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A NOTE ON LINEAR SYSTEMS ON K-3 SURFACES

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ABSTRACT. A simple necessary and sufficient condition is given for a general member of the complete linear system $|Y|$ to be irreducible and nonsingular where Y is a reduced, connected curve on a K-3 surface.

1. Introduction. It is well known that for F a nonsingular rational projective irreducible surface (all our schemes will be defined over a fixed algebraically closed field of characteristic 0), given a reduced connected curve Y on F all of whose irreducible components have strictly negative intersection number with a canonical divisor K_F, a general member of the complete linear system $|Y|$ is nonsingular and irreducible.

In this note we will prove the following analogous theorem for linear systems on K-3 surfaces.

THEOREM. *Let F be a K-3 surface (i.e. a nonsingular projective irreducible surface with $H^1(F, \mathcal{O}_F) = 0$ and $\mathcal{O}_F \cong \mathcal{O}_F(K_F)$), and let Y be a reduced, connected curve on F. Then a general member of $|Y|$ is nonsingular and irreducible if and only if for every expression $Y = Y_1' + \cdots + Y_q'$ of Y as a sum of connected curves ($q \geq 2$), $\Sigma_{i<j} Y_i' \cdot Y_j' \geq q$.*

The author would like to thank the referee for suggesting the proof of Lemma (2.1) below for the case $C^2 = 0$.

2. The proof of the theorem. We begin with some well-known facts about curves on a K-3 surface F. First note that since $\mathcal{O}_F(K_F) \cong \mathcal{O}_F$, from the adjunction formula, $C^2 = 2p_a(C) - 2$ for $C \subset F$ a curve. Moreover if C is a reduced connected curve, then by Riemann-Roch $\dim |C| = p_a(C)$. Indeed we have that

$$\begin{aligned} h^0(\mathcal{O}_F(C)) - h^1(\mathcal{O}_F(C)) + h^2(\mathcal{O}_F(C)) &= C \cdot (C - K_F)/2 + 1 + h^2(\mathcal{O}_F) \\ &= p_a(C) + h^2(\mathcal{O}_F) \\ &= p_a(C) + 1. \end{aligned}$$

By Serre duality $h^2(\mathcal{O}_F(C)) = h^0(\mathcal{O}_F(-C)) = 0$, and so we must show that $h^1(\mathcal{O}_F(C)) = 0$ or equivalently (again by Serre duality) that $h^1(\mathcal{O}_F(-C)) = 0$. Note however since C is reduced, connected $H^0(\mathcal{O}_C) \cong H^0(\mathcal{O}_F) \cong k$, and therefore applying the long exact cohomology sequence to the exact sequence $0 \to \mathcal{O}_F(-C) \to \mathcal{O}_F \to \mathcal{O}_C \to 0$ and using the fact that $h^1(\mathcal{O}_F) = 0$, we get the desired conclusion.

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We can now state the following lemma.

LEMMA (2.1). *Let C be an arbitrary integral curve on a K-3 surface F. Then a general member of $|C|$ is nonsingular.*

PROOF. If $C^2 < 0$, then it is trivial to show that C is nonsingular, rational. If $C^2 > 0$, then by [**2**, (3.1), p. 611], we have that $|C|$ is base point free, and hence we get the required conclusion from Bertini's theorem (recall that we are working over an algebraically closed field of characteristic 0). So we are left with the case $C^2 = 0$. But here we have that $\dim |C| = p_a(C) = 1 > 0$ so that two general elements of $|C|$ intersect properly. Since $C^2 = 0$ they are disjoint, so again by Bertini a general element of $|C|$ is nonsingular. Q.E.D.

PROOF OF THE THEOREM (2.2). By (2.1) it is enough to show that for Y as in the theorem a general member of $|Y|$ is irreducible if and only if for every expression $Y = Y_1' + \cdots + Y_q'$ of Y as a sum of connected curves ($q \geqslant 2$), $\Sigma_{i<j} Y_i' \cdot Y_j' \geqslant q$. Suppose first that a general member of $|Y|$ is irreducible. Let $Y = Y_1' + \cdots + Y_q'$ be an expression of Y as a sum of connected curves. Then

$$\begin{aligned} \dim|Y| &= p_a(Y) = p_a(Y_1' + \cdots + Y_q') \\ &= \sum_{i=1}^{q} p_a(Y_i') + \sum_{i<j} Y_i' \cdot Y_j' - q + 1 \qquad (+) \\ &= \sum_{i=1}^{q} \dim|Y_i'| + \sum_{i<j} Y_i' \cdot Y_j' - q + 1. \end{aligned}$$

Clearly since a general member of $|Y|$ is irreducible, we must have $\dim|Y| > \Sigma_{i=1}^{q} \dim|Y_i'|$ so that $\Sigma_{i<j} Y_i' \cdot Y_j' \geqslant q$.

Conversely suppose that a general member $Z \in |Y|$ is reducible, say $Z = Z_1 + \cdots + Z_q$ with the Z_i irreducible, $i = 1, \ldots, q$. (Note that since Y is reduced, Z is reduced, and hence $Z_i \neq Z_j$ for $i \neq j$.) Then there exist Y_i' connected sums of irreducible components of Y such that Z_i is a generalization of Y_i' in the linear system $|Z_i| = |Y_i'|$ for each $i = 1, \ldots, q$. But then certainly

$$\dim|Y| = \sum_{i=1}^{q} \dim|Z_i| = \sum_{i=1}^{q} \dim|Y_i'|.$$

On the other hand from $(+)$ above, this would imply that $\Sigma_{i<j} Y_i' \cdot Y_j' = q - 1 < q$ completing the proof of the theorem. Q.E.D.

EXAMPLE (2.3). In case $Y = Y_1 + Y_2$ has two irreducible components, the theorem means that a general member of $|Y|$ is irreducible if and only if $Y_1 \cdot Y_2 \geqslant 2$. We give here an example of a curve $Y = Y_1 + Y_2$ on a K-3 surface F such that $Y_1 \cdot Y_2 = 1$, and therefore is not a degeneration of a nonsingular irreducible curve on F.

Indeed, let F be a nonsingular quartic surface in $\mathbf{P}^3$ which contains a line l (F is well known to be a K-3 surface). Let H be a general hyperplane section on F, so that H is isomorphic to a nonsingular plane quartic curve. Then setting $Y = H + l$, we get that $H \cdot l = 1$, and so Y cannot be flatly smoothed on F.

Actually more is true. Y is not even the flat degeneration of an integral curve in $\mathbf{P}^3$ (and therefore in particular a degeneration of a nonsingular curve). Indeed it is easy to compute that $p_a(Y) = 3$, degree $Y = 5$. But from the Castelnuovo bound **[1]** the maximal possible arithmetic genus for an integral curve of degree 5 in $\mathbf{P}^3$ is 2.

References

1. P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.
2. B. Saint-Donat, *Projective models of K-3 surfaces*, Amer. J. Math. **96** (1974), 602–639.

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