PROBLEMS IN CLASSICAL BANACH SPACES

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I dedicate this thesis to my husband Ron for his support and confidence in me "through it all." Without him on the home front and faith, though sometimes the size of a mustard seed, this work would not have been possible.

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SUMMARY

It is well known that every finite dimensional subspace Y of a Banach space X is complemented in X. In general, this property does not hold for infinite dimensional subspaces. For example, no isomorph of c_0 is complemented in ℓ_∞. In many cases the fact that a subspace Y is not complemented in a Banach space X depends on isomorphic properties of X and Y. Hence the following problem arises naturally: Given a Banach space X, what are the isomorphic types of complemented subspaces of X?

If K is a compact, metric, uncountable space then C(K) is separable and the isomorphic properties are well understood. But, if K is not metrizable C(K) is not separable and very little is known. In the first chapter a non-separable C(K) space is examined. Investigation of this space suggests that c_0 is always complemented and that this space may even be primary as is the case with separable C(K) spaces.

In Chapter II the classical Banach spaces ℓ_p ($1 < p < \infty$, $p \neq 2$), L_p ($1 \leq p < \infty$, $p \neq 2$), c_0 and some C(K) spaces are examined to determine if there exist operators A,B on Z (where Z is one of the above named spaces) with the range of A contained in the range of B and yet $A \neq BC$

for any operator C on Z. The answer is positive and is based on the (isomorphic) types of complemented (and non-complemented) subspaces of the space Z.

CHAPTER I

THE GEOMETRY OF A NONSEPARABLE C(K) SPACE

1.1. Background and Preliminaries

Let K be a compact Hausdorff space; then the space C(K) is the set of continuous real-valued functions on K with norm: $\|f\| = \sup_{x \in K} |f(x)|$. A. Milutin [13, page 174] and later A. Pelczynski has shown that for every compact, metric, uncountable space K, C(K) is isomorphic to C[0,1]. Thus, the isomorphic properties of such spaces C(K) may be studied through the more familiar spaces C[0,1] and C(Δ) where Δ denotes the Cantor set. Also, all of these spaces where K is compact, metric and uncountable are separable.

J. Lindenstrauss and A. Pelczynski proved in [11] that whenever K is compact, metric and uncountable, C(K) is primary. In their proof they used the space C(Δ) and particularly its basis, the Haar system.

A. Sobczyck [18] showed that c_0 is always complemented in any separable Banach space in which it embeds. As was pointed out in the introduction, c_0 is not complemented in the nonseparable space ℓ_∞ [15]. Veech's proof of Sobczyck's theorem (done much later in [19]) is based on fact that since X is separable, the w^*-topology on the unit ball of X^* is metrizable. His proof extends to the case where X is weakly compactly generated. The nonseparable space C(TL), defined in Section 1.2, is not weak compactly generated. However, this work offers evidence that every isomorph of c_0 contained in C(TL) is complemented and that C(TL) is primary.

Throughout this work all Banach spaces are over the reals. We will need the following definitions.

Definition 1.1.1. A subspace is a closed linear manifold of a Banach space.

Definition 1.1.2. An operator is a bounded linear map.

Definition 1.1.3. Two Banach spaces X and Y are isomorphic if there is an invertible operator from X onto Y.

Definition 1.1.4. The Banach-Mazur distance between two Banach spaces X and Y is defined to be $\inf\{\|T\|\,\|T^{-1}\|: T$ is an invertible operator from X onto Y$\}$.

Definition 1.1.5. A subspace Y of a Banach space X is said to be complemented if there is a bounded linear projection P from X onto Y. In this case there is a subspace Z such that X is the direct sum of Y and Z, i.e., $X = Y \oplus Z$ (where $Z = (I-P)X$).

Definition 1.1.6. A partially ordered set is a set for which a transitive and reflexive binary relation is defined.

Definition 1.1.7. A partially ordered set is totally ordered if the ordering is antisymmetric and all elements are comparable.

Definition 1.1.8. Let $(X,\|\ \|)$ be a normed linear space and M a subspace of X. The quotient space of X modulo M is denoted X/M and defined to be the normed linear space $(\{x+M\colon x \in X\},\|\cdot\|_1)$ where $\|x+M\|_1 = \inf_{m\in M}\|x-m\|$.

Definition 1.1.9. Let $(X_n,\|\cdot\|_n)$ be a sequence of Banach spaces. Then the infinite direct sum $\left(\sum_{n=1}^{\infty}\oplus X_n\right)_{c_0}$ is the normed linear space consisting of the set $X = \{(x_n)_{n=1}^{\infty}\colon x_n \in X_n \text{ and } \|x_n\|_n \to 0 \text{ as } n \to \infty\}$ with the norm: $\|(x_n)_{n=1}^{\infty}\| = \sup_n\|x_n\|_n$.

Definition 1.1.10. Let $(X_n,\|\cdot\|_n)$ be a sequence of Banach spaces. Then the infinite direct sum $\left(\sum_{n=1}^{\infty}\oplus X_n\right)_{\ell_1}$ is the normed linear space consisting of the set $X = \{(x_n)_{n=1}^{\infty}\colon x_n \in X_n \text{ and } \sum_{n=1}^{\infty}\|x_n\|_n < \infty\}$ with the norm: $\|(x_n)_{n=1}^{\infty}\| = \sum_{n=1}^{\infty}\|x_n\|_n$.

1.2. Description and Topological Properties of C(TL)

Consider two copies of the unit interval, say I_0 and I_1. Let $X = I_0 \cup I_1$ where

$$I_0 \equiv \{(x,0): x \in [0,1]\} \text{ and } I_1 \equiv \{(x,1): x \in [0,1]\}.$$

For each point $p \in I_0$, that is $p \equiv (p,0)$ define

$$B(p;\epsilon) = \{(x,0) \in X: p-\epsilon < x \leq p\} \cup \{(x,1) \in X: p-\epsilon < x < p\}$$

and for each point $q \in I$, that is $q \equiv (q,1)$ define

$$B(q;\epsilon) = \{(x,0) \in X: q < x < 1+\epsilon\} \cup \{(x,1) \in X: q \leq x < q+\epsilon\}.$$

The set $\mathfrak{B} = \{B(p;\epsilon): p \in X\}$ is a base for a topology on X with the two conditions (i) $\cup\{B: B \in \mathfrak{B}\} = X$ and (ii) $U,V \in \mathfrak{B}$ and $x \in U \cap V \Rightarrow$ there is a $W \in \mathfrak{B}$ with $W \subseteq U \cap V$ and $x \in W$, being trivially satisfied. Also, note that

$$\begin{aligned} B((1,1);\epsilon) &= \{(x,0) \in X: 1 < x < x+\epsilon\} \cup \{(x,1): 1 \leq x < 1+\epsilon\} \\ &= \{(1,1)\}. \end{aligned}$$

and that $B((0,0);\epsilon) = \{(0,0)\}$.

Let TL be defined to be the space $X = I_0 \cup I_1$ with the topology generated by $\mathfrak{B}$.

Define a relation "<" on TL as follows: $(x,i) < (y,j)$ if and only if $x < y$ or $x = y$ with $i < j$.

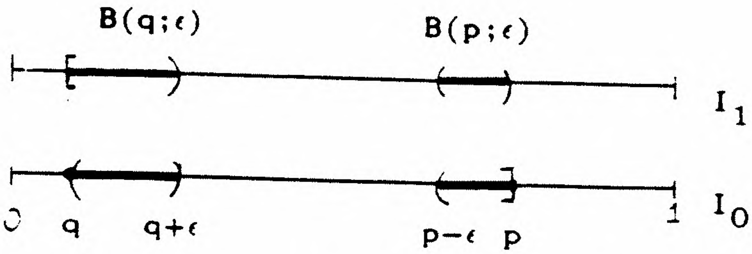


Figure 1. The Open Balls of TL.

Theorem 1.2.1. TL is totally ordered.

Proof. The relation "<" defined above is the well known lexicographic ordering and is clearly a total ordering for this set just as it is for $\mathbf{R}^2$.

The open intervals of TL are the non-empty sets $\langle(a,j),(b,k)\rangle \equiv \{(x,i) \in TL: (a,j) < (x,i) < (b,k)\}$ (see Figure 2). These are all open sets since each one is either an open ball (Figures 2a and 2b), the intersection of two open balls (Figure 2c) or the union of two open balls (Figure 2d).

The following theorem is well known for open sets of real numbers [8, page 207]. Because of the lexicographic ordering the same can be proven for open subsets of TL.

Theorem 1.2.2. If G is an open set in TL then G may be expressed as a countable union of open intervals, say $G = \bigcup_{n=1}^{\infty} J_n$ where for all $n \neq m$, $J_n = J_m$ or $J_n \cap J_m = \emptyset$.

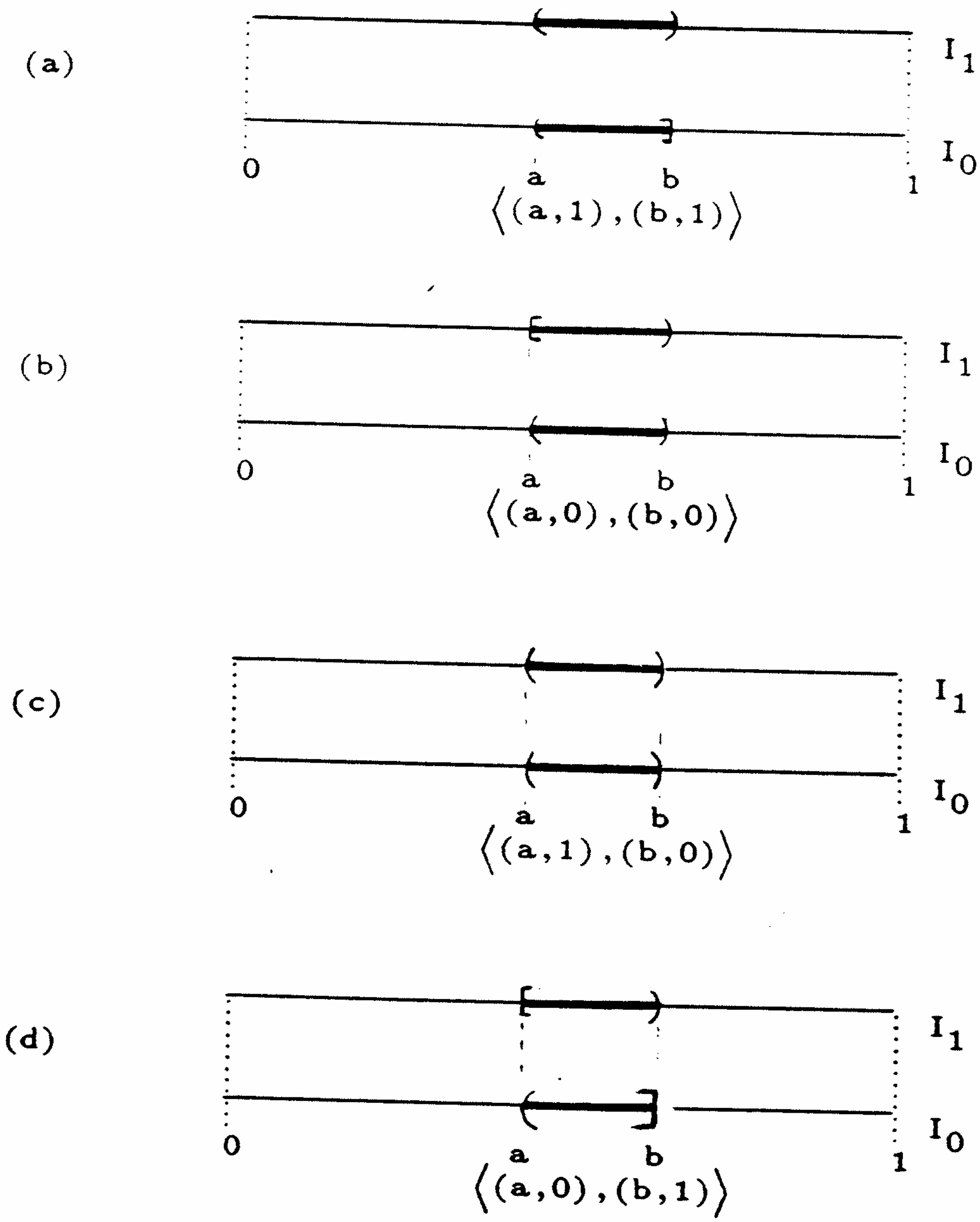


Figure 2. The Open Intervals of TL.

Proof. First of all observe that the set Q_{TL} = $\{(q,i) \in TL$: q is a rational number$\}$ is countable and meets every open ball of TL, i.e., is dense in TL.

If $(p,i) \in G$ then for some $\epsilon > 0$ $B((p,i);\epsilon) \subseteq G$, thus G contains an interval $\langle(a,j),(b,k)\rangle$ containing (p,i). Now, let $r_1, r_2, \ldots$ be a listing of the points of $G \cap Q_{TL}$ and define, for each $n \in \mathbb{N}$, $J_n = \cup\{\langle(a,j),(b,k)\rangle : r_n \in \langle(a,j),(b,k)\rangle \subseteq G\}$. Then clearly each J_n $(n \in \mathbb{N})$ is an open interval contained in G and $\bigcup_{n=1}^{\infty} J_n \subseteq G$.

It remains to be shown that $G \subseteq \bigcup_{n=1}^{\infty} J_n$. Suppose $(s,i) \in G$ where s is an irrational number. Then (s,i) is contained in some ball $B((s,i);\epsilon) \subseteq G$ which must contain a member r_k of Q_{TL} which implies that $(s,i) \in J_k$. Therefore $G = \bigcup_{n=1}^{\infty} J_n$; moreover, by construction if J_n and J_m $(n \neq m)$ share any point they must coincide, that is $J_n = J_m$ or $J_n \cap J_m = \emptyset$.

Theorem 1.2.3. TL is a separable, compact Hausdorff space which is not metrizable.

Proof. It has already been observed that the countable set Q_{TL} is dense in TL, hence its separability.

To show that TL is Hausdorff, let (p,i) and (q,j) be two distinct points in TL. If $p \neq q$, take $\epsilon = |p-q|$ and then $B((p,i);\epsilon/3)$ and $B((q,j);\epsilon/3)$ separate (p,i) and

(q,j). If $p=q$ then $i \neq j$, thus for any $\epsilon > 0$ $B((p,i);\epsilon) \cap B((q,j);\epsilon) = \emptyset$.

In order to show that TL is compact consider a net $\{x_\alpha;D\}$ in TL. There is a cofinal subset D' of D such that either $\{x_\alpha;D'\} \subseteq I_0$ or $\{x_\alpha;D'\} \subseteq I_1$. Consider $\{t_\alpha \in \mathbb{R}: (t_\alpha,i) = x_\alpha$ for some $\alpha \in D'\}$. Then $\{t_\alpha;D'\}$ is a net in the compact space $[0,1]$ and therefore has a cluster point, say $t_0 \in [0,1]$. There is either a net $\{t'_\alpha\} \downarrow t_0$ or a net $\{t''_\alpha\} \uparrow t_0$ which will be a subnet of $\{t_\alpha;D'\}$. In the case where $\{t'_\alpha\} \downarrow t_0$ the point $(t_0,1)$ will be a cluster point of $\{x_\alpha;D\}$ since all open balls $B((t_0,1);\epsilon)$ will contain infinitely many points of the net. In the other case where $\{t''_\alpha\} \uparrow t_0$ to the point $(t_0,0)$ will be a cluster point of $\{x_\alpha;D\}$.

Lastly to show that TL is not metrizable it suffices to show that it is not second countable. The proof of this fact is essentially the same as found in [20, page 76] where the Right Half Open topology on $[0,1]$ is shown to be non-metrizable.

Suppose $\mathfrak{B}'$ is a countable base for TL. Let $(x,1)$ and $(y,1)$ be distinct points of TL so that either $x < y$ or $y < x$. For any $\epsilon > 0$ $B((x,1);\epsilon)$ and $B((y,1);\epsilon)$ are neighborhoods of x and y respectively. Therefore, there are sets B_x and B_y in $\mathfrak{B}'$ with $x \in B_x \subseteq B((x,1);\epsilon)$ and $y \in$

$B_y \subseteq B((y,1);\epsilon)$. However, $B_x \neq B_y$ since $(x,1)$ and $(y,1)$ are their smallest elements; hence $\mathfrak{B}'$ must contain uncountably many sets just to accommodate the points of I_1. This is a contradiction and leads to the conclusion that there is no countable base for TL.

The Space C(TL)

Since the space TL is not metrizable the space of continuous functions on TL, C(TL), is not separable. The following are examples of functions in C(TL).

Example 1.2.3. If $f \in C[0,1]$ then clearly f may be thought of as a member of C(TL) where $f(x,i) = f(x)$ for $i = 0,1$. Hence we may consider $C[0,1]$ to be naturally isometric to a subspace of C(TL).

Example 1.2.4. Consider the function

$$f(x,i) = \begin{cases} \frac{1}{2} & \text{for } x \leq \frac{1}{2} \\ 1 & \text{for } x > \frac{1}{2} \end{cases}$$

Clearly f is continuous at all points of TL other than $(\frac{1}{2},i)$. Even at $(\frac{1}{2},0)$ f is continuous since given any $\epsilon > 0$ we may take $0 < \delta < \frac{1}{2}$, say $\delta = \frac{1}{4}$ and for all (t,i) in $B((\frac{1}{2},0);\delta)$, $|f(t,i)-f(\frac{1}{2},0)| = 0 < \epsilon$. However, at $(\frac{1}{2},1)$ f is not continuous. If $\epsilon = \frac{1}{4}$ no matter how small δ is chosen, $B((\frac{1}{2},1);\delta)$ will contain $(\frac{1}{2},1)$ and (t_1,i) for some $t_1 > \frac{1}{2}$ and $|f(t_1,i)-f(\frac{1}{2},1)| = \frac{1}{2} > \epsilon$ (see Figure 3).

If we redefine the function above to be 1 at $(\frac{1}{2},i)$ the resulting function would be continuous at all points of TL except $(\frac{1}{2},0)$. In order to get a member of C(TL) f must be redefined as

$$g(x,i) = \begin{cases} \frac{1}{2} & \text{if } x < \frac{1}{2} \\ 1 & \text{if } x > 1 \\ & \\ \frac{1}{2} & \text{if } (x,i) = (\frac{1}{2},0) \\ 1 & \text{if } (x,i) = (\frac{1}{2},1) \end{cases}$$

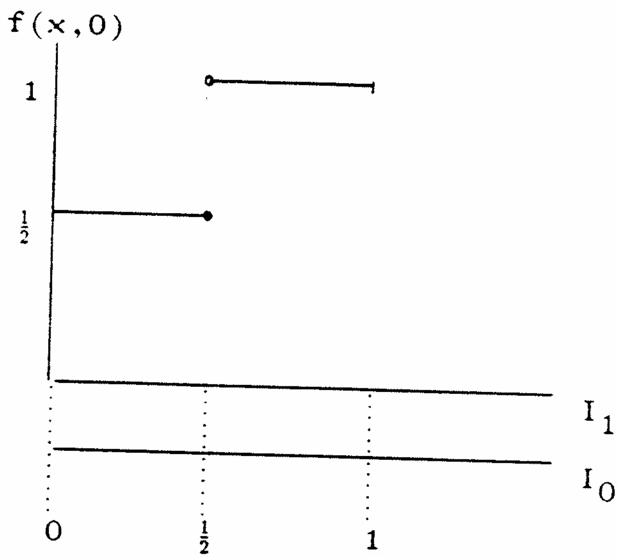
Recall the notion of right- and left-hand limits of real valued functions on $\mathbb{R}$ denoted $f(x^+)$ and $f(x^-)$ respectively and defined

$$f(x^+) = \lim_{t\to x^+} f(t) \quad \text{and} \quad f(x^-) = \lim_{t\to x^-} f(t)$$

where $t \to x^+$ ($t \to x^-$) means t approaches x through values greater (less) than x. A function $f\colon \mathbb{R}\to\mathbb{R}$ is right continuous at x_0 if $f(x_0) = f(x_0^+)$ and left continuous at x_0 if $f(x_0) = f(x_0^-)$. Continuity on [0,1] implies both right and left continuity.

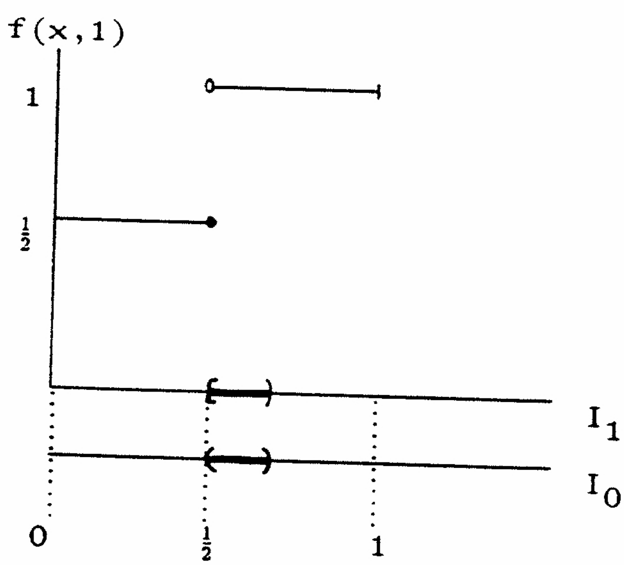
Left- and right-hand limits for real-valued functions on TL may be defined in an analogous manner. From Example

(a)



$\forall\ (x,i) \in B((\tfrac{1}{2},0);\tfrac{1}{4})\ f(x,i) = \tfrac{1}{2}$

(b)



$f(\tfrac{1}{2},1) = \tfrac{1}{2}$ but for all other $(x,i) \in B((\tfrac{1}{2},1);\delta)$ $f(x,i) = 1$.

Figure 3. The Graph of $f(x,i)$.

1.2.4 it is clear that continuity of a function g on TL requires that $g_1(x) = g(x,1)$ be right continuous on $[0,1]$ and $g_0(x) = g(x,0)$ be left continuous on $[0,1]$.

<u>Theorem</u> <u>1.2.5</u>. If $f \in C(TL)$ then the function f_1: $[0,1] \rightarrow \mathbb{R}$ defined by $f_1(x) = f(x,1)$ is right continuous and $f_1(x^-) = f(x,0)$ for all $x \in [0,1]$. Also the function f_0: $[0,1] \rightarrow \mathbb{R}$ defined by $f_0(x) = f(x,0)$ is left continuous and $f_0(x^+) = f(x,1)$ for all $x \in [0,1]$.

<u>Proof</u>. To see that f_1 is right continuous at all points of $[0,1]$, let x_0 be an arbitrary point in $[0,1]$. Since $f \in C(TL)$, it is continuous at $(x_0,1)$ so that given any $\epsilon > 0$ we can find $\delta > 0$ such that $|f(x,i)-f(x_0,1)| < \epsilon$ whenever $(x,i) \in B((x_0,1);\delta) \Rightarrow |f_1(x)-f_1(x_0)| < \epsilon$ whenever $x_0 < x < x_0+\delta$, hence the right continuity of f_1. Since f is continuous at $(x_0,0)$, given any $\epsilon > 0$, we can find δ such that $|f(x_0,0)-f(x,1)| < \epsilon$ whenever $(x,1) \in B((x_0,0);\delta)$, i.e., whenever $x_0-\delta < x < x_0$; or in terms of f_1 we have $|f_1(x)-f(x_0,0)| < \epsilon$ whenever $x_0-\epsilon < x < x_0$, hence $f_1(x^-) = f(x_0,0)$.

The proof that f_0 is left continuous and $f_0(x^+) = f(x,1)$ follows the same line of reasoning.

Let $D[0,1]$ be the set of all functions f on $[0,1]$ that are right continuous with left hand limits and $\|f\| =$

$\sup_{t\in[0,1]} |f(t)|$. Then by Theorem 1.2.5 each member of C(TL) determines a member of D. That is, given $f \in C(TL)$ the function $f_1(x) = f(x,1)$ for all $x \in [0,1]$ is a member of D, with $f_1(x^-) = f(x,0)$. The mapping $T: C(TL) \to D$ defined by $Tf = f_1$ is clearly an isometry. Moreover, T is onto D since given any function g in D, the function

$$f(x,i) = \begin{cases} g(x) & \text{if } i = 1 \\ g(x^-) & \text{if } i = 0 \end{cases}$$

will be mapped to g. Therefore C(TL) is isometrically isomorphic to the space D[0,1]. In the remainder of this work a member of C(TL) will often be defined by its right continuous counterpart in D.

Note that if a function f in C(TL) has image g in D then g will be continuous at all points of [0,1] except those where $f(x,0) \neq f(x,1)$. Also, the identification of C(TL) with D in conjunction with the following lemma taken from [2] gives key properties of functions in C(TL).

<u>Lemma</u> <u>1.2.6</u>. For each g in D and each $\epsilon > 0$ there exists points $t_1, t_2, \ldots, t_k$ such that $0 = t_0 < t_1 < \cdots < t_k = 1$ and for each $i = 1,2,3,\ldots,k$

$$\sup\{|g(s)-g(t)| : s,t \in [t_{i-1},t_i)\} < \epsilon \qquad (1)$$

<u>Proof</u>. Let τ be the supremum of those t for which $[0,t)$ can be decomposed into finitely many subintervals $[t_{i-1},t_i)$ satisfying (1). Since $g(0) = g(0^+)$, $\tau > 0$. Since $g(\tau^-)$ exists $[0,\tau)$ itself can be so decomposed. Also, if $\tau < 1$, since $g(\tau) = g(\tau^+)$ we would always be able to find δ such that $|g(s)-g(\tau)| < \frac{\epsilon}{2}$ whenever $0 < s-\tau < \delta$, i.e., whenever $\tau < s < \tau+s$ and then for all $s,t \in [\tau,\tau+s)$,

$$|g(s)-g(t)| \leq |g(s)-g(\tau)| + |g(\tau)-g(t)| < \epsilon$$

which implies that $[0,\tau+\delta)$ can be decomposed into finitely many intervals satisfying (1). Thus $\tau < 1$ is impossible.

From the above lemma we may conclude that all functions g in D[0,1] have the following properties:

(i) There can be at most finitely many points t at which the jump $|g(t)-g(t^-)|$ exceeds a given positive number.

(ii) g can have at most countably many discontinuities.

(iii) g may be approximated arbitrarily closely by a step function.

By the above identification of C(TL) with D the above statements (i, ii, and iii) yield three key properties of C(TL) stated below:

(i′) If $f \in C(TL)$ then there can be at most finitely many points (t,i) at which $|g(t,1)-g(t,0)|$ exceeds a given positive number.

(ii′) A function f in C(TL) can have at most countably many values of t such that $f(t,0) \neq f(t,1)$.

(iii′) The step functions are dense in C(TL).

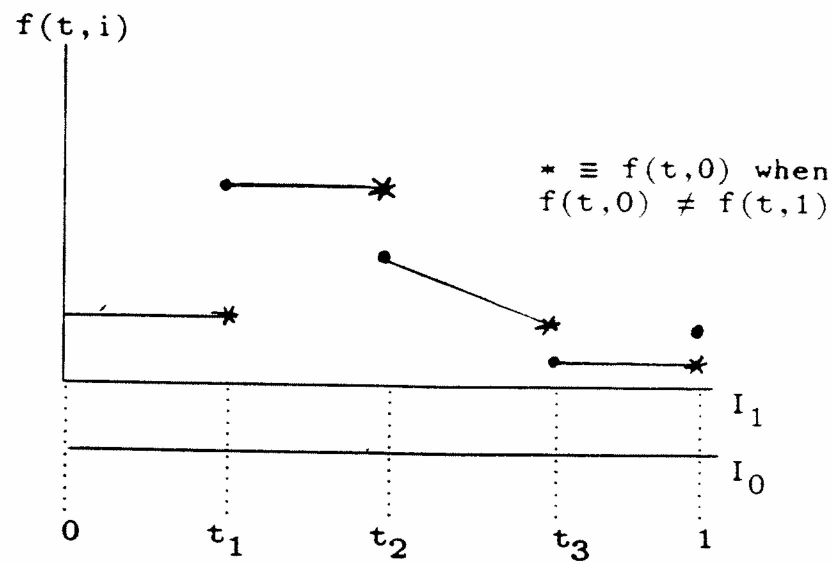
From this point on when $f \in C(TL)$ is referred to as $f = f(x)$ (vs. $f = f(x,i)$) it is being defined by its image in D (see Figure 4).

1.3. <u>Complementation of C[0,1] and c_0-Subspaces of C(TL)</u>

In this section some of the subspaces of C(TL) isomorphic to C[0,1] and c_0 are examined to determine if they are complemented. As has been previously stated C[0,1] is isomorphic to every C(K) space where K is metric, uncountable and compact and c_0 is complemented in every separable C(K) space. It is shown below that the natural image of C[0,1] in C(TL) (see Example 1.2.3) is a closed but non-complemented subspace of C(TL). Evidence is given that every isomorph of c_0 is complemented with a proof for a special class of isomorphs.

<u>Theorem 1.3.1</u>. C[0,1] is a closed subspace of C(TL) which is not complemented in C(TL).

(a) The graph of $f \in C(TL)$



(b) The image of f in D: $Y = f(x,1)$

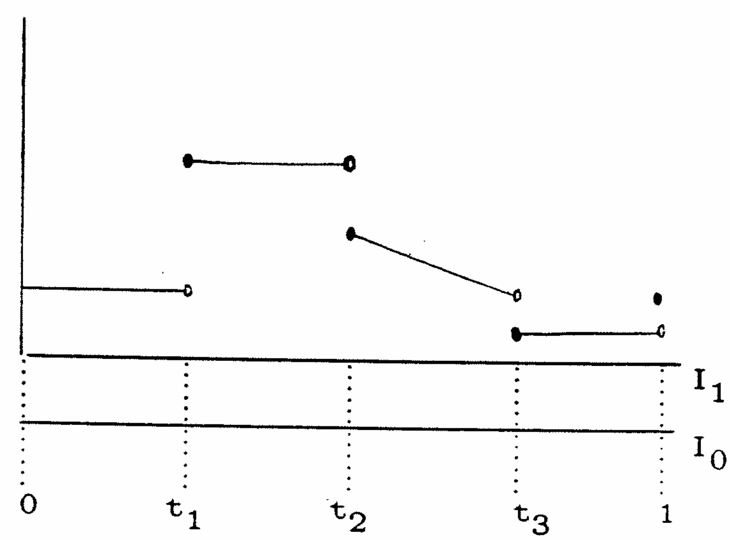


Figure 4. A Function f in C(TL) and Its Image in D.

That $C[0,1]$ is a linear manifold in $C(TL)$ has already been established in Example 1.2.3 and clearly $C[0,1] = \{f \in C(TL)\colon f(t,0) = f(t,1) \text{ for all } t \in [0,1]\}$ is closed in $C(TL)$. One needs to show that $C(TL)$ is not complemented in $C(TL)$. The following lemmas will be needed.

<u>Lemma 1.3.2</u>. If F is a complemented subspace of the Banach space X then $X = F \oplus G$ where G is isomorphic to X/F.

<u>Proof</u>. Consider the quotient map restricted to G, that is, $T\colon G \to X/F$ defined by $Tg = g + F$.

If $x + F$ is an arbitrary element of X/F then $x = f+g$ where $f \in F$ and $g \in G$. Thus $x+F = (f+g) + F = g + (f+F) = g+F$ so that $Tg = x+F$ which shows that T is onto X/F.

If g_1 and g_2 are arbitrary elements of G with $Tg_1 = Tg_2$ this means that $g_1 + F = g_2 + F$. Let f_1 be a member of F. Then for some f_2, also in F, $g_1 + f_1 = g_2 + f_2$ which implies that $g_1 - g_2 = f_2 - f_1$. So either F and G have a common non-zero element, which is impossible since we have expressed X as $F \oplus G$, or $g_1 - g_2 = 0$ which implies that $g_1 = g_2$; therefore, T is one-to-one.

By these arguments the map T is known to be a one-to-one bounded linear operator of G onto X/F and thus by the open mapping theorem [17, page 195] these spaces are isomorphic.

<u>Lemma 1.3.3</u>. $C(TL)/C[0,1]$ is isometrically isomorphic to the Banach space $c_0[0,1]$ where $c_0[0,1]$ is the Banach space $\{f\colon f\colon [0,1]\to\mathbf{R}$ and for all $\epsilon > 0$, $\#\{t\colon |f(t)| > \epsilon\} < \infty\}$ with $\|f\| = \sup_{t\in[0,1]} |f(t)|$ for all f in the space.

<u>Proof</u>. We can define a linear map $T\colon C(TL)/C[0,1] \to c_0[0,1]$ by $T(g+C[0,1])(\alpha) = \frac{1}{2}[g(\alpha,1)-g(\alpha,0)]$.

To see that T is well-defined suppose $g_1 + C[0,1] = g_2 + C[0,1]$ where $g_1 \neq g_2$. Then $g_1 - g_2$ is a continuous function thus

$$
\begin{aligned}
2(Tg_1-Tg_2)(\alpha) &= (g_1(\alpha,1)-g_1(\alpha,0))-(g_2(\alpha,1)-g_2(\alpha,0)) \\
&\qquad\qquad \forall\ \alpha \in [0,1] \\
&= (g_1-g_2)(\alpha,1) - (g_1-g_2)(\alpha,0) \\
&= g(\alpha,1) - g(\alpha,0) \text{ for some } g \in C[0,1] \subseteq D \\
&= 0.
\end{aligned}
$$

As for continuity

$$
\begin{aligned}
\|g+C[0,1]\| \inf_{f\in C[0,1]}\|g-f\| &\geq \tfrac{1}{2}\sup\{|g(t,1)-g(t,0)|\colon t\in[0,1]\} \\
&= \|T(g+C[0,1])\| \qquad (2)
\end{aligned}
$$

In fact for the class of step functions, which are dense in $C(TL)$ we have equality in (2) so that T is an isometry.

Next, suppose $T(g_1+C[0,1]) = T(g_2+C[0,1])$. Then the two functions g_1 and g_2 have exactly the same jumps at the same values of t. Moreover, this means that $g_1-g_2 \in C[0,1]$. To see this, let t_0 be one of the places where $g_1(t_0,0)-g_1(t_0,1) = g_2(t_0,0)-g_2(t_0,1) \neq 0$. From this equation we get the following one for g_1-g_2 at (t_0,i):

$$(g_1-g_2)(t_0,0) = g_1(t_0,0)-g_2(t_0,0) = g_1(t_0,1)-g_2(t_0,1)$$
$$=(g_1-g_2)(t_0,1).$$

Thus, the function g_1-g_2 does not have a jump at t_0. So g_1-g_2 is a continuous function which implies that $(g_1-g_2) + C[0,1] = C[0,1]$, i.e., $g_1 + C[0,1] = g_2 + C[0,1]$.

To show that T is onto $c_0[0,1]$ one must first observe that the set of finitely supported functions are dense in $c_0[0,1]$. Given any $x \in c_0[0,1]$ which is finitely supported, list the points $t_1,\ldots,t_k$ of $[0,1]$ where x does not vanish. Then the function g in $C(TL)$ defined on I, by $g(t,1) = \sum_{n=1}^{k} \ell_n(t)\chi_{[t_n,t_{n+1})}t_{k+1} = 1$ and $\ell_n(t) = \frac{2x(t_n)}{t_n-t_{n+1}}(t-t_{n+1})$, will be mapped to x by T. Thus T is onto.

<u>Proof of Theorem 1.3.1</u>. Suppose $C[0,1]$ is complemented in $C(TL)$. Then $C(TL) = C[0,1] \oplus G$ where G is isomorphic to $C(TL)/C[0,1]$. Let $T\colon G \to C(TL)/C[0,1]$ be

the restriction of the usual quotient map to G. That is $Tg = g + C[0,1]$ for all $g \in G$. $Tg = \bar{0} = C[0,1]$ if and only if $g = 0$. Moreover, since $C(TL)/C[0,1]$ is isometrically isomorphic to $c_0[0,1]$ we may consider $T: G \to c_0[0,1]$ and from the above statements there is a bounded operator U from $c_0[0,1]$ to G such that TU is the identity on $c_0[0,1]$.

Now let $\{r_i\}_{i=1}^{\infty}$ be a listing of the rationals in $[0,1]$. Let x_n be the image in $c_0[0,1]$ of the function in $C(TL)$ which has a jump of +2 at exactly one point, namely r_n. Note that each of these images in $C(TL)/C[0,1]$ and thus also in $c_0[0,1]$ will have norm 1 since $\|x_n\|_{C_0[0,1]/C[0,1]} = \inf_{g \in C[0,1]} \|x_n - g\|$.

Next, take $n_1 = 1$ and let $f_1 = Ux_1$. By passing to $-x_1$ (and thus to $-f_1$) if necessary assume, without loss of generality that $\max(f_1(r_1^+), f_1(r_1^-)) \geq 1$. It follows that there is a non-degenerate interval J_1 with r_1 as one endpoint such that $f_1(x) \geq \frac{1}{2}$ for all $x \in J_1$. Let n_2 be such that r_{n_2} is an interior point of J_1. Then $f_2 = Ux_{n_2}$ is such that $\max(f_2(r^+), f_2(r^-)) \geq 1$ and there is a non-degenerate interval $J_2 \subsetneq J_1$ with $f_2(x) \geq \frac{1}{2}$ for all $x \in J_2$. Continuing this inductively, one constructs a decreasing sequence of intervals such that for every k and every $x \in J_k$ we have that

$$\frac{k}{2} \le (f_1 + f_2 + \cdots + f_k)(x) \le \|U(x_{n_1} + x_{n_2} + \cdots + x_{n_k})\| \le \|U\|$$

which contradicts the fact that U is a bounded map.

$\underline{c_0}$ Subspaces of C(TL)

Example 1.3.4. Let $\{J_n\}_{n=1}^{\infty}$ be the collection of intervals $J_n = [\frac{1}{2^n}, \frac{1}{2^{n-1}})$ for $n \ge 2$ and $J_1 = [\frac{1}{2}, 1]$. Let $f_n(t) = \chi_{J_n}$, that is

$$f_n(t) = \chi_{J_n} = \begin{cases} 1 & \text{if } t \in J_n \\ 0 & \text{if } t \notin J_n \end{cases}$$

be the I_1 definition of f_n, that is $f_n \in D[0,1]$, (from which its I_0 definition may be derived). Let

$$F = \left\{ \sum_{n=1}^{\infty} a_n f_n : \{a_n\} \in c_0 \right\}.$$

Consider the map $T: c_0 \to F$ defined by

$$Tx = \sum_{n=1}^{\infty} a_n f_n \text{ where } \{a_n\} \in c_0.$$

Then clearly T is linear, one-to-one, and onto $F \subseteq C(TL)$. Moreover, it is clear that $\|Tx\| = \|x\|_{c_0}$ since the intervals I_n are disjoint so that T is an isometric isomorphism of c_0 onto F.

Define a map $P: C(TL) \to F$ by $Pf = \sum_{k=1}^{\infty} f(t_k) f_k$ where

t_k is a point chosen from J_k. Then

$$P^2 f = P(Pf) = P\Big(\sum_{j=1}^{\infty} f(t_j) f_j\Big) = \sum_{k=1}^{\infty}\Big(\sum_{j=1}^{\infty} f'(t_j) f_j(t_k)\Big) f_k$$

$$= \sum_{k=1}^{\infty} f(t_k) f_k = Pf$$

and

$$\|Pf\| = \sup_t |(Pf)(t)| = \sup_t \Big|\sum_{k=1}^{\infty} f(t_k) f_k(t)\Big| = \sup_k |f(t_k)| \leq \|f\|.$$

Thus, P is a bounded projection of C(TL) onto F and F is therefore a subspace of C(TL) isometrically isomorphic to c_0 and complemented in C(TL). In fact, all isometric isomorphs of c_0 are complemented in C(TL). To show this we need the following simultaneous extension theorem.

Lemma 1.3.5. Let Z be a closed subset of TL. Then there is a linear isometry T from C(Z) into C(TL) so that for every $f \in C(Z)$ the restriction of Tf to Z is equal to f.

Proof. Since Z is closed $Z' = TL\backslash Z$ is open and by Lemma 1.2.2 this means that $Z' = \bigcup_{n=1}^{\infty}\big\langle (a_n, j_n), (b_n, k_n)\big\rangle$ where any two of these intervals are either disjoint or coincide at all points. Define $U: C(Z) \to C(TL)$ by $Uf = f$ on Z and on $Z' = \bigcup_{n=1}^{\infty}\big\langle (a_n, j_n), (b_n, k_n)\big\rangle$ define Uf as follows:

$$f(a_n,j_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n)$$
$$\text{for } a_n < x \leq b_n \text{ if } j_n = k_n = 1$$

$$(Uf)(x,i) = \begin{cases} f(a_n,j_n) + \dfrac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n) \\ \qquad \text{for } a_n \leq x < b_n \text{ if } j_n = k_n = 0 \\ \\ f(a_n,j_n) + \dfrac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n) \\ \qquad \text{for } a_n < x < b_n \text{ if } j_n = 1 \text{ and } k_n = 0 \end{cases}$$

$$f(a_n,j_n) + \frac{f(b_n,k_n)-f(a_n,j_n)}{b_n-a_n}(x-a_n)$$
$$\text{for } a_n \leq x \leq b_n \text{ if } j_n = 0 \text{ and } k_n = 1$$

Note that the four cases above determined by the values of j_n and k_n parallel the four cases given in Figure 2 which shows the open intervals of TL.

Example 1.3.6. Let $Z = \{(0,1), (\frac{1}{3},1), (\frac{2}{3},1), (1,1), (0,0), (\frac{2}{3},0), (0,1)\}$. Suppose $f(x,i)$ is known for all $(x,i) \in Z$ and we wish to use the scheme of Lemma 1.3.5 to define its extension, Uf, to all of TL. First of all $Z' = \left\langle(0,1),(\frac{1}{3},1)\right\rangle \cup \left\langle(\frac{1}{3},1),(\frac{2}{3},0)\right\rangle \cup \left\langle(\frac{2}{3},1),(1,0)\right\rangle$.

$$(Uf)(x) = \begin{cases} f(x,i) & \text{if } (x,i) \in Z \\ f(0,1) + 3(f(\tfrac{1}{3},1)-f(0,1)(x) & \text{for } 0 < x \le \tfrac{1}{3} \\ f(\tfrac{1}{3},1) + 3(f(\tfrac{2}{3},0)-f(\tfrac{1}{3},1))(x-\tfrac{1}{3}) & \text{for } \tfrac{1}{3} < x < \tfrac{2}{3} \\ f(\tfrac{2}{3},1) + 3(f(1,0)-f(\tfrac{2}{3},1))(x-\tfrac{2}{3}) & \text{for } \tfrac{2}{3} < x < 1 \end{cases}$$

(see Figure 5).

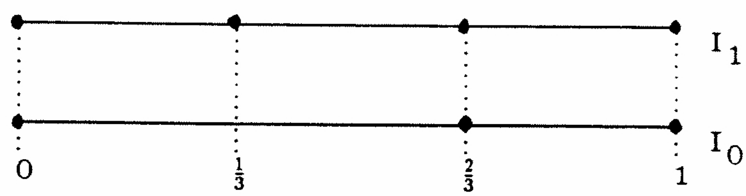
Theorem 1.3.7. If F is a subspace of C(TL) which is isometrically isomorphic to c_0 then F is complemented in C(TL).

Proof. Let T be an isometric isomorphism mapping c_0 onto F. For each $n \in \mathbf{N}$, let $f_n = Te_n$ where $\{e_i\}_{i=1}^{\infty}$ is the unit vector basis of c_0. Since $\|f_n\| = 1$ for each n, there is a number $t_n \in [0,1]$ and $i_n \in \{0,1\}$ such that $|f(t_n,i_n)| = 1$ and let Z be the set of limit points of the sequence $\{t_n,i_n)\}$. Then $\|f_n+f_m\| = \|T(e_n \pm e_m)\| = \|e_n \pm e_m\| = 1$ for all $n \neq m$ which implies that

$$f_n(t_m,i_m) = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Hence, if $(t,i) \in Z$ then $f_n(t,i) = \lim_{k\to\infty} f_n(t_{m_k},i_{m_k}) = 0$ and since f_n is a basis for F this means that $f(t,i) = 0$ for all $(t,i) \in Z$. Now let $Z^{\perp}$ be the subspace of C(TL) consisting of all functions vanishing at each point of Z.

(a) The Set Z



(b) The Graph of $Z \to \mathbf{R}$

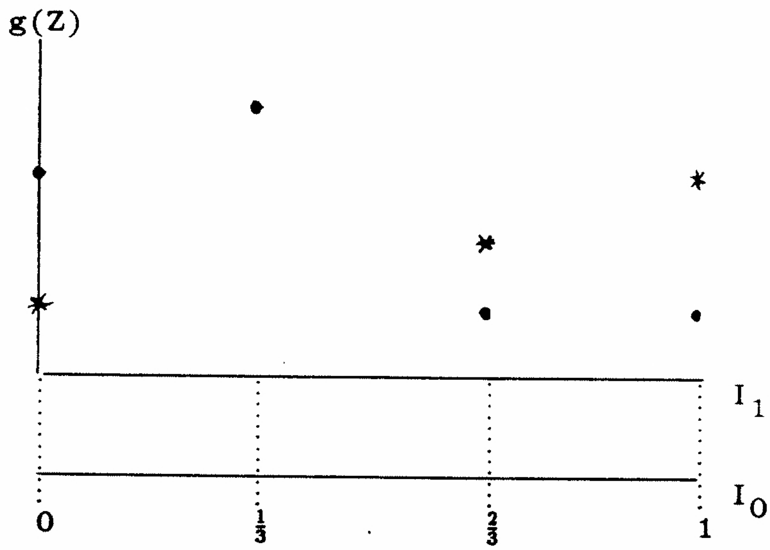
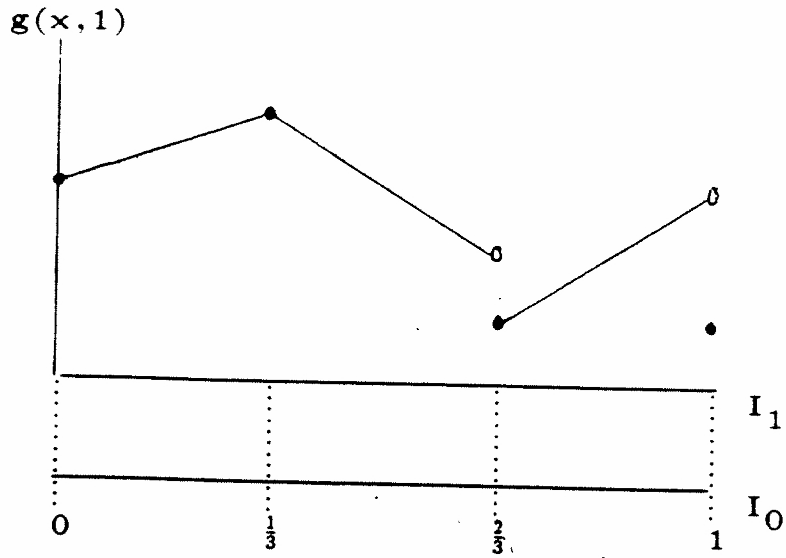


Figure 5. The Extension of f in C(Z) to C(TL).

(c) The Graph of Uf on I_1



(d) The Graph of Uf on I_0

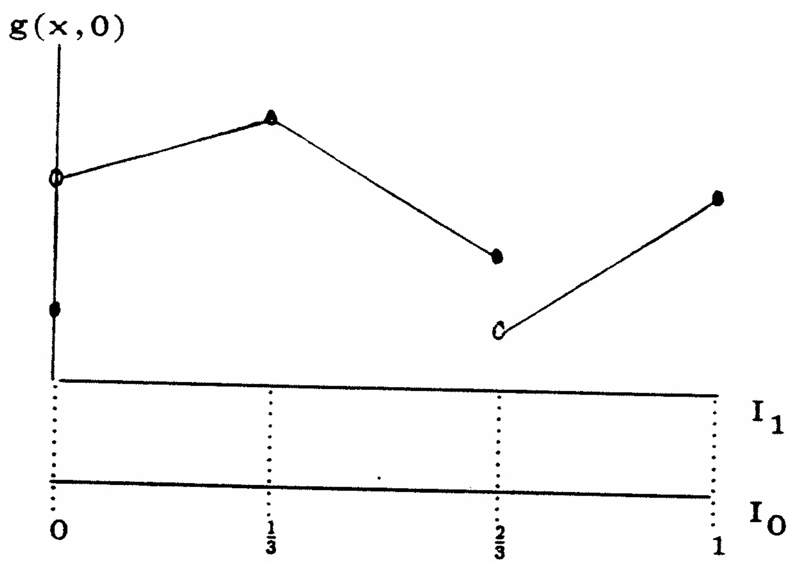


Figure 5. (Continued)

Define a map $P: Z^{\perp} \to F$ by

$$Px = \sum_{n=1}^{\infty} x(t_n, i_n) \operatorname{sgn} f_n(t_n, i_n) f_n .$$

Then it follows that P is a projection of $Z^{\perp}$ onto F since $Pf_n = f_n$ for all $n \in \mathbb{N}$. P is bounded since

$$\begin{aligned}
\|Px\| &= \left\| \sum_{n=1}^{\infty} x(t_n, i_n) \operatorname{sgn} f_n(t_n, i_n) f_n \right\| \\
&= \left\| T\Big(\sum_{n=1}^{\infty} x(t_n, i_n) \operatorname{sgn} f_n(t_n, i_n) e_n \Big) \right\| \\
&= \left\| \sum_{n=1}^{\infty} x(t_n, i_n) \operatorname{sgn} f_n(t_n, i_n) e_n \right\|_{c_0} \\
&= \sup_n |x(t_n, i_n)| \\
&\leq \sup |x(t, i)| \\
&= \|x\| .
\end{aligned}$$

Therefore P is a bounded projection of $Z^{\perp}$ onto F.

To finish the proof a projection $Q: C(TL) \to Z^{\perp}$ must be defined and then the map PQ will be the sought after projection of $C(TL)$ onto F.

Let U be the extension map defined in Lemma 1.3.5. By the definition of Uf, U is an isometry. Define $Qg =$

$g - URg$ where R is the restriction operator, i.e., $Rg = g|_Z$. Then Q is clearly linear and if $g \in Z^\perp$ then $Rg = 0$ which implies $URg = 0$. Therefore, for all $g \in Z^\perp$ $Qg = g$ which means Q is a projection onto $Z^\perp$. Also

$$\begin{aligned}
\|Qx\| &= \|x - URx\| \\
&\leq \|x\| + \|URx\| \\
&\leq \|x\|(1 + \|U\|\|R\|) \\
&\leq 2\|x\|.
\end{aligned}$$

<u>Theorem 1.3.8</u>. (The Isomorphic Case). Let $\{f_n\}$ be a sequence from $C(TL)$ such that:

(i) There exists $\lambda_0 > 0$ with

$$\frac{1}{\lambda}\|\Sigma a_n e_n\|_{c_0} \leq \|\Sigma a_n f_n\| \leq \lambda\|\Sigma a_n e_n\|_{c_0}$$

for some $\lambda < \lambda_0$ and all $\{a_n\} \in C_0$.

(ii) There is a sequence $\{t_n, i_n\} \subset TL$ with

(a) $f_n(t_n, i_n) > \frac{1}{\lambda}$

(b) $\lim_{n\to\infty} f_k(t_n, i_n) = 0$ for all k.

Then $F = \overline{\text{span}}\ \{f_n\}$ is complemented in $C(TL)$.

<u>Proof</u>. Let $X \equiv \{f \in C(TL): \lim_{n\to\infty} f(t_n, i_n) = 0\}$ where for each $n \in \mathbf{N}$, (t_n, i_n) is chosen so that $|f_n(t_n, i_n)| \geq \frac{1}{\lambda}$. This is possible by condition (ii). Clearly X is a

subspace of $C(TL)$. Next, consider the map $Q: X \to F$ defined by $Qg = \sum_{n=1}^{\infty} g(t_n, i_n) f_n$. Since $\lim_{n\to\infty} g(t_n, i_n) = 0$ for all $g \in X$ we have $\{g(t_n, i_n)\}_{n=1}^{\infty} \in c_0$ which implies that $Qg \in F$ for all g, and

$$\|Qg\| = \sup_t \left| \sum_{n=1}^{\infty} g(t_n, i_n) f_n(t, i) \right| \leq \lambda \sup_n |g(t_n, i_n)| \leq \lambda \|g\|$$

so that Q is a bounded linear map of X into F. Let $\tilde{Q}$ be the restriction of Q to F. It shall be shown below that $\tilde{Q}$ is invertible whenever (i) holds. First, an invertible map $\tilde{D}$ will be defined. Next it shall be shown that $\|\tilde{Q} - \tilde{D}\| \leq \frac{1}{\|\tilde{D}^{-1}\|}$ whenever (ii) holds from which it follows that $\tilde{Q}$ is invertible also [8, page 147].

Define an operator

$$\tilde{D}: F \to F \text{ by } \tilde{D}f = \sum_{n=1}^{\infty} (a_n f_n(t_n, i_n)) f_n$$

where $f = \Sigma a_j f_j \in F$. Then

$$\|\tilde{D}f\| = \left\| \sum_{n=1}^{\infty} (a_n f_n(t_n, i_n)) f_n \right\| \leq \lambda \sup_n |a_n f_n(t_n, i_n)|$$

$$\leq \lambda \sup_n |a_n| \cdot \sup_n |f_n(t_n, i_n)| \leq \lambda \left\| \sum_n a_n e_n \right\|_{c_0} \cdot \lambda \leq \lambda^3 \left\| \sum_n a_n f_n \right\|$$

$$= \lambda^3 \|f\|.$$

The inverse of $\tilde{D}$ is the operator $\tilde{D}^{-1}$ which is defined by

$$\tilde{D}^{-1}f = \sum_{n=1}^{\infty} \frac{a_n}{f_n(t_n, i_n)} f_n,$$

and

$$\|\tilde{D}^{-1}f\| = \left\|\sum_n \frac{a_n}{f_n(t_n, i_n)} f_n\right\| \leq \lambda \sup_n \left|\frac{a_n}{f_n(t_n, i_n)}\right|$$

$$\leq \lambda^2 \sup_n |a_n| \leq \lambda^3 \|f\|$$

so that $\tilde{D}^{-1}$ is also a bounded linear map. Also

$$\|\tilde{Q}f - \tilde{D}f\| = \left\|\sum_{n=1}^{\infty} f(t_n, i_n) f_n - \sum_{k=1}^{\infty} (a_k f_k(t_k, i_k) f_k\right\|$$

$$= \left\|\sum_{n=1}^{\infty}\Big(\sum_{j=1}^{\infty} a_j f_j(t_n, i_n)\Big) f_n - \sum_{k=1}^{\infty} a_k f_k(t_k, i_k) f_k\right\|$$

$$= \left\|\sum_{n=1}^{\infty}\Big(\sum_{j \neq n} a_j f_j(t_n, i_n)\Big) f_n\right\|$$

$$\leq \lambda \sup_n \left|\sum_{j \neq n} a_j f_j(t_n, i_n)\right|$$

$$\leq \lambda \sup_j |a_j| \sup_n \sum_{j \neq n} |f_j(t_n, i_n)|$$

$$\leq \lambda^2 \|f\| \sup_n \sum_{j \neq n} |f_j(t_n, i_n)| \tag{3}$$

But

$$\sum_{j\neq n}|f_j(t_n,i_n)| = \sum_{j=1}^{\infty}|f_j(t_n,i_n)| - |f_n(t_n,i_n)|$$

$$\leq \sup_t \sum_{j=1}^{\infty}|f_j(t,i)| - \frac{1}{\lambda} \qquad i = 1,2$$

$$\leq \lambda - \frac{1}{\lambda} \tag{4}$$

Therefore, putting (3) and (4) together

$$\|(\tilde{Q}-\tilde{D}f)\| \leq \lambda^2(\lambda - \tfrac{1}{\lambda})\|f\|$$

i.e., $\|(\tilde{Q}-\tilde{D})\| \leq \lambda^3 - \lambda$. Therefore, if $\lambda_0 > 1$ is chosen so that it is a root of $\lambda^6-\lambda^4-1 = 0$ then $\|(\tilde{Q}-\tilde{D})\| \leq \frac{1}{\|\tilde{D}^{-1}\|}$. For these values of λ, $\tilde{Q}$ will be an invertible operator and the map $\tilde{Q}^{-1}Q$: $X \to F$ is a projection of X onto F. It is bounded and linear because both $\tilde{Q}$ and $\tilde{Q}^{-1}$ are bounded linear maps and clearly $\tilde{Q}^{-1}\tilde{Q} = \tilde{Q}^{-1}\tilde{Q} = I$ on F, is idempotent.

To finish the proof a projection P of C(TL) onto X must now be defined. Then the projection QP will map C(TL) onto F. To this end, let Z $\equiv$ the cluster points of the set $\{(t_1,i_1),(t_2,i_2),\ldots\}$. Then Z is a closed subset of TL. For each $g \in C(TL)$ let Rg be the restriction of g to Z and let U(Rg) be the linear extension of Rg to all of TL as defined in Lemma 1.3.6. Then Pg = g−URg is clearly a linear map and is bounded since $\|Pg\| \leq 2\|g\|$ and

$\lim_{n\to\infty}(Pg)(t_n,i_n) = 0$ because g and URg agree on Z. Also, if $g \in X$, then $Rg \equiv 0$ so that $Pg = g$ for all $g \in X$, hence $P^2g = Pg$ and P is a bounded projection of C(TL) onto X.

There are many subspaces of TL isomorphic to c_0 which do not satisfy conditions (i) and (ii) of Theorem 1.3.9. The following lemma by R. C. James [9] gives that every subspace of C(TL) containing an isomorph of C_0 contains a subspace satisfying condition (i). So if condition (ii) which is more restrictive were removed the class of c_0-subspaces proven to be complemented in C(TL) would be greatly enlarged.

Lemma 1.3.9. (James) If a normed linear space contains a subspace isomorphic to c_0 then for any number $\delta>0$, there is a sequence $\{u_i\}$ of members of the unit ball such that

$$(1-\delta)\sup|a_i| < \|\Sigma a_i u_i\| \leq \sup|a_i|$$

for all finite sequences of numbers that are not all zero.

Theorem 1.3.10. If F is a subspace of C(TL) isomorphic to c_0 then F contains a subspace F_0 also isomorphic to c_0 and complemented in C(TL).

Proof. Choose $\delta > 0$ so that $1-\delta = \frac{1}{\lambda}$ where $1 < \lambda <$

λ_0. Then by Lemma 1.3.9 there is a sequence $\{f_i\}$ of members of the unit ball of F such that

$$\frac{1}{\lambda} \sup|a_i| \leq \|\Sigma a_i f_i\| \leq \lambda \sup|a_i| \tag{5}$$

for all finite sequences of numbers, and thus for all members of c_0. Thus $\overline{\text{span}}$ $\{f_i\}$ is a subspace of F isomorphic to c_0 satisfying condition (i) of Theorem 1.3.8. For the remainder of this proof the functions f_i shall be referenced by their image in D, that is we shall speak of them as functions f(t) (vs. f(t,i)) where f(t) = f(t,1) with left- hand limits $f(t^-) = f(t,0)$.

For each i, let $t_i \in [0,1]$ such that $|f(t_i)| > \frac{1}{2}$ and let t_0 be a cluster point of $\{t_i\}$. There is either a decreasing subsequence $\{t_{n_i}\} \downarrow t_0$ or an increasing sequence $\{t_{n_i}\} \uparrow t_0$. Without loss of generality assume that $\{t_{n_i}\} \downarrow t_0$, and from here on consider this sequence in [0,1] renaming it $\{t_i\}$ and its corresponding functions $\{f_i\}$.

By (5), $\sum_{n=1}^{\infty} |f_n(t_0)| < \infty$ so that given any $\epsilon < 0$ there is a positive integer n_1 such that $|f_n(t_0)| < \epsilon/2$ and by right continuity $|f_{n_1}(t)| < \epsilon$ for all $t \in I_1 = [t_0, t_0+\eta_1)$ for some $\eta_1 > 0$. Let $s_1 = t_{n_1}$ and define a function g_1 in C(TL) as follows

$$g_1(t) = \begin{cases} f_{n_1}(t) & \text{for } t \notin I_1 \\ 0 & \text{for } t \in I_1 \end{cases}$$

Next, by (5) there is an integer $n_2 > n_1$ such that $|f_{n_2}(t_0)| < \epsilon/4$ and by right continuity an interval $I_2 = [t_0, t_0+\eta_2) \subseteq I_1$ such that for all $t \in I_2$ $|f_{n_2}(t)| < \epsilon/2$. Let $s_2 = t_{n_2}$ and define $g_2 = 0$ on I_2 and $g_2 = f_{n_2}$ on $[0,1]\backslash I_2$. Containing this process one obtains a sequence $\{g_i\}$ in C(TL) where $\|g_i - f_{n_i}\| < \epsilon/2^i$ and yet $\lim\limits_{i\to\infty} g_n(s_i) = 0$ for each integer n. Moreover if ϵ is chosen to be less than $\frac{1}{32K}$ where K is the basis constant for the sequence $\{f_i\}$, then

$$\sum_{i=1}^{\infty} \|g_i - f_{n_i}\| \leq \frac{1}{32K} \sum_{i=1}^{\infty} 2^{-i} = \frac{1}{16K} \tag{6}$$

The sequence $\{g_i\}$ is then equivalent to the sequence $\{f_{n_i}\}$ and satisfies conditions (i) and (ii) of Theorem 1.3.8. From the proof of Theorem 1.3.8 there is a projection P of C(TL) onto $\overline{\text{span}}\ \{g_i\}$ with $\|P\| \leq 2$. Thus by (6)

$$\sum_{i=1}^{\infty} \|g_i - f_{n_i}\| \leq \frac{1}{16K} \leq \frac{1}{8K\|P\|}$$

which implies that $F_0 = \overline{\text{span}}\ \{f_{n_i}\}$ is not only isomorphic to the complemented subspace $\overline{\text{span}}\ \{g_i\}$ but is also complemented [14, page 6].

Corollary 1.3.11. C(TL) does not contain ℓ_∞.

Proof. Since it is known [18] that no isomorph of c_0 is complemented in ℓ_∞ this result follows immediately from Theorem 1.3.10.

1.4. Identifying the Complement of c_0-Subspaces of C(TL)

If F is a complemented subspace of C(TL) isomorphic to c_0 then we would like to identify the space G such that $C(TL) = F \oplus G$. Consider the following example.

Example 1.4.1. Consider the space $F = \{\Sigma a_n f_n : \{a_n\} \in c_0\}$ where $f_n = \chi_{[\frac{1}{2^n}, \frac{1}{2^{n-1}})}$ for $n > 1$ and $f_1 = \chi_{[\frac{1}{2}, 1]}$. F is an isometric isomorph of c_0 which is complemented in C(TL). Using the projection

$$Pg = \sum_{n=1}^{\infty} g(t_n) f_n \text{ where } t_n = 1 - \frac{1}{2^{n-1}}$$

it can be shown that $C(TL) = F \oplus G$ where $G = \ker P$ contains an isomorph of C(TL).

Consider the interval $[\frac{2}{3}, \frac{3}{4}]$. If I is an interval define $D(I) = \{f \in D[0,1]: f(t) = 0 \text{ for all } t \notin I\}$. Then $D[\frac{2}{3}, \frac{3}{4}]$ is naturally isomorphic to D[0,1] thus also to C(TL). Moreover, for all $g \in D[\frac{2}{3}, \frac{3}{4}]$, $Pg = \Sigma g(t_n) f_n \equiv 0$ since there are no t_n's in $[\frac{2}{3}, \frac{3}{4}]$. Thus $D[\frac{2}{3}, \frac{3}{4}]$ is contained in the space G.

The following lemma shall be instrumental in using the Pelczynski Decomposition method [12, page 54] to show that if $F \subseteq C(TL)$ is an isometric copy of c_0, not only is it complemented in $C(TL)$ but that its complement is in fact $C(TL)$ (up to an isomorphism).

<u>Lemma 1.4.2</u>. $C(TL)$ is isomorphic to the infinite direct sum $(C(TL)\oplus C(TL)\oplus\cdots)_{c_0}$.

<u>Proof</u>. Let $X = \{g \in D[0,1]: g(0) = 0\}$ and $J_n = [\frac{1}{2^n},\frac{1}{2^{n-1}})$.

$$\begin{aligned} X &\sim [\Sigma\oplus D(J_n)]_{c_0} \\ &\sim [\Sigma\oplus D[0,1)]_{c_0} \\ &\sim [\Sigma\oplus D[0,1)]_{c_0} \oplus [\Sigma\oplus D[0,1)]_{c_0} \\ &\sim X \oplus X. \end{aligned} \tag{7}$$

Also,

$$D[0,1) \sim X \oplus \mathbf{R} \tag{8}$$

since $T\colon D[0,1) \to (X\oplus\mathbf{R})_\infty$ by $T\colon f \mapsto (f(x)-f(0),\ f(0))$ satisfies

$$\|Tf\| = \max(\|f(x)-f(0)\|,|f(0)|) \le 2\|f(x)\|.$$

Further $T^{-1}\colon (g(x),a) \mapsto g(x) + a \in D[0,1)$ and

$$\|g(x)+a\| = \sup|g(x) + a| \le \sup|g(x)| + |a|$$
$$\le \|g\| + |a| \le 2\|(g(x),a)\|.$$

Next,

$$X \sim X \oplus D[0,1) \tag{9}$$

by the operator $f \mapsto (f|_{[0,\frac{1}{2})}, f|_{[\frac{1}{2},1)})$. Putting (7), (8), and (9) together yields

$$\begin{aligned} X &\sim X \oplus D[0,1\} \\ &\sim X \oplus (X \oplus \mathbf{R}) \qquad \text{(by (8))} \\ &\sim X \oplus \mathbf{R} \qquad \text{(by (7))} \\ &\sim D[0,1). \end{aligned}$$

Thus $X \oplus \mathbf{R} \sim D[0,1) \oplus \mathbf{R} \sim D[0,1]$ and

$$\begin{aligned} C(TL) &\sim D[0,1] \sim D[0,1) \sim X \\ &\sim [\Sigma \oplus D[0,1)]_{c_0} \\ &\sim [\Sigma \oplus D[0,1]]_{c_0} \\ &\sim [\Sigma \oplus C(TL)]_{c_0}. \end{aligned}$$

<u>Theorem</u> <u>1.4.3</u>. If F is a subspace of C(TL) isometrically isomorphic to c_0 then $C(TL) = F \oplus G$ where G is isomorphic to C(TL).

<u>Proof</u>. From Theorem 1.3.7 the composition map PQ will be a projection of C(TL) onto F where P: $Z^{\perp} \to F$ and Q: $C(TL) \to Z^{\perp}$ are defined

$$Pg = \sum_{n=1}^{\infty} g(t_n, i_n) \operatorname{sgn} f_n(t_n, i_n) f_n \text{ and } Qg = g - URg.$$

Let [a,b] be an interval of [0,1] such that the set $\{(t,i): t \in [a,b]\}$ does not meet the set $\{(t_1,i_1),(t_2,i_2),\dots\}$ and consider D[a,b] which is isomorphic to C(TL) and $P|_{D[a,b]_{C(TL)}}$ (where $D[a,b]_{C(TL)} = \{f \in C(TL): f(t,1) = g(t)$ and $f(t,0) = g(t^-)$ for some $g \in D[a,b]\}$) will be the zero map. Thus $D[a,b]_{C(TL)}$ is completely contained in G since for all $g \in D[a,b]_{C(TL)}$, $g|_Z = 0$ which means that $URg = 0$ and implies that $Qg = g$, therefore $(PQ)g = Pg = 0$.

Let R map D[0,1] to D[a,b] naturally. $R|_G$ mapping G onto D[a,b] and thus onto $D_{[a,b]_{C(TL)}}$ is a bounded linear projection and thus $D[a,b]_{C(TL)}$ is complemented in G.

At this point it has been shown that $C(TL) = F \oplus G$ where $F \sim c_0$ and $G = \ker QP$ is isomorphic to $C(TL) \oplus U$. Using the Pelczynski Decomposition method and Lemma 1.4.2

$$\begin{aligned} C(TL) \oplus G &\sim C(TL) \oplus (C(TL) \oplus U) \\ &\sim (C(TL) \oplus C(TL)) \oplus U \\ &\sim C(TL) \oplus U \\ \therefore\ C(TL) \oplus G &\sim G \end{aligned}$$

and

$$
\begin{aligned}
C(TL) \oplus G &\sim [\Sigma \oplus C(TL)]_{c_0} \oplus G \\
&\sim [\Sigma \oplus (F \oplus G)]_{c_0} \oplus G \\
&\sim [\Sigma \oplus F]_{c_0} \oplus [\Sigma \oplus G]_{c_0} \oplus G \\
&\sim [\Sigma \oplus F]_{c_0} + [\Sigma \oplus G]_{c_0} \\
&\sim [\Sigma \oplus (F \oplus G)]_{c_0} \\
&\sim [\Sigma \oplus C(TL)]_{c_0} \\
&\sim C(TL).
\end{aligned}
$$

Thus $C(TL) \sim C(TL) \oplus G \sim G$. Note that by Lemma 1.3.2 this is true not only for $G = \ker QP$ where QP is as defined above, but for any subspace G' where F is isometrically isomorphic to c_0 if $C(TL) = F \oplus G'$ then G' is isomorphic to $C(TL)$.

The remainder of this section shall be devoted to showing that whenever $C(TL)$ is projected onto an isomorph of c_0 then the complementary space contains an isomorph of $C(TL)$. We begin with a discussion of $C(\Delta)$ and an isometric copy of $C(\Delta)$ contained in $D[0,1]$. All of these theorems will use $D[0,1]$ (vs. $C(TL)$) as the background space but will deal with isomorphic properties that carry over to $C(TL)$.

Recall the Haar system, which is known to be a monotone basis for $C(\Delta)$. Let

$$\Delta_{n,k} \qquad 0 \leq k \leq 2^n-1 \qquad n = 0,1,2,\ldots$$

be a collection of sets such that

$$\Delta = \bigcup_{k=1}^{2^n-1} \Delta_{n,k}$$

$$\Delta_{n,k} \cap \Delta_{n,\ell} = \phi \qquad \text{for } k \neq \ell$$

$$\Delta_{n,k} = \Delta_{n+1,2k} \cup \Delta_{n+1,2k+1}$$

The Haar functions $\{\phi_n\}$ are defined by

$$\phi_0 = \chi_{\Delta_{0,0}}; \qquad \phi_{2^{n-1}+k} = \chi_{\Delta n,2k} - \chi_{\Delta n,2k+1}$$

$$0 \leq k \leq 2^n-1, \; n = 1,2,\ldots$$

For example, choose $I_{0,0} = [0,1]$ and for each $n = 1,2,3,\ldots$ let $I_{n,k} = [\frac{k}{2^n}, \frac{k+1}{2^n})$ for $0 \leq k < 2^n-1$ and $I_{n,2^n-1} = [\frac{2^n-1}{2^n}, 1]$. Then put $\Delta_{0,0} = I_{0,0} \cap \Delta$ and $\Delta_{n,k} = I_{n,k} \cap \Delta$ for $n = 1,2,3,\ldots$ $0 \leq k < 2^n-1$. Now, we can define the space $D(\Delta)$ to be the closed linear span of the functions $\{\psi_k\}_{k=0}^{\infty}$ where $\psi_0 = \chi_{I_{0,0}}$; $\psi_{2^{n-1}+k} = \chi_{I_{n,2k}} - \chi_{I_{n,2k+1}}$; $0 \leq k \; 2^{n-1}-1$; $n = 0,1,2,\ldots$ Clearly the map which takes ϕ_i to ψ_i is an isometric isomorphism from $C(\Delta)$ onto $D(\Delta) \subseteq D[0,1]$.

The next three theorems were proven by J. Lindestrauss and A. Pelczynski in [11] for the space $C(\Delta)$. The arguments used to prove then for $D(\Delta)$ parallel those of Lindenstrauss and Pelczynski.

Lemma 1.4.4. Let $\{g_i\}_{1\leq i\leq 2}$ be step functions in $D[a,b)$. Put (for $0 \leq k \leq 2^{n-1}$, $n = 1,2,\dots,p$)

$$A_{0,0} = \bar{0}^{1}(1),\ A_{n,2k} = g^{-1}_{2^{n-1}+k}(1),\ A_{n,2k+1} = g^{-1}_{2^{n-1}+k}(-1) \quad (10)$$

Assume that $-1 \leq g_i \leq 1$ and that each g_i assumes the values 1 and -1 (11)

Assume that for each $0 \leq k < 2^n$, $n = 1,2,\dots,p$ $\operatorname{supp} g_1 \subset A_{0,0}$, $\operatorname{supp} g_{2^n+k} \subset A_{n,k}$ (12)

where $\operatorname{supp} f = \{s: |f(s)| \neq 0\}$. Then the sequence $\{g_i\}_{0\leq i<2^p}$ and $\{\phi_i\}_{0\leq i<2^p}$ are isometrically equivalent.

Proof. Observe first by (10) and (12)

(i) the functions of the same level n have disjoint supports, i.e., $g_{2^n+k}g_{2^n+\ell} = 0$ for $0 \leq k < \ell < 2^n$.

(ii) if, $0 \leq i < 2^n$ then g_i is constant on $A_{n,k}$; that constant being either 1, -1 or 0.

It shall be shown by induction on n that

(iii) for any choice of scalars $\{t_i\}_{0\leq i<2^n}$, the maximum of the function $\left|\sum_{i=0}^{2^n-1} t_i g_i(\cdot)\right|$ is attained on one of the sets $A_{n,k}$, $0 \le k < 2^n$.

For $n = 0$, (iii) is obvious. Let $n \le p$ and let $\{t_i\}_{0 \le i < 2^n}$ be scalars. By (ii) the function $\sum_{i=1}^{2^{n-1}-1} t_i g_i(\cdot)$ is constant on each of the sets $A_{n-1,k}$. Denote this constant by $c_{n-1,k}$. By (10), (12), and (i)

$$\sup_{s \in A_{n-1,k}} \left|\sum_{i=1}^{2^n-1} t_i g_i(s)\right| = \sup_{s \in A_{n-1,k}} \left|c_{n-1,k} + t_{2^{n-1}+k} g_{2^{n-1}+k}(s)\right|$$

$$= \sup_{s \in (A_{n-1,2k} \cup A_{n-1,2k+1})} \left|c_{n-1,k} + t_{2^n+k} g_{2^{n-1}+k}(s)\right|$$

$$= |c_{n-1,j}| + |t_{2^{n-1}+k}| \tag{13}$$

If $s \notin \bigcup_{k=1}^{2^{n-1}-1} A_{n-1,k}$ then by (12) $\sum_{i=0}^{2^n-1} t_i g_i(s) = \sum_{i=1}^{2^{n-1}-1} t_i g_i(s)$ and by the induction hypothesis, there is a k, $0 \le k < 2^{n-1}$ such that $\left|\sum_{i=0}^{2^n-1} t_i g_i(s)\right| \le c_{n-1,k}$. If $s \in A_{n-1,k}$ then by equation (13) the max will be attained on one of the sets $A_{n,k}$.

The lemma now follows from (iii) and the observation that g_i on $A_{p,k}$ is equal to ψ_i on $I_{p,k}$ so that $\left\|\sum_{i=0}^{m} t_i g_i(\cdot)\right\| = \left\|\sum_{i=0}^{m} t_i \psi_i(\cdot)\right\|$.

<u>Note</u>: In the proof by Lindesntrauss and Pelczynski the sets $A_{p,k}$ were clopen which insured that the maximum of the functions g_i would be attained on a non-empty set with

non-empty interior. Because our functions are step functions assuming only finitely many values these properties still hold.

<u>Proposition 1.4.5</u>. Let $T\colon D[0,1] \to X$ be such that (*) for every $\epsilon > 0$ and for every $[a,b)$ of $[0,1]$ there is an $f \in D[a,b)$ such that $\|f\| = 1$ and $\|Tf\| \leq \epsilon$.

Then for each $\epsilon > 0$ there is a sequence $\{g_i\}_{i=0}^{\infty}$ in $D[a,b)$ which is isometrically equivalent to the Haar system $\{\psi\}_{i=1}^{\infty}$ and such that $\sum_{i=1}^{\infty} \|Tg\| < \epsilon$.

<u>Proof</u>. First observe that since the step functions are dense in $D[a,b)$ (by Lemma 1.2.6) we may assume that f is a step function. Next, note that we may select a step function which assumes its norm. For if f does not attain one of the values 1 or -1 in the interval $[a,b)$ there is an $x_0 \in [a,b)$ such that either $\lim_{x \to x_0^-} f(x) = 1$ or $\lim_{x \to x_0^-} f(x) = -1$. Assume, without loss of generality, that $\lim_{x \to x_0^-} f(x) = 1$. Then given $\epsilon' > 0$ there is an interval $[x_0 - \delta, x_0)$ such that $|f(x)-1| < \epsilon'/2$. The function f may be approximated by a step function defined to be 1 on $[x_0 - \delta, x_0)$.

Now, if the step function above does not assume both valus +1 or -1 divide the interval $[a,b)$ into two subintervals $[a,b_1)$ and $[b_1,b)$. Then pick a step function f_1 on $[a,b_1)$ and a step function f_2 on $[b_1,b)$ each of which attains the values 1 or -1. Either f_1+f_2 or f_1-f_2 will then be a step function satisfying $(*)$ and achieving both values $+1$ and -1 on the given interval $[a,b)$.

By using $(*)$ and taking f to be a step function attaining both values 1 and -1 as outlined above a sequence $\{g_i\}$ of step functions may be defined for any given $\epsilon > 0$ so that for $i = 0,1,2,\ldots$ $\|Tg_i\| < 2^{-i-1}\epsilon$; and for $0 \leq k < 2^{n-1}$, $n = 1,2,\ldots$ $g_1 \in C(g_0^{-1}(1))$, $g_{2^n+2k} \in C(g_{2^{n-1}+k}^{-1}(1))$, $g_{2^n+2k+1} \in C(g_{2^{n-1}+k}^{-1}(-1))$. Thus by Lemma 1.4.4 the sequence $\{g_i\}$ is isometrically equivalent to $\{\psi_i\}$ and by construction $\Sigma\|Tg_i\| \leq \epsilon$.

Theorem 1.4.6. If $C(TL) = F \oplus G$ then either F contains an isomorph of $C(\Delta)$ or G contains an isomorph of $C(TL)$.

Proof. Suppose $C(TL) = F \oplus G$. Let $Q\colon C(TL) \to G$ be the projection of $C(TL)$ onto G. If Q satisfies $(*)$ then by Proposition 1.4.5 there is for every $\epsilon > 0$ a sequence $\{g_i\}$ in $C(TL)$ such that $\{g_i\}$ is isometrically equivalent to the Haar basis of $D(\Delta)$ and $\sum_{i=1}^{\infty} \|g_i - Pg_i\| = \sum_{i=1}^{\infty} \|Qg_i\| < \epsilon$ (where $P = I - Q$ is a projection of $C(TL)$ onto F). $\overline{\text{span}}\ \{Pg_i\} \subseteq F$ and is equivalent to $\overline{\text{span}}\ \{g_i\} \sim D(\Delta)$ by the Paley-Wiener type stability theorem [12, page 5]. Thus F has a subspace isomorphic to $D(\Delta)$.

If Q does not satisfy $(*)$, then there is an interval $[a,b) \subseteq [0,1]$ such that $\|Qf\| > \epsilon\|f\|$ for all $f \in D[a,b)_{C(TL)}$

with $\|f\|=1$. Thus $Q|_{D[a,b)_{C(TL)}}$ is an isomorphism and hence defines a subspace of G isomorphic to D[a,b) and thus to D[0,1] and C(TL).

<u>Corollary</u> <u>1.4.7</u>. If C(TL) = $F \oplus G$ where F is isomorphic to c_0 then G contains an isomorph of C(TL).

CHAPTER II

RANGE INCLUSION AND FACTORIZATION OF OPERATORS ON BANACH SPACES

2.1. Background Theorems

The following theorem by Douglas [5] is the motivation for the essence of this chapter.

Theorem 2.1.1. (Douglas) If A and B are operators on a Hilbert space H then the following are equivalent:

(i) $A = BC$ for some operator C on H

(ii) $\|A^*x\| \leq k\|B^*x\|$ for some $k \geq 0$ and all $x \in H$

(iii) Range A $\subseteq$ Range B.

Terminology. If (i) holds it is said that C is a right factor of A and that there is a factorization of the operator A. If (ii) holds it is said that B majorizes A. If (iii) holds it is said that there is range inclusion for operators A and B. The space $\mathbf{R}^n$ with ℓ_1-norm is denoted by ℓ_1^n.

Douglas' theorem is generally not true in an arbitrary Banach space. Clearly (i) implies each of (ii) and (iii) even in Banach space and it can be shown that (iii) implies (ii) [7]. R. Bouldin [16] gave an example illustrating that (ii) does not imply (iii). The correct generalization of Douglas' theorem to Banach spaces was given by Embry in [7] and is as follows.

Theorem 2.1.2. (Embry) Let D and E be operators on a Banach space X. The following conditions are equivalent:

(i$'$) $D = FE$ for some operator F: range $E \to X$

(ii$'$) $\|Dx\| \leq k\|Ex\|$ for some $k \geq 0$ and all x in X

(iii$'$) range $D^* \subseteq$ range E^*.

To see that Embry's theorem is a generalization of Douglas' theorem let $A = D^*$ and $B = E^*$ which is possible since every Hilbert space operator is an adjoint. Condition (i$'$) then becomes $A^* = FB^*$ for some operator F: Range $B^* \to X$. Since X is a Hilbert space we may extend F to all of X which yields (i) of Douglas' theorem. With the same assignment: $A = D^*$ and $B = E^*$, statements (ii$'$) and (iii$'$) become (ii) and (iii) of Douglas' theorem.

Embry also presented a counterexample in [7], due to Douglas, of operators A and B on a non-separable, non-reflexive Banach space for which range inclusion (iii) does not imply factorization (i). In this chapter Douglas' counterexample will be simplified and extended. On many of the classical Banach spaces counterexamples to (iii) $\Rightarrow$ (i) are shown to exist.

2.2. Classical Banach Spaces in which Range Inclusion Does not Imply Factorization

The following lemma consists of a pair of sufficient conditions for a counterexample to be found in a given space; one of which is based on Douglas' counterexample [7].

Lemma 2.2.1. Let X and W be a Banach spaces. Let $T: X \to W$ be a surjective map and let $Z = X \oplus W$. If range inclusion implies factorization for operators on Z, then

a) W is isomorphc to a subspace of X, and

b) $\ker T$ is complemented in X.

Proof. Define operators A and B on Z by

$$A(x,w) = (0,w) \quad \text{and} \quad B(x,w) = (0,Tx).$$

then range $A \subset$ range B, so by assumption there exists an operator C on Z such that $A = BC$. Now define

$$Q: Z \to X \text{ by } Q(x,w) = x$$

$$P: Z \to W \text{ by } P(x,w) = w$$

and $$i: W \to Z \text{ by } iw = (0,w).$$

To prove a) define a map $S = QCi$ from W into X. For any $w \in W$,

$$(0,w) = A(0,w) = BC(0,w)$$
$$= B(QCiw,\ PCiw) = (0,TQCiw) = (0,TSw)$$

so that $\|w\| \le \|T\|\|Sw\|$. Hence $\frac{1}{\|T\|}\|w\| \le \|Sw\| \le \|Q\|\|C\|\|w\|$, so S is an isomorphism from W into X.

As for b), consider the operator $R = I-QCiT$. Since

$$(0,Tx) = A(0,Tx) = BC(0,Tx) = B(QCiTx,PCiTx) = (0,TQCiTx)$$

it follows that

$$T(Rx) = T(I - QCiT)(x) = Tx - TQCiTx = 0.$$

Thus R maps into ker T, i.e., range $R \subseteq$ ker T. Also, if $y \in$ ker T, $(I-QCiT)y = y$. Thus ker T = range R and $R^2x = Rx$ so that R is a projection of X onto ker T.

The remainder of this section is devoted to showing that certain classical Banach spaces are not of the form discussed in the lemma which leads to the conlusion that range inclusion does not imply factorization in these spaces.

<u>Theorem</u> <u>2.2.2</u>. Let Z be L_1, ℓ_p $(1 < p < \infty,\ p \neq 2)$ or c_0. Then there exists operators A and B on Z with range $A \subset$ range B, yet $A \neq BC$ for any operator C on Z.

Proof. In the case $Z = L_1$, we write $Z = X \oplus W$ where $X = \ell_1$ and $W = L_1$ [6]. Since L_1 is separable, there exists an operator T from ℓ_1 onto L_1 [13, page 37]. Since L_1 is not isomorphic to any subspace of ℓ_1, range inclusion does not imply factorization for operators on L_1.

If $Z = \ell_p$, let Y be a subspace of $\ell_p^* = \ell_q$ which is ismorphic to ℓ_q and not complemented in ℓ_q. The existence of such a Y was proven in [16] for $2 < p < \infty$ and in [1] for $1 < p < 2$. Now consider the space ${}^{\perp}Y = \{x \in \ell_p: \langle x,y \rangle = 0\}$. Suppose ${}^{\perp}Y$ is complemented in ℓ_p. Then there is a projection $P: \ell_p \to \ell_p$ with range ${}^{\perp}Y$ and $P^*: \ell_q \to \ell_q$ is also a projection. Moreover,

$$\begin{aligned} P^*x^* = 0 &\Leftrightarrow \text{for all } z \in Z,\ \langle P^*x^*, z \rangle = 0 \\ &\Leftrightarrow \langle x^*, Pz \rangle = 0 \\ &\Leftrightarrow x^* \in ({}^{\perp}Y)^{\perp} = Y \end{aligned}$$

Thus range $(I-P^*) = \ker P^* = Y$ which contradicts the fact that Y is uncomplemented. Therefore ${}^{\perp}Y$ is not complemented in ℓ_p.

Now, by part b) of Lemma 2.2.1 range inclusion does not imply factorization for operators on $\ell_p \oplus (\ell_p/{}^{\perp}Y)$ since the canonical quotient map $\pi: \ell_p \to \ell_p/{}^{\perp}Y$ is surjective. Since the subspace Y is weakly closed

$$(\ell_p/{}^{\perp}Y)^* \sim ({}^{\perp}Y)^{\perp} \sim Y \sim \ell_q$$

so that

$$(\ell_p \oplus \ell_p/{}^{\perp}Y)^* \sim \ell_q \oplus \ell_q \sim \ell_q$$

which implies that ,

$$\ell_p \oplus (\ell_p/{}^{\perp}Y) \sim \ell_p .$$

So range inclusion does not imply factorization for operators on ℓ_p $(1 < p < \infty,\ p \neq 2)$.

The case of $Z = c_0$ is basically the same as that of ℓ_p. Bourgain [4] has proven the existence of finite dimensional subspaces $E_n \subset \ell_1^{d(n)}$ with E_n uniformly isomorphic to $\ell_1^{\dim E_n}$ yet

$$\lim_{n\to\infty} \inf\{\|P\|: P: \ell_1^{d(n)} \to E_n \text{ is a projection}\} = \infty. \quad (1)$$

If $Y = \left(\sum_{n=1}^{\infty} \oplus E_n\right)_{\ell_1}$ is complemented in ℓ_1, there is a bounded linear projection $P: \ell_1 \to Y$. Let $P_n: Y \to E_n$ be the natural projection of Y onto E_n and $Q_n: \ell_1^{d(n)} \to \left(\sum_j \oplus \ell_1^{d(j)}\right)_{\ell_1}$ be the natural embedding of $\ell_1^{d(n)}$ into $\left(\sum_j \oplus \ell_1^{d(j)}\right)_{\ell_1}$ then a projection $R_n: \ell_1^{d(n)} \to E_n$ may be defined as the composition map $P_n P Q_n$. Since $\|Q_n\| = 1$ and $\|P_n\| \leq$

$\|P\|$, for each n R_n is a bounded linear projection $\ell_1^{d(n)}$ onto E_n whose norm is dominated by $\|P\|^2$ which contradicts (1). Therefore $Y = \left(\sum\oplus E_n\right)_{\ell_1}$ is a noncomplemented subspace of $\left(\sum\oplus\ell_1^{d(n)}\right)_{\ell_1} \sim \ell_1$.

Since each E_n is finite dimensional and weak* closed in $\ell_1^{d(n)}$ and the weak* topology of $\left(\sum\oplus\ell_1^{d(n)}\right)_{\ell_1}$ is the product topology, i.e., the topology of pointwise convergence, $\left(\sum\oplus E_n\right)_{\ell_1}$ is a weak* closed subspace of $\left(\sum\oplus\ell_1^{d(n)}\right)_{\ell_1}$.

As before ${}^{\perp}Y \subsetneq c_0$ is noncomplemented also. It follows then that range inclusion does not imply factorization for operators on $c_0\oplus(c_0/{}^{\perp}Y)$. Now

$$(c_0/{}^{\perp}Y)^* \sim ({}^{\perp}Y)^{\perp} = \{f \in \ell_1 : \langle z,f\rangle = 0 \text{ for all } z \in {}^{\perp}Y\} = Y,$$

since Y is weak* closed. Since $(c_0/{}^{\perp}Y)^* \sim \ell_1$, $(c_0/{}^{\perp}Y)$ is a $\mathcal{L}_\infty$-space [see 13], and since each quotient of c_0 is isomorphic to a subspace of c_0 [10], $(c_0/{}^{\perp}Y)$ is isomorphic to a $\mathcal{L}_\infty$ subspace of c_0. However, every $\mathcal{L}_\infty$ subspace of c_0 is itself isomorphic to c_0 [10]. It follows that

$$c_0 \oplus (c_0/{}^{\perp}Y) \sim c_0 \oplus c_0 \sim c_0,$$

so range inclusion does not imply factorization for operators on c_0.

Proposition 2.2.3. If Y is a complemented subspace of X and if range inclusion does not imply factorization for operators on Y then range inclusion does not imply factorization for operators on X.

Proof. Suppose range inclusion implies factorization on X. Let A,B be operators on Y such that range A $\subset$ range B. Then

$$\begin{aligned} AY \subseteq BY &\Rightarrow A(PX) \subseteq B(PX) \text{ where } P\colon X \to Y \text{ is a projection} \\ &\Rightarrow \text{range } AP \subseteq \text{range } BP \\ &\Rightarrow AP = (BP)\tilde{C} \text{ for some operator } \tilde{C} \text{ on } X \\ &\Rightarrow Ay = APy = B(P\tilde{C})y \text{ for all } y \in Y \\ &\Rightarrow A = BC \text{ where } C = P\tilde{C}|_Y. \end{aligned}$$

Since each separable C(K) and also C(TL) contains a complemented isomorph of c_0, and since each $\mathcal{L}_p$ space, $1 \leq p < \infty$ contains a complemented subspace isomorphic to ℓ_p range inclusion does not imply factorization on any of these spaces. Also, it is clear that the proof presented above for $Z_1 = L_1$ applies to any separable space X which is not isomorphic to a subspace of ℓ_1 and which contains a complemented subspace isomorphic to ℓ_1.

The case of $Z = \ell_1$ has also been considered in the above setting. The lifting property of ℓ_1 [13, page 38] is: if X and Y are Banach spaces, if $B: X \to Y$ is a surjective linear operator and if $A: \ell_1 \to Y$ then there exists $\widetilde{A}: \ell_1 \to X$ such that $A = B\widetilde{A}$. Thus if B is an operator on ℓ_1 with closed range, and range $A \subseteq$ range B, put $X = \ell_1$ and $Y =$ range B then by the lifting property of ℓ_1 there does exist an operator C with $A = BC$. Since the lemma always produces operators with closed ranges, it is not sufficient to determine the equivalence of range inclusion and factorization for operators on ℓ_1.

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