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MELTING OF A SEMI-INFINITE SOLID

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NOMENCLATURE

Arabic Symbols

A	one half the period of a heating cycle
c	specific heat of solid
f	value of r at time y_m
f_1, f_2	values of r at times y_{m1} and y_{m2} respectively
H	time dependent heat flux at exposed face of solid
H^*	constant heat flux at exposed face of solid or amplitude of heat pulse (see Figure 1, page 13)
H_r	H/H^*: dimensionless heat flux
h	convective heat transfer coefficient
i	integer index on space mesh
j	integer index on time mesh
k	thermal conductivity of solid
L	latent heat of solid
M	$\sqrt{\pi}\, c(T_m - T_o)/2L$: dimensionless property parameter
P	$1/M + 2/\sqrt{\pi}$: dimensionless property parameter
r	$(y - y_o)/A$: fraction of heat pulse that has occurred at time y
s	thickness melted
T	time dependent temperature of solid
T_g	environmental temperature
T_m	melting temperature of solid (constant)
T_o	initial, constant temperature of solid
T_s	time dependent surface temperature

t	time
t_m	time melting first commences when $H = H(t)$
t_m^*	time melting commences when $H(t) = H^*$
v	$(T - T_o)/\sqrt{\pi}\,(T_m - T_o)$: dimensionless temperature
v_e^*	steady state dimensionless temperature when $H(t) = H^*$
x	distance measured from position of exposed face at $t = 0$
y	t/t_m^*: dimensionless time
y_o	value of y when heat pulse begins during a cycle
y_1	value of y when the heat pulse ends during a cycle
y_m	t_m/t_m^*: value of y when melting first commences if $H = H(t)$
y_{m1}, y_{m2}	quasi-steady state values of y when melting commences during two successive heat pulses
y_{10}, y_{20}	quasi-steady state values of y at the beginning of two successive heat pulses
y_{11}, y_{21}	quasi-steady state values of y at the end of two successive heat pulses
z	$(x - s)/\sqrt{\alpha t_m^*}$: dimensionless distance

<u>Greek Symbols</u>

α	$k/\rho c$, thermal diffusivity of solid
μ	$\rho L(ds/dt)/H^*$: dimensionless melting rate
μ_e^*	steady state value of μ when $H(t) = H^*$
ρ	density of solid
σ	$\rho Ls/H^* t_m^*$: dimensionless thickness melted
σ_o	value of σ at $y = y_o$
σ_1	value of σ at $y = y_1$

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SUMMARY

General equations were obtained for use in the solution of the problem of melting of a semi-infinite solid exposed to a heat flux which is a known function of time. Only one-dimensional heat conduction was considered, and thermophysical properties were assumed constant. The liquid melt was assumed to be removed, by some unspecified mechanism, immediately upon formation. The general equations were used to study the problem in detail when the heat flux is a square wave type.

The temperature at the exposed face during the initial pre-melting stage and the time that melting first commences was determined for a square wave flux. It was found that the melting time for the case of a very small heating cycle period was four times greater than that of the case when the heat flux is constant for all times and equal to the peak of the cyclic heat flux.

A numerical finite difference method was employed to solve the boundary value problem after melting first commenced. As was expected, the thickness of solid melted, during a given heating time, was found to decrease with decreasing heating cycle period. Its magnitude was found to be less than one half that melted when the heat flux is constant and equal to the peak cyclic flux.

The numerical solution predicted that the rate of melting could exceed the steady-state rate attained when the heat flux is constant for all times at the peak cyclic value, and could pass through a maximum during a heat pulse. These same characteristics of the melting rate must

occur if the system achieves a quasi-steady state where the temperatures are periodic with heating and steady-state conditions are approached in each cycle before a heat pulse ends. Since it appeared logical that the system may approach this quasi-steady state, the time that melting begins anew during a heat pulse and the thickness melted at the end of a pulse were determined for such a condition.

CHAPTER I

INTRODUCTION

Transient heat-conduction problems involving phase changes are becoming increasingly important to modern technology. The melting or freezing of solids has applications in the thermal protection of high-speed vehicles, food processing, and casting of metals, to mention only a few. These problems are nonlinear, because they involve a moving boundary (the melting front) whose location is unknown.

Generally, previously published work considers unidimensional heat-conduction, the melt either retained or removed immediately upon formation, and constant thermophysical properties in the solid and/or melt. Methods of solutions can be grouped into analytical solutions, complete numerical difference solutions, and "lumped parameter" solutions. For complete difference solutions, both the space and time variables are written in difference form for solution either with a digital computer or by graphical methods. In the lumped parameter solutions, the solid region is divided into discrete space lumps, while time is retained as a continuous variable. By doing this, the boundary value problem can be reduced to a set of coupled ordinary differential equations which can be conveniently solved on an analog computer.

Temperatures in the solid, before melting first commences, can be readily calculated [1].* Landau [2] and Masters [3] employed a numerical

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*Numbers in brackets designate references cited in the Bibliography.

method to solve the transient heat conduction problem after melting commences in a semi-infinite solid, and a slab, respectively. Both considered only the case where the heat flux to the exposed surface is large and constant and the thermophysical properties are constant. Landau assumed the melt to be removed immediately upon formation, and he obtained an analytical solution for the limiting case of infinite latent heat. He also found the steady state solution. Masters differed from Landau in obtaining the same steady state solution by simultaneously solving the heat and wave equations, instead of considering only the heat conduction equation. Masters also studied the effect of a film of melt on the heat transfer across the melting front.

Few solutions have been obtained for cases where the heat flux at the exposed face varies with time. Sunderland and Grosh [4] considered the melting of a semi-infinite solid heated by convection from a constant temperature environment. They presented both a numerical and a graphical method of solution of the transient process. Both of these methods appear applicable to situations where the environment temperature varies with time. Lotkin [5] derived difference equations for melting of an infinite slab (with variable thermophysical properties) that is heated by a flux which varies with time. He used unequal subdivisions in both space and time variables. Finally, Goodman [6, 7] defined a thermal layer which is analogous to the boundary layer in fluid flow. By utilizing this concept and employing techniques similar to those used by Karman and Pohlhousen in boundary-layer theory, he was able to develop approximate solutions for several different cases. His method of solution is applicable to cases where the heat flux varies with time. However, the method is complex

unless the heat flux is either monotonically increasing with time or constant.

Murray and Landis [8] briefly reviewed current methods of solution of problems with melting or freezing. A more comprehensive survey of published literature on this subject was given by Sunderland and Grosh [9].

The problem considered in the present work is the melting of a semi-infinite solid exposed at its face to a heat flux of the square wave type. Melted material is assumed to be removed immediately upon formation. This type of problem is encountered in environments of a cyclic nature. For instance, the specific heat flux considered is related in a general way to the energy output of a laser.

CHAPTER II

THE GENERAL PROBLEM OF MELTING OF A SEMI-INFINITE SOLID

In this chapter, the problem considered is that of one-dimensional heat conduction in a semi-infinite solid that is exposed at its face to a time varying heat flux $H(t)$. Initially the solid extends from $x = 0$ to $x = \infty$ and is at a uniform temperature T_o. If heating continues long enough, the exposed face of the solid reaches the melting temperature T_m and melting commences. At some later time melting may stop if the heat flux becomes insufficient to sustain a surface temperature equal to T_m and then begin again if the heat flux rises sufficiently. The liquid melt is assumed to be removed immediately upon formation, so that the exposed face of the solid moves inward from a position at $x = 0$ initially to a position $x = s(t)$ at time t. The thickness melted $s(t)$ is a quantity of primary interest. This problem is the same as that considered by Landau, except that a different heat flux is considered. Landau assumed H to be constant and equal to H^*.

If the density ρ, specific heat c, thermal conductivity k, and latent heat of fusion L of the solid are constant, the boundary value problem describing the process is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2} , \tag{II.1}$$

$$T(x, 0) = T_o , \tag{II.2}$$

$$\lim_{x \to \infty} T(x,t) = T_o \ , \tag{II.3}$$

$$H(t) = -k \left(\frac{\partial T}{\partial x}\right)_{x=s} + \rho L \frac{ds}{dt} \ . \tag{II.4}$$

The last equation is an energy balance at the exposed surface, which equates the heat flux $H(t)$ at the surface to the sum of the rate of heat conduction into the solid and the rate of latent energy absorption at the surface. This boundary condition is seen to hold true during both melting and non-melting if the conditions

$$\frac{ds}{dt} \geq 0 \quad \text{when} \quad T(s,t) = T_m \ , \tag{II.5}$$

$$\frac{ds}{dt} = 0 \quad \text{when} \quad T(s,t) < T_m \tag{II.6}$$

are imposed.

The solution of the boundary value problem is simplified by the introduction of the new dimensionless variables

$$v = v(z,y) = \frac{T - T_o}{\sqrt{\pi}\ (T_m - T_o)} \ , \tag{II.7}$$

$$z = \frac{x - s(t)}{\sqrt{\alpha\ t_m^*}} \ , \tag{II.8}$$

$$y = \frac{t}{t_m^*} \ , \tag{II.9}$$

$$\sigma = \sigma(y) = \frac{\rho L}{H^*} \ \frac{s}{t_m^*} \ , \tag{II.10}$$

$$\mu = \mu(y) = \frac{d\sigma}{dy} = t_m^* \frac{d\sigma}{dt} = \frac{\rho L}{H^*} \frac{ds}{dt} . \qquad \text{(II.11)}$$

These variables are essentially the same as those used by Landau. The quantity t_m^* is the time at which the surface temperature reaches the melting point when the heat flux at the exposed face is constant and equal to H^*. According to Landau,

$$t_m^* = \frac{\pi}{4} \frac{k\rho c}{(H^*)^2} (T_m - T_o)^2 = \alpha \left(\frac{\rho L M}{H^*}\right)^2 \qquad \text{(II.12)}$$

$$M = \frac{\sqrt{\pi}}{2} \frac{c}{L} (T_m - T_o) . \qquad \text{(II.13)}$$

The relationships

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial z} \frac{\partial z}{\partial t} + \frac{\partial T}{\partial y} \frac{dy}{dt} \qquad \text{(II.14)}$$

$$= - \sqrt{\frac{\pi}{\alpha t_m^*}} (T_m - T_o) \frac{\partial v}{\partial z} \frac{ds}{dt} + \frac{\sqrt{\pi} (T_m - T_o)}{t_m^*} \frac{\partial v}{\partial y} ,$$

$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial z} \frac{\partial z}{\partial x} = \sqrt{\frac{\pi}{\alpha t_m^*}} (T_m - T_o) \frac{\partial v}{\partial z} , \qquad \text{(II.15)}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial z} \left(\frac{\partial T}{\partial x}\right) \frac{\partial z}{\partial x} = \frac{\sqrt{\pi} (T_m - T_o)}{\alpha t_m^*} \frac{\partial^2 v}{\partial z^2} , \qquad \text{(II.16)}$$

are now substituted into equations (II.1) through (II.6). The result, after simplification, is

$$\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial z^2} + M\mu \frac{\partial v}{\partial z} , \qquad \text{(II.17)}$$

$$v(z, 0) = 0 , \tag{II.18}$$

$$\lim_{z \to \infty} v(z,y) = 0 , \tag{II.19}$$

$$H_r(y) = - 2\left(\frac{\partial v}{\partial z}\right)_{z=0} + \mu , \tag{II.20}$$

$$\mu \geq 0 \quad \text{when } v(0,y) = \frac{1}{\sqrt{\pi}} , \tag{II.21}$$

$$\mu = 0 \quad \text{when } v(0,y) < \frac{1}{\sqrt{\pi}} . \tag{II.22}$$

In these equations M is a dimensionless property parameter given by (II.13) and $H_r(y)$ is a dimensionless heat flux defined by

$$H_r(y) = \frac{H(y)}{H^*} . \tag{II.23}$$

Values of M for some metals are given in Table 1, page 8.

The dimensionless form of the boundary value problem is convenient because only one property parameter occurs in the equations. With the equations in this form, it is clear that the melting problem is non-linear since μ, which occurs as a coefficient in the differential equation (II.17), depends on the temperature gradient at the surface.

A complete solution of the problem cannot be obtained until $H(t)$ is specified. However, the temperature distribution in the solid before melting first begins and the energy balance for the system can be found for any $H(t)$. The energy balance can be written directly or derived by application of Gauss's theorem to the heat conduction equation (II.1). The expression derived in Appendix A, page 36, is

Table 1. Values of M for T_o = 70° F

Material	M
Aluminum	2.15
Beryllium	2.85
Chromium	2.80
Copper (pure)	2.29
Iron (pure)	2.86
Magnesium	2.04
Manganese	2.50
Molybdenum	2.57
Nickel	2.31
Platinum	2.34
Silver	2.38
Titanium	2.53
Tungsten	2.78

$$\int_0^t H(t)dt = \rho c \int_{s(t)}^{\infty} (T - T_o)dx \qquad \text{(II.24)}$$

$$+ \rho[L + c(T_m - T_o)]\, s\,.$$

This equation equates the total amount of heat that has reached the exposed surface at time t to the amount of energy that has passed through the surface and is stored in the remaining solid plus the amount of energy that was carried away by the liquid melt as latent heat and as increased energy content of the material which has melted. In many cases, the energy balance can be used to draw worthwhile conclusions about the melting process.

Carslaw and Jaeger [1] have solved the boundary value problem for the temperature distribution for time $t \leq t_m$, where t_m is the time at which melting commences for the first time after heating has begun. Their solution can be written as

$$T - T_o = \frac{1}{\sqrt{\pi k \rho c}} \int_0^t \frac{H(t-\tau)}{\tau^{1/2}} \exp\left(-\frac{x^2}{4\alpha\tau}\right) d\tau\,,\ t \leq t_m. \qquad \text{(II.25)}$$

Equation (II.12) is readily derived from (II.25) by substituting $T = T_m$, $H(t - \tau) = H^*$, and $t = t_m^*$ into (II.25).

Written in dimensionless form, (II.24) and (II.25) become

$$\int_0^y H_r(y)dy = 2\int_0^{\infty} v(z,y)dz + \frac{\sigma(y)}{\mu_e^*}\,, \qquad \text{(II.26)}$$

$$v = \frac{1}{2\sqrt{\pi}} \int_0^y \frac{H_r(y-\tau)}{\tau^{1/2}} \exp\left(-\frac{z^2}{4\alpha\tau}\right) d\tau,\ y \leq y_m\,. \qquad \text{(II.27)}$$

In these equations μ_e^* is the steady state value of μ for a system where $H(t) = H^*$ and is given by Landau as

$$\mu_e^* = 1/(1 + \frac{2M}{\sqrt{\pi}}) , \qquad \text{(II.28)}$$

and y_m is defined as

$$y_m = t_m/t_m^* . \qquad \text{(II.29)}$$

Landau reasoned that, if the heat flux is constant, the system will approach a steady state where μ is constant and v does not vary with time. He solved (II.17), (II.19), and (II.21) for the dimensionless temperature v_e^* at this steady state. His resulting solution is

$$v_e^* = \frac{1}{\sqrt{\pi}} \exp(-M\mu_e^* z) . \qquad \text{(II.30)}$$

He obtained (II.28) by substituting (II.30) and $H(y) = H^*$ into (II.20).

The equations given in this chapter apply to any melting process that is governed by (II.1) through (II.6). These equations can also be applied to a freezing process if $H(t)$ and L are taken to be negative quantities and the inequality in (II.5) is reversed. The dimensionless forms of the equations are convenient because the properties of the solid are lumped into one parameter M. The only other parameters that will be of concern are those needed to describe the heat flux $H(t)$. For this reason the equations should be desirable for any number of problems where the net heat flux at the surface is a function of time only. For instance, if heating is due to convection and the surface temperature remains constant at the melting temperature

$$H(t) = h(t)\,[T_g(t) - T_m]\ , \qquad \text{(II.31)}$$

where $h(t)$ is the convective heat transfer coefficient and $T_g(t)$ is the temperature of the environment.

CHAPTER III

MELTING OF A SEMI-INFINITE SOLID EXPOSED TO A SQUARE WAVE TYPE HEAT FLUX

The remainder of this work will be devoted to the solution of the boundary value problem of Chapter II for a cyclic heat flux of the square wave type shown in Figure 1, page 13. The period before melting first commences, the period of transient conditions, and a postulated quasi-steady state condition will be investigated.

Pre-melting Period

Before melting first commences the temperature distribution is given by (II.25). This equation has been integrated in Appendix B, page 38, to obtain the surface temperature history,

$$\frac{\sqrt{\pi k \rho c}\,(T_s - T_o)}{2H^* \sqrt{A}} = \sum_{n=0}^{\infty} (-1)^n S_n \left(\frac{t}{A}\right) \sqrt{\frac{t}{A} - n}\ , \quad t \leq t_m , \qquad \text{(III.1)}$$

$$S_n(t/A) = 0 \quad \text{when } 0 < t/A < n\ , \qquad \text{(III.2)}$$
$$= 1 \quad \text{when } t/A > n\ ,$$

where T_s is the surface temperature and A is the time duration of a heat pulse or one half the period of the heating cycle.

Equation (III.1) has been plotted in Figure 2, page 14, for values of t/A to 51. For $50 < t/A < 100$ the surface temperature at the beginning and end of the heat pulses have been plotted with straight lines

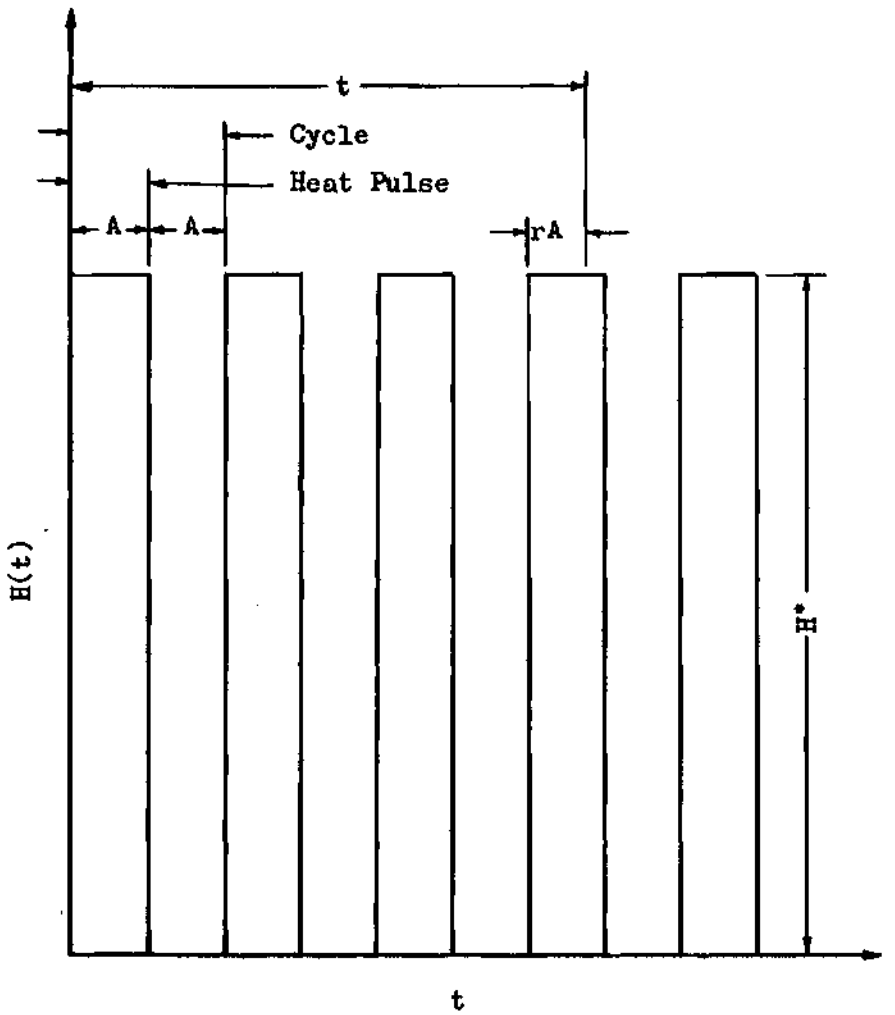

Figure 1. Surface Heating History

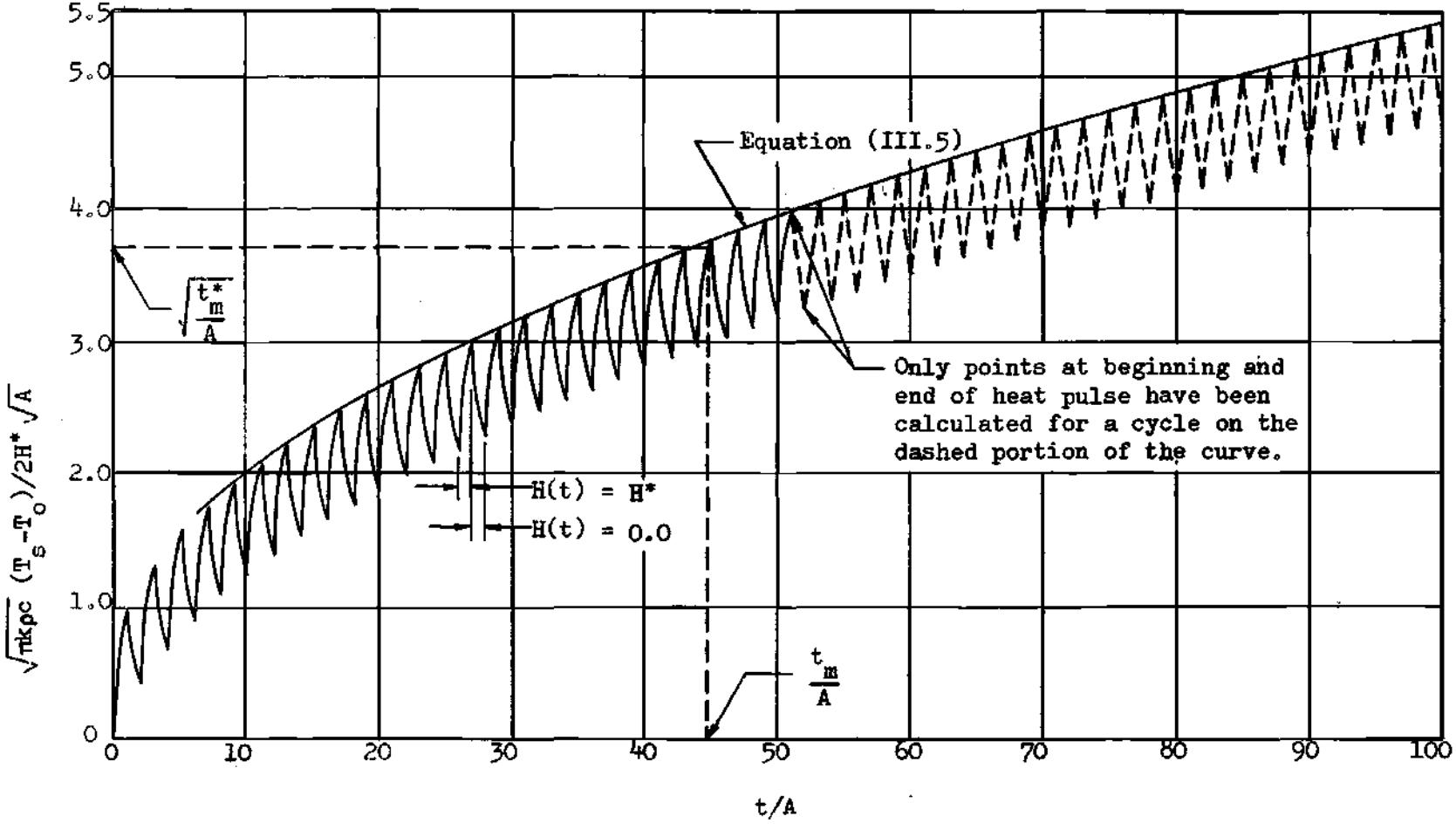

Figure 2. Surface Temperature before Melting

drawn between these points. When $T_s = T_m$,

$$\frac{\sqrt{\pi k \rho c}\ (T_s - T_o)}{2H^* \sqrt{A}} = \frac{\sqrt{\pi k \rho c}\ (T_m - T_o)}{2H^* \sqrt{A}} = \sqrt{\frac{t_m^*}{A}}\ , \qquad \text{(III.3)}$$

and it is seen that the time t_m that melting first commences is a function of t_m^* and A. The value of t_m can be found from the curve as illustrated in Figure 2 by finding the value of t/A at which $\sqrt{\pi k \rho c}\ (T_s - T_o)/2H^* \sqrt{A}$ first becomes equal to $\sqrt{t_m^*/A}$.

When t_m^*/A is large the solution of (III.1) is time-consuming since t_m/A is also large. An approximation for large values of t/A can be found by using (III.1) to determine the difference in the temperature T_{s1} at the end of a heat pulse and the temperature T_{s2} at the end of the same heating cycle. The result, after simplification, is

$$\frac{\sqrt{\pi k \rho c}\ (T_{s1} - T_{s2})}{2H^* \sqrt{A}} = \frac{2\sqrt{\pi k \rho c}\ (T_{s1} - T_o)}{2H^* \sqrt{A}} - \sqrt{\frac{t_1}{A} + 1}\ . \qquad \text{(III.4)}$$

Calculations used to plot Figure 2 indicate that $\sqrt{\pi k \rho c}\ (T_{s1} - T_{s2})/2H^* \sqrt{A}$ converges to a value greater than 0.734 while $\sqrt{\pi k \rho c}\ (T_{s1} - T_{s0})/2H^* \sqrt{A}$, where T_{s0} is the surface temperature at the beginning of the given heat pulse, converges to a value less than 0.786. Since the surface temperature rise during a heat pulse must always be greater than or equal to the temperature decrease during the remaining part of the cycle, $\sqrt{\pi k \rho c}\ (T_{s1} - T_{s2})/2H^* \sqrt{A}$ must converge to some value between 0.734 and 0.786. As a first approximation, $\sqrt{\pi k \rho c}\ (T_{s1} - T_{s2})/2H^* \sqrt{A}$ will be taken as 0.75. Then for large t_m^*/A, noting from Figure 2 that, when $t = t_m$, $T_{s1} \approx T_m$,

$$\frac{\pi k \rho c\ (T_m - T_o)}{2H^* \sqrt{A}} = \sqrt{\frac{t_m^*}{A}} \approx \left(0.75 + \sqrt{\frac{t_m}{A} + 1}\right) \Big/ 2. \qquad \text{(III.5)}$$

Equation (III.5) is plotted in Figure 2. The good agreement with the numerical calculations suggests that (III.5) can be used to obtain accurate values of t_m for $\sqrt{t_m^*/A}$ greater than 5.4. Figure 2 shows that t_m/A is always nearly equal to an odd number such as 97, 99, or 101 for $\sqrt{t_m^*/A}$ greater than 5.4. Since this is true, (III.5) can be used to predict a first approximation of t_m/A for large t_m^*/A, and the value of t_m/A can then be more accurately found by taking its value to be the next odd number that is greater than the value obtained from (III.5). This method will yield more accurate results as t_m^*/A increases.

It can be seen from (III.5) that, as t_m^*/A becomes very large, $\sqrt{t_m/A} \rightarrow 2\sqrt{t_m^*/A}$ or $t_m \rightarrow 4t_m^*$. Thus, when the period of the heating cycle is very small, the time t_m that melting first commences with a heat input of a square wave type is approximately four times greater than the time t_m^* that melting commences with a constant heat input H^*.

The Period of Transient Conditions

For a time after melting first commences, each succeeding heating cycle will affect the solid differently. During this period of time the boundary value problem is non-linear except for that portion of time during a heating cycle when the rate of melting ds/dt is zero. A numerical finite difference solution of the transient boundary value problem was employed.

Numerical Solution

Approximate Equations. In writing the finite difference equations

that approximate the boundary value problem equations, "i" and "j" are used as indices on the time and space mesh and are defined such that

$$z = (i-1)\,\Delta z\ ,\ i = 1,\ 2,\ 3,\ \ldots\ , \qquad \text{(III.6)}$$

$$y = (j-1)\,\Delta y\ ,\ j = 1,\ 2,\ 3,\ \ldots\ , \qquad \text{(III.7)}$$

where Δz and Δy are intervals in z and y. Shorthand notations such as $v(i,j)$ in place of $v(z,y) = v\,[(i-1)\,\Delta z,\ (j-1)\,\Delta y]$ and $\mu(j)$ in place of $\mu(y) = \mu[(j-1)\,\Delta y]$ are used.

The differential equation was approximated by a first order forward difference ratio for the time derivative and by first order central difference ratios for the space derivatives,

$$\frac{\partial v(i,j)}{\partial y} \approx \frac{1}{\Delta y}\,[v(i,j+1) - v(i,j)]\ , \qquad \text{(III.8)}$$

$$\frac{\partial v(i,j)}{\partial z} \approx \frac{1}{2\Delta z}\,[v(i+1,j) - v(i-1,j)]\ , \qquad \text{(III.9)}$$

$$\frac{\partial^2 v(i,j)}{\partial z^2} \approx \frac{1}{(\Delta z)^2}\,[v(i+1,j) - 2v(i,j) + v(i-1,j)] \qquad \text{(III.10)}$$

so that (II.18) is approximated by the difference equation

$$v(i,j+1) - v(i,j) = \frac{\Delta y}{\Delta z^2}\Big\{v(i+1,j) - 2v(i,j) + v(i-1,j) \qquad \text{(III.11)}$$

$$+ M\mu(j)\,\frac{\Delta z}{2}\,[v(i+1,j) - v(i-1,j)]\Big\}\ .$$

It is desirable to use a forward difference ratio of higher order than one for $\partial v/\partial z$ at $z = 0$ in (II.21) in order to approximate μ as

accurately as possible. A third order approximation for this derivative yielded negative values of μ during the solution because of the coarseness of the time and space grid used. For a finer grid than that used in this work, the third order approximation would be desirable. However, since the computer solution of this problem is long, the time and space grid was made as coarse as possible for reasonable accuracy and a second order approximation for $\partial v/\partial z$ at $z = 0$ was used. This approximation as derived in Appendix C, page 40, is

$$\left(\frac{\partial v}{\partial z}\right)_{z=0} = \frac{\partial v(1,j)}{\partial z} \approx - \left[\frac{3v(1,j) - 4v(2,j) + v(3,j)}{2\,\Delta z}\right] . \qquad \text{(III.12)}$$

In solving differential equations by finite difference techniques, the time and space grid must be chosen so that the solution is stable. Stability of the solution means that numerical inaccuracies which occur during the process of solution, such as round-off errors, are damped out instead of being magnified. According to Sunderland and Grosh [9], the stability requirements for (III.11) are

$$\frac{\Delta y}{(\Delta z)^2} \leq 1/2 , \qquad \text{(III.13)}$$

$$[M\mu(j)]_{\text{maximum}} \frac{\Delta z}{2} \leq 1 . \qquad \text{(III.14)}$$

In the solution, $\Delta y/(\Delta z)^2$ was taken as 1/2. The approximation of the boundary value problem is, for $\Delta y/(\Delta z)^2 = 1/2$,

$$v(i,j+1) = \frac{1}{2}\left\{\left[1 + \frac{M\mu(j)\,\Delta z}{2}\right] v(i+1,j) + \left[1 - \frac{M\mu(j)\,\Delta z}{2}\right] v(i-1,j)\right\}, \qquad \text{(III.15)}$$

$$v(i,1) = 0 \,, \tag{III.16}$$

$$\lim_{i \to \infty} v(i,j) = 0 \,, \tag{III.17}$$

$$\mu(j) = H_r(j) - [3v(1,j) - 4v(2,j) + v(3,j)]/\Delta z \,, \tag{III.18}$$

$$\mu(j) \geq 0 \quad \text{for } v(1,j) = 1/\sqrt{\pi} \,, \tag{III.19}$$

$$\mu(j) = 0 \quad \text{for } v(1,j) < 1/\sqrt{\pi} \,. \tag{III.20}$$

An expression for the dimensionless melted thickness can be found by noting that, by virtue of its definition, equation (II.10),

$$\sigma(y) = \int_0^y \mu \, dy \,. \tag{III.21}$$

Using an approximate numerical integration, (III.21) becomes

$$\sigma(j+1) \approx \sigma(j) + [\mu(j+1) + \mu(j)] \, \Delta y/2 \,. \tag{III.22}$$

<u>Method of Solution</u>. The computer program used in this solution is included in Appendix D, page 41. Values of 0.005 and 0.1 were used for Δy and Δz, respectively. The selection of this grid size was based on (III.13) and (III.14) and on an inspection of the temperature and melting rate gradients found by Landau for $H(t) = \text{const.} = H^*$. Landau's work was used as a criterion for selecting the grid sizes, because the heat flux is constant during the first heat pulse.

Computations were carried to values of i large enough to make $v(i,j) < 5 \times 10^{-7}$ before going on to computations for $j+1$. The initial values $v(i,1) = 0$, $\mu(1) = 0$, and $\sigma(1) = 0$ were used to begin computations,

and melting was assumed to begin at the time $y = (j-1)\,\Delta y$, if $|v(1,j) - 1/\sqrt{\pi}\,|$ was less than both $|v(1,j-1) - 1/\sqrt{\pi}\,|$ and $|v(1,j+1) - 1/\sqrt{\pi}\,|$.

Results of Solution. The results of the numerical solution are shown in Figures 3a through 7, pages 21 - 26. A check on the accuracy of the computer program was made by solving the boundary value problem for $H(t) = H^*$ and values of two and ten for M and comparing the results with those obtained by Landau. Agreement between Landau's results and those obtained here is generally good even at small values of melting rate where the difference is largest (Figure 3a and 3b). This close agreement supports the choice of the time and space grid used in this work, at least for conditions where $H(t)$ is a continuous function. For periodic heat input functions such as that studied in this work, errors such as those at low melting rates may accumulate. However, numerical predictions of the time that melting first commences, for the square wave type heat input, agreed with values obtained from Figure 2 within the accuracy of the plot. This indicates that the time and space grid chosen for this work is fine enough to yield results of adequate accuracy.

The predicted melted thickness σ for $M = 2.0$ and values of A/t_m^* from 0.25 to 2.0 are shown in Figure 4. It was found that, for a given heating time, the dimensionless thickness melted decreased with decreasing A/t_m^* and appeared to approach a limit with $A/t_m^* = 0$. A comparison of the results in Figure 4 with those shown in Figure 3a shows that the thickness melted when $H(t)$ is of the square wave type is always less than one half the thickness melted when $H(t) = \text{const.} = H^*$. The curves are plotted as straight lines during the times melting is occurring,

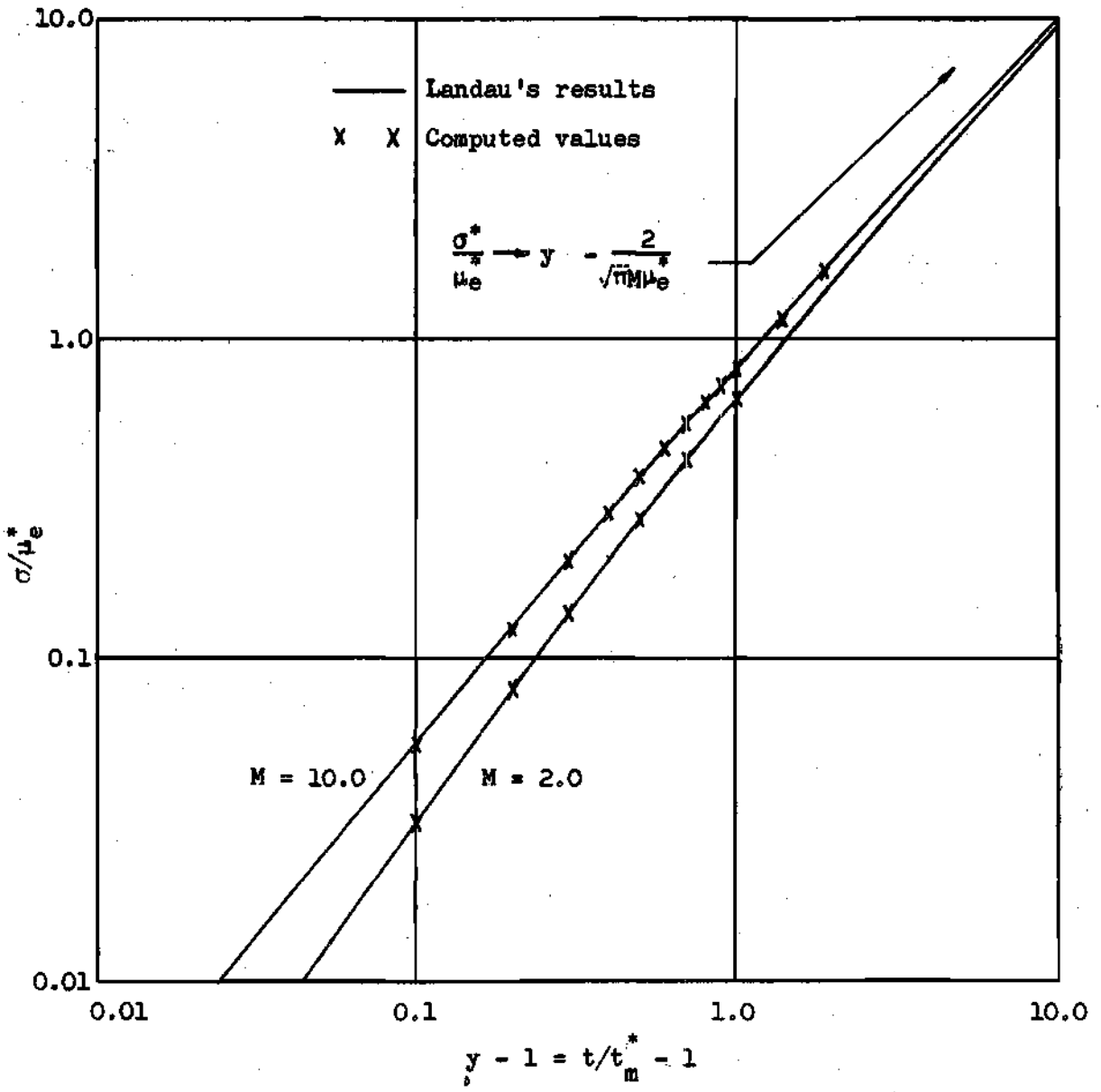

Figure 3a. Accuracy of Solution ($H(t) = H^*$)

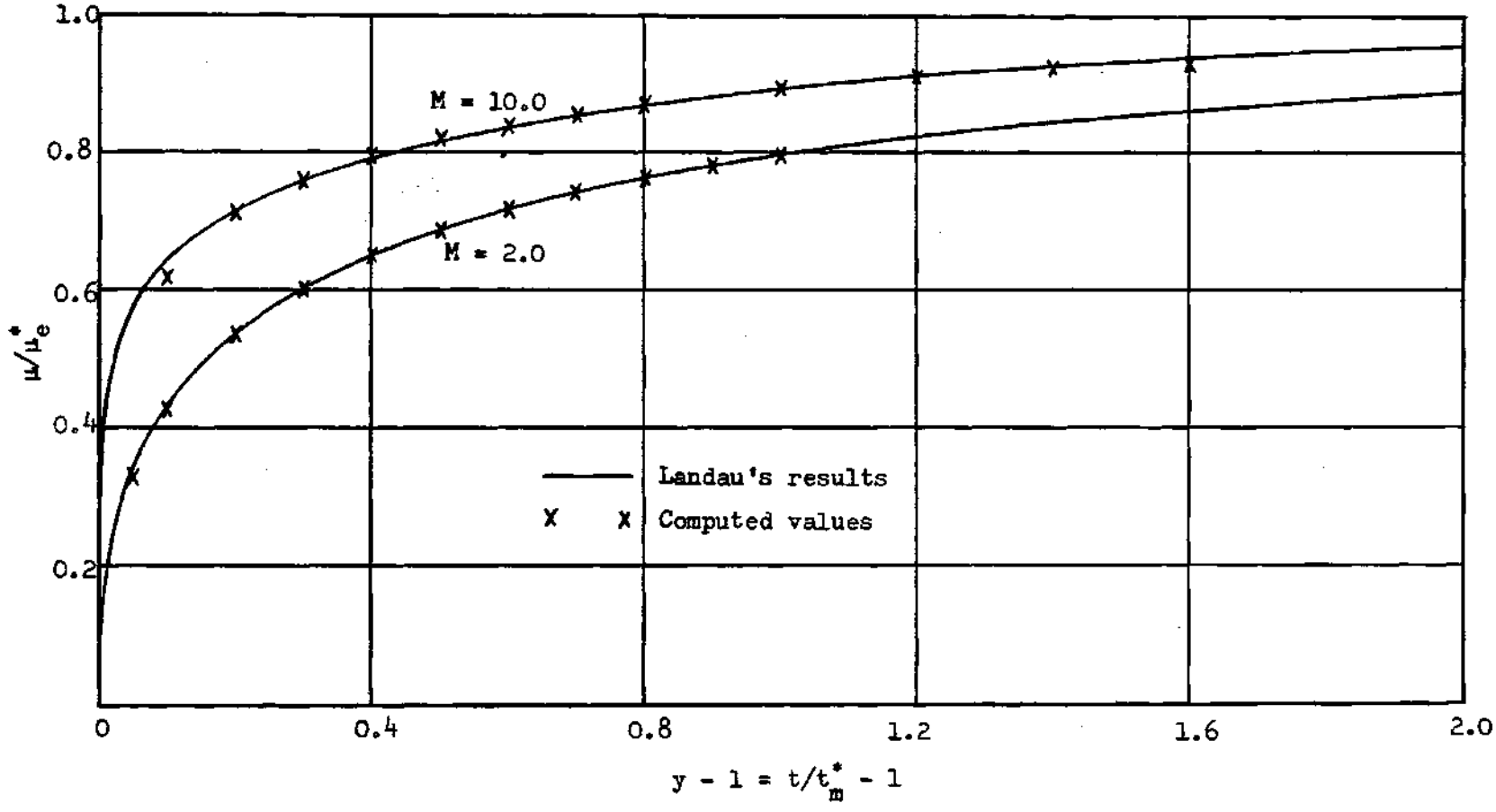

Figure 3b. Check on Accuracy of Solution ($H(t) = H^*$)

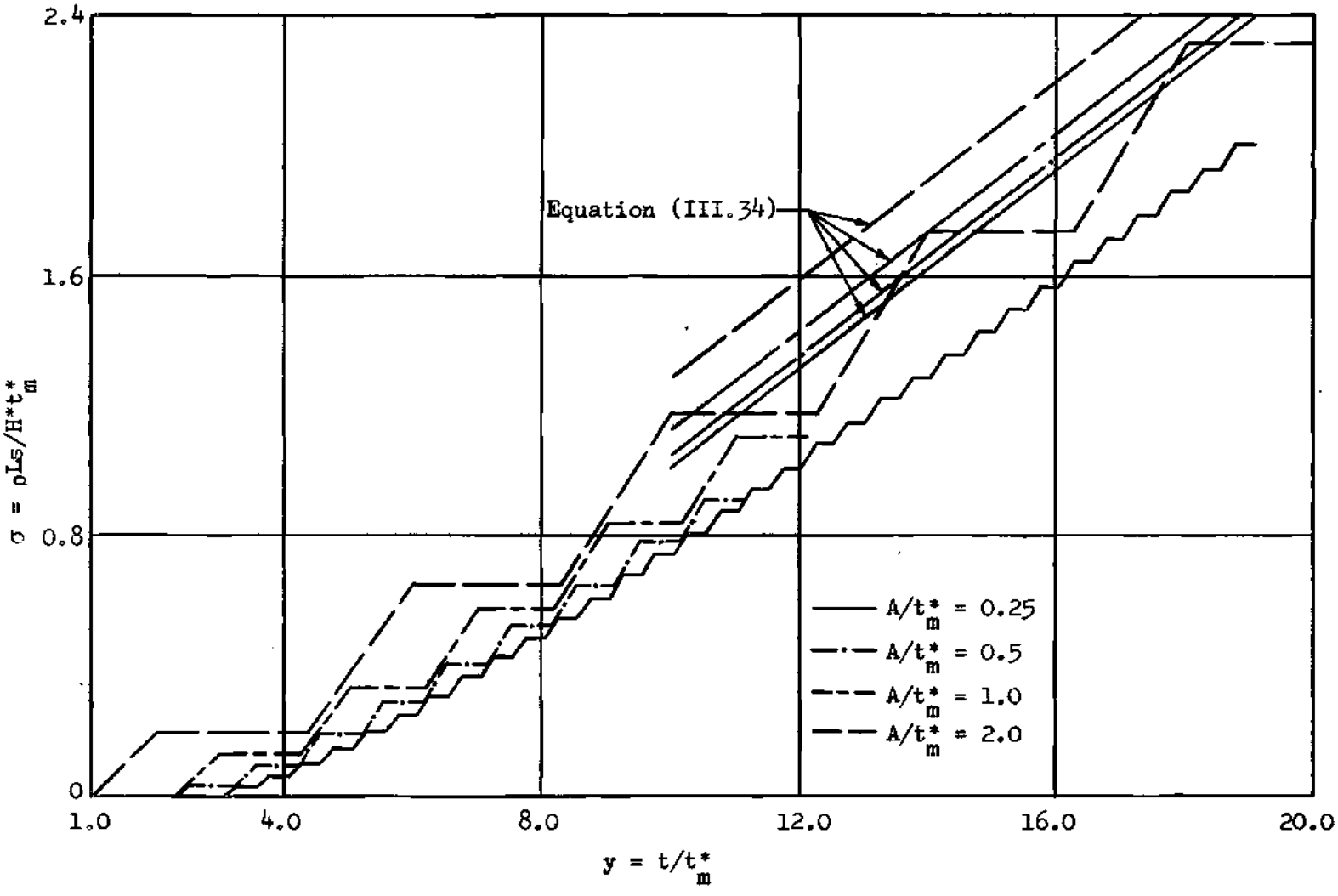

Figure 4. Thickness Melted (M = 2.0)

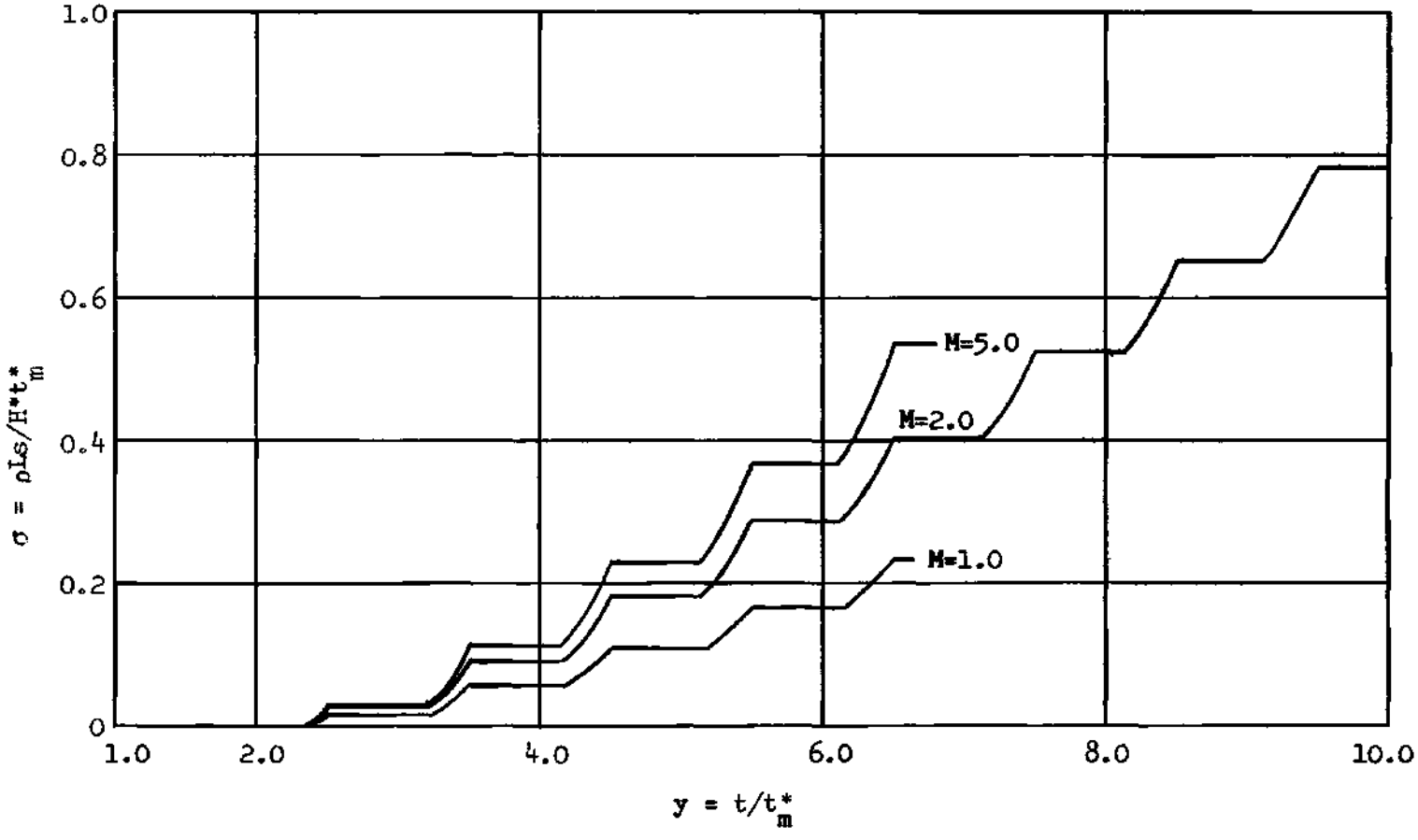

Figure 5. Variation of Thickness Melted with M ($A/t_m^* = 1.0$)

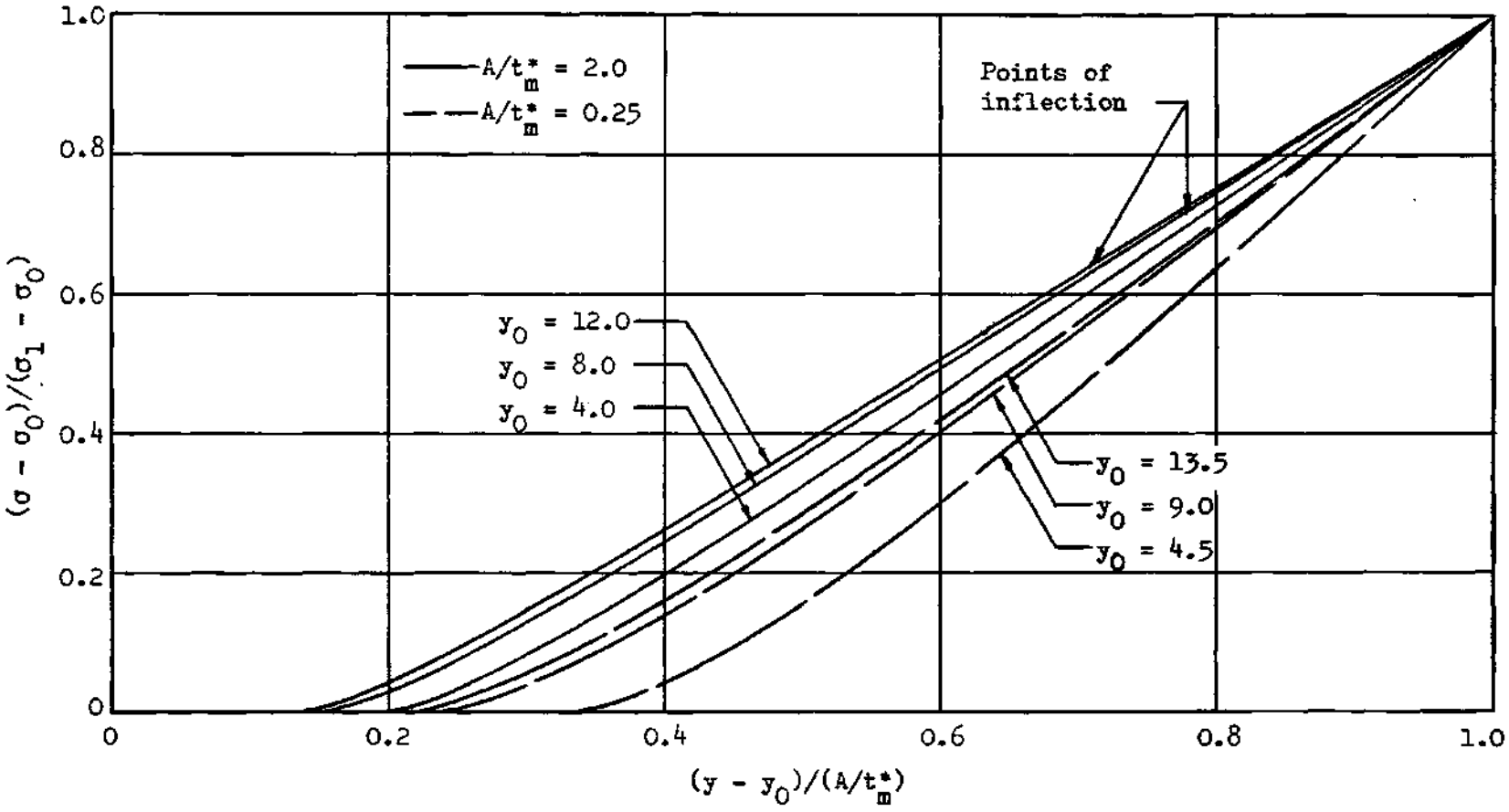

Figure 6. Reduced Thickness Melted During a Heat Pulse (M = 2.0)

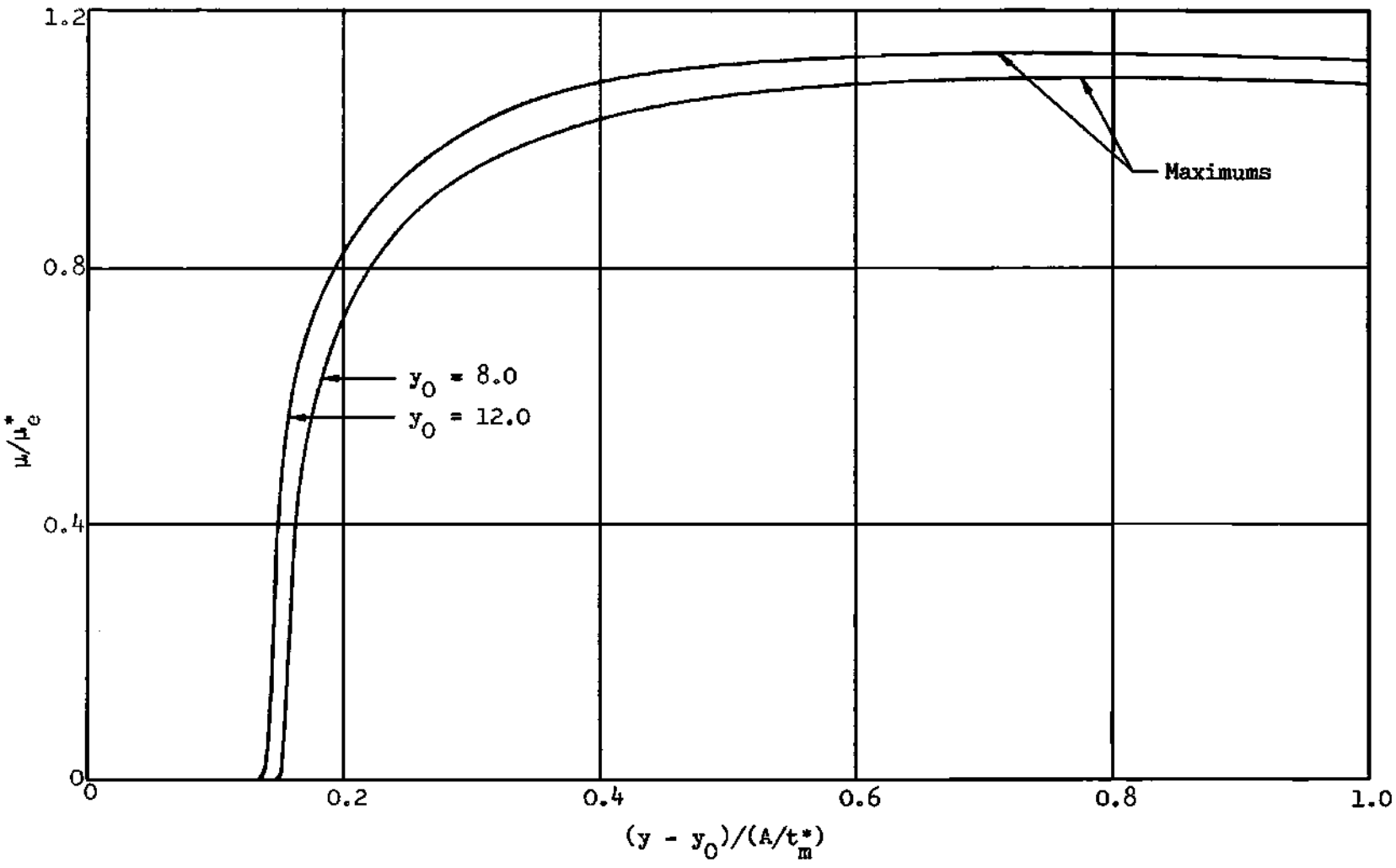

Figure 7. Reduced Rate of Melting During a Heat Pulse ($M = 2.0$, $A/t_m^* = 2.0$)

because the scale of the plot is too small to show the actual gradients that exist during this time. The actual gradients are shown in Figure 5 which indicates that an increase in M increases the thickness melted, and that this increase becomes smaller as M increases.

Figure 6 is a plot of the thickness melted during three heat pulses in the time from $y = 4.0$ to $y = 14.0$. Three successive pulses are shown for $A/t_m^* = 2.0$ and three pulses spaced approximately even in time are shown for $A/t_m = 0.25$. As time increases, the amount of melting during each pulse tends to become equal for successive pulses. The tendency to approach this condition is greater as A/t_m^* decreases, although, at the end of the time range considered, the system appears closer to this condition when $A/t_m^* = 2.0$ than when $A/t_m^* = 0.25$. Inflection points in the curves for two of the pulses indicate that the melting rate reached a maximum at these times. The melting rate curves for these two pulses are shown in Figure 7 where another seemingly peculiar result is indicated. During the heat pulse, the melting rate for these pulses and, although data is not shown, all except the first few of the pulses for all values of A/t_m^* exceeds the maximum rate attained when $H(t) = \text{const.} = H^*$ [the steady state value given by (II.28)]. Figure 7 also shows that the maximum melting rate occurs earlier in each successive pulse and that at the end of the pulse the rate is decreasing. This indicates that for large enough times, the ratio μ/μ_e^* may approach a value of one near the end of the pulse, which seems physically reasonable.

Period of Quasi-Steady Conditions

The results of the numerical solution suggest that the system may eventually approach a state where the amount of melting during successive

pulses is equal and the temperatures and melting rate are periodic functions of time. This will be postulated to be the quasi-steady state that the system approaches.

In the investigation of this postulated state, the quantities defined in Figure 8, page 29, are used. As indicated in the figure, y_{10} and y_{20} are the values of y at the beginning of two successive heat pulses, y_{11} and y_{21} are the values of y at the end of the same heat pulses, y_{m_1} and y_{m_2} are the values of y at the instant melting begins anew during the pulses, r is the fraction of the heat pulse that has occurred at time y, and f_1 and f_2 are the values of r at the times y_{m1} and y_{m2} respectively.

Using (II.26), the increase in thickness melted during a cycle is

$$\frac{\sigma(y_{21}) - \sigma(y_{m2})}{\mu_e^*} = \frac{\sigma(y_{21}) - \sigma(y_{20})}{\mu_e^*} = \frac{\sigma(y_{21}) - \sigma(y_{11})}{\mu_e^*} \qquad \text{(III.23)}$$

$$= \int_{y_{11}}^{y_{21}} H_r(y)dy - 2\left[\int_0^\infty v(z,y_{21})dz - \int_0^\infty v(z,y_{11})dz\right]$$

$$= \frac{A}{t_m^*} - 2\int_0^\infty [v(z,y_{21}) - v(z,y_{11})]\, dz \, .$$

But since the temperatures are periodic, this last integral is zero and, for any heat pulse occurring after quasi-steady conditions have been reached,

$$\frac{\sigma(y_{11}) - \sigma(y_{m1})}{\mu_e^*} \rightarrow \frac{A}{t_m^*} \qquad \text{(III.24)}$$

The value of f_2 can be found from (II.26) and the relation

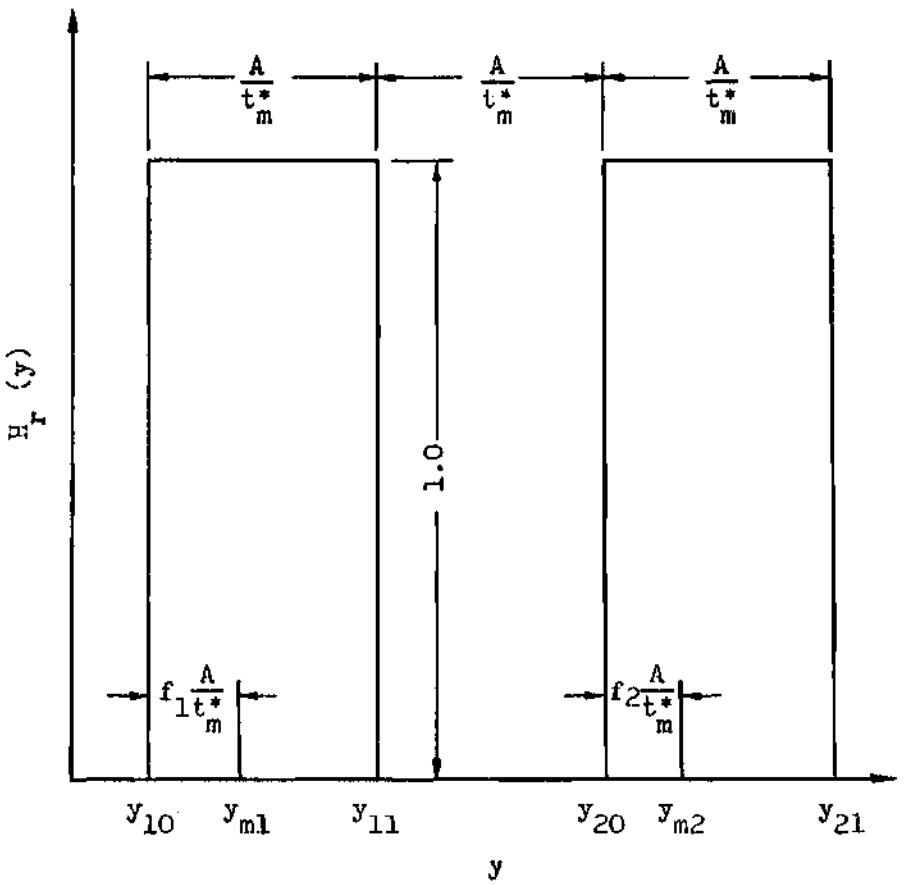

Figure 8. Quasi-Steady State Nomenclature

$\sigma(y_{m2}) - \sigma(y_{11}) = 0$. The result, after substituting $v(z,y_{11}) = v(z,y_{21})$, is

$$f_2 \frac{A}{t_m^*} = 2\int_0^\infty [v(z,y_{m2}) - v(z,y_{21})]\, dz\,. \qquad \text{(III.25)}$$

Then, since f_2 is positive or zero,

$$\int_0^\infty v(z,y_{m1})\, dz \geq \int_0^\infty v(z,y_{11})\, dz\,. \qquad \text{(III.26)}$$

This is a seemingly peculiar characteristic, since the energy content of the solid is less at the end of a heat pulse than at the time melting began during the pulse. To investigate the situation further, consider the melting speed

$$\frac{\mu}{\mu_e^*} = \frac{1}{\mu_e^*}\frac{d\sigma}{dy} = 1 - 2\frac{d}{dy}\int_0^\infty v(z,y)dz,\ y_0 \leq y \leq y_1, \qquad \text{(III.27)}$$

obtained from (II.11) and (II.26). As melting begins in a heat pulse, $\mu = 0$ and, by (III.27),

$$\frac{d}{dy}\int_0^\infty v dz = 1/2\ .$$

As μ increases from zero, the derivative of the energy content begins to decrease and the energy content rises for a period of time. Since the energy content at the end of the pulse must be less than or equal to that at the time melting started, it must have a maximum value sometime during the pulse. As the heat content passes through its maximum and decreases, μ/μ_e^* becomes greater than unity as predicted by the numerical solution.

Although the postulated conditions of periodic temperatures and melting rates require that μ/μ_e^* be greater than unity sometime during a heat pulse, they do not by themselves require μ/μ_e^* to be a maximum during any pulse. Actually, the melting rate becomes a maximum only when the energy content passes through an inflection [equation (III.27)] as it would if the temperatures in the solid approach steady values during the latter stage of the heat pulse. It may be possible that the temperature distribution at the beginning of a heat pulse is such that the system will have time to approach a steady state before the pulse ends. If this happens, μ/μ_e^* will approach unity according to (III.27), the temperatures will approach the steady state values

$$v_e^* = \frac{1}{\sqrt{\pi}} \exp\left(-M\mu_e^* z\right) \qquad \text{(III.28)}$$

found by Landau, and typical conditions during a cycle will be as shown in Figure 9, page 32.

For a cycle such as that in Figure 9, the thickness melted at the end of the heat pulse can be found by substituting $y = y_{21}$ and $v = v_e^*$ into (II.26). The result, after integrating, is

$$\frac{\sigma(y_1)}{\mu_e^*} = \frac{y_1 + A/t_m^*}{2} - \frac{2}{\sqrt{\pi}\ M\mu_e^*} . \qquad \text{(III.29)}$$

Equation (III.29) is valid only at times y_1 at the end of heat pulses. Values computed from this equation have been plotted in Figure 4, and straight lines have been drawn through them. The values of σ at the end of heat pulses should approach these lines asymptotically if the system

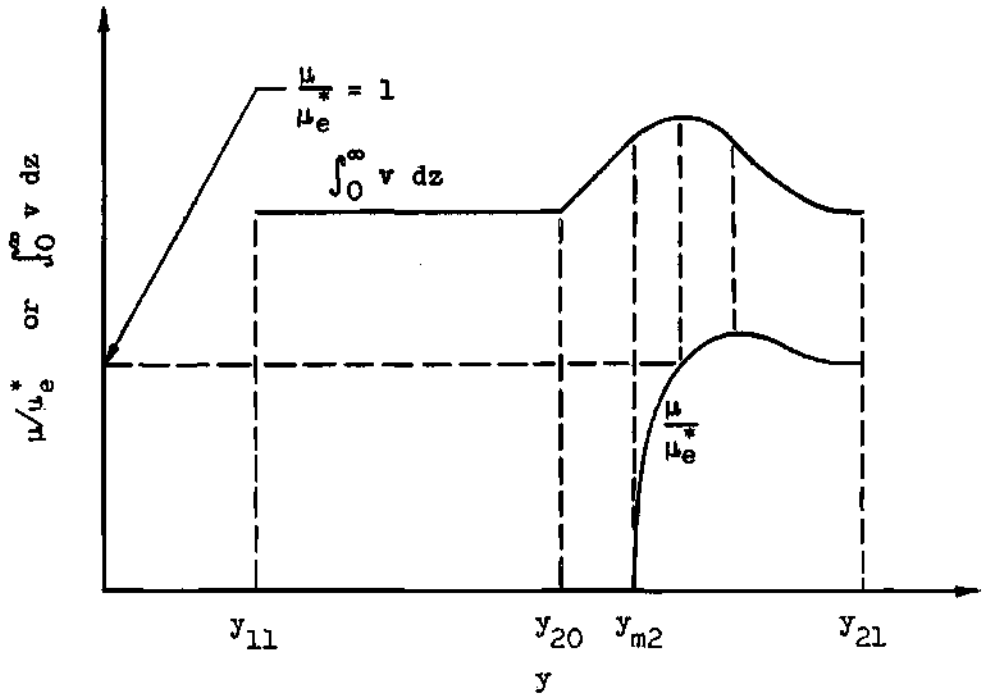

Figure 9. Conditions During a Quasi-Steady State Heating Cycle

is approaching the quasi-steady state postulated here. The heat pulse end points may approach these lines in the actual case for large enough y. If the system achieves a quasi-steady state where the temperatures become periodic, but steady conditions are not approached at the end of the heat pulses; the asymptotes would be displaced either upward or downward, but the slopes would remain equal to $\mu_e^*/2$.

Equation (III.28) can be used to obtain an implicit expression for the fraction f of the heat pulse that has expired at time y_m. The result, as derived in Appendix E, page 48, is

$$1 = \frac{\exp\left[(1+f)\,\frac{A/t_m^*}{p^2}\right]}{f}\int_{-\infty}^{\infty}\exp\left\{-\frac{\left[\delta - \frac{A/t_m^*}{p}(1+f)\right]^2}{f}\right\}\operatorname{erfc}(\delta)\,d\delta \quad \text{(III.30)}$$

$$+\sqrt{fA/t_m^*}$$

Equation (III.30) was solved for values of A/t_m^* of 0.25, 0.5, 1.0, and 2.0 and the results were fitted with a second degree polynomial in A/t_m^*. The resulting equation is, for $0.25 \leq A/t_m^* \leq 2.0$,

$$f = 0.343 - 0.109\,\frac{A}{t_m^*} + 0.0064\,\frac{A}{t_m^*}^{\,2}. \qquad \text{(III.31)}$$

Equation (III.31) predicts values of f within 0.4 per cent of those calculated from (III.30). The values of f, as predicted by the numerical solution, decrease to a value slightly less than those predicted by (III.30).

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

Primary conclusions reached in this study and recommendations for an extended study of the problem are:

(1) The time and space grid size used in the numerical solution is fine enough to yield results of adequate accuracy, at least for the case where the heat flux is such that melting continues once it commences. If melting stops and then begins anew several times, errors in choosing the time that melting begins anew as the nearest indexed time on the grid may become cumulative if the time choice is not random. A comparison of results obtained using a finer time and space grid with those obtained here should indicate whether the error is cumulative or not.

(2) As was expected, the thickness melted when the heat flux is a square wave type was found to be always less than one half that melted when the heat flux is constant and equal to the peak of the square wave type flux.

(3) Results of the numerical solution indicate that the system may approach a physically reasonable quasi-steady state where the melting rate and the temperatures are periodic and the amount of melting is the same for each cycle.

(4) When periodic conditions exist in a solid exposed to a square wave type heat flux, the ratio μ/μ_e^* must become greater than unity sometime during a heat pulse.

(5) When periodic conditions exist and a steady state is approached

before a heat pulse ends, μ/μ_e^* must pass through a maximum sometime during a pulse.

(6) The postulated quasi-steady state, where periodic conditions exist and where steady state conditions are approached before a heat pulse ends, appears logical since the numerical solution indicates that μ/μ_e^* does become greater than unity and passes through a maximum. However, other conditions can possibly cause μ/μ_e^* to behave in this way. For this reason, the results of the numerical solution should be extended to larger exposure times to determine if the system does approach the postulated quasi-steady state.

APPENDIX A

DERIVATION OF ENERGY BALANCE

According to Landau [2], Gauss's theorem applied to the heat conduction equation (II.1) gives

$$\iint_D \left[k \frac{\partial^2 T}{\partial x^2} - \rho c \frac{\partial T}{\partial t} \right] dxdt \tag{A.1}$$

$$= \int_C \left[k \frac{\partial T}{\partial x} dt + \rho c T dx \right] = 0 ,$$

where D is any region in the x, t - plane to which the theorem applies, and C is its boundary. When D is the semi-infinite region bounded by the lines $t = 0$ and $t = t_1$ and the curve $x = s(t)$, (A.1) becomes

$$\int_0^{t_1} k\left(\frac{\partial T}{\partial x}\right)_{x=s(t)} dt + \lim_{a \to \infty} \int_{t_1}^{0} k\left(\frac{\partial T}{\partial x}\right)_{x=a} dt \tag{A.2}$$

$$+ \lim_{a \to \infty} \left[\int_{s(t_1)}^{a} \rho c T(x,t_1) dx + \int_a^0 \rho c T_0 dx \right]$$

$$+ \int_0^{t_1} \rho c T\,(s(t),t) \frac{ds}{dt} dt = 0 .$$

The second integral is zero at all times, since the temperature of the solid is constant initially. In the last integral $ds/dt = 0$ except when $T(s(t),t) = T_m$, and this integral, therefore, equals $\rho c T_m s(t_1)$. After simplification, (A.2) becomes

$$\int_0^t H(t)\, dt = \rho c \int_{s(t)}^{\infty} (T(x,t) - T_o)\, dx \tag{A.3}$$

$$+ \rho\, [L + c(T_m - T_o)]\, s(t)\ ,$$

where (II.4) has been used to evaluate the first integral in (A.2) and the subscript has been dropped from t_1.

APPENDIX B

SURFACE TEMPERATURE BEFORE INITIAL MELTING

The surface temperature before melting first commences is given by (II.25) with $x = 0$ as

$$T_s - T_o = \frac{1}{\sqrt{\pi k \rho c}} \int_o^t \frac{H(t-\tau)}{\tau^{1/2}} \, d\tau \, , \tag{B.1}$$

where $T_s = T(o,t)$ is the surface temperature. Using the convolution of $t^{-1/2}$ and $H(t)$, $t^{-1/2} * H(t)$, for $\int_o^t \tau^{-1/2} H(t-\tau) \, d\tau$, (B.1) can be rewritten as

$$T_s - T_o = \frac{t^{-1/2} * H(t)}{\sqrt{\pi k \rho c}} \, . \tag{B.2}$$

The Laplace transform of (B.2) is

$$\mathcal{L}[T_s - T_o] = \frac{\mathcal{L}[t^{-1/2}]\mathcal{L}[H(t)]}{\sqrt{\pi k \rho c}} \tag{B.3}$$

$$= \frac{\sqrt{\pi/s} \left\{ H^*[1 + \exp(-As)]/s \right\}}{\sqrt{\pi k \rho c}}$$

$$= \frac{H^*}{\sqrt{k \rho c}} \frac{1}{s^{3/2} [1 + \exp(-As)]} \, .$$

Using a Maclaurin series expansion for $1/[1 + \exp(-As)]$, (B.3) becomes

$$\mathcal{L}[T_s - T_o] = \frac{H^*}{\sqrt{k \rho c}} \sum_{n=0}^{\infty} (-1)^n \frac{\exp(-nAs)}{s^{3/2}} \, . \tag{B.4}$$

Then, taking the inverse transform of (B.4),

$$T_s - T_o = \frac{H^*}{\sqrt{k\rho c}} \sum_{n=0}^{\infty} (-1)^n \frac{S_{nA}(t)\sqrt{t-nA}}{\Gamma(3/2)}, \qquad (B.5)$$

where $\Gamma(n)$ is the Gamma function and $S_{nA}(t)$ is the unit step function defined by

$$S_{nA}(t) = 0 \quad \text{when} \quad 0 < t < nA ,$$
$$= 1 \quad \text{when} \quad t > nA .$$

Equation (B.5) can be made dimensionless by rearrangement to give

$$\frac{\sqrt{\pi k\rho c}\,(T_s - T_o)}{2H^*\sqrt{A}} = \sum_{n=0}^{\infty} (-1)^n S_n(t/A) \sqrt{\frac{t}{A} - n} \quad . \qquad (B.6)$$

APPENDIX C

FORWARD DIFFERENCE APPROXIMATION

The equation of an "n^{th}" order forward difference approximation of a first order derivative is given by Sunderland and Grosh [9] as

$$\left.\frac{\partial v}{\partial z}\right|_{\substack{z=i\Delta z\\ y=j\Delta y}} = \frac{1}{\Delta z}\left[\Delta v_k - \frac{1}{2}\Delta^2 v_k + \frac{1}{3}\Delta^3 v_k - \ldots \pm \frac{1}{n}\Delta^n v_k\right] , \qquad (C.1)$$

where

$$\Delta v_k = v(i+1,j) - v(i,j) ,$$

$$\Delta^2 v_k = v(i+2,j) - 2v(i+1,j) + v(i,j) ,$$

$$\Delta^3 v_k = v(i+3,j) - 3v(i+2,j) + 3v(i+1,j) - v(i,j) .$$

Combining these terms for a second order approximation,

$$\frac{\partial v(i,j)}{\partial z} = \frac{1}{\Delta z}\left[-\frac{1}{2}v(i+2,j) + 2v(i+1,j) - \frac{3}{2}v(i,j)\right] , \qquad (C.2)$$

$$\left.\frac{\partial v}{\partial z}\right|_{z=0} = \frac{\partial v(1,j)}{\partial z} = -\left[\frac{3v(1,j) - 4v(2,j) + v(3,j)}{2\Delta z}\right] . \qquad (C.3)$$

APPENDIX D

COMPUTER PROGRAM

The complete computer program used in this study is presented below in ALGOL. Symbols used and special characters employed for the computer program used for this work are:

V	dimensionless temperature v
Z	dimensionless distance z
MU	dimensionless rate of melting
SIGMA	dimensionless thickness melted
ATM	dimensionless heat pulse time A/t_m^*
HR	dimensionless heat flux $H(t)/H^*$
VOM	dimensionless melting temperature
M	property parameter M
I	distance subscript integer
J	time subscript integer
K	integer limit of I
P	integer limit of J
N	integer number of increments of time Δy in each heat pulse
S	integer number of increments of time Δy up to end of heat pulse in the cycle calculations are proceeding for
R,T	integers used to control print out
⊄	indicates end of statement or used as a prefix in a command statement
**	indicates powers of ten (5.0**-7 = 5.0×10^{-7})

BAC-220 STANDARD VERSION

```
COMMENT MELTING OF SEMI INFINITE SOLID, MELT REMOVED                    $
INTEGER I,J,K,N,P,R,S,T                                                 $
TRANS.. READ ($$DATA)                                                   $
        ARRAY V(K+5,3),MU(3),SIGMA(3),Z(K+5)                            $
        FOR I=(1,1,K+1)                                                 $
        V(I,1)=0.0                                                      $
        SIGMA(1)=0.0                                                    $
        WRITE ($$TITLE1)                                                $
        WRITE ($$ANS1,FMT1)                                             $
        WRITE ($$TITLE2)                                                $
        WRITE ($$TITLE3)                                                $
        VOM=(1.0)/(SQRT(3.14159))                                       $
        N=FIX((200.0)(ATM))                                             $
        S=N                                                             $
        T=1                                                             $
        J=1                                                             $
LOOP1.. I=1                                                             $
LOOP2.. V(I+1,2)=(V(I+2,1)+V(I,1))/(2.0)                                $
        IF V(I+1,2) GTR (5.0**-7)                                       $
BEGIN   I=I+1                                                           $
        IF I LSS K                                                      $
        GO TO LOOP2                                                     $
        WRITE ($$TITLEA)                                                $
        GO TO TRANS                                 END                 $
```

```
        FOR I=(I+2,1,K+1)                                          $
        V(I,2)=0.0                                                 $
        IF J LEQ S                                                 $
        HR=1.0                                                     $
        IF J GTR S                                                 $
        HR=0.0                                                     $
        V(1,2)=((3.0)(DZ)(HR)+(18.0)(V(2,2))-(9.0)(V(3.2))
              +(2.0).(V(4,2)))/(11.0)                              $
        IF J EQL (S+N)                                             $
        S=S+2N                                                     $
        J=J+1                                                      $
        IF J GTR P                                                 $
        GO TO TRANS                                                $
        IF V(1,2) LSS VOM                                          $
BEGIN   FOR I=(1,1,K+1)                                            $
        V(I,1)=V(I,2)                                              $
        GO TO LOOP1                                     END        $
        ERR1=ABS(V(1,1)-VOM)                                       $
        ERR2=ABS(V(1,2)-VOM)                                       $
        IF ERR2 LSS ERR1                                           $
BEGIN   FOR I=(1,1,K+1)                                            $
        V(I,1)=V(I,2)                                              $
        IF (J-1) EQL S                                             $
BEGIN   Y=(DY)(FLOAT(J)-1.0)                                       $
        IF T EQL 1                                                 $
```

```
BEGIN   T=2                                                          $
        FOR I=(1,1,K+1)                                              $
BEGIN   Z(I)=(DZ)(FLOAT(I)-1.0)                                      $
        WRITE ($$ANS2,FMT2)                              END         $
        WRITE ($$TITLE4)                                             $
        WRITE ($$TITLE5)                                             $
        WRITE ($$TITLE6)                                             $
        WRITE ($$TITLE7)                                 END         $
        IF T GTR 1                                                   $
        WRITE ($$ANS3,FMT3)                                          $
        GO TO LOOP1                                      END         $
        MU(1)=0.0                                        END         $
        IF ERR2 GEQ ERR1                                             $
BEGIN   J=J-1                                                        $
        MU(1)=HR-((3.0)(V(1,1))-(4.0)(V(2,1))+V(3,1))/(DZ)     END   $
        Y=(DY)(FLOAT(J)-1.0)                                         $
        IF T EQL 1                                                   $
BEGIN   T=2                                                          $
        FOR I=(1,1,K+1)                                              $
BEGIN   Z(I)=(DZ)(FLOAT(I)-1.0)                                      $
        WRITE ($$ANS2,FMT2)                              END         $
        WRITE ($$TITLE4)                                             $
        WRITE ($$TITLE5)                                             $
        WRITE ($$TITLE6)                                             $
        WRITE ($$TITLE7)                                 END         $
        IF T GTR 1                                                   $
```

```
        WRITE ($$ANS3,FMT3)                                          $
        R=J                                                          $
LOOP3.. I=1                                                          $
LOOP4.. V(I+1,2)=(0.5+((M)(MU(1))(DZ)/4.0)))(V(I+2,1))+(0.5-((M).
                (MU(1))(DZ)/(4.0)))(V(I,1))                          $
        V(1,2)=V(1,1)                                                $
        IF V(I+1,2) GTR (5.0**-7)                                    $
BEGIN   I=I+1                                                        $
        IF I LSS K                                                   $
        GO TO LOOP4                                                  $
        WRITE ($$TITLEA)                                             $
        GO TO TRANS                                    END           $
        FOR I=(I+2,1,K+1)                                            $
        V(I,2)=0.0                                                   $
        MU(2)=HR-((3.0)(V(1,2))-(4.0)(V(2,2))+V(3,2))/(DZ)           $
        SIGMA(2)=SIGMA(1)+(MU(2)+MU(1))(DY)/(2.0)                    $
        FOR I=(1,1,K+1)                                              $
        V(I,1)+V(I,2)                                                $
        MU(1)=MU(2)                                                  $
        SIGMA(1)=SIGMA(2)                                            $
        J=J+1                                                        $
EITHER  IF (J-1) EQL S                                               $
BEGIN   Y=(DY)(FLOAT(J)-1.0)                                         $
        WRITE ($$ANS4,FMT4)                                          $
        GO TO LOOP1                                    END           $
     OR IF J EQL FIX((S+1+R)/(2))                                    $
BEGIN   Y=(DY)(FLOAT(J)-1.0)                                         $
```

```
        WRITE ($$ANS4,FMT4)                                          $
        R=R+10                                   END    END         $
EITHER  IF MU(1) LSS 0.0                                             $
BEGIN   WRITE ($$TITLEB)                                             $
        GO TO TRANS                                END               $
     OR IF MU(2) LSS 0.0                                             $
BEGIN   WRITE ($$TITLEB)                                             $
        GO TO TRANS                              END    END         $
        IF J GTR P                                                   $
        GO TO TRANS                                                  $
        GO TO LOOP3                                                  $
INPUT   DATA(K,M,P,ATM,DY,DZ)                                        $
OUTPUT  ANS1(K,M,P,ATM,DY,DZ)                                        $
OUTPUT  ANS2(Y,Z(I),V(I,1))                                          $
OUTPUT  ANS3(Y,V(1,1))                                               $
OUTPUT  ANS4(Y,MU(1),SIGMA(1))                                       $
FORMAT  TITLEA(B31, *K TOO SMALL*,W4)                                $
FORMAT  TITLEB(B31,*NEGATIVE MU* ,"4)                                $
FORMAT  TITLE1(B19, K ,B5, M ,B5, P ,B5, ATM ,B5, DY ,B5, DZ ,W6)    $
FORMAT  TITLE2(B26,*INITIAL MELTING THRESHOLD*,W6)                   $
FORMAT  TITLE3(B23,*Y*,B12,*Z*,B13,*V*,W2)                           $
FORMAT  TITLE4(B26,*MELTING THRESHOLD VALUES*,W6)                    $
FORMAT  TITLE5(B23,*Y*,B26,(V1*,W2)                                  $
FORMAT  TITLE6(B28,*VALUES DURING MELTING*,W6)                       $
FORMAT  TITLE7(B16,*Y*,B20,*MU*,B18,*SIGMA*,W2)                      $
```

```
FORMAT  FMT1(B18,1I3,1X6.1,1I7,1X6,2,1X8.4,1X6.2,W2)   $
FORMAT  FMT2(B20,1X6.4,1X12.2,1X17.8,W0)               $
FORMAT  FMT3(B20,1X6.3,1X29.8,W6)                      $
FORMAT  FMT4(B12,1X7.4,1X22.6,1X22.6,W0)               $
        FINISH                                         $
```

APPENDIX E

QUASI-STEADY STATE SOLUTION FOR MELTING TIME DURING A PULSE

For the time interval $y_{11} \leq y_I \leq y_{20}$, the temperature distribution as given by Carslaw and Jaeger [10] can be written as

$$v(z,y_I) = \frac{1}{2\sqrt{\pi y_I}} \int_o^\infty v(z',y_{11}) \left\{ \exp\left[-\frac{(z-z')^2}{4y_I}\right] + \exp\left[-\frac{(z+z')^2}{4y_I}\right]\right\} dz' \quad , \tag{E.1}$$

where $y_I = y - y_{11}$ and $v(z,y_{11}) = v_e^*$ as given by (III.28). Equation (E.1) can be rearranged to give

$$v(z,y_I) = \frac{1}{2\sqrt{\pi y_I}} \left\{ \int_o^\infty \exp\left[-\frac{(z-z')^2}{4y_I} - \frac{z'}{P}\right] dz' + \int_o^\infty \exp\left[-\frac{(z+z')^2}{4y_I} - \frac{z'}{P}\right] dz' \right\} \quad , \tag{E.2}$$

where $P = 1/M + 2/\sqrt{\pi}$. Substituting $z' = z + 2\sqrt{y_I}\, u - 2y_{I/P}$ in the first integral and $z' = -z + 2\sqrt{y_I}\, u - 2y_I/P$ in the second, the solution of (E.2) for $y_I = A/t_m^*$ is

$$v(z,y_{20}) = \frac{1}{2\sqrt{\pi}} \left\{ \exp\left(\frac{A}{P^2 t_m^*} - \frac{z}{P}\right) \operatorname{erfc}\left(\frac{\sqrt{A/t_m^*}}{P} - \frac{z}{2\sqrt{A/t_m^*}}\right) + \exp\left(\frac{A}{P^2 t_m^*} + \frac{z}{P}\right) \operatorname{erfc}\left(\frac{\sqrt{A/t_m^*}}{P} + \frac{z}{2\sqrt{A/t_m^*}}\right)\right\} . \tag{E.3}$$

An expression for the temperature distribution for $y_{20} \leq y \leq y_{m2}$ can be found by combining equations from Carslaw and Jaeger [11] by the principle of superposition. If $y_{II} = y - y_{20}$, the resulting equation is

$$v(z,y_{II}) = \frac{1}{2\sqrt{\pi y_{II}}} \int_o^\infty v(z',y_{20}) \left\{ \exp\left[-\frac{(z-z')^2}{4y_{II}}\right] + \exp\left[-\frac{(z+z')^2}{4y_{II}}\right]\right\} dz' + \sqrt{y_{II}}\ \mathrm{ierfc}\left(\frac{z}{2\sqrt{y_{II}}}\right); \tag{E.4}$$

where ierfc is the notation for the integral of the error function complement, and

$$\mathrm{ierfc}(x) = \pi^{-1/2} \exp(-x^2) - x\ \mathrm{erfc}(x)\ . \tag{E.4}$$

Substituting (E.3) into (E.4) with $z = 0$ and $y_{II} = f_2\ A/t_m^*$ gives

$$v(o,y_{m2}) = \frac{1}{\sqrt{\pi}} = \frac{1}{2\pi\sqrt{f_2 A/t_m^*}} \left\{ \int_o^\infty \exp\left[-\frac{(z')^2}{4f_2 A/t_m^*} - \frac{z'}{P} + \frac{A/t_m^*}{P^2}\right] \mathrm{erfc}\left(\frac{\sqrt{A/t_m^*}}{P} - \frac{z'}{2\sqrt{A/t_m^*}}\right) dz' + \int_o^\infty \exp\left[-\frac{(z')^2}{4f_2 A/t_m^*} + \frac{z'}{P} + \frac{A/t_m^*}{P^2}\right] \mathrm{erfc}\left(\frac{\sqrt{A/t_m^*}}{P} + \frac{z'}{2\sqrt{A/t_m^*}}\right) dz'\right\} + \frac{f_2 A/t_m^*}{\sqrt{\pi}}\ . \tag{E.5}$$

Substituting $z' = -2\left(\sqrt{A/t_m^*}\ \delta - \frac{A/t_m^*}{P}\right)$ into the first integral and $z' = 2\left(\sqrt{A/t_m^*}\,\delta - \frac{A/t_m^*}{P}\right)$ into the second integral and dropping the subscript on f_2, (E.5) simplifies to

$$1 = \frac{\exp\left[(1+f)\ \frac{A/t_m^*}{P^2}\right]}{\sqrt{\pi f}} \int_{-\infty}^{\infty} \exp\left\{-\frac{\left[\delta - \frac{A/t_m^*}{P}\ (1+f)\right]^2}{f}\right\} \operatorname{erfc}(\delta)d\delta \quad \text{(E.6)}$$

$$+ \sqrt{f\ A/t_m^*}\ .$$

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