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AN IMPROVED ALGORITHM FOR THE COMBINED

ESTIMATION AND CONTROL OF NONGAUSSIAN STOCHASTIC SYSTEMS

A THESIS

Presented to

The Faculty of the Graduate Division

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Ch

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SUMMARY

This dissertation describes an improved algorithm for the estimation and control of nongaussian stochastic systems. It is assumed that the plant and measurement noises are bounded with the specific bounds and density functions known. The system is to be controlled to minimize a cost criterion that encompasses both the standard quadratic performance index and the error in the estimation of the plant states.

The development of the combined estimation and control algorithm for a noisy, discrete linear system depends on the applicability of the Separation Theorem. Its validity for this case is demonstrated in a proof by dynamic programming resulting in a Riccati controller operating on the least-mean-square estimate. A moment technique is used in applying Bayes-law computation to obtain this estimate. The conditional density functions required in the Bayes-law computation are either expressed directly in terms of their moments or approximated by polynomials whose coefficients are functions of the moments. To evaluate the expected value of cross-products of the plant state and estimate, the estimate is expanded into a truncated polynomial. A rather complex relationship depending on the value of the measurement is obtained for this cross-product. The estimate of the plant state is then combined with the Riccati controller to yield the improved estimation and control algorithm.

The approximately optimal algorithm is applied both to linear and

nonlinear systems. To implement the algorithm for the nonlinear plant, linear perturbations about a nominal trajectory are assumed. In both linear and nonlinear cases, the use of the algorithm improves the performance and estimation error over that obtained from the combination of the Riccati controller and the Kalman filter. From further considerations of the approximately optimal algorithm, a specific controller is synthesized which improves system performance for a fixed nonoptimal filter over the use of a Riccati controller with the nonoptimal filter.

The basic algorithm for the linear, nongaussian, stochastic system was shown to be sensitive to incorrect data statistics. However, when correct data was used, particular examples demonstrated that the primary improvement was the resulting lower estimation error. It is expected that for other systems improvement in the standard quadratic performance index can be achieved by using this basic algorithm.

CHAPTER I

INTRODUCTION

Motivation

Modern design approaches for deterministic control systems have utilized such variational methods as Bellman's Dynamic Programming and Pontryagin's Maximum Principle. Because the optimal closed-loop control law found by these methods is a function of the plant states, it is often assumed that these states are exactly measurable. This assumption is usually not justified in practical systems due to instrumentation errors and external disturbances. Thus, it becomes necessary in such cases to estimate the plant states for closed-loop control purposes. This estimation and its subsequent use for control is referred to as the combined estimation and control problem.

The usual approach to the combined estimation and control problem has been to use the Kalman-Bucy (linear) filter with the deterministic optimal controller to obtain a closed-loop solution. For this solution to be optimal, the system must be linear and the disturbances gaussian. Nevertheless, even though these assumptions are rarely satisfied, the number of applications of the Kalman-Bucy filter in physical systems has rapidly increased in recent years. For example, the Kalman-Bucy filter has been used in the guidance of Rangers VI and VII as well as in the analysis of test data from the Boeing 747. In this dissertation the gaussian assumption is relaxed, and the resulting optimal estimator is nonlinear. Previous approaches to the optimal nongaussian combined problem have not resulted in computationally feasible solutions.

The objective of this dissertation is to develop a new computational technique to yield an approximately optimal solution for the linear, nongaussian combined estimation and control problem. Using this solution, the nonlinear plant will also be investigated by linearizing about a deterministic nominal trajectory.

Background

The historical background relevant to this dissertation emphasizes recent developments in stochastic system theory, the Separation Theorem, approximate solutions for nonlinear systems, estimation theory, and specific optimal estimation and control.

Stochastic System Theory

General nonlinear stochastic systems have been investigated within the last decade with some success. The major difficulty has been the associated computational problem. The estimation of the states of a nonlinear plant disturbed by gaussian noise was considered by Kushner [1,2]. His result was the first rigorous treatment of the continuous-time nonlinear filter and yielded an infinite set of partial stochastic differential equations. The equations were the result of application of the Kolmogorov forward and backward equations. This nonlinear filtering algorithm was also derived with appropriate conditions by Bucy and Joseph [3]. W. M. Wonham [4] developed a procedure for analyzing the stochastic optimal control problem. He derived an equation analogous to the Hamilton-Jacobi equation in deterministic control that reduced the control problem to the solution of a functional equation. However, this stochastic Hamilton-Jacobi equation contained the partial differential operator of the Kolmogorov equations. Wonham applied his equation to a linear system disturbed by gaussian noise and obtained a solution that could be verified by the Separation Theorem.

Florentin [5] used the method of dynamic programming to obtain a nonlinear integro-partial differential equation whose solution would yield both the optimal control and the value of the performance index. Because the resulting equations could not easily be solved in general, Florentin applied the method only to a linear plant with a quadratic performance index and gaussian disturbances. His approach to this problem yielded a set of ordinary differential equations suited for computer solution. Florentin concluded that for certain examples the separate optimization of control and estimation functions would also provide the optimal control policy. Because of the computational difficulties in these approaches, the nonlinear problem has often been linearized and the higher order moments neglected.

The Separation Theorem

The possibility of the separate optimization of a statistical estimator and the performance criterion of the plant to yield a system which would be optimal in an overall sense was suggested in 1958 by Kalman and Koepcke [6]. Booton [7] showed for a linear terminal control problem corrupted by gaussian noise that the separate optima imply an overall optimum system. With the advent of the Kalman estimator and the Duality Principle [8], considerable attention has been devoted to the separate optimization of the estimation and control functions. The first proofs of the Separation Theorem were given by Joseph and Tou [9] and by Gunckel and Franklin [10]. For the linear discrete multivariable control system subjected to additive white noise, the dynamic programming technique was used to show that the independent optimization of the controller and the estimator results in an optimum control system with respect to a quadratic performance index. For a similar system, A. R. M. Noton [11] showed that the Separation Theorem is valid when the measurements are a mixture of both continuous and discrete data. The extension of the Separation Theorem to continuous linear multivariable plants disturbed by white noise has been shown by Sage [12] and Lee [13]. Bryson and Ho [14] proved via dynamic programming that the Separation Theorem is valid for both continuous and discrete linear plants having quadratic performance indices with gaussian noise. Alspach and Sorenson [15] demonstrated that for a linear discrete system with nongaussian disturbances the separate optimization of the estimation and control functions results in an overall optimal scheme. Curry [16] indicated that the Separation Theorem is valid for a linear discrete system with nonlinear measurements. However, Alspach and Sorenson [15] asserted that his result does not yield the optimal solution. In a paper more recent than [4], Wonham [17] used dynamic programming and the Ito-Nisio-Fleming theory of functional stochastic differential equations to determine results for more general controllers and cost criteria. Specifically, he showed that, for a linear continuous plant disturbed by white gaussian noise, the independent control and estimation of the plant is correct regardless of whether the optimal control is linear in the plant state or the cost criterion is quadratic. In a generalization of the combined estimation and control problem, Meier, Peschon, and Dressler [18] considered a system which had available a control input to the plant as well as to a measurement subsystem. In this class of problems referred to as measurement adaptive systems, a linear plant, a quadratic performance index, and gaussian disturbances were considered. The authors demonstrated that measurement control may be computed a priori and that the plant control and state estimation may be performed independently. Koivo [19] showed that the separate optimization of the plant control and state estimation holds for linear systems containing delayed state variables and having a quadratic cost functional.

<u>Approximate Solutions for Nonlinear Systems</u>

The computational difficulties accompanying the exact solution of nonlinear stochastic systems has led to approximate solutions for these systems. Sage [20] presented a method for applying the Kalman filtering theory to nonlinear systems. By forming approximate linear perturbational equations about the nominal solution of the nonlinear differential equations, an approximate estimate may be found by the addition of the nominal state and the Kalman estimate of the linear perturbation. Wells [21] applied the same technique in the control of a nonlinear reactor. Sunahara [22] described another approximation for nonlinear systems called stochastic linearization. The method involves the expansion of the nonlinearity into a linear function whose coefficients are selected to minimize mean square error. Sunahara and Ohsumi [23] used this linearization technique and a computational approach from dynamic programming to yield a suboptimal approach to the nonlinear, stochastic control problem. Regardless of the particular approximations used, nonlinear systems are usually handled more expediently by some linearizing technique.

Estimation Theory

The estimation of the states of a plant corrupted by additive noise is an important aspect in stochastic control. Since the basic work of Wiener [24], the major contribution in estimation theory was developed by Kalman and Bucy [8,25]. They converted the Wiener-Hopf integral equation into a nonlinear differential equation containing the necessary information for the design of the optimal filter. Their procedure applies to linear systems corrupted by additive white noise with stationary or nonstationary statistics and finite or infinite smoothing intervals. Ho and Lee [26] formulated the nonlinear, nongaussian estimation problem from a Bayesian decision viewpoint. However, because of the difficulties in finding the associated marginal and conditional densities, the problem as formulated was intractable except for certain very special cases. Schweppe [27] developed a reachable set technique which resulted in a recursive algorithm to calculate a time-varying ellipsoid that always contained the system's actual state. The procedure permitted the input to the dynamical system and the observation errors to be completely unknown except for bounds on their magnitude and energy. Kuo and Rowland [28,29] combined a moment technique with the reachable set concept in applying the Bayesian decision rule and the least-mean-square error criterion to a linear stationary system having nongaussian disturbances. This estimator was adaptive because the filter learned the moments of the input noise from the data received and suboptimal because densities were approximated by truncated polynomials.

Bucy and Senne [30] also approached the nonlinear filtering problem for discrete, nongaussian systems by Bayes-law computation. Density storage was accomplished in [30] by a point mass representation on a floating grid of indices. Alspach and Sorenson [31] approximated conditional density functions by a sum of gaussians for nonlinear Bayesian estimation. The results in [30,31] represent alternate approaches to the basic computational problem considered by Kuo and Rowland [28,29]. Earlier, Bryan [32] had developed a zero-order nonlinear estimator which applied to discrete nonstationary systems. The difficult problem of finding density functions to use in the Bayesian decision approach when applied to nongaussian and nonlinear problems inevitably results in such approximate filtering algorithms.

Specific Optimal Estimation and Control

In those cases where the independent optimization of the control and estimation functions is valid, some interest has been devoted to the investigation of suboptimal control and estimation techniques. This has been necessary because of the inherent implementation problem for the optimal scheme. Sims and Melsa [33] considered problems of specific optimal estimation for linear and nonlinear systems. Their estimation scheme was achieved by preselecting the filter configuration with some unspecified parameters and optimizing the filter performance by the selection of these parameters. The solution of the associated two-point boundary value problem for the specific optimal estimation gave nearly optimal results with considerable reduction in the complexity of implementation. The design and use of specific optimal controllers was also extensively investigated by Agarwal and Sridhar [34] and Murtuza [35]. Their research was motivated by the undesirable aspects of time-varying parameters in the optimal controller structure and by the implementation problem mentioned above. A systematic approach for the selection of unknown parameters of a fixed controller configuration was presented by Murtuza such that the behavior of a class of single input, time-varying systems would be nearly optimal. The sensitivity of specific optimal controllers to variations in time intervals of optimization and initial conditions was investigated by Sims and Melsa [36]. They indicated that the use of a dynamic controller involving an intermediate system improves the performance of the system while reducing the sensitivity to variations.

Specific optimal control is currently being investigated for linear systems disturbed by nongaussian noise. In many specific optimal control and estimation schemes found in the literature, the parameters of either fixed configuration filters or controllers are adjusted to optimize system performance. Raphael Sivan [37] has shown for the above mentioned system with a quadratic performance index that using a linear estimator with a linear controller would not be optimal. In fact, Sivan [38] demonstrated that for a polynomial controller using the first four moments of all random variables, coupled with a linear estimator, better results were obtained than for a linear controller with a linear estimator. Thus, it is possible that in those situations where the estimator is fixed, the corresponding controller obtained by separate optimization would not result in the best performance.

Development of the Separation Theorem

The Separation Theorem permits the separate optimization of the control and estimation functions to give an overall optimal solution. The following proof of the Separation Theorem for linear systems with nongaussian disturbances was developed concurrently with, but independently of, the version given in [15].

Problem Statement

Consider the linear system given by

$$x_{k+1} = A_k x_k + B_k u_k + w_k \tag{1.1}$$

with a linear measurement of the plant state given as

$$z_{k+1} = x_{k+1} + v_{k+1} \tag{1.2}$$

where x_k represents the n-dimensional state vector, u_k is the control input vector, w_k is the system noise vector, v_k is the measurement noise vector, and z_k is the measurement vector of the corrupted state vector. The plant and measurement noises are assumed to be zero-mean nongaussian white noises with a covariance matrix specified as

$$E\begin{bmatrix} w_i \\ v_i \end{bmatrix} [w_j^T, v_j^T] = \begin{bmatrix} Q_i & 0 \\ 0 & R_i \end{bmatrix} \delta_{ij} \tag{1.3}$$

The problem is to minimize

$$J = \frac{1}{2} E\left\{ \sum_{k=0}^{N} x_k^T Qx_k + u_k^T Ru_k + (x_k - \hat{x}_k)^T D(x_k - \hat{x}_k)\right\} \tag{1.4}$$

by filtering the measurement data to yield the best state estimate $\hat{x}_k$ and then to use that estimate in a suitable control algorithm.

Problem Solution

Using the dynamic programming approach, one may define $V_1(Z_{N-1})$ as the optimal expected value of J for a single-stage control process starting at k = N-1 obtained by using an optimal control u_{N-1} and by knowing the measurements $Z_{N-1} = \{z_o, z_1, \ldots, z_{N-1}\}$. Therefore, using (1.4)

$$V_1(Z_{N-1}) = \underset{u_{N-1}}{\mathrm{Min}} \ \frac{1}{2} \ E\Big\{x_{N-1}^T Qx_{N-1} + u_{N-1}^T Ru_{N-1} + (x_{N-1} - \hat{x}_{N-1})^T D(x_{N-1} - \hat{x}_{N-1}) \,\big|\, Z_{N-1}\Big\} \tag{1.5}$$

Differentiating (1.5) with respect to u_{N-1} and setting the result equal to zero to minimize $V_1(Z_{N-1})$ yields u_{N-1} equals zero, because neither x_{N-1} nor $\hat{x}_{N-1}$ are functions of u_{N-1}. Using the principle of optimality, one may express $V_2(Z_{N-2})$ as

$$V_2(Z_{N-2}) = \underset{u_{N-2}}{\mathrm{Min}} \ \frac{1}{2} \ E\Big\{V_1(Z_{N-1}) + x_{N-2}^T Qx_{N-2} + u_{N-2}^T Ru_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T D(x_{N-2} - \hat{x}_{N-2}) \,\big|\, Z_{N-2}\Big\} \tag{1.6}$$

However, letting $L_{N-1} = Q$, one may rewrite $V_1(Z_{N-1})$ as

$$V_1(Z_{N-1}) = x_{N-1}^T L_{N-1} x_{N-1} + (x_{N-1} - \hat{x}_{N-1})^T D(x_{N-1} - \hat{x}_{N-1}) \tag{1.7}$$

By substitution of (1.7) into (1.6), $V_2(Z_{N-2})$ becomes

$$V_2(Z_{N-2}) = \underset{u_{N-2}}{\text{Min}} \ \frac{1}{2} \ E\Big\{x_{N-1}^T L_{N-1} x_{N-1} + (x_{N-1} - \hat{x}_{N-1})^T D (x_{N-1} - \hat{x}_{N-1}) \tag{1.8}$$

$$+ x_{N-2}^T \, Q x_{N-2} + u_{N-2}^T \, R u_{N-2}$$

$$+ (x_{N-2} - \hat{x}_{N-2})^T D (x_{N-2} - \hat{x}_{N-2}) \,|\, Z_{N-2}\Big\}$$

Note that x_{N-1} may be replaced by the plant equation in (1.1) by letting $k = N-2$ to give

$$V_2(Z_{N-2}) = \underset{u_{N-2}}{\text{Min}} \ \frac{1}{2} \ E\Big\{x_{N-2}^T A_{N-2}^T L_{N-1} A_{N-2} x_{N-2} + u_{N-2}^T B_{N-2}^T L_{N-1} B_{N-2} u_{N-2} \tag{1.9}$$

$$+ w_{N-2}^T L_{N-1} w_{N-2} + u_{N-2}^T B_{N-2}^T L_{N-1} A_{N-2} x_{N-2} + u_{N-2}^T B_{N-2}^T L_{N-1} w_{N-2}$$

$$+ w_{N-2}^T L_{N-1} (A_{N-2} x_{N-2} + B_{N-2} u_{N-2}) + x_{N-2}^T A_{N-2}^T L_{N-1} B_{N-2} u_{N-2}$$

$$+ x_{N-2}^T A_{N-2}^T L_{N-1} w_{N-2} + x_{N-2}^T \, Q x_{N-2} + u_{N-2}^T \, R u_{N-2}$$

$$+ (x_{N-2} - \hat{x}_{N-2})^T D (x_{N-2} - \hat{x}_{N-2}) + (x_{N-1} - \hat{x}_{N-1})^T D (x_{N-1} - \hat{x}_{N-1}) \,|\, Z_{N-2}\Big\}$$

Recognizing both that the estimator minimizes $(x_{N-1} - \hat{x}_{N-1})^T D (x_{N-1} - \hat{x}_{N-1})$ for any positive definite matrix D and that $(x_{N-2} - \hat{x}_{N-2})^T D (x_{N-2} - \hat{x}_{N-2})$ is independent of u_{N-2}, then differentiation inside the expected value of (1.9) with respect to u_{N-2} yields

$$E\Big\{2B_{N-2}^T L_{N-1} B_{N-2} u_{N-2} + 2B_{N-2}^T L_{N-1} A_{N-2} x_{N-2} + 2B_{N-2}^T L_{N-1} w_{N-2} \tag{1.10}$$

$$+ 2Ru_{N-2} \,|\, Z_{N-2}\Big\} = 0$$

Since $E[w_{N-2}] = 0$, then from (1.10)

$$u_{N-2} = E\left\{-(B^T_{N-2}L_{N-1}B_{N-2} + R)^{-1}B^T_{N-2}L_{N-1}A_{N-2}x_{N-2}|Z_{N-2}\right\} \tag{1.11}$$

$$u_{N-2} = -(B^T_{N-2}L_{N-1}B_{N-2} + R)^{-1}B^T_{N-2}L_{N-1}A_{N-2}\hat{x}_{N-2} \tag{1.12}$$

Therefore,

$$u_{N-2} = -C_{N-2}\,\hat{x}_{N-2} \tag{1.13}$$

where $C_{N-2} = (B^T_{N-2}L_{N-1}B_{N-2} + R)^{-1}B^T_{N-2}L_{N-1}A_{N-2}$ and $\hat{x}_{N-2} = E[x_{N-2}|Z_{N-2}]$ which is the least mean-square estimate of $x_{N-2}|Z_{N-2}$.

Rewriting $V_2(Z_{N-2})$ in (1.9) yields

$$\begin{aligned} V_2(Z_{N-2}) = \underset{u_{N-2}}{\text{Min}}\ \frac{1}{2}\ E\Big\{&x^T_{N-2}A^T_{N-2}L_{N-1}x_{N-2} - \hat{x}^T_{N-2}C^T_{N-2}B^T_{N-2}L_{N-1}A_{N-2}x_{N-2} \\ &-x^T_{N-2}A^T_{N-2}L_{N-1}B_{N-2}C_{N-2}\hat{x}_{N-2} + \hat{x}^T_{N-2}C^T_{N-2}(B^T_{N-2}L_{N-1}B_{N-2} + R)C_{N-2}\hat{x}_{N-2} \\ &+x^T_{N-2}Qx_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T D(x_{N-2} - \hat{x}_{N-2}) + (x_{N-1} - \hat{x}_{N-1})^T D \cdot \\ &(x_{N-1} - \hat{x}_{N-1})|Z_{N-2}\Big\} + \text{Constant Terms} \end{aligned} \tag{1.14}$$

However, if one recognizes from (1.12) that

$$B^T_{N-2}L_{N-1}A_{N-2} = (B^T_{N-2}L_{N-1}B_{N-2} + R)C_{N-2} \tag{1.15}$$

then (1.14) may be rewritten as

$$V_2(Z_{N-2}) = \frac{1}{2}\, E\Big\{x_{N-2}^T(A_{N-2}^T L_{N-1} + Q)x_{N-2} + \hat{x}_{N-2}^T C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} \tag{1.16}$$

$$+ R)C_{N-2}\hat{x}_{N-2} - \hat{x}_{N-2}^T C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} + R)C_{N-2}x_{N-2} - x_{N-2}^T C_{N-2}^T$$

$$(B_{N-2}^T L_{N-1} B_{N-2} + R)C_{N-2}\hat{x}_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T D(x_{N-2} - \hat{x}_{N-2})$$

$$+ (x_{N-1} - \hat{x}_{N-1})^T D(x_{N-1} - \hat{x}_{N-1})\,\big|Z_{N-2}\Big\} + \text{Constant Terms}$$

Completing the square, (1.16) becomes

$$V_2(Z_{N-2}) = \frac{1}{2}\, E\Big\{x_{N-2}^T(A_{N-2}^T L_{N-1} A_{N-2} + Q - C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} + R)\cdot \tag{1.17}$$

$$C_{N-2})x_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} + R)C_{N-2}\,\cdot$$

$$(x_{N-2} - \hat{x}_{N-2}) + (x_{N-2} - \hat{x}_{N-2})^T D(x_{N-2} - \hat{x}_{N-2})\,\big|Z_{N-2}\Big\} + \text{Constant Terms}$$

By setting

$$L_{N-2} = A_{N-2}^T L_{N-1} A_{N-2} + Q - C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} + R)C_{N-2} \tag{1.18}$$

$$K_{N-2} = C_{N-2}^T(B_{N-2}^T L_{N-1} B_{N-2} + R)C_{N-2} + D$$

then (1.16) may be written as

$$V_2(Z_{N-2}) = \frac{1}{2}\, E\Big\{x_{N-2}^T L_{N-2} x_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T K_{N-2}(x_{N-2} - \hat{x}_{N-2})\,\big|Z_{N-2}\Big\} \tag{1.19}$$

$$+ \text{Constant Terms}$$

Using the principle of optimality, one may express $V_3(Z_{N-3})$ as

$$V_3(Z_{N-3}) = \frac{1}{2} \underset{u_{N-3}}{\mathrm{Min}} \; E\Big\{x_{N-2}^T L_{N-2} x_{N-2} + (x_{N-2} - \hat{x}_{N-2})^T K_{N-2} (x_{N-2} - \hat{x}_{N-2}) \tag{1.20}$$

$$+ x_{N-3}^T Q x_{N-3} + u_{N-3}^T R u_{N-3} + (x_{N-3} - \hat{x}_{N-3})^T D (x_{N-3} - \hat{x}_{N-3}) \,|\, Z_{N-3}\Big\}$$

$$+ \text{ Constant Terms}$$

After substitution of (1.1), (1.20) becomes

$$V_3(Z_{N-3}) = \frac{1}{2} \underset{u_{N-3}}{\mathrm{Min}} \; E\Big\{(A_{N-3}x_{N-3} + B_{N-3}u_{N-3} + w_{N-3})^T L_{N-2} (A_{N-3}x_{N-3} \tag{1.21}$$

$$+ B_{N-3}u_{N-3} + w_{N-3}) + x_{N-3}^T Q x_{N-3} + u_{N-3} R u_{N-3} + (x_{N-3} - \hat{x}_{N-3})^T D \cdot$$

$$(x_{N-3} - \hat{x}_{N-3}) + (x_{N-2} - \hat{x}_{N-2})^T K_{N-2} (x_{N-2} - \hat{x}_{N-2}) \,|\, Z_{N-3}\Big\} + \text{Constant Terms}$$

Differentiating with respect to u_{N-3} inside the expected value sign yields

$$u_{N-3} = -(B_{N-3}^T L_{N-2} B_{N-3} + R)^{-1} B_{N-3}^T L_{N-2} A_{N-3} \hat{x}_{N-3} \tag{1.22}$$

where again it is recognized that the estimator minimizes $(x_{N-2} - \hat{x}_{N-2})^T \cdot (x_{N-2} - \hat{x}_{N-2})$ and, in addition, that $(x_{N-3} - \hat{x}_{N-3})^T D (x_{N-3} - \hat{x}_{N-3})$ is independent of the control u_{N-3}. Rewriting (1.21) and collecting terms, one obtains

$$V_3(Z_{N-3}) = \frac{1}{2} E\Big\{x_{N-3}^T (A_{N-3}^T L_{N-2} A_{N-3} + Q) x_{N-3} + \hat{x}_{N-3} C_{N-3}^T (B_{N-3}^T L_{N-2} B_{N-3} \tag{1.23}$$

$$+ R) C_{N-3}^T \hat{x}_{N-3} - \hat{x}_{N-3}^T C_{N-3}^T B_{N-3}^T L_{N-2} A_{N-3} x_{N-3} - x_{N-3}^T A_{N-3}^T L_{N-2} B_{N-3} C_{N-3} \cdot$$

(See next page)

$$\hat{x}_{N-3} + (x_{N-3} - \hat{x}_{N-3})^T D (x_{N-3} - \hat{x}_{N-3}) + (x_{N-2} - \hat{x}_{N-2})^T K_{N-2} \cdot$$

$$(x_{N-2} - \hat{x}_{N-2}) \left| Z_{N-3} \right\} + \text{Constant Terms}$$

Completing the square, (1.23) may be written as

$$V_3(Z_{N-3}) = \frac{1}{2} E\left\{ x_{N-3}^T (A_{N-3}^T L_{N-2} A_{N-3} + Q - C_{N-3}^T (B_{N-3}^T L_{N-2} B_{N-3} + R) \cdot \right. \tag{1.24}$$

$$C_{N-3}) x_{N-3} + (x_{N-3} - \hat{x}_{N-3})^T C_{N-3}^T (B_{N-3}^T L_{N-2} B_{N-3} + R) C_{N-3} \cdot$$

$$(x_{N-3} - \hat{x}_{N-3}) + (x_{N-3} - \hat{x}_{N-3})^T D (x_{N-3} - \hat{x}_{N-3}) \left| Z_{N-3} \right\} + \text{Constant Terms}$$

It is easy to see the repetition in (1.24) of the terms appearing in (1.17). Therefore, by defining

$$L_{k-1} = A_{k-1}^T L_k A_{k-1} + Q - C_{k-1} (B_{k-1}^T L_k B_{k-1} + R) C_{k-1} \tag{1.25}$$

with

$$L_{N-1} = Q \tag{1.26}$$

and

$$u_{k-1} = -C_{k-1} \hat{x}_{k-1} \tag{1.27}$$

with

$$C_{k-1} = (B_{k-1}^T L_k B_{k-1} + R)^{-1} B_{k-1}^T L_k A_{k-1} \tag{1.28}$$

where

$$\hat{x}_{k+1} = E[x_{k+1}|Z_{k+1}] \tag{1.29}$$

one recognizes this as the solution to the independent optimization problem, namely, a Riccati controller operating on the least-mean-square estimate. This constitutes the proof by induction of the Separation Theorem.

It has been shown that the Separation Theorem is valid for a linear discrete plant disturbed by nongaussian, white noise for the performance index given by (1.4). By letting D = 0, one obtains the special case given as

$$J = \frac{1}{2} E\left\{\sum_{k=1}^{N} x_k^T Q x_k + u_k^T R u_k\right\} \tag{1.30}$$

Therefore, the optimal control policy for a linear nongaussian stochastic system may be determined by the plant control (1.29) acting on the optimal nonlinear estimate (1.29). This calculation may be performed by considering the separate optimization of the control and estimation functions.

Method of Attack

The problem investigated in this thesis research is the combined estimation and control of noisy dynamical systems. Both linear and nonlinear discrete systems were considered with the input noise and the measurement noise assumed to be ergodic white noises with known nongaussian density functions.

The solution to the linear system with quadratic performance index is specified by the results of the Separation Theorem, given by (1.25)-(1.29). However, for nongaussian disturbances the resulting least-mean-square estimate is nonlinear. A Bayes-law computation permits the determination of the estimate with the associated difficulty of obtaining the required density functions with application of the moment technique was selected because of the resulting simplifications in computer programming. By applying this technique, formulas relating the moments of the observed data, the states, and the control were established and utilized in first-order examples.

An approximate approach to the nonlinear combined estimation and control problem was shown to be computationally feasible with acceptible accuracy. The approach was to linearize the noise perturbations about the deterministic nominal trajectory and to form a variational control by using those techniques established in the linear case. This approximate approach was selected because the exact solution to the nonlinear problem was not known.

The sensitivity of the estimation and control algorithms to erroneous bound and moment data was investigated. Fixing either the measurement or the plant noise, the density of the other noise was changed to one with the same second moment. However, since the original bound and moment data were supplied to the filter, the sensitivity of the algorithms to incorrect input data was observed.

A method for selecting the form and parameters of a specific optimal controller was determined. The controller selection was accomplished by a comparison technique with the algorithms established for the linear system. The controller was used with a fixed, nonlinear estimator to yield a better performance than that obtained by using the same nonlinear estimator with a Riccati controller. This controller selection further demonstrated the sensitivity of the algorithms to incorrect noise statistics.

The combined control and estimation algorithms were formulated, and the value of each for specific examples was verified by computer simulations. The results were compared with the combination of the Kalman-Bucy filter and the Riccati controller which was shown to be the exact solution specified by the Separation Theorem for the linear, gaussian system.

Thesis Contributions

The thesis research reported here has contributed to the state-of-the-art in stochastic control theory in four specific ways:

1. The derivation of the Separation Theorem for linear systems disturbed by nongaussian noises.
2. The development of nearly optimal combined estimation and control algorithms for linear nongaussian stochastic systems.
3. The application of these nearly optimal algorithms for an approximate analysis of nonlinear stochastic systems.
4. The synthesis of specific optimal controllers for a fixed nonoptimal estimator for nongaussian stochastic systems.

Outline of the Thesis

The Linear Stochastic Control Problem is treated in detail in Chapter II. The least-mean-square estimator incorporating the control function is derived in detail using the moment technique. Computer results for a particular example demonstrate the effectiveness of this result. Several approximations to the suboptimal estimator are then presented and the simulation results included. Chapter III applies the methods of Chapter II to a nonlinear example by linearizing about a nominal trajectory to form a variational controller. In Chapter IV a sensitivity analysis is presented to show the variation of the resulting estimate with incorrect noise statistics. The design of a specific optimal controller to be used with a fixed nonoptimal filter is also discussed. Simulation results demonstrate the usefulness of this approach. Finally, Chapter V presents some conclusions as well as some recommendations for further research.

CHAPTER II

THE LINEAR STOCHASTIC CONTROL PROBLEM

Introduction

The analysis of a linear control system disturbed by noise is commonly referred to as the Linear Stochastic Control Problem. This problem was considered in the thesis research both as a problem whose solution itself is useful and as a building block for the subsequent consideration of nonlinear stochastic control systems. The principal tool used in investigating this problem was Bayes-law computation based on the moment technique. As mentioned previously, Kuo and Rowland [28,29] had successfully applied the method of moments to the linear nongaussian estimation problem. However, the application of the moment technique to the combined control and estimation problem proved to be considerably more complicated because of the presence of feedback control.

In the following sections the problem is formulated and the moment technique applied to the estimation of the plant states. After a suitable comparison of the results obtained with previous methods, several approximations to the new filtering algorithm are presented.

Mathematical Formulation

A model of the linear stochastic control system is given in Figure 2.1.

For the given first-order system the linear discrete plant is:

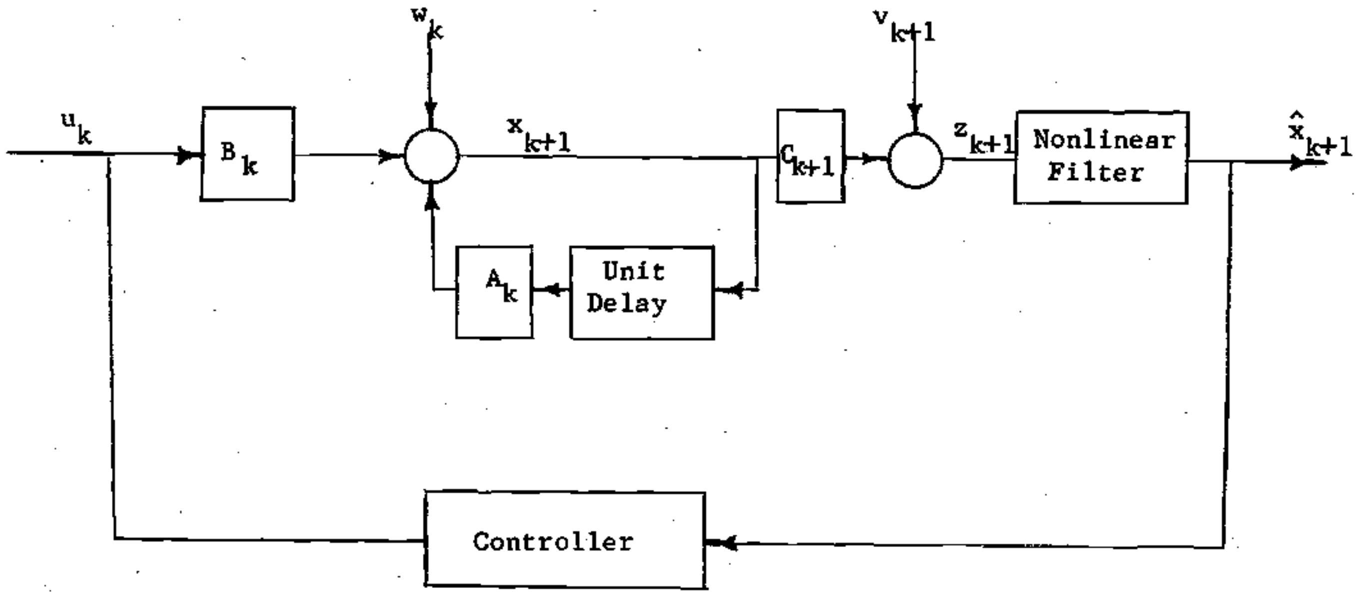

Figure 2.1 Diagram of the Linear Stochastic Control Problem.

$$x_{k+1} = A_k x_k + B_k u_k + w_k \tag{2.1}$$

$$z_k = C_k x_k + v_k \tag{2.2}$$

where x(k) represents the state of the plant at the kth sampling instant, u(k) represents the control supplied to the plant, w(k) is the nongaussian noise input to the plant, z(k) is the measurement of the plant state intermixed with noise, and v(k) is the nongaussian noise corrupting the measurement of the plant state. The performance of the systems is measured by

$$J = \frac{1}{2} E\left\{ \sum_{k=0}^{N} x_k^T Q x_k + u_k^T R u_k + (x_k - \hat{x}_k)^T D (x_k - \hat{x}_k) \right\} \tag{2.3}$$

In some applications, large-scale systems are arbitrarily designed on a subsystem basis. The performance criterion (2.3) which penalizes estimation error in addition to the normal quadratic cost is important in such cases when the subsystem under consideration is being used as a link within the larger system. Accurate estimates of the subsystem states are needed for use as inputs to the following subsystem.

<u>Assumptions</u>

The following basic assumptions about the given system were made:

1. The input signal w and the measurement noise v are discrete time series composed of mutually independent random variables.
2. Both w and v have known density functions with finite bounds.

3. The random processes x and v are independent.

4. At any particular stage k, the noise w is independent of x and u.

The problem was to find a nearly optimal combined control and estimation scheme which minimized the performance index (2.3) for the specified linear system (2.1)-(2.2).

Application of the Moment Technique

The optimal combined control and estimation algorithm for the basic Linear Stochastic Control Problem given above was specified by the Separation Theorem developed in Chapter I. The resulting optimal scheme was derived by the separate optimization of the control and estimation functions. The following derivation is for a first-order system although the application of the moment technique to higher-order systems can be achieved. However, as discussed in a later chapter, computational difficulties involved might suggest alternative approaches. It has been shown in Chapter I that the resulting estimator for this problem is nonlinear in form and given by

$$\hat{x}_{k+1} = E[x_{k+1} | Z_{k+1}] \tag{2.4}$$

where $\hat{x}_{k+1}$ is the least-mean-square estimate of x_{k+1}, and Z_{k+1} denotes the complete set of measurements, i.e. $Z_{k+1} = (z_1, z_2, \ldots, z_{k+1})$. The controller is the Riccati controller, which is a linear combination of the estimated states of the system, e.g. for the linear time-varying case

$$u_k = (B_k^T L_{k+1} B_k + R)^{-1} B_k^T L_{k+1} A_k \hat{x}_k = K_k \hat{x}_k \tag{2.5}$$

where

$$L_k = A_k^T L_{k+1} A_k + Q - K_k(B_k^T L_{k+1} B_k + R)K_k \tag{2.6}$$

$$L_{N-1} = Q \tag{2.7}$$

Let B and C both equal unity for the problem under consideration. The minimum mean-square estimate can be written as

$$\hat{x}_{k+1} = E[x_{k+1}|Z_{k+1}] = \int_{\ell_{k+1}}^{f_{k+1}} x_{k+1} f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1})\ dx_{k+1} \tag{2.8}$$

where ℓ_{k+1} and f_{k+1} are the lower and upper bounds respectively of $x_{k+1}|Z_{k+1}$. By the Bayesian rule, the conditional density function $f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1})$ may be expressed as

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) \cdot f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)}{f_{z_{k+1}|Z_k}(z_{k+1}|Z_k)} \tag{2.9}$$

This Bayesian rule is considered in Appendix I, and for v uniformly distributed on (-1,1) is shown to be

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)}{\int_{\ell_{k+1}}^{f_{k+1}} f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)\ dx_{k+1}} \tag{2.10}$$

The upper and lower bounds on $x_{k+1}|Z_k$ are given from the appendix as

$$f_{k+1,0} = A_k f_k + u_k + w_{max} \tag{2.11}$$

$$\ell_{k+1,0} = A_k \ell_k + u_k + w_{min} \tag{2.12}$$

The density function $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$ is represented by a polynomial given as

$$f_{x_{k+1}|Z_k}(x_{k+1}|Z_k) = \sum_{i=0}^{M} a_i \, x_{k+1}^{i} \tag{2.13}$$

where the coefficients are expressed as functions of the moments of $x_{k+1}|Z_k$. Assuming that the moments of $x_{k+1}|Z_k$ are known, one may use the polynomial approximation to the density function of $x_{k+1}|Z_k$ to express the moments of $x_{k+1}|Z_{k+1}$. The resulting expression from Appendix I is

$$m^j_{x_{k+1}|Z_{k+1}} = \frac{\sum_{i=0}^{M} a_i (f_{k+1}{}^{i+j+1} - \ell_{k+1}{}^{i+j+1})/i+j+1}{\sum_{i=0}^{M} a_i (f_{k+1}{}^{i+1} - \ell_{k+1}{}^{i+1})/i+1} \tag{2.14}$$

The upper and lower bounds respectively on $x_{k+1}|Z_{k+1}$ may be expressed as

$$f_{k+1} = \begin{cases} f_{k+1,0} & \text{when } z_{k+1} \geqq f_{k+1,0}{-1} \\ z_{k+1}{+1} & \text{when } z_{k+1} < f_{k+1,0}{-1} \end{cases} \tag{2.15}$$

and

$$\ell_{k+1} = \begin{cases} z_{k+1}{-1} & \text{when } z_{k+1} > \ell_{k+1,0}{+1} \\ \ell_{k+1,0} & \text{when } z_{k+1} \leqq \ell_{k+1,0}{+1} \end{cases} \tag{2.16}$$

By letting $j = 1$, the expected value of $x_{k+1}|Z_{k+1}$ which is the estimate $\hat{x}_{k+1}$ may be determined as

$$\hat{x}_{k+1} = \frac{\sum_{i=0}^{M} a_i (f_{k+1}^{\,i+2} - \ell_{k+1}^{\,i+2})/i+2}{\sum_{i=0}^{M} a_i (f_{k+1}^{\,i+1} - \ell_{k+1}^{\,i+1})/i+1} \tag{2.17}$$

The order of the filter is then referred to as the m-th order nonlinear filter depending on the number of terms in the polynomial density representation. At this point, the solution for the combined estimation and control problem can be specified, except for the evaluation of the moments of $x_{k+1}|Z_k$.

Evaluation of Moments of $x_{k+1}|Z_k$

The moments of $x_{k+1}|Z_k$ were obtained by taking the expected value of

$$(x_{k+1}|Z_k)^i = (A_k x_k + u_k + w_k|Z_k)^i \tag{2.18}$$

$$\text{for } i = 1,2,3,\ldots,N$$

The resultant expected value is

$$E(x_{k+1}|Z_k)^i = E\left\{\sum_{j=0}^{i} w_k^{i-j} \sum_{p=0}^{j} (A_k x_k|Z_k)^{j-p} (u_k|Z_k)^p\right\} \tag{2.19}$$

$$i = 1,2,\ldots,N$$

Recognizing that $u_k = K_k \hat{x}_k$, the above expression can be evaluated except for the terms involving

$$E[x_k^i \hat{x}_k^{4-i} | Z_k]$$

for i = 1,2,3,4.

Appendix II considered this expected value by expanding the estimate (2.17) by long division to yield

$$\hat{x}_k = .5(f_k + \ell_k) + \frac{a_1}{a_0}\frac{f_k^2}{12} + \frac{a_1}{a_0}\frac{\ell_k^2}{12} - \frac{a_2}{a_0}\frac{f_k^2 \ell_k}{6} + \ldots \quad (2.20)$$

Retaining the first two terms of (2.20), one is able to express $\hat{x}_k$ to permit the evaluation of $E[x_k \hat{x}_k | Z_k]$. From Appendix II, the value of $\hat{x}_k$ has to be expressed differently for four ranges of z_k. For example, the second moment of $x_{k+1} | Z_k$ is expressed for Range B when $z_k \leqq f_{k,0}-1$ and $z_k \leqq \ell_{k,0}+1$ as

$$\begin{aligned} E[x_{k+1}^2 | Z_k] = {} & E(w_k^2) + A_k^2 E[x_k^2 | Z_k] + 2A_k E[w_k] E[x_k | Z_k] \quad (2.21) \\ & + K_k E[w_k] \left\{ E[x_k | Z_k] + E[v_k] + \ell_{k,0}+1 \right\} \\ & + \frac{K_k^4}{4} \left\{ E[x_k^2 | Z_k] + 2E[x_k | Z_k](E[v_k] + \ell_{k,0}+1) \right. \\ & \left. + E[v_k^2] (\ell_{k,0}+1) + (\ell_{k,0}+1)^2 \right\} \\ & + A_k K_k \left\{ E[x_k^2 | Z_k] + E[x_k | Z_k] (E[v_k] + \ell_{k,0}+1) \right\} \end{aligned}$$

The other moments of $x_{k+1} | Z_k$ can be similarly calculated for the four ranges of z_k. These are included in Appendix II. Having obtained the moments of $x_{k+1}|Z_k$, the necessary algorithms are complete. A computer simulation flow chart is presented in Figure 2.2 to explain the detailed steps of the complete filtering algorithms. After an initial assignment of values to $E[x_k^i|Z_k]$, ℓ_k, f_k and the formation of z_k, the bounds and moments of $x_{k+1}|Z_k$ are calculated. Then by evaluating the coefficients a_i in (2.13) and the bounds of $x_{k+1}|Z_{k+1}$, the density of $x_{k+1}|Z_{k+1}$ is formed. With this resulting density function, one is able to calculate the moments of $x_{k+1}|Z_{k+1}$ and, therefore, the least-mean-square estimate $\hat{x}_{k+1}$. The entire process is then repeated as shown in Figure 2.2.

Comparison of Filtering Algorithms

The comparison of the nearly optimal combined estimation and control algorithm with other filtering algorithms was accomplished by a specific example.

Example 1 A Linear Nongaussian System. Consider a linear system given as

$$x_{k+1} = 0.1x_k + u_k + w_k \tag{2.22}$$

with a linear measurement

$$z_{k+1} = x_{k+1} + v_{k+1} \tag{2.23}$$

subject to the performance indices given in (2.3) with $Q = R = 1$ and $D = 5$. The density function of the input noise was given as

$$f_w(w) = \begin{cases} \frac{11}{2} \quad w^{10} & \text{when } |w| < 1 \\ 0 & \text{elsewhere} \end{cases}$$

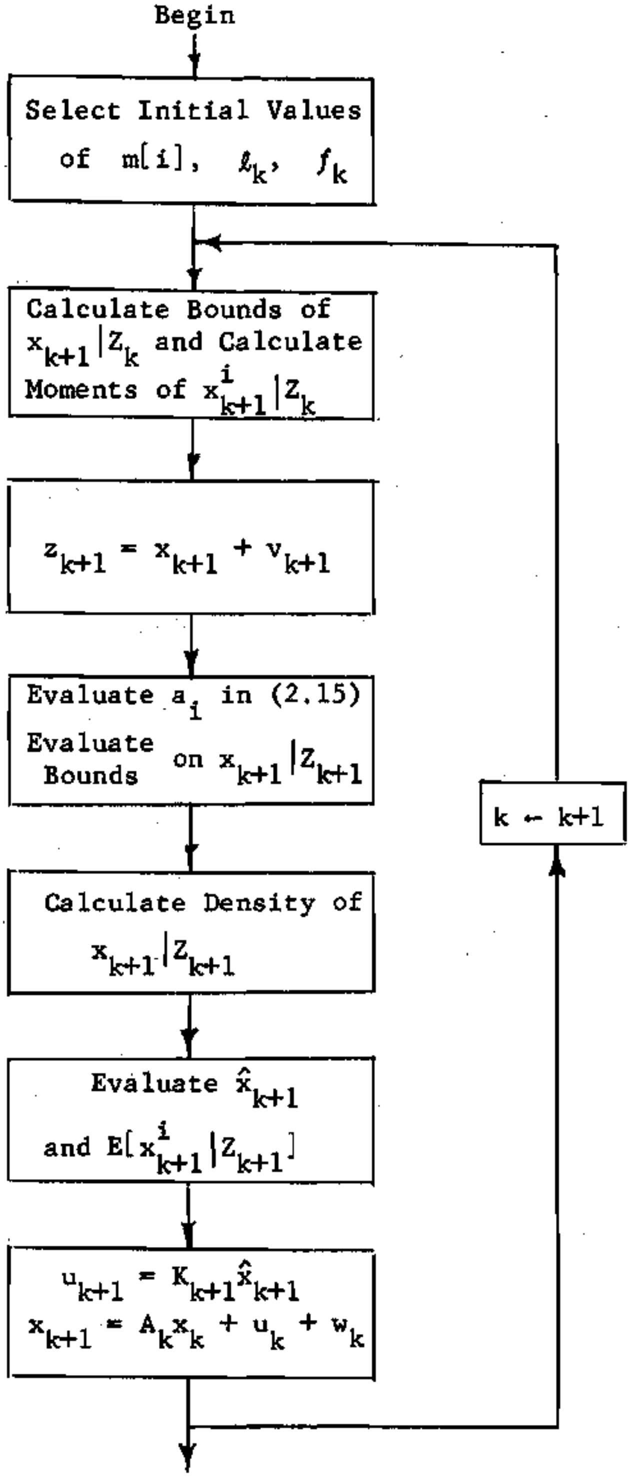

Figure 2.2 Flow Chart for the Combined Estimation and Control Algorithm.

The measurement noise was assumed to be uniform on (-1,1). Table 2.1 shows the performance of the linear system using the cost criterion of (2.3) with D = 0.

Table 2.1. Linear System Performance for N=1000

	Performance (6×10^{-3})	Estimation Error
Kalman Filter and Riccati Controller	5.117141	0.240
Fourth-Order Nonlinear Filter and Riccati Controller	5.110580	0.138

It is noted that the performance index used is relatively insensitive to the difference in estimation error in the above examples. Although there was improvement in performance, the magnitude was small for the given simple system. However, by using the cost criterion (2.3) with D = 5 that equally weighted estimation performance and control performance, an appreciable difference in performance was obtained. This is seen in Figure 2.3 with the corresponding estimation seen in Figure 2.4.

Approximations to the Supoptimal Filter

The dependence of the estimate $\hat{x}_{k+1}$ on the ranges of z_{k+1} and the complexity of calculating the moments and cross-moments of the system states pointed out the desirability of finding an approximation to this highly nonlinear estimate at the possible expense of some accuracy.

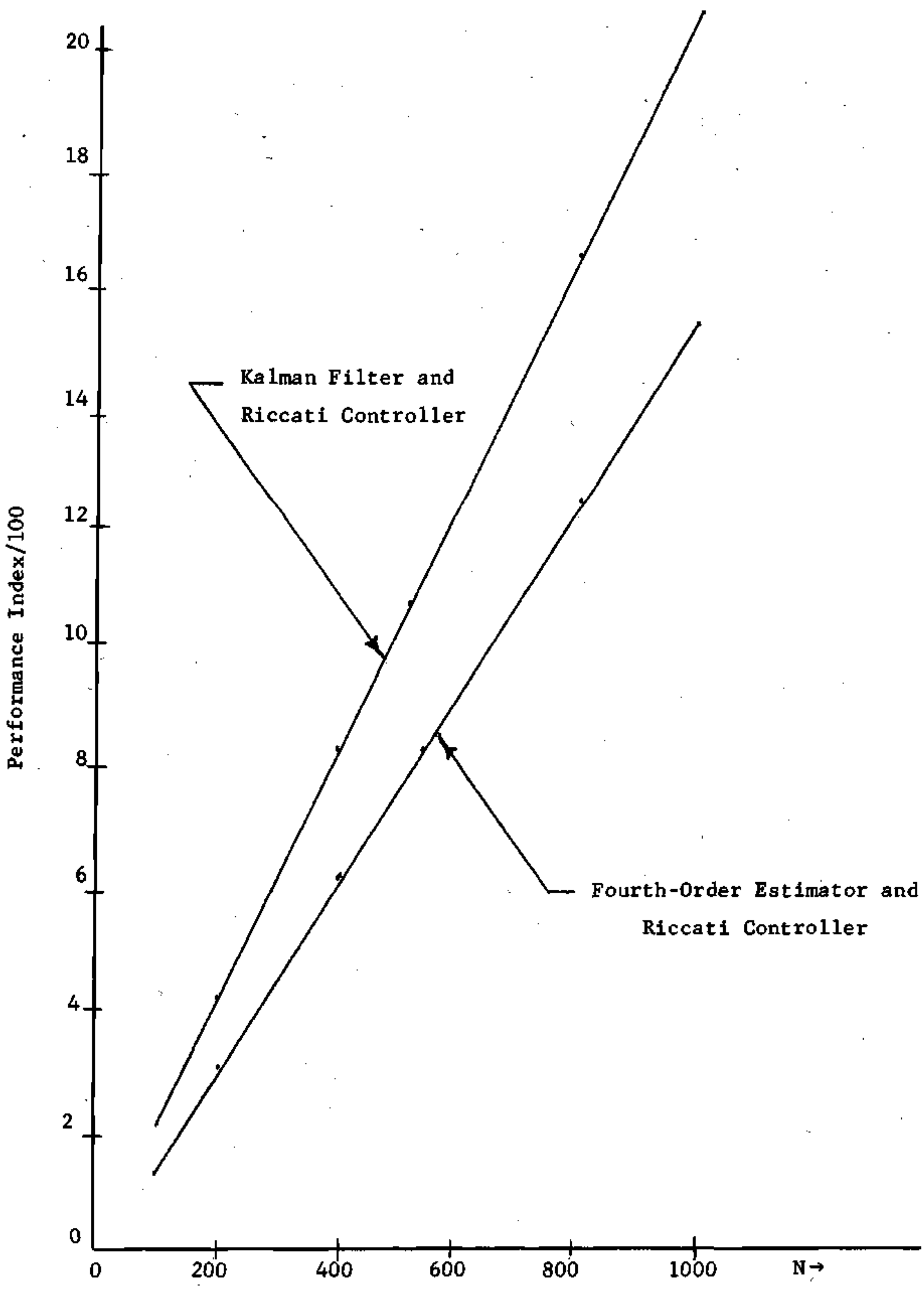

Figure 2.3. Plot of System Performance

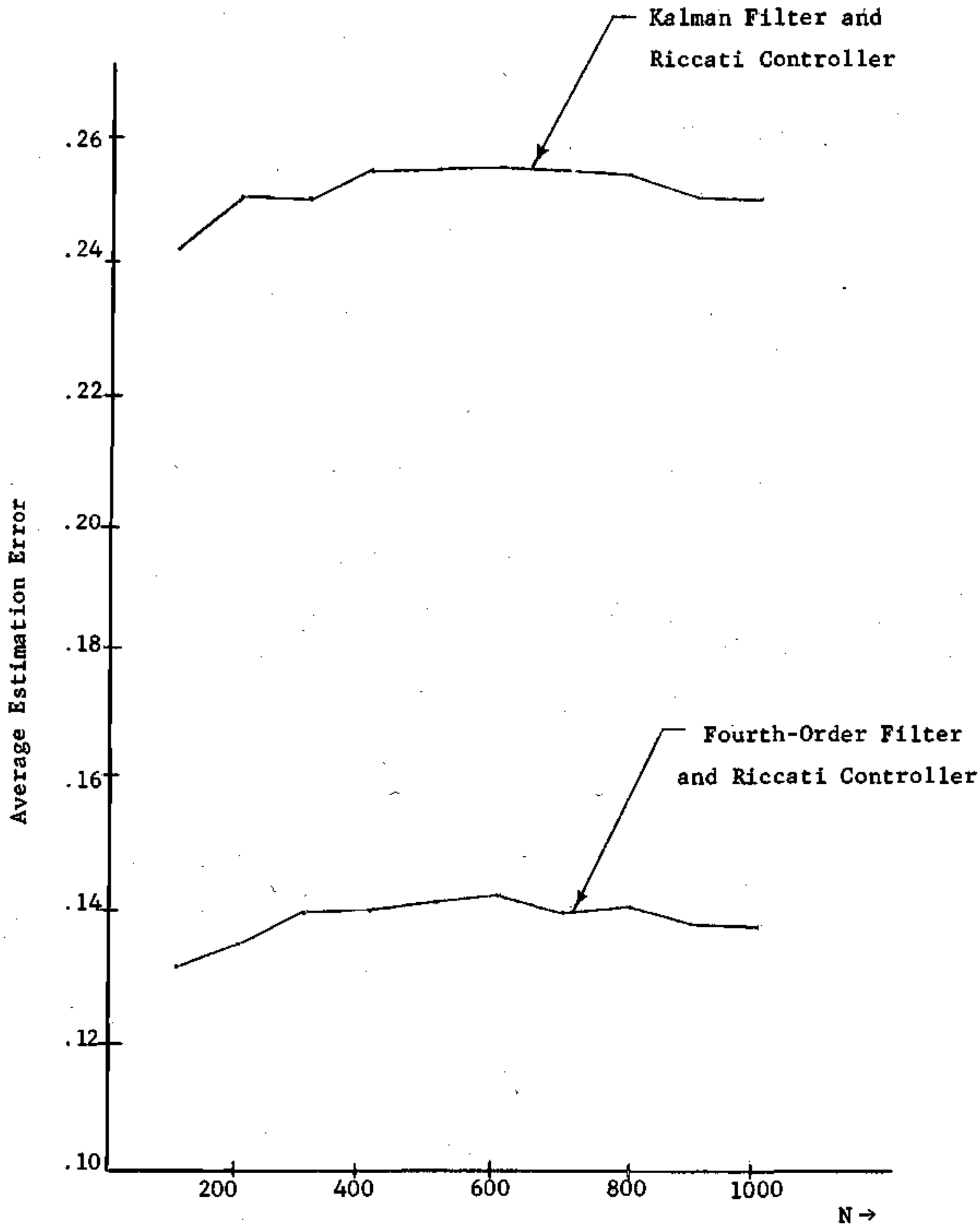

Figure 2.4. Estimation Error for the Nearly Optimal System

The method of analysis was to examine the geometric consideration found in the first term of $\hat{x}_{k+1}$. Consider the density distribution of $x_{k+1}z_{k+1}$, as shown in Figure 2.5. The term $\hat{x}_{k+1} = \frac{1}{2}(\ell_{k+1} + f_{k+1})$ which represents the first two terms of (2.20) appears on this figure as the dotted line. Three approximations of the estimate were generated by using the expected value of $f_{k+1,0}$ and $\ell_{k+1,0}$. One may write $f_{k+1,0}$ as

$$f_{k+1,0} = Af_k + u_k + w_{max} \tag{2.24}$$

Taking the expected value of (2.24)

$$E[f_{k+1,0}] = A\ E[f_k] + E[u_k] + w_{max} \tag{2.25}$$

But $E[u_k] = K_k E[\hat{x}_k] = 0$, and $E[f_k] \approx 1$ for the designated noises. Therefore

$$E[f_{k+1,0}] = A(1) + w_{max} \tag{2.26}$$

which for $A = .1$ is $E[f_{k+1,0}] = 1.1$. Similarly, $E[\ell_{k+1,0}] = -1.1$. Constructing these points on the density diagram of $x_{k+1}z_{k+1}$, the approximate curve appears as shown in Figure 2.5.

Least-Squares Approximation

By considering this figure, values of $\hat{x}_{k+1}$ corresponding to the first two terms of (2.20) for several z_{k+1} can be obtained. The method of least squares is to determine a polynomial that fits the data points. For the data points $\hat{x}_{k+1} = 0,.1,-.1,1.05,-1.05$, respectively, a polynomial

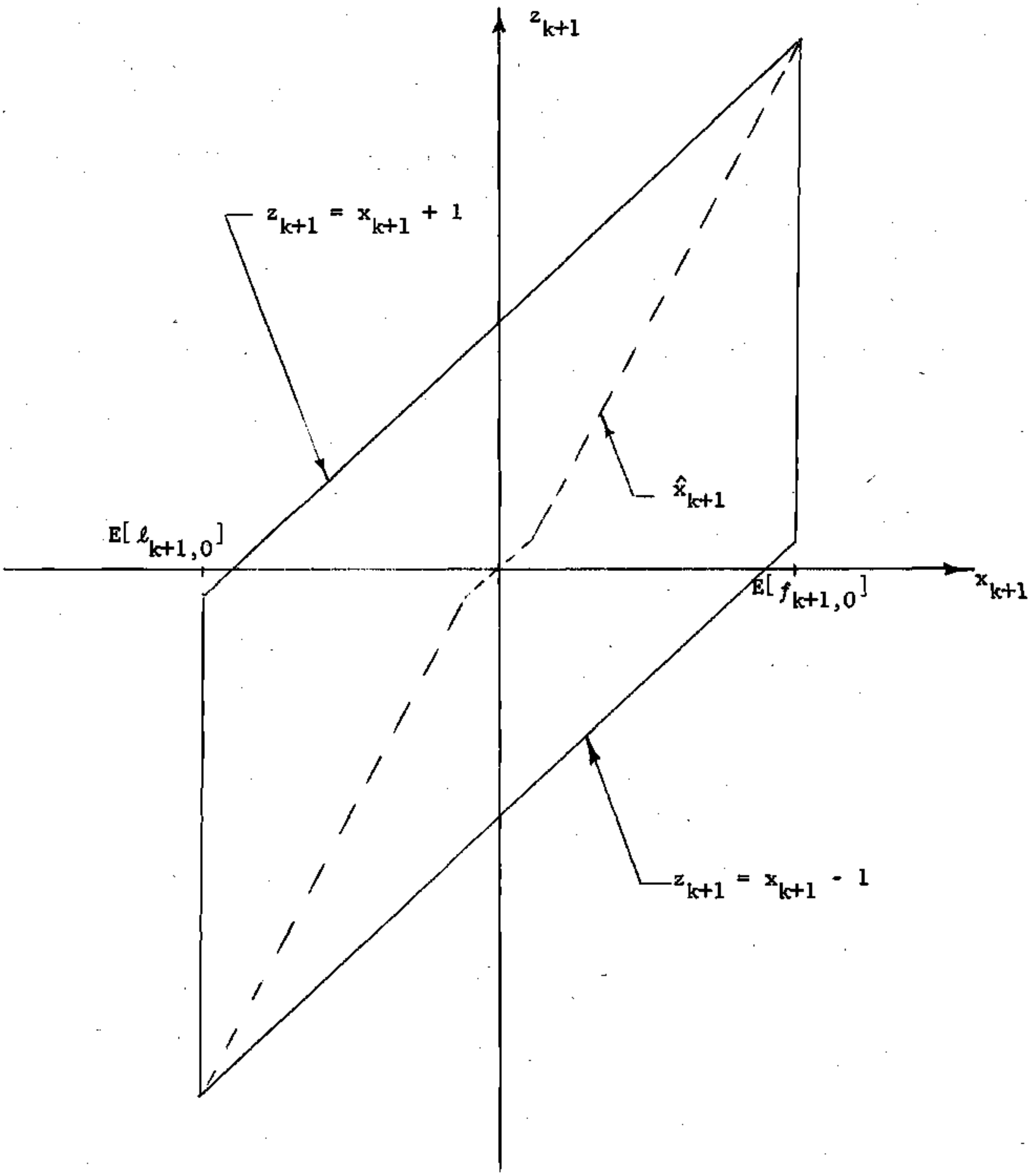

Figure 2.5. Approximation of Estimate $\hat{x}_{k+1}$ and Distribution Region $x_{k+1}z_{k+1}$

of the form

$$\hat{x}_{k+1} = a + bz_{k+1} + cz_{k+1}^2 + dz_{k+1}^3 \tag{2.27}$$

was selected. By forming the residuals and solving the normal equations in [39], the values of a,b,c,d were determined. The resulting polynomial was

$$\hat{x}_{k+1} = z_{k+1} - .119\ z_{k+1}^3 \tag{2.28}$$

By checking the data points it is seen that this approximation is a good one.

Straight Line Approximation

Another approximation to the estimate of the plant state described by the first two terms of (2.20) was the straight line approximation. By considering the two extreme data points $z_{k+1} = 2,-2$ and $\hat{x}_{k+1} = 1.05,-1.05$, respectively, a straight line relationship between $\hat{x}_{k+1}$ and z_{k+1} was determined as

$$\hat{x}_{k+1} = 0.525\ z_{k+1} \tag{2.29}$$

Obviously because of symmetry, (2.29) necessarily satisfies the data point at the origin.

Hyperbolic Approximation

The third approximation was obtained by inspection in Figure 2.5. The relationship between z_{k+1} and $\hat{x}_{k+1}$ was seen to resemble the sum of a hyperbolic sine and a hyperbolic tangent. This resemblance was also apparent in (2.20) because of the increasing powers of f_{k+1} and ℓ_{k+1}.

Therefore, the approximation of the estimate of the plant state was arbitrarily selected as

$$\hat{x}_{k+1} \doteq \frac{1}{2} [\sinh z_{k+1} + \tanh z_{k+1}] \tag{2.30}$$

which may be expressed as

$$\hat{x}_{k+1} \cong z_{k+1} - \frac{z_{k+1}^3}{12} - \frac{15\, z_{k+1}^5}{240} - \frac{17\, z_{k+1}^7}{630} \tag{2.31}$$

Approximations to the Suboptimal Filter

Consider again Example 1 with the performance index (2.3). Each of the above three approximations was compared to the Kalman filter and the fourth-order nonlinear filter described in Appendix I. As seen in Figure 2.6, the straight-line approximation showed no improvement in estimation accuracy while the least-squares polynomial and the hyperbolic approximations yielded improved estimates. As in Example 1, the effect of the three different approximations on the performance index (2.3) for D = 0 was negligible.

Summary and Conclusions

The application of the method of moments to Bayes-law computation for the Linear Stochastic Control Problem has been presented. The application of the computational method to a specific example was given. It was seen that the increase in performance for a cost criterion which penalized estimation error as well as control variables was significant. However, the principal improvement was due to the increased estimation accuracy.

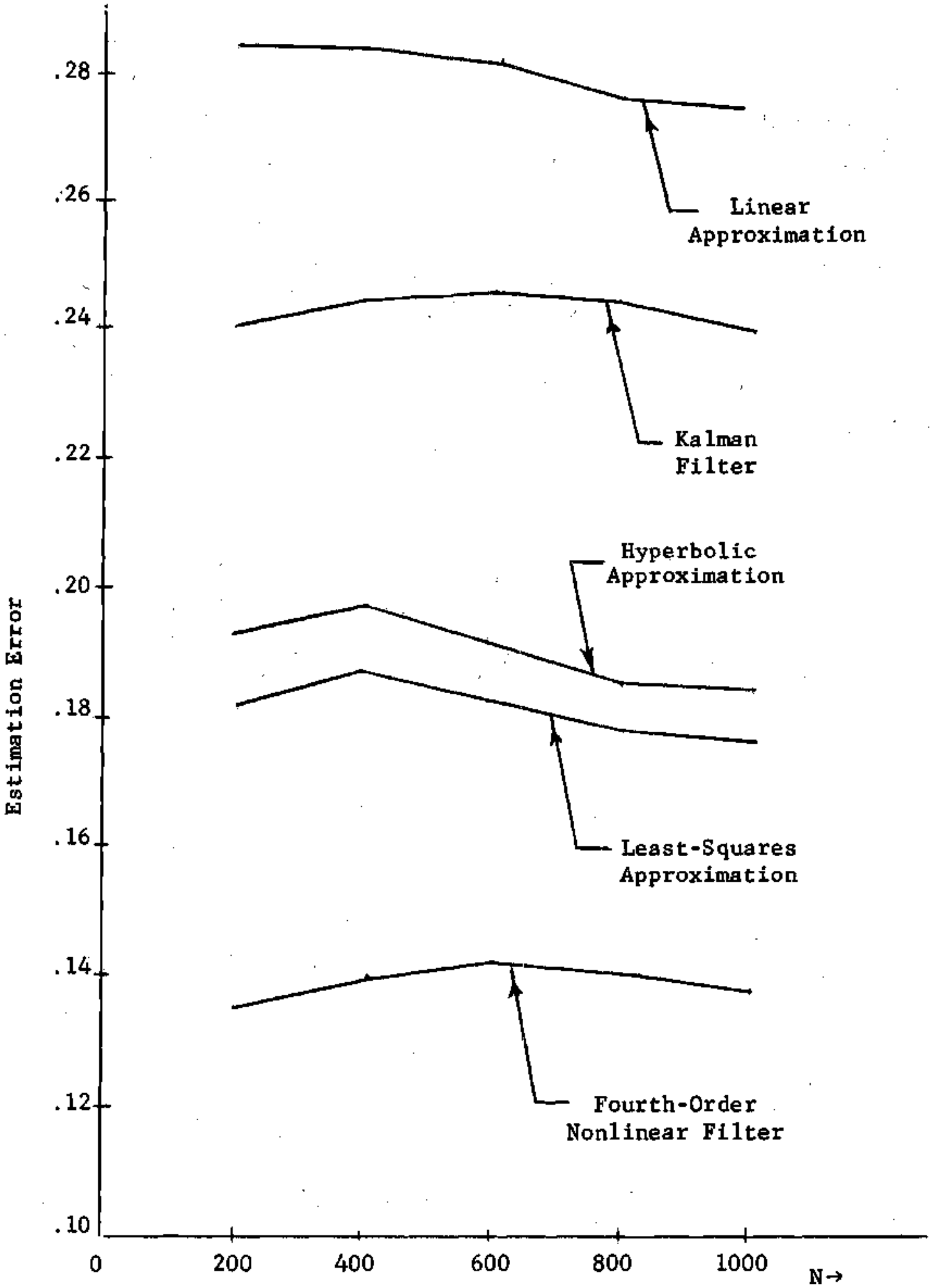

Figure 2.6. Estimation Error for the Three Approximations to the Fourth-Order Filter

Several approximations to the suboptimal filter were given which might reduce computational difficulties. It was seen that two of the approximations, i.e. the least-squares polynomial and the hyperbolic approximation, reduced the cost criterion somewhat, while the straight line approximation showed no improvement.

The significance of the development of the nearly optimal scheme is that it gives an accurate basis to which one may compare newly developed approximate solutions for higher order linear systems. Moreover, the algorithms developed may also be applied to an approximate analysis of nonlinear stochastic systems.

CHAPTER III

APPROXIMATE ANALYSIS OF NONLINEAR STOCHASTIC SYSTEMS

Background

Exact solutions for the combined estimation and control of nonlinear stochastic systems are not yet available. The traditional approaches used to analyze these stochastic control problems have been approximate. This chapter illustrates an application of improved estimation by Bayes-law computation for the closed-loop nonlinear stochastic control problem.

The approach utilized here was to assume that the deterministic nominal solution to the nonlinear difference equations of the plant provided a good approximation to the actual system behavior, i.e. the deviations from the nominal solution could be described by a set of linear difference equations. For those cases where this approximation is valid, the Separation Theorem may be applied to the resulting set of linear difference equations. This permits the estimation of the deviation from the deterministic nominal solution to be used in the formation of a variational feedback controller. The purpose of this chapter is to present the results of that application of the combined estimation and control algorithms of Chapter II in the analysis of nonlinear stochastic systems.

Derivation of the Variational Equations

The plant for the nonlinear stochastic control problem is specified as

$$x_{k+1} = f(x_k, u_k) + w_k \tag{3.1}$$

where x_k represents the plant state at the kth sampling instant, u_k represents the control supplied to the plant at the kth sampling instant, and w_k is the nongaussian noise input to the plant. A linear measurement of the plant state corrupted by noise is available as

$$z_k = x_k + v_k \tag{3.2}$$

where v_k is the nongaussian measurement disturbance. The system was to be controlled to minimize the performance index measured by

$$J = \frac{1}{2} E\left\{ \sum_{k=0}^{N} Qx_k^2 + Ru_k^2 + D(x_k - \hat{x}_k)^2 \right\} \tag{3.3}$$

The scalar system was considered although the resulting equations may be applied with modification to the vector case.

A method of feedback control about the optimal trajectory which minimized the deviation from the nominal trajectory and control was developed. The linearized variational equations about the nominal trajectories were determined first. The plant state and control were described as

$$x_k = \bar{x}_k + \delta x_k \tag{3.4}$$

$$u_k = \bar{u}_k + \delta u_k \tag{3.5}$$

where $\bar{x}_k$ and $\bar{u}_k$ represent the nominal plant state and control, respectively, at the kth sampling instant. Using δx_k and δu_k, the variations from the nominal state and nominal control, one is able to determine perturbational difference equations. Expanding the state equation (3.1) in a Taylor Series, one obtains

$$x_{k+1} = f(\bar{x}_k+\delta x_k,\bar{u}_k+\delta u_k) + w_k = f(\bar{x}_k,\bar{u}_k) + \left.\frac{\partial f(x_k,u_k)}{\partial x_k}\right|_{\bar{x}_k} \delta x_k \tag{3.6}$$

$$+ \left.\frac{\partial f(x_k,u_k)}{\partial u_k}\right|_{\bar{u}_k} \delta u_k + w_k + \text{Higher Order Terms}$$

Recognizing that $\bar{x}_{k+1} = f(\bar{x}_k,\bar{u}_k)$ and neglecting all higher order terms beyond the first, (3.6) becomes

$$x_{k+1} = \bar{x}_{k+1} + \left.\frac{\partial f(x_k,u_k)}{\partial x_k}\right|_{\bar{x}_k} \delta x_k + \left.\frac{\partial f(x_k,u_k)}{\partial u_k}\right|_{\bar{u}_k} \delta u_k + w_k \tag{3.7}$$

From (3.4) one may define

$$\delta x_{k+1} = \left.\frac{\partial f(x_k,u_k)}{\partial x_k}\right|_{\bar{x}_k} \delta x_k + \left.\frac{\partial f(x_k,u_k)}{\partial u_k}\right|_{\bar{u}_k} \delta u_k + w_k \tag{3.8}$$

$$\delta x_{k+1} = A_k \delta x_k + B_k \delta u_k + w_k \tag{3.9}$$

which is governed by a new performance index given by

$$J_{var} = \frac{1}{2} \; E\left\{ \sum_{k=0}^{N} Q\delta x^2 + R\delta u^2 \right\} \tag{3.10}$$

Extensions of the Basic Technique

At this point one may identify (3.9) and (3.10) as representing the same linear stochastic control problem analyzed in Chapter II. Thus, the solution to the variational control δu_k may be determined from the Separation Theorem as

$$\delta\bar{x}_k = E[\delta x_k | \delta Z_k] \tag{3.11}$$

$$\delta u_k = -R^{-1}B_k[P_{k+1}^{-1} + B_k R^{-1} B_k]^{-1} A_k \delta\hat{x}_k$$

$$P_k = Q + A_k[P_{k+1}^{-1} + B_k R^{-1} B_k]^{-1} A_k \tag{3.12}$$

$$P_{k_f} = 0$$

The perturbational measurement δz_k is

$$\delta z_k = z_k - \bar{x}_k \tag{3.13}$$

which may be shown to be

$$\delta z_k = x_k - \bar{x}_k + v_k \tag{3.14}$$

$$= \delta x_k + v_k$$

Therefore, after evaluating the variational control (3.12), the overall control of the actual plant is determined from (3.5). The approximate method of analyzing the nonlinear stochastic system is represented in Figure 3.1. The estimate $\delta\hat{x}_k$ of the perturbed state may then be determined by using the estimation algorithms described in Chapter II.

A Nonlinear Example

The algorithms for the approximate analysis previously presented were applied to a particular nonlinear example. Comparisons were made with a Kalman filter used in the perturbation loop. The nonlinear system model was given by

$$x_{k+1} = 0.995x_k + .0025x_k^3 - .00035x_k^5 + .01u_k + w_k$$

with Q = R = 2. The measurement noise v_k was uniform on (-1,1) and the plant noise w_k had a density function given by

$$f_w(w) = \begin{cases} \frac{11}{2} \quad w^{10} & \text{when } |w| < 1 \\ 0 & \text{otherwise} \end{cases}$$

A plot of the system nonlinearity is seen to have the characteristic shown in Figure 3.2.

By observing Figures 3.3 - 3.5 and Table 3.1, the improvement in estimation error and performance is evident. Figure 3.3 demonstrates that the estimation error obtained was considerably lower using the fourth-order filter in the variational loop as opposed to using the Kalman filter. This

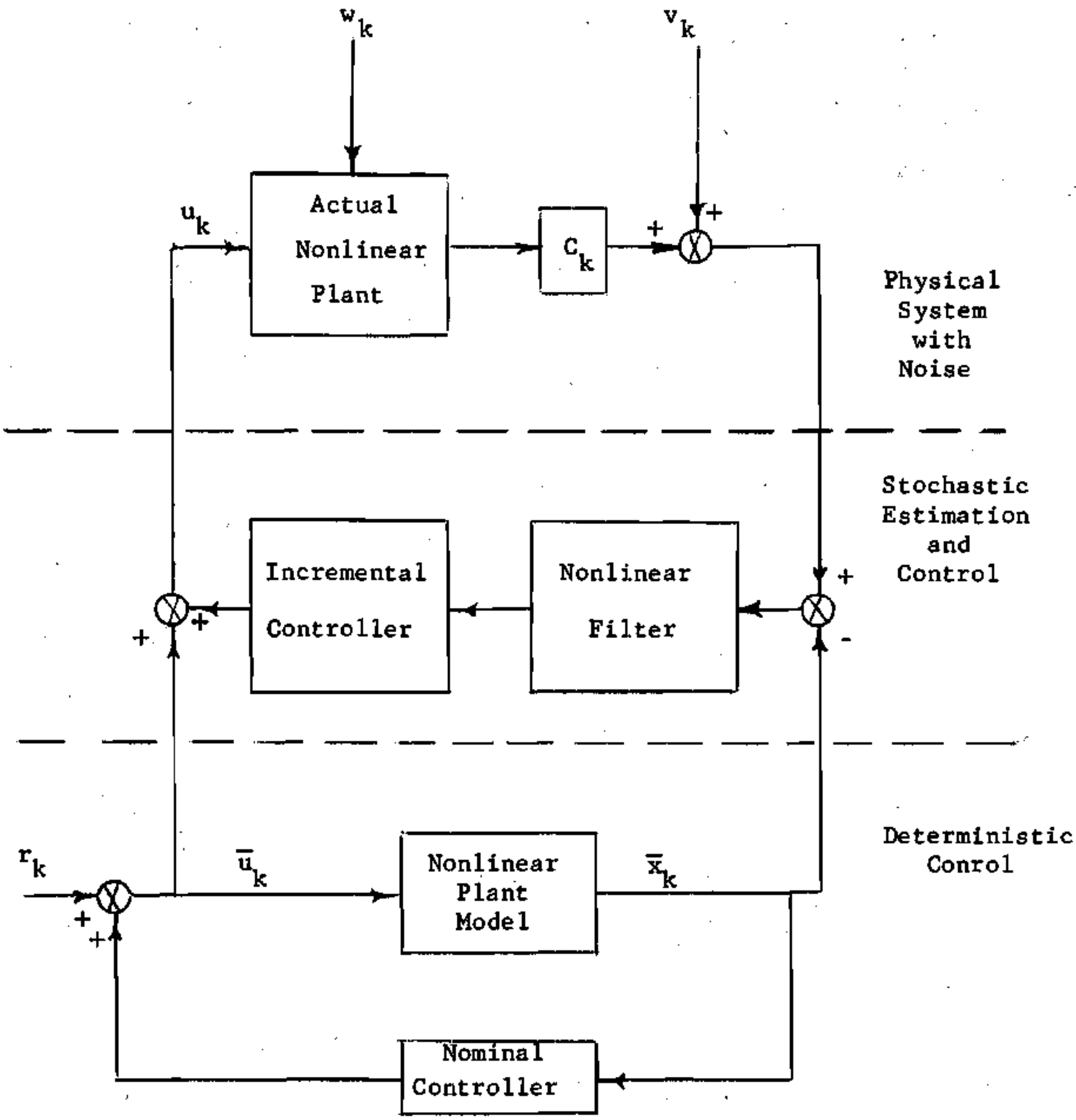

Figure 3.1. The Nonlinear Stochastic Control Problem.

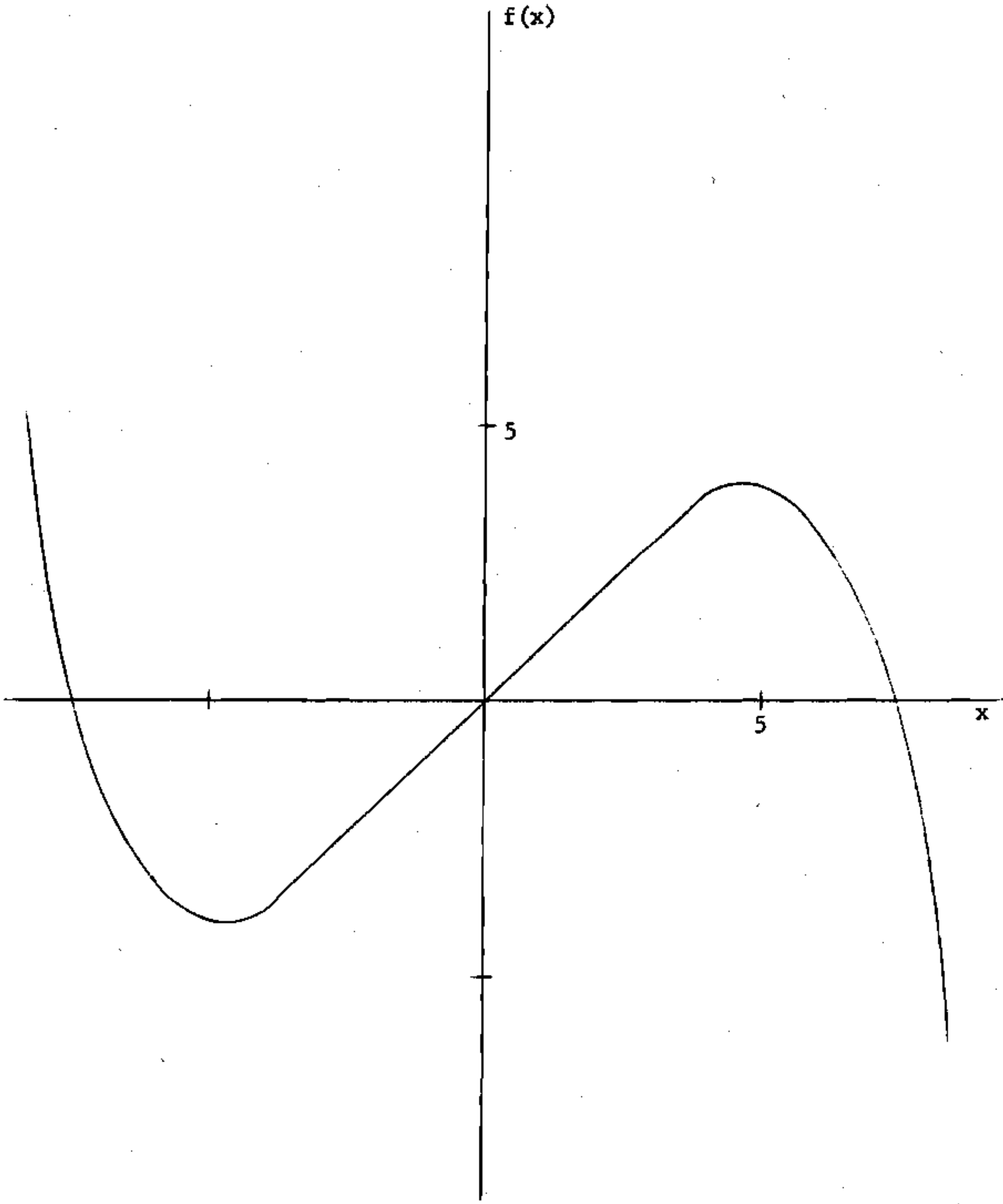

Figure 3.2. System Nonlinearity.

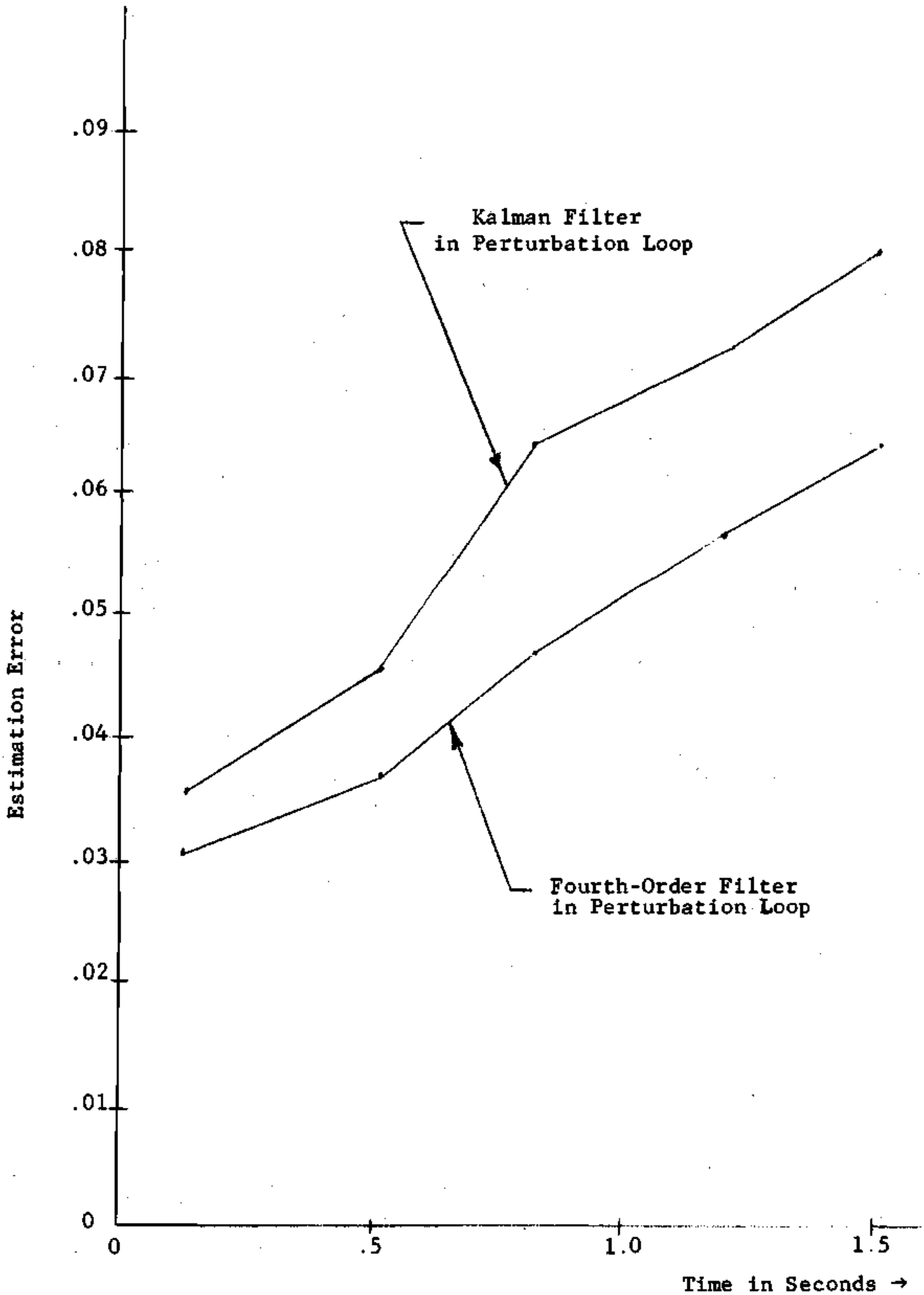

Figure 3.3. Estimation Error Versus Time for Nonlinear System.

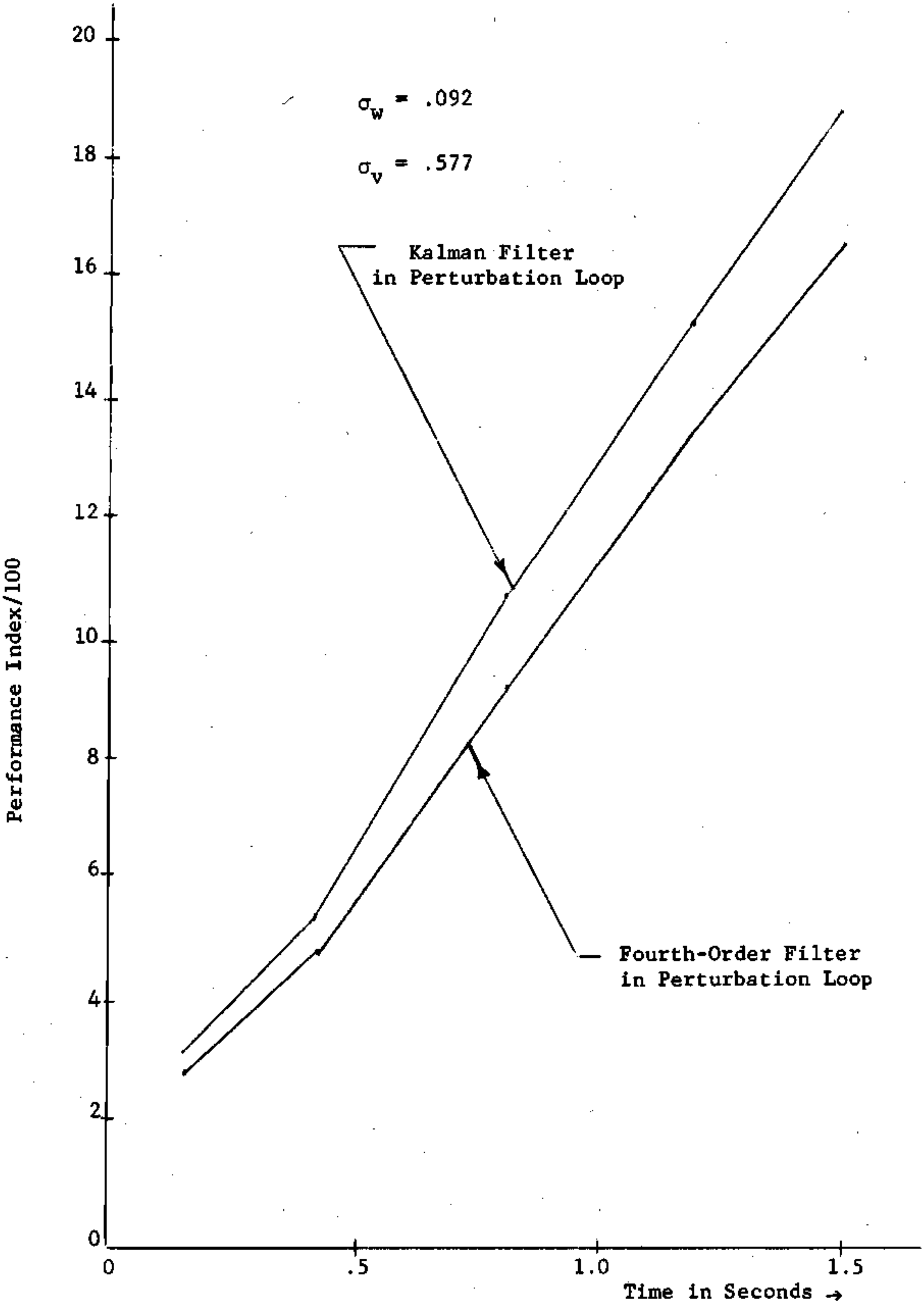

Figure 3.4. Performance Index Versus Time for Nonlinear System.

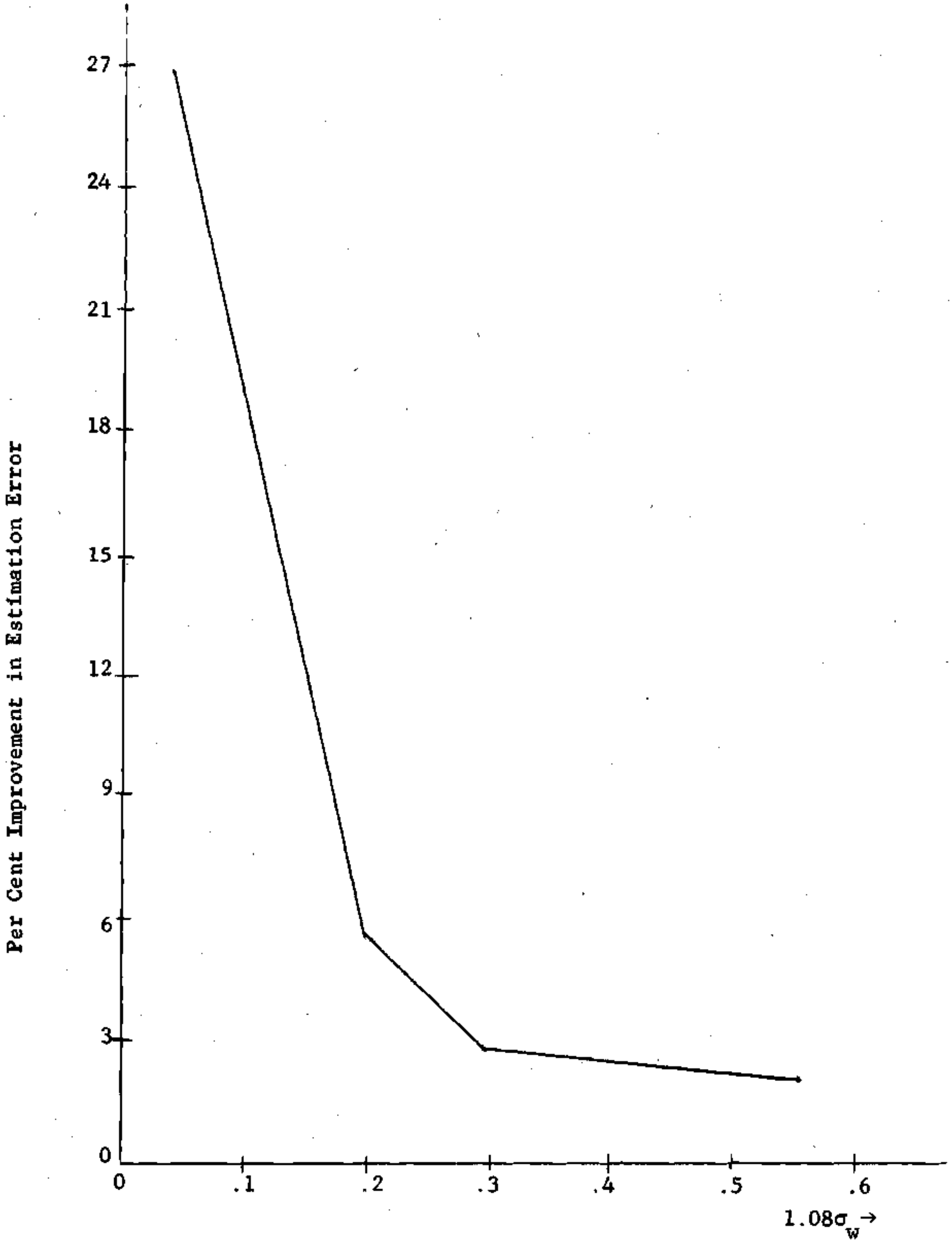

Figure 3.5. Per Cent Improvement in Estimation by Using a Fourth-Order Filter Rather than the Kalman in the Perturbation Loop.

reduction is also reflected in Figure 3.4 which shows improvement in performance as measured by (2.3). This improvement again primarily results from improvement in estimation error. This result supports the conclusions drawn in Chapter II concerning the relative insensitiveness of the quadratic performance index to improvement in estimation. Table 3.1 and Figure 3.5 illustrate reduction in estimation error for various ranges of plant noise. As the plant noise increased, the improvement decreased. This decrease occurred because the linear perturbation assumption was no longer valid for higher noises. Consequently, no estimator in a linear perturbation loop will behave adequately in this range because the variations from the nominal trajectory are no longer linear.

Table 3.1. Estimation Errors for Various σ_w

$1.08\ \sigma_w$	.05	.1	.2	.3	.545
Kalman Filter in Perturbation Loop	.0256466	.0794721	.473700	1.5463169	8.6842
Fourth-Order Filter in Perturbation Loop	.0186825	.0644985	.446892	1.5033834	8.4899

A Different Variational Performance Index

For linear perturbations about a deterministic nominal trajectory, it had been suggested in [21] that a different weighting matrix Q be used in the variational performance index. This new Q is specified as

$$Q_{var} = Q + \lambda_{k+1} \left. \frac{\partial^2 f(x_k,u_k)}{\partial x_k^2} \right|_{\bar{x}_k,\bar{u}_k} \tag{3.15}$$

where λ_{k+1} is the adjoint variable in the optimization problem and Q is the weighting matrix for the nonlinear plant state. Using this new variational Q, the estimation and performance of the system are degraded as seen in Table 3.2.

Table 3.2. Degradation for Q of (3.15) in Variational Performance Index

	Estimation Error Original Q	Estimation Error New Q	J Using Original Q	J Using New Q
Kalman Filter in Perturbation Loop	0.079	0.084	18.74	19.43
Fourth-Order Filter in Perturbation Loop	0.064	0.068	16.50	17.03

A New Nominal Trajectory

It was also observed that the use of a properly selected nominal trajectory other than the deterministic nominal trajectory improved system performance [40]. This new nominal trajectory was selected to optimize the Kalman-Bucy filter gain while shaping the trajectory to minimize the performance index (2.3).

By inspection of Table 3.3, one may observe the effect of selecting the new nominal trajectory. The dramatic improvement demonstrated by the new selection of a trajectory is misleading because the specific case illustrated was a high noise case which invalidated the linear perturbation approach. The new trajectory is useful in certain special cases. The effect of the shaped trajectory on the variational Riccati controller gain can be seen in Figure 3.6. This trajectory also stabilized the linear perturbational equations.

Table 3.3. Performance Improvement for New Nominal with $\sigma_w = 0.500$ and $\sigma_v = 3.14$

	Performance Index Shaped Trajectory	Performance Index Deterministic Trajectory
Kalman Filter in Perturbation Loop	14.24	19.80
Fourth-Order Filter in Perturbation Loop	14.32	19.84

Conclusions

The combined estimation and control algorithms developed in Chapter II were applied in an approximate analysis of nonlinear stochastic systems. By assuming linear perturbations about a deterministic nominal trajectory, a variational feedback control scheme was developed. At low noise levels, improvement was noticeable when the fourth-order filter was used in the perturbational loop. At higher noise levels, the linear perturbation

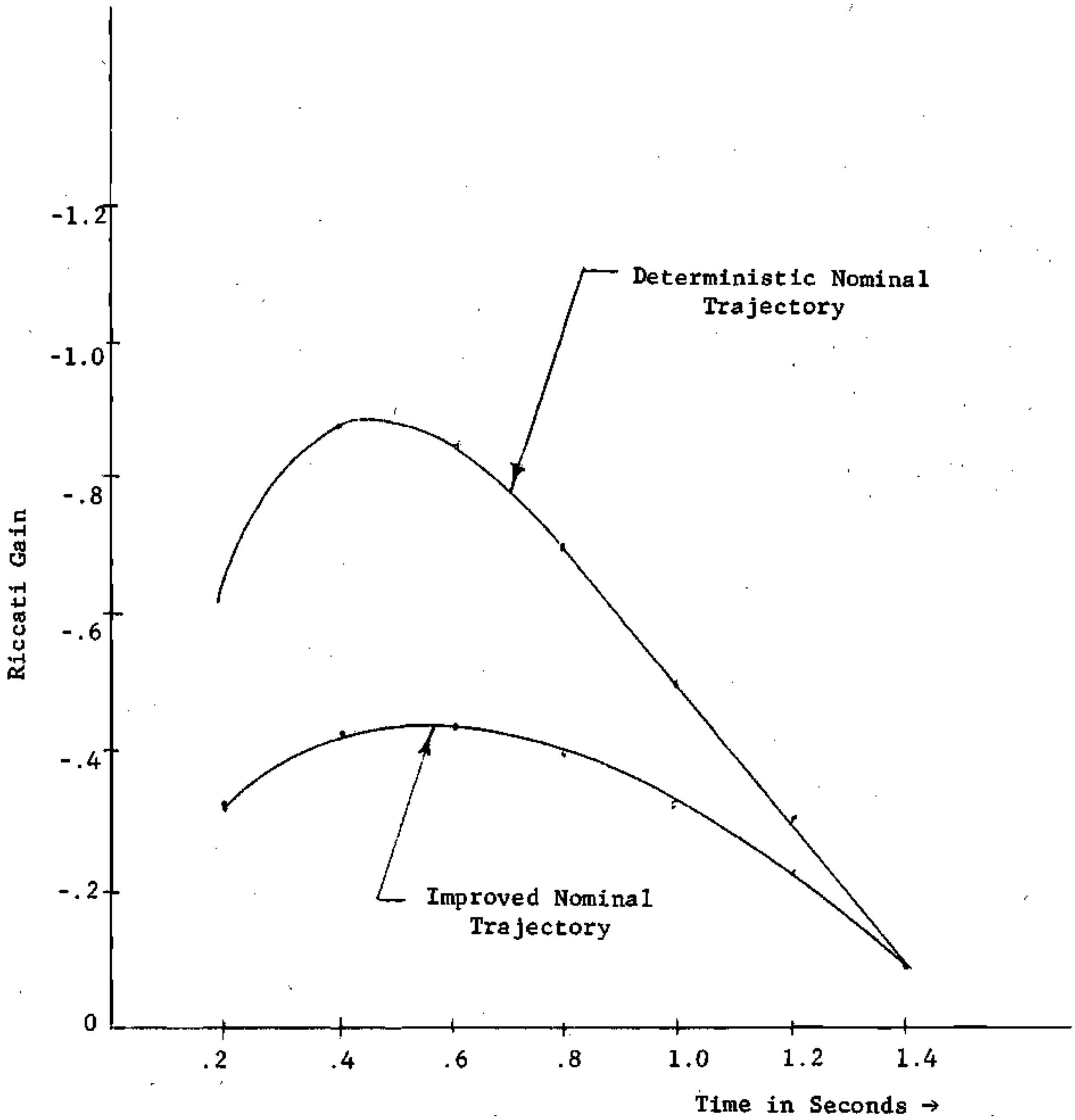

Figure 3.6. The Variation of Gain with Nominal Trajectories.

scheme was not valid, and the resulting performance was poor. A new nominal trajectory was shown to have a desirable effect for those high noise cases. However, estimation accuracy was unacceptable at all high noise cases.

CHAPTER IV

SENSITIVITY ANALYSIS AND SPECIFIC OPTIMAL CONTROLLER DESIGN

Introduction

The sensitivity of the estimation and control algorithms to variations in input data is a practical consideration in the use of the algorithms. The effects of incorrect modeling for Kalman-Bucy filtering and erroneous input data are well known [41,42]. Throughout this research it was necessary to supply the fourth-order estimator with the first four moments and the bounds of both noise disturbances. Because of this dependency, the question of sensitivity to data on moments and bounds was investigated.

Another sensitivity problem was implicit in a specific optimal controller formulation. The problem was to select a controller to use with a fixed, nonoptimal, nonlinear filter that yielded a better estimate of the plant state and improved system performance. The sensitivity problem occurred in forming an estimate with inaccurate data. A new estimate was obtained by operating on the estimate of a nonoptimal filter as though the measurement had been available.

Sensitivity to Noise Variations

The sensitivity in estimation for the fourth-order nonlinear filter presented in Chapter II for incorrect noise statistics was investigated by a specific example. The example was the same as in Chapter II, i.e.,

$$x_{k+1} = 0.1x_k + u_k + w_k \tag{4.1}$$

with the linear measurement given as

$$z_{k+1} = x_{k+1} + v_{k+1} \tag{4.2}$$

The original data supplied to the estimator given from Chapter II as

$$\hat{x}_{k+1} = \frac{\sum_{i=1}^{4} a_i(f_{k+1}^{\ i+2} + \ell_{k+1}^{\ i+2})/i+2}{\sum_{i=1}^{4} a_i(f_{k+1}^{\ i+1} + \ell_{k+1}^{\ i+1})/i+1} \tag{4.3}$$

consisted of the upper and lower bounds and the first four moments of w and v. The random variables w_k and v_k were both assumed to be bounded and the density functions for both variables were obtained from

$$f_y(y) = \begin{cases} \frac{\rho + 1}{2} \ y^{\rho} & |y| \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

where y in (4.4) is a dummy variable used to represent either w or v. The density of w was originally given for $\rho = 10$, i.e., approximately an inverted bell-shaped distribution on the range (-1,1). The density of v was specified as uniform, which corresponds to $\rho = 0$, and was also bounded on (-1,1). The corresponding moments of w and v with $\rho = 10$ and $\rho = 0$, respectively, were given by

$$m_y^i = \int_{-1}^{1} y^i \ \frac{\rho + 1}{2} \ y^{\rho} \ dy \tag{4.5}$$

$$m_y^i = \begin{cases} \dfrac{\rho + 1}{\rho + i + 1} & \text{when } i \text{ is even} \\ 0 & \text{when } i \text{ is odd} \end{cases}$$

The sensitivity in estimation was studied by varying the input noise and measurement noise density functions, while keeping the second moment fixed. This requirement then changed the bounds on the random variables w and v. However, the original bounds and moments, which were then incorrect, were supplied to the estimator.

Variations in the Plant Noise

The sensitivity to incorrect statistics in the plant noise w was considered first. The density function of the measurement noise was uniform on (-1,1) and held constant as the density function of the plant noise was varied. The original plant noise was specified by (4.4) for $\rho = 10$. The two nonzero moments of the input data were given as $m_w^2 = 11/13$ and $m_w^4 = 11/15$ with bounds of (-1,1). The plant noise was allowed to vary from $\rho = 10$ to $\rho = 0$. The input data supplied to the estimator was assumed to be the same as the original data, although actually the second moment was held constant and the bounds and other moments were changed. The cases that were considered are shown in Table 4.1.

The plots of estimation error for the various density functions of w is shown in Figure 4.1. One can see from this figure the estimation error increased as the input noise became closer to uniform. This increase in estimation error resulted because the bounds were becoming larger than the input bound to the filter. One may conclude that estimation accuracy is highly dependent on the correct bounds and moments of the plant noise.

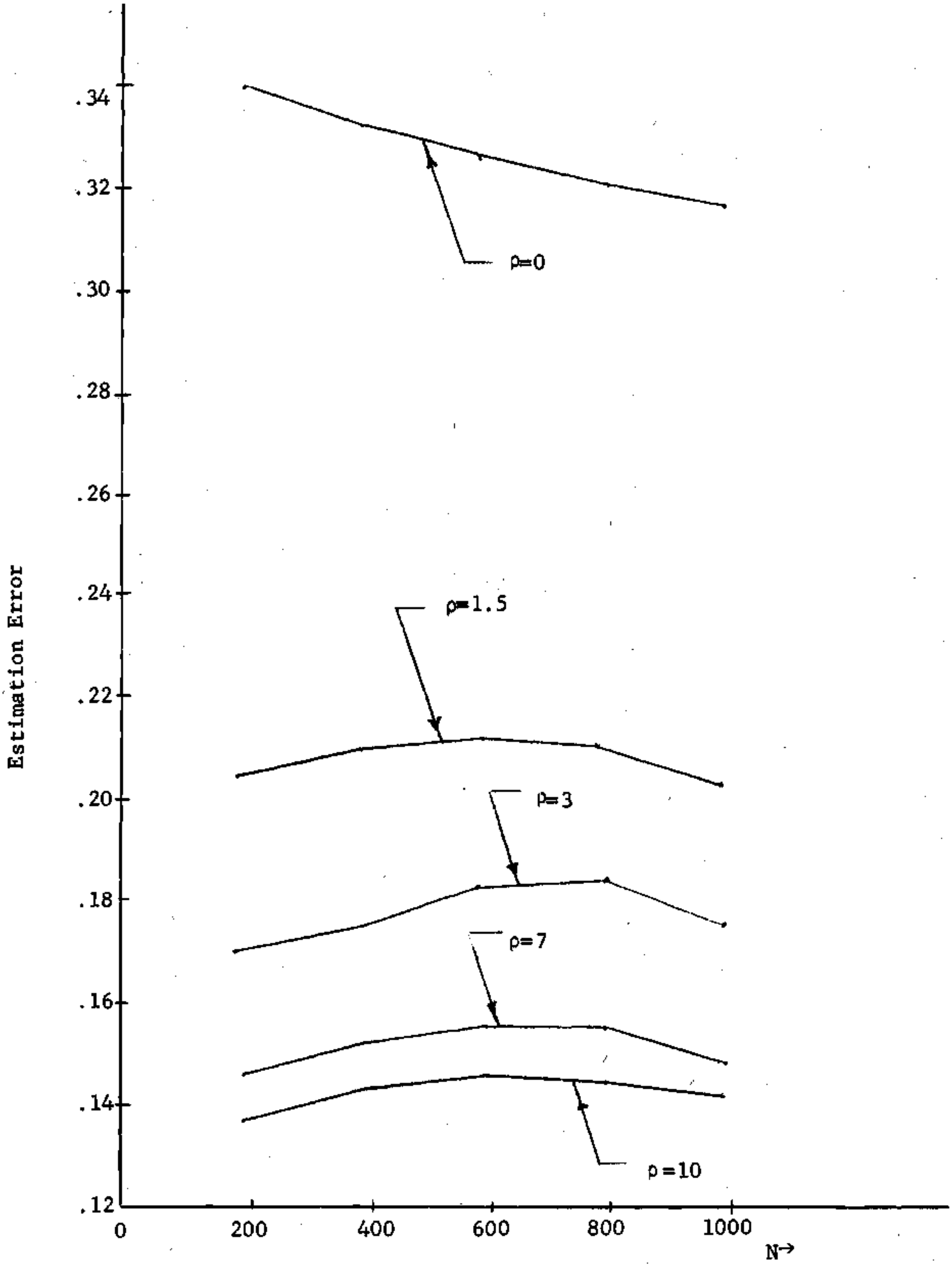

Figure 4.1. Estimation Error for Incorrect Plant Noise.

Table 4.1. Variations in Plant Noise

ρ	Bounds	m_x^2	m_x^4
$\rho = 10$	(-1,1)	.847	.734
$\rho = 7$	(-1.03,1.03)	.847	.834
$\rho = 3$	(-1.13,1.13)	.847	1.19
$\rho = 1.5$	(-1.23,1.23)	.847	1.67
$\rho = 0$	(-1.67,1.67)	.847	5.76

Variations in the Measurement Noise

The sensitivity in estimation to incorrect statistics in the measurement noise was then considered. The plant noise density function specified by (4.4) with $\rho = 10$ was held constant. The original measurement was uniformly distributed on (-1,1). The two nonzero moments of the input data were given as $m_v^2 = 1/3$ and $m_v^4 = 1/5$ with bounds (-1,1). The measurement noise was allowed to vary from $\rho = 0$ to $\rho = 10$. The input data was again identical to the original data although the bounds and the other moments, except the second, were changed. Table 4.2 shows the cases that were considered. The data supplied to the filter consisted of bounds (-1,1), and $m_v^2 = .33$, $m_v^4 = .20$. The results of this variation in noise show from Figure 4.2 and Table 4.3 that the estimation error decreased until $\rho = 2$. This indicates that the magnitude of the bounds on the noise decreased rapidly until this point and, therefore, a decrease in estimation error resulted. However, for $\rho = 3,7$, and 10, the bounds

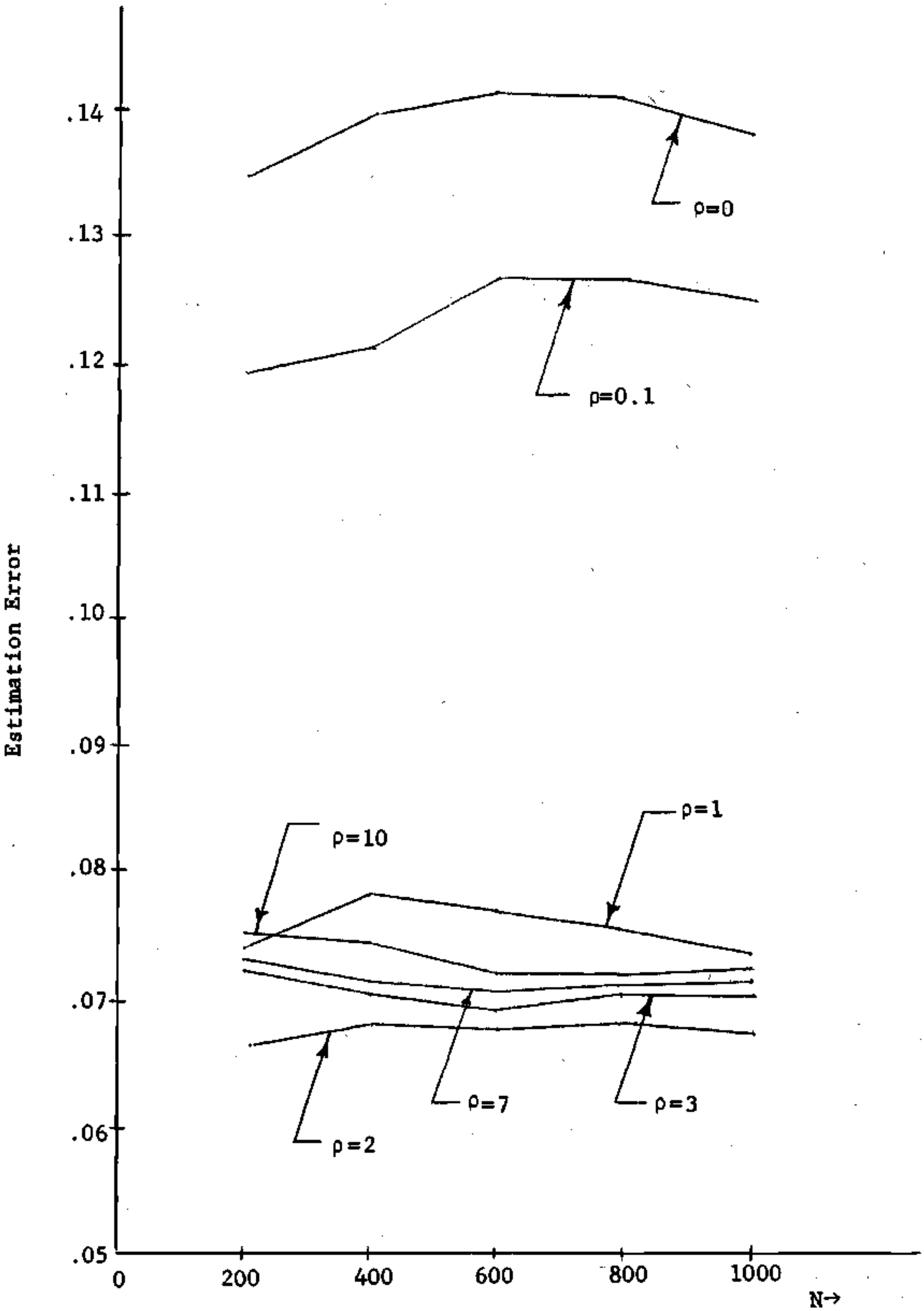

Figure 4.2. Estimation Error for Incorrect Measurement Statistics.

Table 4.2. Variations in Measurement Noise

ρ	Bounds	m_v^2	m_v^4
0	(-1,1)	.33	.200
0.1	(-96,.96)	.33	.170
1	(-.82,82)	.33	.091
2	(-.746,.746)	.33	.062
3	(-.707,.707)	.33	.050
7	(-.645,.645)	.33	.034
10	(-.626,.626)	.33	.030

remained nearly the same with the density function becoming more concentrated on the bounds. Consequently, the estimation error increased in these regions, and one would expect even further increases in error with higher values of ρ.

Specific Optimal Controller Design

In most specific optimal control and estimation schemes, the parameters of either fixed configuration filters or controllers are adjusted to optimize system performance. This section presents a basis for selecting a specific optimal controller to optimize the performance of a stochastic system having a fixed nonlinear filter.

Problem Formulation

For the linear system given in Chapter II as

Table 4.3. Estimation Error for Various Measurement Noises

ρ	0	0.1	1	2	3	7	10
Estimation Error at N = 1000	.138	.126	.073	.067	.070	.072	.0725

$$x_{k+1} = A_k x_k + u_k + w_k \tag{4.6}$$

subject to the performance index (2.3), a nonoptimal estimate $\hat{x}_{k+1}$ of the plant state was assumed to be available. This nonoptimal nonlinear estimate was determined by

$$\hat{x}_{k+1} = \frac{1}{2} (f_{k+1} + \ell_{k+1}) \tag{4.7}$$

where f_{k+1} and ℓ_{k+1} represented the upper and lower bounds of $x_{k+1}|Z_{k+1}$. The problem was to determine a controller other than the Riccati controller to improve the performance of the system according to (2.3).

Development of the Controller

The nearly optimal combined control and estimation scheme for the linear system was given in Chapter II as (2.5) and (2.17). It is seen that for M = 4 the optimal control is expressed as

$$u_{k+1} = K_{k+1}\hat{x}_{k+1} = K_{k+1} \frac{\sum_{i=0}^{4} a_i (f_{k+1}^{i+2} - \ell_{k+1}^{i+2})/i+2}{\sum_{i=0}^{N} a_i (f_{k+1}^{i+2} - \ell_{k+1}^{i+2})/i+2} \tag{4.8}$$

Dividing the polynomials in (4.8), one obtained

$$u_{k+1} = K_{k+1}\Big[0.5\ (f_{k+1} + \ell_{k+1}) + \frac{a_1}{a_0}\ \frac{f_{k+1}^2}{12} + \frac{a_1}{a_0}\ \frac{\ell_{k+1}^2}{12} \tag{4.9}$$

$$- \frac{a_2}{a_0}\ \frac{f_{k+1}^2\ \ell_{k+1}}{6} - \frac{a_2}{a_0}\ \frac{\ell_{k+1}^2\ f_{k+1}}{6} + \ldots \Big]$$

The problem was to find a controller to optimize system performance using a nonoptimal estimate of the plant state (4.7). If a Riccati controller were selected with time-varying gain K_k, then the control would be given as

$$u_{k+1} = K_{k+1}\hat{x}_{k+1} = K_{k+1}[0.5(f_{k+1} + \ell_{k+1})] \tag{4.10}$$

One recognizes this term as being the first term of the nearly optimal expansion (4.9). However, since the measurement z_{k+1} was not available, the terms f_{k+1} and ℓ_{k+1} in the optimal expansion were not known exactly. By selecting the controller for the nonoptimal estimator as

$$u_{k+1} = K_k\hat{x}_k + K_k\Big(\frac{a_1}{a_0}\ \frac{f_{k+1}^2}{12} + \frac{a_1}{a_0}\ \frac{\ell_{k+1}^2}{12} - \frac{a_2}{a_0}\ \frac{f_{k+1}^2\ \ell_{k+1}}{6}$$

$$- \frac{a_2}{a_0}\ \frac{\ell_{k+1}^2\ f_{k+1}}{6} + \ldots\Big) \tag{4.11}$$

where f_{k+1} and ℓ_{k+1} were determined by using the approximation

$$z_{k+1} = \hat{x}_{k+1}, \tag{4.12}$$

then a better performance could be obtained. This expression resulted from taking the expected value of (4.2).

The net effect of this solution was the formation of a new estimate and the use of this estimate through a Riccati controller. This idea is illustrated in Figure 4.3.

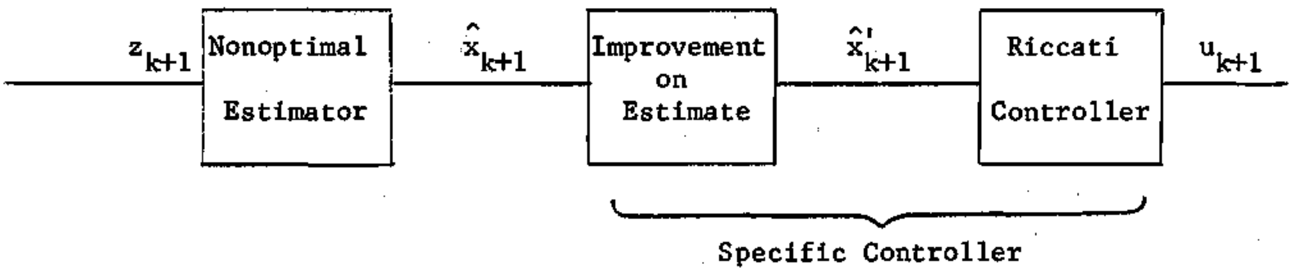

Figure 4.3. Specific Optimal Controller.

Referring to Figure 4.3, one may identify the new estimate $\hat{x}'_{k+1}$ as being formed from a nonoptimal estimate rather than from the measurement. This new estimate may be written as

$$\hat{x}'_{k+1} = \hat{x}_{k+1} + \frac{a_1}{a_0}\,\frac{f_{k+1}^2}{12} + \frac{a_2}{a_0}\,\ell_{k+1}^2 - \frac{a_2}{a_0}\,f_{k+1}^2\,\ell_{k+1} + \cdots \tag{4.13}$$

where the approximation $\hat{x}_{k+1} = z_{k+1}$ has been used. Therefore, the specific controller problem has provided information about the sensitivity of the optimal filter of Chapter II by demonstrating the increase in estimation error with an inaccurate measurement.

The example in the following section demonstrated the effectiveness of the specific optimal controller selection.

Simulation Results for a Particular Example

Consider the linear system from the example in Chapter II given by

$$x_{k+1} = 0.1x_k + u_k + w_k \tag{2.12}$$

It was assumed that a linear measurement of the plant state corrupted by noise was not available. Rather, a nonlinear nonoptimal estimate was used with the specific optimal controller. The density function of v was uniform on (-1,1) and the density function of w was

$$f_w(w) = \begin{cases} \frac{11}{2} \quad w^{10} & \text{when } |w| \leqq 1 \\ 0 & \text{elsewhere.} \end{cases}$$

From Figure 4.4, it is evident evident that the average estimation error was reduced by the specific controller by 40%. However, the specific optimal controller did not perform as well as a fourth-order filter operating on the measurement. Thus, the sensitivity of the estimator to incorrect input data was demonstrated. Again, the effect on system performance with D = 0 was negligible in all cases.

Conclusions

The results of this chapter have shown the sensitivity of the

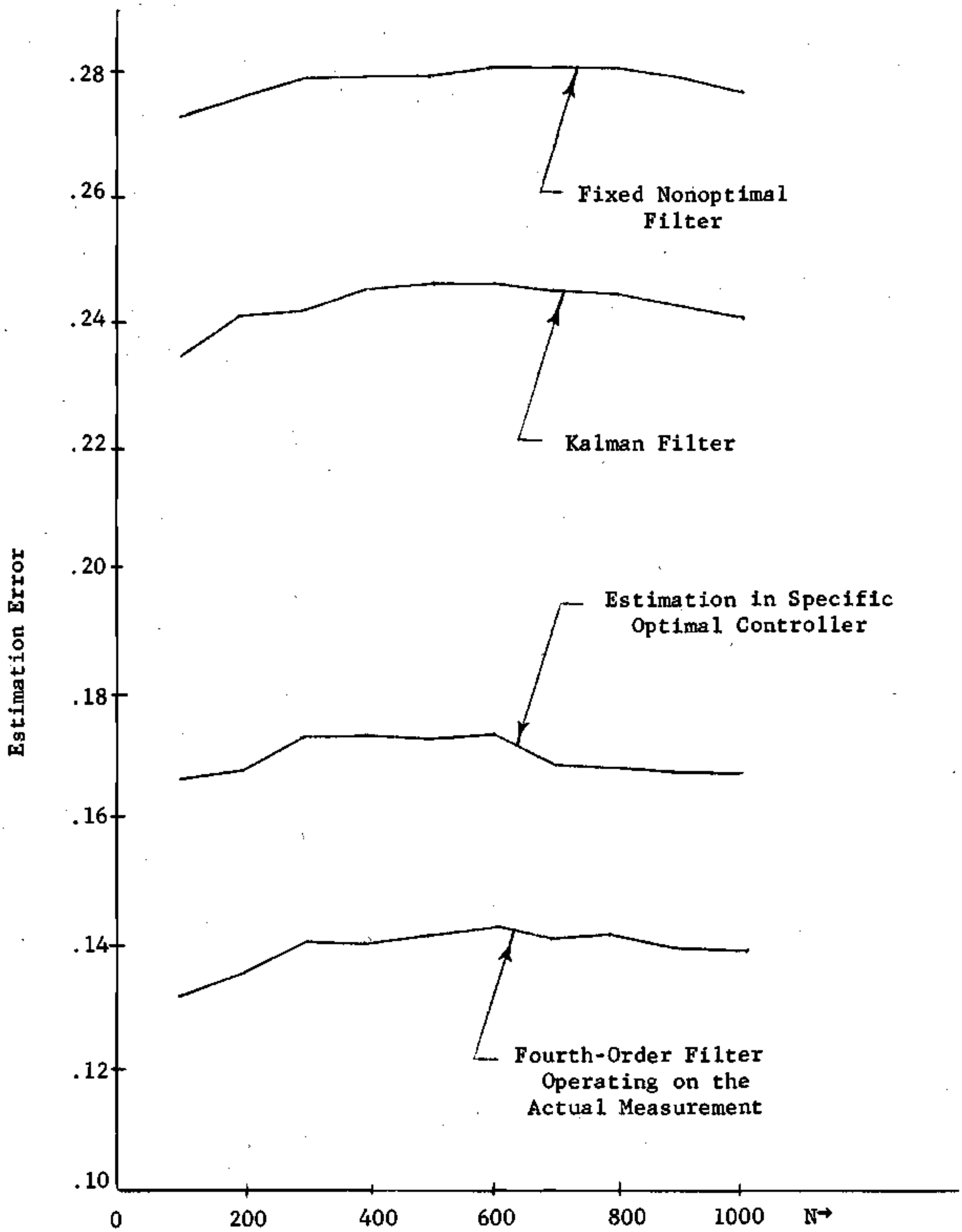

Figure 4.4. Estimation Error for Specific Controller.

fourth-order filter in closed-loop operation to variations in the input data. Specifically, as the incorrect bounds became large, the estimation error deteriorated rapidly and might possibly diverge.

A method of selecting a specific controller was shown to be effective in improving performance as measured by (2.3). This problem also contained information on performance sensitivity, since the controller operated without accurate knowledge of the measurement.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

This dissertation has considered the combined estimation and control of time-varying, discrete stochastic systems. For a linear stochastic system, the separate optimization of the estimation and control functions resulted in an overall optimal system as shown by the Separation Theorem derived in Chapter I. Estimation by Bayes-law computation was then developed by application of the moment technique. An approximate analysis of nonlinear stochastic systems was presented, assuming linear perturbations about the deterministic nominal trajectory, which permitted the formation of estimation and control algorithms.

Conclusions

The estimation and control algorithms developed for linear stochastic systems were nearly optimal because of the basic assumptions made in forming the moments. Only the linear term was used to represent the estimate in obtaining cross-moments of the state and estimate. Because of the computational difficulties involved in calculating the moments using only the single term in the estimate expansion, the use of additional terms for the moment calculations was not feasible.

The results of using the estimation and control algorithms on a particular linear stochastic system demonstrated several points of interest. The fourth-order filter, when used with a Riccati controller as specified by the Separation Theorem, decreased the mean-square estimation error and reduced system cost as measured by (2.3). This improvement over the commonly used Kalman filter and Riccati controller combinations was due to the additional information about the plant and measurement noises which was supplied to the fourth-order filter. The decrease in system cost was primarily caused by including the estimation error in the performance index. From the observation, it is evident that the fourth-order algorithm possesses important advantages for systems where estimation is an important consideration. However, if the performance is measured by the standard quadratic performance index, then the Kalman filter and Riccati controller appear to be adequate in most applications. However for higher order systems, it is possible that the fourth-order filter might reduce the standard quadratic performance index sufficiently so that the Kalman filter is no longer acceptable with the Riccati controller. This situation could result because the feedback controller operates on more than one state requiring greater precision in estimation.

Although the machine execution time was unusually high for the Bayes-law computation in the first-order example considered, an important advantage was realized by its use. For low-order systems, the Bayes-law scheme is a nearly optimal algorithm to which one may compare the accuracy of faster estimation schemes. This advantage is particularly useful in selecting approximate algorithms for use on higher-order systems. These approximate algorithms are necessary because of the extreme difficulty in extending the Bayes-law scheme to higher-order systems.

The extension of the basic algorithms to a nonlinear stochastic system demonstrated an approximately optimal method of handling a problem which has not been solved. The method presented was valid only at low noise levels, and acceptable accuracy was obtained for those cases. The extension to the problem of selecting a specific optimal controller was useful in delineating the structure and sensitivity considerations of the estimation and control algorithms.

In summary, the most attractive feature of the combined estimation and control algorithms was the improvement in estimation demonstrated for both linear and nonlinear stochastic systems.

Recommendations for Further Work

Three problems related to this thesis research are suggested for further study. The first recommendation is that the consideration of bounds in the Bayes-law computation be eliminated. Secondly, to avoid detailed moment calculations, another representation of the density function is suggested. Finally, it is recommended that the method be extended to higher-order systems by selecting a suitable specific filter.

The sensitivity analysis of the fourth-order filter revealed that the knowledge of the correct state and noise bounds was essential in achieving accurate estimation. This sensitivity to data on the bounds and moments might be reduced by the addition of higher-order terms to the polynomial expansion. However, the implementation of these terms would result in increased computation.

An important restriction in the extension of the combined estimation and control algorithms developed is the difficulty in theoretically evaluating the expected values of cross-products of states and estimates of the states. To eliminate this difficulty, it is suggested that the polynomial density representation be removed and replaced with some other form of density representation. Specifically, the representation of a density by a sum of gaussians [32] seems to offer important advantages.

Finally, another possibility in the extension to higher-order systems is to select a specific nonlinear filter which compares favorably with the Bayes-law computation for the first-order systems.

This thesis has presented new algorithms for the combined estimation and control of nongaussian stochastic systems. The resulting algorithms have been compared favorably with the Kalman filter and Riccati controller combination, which are optimal for gaussian disturbances. The problems outlined in this section are recommended as fruitful areas for further work.

APPENDIX I

DERIVATION OF THE BAYESIAN ESTIMATOR

This appendix presents the derivation of the least-mean-square estimator by Bayes-law computation. the approach differs from the development in [28,29] because of the feedback control in the moment calculations, and the resulting estimator presented here is not adaptive.

The least-mean-square estimate for the linear system given in (2.1) may be expressed as

$$\hat{x}_{k+1} = E[x_{k+1}|Z_{k+1}] = \int_{\ell_{k+1}}^{f_{k+1}} x_{k+1} f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1})\,dx_{k+1} \tag{A1.0}$$

where ℓ_{k+1} and f_{k+1} represent the lower and upper bounds, respectively, of $x_{k+1}|Z_{k+1}$. By the Bayesian rule, one has

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) \cdot f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)}{f_{z_{k+1}|Z_k}(z_{k+1}|Z_k)} \tag{A1.1}$$

The density function $f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1})$ can be written as

$$f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) = f_{v_{k+1}}(v_{k+1} = z_{k+1} - x_{k+1}) \tag{A1.2}$$

The denominator of (A1.1) may be expressed as

$$f_{z_{k+1}|Z_k}(z_{k+1}|Z_k) = \int_{\ell_{k+1}}^{f_{k+1}} f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)\,dx_{k+1} \tag{A1.3}$$

Therefore, one needs only to evaluate $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$ to obtain the density $x_{k+1}|Z_k$, which may be expressed as a polynomial. However it is necessary first to determine certain bounds on $x_{k+1}|Z_k$.

Bounds and Moments of $x_{k+1}|Z_k$

Assuming the bounds and moments of $x_k|Z_k$ have been found in the last sampling period, one may evaluate the bounds of $x_{k+1}|Z_k$ as

$$\ell_{k+1,0} = A_k \ell_k + u_k + w_{min} \tag{2.11}$$

$$f_{k+1,0} = A_k f_k + u_k + w_{max} \tag{2.12}$$

where it may be assumed that $A_k > 0$ without loss of generality. Furthermore let

$$\begin{aligned} \ell_k &: \text{ lower bound of } x_k|Z_k \\ f_k &: \text{ upper bound of } x_k|Z_k \\ \ell_{k+1,0} &: \text{ lower bound of } x_{k+1}|Z_k \\ f_{k+1,0} &: \text{ upper bound of } x_{k+1}|Z_k \end{aligned} \tag{A1.4}$$

One may obtain the moments of $x_{k+1}|Z_k$ by taking the expected value of (1.1) as

$$\begin{aligned} E[x^i_{k+1}|Z_k] &= E[A_k x_k + u_k + w_k|Z_k]^i \\ &= E\left\{ \sum_{j=0}^{i} w_k^{i-j} \sum_{p=0}^{j} (A_k x_k|Z_k)^{j-p} (u_k|Z_k)^p \right\} \end{aligned} \tag{A1.5}$$

$$i=1,2,\ldots,M$$

To evaluate the above expression, the form of the estimate is required. This expression will be assumed known for the remainder of this section and will be developed in Appendix II.

Approximation of $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$

To approximate a density function by a polynomial of the form $f_y(y) \cong \sum_{i=0}^{N} a_i P_i(y)$, where $y \in (-1,1)$, the coefficients a_i must be selected to minimize the mean square error of this approximation. Specifically, from [28] for a density function

$$f_y(y) = \sum_{i=0}^{4} a_i y^i \tag{A1.6}$$

the coefficients a_i for i=1,2,3, and 4 are

$$\begin{aligned}
a_0 &= 7.3828125 m_y^{(4)} - 8.203125 m_y^{(2)} + 1.7578125 \\
a_1 &= -13.125 m_y^{(3)} + 9.375 m_y^{(1)} \\
a_2 &= -73.828125 m_y^{(4)} + 68.90625 m_y^{(2)} - 8.203125 \\
a_3 &= 21.875 m_y^{(3)} - 13.125 m_y^{(1)} \\
a_4 &= 86.1328125 m_y^{(4)} - 73.828125 m_y^{(2)} + 7.3828125
\end{aligned} \tag{A1.7}$$

To approximate $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$ by a polynomial as given by (A1.6), a transformation must be made such that the new variable will be distributed on (-1,1). Such a transformation is given as

$$s_{k+1} = \frac{2}{f_{k+1,0} - \ell_{k+1,0}}\left(x_{k+1} - \frac{f_{k+1,0} + \ell_{k+1,0}}{2}\right) \tag{A1.8}$$

where s_{k+1} is the new random variable. Let

$$c = \frac{2}{f_{k+1,0} - \ell_{k+1,0}} \tag{A1.9}$$

$$d = -\frac{f_{k+1,0} + \ell_{k+1,0}}{f_{k+1,0} - \ell_{k+1,0}}$$

Having determined a polynomial representation of $f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)$, one is able to determine the density function of $x_{k+1}|Z_k$.

The Bounds and Density Function of $x_{k+1}|Z_{k+1}$

Since it is known that $x_{k+1}|Z_k \in (\ell_{k+1,0}, f_{k+1,0})$ and assuming that v is uniform $\in$ (-1,1) without any loss in generality, the distribution of the joint random variable $z_{k+1}x_{k+1}|Z_{k+1}$ may be represented by a parallelogram as shown in Figure A1.1.

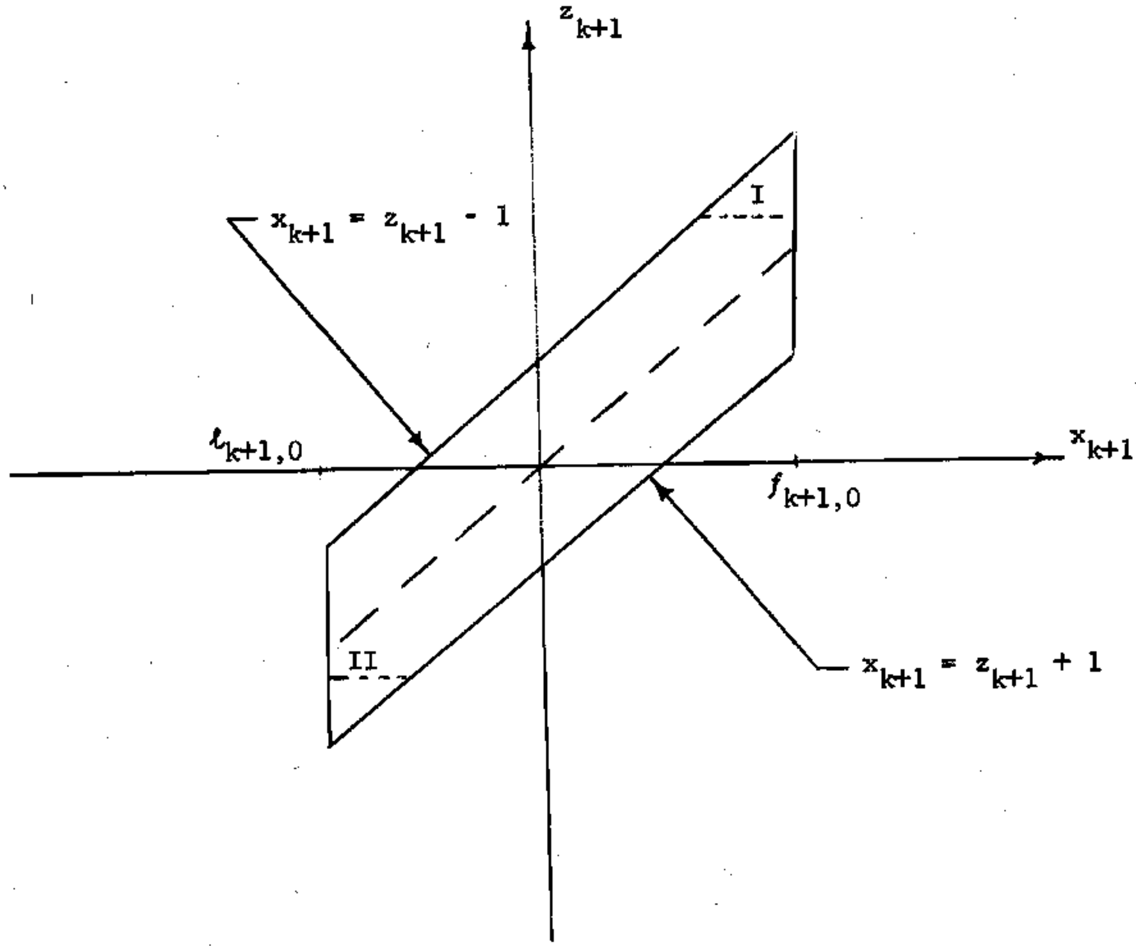

Figure A1.1 Distribution Region of $x_{k+1}z_{k+1}$.

then (A1.8) becomes

$$s_{k+1} = cx_{k+1} + d \tag{A1.10}$$

Taking expected values of (A1.10), the moments of $s_{k+1}|Z_k$ are found as

$$m^{(i)}_{s_{k+1}|Z_k} = \sum_{j=0}^{i} \binom{i}{j} c^{i-j} d^j m^{(i-j)}_{x_{k+1}|Z_k} \tag{A1.11}$$

Thus, the density function of $s_{k+1}|Z_k$ can be approximated as a polynomial

$$f_{s_{k+1}|Z_k}(s_{k+1}|Z_k) \cong \sum_{i=0}^{M} b_i s^i_{k+1} \tag{A1.12}$$

Making use of (A1.10) again, one has

$$f_{x_{k+1}|Z_k}(x_{k+1}|Z_k) = \left|\frac{ds_{k+1}}{dx_{k+1}}\right| f_{s_{k+1}|Z_k}(s_{k+1} = cx_{k+1} + d|Z_k) \tag{A1.13}$$

$$f_{x_{k+1}|Z_k} = c \sum_{i=0}^{M} b_i (cx_{k+1} + d)^i \tag{2.13}$$

Therefore, for M = 4

$$f_{x_{k+1}|Z_k} = \sum_{i=0}^{4} a_i x^i_{k+1} \tag{2.15}$$

where

$$a_0 = b_0 + b_1 d + b_2 d^2 + b_3 d^3 + b_4 d^4$$

$$a_1 = b_1 c + 2b_2 cd + 3b_3 cd^2 + 4b_4 cd^3$$

$$a_2 = b_2 c^2 + 3b_3 c^2 d + 6b_4 c^2 d^2$$

$$a_3 = b_3 c^3 + 4b_4 c^3 d$$

$$a_4 = b_4 c^4$$

Observing the right ends of Lines I and II, it is seen that

$$f_{k+1} = \begin{cases} f_{k+1,0} & \text{when } z_{k+1} \geqq f_{k+1,0} - 1 \\ z_{k+1} + 1 & \text{when } z_{k+1} < f_{k+1,0} - 1 \end{cases} \tag{2.15}$$

Similarly, from the left ends of Lines I and II, one has

$$\ell_{k+1} = \begin{cases} z_{k+1} - 1 & \text{when } z_{k+1} > \ell_{k+1,0} + 1 \\ \ell_{k+1,0} & \text{when } z_{k+1} \leqq \ell_{k+1,0} + 1 \end{cases} \tag{2.16}$$

The density of $x_{k+1}|Z_{k+1}$ can be written from (A1.1) and (A1.3) as

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) \cdot f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)}{\int_{\ell_{k+1}}^{f_{k+1}} f_{z_{k+1}|x_{k+1}}(z_{k+1}|x_{k+1}) f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)\, dx_{k+1}} \tag{A1.14}$$

Making use of the assumption that v_{k+1} is uniformly distributed on (-1,1) and (A1.2), one has

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)}{\int_{\ell_{k+1}}^{f_{k+1}} f_{x_{k+1}|Z_k}(x_{k+1}|Z_k)\, dx_{k+1}} \tag{2.10}$$

Inserting (2.15) into (2.10), one obtains

$$f_{x_{k+1}|Z_{k+1}}(x_{k+1}|Z_{k+1}) = \frac{\sum_{i=0}^{M} a_i x_{k+1}^{i}}{\sum_{i=0}^{M} a_i (f_{k+1}^{i+1} - \ell_{k+1}^{i+1})/(i+1)} \tag{A1.15}$$

Therefore, the moments of $x_{k+1}|Z_{k+1}$ are given for the computation in the next sampling period as

$$E[x^j_{k+1}|Z_{k+1}] = m^{(j)}_{x_{k+1}|Z_{k+1}} = \int_{\ell_{k+1}}^{f_{k+1}} x^j_{k+1} \frac{\sum_{i=0}^{M} a_i x^i_{k+1}}{\sum_{i=0}^{M} a_i (f^{i+1}_{k+1} - \ell^{i+1}_{k+1})/i+1} \tag{A1.16}$$

$$m^j_{x_{k+1}|Z_{k+1}} = \frac{\sum_{i=0}^{M} a_i (f^{i+j+1}_{k+1} - \ell^{i+j+1}_{k+1})/(i+j+1)}{\sum_{i=0}^{M} a_i (f^{i+1}_{k+1} - \ell^{i+1}_{k+1})/(i+1)} \tag{A1.17}$$

for $j=1,2,\ldots,M$

This suboptimal estimate is the conditional mean value, i.e. the first moment of $x_{k+1}|Z_{k+1}$, and may be obtained from (A1.17) by setting $j=1$. Hence, one has

$$\hat{x}_{k+1} = \frac{\sum_{i=0}^{M} a_i (f^{i+2}_{k+1} - \ell^{i+2}_{k+1})/(i+2)}{\sum_{i=0}^{M} a_i (f^{i+1}_{k+1} - \ell^{i+1}_{k+1})/(i+1)} \tag{2.17}$$

Thus the estimate of the plant state is completely specified except for the moments of $x_{k+1}|Z_{k+1}$, which are to be evaluated in Appendix II.

APPENDIX II

CALCULATION OF THE MOMENTS OF $x_{k+1}|Z_k$

The purpose of Appendix II is to derive the moments (A1.7) required in the Bayes-law formulation of the estimate. M=4 was selected arbitrarily because of the ease in implementation and the relative accuracy demonstrated in [28,29].

The moments of $x_{k+1}|Z_k$ are obtained by taking the expected value of (2.1) as seen in (A1.7). By using this expression, one is able to evaluate directly all expected cross-products except those given by

$$E[u_k^i x_k^{4-i}|Z_k] \quad \text{for } i=1,2,3, \text{ and } 4. \tag{A2.1}$$

However, because $u_k = K_k\hat{x}_k$, (A2.1) becomes

$$E[u_k^i x_k^{4-i}|Z_k] = K_k^i E[\hat{x}_k^i x_k^{4-i}|Z_k] \quad \text{for } i=1,2,3,4. \tag{A2.2}$$

To evaluate these expected values, one must first express the estimate $\hat{x}_k$ in terms of the variables of the system. Since from (A2.24) with M=4, x_k may be written as

$$\hat{x}_k = \frac{\sum_{i=0}^{4} a_i(f_k^{i+2} - \ell_k^{i+2})/i+2}{\sum_{i=0}^{4} a_i(f_k^{i+1} - \ell_k^{i+1})/i+1} \tag{A2.3}$$

which may be expanded by long division to give

$$\hat{x}_k = 0.5(f_k + \ell_k) + \frac{a_1 f_k^2}{a_0 12} + \frac{a_1 \ell_k^2}{a_0 12} - \frac{a_2 f_k^2 \ell_k}{a_0\ 6} + \cdots \tag{A2.4}$$

Retaining the first two terms, the expected value of the products in (A2.2) may be expressed as

$$E[\hat{x}_k^i \, x_k^{4-i} | Z_k] = E[(.5[f_k + \ell_k])^i \, x_k^{4-i} | Z_k] \tag{A2.5}$$

It has been shown from earlier considerations that f_k and ℓ_k are functions of the measurement z_k. Again by examination of Figure A1.1, it can be seen that the values of f_k and ℓ_k depend explicitly on four ranges of z_k. The ranges of z_k and the corresponding values of f_k and ℓ_k are given as:

Range A: $z_k < f_{k,0}-1$ and $z_k \geqq \ell_{k,0}+1$ (A2.6)

$$f_k = z_{k+1} \tag{A2.7}$$

$$\ell_k = z_{k+1}$$

Range B: $z_k < f_{k,0}-1$ and $z_k \leqq \ell_{k,0}+1$ (A2.8)

$$f_k = z_{k+1} \tag{A2.9}$$

$$\ell_k = \ell_{k,0}$$

Range C: $z_k \geqq f_{k,0}-1$ and $z_k > \ell_{k,0}+1$ (A2.10)

$$f_k = f_{k,0} \tag{A2.11}$$

$$\ell_k = z_{k+1}$$

Range D: $z_k \geqq f_{k,0}-1$ and $z_k \leqq \ell_{k,0}+1$ (A2.12)

$$f_k = f_{k,0} \tag{A2.13}$$

$$\ell_k = \ell_{k,0}$$

The approach will be to find expressions for $E[x_k^j \hat{x}_k^i]$ for j = 1 and i = 1,2,3, and 4 for all four ranges and then to generalize to the other required cases, i.e. j = 2,3, and 4.

$E[x_k\hat{x}_k|Z_k]$

The expected value of $x_k\hat{x}_k|Z_k$ may be written from (A2.5) as

$$E[x_k\hat{x}_k|Z_k] = 0.5\ E[x_k(f_k+\ell_k)|Z_k] \tag{A2.14}$$

For Range A, (A2.14) becomes

$$E[x_k\hat{x}_k|Z_k] = 0.5\ E[x_k|Z_k(2z_k)] \tag{A2.15}$$

From 2.1 this may be expressed as

$$E[x_k\hat{x}_k|Z_k] = E[x_k^2|Z_k] + E[x_kv_k|Z_k] \tag{A2.16}$$

Because of the independence of x_k and v_k, (A2.16) may be written as

$$E[x_k\hat{x}_k|Z_k] = E[x_k^2|Z_k] + E[x_k|Z_k]E[v_k] \tag{A2.17}$$

Similarly expressions may be derived for the remaining three regions and may be written as

Range B:

$$E[x_k\hat{x}_k|Z_k] = \frac{1}{2}\ (E[x_k^2|Z_k] + E[x_k|Z_k]E[v_k] + E[x_k|Z_k] \tag{A2.18}$$

$$+\ \ell_{k,0}E[x_k|Z_k])$$

Range C:

$$E[x_k \hat{x}_k | Z_k] = \frac{1}{2} \left(E[x_k^2 | Z_k] + E[x_k | Z_k] E[v_k] - E[x_k | Z_k] + f_{k,0} E[x_k | Z_k] \right) \tag{A2.19}$$

Range D:

$$E[x_k \hat{x}_k | Z_k] = \frac{1}{2} E[x_k | Z_k] \; (f_{k,0} + \ell_{k,0}) \tag{A2.20}$$

At this point, it is easy to generalize to $E[x_k^j \hat{x}_k]$ for $j = 2,3,4$ by increasing the corresponding power of x_k in (A2.17) and (A2.20) for all four ranges. The other cross-products of $E[x_k^j \hat{x}^i]$ for $i = 2,3,4$ are calculated by suitable substitution for $\hat{x}^i$.

Having obtained the expressions for $E[x_k^i \hat{x}_k^{4-i}]$ where $i = 0,1,2,3,4$ one may proceed to find an expression for the moments of $x_{k+1} | Z_k$. By expanding (A1.7) the moments of $x_{k+1} | Z_k$ may be expressed as

$$m_{x_{k+1} | Z_k} = E[w_k] + E[u_k | Z_k] + AE[x_k | Z_k] \tag{A2.21}$$

$$m^2_{x_{k+1} | Z_k} = E[w_k^2] + E[u_k^2 | Z_k] + A^2 E[x_k^2 | Z_k] + 2A \, E[x_k u_k | Z_k] \tag{A2.22}$$

$$+ 2AE[x_k | Z_k] E[w_k] + 2E[w_k] E[u_k | Z_k]$$

$$m^3_{x_{k+1} | Z_k} = A^3 E[x_k^3 | Z_k] + E[w_k^3] + E[u_k^3 | Z_k] + 3AE[x_k | Z_k] E[w_k^2] \tag{A2.23}$$

$$+ 3AE[x_k u_k^2 | Z_k] + 3A^2 E[x_k^2 | Z_k] E[w_k] + 3A^2 E[x_k^2 u_k | Z_k]$$

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$$+ 3E[w_k]E[u_k^2|Z_k] + 3E[w_k^2]E[u_k|Z_k] + 6AE[x_k u_k|Z_k]E[w_k]$$

$$m^4_{x_{k+1}|Z_k} = A^4E[x_k^4|Z_k] + E[w_k^4] + E[u_k^4|Z_k] + 4A^3E[x_k^3 u_k|Z_k] + 4A^3E[x_k^3|Z_k]E[w_k]$$

$$+ 4AE[x_k u_k^3|Z_k] + 4AE[x_k|Z_k]E[w_k^3] + 4E[u_k|Z_k]E[w_k^3] \qquad \text{(A2.24)}$$

$$+ 4E[u_k^3|Z_k]E[w_k] + 6E[u_k^2|Z_k]E[w_k^2] + 6A^2E[x_k^2 u_k^2|Z_k]$$

$$+6A^2E[x_k^2|Z_k]E[w_k^2] + 12AE[x_k u_k^2|Z_k]E[w_k] + 12AE[x_k u_k|Z_k]E[w_k^2]$$

$$+ 12A^2E[x_k^2 u_k|Z_k]E[w_k]$$

Recalling $u_k = K_k\hat{x}_k$ and the expressions for $E[x_k^i\hat{x}_k^{4-i}|Z_k]$ where $i=0,1,2,3,4$, one is able to write down the moments required directly for the four ranges of z_k. For convenience, one may define

$$E[x_k^i|Z_k] = m[i]$$

$$E[w_k^i] = mwi \qquad \text{for } i=1,2,3,4$$

$$E[v_k^i] = mvi$$

For Range A : $z_k < \ell_{k,0} - 1$ and $z_k > \ell_{k,0} + 1$

$$m_{x_{k+1}|Z_k} = mw1 + Am[1] + K_k(m[1] + mv1) \qquad \text{(A2.25)}$$

$$m^2_{x_{k+1}|Z_k} = mw2 + A^2m[2] + 2Amw1m[1] + K_k mw1(m[1] + mv1) \tag{A2.26}$$
$$+ K_k^2(m[2] + 2m[1]mv1 + mv2) + 2AK_k(m[2[+ m[1]mv1)$$

$$m^3_{x_{k+1}|Z_k} = mw3 + A^3m[3] + 3Amw2m[1] + 3A^2mw1m[2] + K_k^3(m[3] \tag{A2.27}$$
$$+ 3m[2]mv1 + 3m[1]mv2 + mv3) + 3AK_k^2(m[3] + 2m[2]mv1$$
$$+ m[1]mv2) + 3A^2K_k(m[3] + m[2]mv1) + 3mw1K_k^2(m[2]$$
$$+ 2m[1]mv1 + mv2) + 3K_k mw2(m[1] + mv1)$$
$$+ 6AK_k mw1(m[2] + m[1]mv1)$$

$$m^4_{x_{k+1}|Z_k} = mw4 + A^4m[4] + 4Am[1]mw3 + 6A^2m[2]mw2 + 4A^3m[3]mw1$$
$$+ K_k^4(m[4] + 4m[3]mv1 + 6m[2]mv2 + 4m[1]mv3 + mv4)$$
$$+ 4AK_k^3mw1(m[4] + 3m[3]mv1 + 3m[2]mv2 + m[1]mv3)$$
$$+ 4K_k^3mw1(m[3] + 3m[2]mv1 + 3m[1]mv2 + mv3) \tag{A2.28}$$
$$+ 6K_k^2mw2(m[2] + 2m[1]mv1 + mv2) + 6A^2K_k^2(m[4]$$
$$+ 2m[3]mv1 + m[2]mv2) + 12AK_k^2mw1(m[3] + 2m[2]mv1$$
$$+ m[1]mv2) + 4A^3K_k(m[4] + m[3]mv1) + 4mw3K_k(m[1]$$
$$+ mv1) + 12AK_k mw2(m[2] + m[1]mv1) + 12A^2K_k mw1(m[3]$$
$$+ m[2]mv1)$$

Consider Range B : $z_k < f_{k,0} -1$ and $z_k \leq \ell_{k,0} +1$

$$m_{x_{k+1}|Z_k} = mw1 + \frac{K_k}{2}(m[1] + mv1 + \ell_{k,0} + 1) \tag{A2.29}$$

$$m^2_{x_{k+1}|Z_k} = mw2 + A^2m[2] + 2Amw1m[1] + \text{(see next page)} \tag{A2.30}$$

$$+ K_k mw1\{m[1] + mv1 + \ell_{k,0} + 1\} + \frac{K_k^2}{4}\{m[2] + 2m[1](mv1 + \ell_{k,0} +1)$$
$$+ mv2 + 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2\} + AK_k\{M[2] + m[1]mv1$$
$$+ m[1](\ell_{k,0} +1)$$

$$m^3_{x_{k+1}|z_k} = A^3m[3] + mw3 + 3Amw2m[1] + 3A^2mw1m[2] + \frac{K_k^3}{8}\{m[3] + 3m[2]\cdot$$
$$(mv1 + \ell_{k,0} + 1) + 3m[1](mv2 + 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2$$
$$+ mv3 + 3mv2(\ell_{k,0} + 1) + 3mv1(\ell_{k,0} + 1)^2 + (\ell_{k,0} + 1)^3)$$
$$+ \frac{3AK_k^2}{4}\{m[3] + 2m[2](mv1 + \ell_{k,0} + 1) + m[1](mv2 + 2mv1\cdot$$
$$(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2)\} + \frac{3A^2K_k}{2}\{m[3] + m[2](mv1 + \ell_{k,0}$$
$$+ 1)\} + \frac{3mw1K_k^2}{4}\{m[2] + 2m[1](mv1 + \ell_{k,0} + 1) + mv2 \qquad \text{(A2.31)}$$
$$+ 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2\} + \frac{3}{2}K_k mw2\{m[1] + mv1$$
$$+ \ell_{k,0} + 1\} + 3AK_k mw1\{m[2] + m[1](mv1 + \ell_{k,0} + 1)\}$$

$$m^4_{x_{k+1}|z_k} = mw4 + A^4m[4] + 4Am[1]mw3 + 6A^2m[2]mw2 + 4A^3m[3]mw1 + \frac{K_k^4}{16}\cdot$$
$$\{m[4] + 4m[3](mv1 + \ell_{k,0} + 1) + 6m[2](mv2 + 2mv1(\ell_{k,0} + 1)$$
$$+ (\ell_{k,0} + 1)^2) + 4m[1](mv3 + 3mv2(\ell_{k,0} + 1) + 3mv1(\ell_{k,0}$$
$$+ 1)^2 + (\ell_{k,0} + 1)^3) + mv4 + 4mv3(\ell_{k,0} + 1) + 6mv2(\ell_{k,0}$$
$$+ 1)^2 + 4mv1(\ell_{k,0} + 1)^3 + (\ell_{k,0} + 1)^4\} + \frac{1}{2}AK_k^3\{m[4]$$
$$3m[3](mv1 + \ell_{k,0} + 1) + 3m[2](mv2 + 2mv1(\ell_{k,0} + 1) \qquad \text{(A2.32)}$$

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$$+ (\ell_{k,0} + 1)^2) + m[1](mv3 + 3mv2(\ell_{k,0} + 1) + 3mv1(\ell_{k,0} + 1)^2$$

$$+ (\ell_{k,0} + 1)^3)\} + \frac{1}{2}K_k^3 mw1\{m[3] + 3m[2](mv1 + \ell_{k,0} + 1) + 3m[1](mv2$$

$$+ 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 0)^2) + mv3 + 3mv2(\ell_{k,0} + 1) + 3mv1(\ell_{k,0}$$

$$+ 1)^2 + (\ell_{k,0} + 1)^3\} + \frac{6}{4}K_k^2 mw2\{m[2] + 2m[1](mv1 + \ell_{k,0} + 1) + mv2$$

$$+ 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2\} + \frac{3}{2}A^2K_k^2\{m[4] + 2m[3](mv1 + \ell_{k,0} + 1)$$

$$+ m[2](mv2 + 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2)\} + 3AK_k^2 mw1\{m[3] + 2m[2]\cdot$$

$$(mv1 + \ell_{k,0} + 1) + m[1](mv2 + 2mv1(\ell_{k,0} + 1) + (\ell_{k,0} + 1)^2)\}$$

$$+ 2A^3K_k\{m[4] + m[3](mv1 + \ell_{k,0} + 1)\} + 2mw3K_k\{m[1] + mv1 + \ell_{k,0} + 1\}$$

$$+ 6AK_k mw2\{m[2] + m[1](mv1 + \ell_{k,0} + 1)\} + 6A^2K_k mw1\{m[3] + m[2]\cdot$$

$$(mv1 + \ell_{k,0} + 1)\}$$

Range C : $z_k \geqq f_{k,0} - 1$ and $z_k > \ell_{k,0} + 1$

$$m_{x_{k+1}|Z_k} = mw1 + \frac{1}{2}K_k(m[1] + mv1 + f_{k,0} - 1) + Am[1] \tag{A2.33}$$

$$m^2_{x_{k+1}|Z_k} = mw2 + A^2m[2] + 2Amw1m[1] + K_k mw1\{m[1] + mv1 + f_{k,0} - 1\} + \frac{1}{4}K_k^2\{m[2] + 2m[1](mv1 + f_{k,0} - 1) + mv2 + 2mv1\cdot(f_{k,0} - 1) + (f_{k,0} - 1)^2\} + AK_k\{m[2] + m[1](mv1 + f_{k,0} - 1)\} \tag{A2.34}$$

$$m^3_{x_{k+1}|Z_k} = mw3 + A^3 m[3] + 3Amw2m[1] + 3A^2 mw1m[2] + \frac{1}{8}K_k^3\{m[3] + 3m[2]\cdot$$
$$(mv1 + f_{k,0} - 1) + 3m[1](mv2 + 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2)$$
$$+ mv3 + 3mv2(f_{k,0} - 1) + 3mv1(f_{k,0} - 1)^2 + (f_{k,0} - 1)^3\}$$
$$+ \frac{3}{4}AK_k^2\{m[3] + 2m[2](mv1 + f_{k,0} - 1) + m[1](mv2 + 2mv1\cdot \qquad \text{(A2.35)}$$
$$(f_{k,0} - 1) + (f_{k,0} - 1)^2)\} + \frac{3}{2}A^2K_k\{m[3] + m[2](mv1 + f_{k,0}$$
$$- 1)\} + \frac{3}{4}mw1K_k^2\{m[2] + 2m[1](mv1 + f_{k,0} - 1) + mv2$$
$$+ 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2\} + \frac{3}{2}K_k mw2\{m[1] + mv1 + f_{k,0}$$
$$- 1\} + 3AK_k mw1\{m[2] + m[1](mv1 + f_{k,0} - 1)\}$$

$$m^4_{x_{k+1}|Z_k} = mw4 + A^4 m[4] + 4Am[1]mw3 + 6A^2 m[2]mw2 + 4A^3 m[3]mw1 \qquad \text{(A2.36)}$$
$$+ \frac{1}{16}K_k^4\{m[4] + 4m[3](mv1 + f_{k,0} - 1) + 6m[2](mv2 + 2mv1(f_{k,0}$$
$$- 1) + (f_{k,0} - 1)^2) + 4m[1](mv3 + 3mv2(f_{k,0} - 1) + 3mv1\cdot$$
$$(f_{k,0} - 1)^2 + (f_{k,0} - 1)^3) + mv4 + 4mv3(f_{k,0} - 1) + 6mv2\cdot$$
$$(f_{k,0} - 1)^2 + 4mv1(f_{k,0} - 1)^3 + (f_{k,0} - 1)^4\}$$
$$+ \frac{A}{2}K_k^3\{m[4] + 3m[3](mv1 + f_{k,0} - 1) + 3m[2](mv2 + 2mv1(f_{k,0}$$
$$- 1) + (f_{k,0} - 1)^2) + m[1](mv3 + 3mv2(f_{k,0} - 1) + 3mv1\cdot$$
$$(f_{k,0} - 1)^2 + (f_{k,0} - 1)^3\} + \frac{1}{2}K_k^3 mw1\{m[3] + 3m[2](mv1 + f_{k,0}$$
$$- 1) + 3m[1](mv2 + 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2) + mv3$$
$$+ 3mv2(f_{k,0} - 1) + 3mv1(f_{k,0} - 1)^2 + (f_{k,0} - 1)^3\}$$

+(see next page)

$$+ \frac{3}{2}K_k^2 mw2\{m[2] + 2m[1](mv1 + f_{k,0} - 1) + mv2 + 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2\} + \frac{3}{2}A^2K_k^2\{m[4] + 3m[3](mv1 + f_{k,0} - 1) + 3m[2](mv2 + 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2)\} + 3AK_k^2 mw1\{m[3] + 2m[2](mv1 + f_{k,0} - 1) + m[1]\cdot(mv2 + 2mv1(f_{k,0} - 1) + (f_{k,0} - 1)^2)\} + 2A^3K_k\{m[4] + m[3](mv1 + f_{k,0} - 1)\} + 2mw3K_k\{m[1] + mv1 + f_{k,0} - 1\} + 6AK_k mw2\{m[2] + m[1](mv1 + f_{k,0} - 1)\} + 6A^2K_k mw1\{m[3] + m[2](mv1 + f_{k,0} - 1)\}$$

For Range D : $z_k \geqq f_{k,0} - 1$ and $z_k \leqq \ell_{k,0} + 1$

$$m_{x_{k+1}|Z_k} = mw1 + \frac{K_k}{2}(f_{k,0} + \ell_{k,0}) + Am[1] \tag{A2.37}$$

$$m^2_{x_{k+1}|Z_k} = mw2 + A^2m[2] + 2Amw1m[1] + K_k mw1(f_{k,0} + \ell_{k,0}) + \frac{K_k^2}{4}(f_{k,0} + \ell_{k,0})^2 + AK_k(m[1](f_{k,0} + \ell_{k,0})) \tag{A2.38}$$

$$m^3_{x_{k+1}|Z_k} = mw3 + A^3m[3] + 3Amw2m[1] + 3A^2mw1m[2] + \frac{1}{8}K_k^3(f_{k,0} + \ell_{k,0})^3 + \frac{3}{4}AK_k^2m[1](f_{k,0} + \ell_{k,0})^2 + \frac{3}{2}A^2K_k m[2](f_{k,0} + \ell_{k,0}) + \frac{3}{4}mw1K_k^2(f_{k,0} + \ell_{k,0})^2 + \frac{3}{2}K_k mw2(f_{k,0} + \ell_{k,0}) + 3AK_k mw1m[1](f_{k,0} + \ell_{k,0}) \tag{A2.39}$$

$$m^4_{x_{k+1}|Z_k} = mw4 + A^4m[4] + 4Am[1] + 6A^2m[2]mw2 + 4A^3m[3]mw1 + \frac{K_k^4}{16}\cdot$$

$$(f_{k,0} + \ell_{k,0})^4 + \frac{A}{2}K_k^3m[1](f_{k,0} + \ell_{k,0})^3 + \frac{1}{2}K_k^3mw1(f_{k,0} \qquad \text{(A2.40)}$$

$$+ \ell_{k,0})^3 + \frac{3}{2}K_k^2mw2(f_{k,0} + \ell_{k,0})^2 + \frac{3}{2}A^2K_k^2m[2](f_{k,0} + \ell_{k,0})^2$$

$$+ 3AK_k^2mw1m[1](f_{k,0} + \ell_{k,0})^2 + 2A^3K_km[3](f_{k,0} + \ell_{k,0})$$

$$+ 2mw3K_k(f_{k,0} + \ell_{k,0}) + 6AK_kmw2m[1](f_{k,0} + \ell_{k,0})$$

$$+ 6A^2K_kmw1m[2](f_{k,0} + \ell_{k,0})$$

Having obtained the moments of $x_{k+1}|Z_k$, the filtering algorithm is complete.

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