}\} \text{ and } Z'_0 := Z_0 \setminus Z'_{q+1}.$$

Let $\mathfrak{P}' := (\mathcal{P}'_i \colon i \in [q+1])$ and $\mathcal{Z}' := (Z'_i \colon i \in \{0\} \cup [q+1])$. Then $(W', \mathfrak{P}', \mathcal{Z}')$ is a $(c, q+1, q+1)$-McGuffin, since (1) follows from the fact that $|\mathcal{P}'_{q+1}| \geq \alpha(q+1, |Z_0| - 1) \geq \alpha(q+1, |Z'_0|)$ and the remaining conditions are easy to check. This contradicts the maximality of (q, p) because $q + 1 \leq p \leq \hat{p}$. Therefore, $p < \rho(|Z_0|) \leq \hat{p}$.

Now let us show that $(W', \mathfrak{P}, \mathcal{Z})$ satisfies the following statement:

($*$) There is some $g \in \langle \bigcup_{i \in [p]} \gamma(\mathcal{P}_i) \rangle$ such that $\pi_j(g) \notin \Omega_j$ for all $j \in Z_0$.

Suppose to the contrary that such g does not exist. Then Z_0 is nonempty. Let Λ be the subgroup of Γ consisting of all $g \in \Gamma$ for which there is $g' \in \langle \bigcup_{i \in [p]} \gamma(\mathcal{P}_i) \rangle$ such that $\pi_j(g) = \pi_j(g')$ for all $j \in Z_0$. Let λ be the induced Γ/Λ-labelling of G. Note that by the negation of ($*$), neither $\langle \bigcup_{i \in [p]} \gamma(\mathcal{P}_i) \rangle$ nor Λ contains an element g such that $\pi_j(g) \notin \Omega_j$ for all $j \in Z_0$. Therefore,

(†) every cycle O of G for which $\pi_j(\gamma(O)) \notin \Omega_j$ for all $j \in [m]$ is λ-nonzero.

Note that W' is a subwall of W of order $c \geq u$. For any $S \subseteq V(G)$ of size at most $u - 1$, there is a component X of $G - S$ containing a row of W', which contains a vertex in $V_{\neq 2}(W')$ because $u \geq 3$. Since $\mathcal{T}_T$ dominates W', the separation $(V(G) \setminus V(X), S \cup V(X))$ is in $\mathcal{T}_T$ and hence X contains a vertex of $V_{\neq 2}(W')$ and at least

$$5t/6 - (u-1) > 5f_{m,\omega}(k, \kappa, \theta)/6 - (u-1) > 4u$$

vertices of T. By (†), every minimal subgraph H with $\nu(H) \geq 1$ is a λ-nonzero cycle. Moreover, if H is a subgraph of G with $\nu(H) < \nu(G) = k - 1$, then by the induction hypothesis,

$$\tau_\nu(H) \le f_{m,w}(k-1,\kappa,\theta) \le 3u.$$

Hence, by Lemma 6.1.3, G has $f_{6.1.4}(f_{6.3.2}(\alpha(1,m),m))$ disjoint λ-nonzero $V_{\neq 2}(W')$-paths. Note that we may assume that the function $w_{6.1.4}$ is increasing in both of its arguments. As $|Z_0|>0$ and $p<\hat{p}$, we have

$$c \ge \beta(p,|Z_0|,|Z|) \ge w_{6.1.4}(f_{6.3.2}(\alpha(p+1,|Z_0|),m),\beta(p+1,|Z_0|,|Z|)).$$

Recall that $|\mathcal{P}_i| \ge \alpha(p,|Z_0|) \ge 4f_{6.3.2}(\alpha(p+1,|Z_0|),m)$ for each $i\in[p]$. Thus, by Lemma 6.1.4 applied to W', there exists a c'-column-slice W'' of W' for some

$$c' \ge \beta(p+1,|Z_0|,|Z|) \ge \beta(q+1,|Z_0|-1,|Z|)$$

and there exists a set $\mathcal{P}'_i$ of $f_{6.3.2}(\alpha(p+1,|Z_0|),m)$ disjoint W''-handles for each $i\in[p+1]$ such that

- for each $i\in[p]$, the set $\mathcal{P}'_i$ is a subset of the row-extension of $\mathcal{P}_i$ to W'' in W',
- the paths in $\bigcup_{i\in[p+1]}\mathcal{P}'_i$ are disjoint, and
- the paths in $\mathcal{P}'_{p+1}$ are λ-nonzero.

Note that since W is $(\gamma',Z,\psi(|Z|+1)+2)$-clean, every N^W-path in W is $(\pi_j\circ\gamma)$-zero for all $j\in Z$ and therefore if P' is the row-extension of a W'-handle P to W'' in W', then $\pi_j(\gamma(P'))=\pi_j(\gamma(P))$ for all $j\in Z$.

Since $|Z|\le m$, by Lemma 6.3.2, there exist a subset $\mathcal{R}$ of $\mathcal{P}'_{p+1}$ and a subset Z' of Z such that

- $|\pi_j(\gamma(\mathcal{R}))|=|\mathcal{R}|$ for all $j\in Z'$,
- $|\pi_j(\gamma(\mathcal{R}))|=1$ for all $j\in Z\setminus Z'$, and

- $|\mathcal{R}| = \alpha(p+1, |Z_0|) \geq \alpha(q+1, |Z_0|-1)$.

Let $p'' := p+1$ and $q'' := q$ if $Z' \cap Z_0$ is empty and let $p'' := q+1$ and $q'' := q+1$ if $Z' \cap Z_0$ is nonempty, and for $i \in \{0\} \cup [p'']$, let

$$Z''_i := \begin{cases} Z_0 \setminus Z' & \text{if } i = 0, \\ Z_i & \text{if } i \in [p''-1], \\ Z' & \text{if } i = p''. \end{cases}$$

For $i \in [p''-1]$, let $\mathcal{P}''_i := \mathcal{P}'_i$ and let $\mathcal{P}''_{p''} := \mathcal{R}$.

We now show that $\big(W'', (\mathcal{P}''_i \colon i \in [p'']), (Z''_i \colon i \in \{0\} \cup [p''])\big)$ is a (c', q'', p'')-McGuffin; if true, then since $p'' \leq \hat{p}$, it contradicts the maximality of (q, p).

To observe property (1), note that $\alpha(p, |Z_0|) \geq \alpha(p+1, |Z_0|)$, and if $Z' \cap Z_0$ is nonempty, then $\alpha(p+1, |Z_0|) \geq \alpha(q+1, |Z_0 \setminus Z'|)$. If P' is the row-extension of a W'-handle P to W'' in W', then $\pi_j(\gamma(P')) = \pi_j(\gamma(P))$ for all $j \in Z$, implying properties (2) and (3) for $i < p''$. By the definition of Z', properties (2) and (3) hold for $i = p''$. Property (4) holds trivially. Property (5) holds because $\emptyset \neq Z' \cap Z_0$ and $Z_0 \cap \bigcup_{i \in [p''-1]} Z_i = \emptyset$ by (4). It remains to check (6) when $Z' \cap Z_0$ is empty, $q < i \leq p$, and $i' = p'' = p+1$. This is implied by the property that the paths in $\mathcal{P}'_{p+1}$ are λ-nonzero. We conclude that $(W', \mathfrak{P}, \mathcal{Z})$ satisfies $(*)$.

If $p = 0$, then by property $(*)$, property (d) holds with $\mathcal{Z}$ and $I := \emptyset$ and properties (a), (b), (c), (e), and (f) hold vacuously.

Therefore, we may assume that $0 < p < \rho(|Z_0|)$. Let $I' := [p]$. Since

$$|\mathcal{P}_i| \geq \alpha(p, |Z_0|) \geq f_{6.2.1}\Big(p, f_{6.3.3}\big(2^{m\omega+1} f_{6.3.5}(m, \omega, k^\star, p), m\big)\Big)$$

for each $i \in [p]$, by Lemma 6.2.1 and (1), there is a family $(\mathcal{P}^*_i \subseteq \mathcal{P}_i \colon i \in [p])$ of pairwise disjoint non-mixing W'-handlebars, each of size $f_{6.3.3}(2^{m\omega+1} \cdot f_{6.3.5}(m, \omega, k^\star, p), m)$. By applying Lemma 6.3.3 to the restriction of $\gamma(\mathcal{P}_i^*)$ to $\prod_{j\in Z_i}\Gamma_i$ for each $i\in[p]$, we deduce that there is a family of subsets $(\mathcal{P}_i'\subseteq\mathcal{P}_i^* : i\in[p])$, each of size $2^{m\omega+1}f_{6.3.5}(m,\omega,k^\star,p)$, satisfying properties (a) and (f) with the set $I'=[p]$. They also satisfy properties (b) and (c) with an arbitrary family $(g_i\in\gamma(\mathcal{P}_i') : i\in[p])$, by (2) and (3). Observe that properties (a), (b), (c), and (f) hold for any subset I of I' (and the corresponding subfamilies $(\mathcal{P}_i' : i\in I)$, $(Z_i : i\in I)$, and $(g_i : i\in I)$) because we may assume that $f_{6.3.5}$ is increasing in its fourth argument. Now property (d) holds for I' by property $(*)$, so taking a minimal subset I of I' satisfying property (d), we have that property (e) is also satisfied by I. $\square$

Claim 6.4.0.2. *Let W'' be a $(c-2)$-column-slice of W' containing $C_2^{W'}$ and $C_{c-1}^{W'}$. Then there is a family $(\mathcal{P}_i''\colon i\in I)$ of pairwise disjoint non-mixing W''-handlebars, each of size $k^\star$, such that*

(a) for each $j\in Z$ and each $(P_i\colon i\in I)$ with $P_i\in\mathcal{P}_i''$ for all $i\in I$, we have $\sum_{i\in I}\pi_j(\gamma(P_i))\notin\Omega_j$,

(b) for each $i\in I$ and each $g\in\langle\gamma(P)\colon P\in\bigcup_{i'\in I\setminus\{i\}}\mathcal{P}_{i'}''\rangle$ there is $j\in Z$ such that $\pi_j(g)\in\Omega_j$,

(c) for each $y\in I$ such that $\mathcal{P}_y''$ is not in series and every function $f\colon\bigcup_{i\in I}\mathcal{P}_i''\to\mathbb{Z}$ for which $\sum_{P\in\mathcal{P}_y''}f(P)$ is even, there is some $j\in Z$ such that $\sum_{i\in I}\sum_{P\in\mathcal{P}_i''}f(P)\pi_j(\gamma(P))\in\Omega_j$.

Proof. If $I=\emptyset$, then $0\notin\Omega_j$ for all $j\in Z$ by Claim 6.4.0.1(d) and therefore this claim is trivially true. Thus we may assume that $I\neq\emptyset$. Let S be a maximal subset of I such that

$$\big\langle\{2g_i\colon i\in S\}\cup\{g_i\colon i\in I\setminus S\}\big\rangle\cap\bigcap_{j\in Z\setminus\bigcup_{i\in I}Z_i}\pi_j^{-1}(\Gamma_j\setminus\Omega_j)\neq\emptyset.$$

Note that such a set S exists, since Claim 6.4.0.1(d) implies that the empty set satisfies this condition. By Lemma 6.3.1, there exist integers $(d_i : i\in I)$ such that $d_i\in[2^{m\omega+1}]$ is even for each $i\in S$, $d_i\in[2^{m\omega}]$ for each $i\in I\setminus S$, and $\pi_j\left(\sum_{i\in I}d_ig_i\right)\notin\Omega_j$ for all $j\in Z\setminus\bigcup_{i\in I}Z_i$. By the choice of S, d_i is odd for all $i\in I\setminus S$. By Lemma 6.2.2, there is a family $(\mathcal{P}_i^*\colon i\in I)$ of pairwise disjoint non-mixing W''-handlebars each of size $f_{6.3.5}(m,\omega,k^\star,|I|)$ such that for each $i \in I$ and $Q \in \mathcal{P}_i^*$, there is a set $\{P_{\ell,Q} \in \mathcal{P}_i \colon \ell \in [d_i]\}$ of size d_i satisfying the following three properties:

- $\bigcup_{\ell=1}^{d_i} P_{\ell,Q} \subseteq Q \subseteq W \cup \bigcup_{\ell=1}^{d_i} P_{\ell,Q}$.
- $\mathcal{P}_i^*$ is in series for each $i \in S$.
- $\mathcal{P}_i^*$ is of the same type as $\mathcal{P}_i'$ for each $i \in I \setminus S$.

Note that $\pi_j(\gamma(\mathcal{P}_i^*)) = \{d_i \pi_j(g_i)\}$ for all $i \in I$ and $j \in Z \setminus Z_i$ by Claim 6.4.0.1(c) and that $|\pi_j(\gamma(\mathcal{P}_i^*))| = |\mathcal{P}_i^*|$ for all $i \in I$ and $j \in Z_i$ by Claim 6.4.0.1(f).

Since $|\mathcal{P}_i^*| = f_{6.3.5}(m, \omega, k^\star, |I|) > m\omega \geq |\bigcup_{i \in I} Z_i|\, \omega$ for each $i \in I$, by Lemma 6.3.4, there is a family $(g_i' \colon i \in I)$ of elements of Γ such that

1. $g_i' \in \gamma(\mathcal{P}_i^*)$ for each $i \in I$ and
2. $\pi_j\left(\sum_{i \in I} g_i'\right) \notin \Omega_j$ for all $j \in \bigcup_{i \in I} Z_i$.

By Lemma 6.3.5, for each $i \in I$ there is a subset $\mathcal{P}_i''$ of $\mathcal{P}_i^*$ of size $k^\star$ such that $(\mathcal{P}_i'' \colon i \in I)$ satisfies property (a). Now $(\mathcal{P}_i'' \colon i \in I)$ satisfies property (b) by Claim 6.4.0.1(e).

To prove property (c), suppose that $y \in I$, $\mathcal{P}_y''$ is not in series, and $f : \bigcup_{i \in I} \mathcal{P}_i'' \to \mathbb{Z}$ is a function such that $\sum_{P \in \mathcal{P}_y''} f(P)$ is even. Since $\mathcal{P}_y''$ is not in series, we have $y \in I \setminus S$. By Claim 6.4.0.1(c),

$$\pi_j(\gamma(P)) = d_i \pi_j(g_i) \text{ for all } i \in I,\, P \in \mathcal{P}_i'',\, j \in Z \setminus \bigcup_{i' \in I} Z_{i'}.$$

In particular, if d_i is even or $\sum_{P \in \mathcal{P}_i''} f(P)$ is even, then $\sum_{P \in \mathcal{P}_i''} f(P)\pi_j(\gamma(P)) \in \pi_j(\langle 2g_i \rangle)$.

Let $S' = S \cup \{y\}$. Then for all $i \in S'$, either d_i or $\sum_{P \in \mathcal{P}_i''} f(P)$ is even. Let $g = \sum_{i \in I} \sum_{P \in \mathcal{P}_i''} f(P)\gamma(P)$. Then there is $g' \in \Gamma$ such that $\pi_j(g) = \pi_j(g')$ for all $j \in Z \setminus \bigcup_{i' \in I} Z_{i'}$ and $g' \in \langle \{2g_i \colon i \in S'\} \cup \{g_i \colon i \in I \setminus S'\} \rangle$. By the maximality of S, $g' \notin \bigcap_{j \in Z \setminus \bigcup_{i' \in I} Z_{i'}} \pi_j^{-1}(\Gamma_j \setminus \Omega_j)$. Therefore, there is some $j \in Z \setminus \bigcup_{i' \in I} Z_{i'}$ such that $\pi_j(g') = \pi_j(g) \in \Omega_j$. This proves property (c). $\square$

Let H be the union of W'' and $\bigcup\{\bigcup\mathcal{P}_i''\colon i\in I\}$. Note that W'' has at least $c_2(|Z|)$ columns and at least $r_2(|Z|)$ rows and therefore the order of W'' is greater than or equal to θ.

We now find a half-integral packing in a similar manner as in the proof of [20, Theorem 1].

Claim 6.4.0.3. *H contains a half-integral packing of $k^\star$ cycles in $\mathcal{O}$. Moreover, if $I=\emptyset$, then H contains a packing of $k^\star$ cycles in $\mathcal{O}$.*

Proof. Since $|\mathcal{P}_i''|=k^\star$ for each $i\in I$, there exists a family $(\mathcal{Q}_x\subseteq\bigcup_{i\in I}\mathcal{P}_i''\colon x\in[k^\star])$ of disjoint sets such that $|\mathcal{Q}_x\cap\mathcal{P}_i''|=1$ for all $i\in I$ and $x\in[k^\star]$. Note that if $I=\emptyset$, then $\mathcal{Q}_x=\emptyset$ for all $x\in[k^\star]$. By Claim 6.4.0.2(a), for each $x\in[k^\star]$ and $j\in Z$, we have $\sum_{P\in\mathcal{Q}_x}\gamma_j(P)\notin\Omega_j$. We remark that if $I=\emptyset$, then $0\notin\Omega_j$ for all $j\in Z$.

Since W'' has at least $c_2(|Z|)$ columns and $c_2(|Z|)\geq k^\star c_{6.1.5}(\hat{p},\psi(|Z|+1)+2,m,\omega)$, there exists a set $\{W_x\colon x\in[k^\star]\}$ of $k^\star$ disjoint $c_{6.1.5}(\hat{p},\psi(|Z|+1)+2,m,\omega)$-column-slices of W''. Note that W'' has at least $r_{6.1.5}(\hat{p},\psi(|Z|+1)+2,m,\omega)$ rows. For each $x\in[k^\star]$, let $\mathcal{Q}_x^*$ be the row-extension of $\mathcal{Q}_x$ to W_x. Note that if $I=\emptyset$, then $\mathcal{Q}_x^*$ is also empty for each $x\in[k^\star]$. Since $|I|\leq\hat{p}$, by Lemma 6.1.5, for each $x\in[k^\star]$, there is a cycle O_x in $W_x\cup\bigcup\mathcal{Q}_x^*$ such that $\gamma_j(O_x)\notin\Omega_j$ for all $j\in[m]$. Observe that no vertex is in more than two of the subgraphs in $\{W_x\cup\bigcup\mathcal{Q}_x^*\,:\,x\in[k^\star]\}$ and therefore no vertex is in more than two of the cycles in $\{O_x\,:\,x\in[k^\star]\}$. Moreover, if $I=\emptyset$, then O_x is contained in W_x for each $x\in[k^*]$, and therefore H contains a packing of $k^\star$ cycles in $\mathcal{O}$. $\square$

By Claim 6.4.0.3, I is nonempty because we assumed that $\nu(G)<k\leq k^\star$. Let $J:=[m]\setminus Z$ and let γ'' be the (Γ/Γ_J)-labelling induced by the restriction of γ to H. Since we assumed that statement (iii) fails and H has a half-integral packing of κ cycles in $\mathcal{O}$ by Claim 6.4.0.3, we have $(H,\gamma'')\notin\mathcal{C}(\kappa,\theta,\Gamma/\Gamma_J,A+\Gamma_J)$. We will find the desired contradiction by constructing in H a packing of k cycles in $\mathcal{O}$.

Recall the properties (1)–(6) of $\mathcal{C}(\kappa,\theta,\Gamma/\Gamma_J,A+\Gamma_J)$ in Definition 3.3.1. With W''

and $(\mathcal{P}''_i\colon i \in I)$, (H, γ'') satisfies (1), (2), (3), (4), and (5) of Definition 3.3.1 by Claim 6.4.0.2. Thus, (6) fails to hold[1].

First consider the case that $(\mathcal{P}''_i\colon i \in I)$ contains a W''-handlebar that is not in series. Then $(\mathcal{P}''_i\colon i \in I)$ contains an even number of crossing W''-handlebars and no W''-handlebar that is in series because (6)(a) and (6)(b) fail to hold respectively. By Lemma 6.2.3, there exist a subwall W^* of W'' with at least $c_1(|Z|)$ columns and at least $r_1(|Z|)$ rows and a nested W^*-handlebar $\mathcal{Q}_1$ of size k such that $\gamma_j(Q) \notin \Omega_j$ for all $Q \in \mathcal{Q}_1$ and $j \in Z$. Let us define $q := 1$.

If the first case does not hold, then all W''-handlebars in $(\mathcal{P}''_i : i \in I)$ are in series. Let $q := |I|$ and observe that $q \in \{0, 1, 2\}$ because (6)(c) fails to hold. For each $i \in [q]$, let us define $\mathcal{Q}_i$ to be $\mathcal{P}''_j$ for the i-th element j of I. Let $W^* := W''$. Note that $c_2(|Z|) \geq c_1(|Z|)$ and $r_2(|Z|) \geq r_1(|Z|)$.

In either case, we can apply Lemma 6.2.4 to obtain, for each $x \in [k]$, an $r_0(|Z|) \times c_0(|Z|)$-subwall W_x of W^* and a set $\mathcal{H}_x = \{H_{x,i}\colon i \in [q]\}$ of q disjoint W_x-handles such that

- for distinct $x, x' \in [k]$, the graphs $W_x \cup \bigcup \mathcal{H}_x$ and $W_{x'} \cup \bigcup \mathcal{H}_{x'}$ are disjoint and
- $\sum_{i \in [q]} \gamma_j(H_{x,i}) \notin \Omega_j$ for each $x \in [k]$ and $j \in Z$.

Finally, we apply Lemma 6.1.5 to obtain a packing of k cycles in $\mathcal{O}$. This contradiction completes the proof. $\square$

[1] Note that the number of handlebars in H is at most $\hat{p}$, which depends only on m and ω. We could strengthen Theorem 3.3.2 by imposing this additional restriction on the class $\mathcal{C}(\kappa, \theta, \Gamma, A)$, which would give us that there are only finitely many types of obstructions to consider in terms of the arrangement of the handlebars around the wall.

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