

VISCOUS CLUTCH APPLICATION OF MECHANICALLY STORED ENERGY
TO A DAMPED OSCILLATORY SYSTEM

A THESIS

Presented to

The Faculty of the Division of Graduate
Studies and Research

By
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In Partial Fulfillment
of the Requirements for the Degree
Master of Science in Mechanical Engineering

Georgia Institute of Technology


March, 1974

VISCOUS CLUTCH APPLICATION OF MECHANICALLY STORED ENERGY
TO A DAMPED OSCILLATORY SYSTEM

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ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. David M. Sanborn for his guidance and assistance in this research. To Dr. Harold Johnson and Dr. David McGill I am gratefully indebted for their careful review of the manuscript.

I also wish to thank Whirlpool Corporation for their support in making this work possible.

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NOMENCLATURE

b_1, b_2, b_3	time constants (sec^{-1})
c	damping coefficient (in-lbf-sec/rad)
c_o	constant of integration (rad/sec)
c_1, \dots, c_8	constants of integration (radians)
d	clutch groove depth (inches)
h	clutch groove clearance (inches)
h_A, h_W	heat transfer coefficient ($\text{BTU/hr-ft}^2\text{-F}^\circ$)
ℓ	characteristic length (inches)
m	time constant (sec^{-1})
n	number of clutch grooves
q_1, q_2	time constant (sec^{-1})
r	intermediate variable (sec^{-2}) and radius (inches)
s	intermediate variable (sec^{-3})
t	time (sec)
t_1	time at maximum basket displacement (sec)
t_2	time of clutch engagement for coupled solution (sec)
t_3	time of clutch engagement
u	clutch temperature rise (F°)
v	characteristic velocity (in/sec)
x	depth of clutch engagement (inches)
Λ	intermediate variable (sec^{-1})
A_A	area of clutch exposed to air (ft^2)
A_i	area of surface 1 of i 'th clutch groove (in^2)

A_w	area of clutch exposed to water (ft^2)
A_1	intermediate variable (sec^{-1})
A_2	intermediate variable (sec^{-2})
A_3	intermediate variable (sec^{-3})
B	intermediate variable (sec^{-1})
BRKDWN	minimum motor shaft speed (rad/sec)
C_h	heat capacity (BTU/°F)
E	variance (rad^2)
F	particular solution of $\theta_2(t)$ (radians)
F_1	force (lbf)
H	basket displacement (radians)
I_F	flywheel moment of inertia ($\text{in-lbf-sec}^2/\text{rad}$)
I_T	basket moment of inertia ($\text{in-lbf-sec}^2/\text{rad}$)
I_W	moment of inertia of basket contents ($\text{in-lbf-sec}^2/\text{rad}$)
K	spring constant (in-lbf/rad)
N	gear ratio (rad/rad)
P	average power through motor shaft (hp)
P_c	average clutch power dissipation (hp)
R	Reynolds number
T	torque through clutch (in-lbf)
T_{\max}	maximum torque through clutch (in-lbf)
T_1	torque transmitted through surfaces 1 (in-lbf)
T_2	torque transmitted through surfaces 2 (in-lbf)
TQ	T intercept of motor-torque curve (in-lbf)
V	clutch constant (in-lbf-sec/rad)
V_s	lower bound on clutch constant (in-lbf-sec/rad)

V_L	upper bound on clutch constant (in-lbf-sec/rad)
W	motor work per cycles (in-lbf)
W_c	clutch dissipated work per cycle (in-lbf)
α	slope of motor torque curve (in-lbf-sec/rad)
μ	viscosity (lbf-sec/in ²)
ρ	clutch fluid density (lbm/in ³)
τ	period (sec)
ω_1	frequency of uncoupled motion (sec ⁻¹)
ω_2	frequency of coupled motion (sec ⁻¹)
θ	basket displacement (radians)
θ_{max}	maximum basket displacement (radians)
θ_{min}	basket displacement at time zero (radians)
θ_1	uncoupled basket displacement (radians)
θ_2	coupled basket displacement (radians)
θ_3	start-up basket displacement (radians)
ϕ	driving clutch member displacement and displacement of I_w (radians)
ϕ_1	uncoupled driving member displacement (radians)
ϕ_2	coupled driving member displacement (radians)
$\dot{\phi}_{max}$	driving member velocity at time zero (rad/sec)
$\dot{\phi}_{max_s}$	lower bound on $\dot{\phi}_{max}$ (rad/sec)
$\dot{\phi}_{max_L}$	upper bound on $\dot{\phi}_{max}$ (rad/sec)

SUMMARY

Consideration of wear and complexity of design led to the proposal that a viscous clutch be substituted for the dry friction clutch which was used in a previous analysis of an agi-basket drive mechanism for a vertical axis, automatic clothes washing machine. Values of horsepower loss in the clutch, maximum transmitted torque, time of clutch release, and physical constants of the clutch are analytically determined to obtain the proper amplitude and period of oscillation of the agi-basket. Gear ratios and spring constants of the mechanism are then adjusted to minimize motor power requirements. An experimental investigation is conducted to compare a previously used model of the oscillating system with alternative models, and values of damping and inertia characteristic of the system are determined.

CHAPTER I

INTRODUCTION

In an effort to increase the load capacity of the vertical axis, automatic clothes washing machine, the idea of the agi-basket has been advanced. The conventional washing machine oscillates a central agitator in a stationary tub to provide the cleansing motion and when this configuration is enlarged to gain a larger wash load, the washing action is diminished at the periphery of the tub due to the distance from the agitator. If the tub is oscillated with an attached agitator to improve washing at the periphery, the increase in oscillating mass requires a heavy motor and transmission. To reduce this inertia, a light-weight perforated plastic tub, the agi-basket, can be agitated within the conventional tub. The agitator is eliminated and the washing motion provided by paddles on the inside of the basket.

The conventional bellcrank transmission used on agitator machines was found to be incapable of dealing with the inertial loads. Another attempt was made in which a large motor was connected through a spring to the basket and was run in one direction and then reversed to give the desired motion.

All later attempts have used a spring to store energy at zero velocity and reverse the motion of the basket. In a design currently under evaluation, the basket rebounds off springs at the last few degrees of either end of its stroke. A continuously running motor applies torque through either of two electric clutches at some point after zero velocity. There is a separate clutch and drive train for each direction of motion.

Horn [1] describes an analytical study in which an agi-basket is attached directly to a rotational spring. The energy dissipated in the washing action is restored by coupling the basket to a flywheel during one direction of oscillation and the flywheel speed is maintained by an electric motor. A dry friction clutch is engaged to transmit energy from the flywheel to the basket and released at a proper time to produce the desired amplitude and period.

Since the flywheel speed is greater than the basket speed, the clutch slips continuously. This requires that the clutch be large in order to effect a reasonable wear life. Also, the mechanism requires a device to apply a large normal force to the clutch surfaces in order to transmit the necessary torque. For these reasons it was decided to carry out a similar investigation to that done by Horn [1], but using a viscous clutch in place of the dry friction clutch. This research is devoted to this analysis and an experimental evaluation of some assumptions made in

this and earlier work [1].

The proposed mechanism is as shown in Figure 1. As before, the agi-basket is attached permanently to the spring. The clutch is situated between the basket and a gear reduction unit which, in turn, is coupled to the flywheel and motor. The motor runs continuously during the agitation.

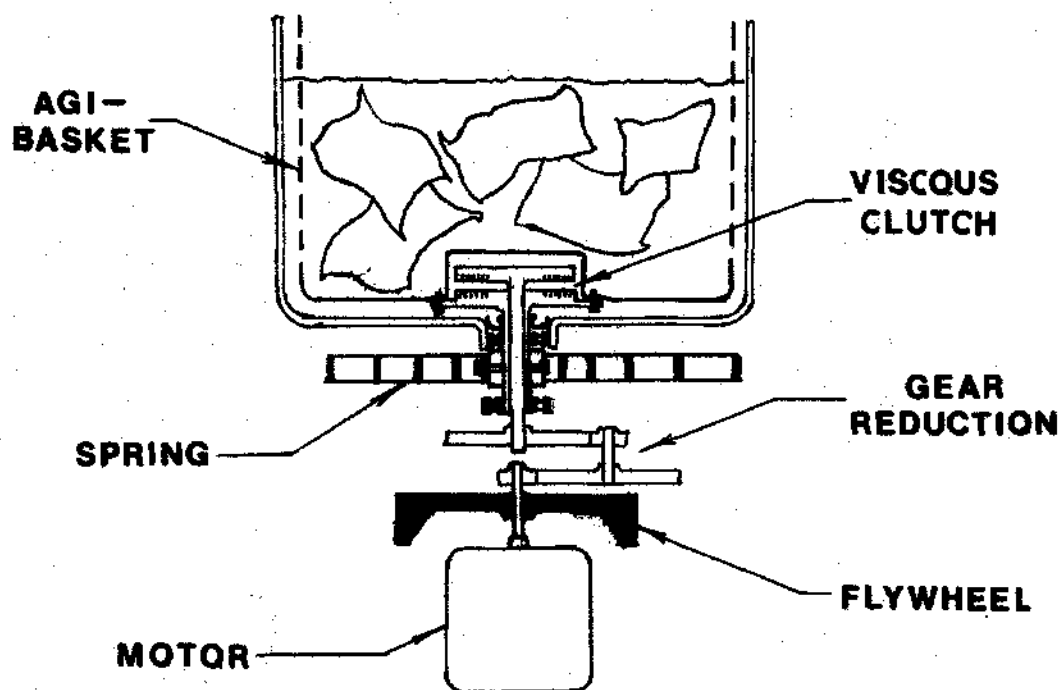


Figure 1. Proposed Mechanism

CHAPTER II

GOVERNING EQUATIONS

The purpose of this analysis is to give evidence that the proposed mechanism will provide a suitable washing action with a practical electric motor output and to specify clutch, spring, and gear ratio design parameters which will minimize power requirements.

Clutch

The proposed viscous clutch consists of two grooved discs with plane surfaces facing and provision for changing the fluid filled clearance between the two, Figure 2. Assuming laminar flow and no edge effects, the force transmitted through surfaces (1) is

$$F_{1_i} = A_i \mu \frac{(r_i - \frac{w}{2})(\dot{\phi} - \dot{\theta})}{h}$$

where A_i is the area of the surface (1) undergoing shear and μ is the absolute viscosity of the fluid.

$$A_i = 2\pi(r_i - \frac{w}{2})x$$

and the torque transmitted through surfaces (1),

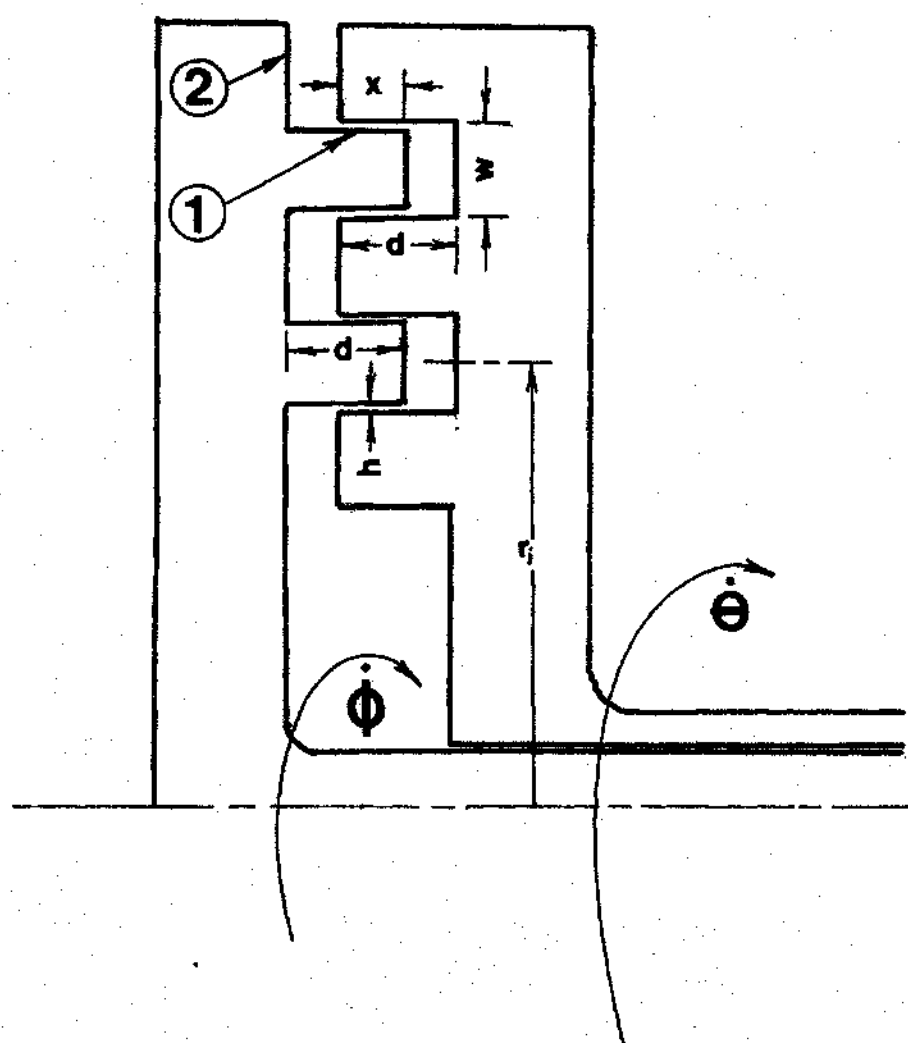


Figure 2. Clutch Detail

$$T_1 = \sum_i F_{1i} (r_i - \frac{w}{2})$$

Making the substitutions,

$$T_1 = \sum_i 2\pi\mu x (r_i - \frac{w}{2})^3 \frac{(\dot{\phi} - \dot{\theta})}{h}$$

If surfaces (2) are thought of as two continuous parallel plates, then

$$dT_2 = 2\pi r \mu r^2 \frac{(\dot{\phi} - \dot{\theta})}{(d-x)} dr$$

$$T_2 = \int_{r_2 - \frac{w}{2}}^{r_n + \frac{w}{2}} 2\pi r^3 \frac{(\dot{\phi} - \dot{\theta})}{(d-x)} dr$$

$$T_2 = [(r_n + \frac{w}{2})^4 - (r_1 - \frac{w}{2})^4] \mu \frac{(\dot{\phi} - \dot{\theta})}{(d-x)}$$

Summing the contributions from surfaces of type (1) and (2), the total transmitted torque becomes

$$T = \pi\mu(\dot{\phi} - \dot{\theta}) \left\{ \frac{2x}{h} \sum_{i=1}^n (r_i - \frac{w}{2})^3 + \frac{1}{2(d-x)} [(r_n + \frac{w}{2})^4 - (r_1 - \frac{w}{2})^4] \right\}$$

for positive values of x . If

$$V = \pi\mu \left\{ \frac{2x}{h} \sum_{i=1}^n (r_i - \frac{w}{2})^3 + \frac{1}{2(d-x)} [(r_n + \frac{w}{2})^4 - (r_1 - \frac{w}{2})^4] \right\} \quad (1)$$

then

$$T = V(\dot{\phi} - \dot{\theta}) \quad (2)$$

To check the validity of the laminar flow assumption take the Reynold's number for a fluid between plates moving one with respect to another [2]

$$R_e = \frac{\rho v l}{\mu}$$

where l is a characteristic length, v is a characteristic velocity, and ρ is the density of the fluid. Since a high Reynold's number is characteristic of turbulent flow the highest possible values of v and l will be used to show that the flow is laminar. Taking the dimensions of a typical clutch to be

$$\begin{array}{ll} r_n = 5 \text{ in.} & d = 0.210 \text{ in.} \\ r_1 = 2.5 \text{ in.} & h = 0.010 \text{ in.} \\ n = 26 \text{ grooves} & x = 0.200 \text{ in. engaged} \end{array}$$

then $w = 0.10$ inches (Figure 2). A typical value of V is about 110 lbf-in-sec (see Chapter IV). Substituting these dimensions and V into equation (1)

$$\mu = \frac{110}{\pi} \frac{1}{40[1601] + 50[611]}$$

$$\mu = 3.71 \times 10^{-4} \text{ lbf-sec/in}^2$$

or

$$0.143 \text{ lbm/in-sec}$$

A typical fluid might have a density of about

$$\rho = 0.035 \text{ lbm/in}^3$$

A maximum characteristic velocity as seen between the clutch surfaces would be the relative velocity of the surfaces and this relative velocity is greatest at r_n . Therefore,

$$v = r_n (\dot{\phi} - \dot{\theta})$$

The largest $(\dot{\phi} - \dot{\theta})$ encountered (Chapter IV) is

$$(\dot{\phi} - \dot{\theta}) \approx 12 \text{ rad/sec}$$

The cross-section of the flow region is a slot 0.10 inches by 0.01 inches. To maximize l , let

$$l = w = 0.10 \text{ inches}$$

Then,

$$R_e = \frac{(60 \text{ in/sec})(0.10 \text{ in})(0.035 \text{ lbm/in}^{-3})}{0.143 \text{ lbm/in-sec}}$$

$$R_e = 1.47$$

which is far below the transition value of about 2300 for turbulent flow.

It is assumed for the system studied that $V(t)$ is a step function of time, Figure 3, V being approximately zero when the clutch is disengaged.

This, of course, is accomplished by manipulation of the clutch engagement depth, x . V can be made as small as desired during disengagement by bringing the plates farther apart.

Definitions

Each agitation cycle of the agi-basket consists of two parts: (1) uncoupled motion, and (2) coupled motion (clutch engaged). To aid in the analysis, zero time will be at the zero velocity point immediately after coasting, Figure 4, and will also be the time of clutch engagement, while $t_3 \equiv t_2$ is the time of clutch disengagement. $\dot{\theta}$ is the angular displacement of the basket and $\dot{\phi}$ is the angular velocity of the flywheel member of the clutch. Subscripts (1) and (2) indicate uncoupled and coupled motion, respectively. In other words,

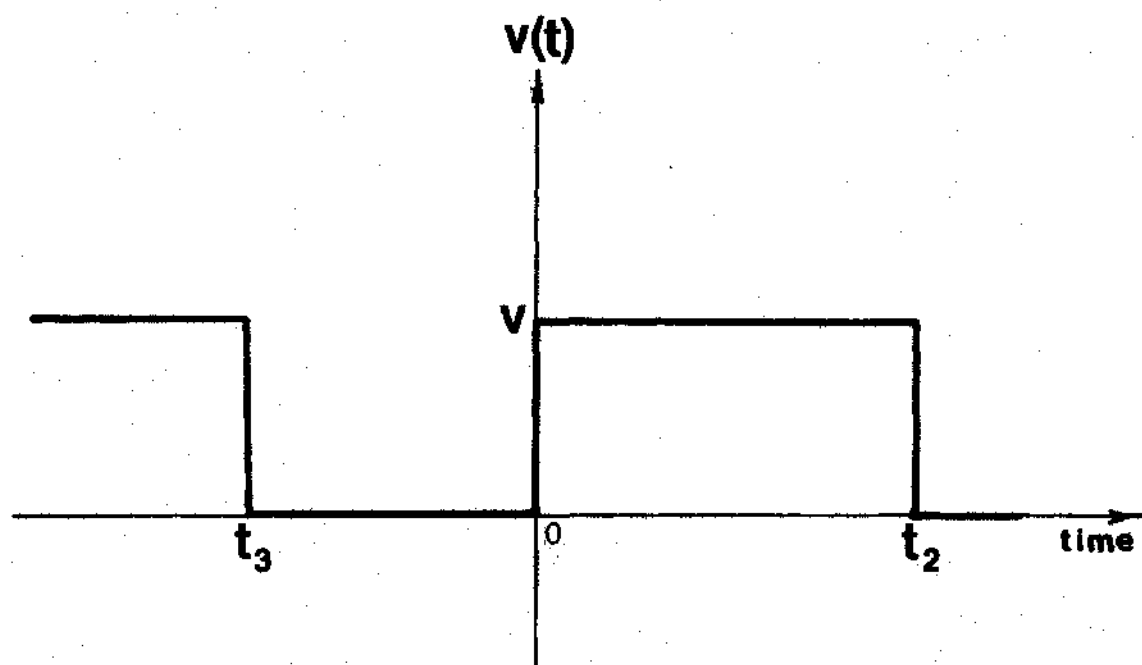


Figure 3. Clutch Parameter vs. Time

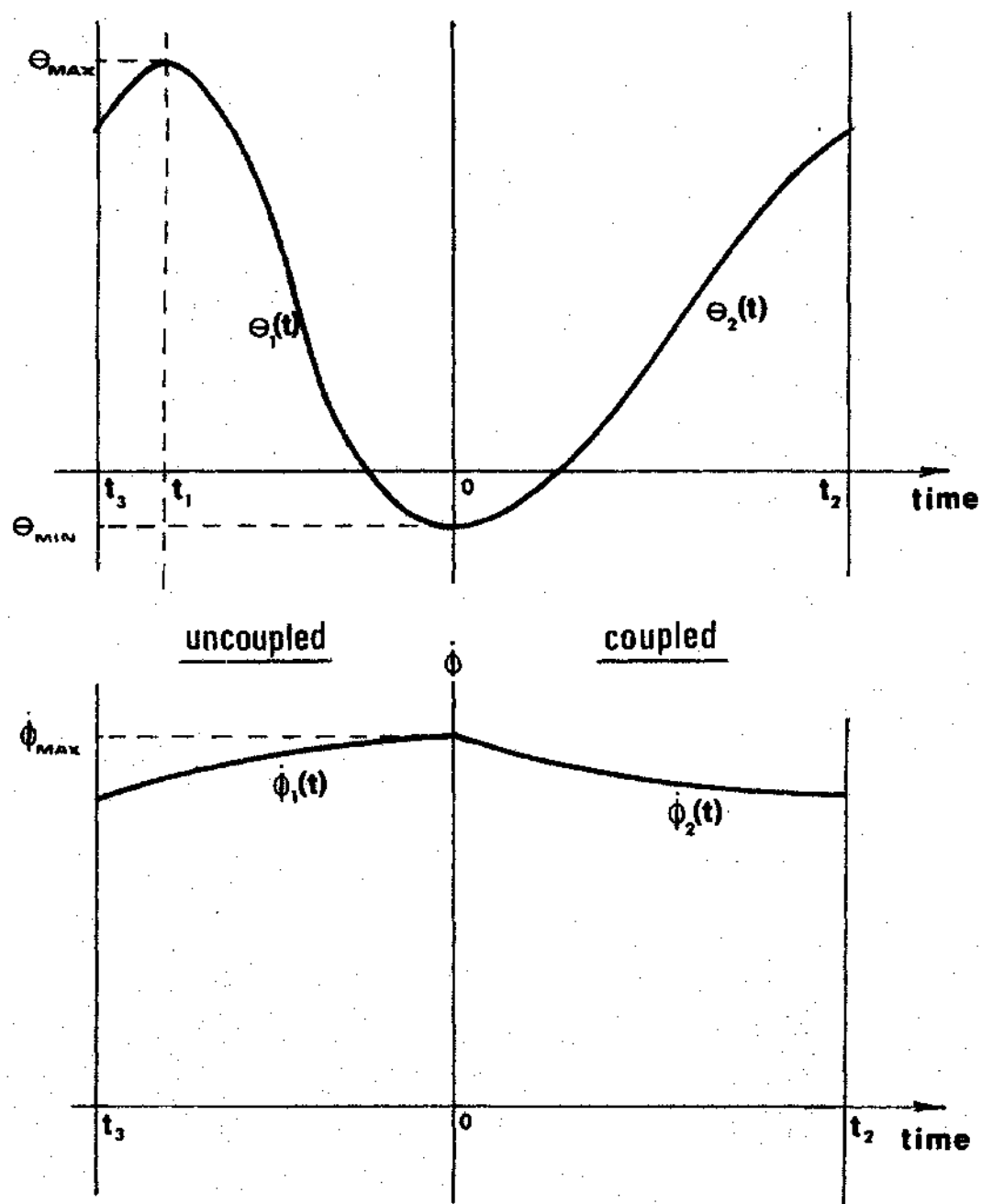


Figure 4. Two Stages of Motion

$$\theta(t) = \begin{cases} \theta_1(t) & \text{if } t_3 < t < 0 \\ \theta_2(t) & \text{if } 0 < t < t_2 \end{cases}$$

$$\dot{\phi}(t) = \begin{cases} \dot{\phi}_1(t) & \text{if } t_3 < t < 0 \\ \dot{\phi}_2(t) & \text{if } 0 < t < t_2 \end{cases}$$

Notice that,

$$\theta_{\min} = \theta_1(0) = \theta_2(0)$$

$$\theta_{\max} = \theta_1(t_1)$$

$$\dot{\phi}_{\max} = \dot{\phi}_1(0) = \dot{\phi}_2(0)$$

The clutch is engaged at $t = 0$ not only to simplify the analysis but to represent a desirable design feature since the change in direction of basket motion that occurs at time zero is easily detected. It is also assumed that the criteria for clutch release is displacement. That is, the clutch releases at $\theta = \theta(t_2)$.

See Figure 5 for a schematic of the system. Notice that for description the mechanism is divided between the clutch members into a driving side and a driven side. A reduction gear assembly is included for reasons to be discussed

later. To keep the flywheel inertia requirements small and in turn keep the flywheel diameter small, assume that the gear assembly is between the clutch and the flywheel. Then the angular velocity of the flywheel and motor shaft is $N\dot{\phi}$ where N is the reduction gear ratio.

The basket motion specified by one washing machine manufacturer [3] is periodic, rotational with a peak to peak displacement of 270° and a period of 1.50 seconds or an arc of 196° with a period of 1.07 seconds or an interpolation between these limits. Horn [1] chose an arc of 270° and a period of 1.40 seconds, a sufficient set of constraints considering the previous criteria.

Uncoupled Motion

Unlike the dry-friction clutch, the transmitted torque for a viscous clutch is a function of slip speed. Therefore, the equations of motion of the drive side and of the driven side are coupled when the clutch is engaged, making the analysis more complex. The agi-basket model used in previous analysis [1] consists of a mass anchored to its support by a spring and a damper, Figure 5. This model will be used in order that comparisons can be made. In Chapter V discrepancies between this model and the real system will be discussed and an alternative model introduced.

The differential equation for the uncoupled basket motion is

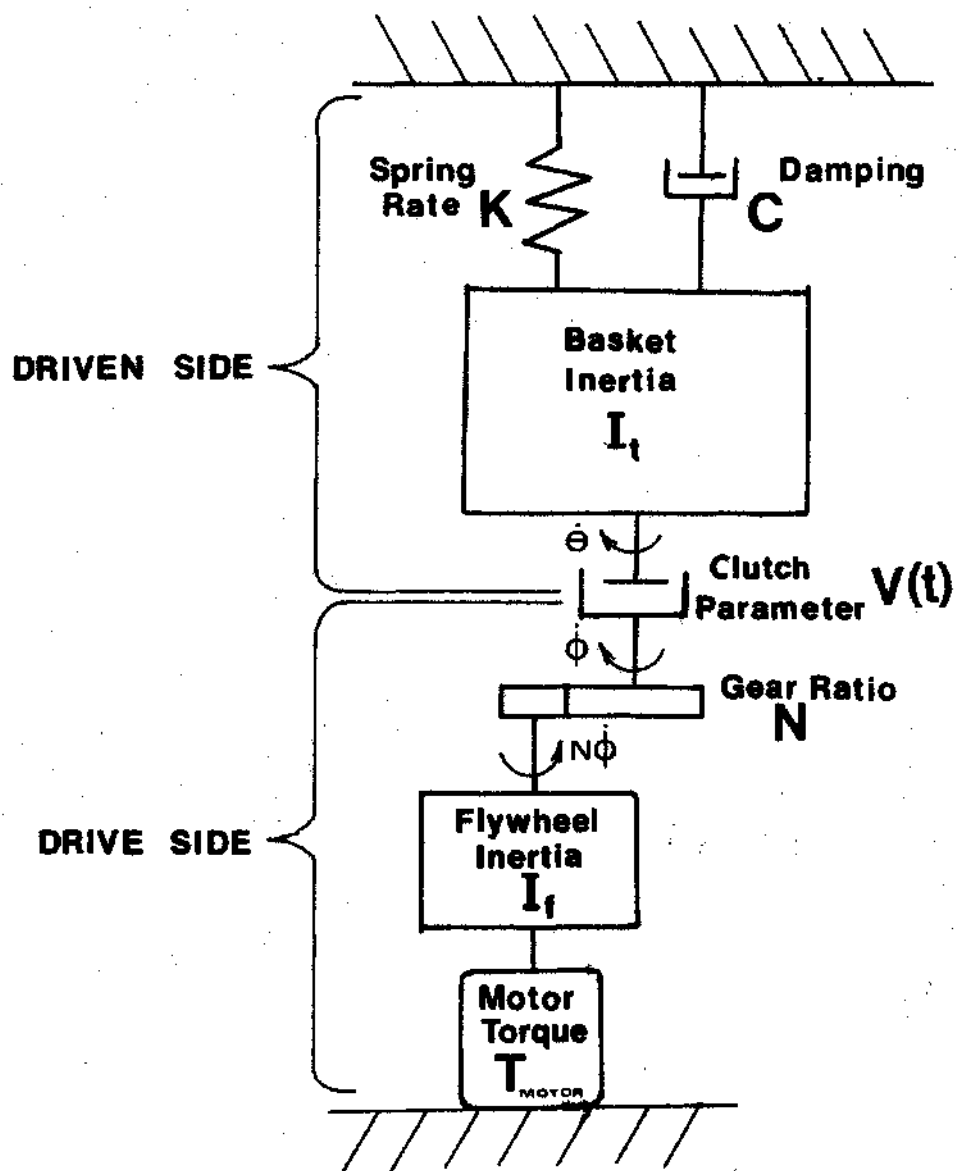


Figure 5. System Schematic

$$I_T \ddot{\theta}_1 + C \dot{\theta}_1 + K \theta_1 = 0 \quad (3)$$

assuming negligible friction in the disengaged clutch and supporting bearings (an allowable assumption due to the large damping component). Where I_T is the moment of inertia of the agi-basket and the driven side of the clutch, c is the damping coefficient corresponding to the energy dissipated in the washing action, and k is the angular spring constant for the particular spring being used, and a dot indicates differentiation with respect to time. Solving for θ_1 ,

$$\theta_1(t) = e^{-q_1 t} [c_7 \cos \omega_1 t + c_8 \sin \omega_1 t] \quad (4)$$

where

$$q_1 = \frac{c}{2I_T}, \quad \omega_1 = \frac{1}{2I_T} \sqrt{4I_T K - c^2}$$

and c_7 and c_8 are constants of integration. Also,

$$\begin{aligned} \dot{\theta}_1(t) = e^{-q_1 t} [-q_1 c_7 \cos \omega_1 t - \omega_1 c_7 \sin \omega_1 t \\ - q_1 c_8 \sin \omega_1 t + \omega_1 c_8 \cos \omega_1 t] \end{aligned}$$

By definition, at $t = 0$,

$$\dot{\theta}_1(t) = 0$$

$$\omega_1 c_8 - q_1 c_7 = 0 \quad (5)$$

Since $\dot{\theta}_1 = 0$ at $\sin \omega_1 t = 0$ or in other words at

$$\omega_1 t = \dots, -2\pi, -\pi, 0, \pi, 2\pi, \dots$$

and $t = t_1$ at the zero velocity ($\dot{\theta}_1 = 0$) point previous to $t = 0$ (see Figure 4),

$$\omega_1 t_1 = -\pi$$

$$t_1 = \frac{-\pi}{\omega_1}$$

It is desired that the total arc be $\frac{3\pi}{2}$ radius, therefore,

$$\theta_1\left(\frac{-\pi}{\omega_1}\right) - \theta_1(0) = \frac{3\pi}{2}$$

From equation (4),

$$-c_7 e^{-q_1 \left(\frac{-\pi}{\omega_1}\right)} - c_7 = \frac{3\pi}{2} \quad (6)$$

$$c_7 = \frac{-3\pi}{2} / \left(1 + e^{\frac{q_1 \pi}{\omega_1}}\right)$$

Inserting C_7 into equation (5),

$$C_8 = \frac{q_1}{\omega_1} \left(\frac{-\frac{3\pi}{2}}{\frac{q_1 \pi}{\omega_1}} \right)$$

$$1 + e$$

Notice that

$$\theta_{\min} = C_7$$

$$\theta_{\max} = C_7 + \frac{3\pi}{2}$$

In the previous analysis [1], a 0.5 hp split-base electric motor was used as the source of mechanical power. The motor torque versus speed was approximated by a straight line (Figure 6) with the relationship

$$T_{\text{motor}} = \alpha(N\dot{\phi}) + TQ$$

where TQ is the T_{motor} intercept and α is the slope of the straight line approximation. Then the no load running speed is

$$\left(\frac{-TQ}{\alpha} \right)$$

Note that below the breakdown speed, BRKDWN, motor torque decreases with decreasing motor speed and the straight line

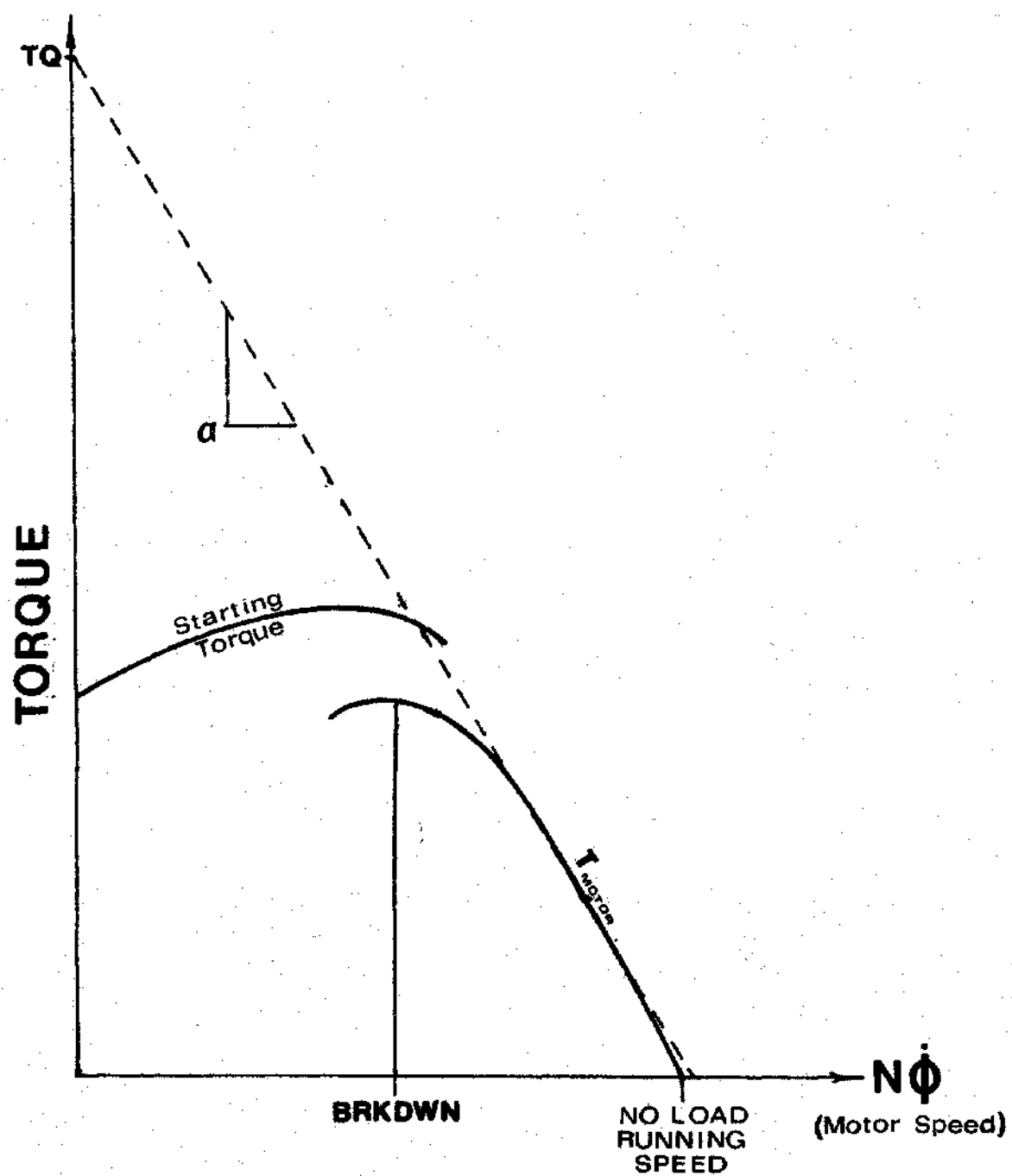


Figure 6. Motor Torque Curve

does not provide a satisfactory fit to the real curve. For the sake of meaningful comparisons, the same motor-torque approximations will be used here and checks will be made to insure that $N\dot{\phi}$ is never smaller than BRKDWN. Here,

$$TQ = 431 \text{ in-lbf}, \alpha = -2.2 \text{ in-lbf-sec/rad},$$

$$\text{BRKDWN} = 168 \text{ rad/sec}$$

The differential equation for the motion of the uncoupled flywheel is

$$I_F(N\ddot{\phi}_1) = \alpha(N\dot{\phi}_1) + TQ$$

Solving for $\dot{\phi}_1(t)$ we have

$$N\dot{\phi}_1(t) = c_0 e^{\alpha/I_F t} - \frac{TQ}{\alpha} \quad (7)$$

Since $\dot{\phi}$ is maximum at $t = 0$,

$$C_0 = \frac{TQ}{\alpha} + N\dot{\phi}_{\max}$$

Coupled Motion

The torque transmitted through the clutch is

$$T = V(\dot{\phi}_2 - \dot{\phi}_1)$$

where V is as defined in equation (1). The differential

equation for the coupled agi-basket and flywheel are respectively,

$$I_T \ddot{\theta}_2 + c \dot{\theta}_2 + K \theta_2 = V(\dot{\phi}_2 - \dot{\theta}_2) \quad (8)$$

$$I_F(N\ddot{\phi}_2) = T_{\text{motor}} \frac{T}{N} \quad (9)$$

where $\frac{T}{N}$ is the torque at the clutch reflected across the gear reduction to the flywheel. Making the proper substitutions, equation (9) becomes

$$I_F(N\ddot{\phi}_2) = \alpha N \dot{\phi}_2 + TQ \frac{V}{N}(\dot{\phi}_2 - \dot{\theta}_2) \quad (9a)$$

Solving equation (8) for $\dot{\phi}_2$ and differentiating with respect to time,

$$\dot{\phi}_2 = \frac{I_T \ddot{\theta}_2}{V} + \frac{C+V}{V} \dot{\theta}_2 + \frac{K}{V} \theta_2 \quad (10)$$

$$\ddot{\phi}_2 = \frac{I_T \ddot{\theta}_2}{V} + \frac{C+V}{V} \ddot{\theta}_2 + \frac{K}{V} \dot{\theta}_2 \quad (11)$$

Substituting equations (10) and (11) into equation (9a) and simplifying,

$$\frac{\ddot{\theta}_2 + I_F(C+V) - I_T \left(\frac{V}{N^2} - \alpha \right) \ddot{\theta}_2 + I_F K + \frac{V}{N^2} C - \alpha(C+V) \dot{\theta}_2}{I_F I_T}$$

$$+ \frac{K(\frac{V}{N^2} - \alpha)}{I_F I_T} \theta_2 = \frac{TQV}{I_F I_T N}$$

This is a third order, linear differential equation in one variable. To solve, let

$$A_1 = \frac{I_F(C+V) - I_T(\frac{V}{N^2} - \alpha)}{I_T I_F}$$

$$A_2 = \frac{I_F K + \frac{V}{N^2} C - \alpha(C+V)}{I_T I_F}$$

$$A_3 = \frac{K(\frac{V}{N^2} - \alpha)}{I_T I_F}$$

Then

$$\ddot{\theta}_2 + A_1 \ddot{\theta}_2 + A_2 \dot{\theta}_2 + A_3 \theta_2 = \frac{TQV}{I_F I_T N}$$

for which the characteristic equation is

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0$$

To solve this cubic equation let,

$$r = \frac{1}{3}(3A_2 - A_1^2)$$

$$s = \frac{1}{27}(2A_1^3 - 9A_1 A_2 + 27A_3)$$

and,

$$A = \sqrt[3]{\frac{-s}{2} + 2\sqrt{\frac{s^2}{4} + \frac{r^3}{27}}}$$

$$B = \sqrt[3]{\frac{-s}{2} - 2\sqrt{\frac{s^2}{4} + \frac{r^3}{27}}}$$

Then the solutions for λ are:

$$\lambda_1 = A+B - \frac{A_1}{3}$$

$$\lambda_2 = \frac{-A+B}{2} - \frac{A_1}{3} + \frac{A-B}{2}\sqrt{-3}$$

$$\lambda_3 = \frac{-A+B}{2} - \frac{A_1}{3} - \frac{A-B}{2}\sqrt{-3}$$

It is obvious that there are two general cases for the solution of θ_2 .

Underdamped

If $\frac{s^2}{4} + \frac{r^3}{27} > 0$, then A and B are real numbers and there will be two imaginary and one real root of λ . The corresponding solution for θ_2^* where superscript star indicates underdamped case is,

$$\theta_2^*(t) = C_1^* e^{mt} + e^{-q_2 t} (C_2^* \cos \omega_2 t + C_3^* \sin \omega_2 t) + F \quad (12)$$

where

$$m = A+B - \frac{A_1}{3}, \quad q_2 = \frac{A+B}{2} + \frac{A_1}{3}, \quad \omega_2 = \frac{\sqrt{3}(A-B)}{2}$$

$$F = \frac{TQV}{I_F I_T N A_3}$$

and c^* , c_2^* , c_3^* are constants of integration.

Overdamped

If $\frac{s^2}{4} + r^3 > 0$ then A and B are complex conjugates and λ has three real solutions. In this case θ_2' , where prime indicates overdamped solution, is

$$\theta_2'(t) = c_1' e^{b_1 t} + c_2' e^{b_2 t} + c_3' e^{b_3 t} + F \quad (13)$$

where

$$b_1 = A + B - \frac{A_1}{3}$$

$$b_2 = -\frac{A+B}{2} - \frac{A_1}{3} + \frac{\sqrt{-3}}{2}(A-B)$$

$$b_3 = -\frac{A+B}{2} - \frac{A_1}{3} - \frac{\sqrt{-3}}{2}(A-B)$$

Initial Conditions

1. $\dot{\theta}_2 = 0$ @ $t = 0$ from the definition of t
2. $\theta_2 = \theta_{\min}$ @ $t = 0$ from the definition of θ_{\min}
3. $\ddot{\theta}_2 = \frac{V}{JT} \dot{\phi}_{\max}$ @ $t = 0$ from equation (8), condition 1. and
2. and the definition of $\dot{\phi}_{\max}$

Solving the three linear simultaneous equations

resulting from the initial conditions, the constants of integration are,

$$c_1^* = \frac{(F - \theta_{\min})(q_2^2 I_T + \omega_2^2 I_T) - V \dot{\phi}_{\max} + K \theta_{\min}}{I_T(m^2 - q_2^2 - \omega_2^2 - 2q_2 m)}$$

$$c_2^* = \theta_{\min} - F - c_1^*$$

$$c_3^* = \frac{-\frac{V}{I_T} \dot{\phi}_{\max} + \frac{K}{I_T} \theta_{\min} + c_1^* m^2 + c_2^* (q_2^2 - \omega_2^2)}{2q_2 \omega_2}$$

$$c_1^i = \frac{(\theta_{\min} - F)(b_2 b_3^2 - b_3 b_2^2) + (b_3 - b_2) \left(\frac{V}{I_T} \dot{\phi}_{\max} - \frac{K}{I_T} (\theta_{\min} - F) \right)}{b_1(b_2^2 - b_3^2) + b_2(b_3^2 - b_1^2) + b_3(b_1^2 - b_2^2)}$$

$$c_2^i = \frac{(\theta_{\min} - F)(b_3 b_1^2 - b_1 b_3^2) + (b_1 - b_3) \left(\frac{V}{I_T} \dot{\phi}_{\max} - \frac{K}{I_T} (\theta_{\min} - F) \right)}{b_1(b_2^2 - b_3^2) + b_2(b_3^2 - b_1^2) + b_3(b_1^2 - b_2^2)}$$

$$c_3^i = \theta_{\min} - c_2^i - c_3^*$$

Underdamped $\dot{\phi}_2$

Differentiating equation (12) twice with respect to time and substituting into equation (10) we have $\dot{\phi}_2^*$ for the underdamped case,

$$\dot{\phi}_2^*(t) = c_4^* e^{mt} + e^{-q_2 t} (c_5^* \cos \omega_2 t + c_6^* \sin \omega_2 t) + \frac{FK}{V} \quad (14)$$

where the constants of integration are

$$c_4^* = c_1 \left(\frac{K}{V} + \frac{C+V}{V} m + \frac{I_T m^2}{V} \right)$$

$$c_5^* = \frac{1}{V} (Kc_2^* - (c+V)q_2 c_2^* + I_T q_2^2 c_2^* - I_T \omega_2^2 c_2^* - 2I_T q_2 \omega_2 c_3^* + (c+V)\omega_2 c_3^*)$$

$$c_6^* = \frac{1}{V} (Kc_3^* - (c+V)q_2 c_3^* + I_T q_2^2 c_3^* - I_T \omega_2^2 c_3^* - 2I_T q_2 \omega_2 c_2^* + (c+V)\omega_2 c_2^*)$$

Overdamped $\dot{\phi}_2$

Differentiating equation (13) twice with respect to time and substituting into equation (10) we have $\dot{\phi}_2^i$ for the overdamped case,

$$\dot{\phi}_2^i(t) = c_4^i e^{b_1 t} + c_5^i e^{b_2 t} + c_6^i e^{b_3 t} + \frac{FK}{V} \quad (15)$$

where

$$c_4^i = \frac{1}{V} (I_T c_1^i b_1^2 + (c+V)c_1^i b_1 + Kc_1^i)$$

$$c_5^i = \frac{1}{V} (I_T c_2^i b_2^2 + (c+V)c_2^i b_2 + Kc_2^i)$$

$$c_6^i = \frac{1}{V} (I_T c_3^i b_3^2 + (c+V)c_3^i b_3 + Kc_3^i)$$

CHAPTER III

NUMERICAL SOLUTIONS

At this point solutions are available for θ_1 , θ_2 , $\dot{\phi}_1$, and $\dot{\phi}_2$ provided we know the clutch parameter V , the maximum flywheel speed $N\dot{\phi}_{\max}$, the reduction ratio N , and the spring rate K . To construct a picture of the entire motion, the time of clutch release, t_2 for θ_2 and t_3 for θ_1 must be known so that the transition from coupled to uncoupled motion can be made. K and N are given assumed values at the beginning of the numerical analysis. These variables (K and N) are design parameters and will be used later in an effort to minimize power requirements. The damping coefficient c and the basket inertia I_T as used by Horn [1] and specified by a manufacturer [3] are

$$\begin{aligned} c &= 28.8 \text{ in-lbf-sec} \\ I_T &= 4.44 \text{ in-lbf-sec}^2 \end{aligned}$$

Values of V , $\dot{\phi}_{\max}$, t_2 and t_3 will be found using the iterative methods of bisection and Newton-Raphson, and appropriate constraints. Three nested loops are employed, Figure 7.

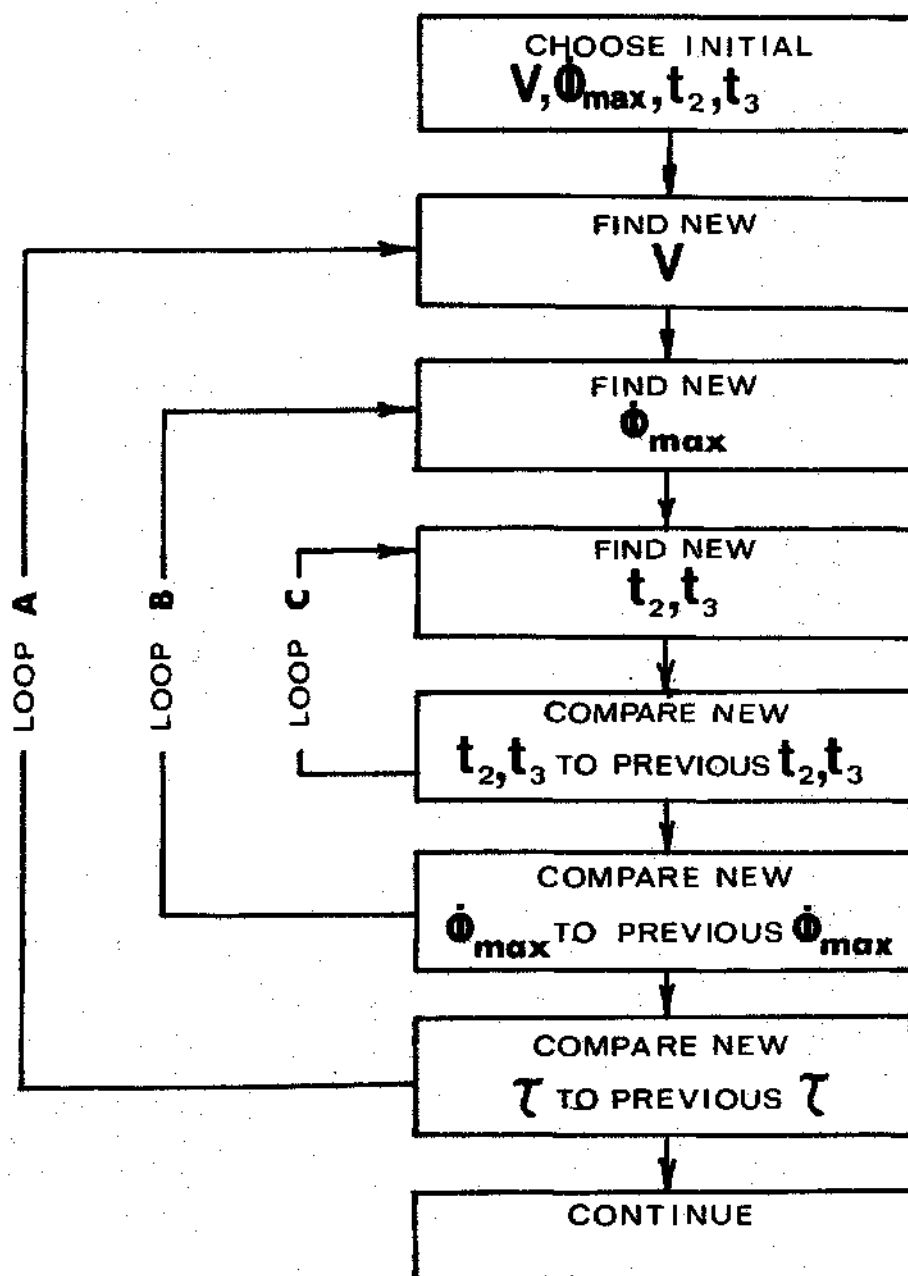


Figure 7. Nested Loops

Loop A

An increase in V , equation (1), whether by using a more viscous fluid, decreasing clearances or increasing the area of the working surfaces, would be expected to produce a higher acceleration of the coupled basket, Figure 8. The time of clutch engagement, effectively t_2 , would then decrease. The period is

$$\tau = t_2 - t_3 \quad (16)$$

Recall from Page 14,

$$\tau = 1.40 \text{ seconds}$$

The bisection method [4] is used to arrive at a value of V which satisfies the criterion of period. In these iterative steps, a variable subscripted with i indicates the value that the variable takes on after the i 'th iteration and zero indicates the initial value.

Choose V_{s_o} and V_{L_o} , the initial upper and lower limits on V_i , respectively, so that

$$V_{s_o} \leq V \leq V_{L_o}$$

Note that V_{s_o} must be reasonably "close" to V or θ_2 will never reach a sufficiently large displacement or velocity to

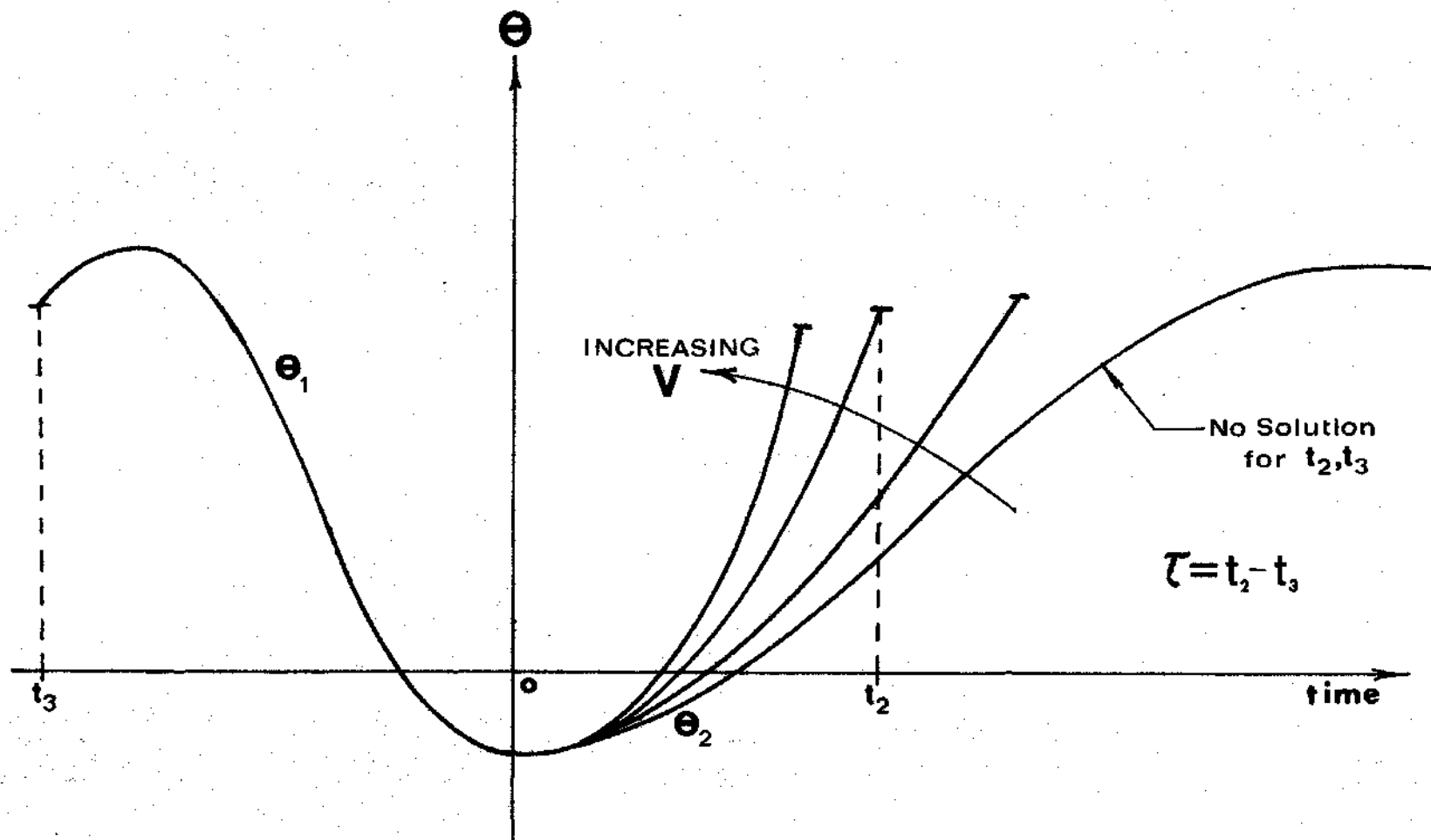


Figure 8. Effect of V on τ (Period)

complete the cycle with the specified arc ($\frac{3\pi}{2}$ radius). In this case loop C would not converge.

Begin the iteration by setting

$$V_i = \frac{V_{s_i} + V_{L_i}}{2}$$

Evaluate $\tau_i = t_{2_i} - t_{3_i}$ (t_{2_i} and t_{3_i} from loop C). If

$$|\tau_i - \tau| < \Delta\tau$$

where $\Delta\tau$ is error in period, take

$$V \approx V_i$$

and stop. $\Delta\tau$ is taken as 0.005 seconds in this program. Smaller $\Delta\tau$ will give greater accuracy at the expense of more computer time. Otherwise, if

$$\tau_i < \tau$$

then V_i is too large. Since it is known that

$$V < V_i$$

set

$$V_{L_{i+1}} = V_i$$

$$V_{S_{i+1}} = V_{S_i}$$

and V will be contained in the interval $[V_{S_{i+1}}, V_{L_{i+1}}]$. If

$$\tau_i > \tau$$

then V_i is too small. Set

$$V_{L_{i+1}} = V_{L_i}$$

$$V_{S_{i+1}} = V_i$$

In either case, we have reduced the interval in which we know V to exist by one-half.

Loop B

In order to calculate τ from equation (16) it is necessary to have t_2 and t_3 which cannot be found without a value for $\dot{\phi}_{\max}$. ($\dot{\phi}_{\max}$ is necessary to calculate the constants of integration for coupled solutions.) In this way all three nested loops are interdependent. An obvious set of constraints is that the solutions of $\dot{\phi}_1$ and $\dot{\phi}_2$ be equal at both the time the clutch engages and the time the clutch disengages, Figure 9. In other words, the velocity of the

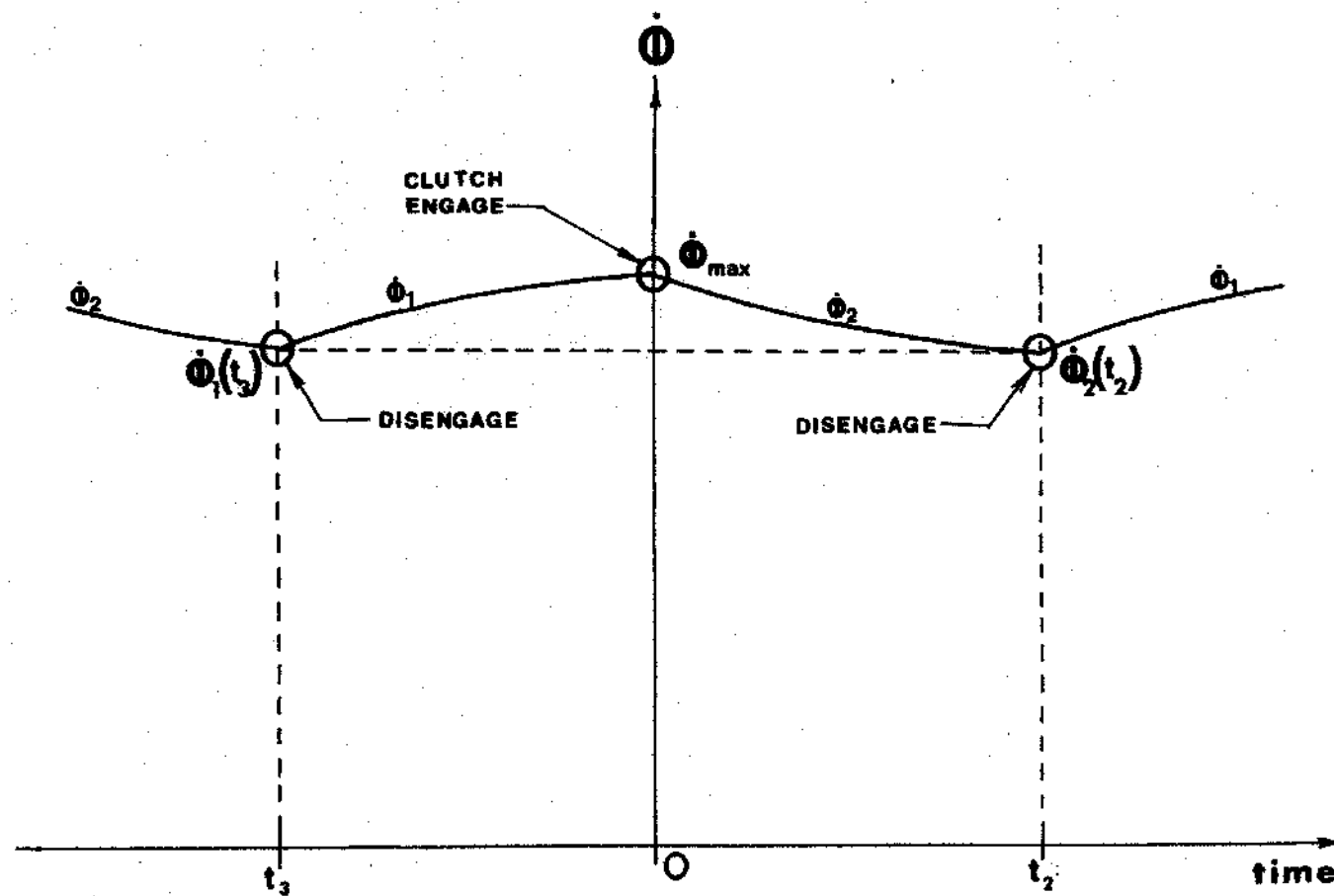


Figure 9. Velocity of Driving Clutch Member Versus Time

clutch driving member $\dot{\phi}$ should not have discontinuities where the solution changes from the coupled case $\dot{\phi}_2$, to the uncoupled case $\dot{\phi}_1$ or from $\dot{\phi}_1$ to $\dot{\phi}_2$.

$$\dot{\phi}_1(t_3) = \dot{\phi}_2(t_2) \quad (17)$$

$$\dot{\phi}_1(0) = \dot{\phi}_2(0) \quad (18)$$

By using the same value of $\dot{\phi}_{\max}$ in calculating the constants of integration for $\dot{\phi}_1(t)$ and $\dot{\phi}_2(t)$ we satisfy the constraint given by equation (18) since $\dot{\phi}_{\max}$ is the clutch driving member speed $\dot{\phi}$ at $t = 0$.

It would seem that a high value of $\dot{\phi}_{\max}$ would cause an increased drop in $\dot{\phi}_2$ from $t = 0$ to t_2 , Figure 10, since the torque transmitted through the clutch would be larger. Recall equation (2). It would also seem that a high value of $\dot{\phi}_{\max}$ would produce a smaller gain in $\dot{\phi}_1$ from t_3 to $t = 0$ since motor torque decreases as motor speed approaches the no-load running speed giving a lesser acceleration of the flywheel (Figure 6). In other words, if a value for $\dot{\phi}_{\max}$ is chosen too high the calculated $\dot{\phi}_1(t_3)$ will be larger than the calculated $\dot{\phi}_2(t_2)$.

Using the bisection method again, choose $\dot{\phi}_{\max_{s_0}}$ and $\dot{\phi}_{\max_{L_0}}$ the upper and lower limits of $\dot{\phi}_{\max_i}$ such that

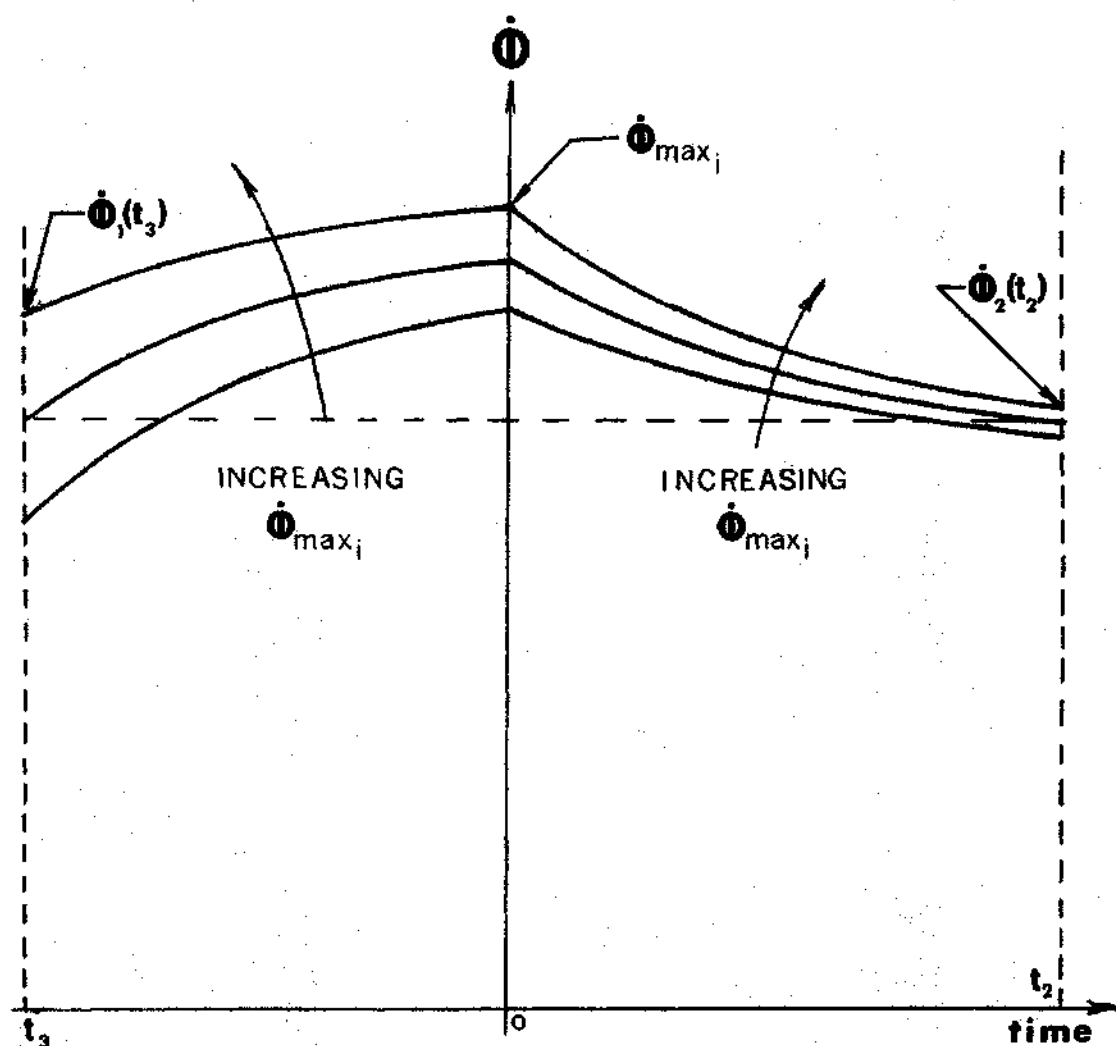


Figure 10. Effect of $\dot{\phi}_{\max_i}$ on $\dot{\phi}_1$ and $\dot{\phi}_2$

$$\dot{\phi}_{\max_{s_o}} \leq \dot{\phi}_{\max} \leq \dot{\phi}_{\max_{L_o}}$$

For $\dot{\phi}_{\max_{L_o}}$ we may take the no load running speed reflected to the driving clutch member, $-\frac{T_a}{N\alpha}$, as this imposes an upper limit on $\dot{\phi}$. Begin by setting

$$\dot{\phi}_{\max_i} = \frac{\dot{\phi}_{\max_{s_i}} + \dot{\phi}_{\max_{L_i}}}{2}$$

Evaluate $\dot{\phi}_1(t_3)_i$ and $\dot{\phi}_2(t_2)_i$ using $\dot{\phi}_{\max_i}$ in equations (7) and (14) or (15).

If

$$|\dot{\phi}_1(t_3) - \dot{\phi}_2(t_2)| < \Delta\dot{\phi}$$

take

$$\dot{\phi}_{\max} \approx \dot{\phi}_{\max_i}$$

as a reasonably good approximation and stop. 0.5 rad/sec is used for $\Delta\dot{\phi}$ here but smaller values can be employed with correspondingly greater accuracy and increased computer time cost. Otherwise, if

$$\dot{\phi}_1(t_3) - \dot{\phi}_2(t_2) > 0$$

then $\dot{\phi}_{\max_i}$ is too large and $\dot{\phi}_{\max_i}$ can become the next upper limit. Set

$$\dot{\phi}_{\max_{L_{i+1}}} = \dot{\phi}_{\max_i}$$

$$\dot{\phi}_{\max_{S_{i+1}}} = \dot{\phi}_{\max_{S_i}}$$

If

$$\dot{\phi}_1(t_3) - \dot{\phi}_2(t_2) < 0$$

set

$$\dot{\phi}_{\max_{L_{i+1}}} = \dot{\phi}_{\max_{L_i}}$$

$$\dot{\phi}_{\max_{S_{i+1}}} = \dot{\phi}_{\max_i}$$

Loop C

In order for the disengaged basket to reach a maximum displacement, θ_{\max} and repeat the uncoupled motion phase, the clutch should be released at a time t_2 for θ_2 such that

$$\theta_2(t_2) = \theta_1(t_3) \quad (19)$$

$$\dot{\theta}_2(t_2) = \dot{\theta}_1(t_3) \quad (20)$$

for a t_3 which must also be determined for θ_1 . To show this is true consider bringing together curves of θ_2 and θ_1 at t_2 and t_3 , respectively, Figure 11. At this time in the agitation cycle the clutch releases and there is a transition in the solution from $\theta_2(t)$ to $\theta_1(t)$. If this change in solutions is to accurately represent events in the real system, two criteria must be met. The displacement of the basket must coincide for both solutions and the velocity of the basket must coincide for both solutions. In other words, the velocity and acceleration must be finite at the time of disengagement. (Since it is assumed that the clutch releases instantaneously the accelerations of θ_1 and θ_2 do not match at this point.) The problem is then one of finding t_2 on the curve $\theta_2(t)$ and t_3 on the curve $\theta_1(t)$ such that equations (19) and (20) hold.

The iterative process used to find t_2 and t_3 is Newton's method for systems of equations [4]. Letting,

$$t_2 = x, t_3 = y$$

and

$$\theta_2(t_2) - \theta_1(t_3) = f(x, y)$$

$$\dot{\theta}_2(t_2) - \dot{\theta}_1(t_3) = g(x, y)$$

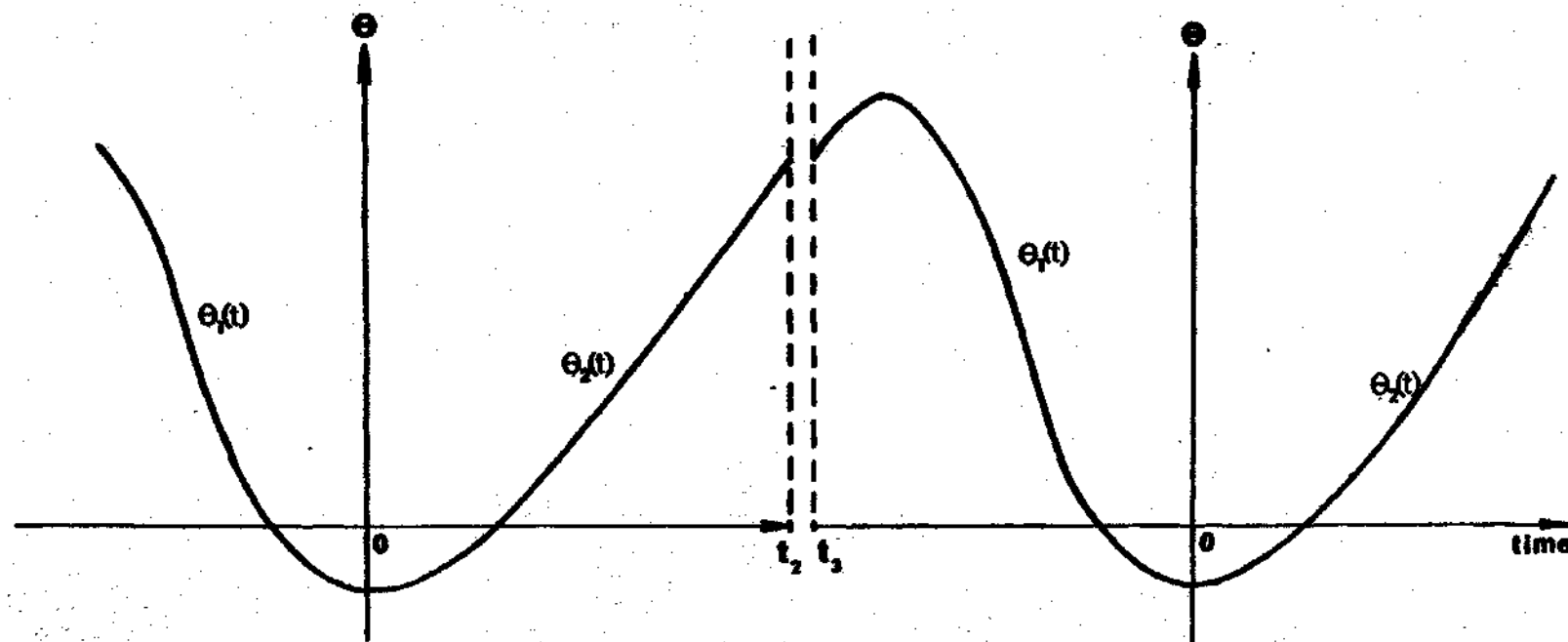


Figure 11. Clutch Release

After choosing an approximation x_0, y_0 to the roots of equations (19) and (20) use the recursion formulas,

$$X_{i+1} = X_i - \frac{f \frac{\partial g}{\partial y} - g \frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}} \quad x=x_i, y=y_i$$

$$Y_{i+1} = Y_i - \frac{g \frac{\partial f}{\partial x} - f \frac{\partial g}{\partial x}}{\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y}} \quad x=x_i, y=y_i$$

to generate successive values of x_i, y_i . Making the substitutions,

$$t_{2i+1} = t_{2i} + \frac{\left[\frac{\partial^2 \theta_1(t_3)}{\partial t_3^2} (\theta_2(t_2) - \theta_1(t_3)) - \frac{\partial \theta_1(t_3)}{\partial t_3} \left(\frac{\partial \theta_2(t_2)}{\partial t_2} - \frac{\partial \theta_1(t_3)}{\partial t_3} \right) \right]}{\frac{\partial \theta_1(t_3)}{\partial t_3} \frac{\partial^2 \theta_2(t_2)}{\partial t_2^2} - \frac{\partial \theta_2(t_2)}{\partial t_2} \frac{\partial^2 \theta_1(t_3)}{\partial t_3^2}} \quad \begin{matrix} t_2 = t_{2i} \\ t_3 = t_{3i} \end{matrix}$$

$$t_{3i+1} = t_{3i} + \frac{\left[\frac{\partial^2 \theta_2(t_2)}{\partial t_2^2} (\theta_2(t_2) - \theta_1(t_3)) - \frac{\partial \theta_2(t_2)}{\partial t_2} \left(\frac{\partial \theta_2(t_2)}{\partial t_2} - \frac{\partial \theta_1(t_3)}{\partial t_3} \right) \right]}{\frac{\partial \theta_1(t_3)}{\partial t_3} \frac{\partial^2 \theta_2(t_2)}{\partial t_2^2} - \frac{\partial \theta_2(t_2)}{\partial t_2} \frac{\partial^2 \theta_1(t_3)}{\partial t_3^2}} \quad \begin{matrix} t_2 = t_{2i} \\ t_3 = t_{3i} \end{matrix}$$

The rate of convergence of this process is a great deal faster than that of the previous bisection method, but care must be taken that the Jacobian $(\frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial g}{\partial x} \frac{\partial f}{\partial y})$ does not vanish or become so small as to vanish due to round off error. It should be mentioned also that x_0, y_0 must be very good approximations to the roots or this method will not converge at all. This requires changes to the initial approximations of t_2, t_3 every time a system parameter is varied significantly. The test,

$$|t_{2_{i+1}} - t_{2_i}| < \Delta t$$

and

$$|t_{3_{i+1}} - t_{3_i}| < \Delta t$$

terminates loop C if true. The value of Δt used here is 0.0001 seconds.

Power Determinations

Now that complete solutions are available for θ_2 and $\dot{\phi}_2$ it is possible to calculate the average power requirements at the motor P and the average power dissipated by the viscous clutch, P_c . See Figure 12. The remaining power, $P - P_c$, is dissipated by the damping on the basket, that is, $P - P_c$ is the power that goes into washing clothes. The work

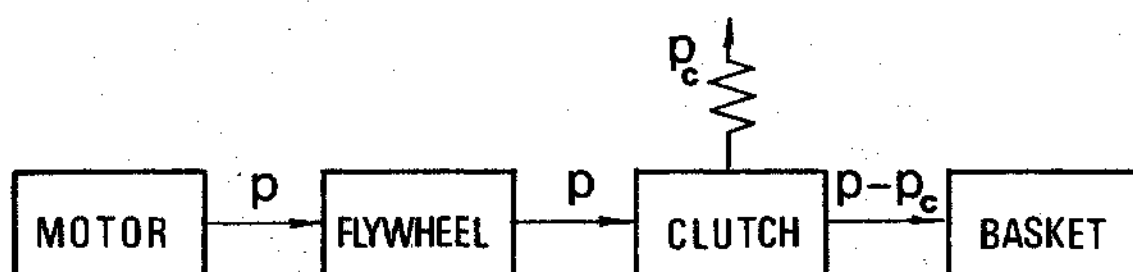


Figure 12. Power Flow

done per cycle by the motor-flywheel assembly is

$$W = \int_0^{t_2} \dot{\phi}_2 T \, dt \quad (21)$$

where torque T is as defined by equation (2)

$$T = V(\dot{\phi}_2 - \dot{\theta}_2) \quad (2)$$

Note that this W is the work done per cycle by the motor since any energy removed from the flywheel by the clutch is restored to the flywheel by the motor before the cycle is completed. In other words, the flywheel merely stores energy and lessens the maximum power draw from the motor. See Figure 13.

Substituting equation (2) into equation (21),

$$W = V \int_0^{t_2} (\dot{\phi}_2^2 - \dot{\phi}_2 \dot{\theta}_2) \, dt$$

The work dissipated by viscous damping in the clutch per cycle is

$$W_c = \int_0^{t_2} (\dot{\phi}_2 - \dot{\theta}_2) T \, dt$$

or

$$W_c = V \int_0^{t_2} (\dot{\phi}_2 - \dot{\theta}_2)^2 \, dt$$

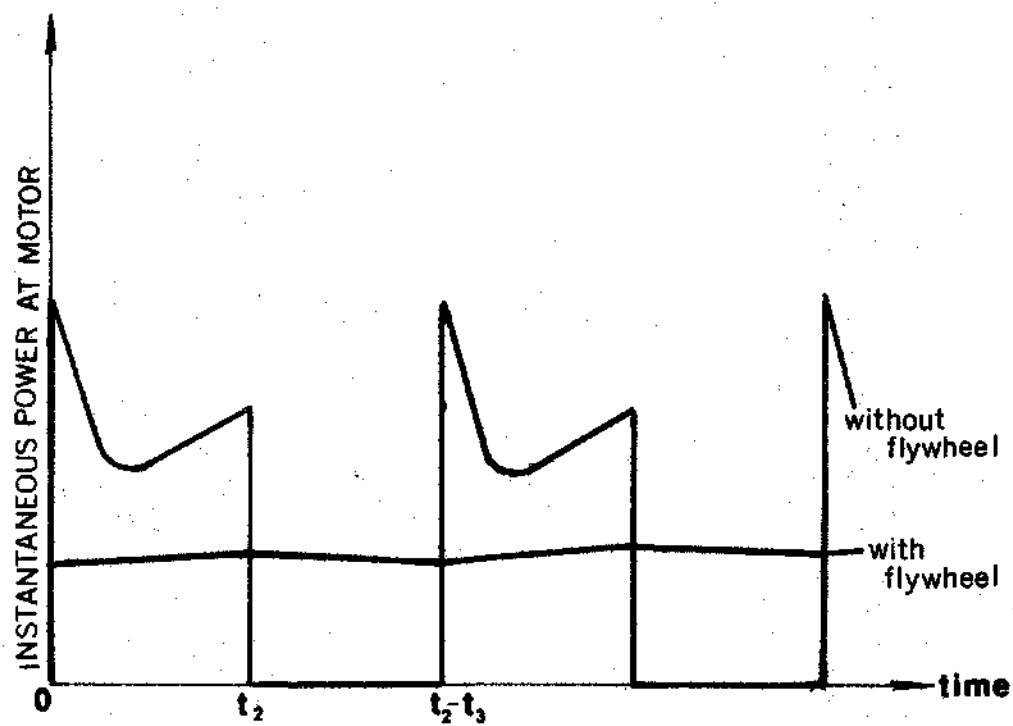


Figure 13. Effect of Flywheel on Power Requirements

Exact solutions for W and W_c would be very difficult to obtain but numerical methods of integration provide sufficiently accurate results. The trapezoid rule [4] is used here (Figure 14). Let

$$f(t) = V(\dot{\phi}_2^2(t) - \dot{\phi}_2(t)\dot{\theta}_2(t))$$

Then

$$W = \int_0^{t_2} f(t) dt$$

and summing the area of the trapezoids in Figure 14,

$$\begin{aligned} W &\cong \sum_{i=0}^{n-1} \frac{f(\frac{i}{n}t_2) - f(\frac{i+1}{n}t_2)}{2} \cdot \frac{t_2}{n} \\ &= \frac{t_2}{2n} [f(0) + f(t_2) + 2 \sum_{i=1}^{n-1} f(\frac{i}{n}t_2)] \end{aligned}$$

W_c is calculated in the same manner with

$$f_c(t) = V(\dot{\phi}_2(t) - \dot{\theta}_2(t))^2$$

Since the duration of a cycle τ is known, P and P_c can be calculated

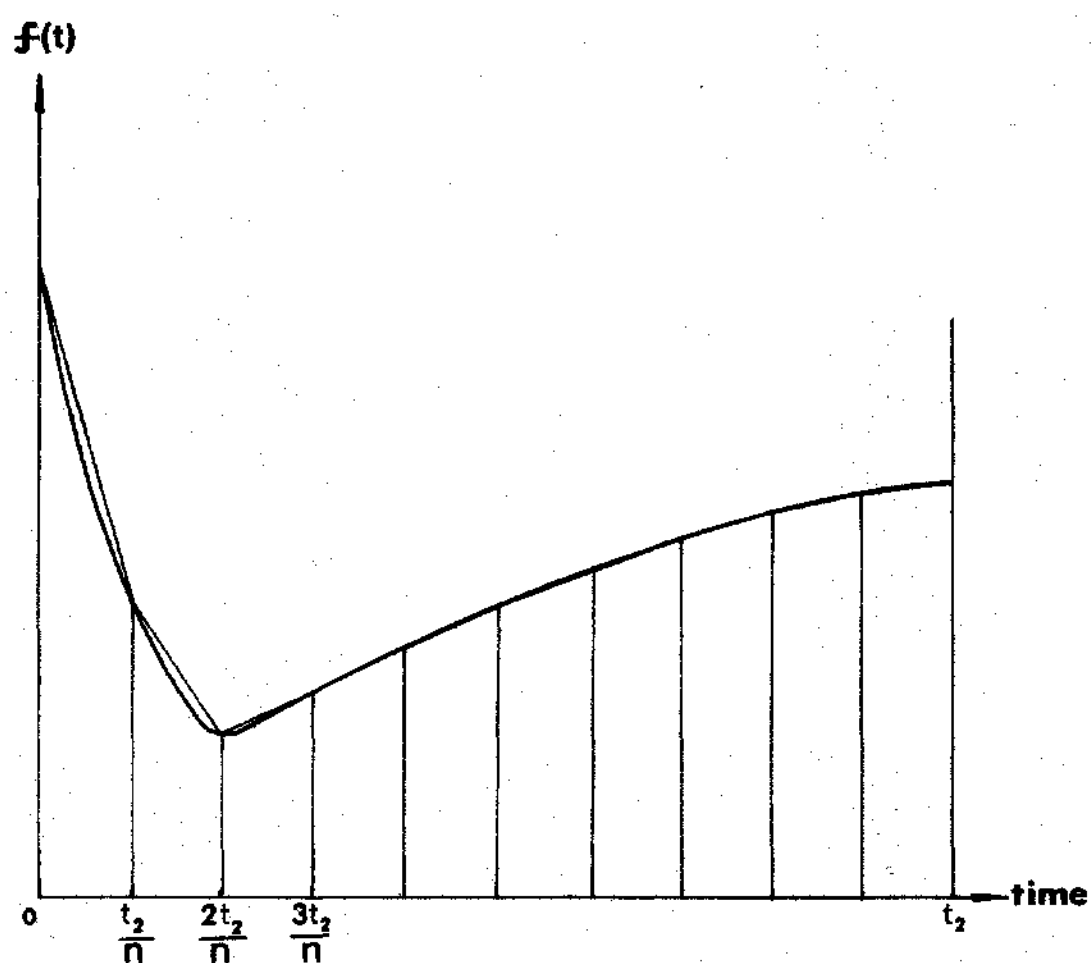


Figure 14. Numerical Integration

$$P = \frac{W}{T}$$

$$P_c = \frac{W_c}{T}$$

Keep in mind that P and P_c are average power values and instantaneous values may be quite different.

Torque Calculation

The torque transmitted through the clutch is

$$T(t) = V(\dot{\phi}_2(t) - \dot{\theta}_2(t)) \quad (2)$$

therefore the maximum torque T_{\max} can be found from equation (2) if t is found that maximizes $(\dot{\phi}_2(t) - \dot{\theta}_2(t))$. ϕ_{\max} occurs at $t = 0$ and $\dot{\theta}_2(0) = 0$ by definition. Since $\dot{\theta}_2$ is always positive, T_{\max} occurs at time zero, or

$$T_{\max} = V \cdot \phi_{\max}$$

This numerical work was accomplished with a computer program written in Fortran IV language and run on the Univac 1108 machine. A flow chart, Figure 15, and a copy of the main program, Appendix A, are included as tools for further investigation. A fourth loop, loop D, is shown in which the reduction ratio N is increased by an increment of 1.0 to some value N_{\max} . This has been included in the flow chart to

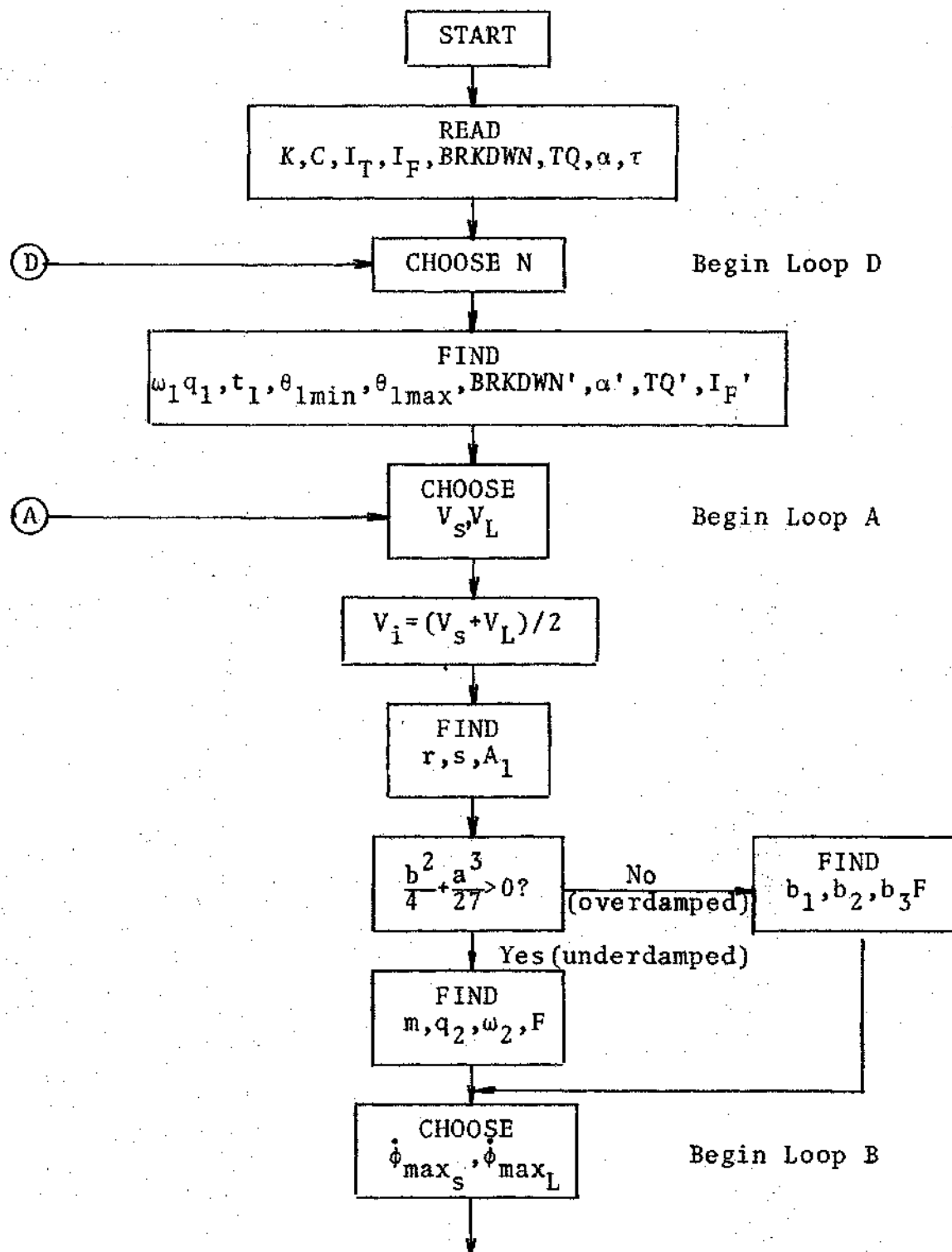


Figure 15. Flow Chart for Main Program

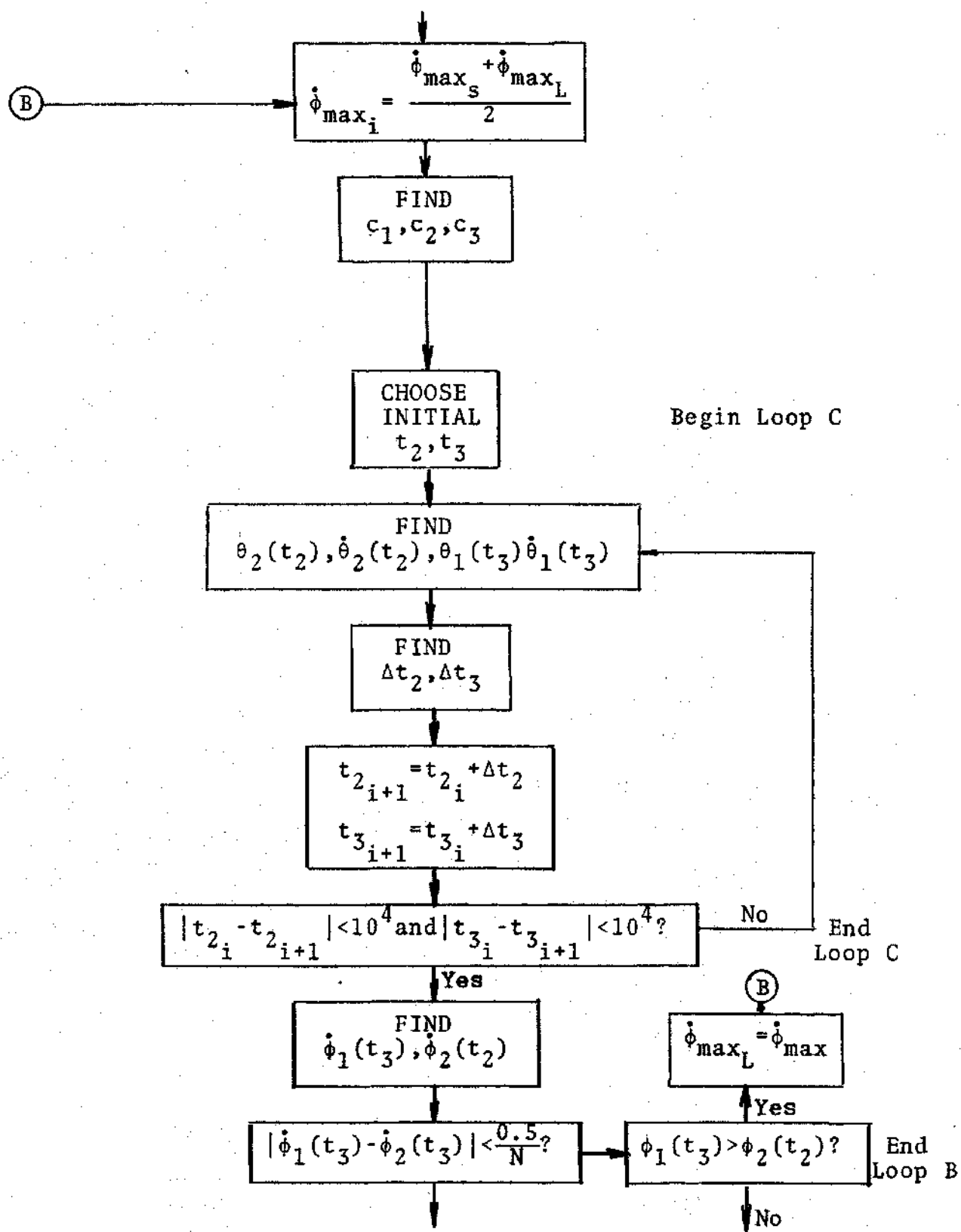


Figure 15 (continued)

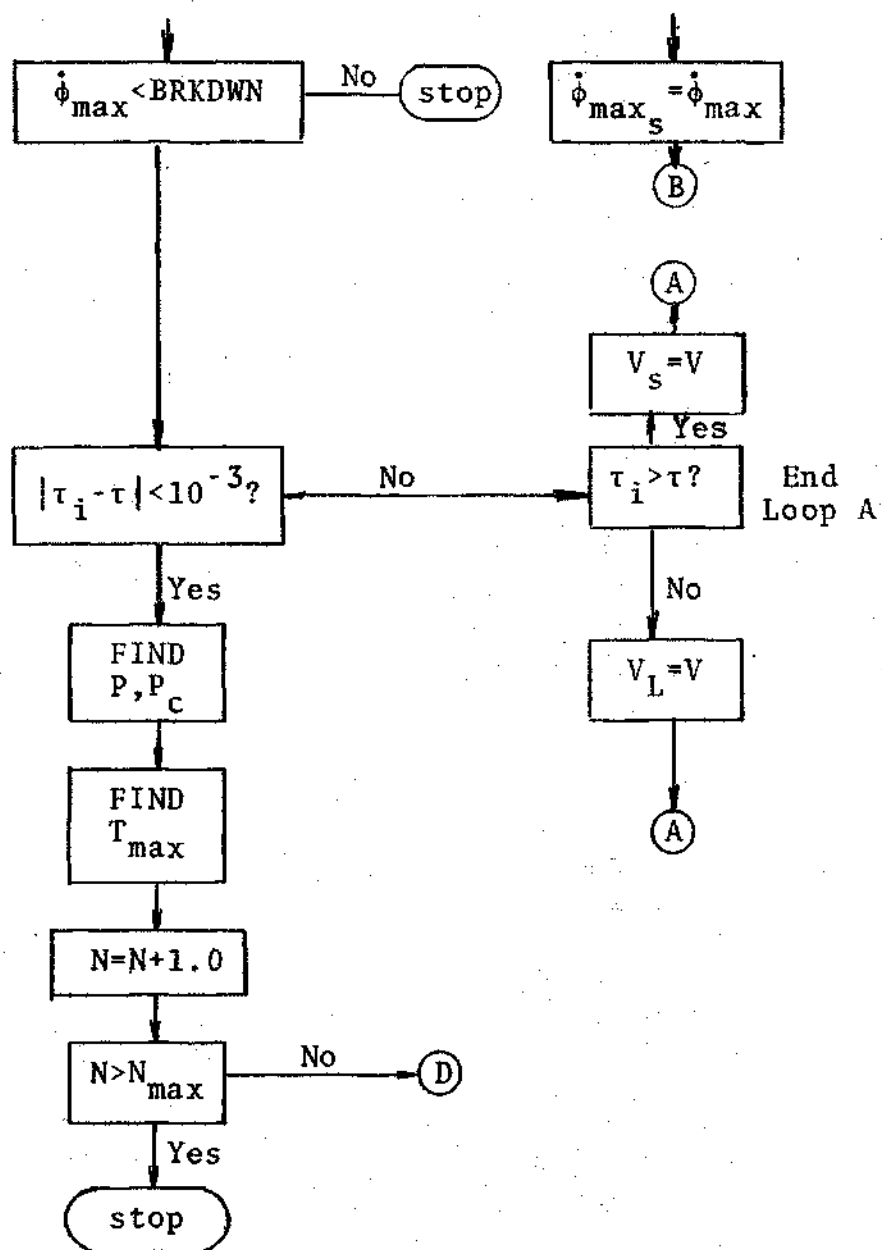


Figure 15 (concluded)

illustrate the steps taken to evaluate the effects of the parameters N and K on the system operation. A similar loop was used for spring constant K while holding N constant.

CHAPTER IV

RESULTS OF ANALYSIS

The numerical methods of the previous chapter yield various power requirements, maximum torque at the clutch, clutch constant, limits of basket motion and flywheel speed, time of clutch disengagement, and the appropriate constants for evaluating the equations of motion. A sample of the computer output is included at the end of Appendix A. Through the analysis, values of

$$I_F = 1.00 \text{ in-lbf-sec}^2$$

$$I_T = 4.44 \text{ in-lbf-sec}^2$$

$$C = 28.8 \text{ in-lbf-sec}$$

have been used for the inertias of the flywheel and basket and the damping coefficient. These are the same values used by Horn [1] so that comparisons are valid. A larger flywheel inertia would result in a smaller drop in flywheel speed when the clutch is engaged. The flywheel inertia used gave a drop in speed of about 5 percent of $N\dot{\phi}_{\max}$. In any future analysis it may be safe to assume a constant flywheel speed, especially if I_F were increased. Spring constant K and

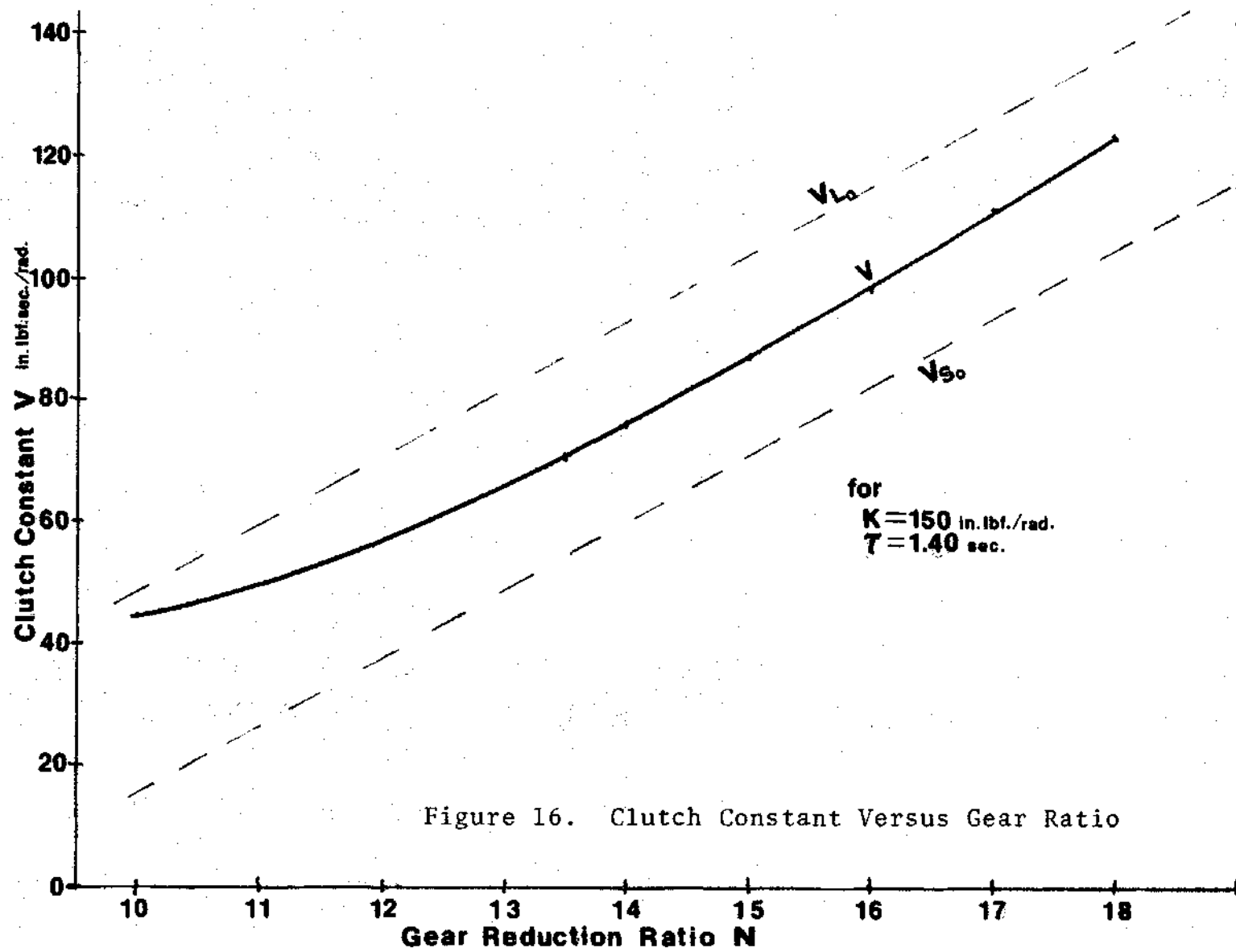
reduction ratio N are the independent variables.

Plots

In order to minimize the number of steps for each iteration (see Figure 7) initial approximations for V , $\dot{\phi}_{\max}$, t_2 , and t_3 were arrived at as experience with the program was gained. Since the clutch constant V appeared to be relatively linear with respect to reduction ratio N , Figure 16, initial values for V_L and V_S (Chapter III, loop A) were replaced by linear equations dependent on N for constant K .

Figure 17 is a plot of power at the motor shaft P and power dissipated in the clutch P_c versus N for $K = 150$ in-lbf/rad. As might be expected $P - P_c$, the power dissipated in the basket due to washing action, remains nearly constant, while P_c decreases with increasing N . Since the speed of the flywheel side of the clutch approaches the speed of the basket side as N increases while transmitted torque remains fairly constant, this is not surprising. There is, however, an upper limit on N . For a large N the difference in velocities of clutch members will be insufficient to drive the basket to a displacement and velocity large enough so that the specified arc and period are achieved. In this case there is no solution for the time of clutch disengagement, t_2 and t_3 .

A plot of P versus K for N constant at 16.0 rad/rad, Figure 18, reveals that a small K minimizes P . However, making K small lengthens the coasting or uncoupled time for



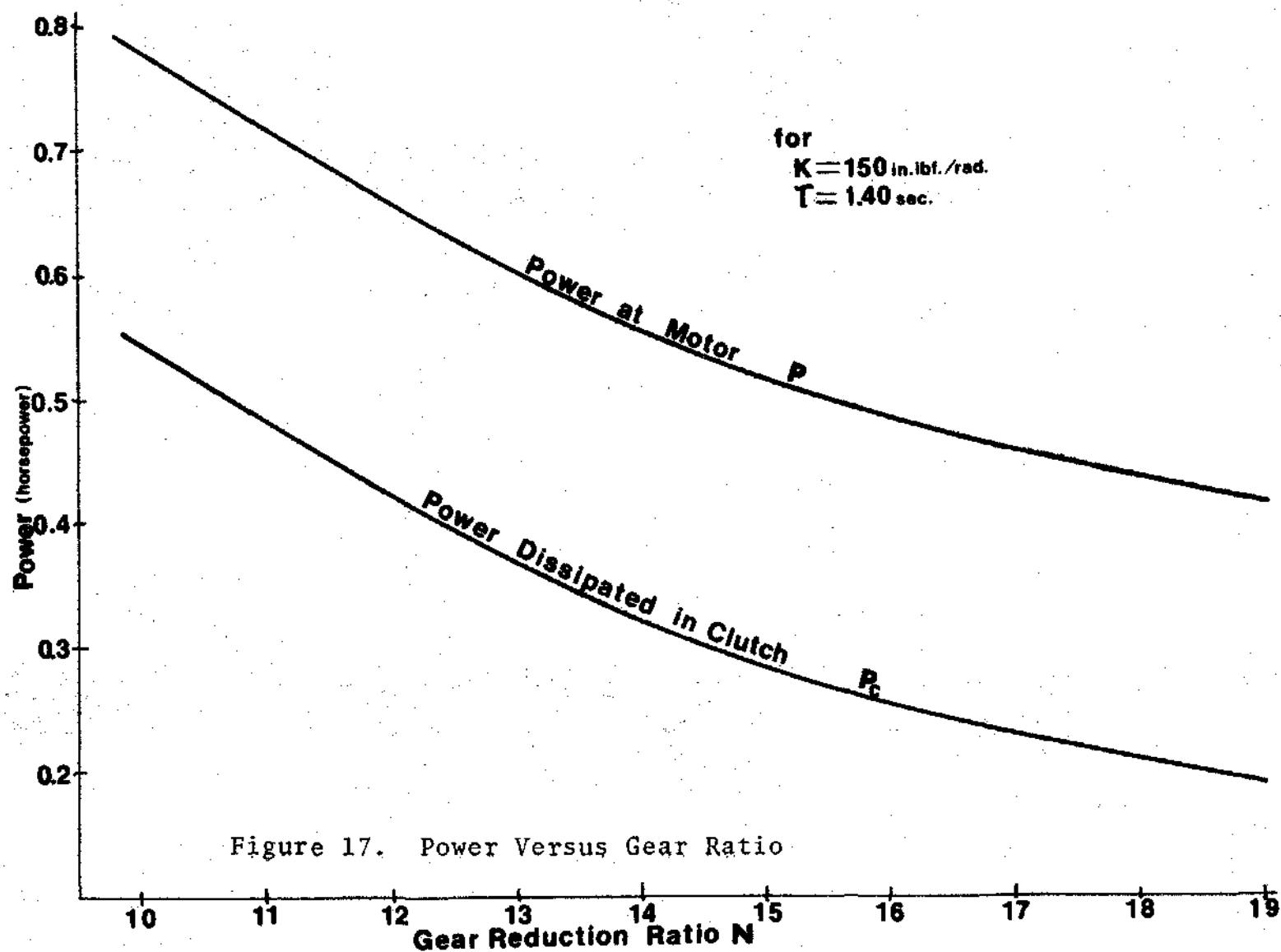


Figure 17. Power Versus Gear Ratio

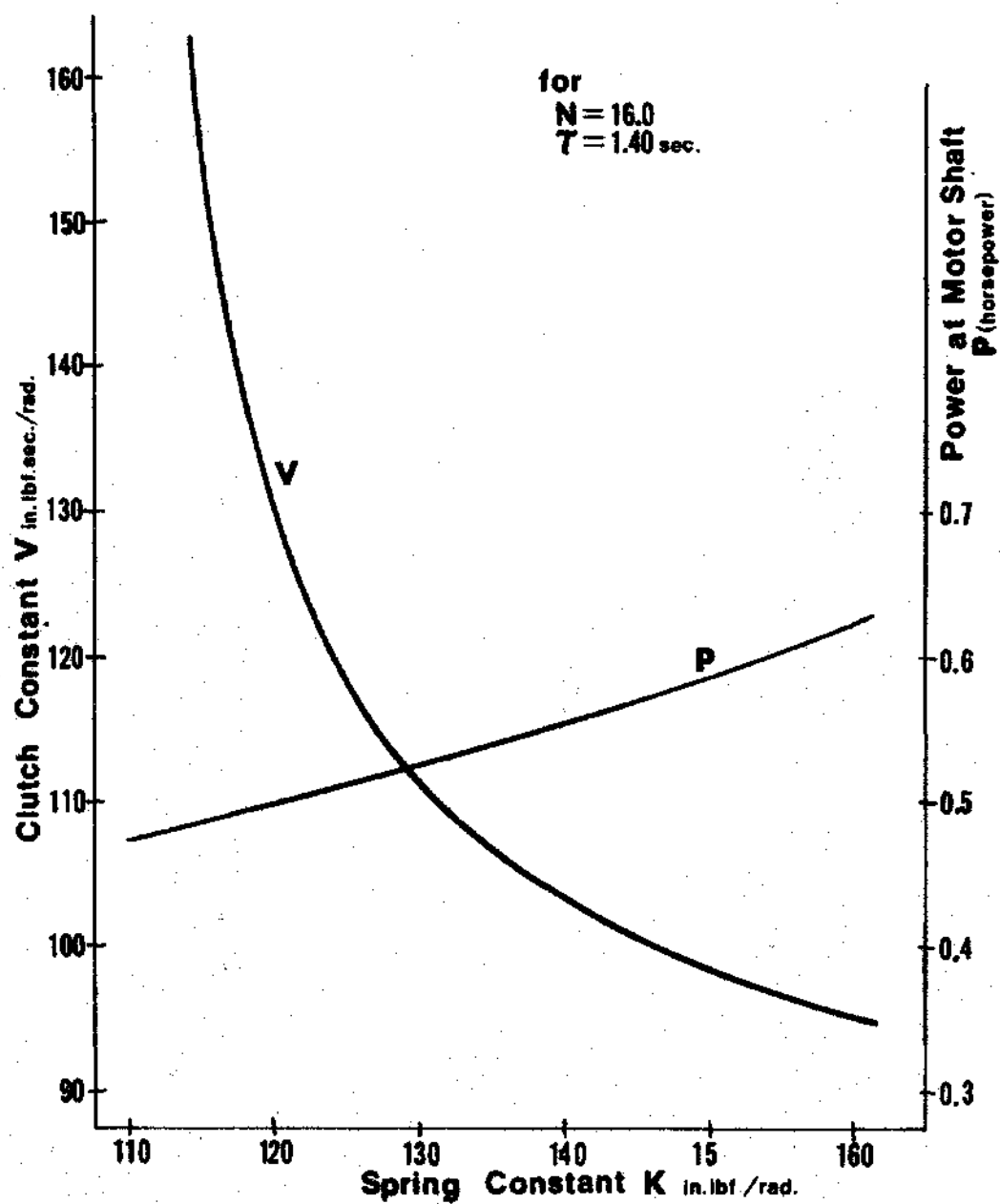


Figure 18. Clutch Constant and Power Versus Spring Constant

the basket and a larger V is required to complete the cycle within the specified period. From Figure 18 it can be seen that there is some practical lower limit to K so that V is not prohibitively large. A large V would necessitate a large clutch, extremely small clearances, or excessively high fluid viscosities. This power draw P at the motor is of primary concern since it determines whether or not the use of the viscous clutch or the dry friction clutch is justified.

The power dissipated in the clutch P_c results in an increase in the temperature of the clutch fluid until the heat transfer rate from the working fluid to the clutch environment is equal to P_c . The rise in temperature would decrease V which is directly proportional to viscosity. In Figure 19, the period τ is plotted against V for three combinations of gear ratio and spring rate. Loop A in the computation scheme shown in Figure 7 provides successive values of V and the resulting τ for the plots in Figure 19. Within loop A, however, the clutch release displacement, $\theta_2(t_2)$ is adjusted by varying t_2 to give the proper arc, while in the real system the clutch release is fixed and the arc will vary. Table 1 shows that $\theta_2(t_2)$ remains fairly constant with respect to V so that the plots in Figure 19 are representative of the effect of temperature on the period of oscillation during warm-up.

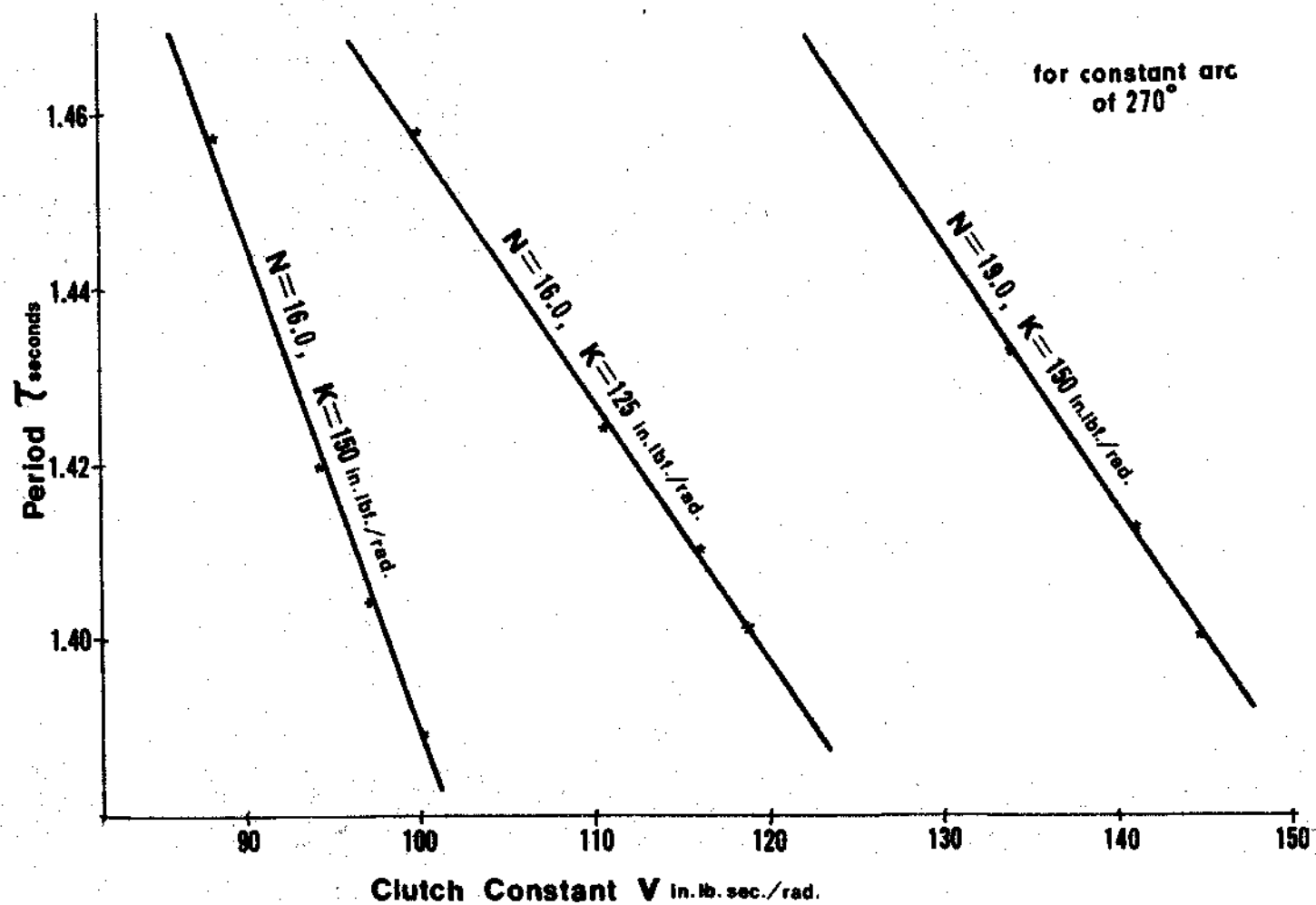


Figure 19. Period Versus Clutch Constant for Warm-Up

Table 1. Main Program Results

K in. lbf./rad.	N	P hp.	P_C hp.	V in. lbf. sec. rad.	Φ_{max} rad./sec.	Φ_{1(t₂)} rad./sec.	T_{max} in. lbf.	t₂ sec.	Θ_{2(t₂)} rad.
150.0	14.0	0.5551	0.3212	76.1	13.75	12.94	1047	0.7230	4.160
150.0	16.0	0.4872	0.2566	98.5	12.05	11.39	1187	0.7196	4.150
150.0	18.0	0.4363	0.2086	128.8	10.73	10.20	1381	0.7162	4.139
125.0	16.0	0.4137	0.1871	118.3	12.11	11.48	1432	0.6138	4.211
112.0	16.0	0.3791	0.1505	163.3	12.14	11.55	1982	0.5272	4.202

Warm-Up

To maintain a more nearly constant clutch fluid temperature, the clutch assembly could be exposed directly to the water in the agi-basket as in Figure 1. For example, consider the case of $N = 16.0$ and $K = 112$ in-lbf/rad. From Table 1

$$P_c = 0.150 \text{ horsepower}$$

$$= 382 \text{ BTU/hr}$$

For this heat input the water temperature will increase 1.3°F during a 20 minute agitation if no heat is transferred out of the water. Assume that the convective heat transfer coefficient for the clutch housing and water is about

$$h_w = 800 \frac{\text{BTU}}{\text{hr-ft}^2\text{-}^\circ\text{F}}$$

Then

$$P_c = h_w A_w U_w$$

where A_w is the area of the clutch housing exposed to water and U_w is the temperature difference between the clutch housing surface and the water. For the clutch example in Chapter I, a typical housing might be 7 inches in radius with

an exposed perimeter 4 inches in height. Then A_w is

$$\begin{aligned} A_w &= \pi \left(\frac{7}{12}\right)^2 + 2\pi \left(\frac{7}{12}\right) \left(\frac{4}{12}\right) \\ &= 2.29 \text{ ft}^2 \end{aligned}$$

Substituting these values into equation (22)

$$382 \text{ BTU/hr} = 800 \text{ BTU/hr-ft}^2\text{-F}^\circ \cdot 2.29 \text{ ft}^2 \cdot U_w$$

$$U_w = 0.209 \text{ F}^\circ$$

Temperature variations in the washing water will then have more effect on period than U_w . Also the initial temperature of the water may vary due to different washing requirements.

If this is the case the clutch may be fined and exposed to air. Now the area exposed for cooling can be

$$A_A = 2\pi \left(\frac{7}{12}\right)^2 + 2\pi \left(\frac{7}{12}\right) \left(\frac{4}{12}\right)$$

Take the heat transfer coefficient to be about

$$h_A = 8 \text{ BTU/hr-ft}^2\text{F}^\circ$$

and the heat capacity of the clutch, about

$$C_h = 8 \text{ BTU/F}^\circ$$

Assuming that the ambient air temperature remains constant, then

$$U_A(t) = \frac{P_c}{h_A A_A} (1 - e^{\frac{-h_A A_A t}{C_h}})$$

where $U_a(t)$ is the temperature difference between the air and the clutch. In Figure 20, τ (period) is plotted against time for Dow Corning 200 Fluid [5], a silicon oil with a low viscosity-temperature gradient.

Start Up

If the mechanism is able to start without external assistance then a sufficient torque must be applied to the basket for a sufficient time to bring the basket to the clutch release displacement ($\theta_2(t_2)$). If this displacement is not reached, the basket will stall at some lower basket displacement since the clutch will never disengage. Suppose the motor and flywheel were brought up to the no load running speed, $\frac{-TQ}{\alpha}$ (Figure 6), and the clutch engaged. If subscript 3 denotes start up,

$$I_T \ddot{\theta}_3 + C_2 \dot{\theta}_3 + K\theta_3 = V(\dot{\phi}_2 - \dot{\theta}_2)$$

where the solution is the same as for $\dot{\theta}_2$ and $\dot{\phi}_2$ except that the initial conditions have changed.

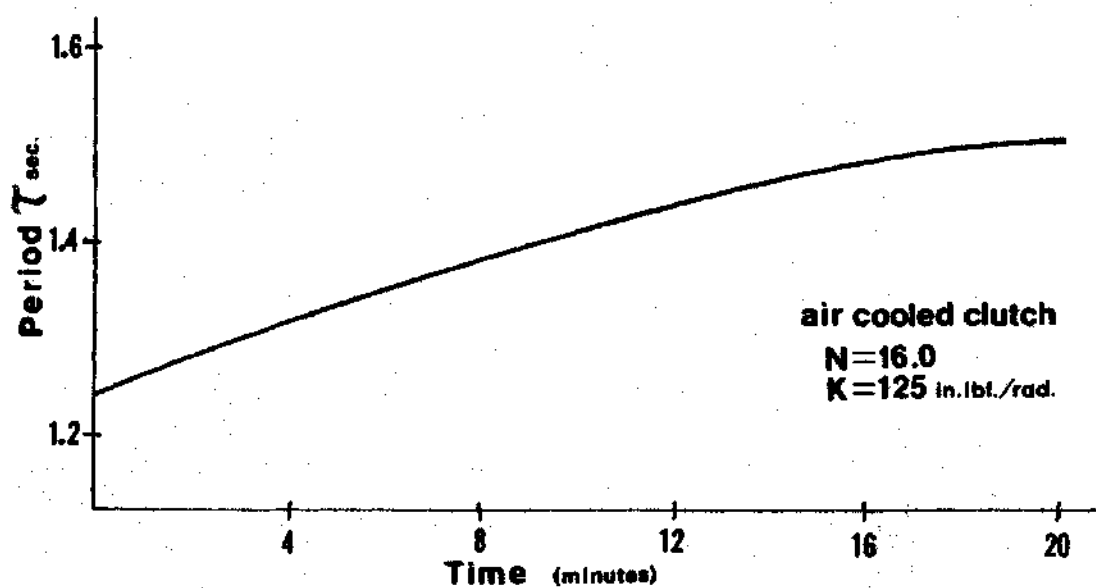


Figure 20. Period Versus Time for Warm-Up

Initial Conditions

$$\dot{\phi}_3 = \frac{-TQ}{N\alpha} \text{ @ } t = 0$$

$$\theta_3 = 0 \text{ @ } t = 0$$

$$\dot{\theta}_3 = 0 \text{ @ } t = 0$$

$$\ddot{\theta}_3 = 0 \text{ @ } t = 0$$

However these initial conditions affect only the constants of integration.

Taking the example of $N = 16.0$,

$K = 112 \text{ in-lbf/rad}$ (Appendix A)

$$b_1 = - .81833$$

$$b_2 = - 2.02242$$

$$b_3 = - 43.25145$$

$$F = 13.836 \text{ radius}$$

where the solution is of the form

$$\theta_2(t) = C_1 e^{b_1 t} + C_2 e^{b_2 t} + C_3 e^{b_3 t} + F$$

Then,

$$\theta_2(\infty) = F$$

$$\theta_2(\infty) = 13.836 \text{ radians}$$

For the example above,

$$\theta_2(t_2) = 4.2 \text{ radians}$$

Therefore,

$$\theta_2(\infty) > \theta_2(t_2)$$

and start-up is easily achieved. Figures 21, 22, 23 and 24 are plots of basket displacement, basket velocity, and driving member velocity for two combinations of reduction ratio and spring rate.

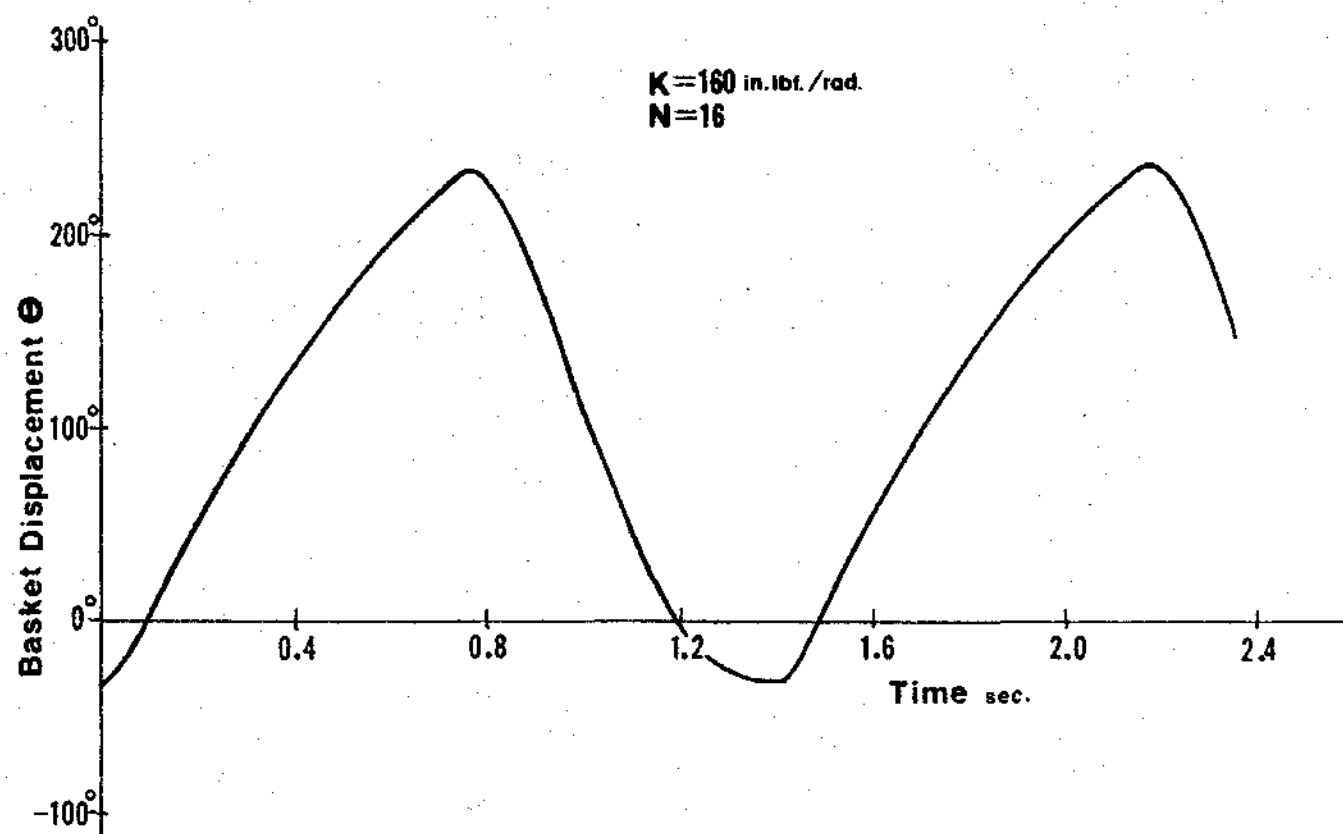


Figure 21. Basket Displacement Versus Time

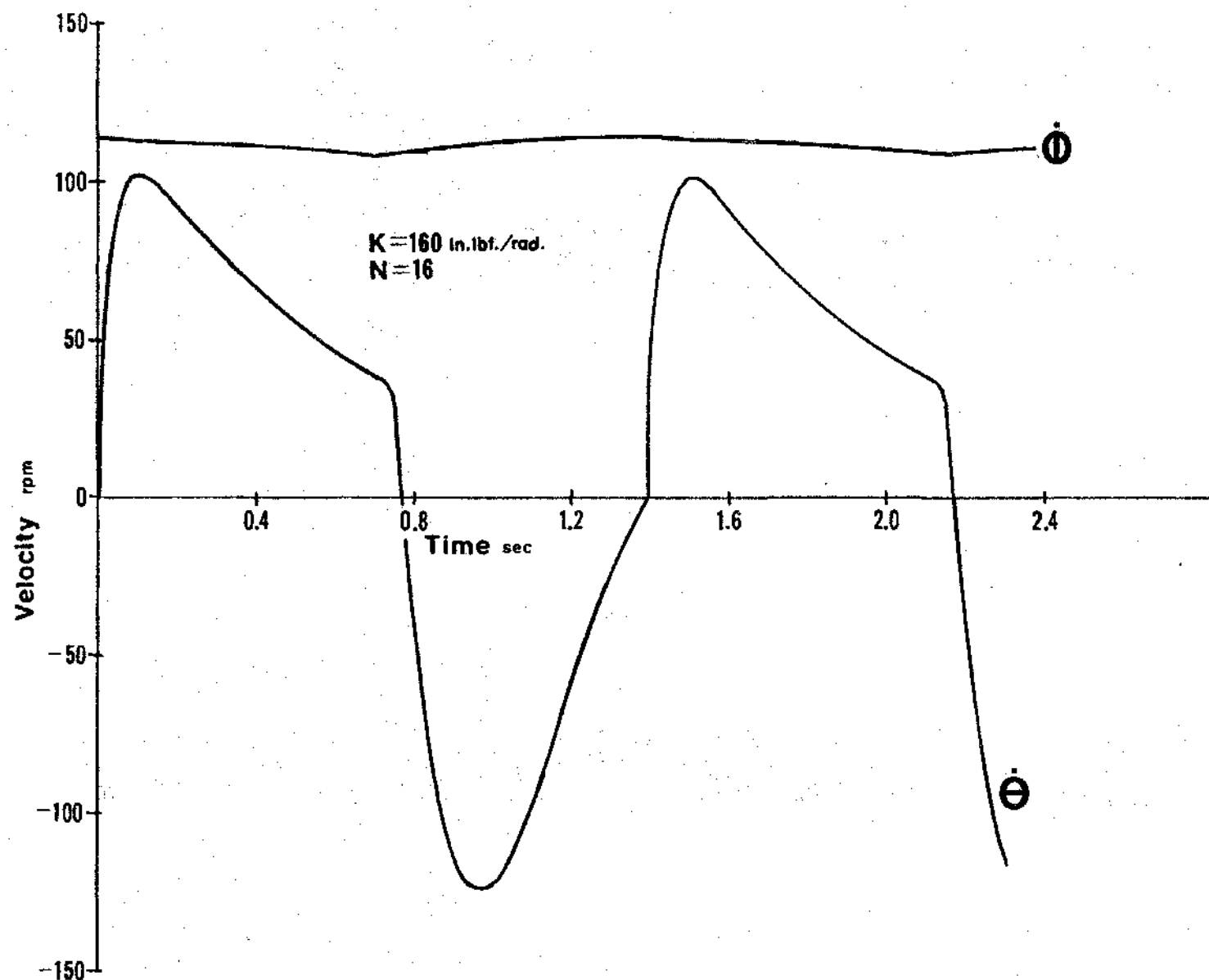


Figure 22. Clutch Member Velocities Versus Time

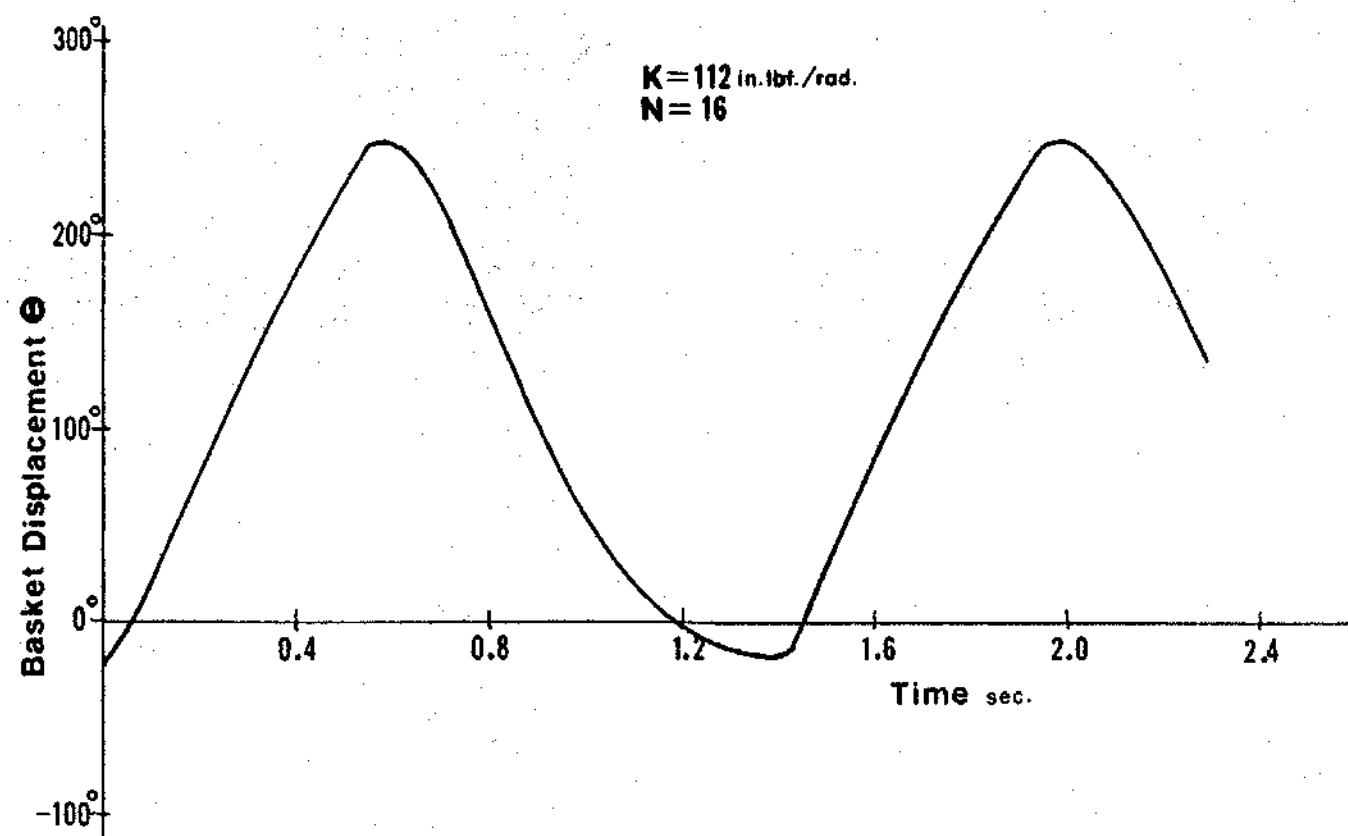


Figure 23. Basket Displacement Versus Time

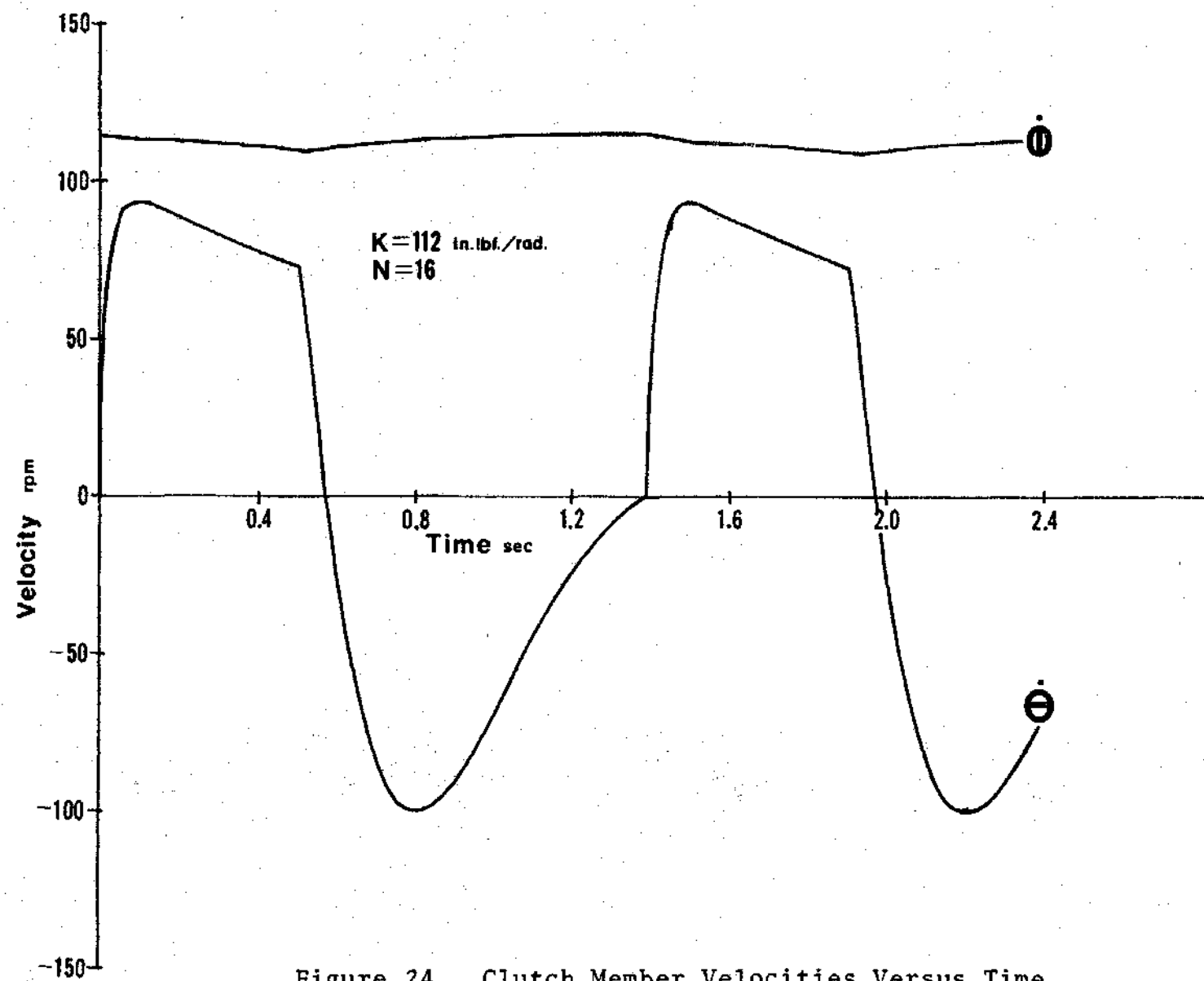


Figure 24. Clutch Member Velocities Versus Time

CHAPTER V

EXPERIMENTAL METHODS

The agi-basket model which has been used until the present is shown schematically in Figure 25(a). The damping in this case is assumed to be between the tub inertia and the support. Since there is no velocity dependent interaction between the tub and the support the placement of the damping in this model lacks physical significance. An experiment is conducted to determine whether this model or another shown schematically in Figure 25(b) is capable of representing basket motion and to determine the appropriate values of damping and inertia representative of the agi-basket. The model in Figure 25(b) has two inertias I_T and I_w connected by a dashpot C and anchored to a support at I_T by a spring with rate K . The inertia, I_w , signifies the contents of the basket (water and clothes). In this model all damping is considered to be between the basket and its contents.

In reality the system is much more complex than either of these models. Water and clothes trapped between paddles are partially carried along with the basket and separate regions of circulation are present at a distance from the paddles as seen from the operation of the experimental equipment. These rigid body models, however, provide a basis

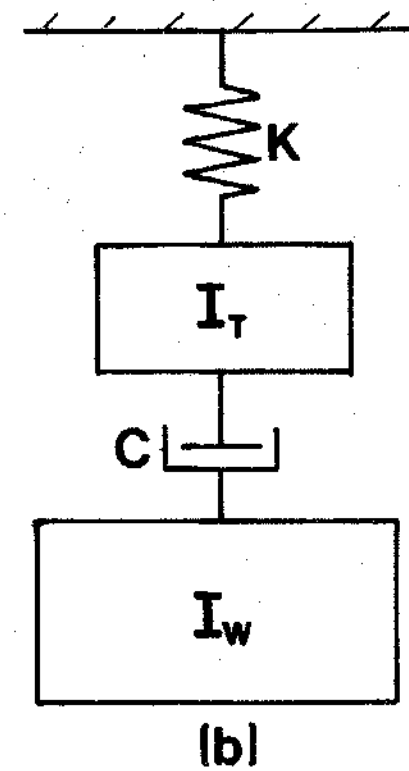
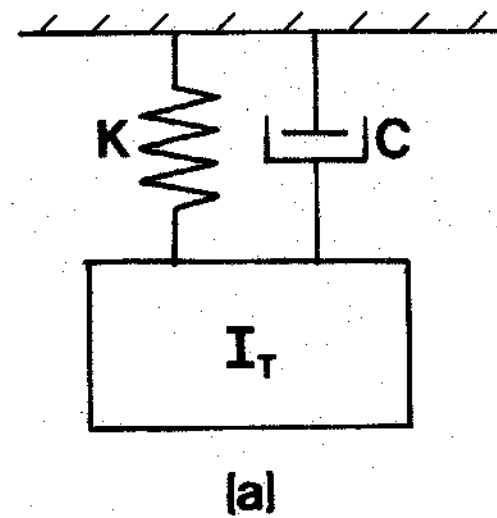


Figure 25. Agi-Basket Models

for analysis.

Equipment

The apparatus used in this experiment is shown in Figure 26. A conventional vertical axis washing machine tub was used for the basket and a conventional central agitator is fixed to the center to provide paddles. (An outer container and a perforated basket were not used in order to simplify construction.) The tub diameter is 20.50 inches, the depth is 13.25 inches and it is filled with 108.5 lbm of water. The empty agi-basket weighs 20.03 pounds. Towels are used to represent the wash load. A displacement transducer is fixed to the bottom of the basket through a fork and pin and the basket is supported at the top by a tapered roller bearing and a bushing. A 6.5 inch diameter drive pulley is also fixed to the basket. A weight applies a torque to the basket by a wire attached to the drive pulley and a release mechanism between the tub and the bushing support serves as a trigger for an oscilloscope. The transducer is incorporated in a voltage divider providing a signal for the oscilloscope. A camera attachment records the displacement-time curve produced on the oscilloscope.

The spring in both agi-basket models in Figure 25 has been replaced by a constant torque due to the weight (if the inertia of the weight is added to the tub inertia I_T) for this experiment. It was found that the constant torque

simplifies the analysis of data since the torque applied by a spring is dependent upon displacement. Since the spring is external to the basket and its rate determined only by the designer, this substitution is justified.

Procedure

Six towels weighing 0.75 pounds each were found to provide an agitation representative of the cleaning action in a vertical axis washing machine when a weight of 20.0 lbf was used to accelerate the tub. The transducer was calibrated by turning the basket through 360° by ten degree increments. Linearity was observed throughout this range.

For the real agi-basket the motion is reversed every 0.7 seconds, therefore the data was recorded for 0.7 seconds after release and read every 0.1 seconds. It is, after all, the transient response of the tub that yields the system variables C , I_T , and I_w and at a longer time from release the steady-state response becomes dominant.

CHAPTER VI

ANALYSIS OF DATA

The data from this experiment is a plot of basket angular displacement versus time for an agi-basket accelerated by a constant torque T . Schematics for the one mass and two mass models with damping are shown for this situation in Figure 27 a and b.

A third system is included (Figure 27c) to facilitate comparison of the other two models. For each model a solution is found for the displacement of the inertia I_T , that is

$$\theta_{IC} = \theta_{IC}(t, I_T, C)$$

$$\theta_{ICI} = \theta_{ICI}(t, I_T, C, I_w)$$

$$\theta_I = \theta_I(t, I_T)$$

Values of inertia and damping (I_{T_1} , C_1 , I_{T_2} , ..., etc.) are found that give the best correlation between the particular solution and the experimental curve at the points where the data is read. The models are then judged by the degree of correlation.

The criterion for judging the degree of correlation between the function θ_{IC} , θ_{ICI} , or θ_I and the data points

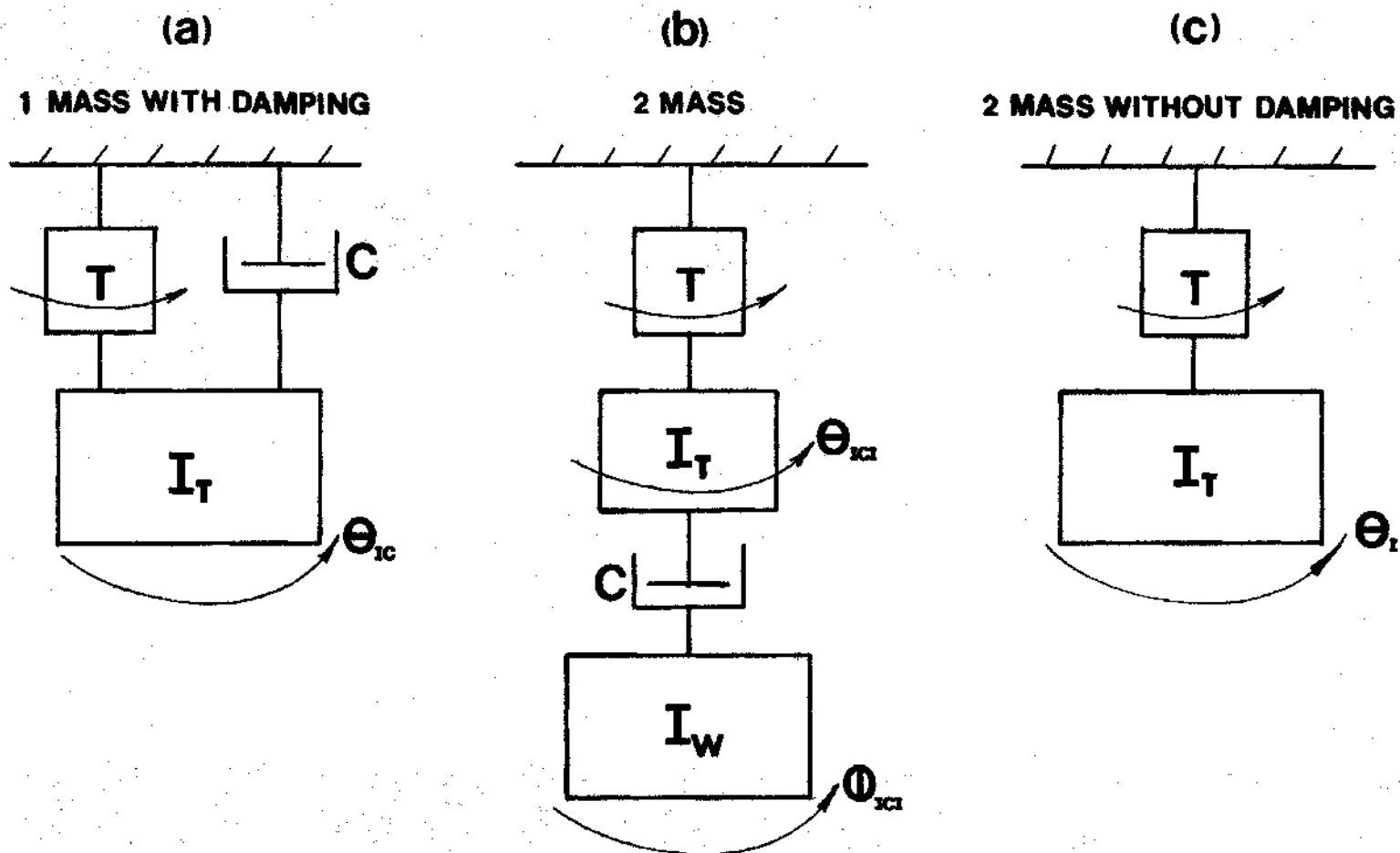


Figure 27. Models of Agi-Basket Used in Experiment

H_0, H_1, \dots, H_7 is the variance, defined by

$$E_{IC}(I_T, C) = \sum_{i=0}^7 [\theta_{IC}(t_i, I_T, C) - H_i]^2 \quad (23)$$

$$E_{ICI}(I_T, C, I_W) = \sum_{i=0}^7 [\theta_{ICI}(t_i, I_T, C, I_W) - H_i]^2 \quad (24)$$

$$E_I(I_T) = \sum_{i=0}^7 [\theta_I(t_i, I_T) - H_i]^2 \quad (25)$$

where $t_i = 0.1(i)$ seconds.

One Mass with Damping

The differential equation for the one mass system with damping is

$$I_T \ddot{\theta}_{IC} + C \dot{\theta}_{IC} = T$$

with initial conditions

$$\theta_{IC} = 0 \text{ @ } t = 0$$

$$\dot{\theta}_{IC} = 0 \text{ @ } t = 0$$

for which the solution is

$$\theta_{IC}(t, C, I_T) = \frac{I_T}{C_1} \frac{1}{2} e^{-\frac{C}{I_T} t} (t-1) + \frac{T}{C} t$$

To minimize the variance E_{IC} , set

$$\frac{\partial E_{IC}}{\partial C} = 0 = -2 \sum_i \{ [\theta_{IC}(t_i, I_T, C) - H_i] \frac{\partial \theta_{IC}(t_i, I_T, C)}{\partial C} \}$$

$$\frac{\partial E_{IC}}{\partial I_T} = 0 = -2 \sum_i \{ [\theta_{IC}(t_i, I_T, C) - H_i] \frac{\partial \theta_{IC}(t_i, I_T, C)}{\partial I_T} \}$$

and solve these two simultaneous equations for I_T and C . Newton's method for simultaneous equations is used where initial values are chosen for C and I_T . Denoting these initial values C_0 and I_{T_0} and each iterative value C_i and I_{T_i} , each successive value is generated by

$$C_{i+1} = C_i + \begin{vmatrix} -\frac{\partial E_{IC}}{\partial C} & \frac{\partial^2 E_{IC}}{\partial C \partial I_T} \\ -\frac{\partial E_{IC}}{\partial I_T} & \frac{\partial^2 E_{IC}}{\partial I_T \partial I_T} \end{vmatrix} \begin{matrix} C=C_i \\ I_T=I_{T_i} \end{matrix}$$

$$C_{i+1} = C_i + \begin{vmatrix} \frac{\partial^2 E_{IC}}{\partial C \partial C} & \frac{\partial^2 E_{IC}}{\partial C \partial I_T} \\ \frac{\partial^2 E_{IC}}{\partial I_T \partial C} & \frac{\partial^2 E_{IC}}{\partial I_T \partial I_T} \end{vmatrix} \begin{matrix} C=C_i \\ I_T=I_{T_i} \end{matrix}$$

$$I_{T_{i+1}} = I_{T_i} + \begin{vmatrix} \frac{\partial^2 E_{IC}}{\partial C \partial C} & -\frac{\partial E_{IC}}{\partial C} \\ \frac{\partial^2 E_{IC}}{\partial I_T \partial C} & -\frac{\partial E_{IC}}{\partial I_T} \\ \frac{\partial^2 E_{IC}}{\partial C \partial C} & \frac{\partial^2 E_{IC}}{\partial C \partial I_T} \\ \frac{\partial^2 E_{IC}}{\partial I_T \partial C} & \frac{\partial^2 E_{IC}}{\partial I_T \partial I_T} \end{vmatrix} \begin{matrix} C=C_i \\ I_{T_{i+1}}=I_{T_i} \\ C=C_i \\ I_T=I_{T_i} \end{matrix}$$

The iteration terminates when

$$|C_{i+1} - C_i| < \Delta C$$

and

$$|I_{T_{i+1}} - I_i| < \Delta I_T$$

For this analysis

$$\Delta C = 0.001 \text{ in-lbf-sec/rad}$$

$$\Delta I_T = 0.001 \text{ in-lbf-sec}^2/\text{rad}$$

The minimized E_{IC} is obtained from the final I_{T_i} and C_i by equation (23).

Two Mass System with Damping

The differential equations for the two mass system

shown in Figure 27 are

$$I_T \ddot{\theta}_{ICI} + I_W \ddot{\phi}_{ICI} = T \quad (26)$$

$$I_W \ddot{\phi}_{ICI} = C(\dot{\theta}_{ICI} - \dot{\phi}_{ICI}) \quad (27)$$

with initial conditions

$$\theta_{ICI} = 0 \text{ @ } t = 0$$

$$\dot{\theta}_{ICI} = 0 \text{ [} t = 0$$

$$\dot{\phi} = 0 \text{ @ } t = 0$$

Differentiating equation (27) with respect to time and rearranging

$$\frac{I_W}{C} \ddot{\phi}_{ICI} + \dot{\phi}_{ICI} - \dot{\theta}_{ICI} = 0$$

Combining with equation (26)

$$\ddot{\phi}_{ICI} + C\left(\frac{1}{I_T} + \frac{1}{I_W}\right)\dot{\phi}_{ICI} = \frac{TC}{I_T I_W}$$

This equation can easily be solved for $\ddot{\phi}_{ICI}$

$$\ddot{\phi}_{ICI} = A'e^{qt} + \frac{T}{I_T + I_W}$$

where Λ' is a constant of integration and

$$q = -C\left(\frac{1}{I_T} + \frac{1}{I_W}\right)$$

Substitute this solution for ICI into equation (26)

$$\ddot{\theta}_{ICI} = \Lambda e^{qt} + \frac{T}{I_T + I_W}$$

Integrating twice with respect to time and applying the initial conditions

$$\theta_{ICI}(t, I_T, C, I_W) = \frac{\Lambda}{q^2} e^{qt} + \frac{Tt^2}{2(I_T + I_W)} - \frac{\Lambda}{q}t - \frac{\Lambda}{q^2}$$

where

$$\Lambda = \frac{I_W T}{I_T(I_T + I_W)}$$

Using Newton's method, as in the previous model, I_T, C , and I_W are found which minimize $E_{ICI}(t, I_T, C, I_W)$. Again the iteration is terminated by the test,

$$|I_{T_{i+1}} - I_{T_i}| < \Delta I_T$$

$$|C_{i+1} - C_i| < \Delta C$$

$$|I_{W_{i+1}} - I_{W_i}| < \Delta I_W$$

where ΔI_T and ΔC are as given for the first model and

$$\Delta I_W = 0.001 \text{ in-lbf-sec}^2/\text{rad}$$

The minimized E_{ICI} is then evaluated from equation (24).

One Mass System Without Damping

The differential equation for the single mass without damping is

$$I_T \ddot{\theta}_I = T$$

with initial conditions

$$\theta_I = 0 \text{ @ } t = 0$$

$$\dot{\theta}_I = 0 \text{ @ } t = 0$$

The solution of the differential equation is

$$\theta_I(t, I_T) = \frac{T}{2I_T} t^2$$

To minimize E_I set

$$\frac{\partial E_I}{\partial I_T} = -2 \sum_i \{ [\theta_I(t_i, I_T) - H_i] \frac{\partial \theta_I(t_i, I_T)}{\partial I_T} \} = 0$$

$$\sum_i \left[\frac{T t_i^2}{2I_T} - H_i \right] \left(\frac{-T t_i^2}{I_T} \right) = 0$$

$$\frac{1}{2I_T} \sum_i T t_i^4 - \sum_i H_i t_i^2 = 0$$

$$I_T = \frac{T}{2} \frac{\sum_i t_i^4}{\sum_i H_i t_i^2}$$

The minimized E_I is then calculated from equation (25).

A summary of these results can be found in Table 2. Values of C and I_T for this single mass with damping are in good correlation with C and I_T used in the analysis of the proposed drive mechanism (4.44 in-lbf-sec²/rad and 28.8 in-lbf-sec/rad, respectively). As judged by the variance E , the two mass model with damping provides a more accurate displacement-time curve than either of the other models. More complex models might be expected to be in even closer argument to experiment. While the analysis will not be carried out again with the two mass model here due to the complexity of the solutions in the coupled portion of basket motion (Chapter III) this is recommended for future research.

Table 2. Experimental Results

MODEL	E rad. ²	I_T in.lbf.sec ² / rad.	C in.lbf.sec/ rad.	I_w in.lbf.sec ² / rad.
1 MASS with DAMPING	0.00179	4.509	31.949	
2 MASS with DAMPING	0.00029	2.197	55.234	30.318
1 MASS w/o DAMPING	0.08223	11.31		

APPENDICES

APPENDIX A

MAIN PROGRAM

```

C VISCIOUS CLUTCH APPLICATION
C
      IMPLICIT REAL (A-Z)
      COMPLEX AC,BC
C
      W(T)=(P(T))**2-TH(T)*P(T)
      L(T)=(P(T)-TH(T))**2
C
      READ (5, 100) K, C, JT, JF, BRKDOWN, TO, ALFA, TAU
100  FORMAT ('F8.0')
C
      K=120.0
050  K=K+20.0
      IF (K .GT. 180.0) STOP
      N=15.0
300  N=N+1.0
      IF (N .GT. 18.0) GO TO 050
C
      JF=JF*N*N
      TO=TO*N
      ALFA=ALFA*N*N
      BRKDOWN=BRKDOWN/N
C
      W1=SQRT(4.0*JT*K-C*C)/(2.0*JT)
      Q1=C/(2.0*JT)
      T1=(-3.14159)/W1
      C7=(-1.5)*3.14159/(1.0+EXP((-Q1)*T1))
      C6=(Q1*C7)/W1
C
      TH1MIN=C7
      TH1MAX=C7+1.5*3.14159
C
C BEGIN LOOP A
C FIND VALUE OF V TO GIVE PROPER TAU
      VS=11.0*N-98.0
      VL=11.0*N-47.0
      NA=0.0
200  V=(VS+VL)/2.0
      WRITE (6,140) V
140  FORMAT ('/1X, 'V=',F8.4)
      A1=(JF*(C+V)+JT*(V-ALFA))/(JF*JT)
      A2=(JF*(V+V*C-ALFA+C*ALFA*V))/(JF*JT)
      A3=K*(V-ALFA)/(JF*JT)
C
      P=13.0*A2-A1*A1/3.0
      S=13.0*(A1**3)-0.0*A2*A1+27.0*A3/27.0
C
C TEST FOR FORM OF SOLUTION OF THETA2
      Z=1.0
      IF ((S**2)/4.0+(R**3)/27.0 .LT. 0.0) GO TO 902
C
C
C EVALUATE CONSTANTS FOR UNDERDAMPED CASE
      A=0.01*(-S)/2.0*SQRT((S**2)/4.0+(R**3)/27.0)
      B=0.01*(-S)/2.0*SQRT((S**2)/4.0+(R**3)/27.0)

```

```

      T2=1/(A1/3.0)
      Q2=(1+B)/2.0+A1/3.0
      W2=(SQRT(3.0)/2.0)*(A-B)
      F=(T0*V)/(UF*JT+A3)
      GO TO 601

C
902 WRITE (6,105)
105 FORMAT (1X,'OVER DAMPED')
      Z=3.0

C
C EVALUATE CONSTANTS FOR OVERDAMPED CASE
      X=-S/2.0
      Y=SQRT(-(1/3*S)/2.0+(R**2)/27.0)
      AC=CMPLX(X,Y)
      BC=CMPLX(X,-Y)

C
      B1=REAL(AC+BC)-1/3.0
      B2=REAL(AC+BC)/(-2.0)+SQRT(3.0)*(AIMAG(AC-BC)/2.0)-A1/3.0
      B3=REAL(AC+BC)/(-2.0)-SQRT(3.0)*(AIMAG(AC-BC)/2.0)-A1/3.0
      F=(T0*V)/(UF*JT+A3)

C
C BEGIN LOOP B
C FIND VALUE OF PHMAX
C
601 CONTINUE
      PHMAXS=PHKDOWN*1.05
      PHMAXL=T0/ALFA
      NB=0.0
201 PHMAX=(PHMAXL+PHMAXS)/2.0

C
      IF (7 .GT. 2.0) GO TO 602
      DEN1=JT*(-W2)*(1.0**2+Q2**2+W2**2+2.0*Q2*M)
      C1=((F-TH1MIN)*Q2*Q2*W2*JT+W2*W2*W2*JT)-J2*V*PHMAX+W2*K*TH1MIN)/
1 DEN1
      C2=TH1MIN-F-C1
      C3=((K/JT)*TH1MIN-(V/JT)*PHMAX+C1*M**2+C2*(Q2**2-W2**2))/(2.0*Q2*
1 W2)
      GO TO 603

C
602 DEN1=B1*((B2**2)-(B3**2))+B2*((B3**2)-(B1**2))+B3*((B1**2)-(B2**2)
      C1=((TH1MIN-F)*(B2*B3*B1-B3*B2*B1)+(B3-B2)*((V/JT)*PHMAX
1 -(K/JT)*TH1MIN)/DEN1
      C2=((TH1MIN-F)*(B3*B1*B1-B1*B3*B1)+(B1-B3)*((V/JT)*PHMAX
1 -(K/JT)*TH1MIN)/DEN1
      C3=TH1MIN-F-C2-C1
      WRITE (6,089) B1, B2, B3, C1, C2, C3
089 FORMAT (5X,6E15.6)

C
C BEGIN LOOP C
C FIND T2 AND T3 BY H. R. METHOD
C
603 CONTINUE
      T2=0.60
      T3=((1-1.22)*3.14159)/W1
      NC=0.0

C
202 IF (7 .GT. 2.0) GO TO 604

```

```

TH2T2=C1*EXP(M**2)+EXP((-W2)*T2)*(C2*COS(W2*T2)+C3*SIN(W2*T2))+F
TH2FT2=C1*M*EXP(M**2)-G2*EXP((-W2)*T2)*(C2*COS(W2*T2)+C3*SIN(W2*T2
1))+EXP((-W2)*T2)*((-C2)*W2*SIN(W2*T2)+C3*W2*COS(W2*T2))
TH2ST2=C1*(M**2)*EXP(M**2)+((G2**2)*EXP((-W2)*T2)*(C2*COS(W2*T2)+C3
1*SIN(W2*T2))-2.0*W2*EXP((-W2)*T2)*((-C2)*W2*SIN(W2*T2)+C3*W2*COS(
2*W2*T2))-EXP((-W2)*T2)*(C2*(W2**2)*COS(W2*T2)+C3*(W2**2)*SIN(W2*T2
3))
GO TO 605
C
604 TH2T2=C1*EXP(B1*T2)+C2*EXP(R2*T2)+C3*EXP(L3*T2)+F
TH2FT2=C1*B1*EXP(B1*T2)+C2*B2*EXP(R2*T2)+C3*B3*EXP(L3*T2)
TH2ST2=C1*B1*B1*EXP(B1*T2)+C2*B2*B2*EXP(R2*T2)+C3*B3*B3*EXP(L3*T2)
C
605 TH1T3=EXP((-Q1)*T3)*(C7*COS(W1*T3)+C8*SIN(W1*T3))
TH1FT3=EXP((-Q1)*T3)*((W1*C8-Q1*C7)*COS(W1*T3)-(W1*C7+Q1*C8)*SIN(
1*W1*T3))
TH1ST3=EXP((-Q1)*T3)*((Q1*C7-W1*C8)*Q1-(W1*C7+Q1*C8)*W1)*COS(W1*T
1)+((W1*C7+Q1*C8)*Q1-(W1*C8-Q1*C7)*W1)*SIN(W1*T3))
C
WRITE (6,150) TH1T3, TH1FT3, TH2T2, TH2FT2
150 FORMAT (1X,'TH1T3=',E16.7,4X,'TH1FT3=',E16.7,4X,'TH2T2=',E16.7,4X,
1,'TH2FT2=',E16.7)
C
DEN2=TH1FT3*TH2ST2-TH2FT2*TH1ST3
KA=(TH2ST2*(TH2T2-TH1T3)-TH2FT2*(TH2FT2-TH1FT3))/DEN2
HA=(TH1ST3*(TH2T2-TH1T3)-TH1FT3*(TH2FT2-TH1FT3))/DEN2
C
T2=T2+HA
T3=T3+KA
WRITE (6,130) T3, T2
130 FORMAT (1X,'T3=',E13.7,4X,'T2=',E13.7)
C
NC=NC+1
IF (NC.GT. 10.) GO TO 905
IF (ABS(HA).GT. 1.E-4) GO TO 202
IF (ABS(KA).GT. 1.E-4) GO TO 202
C
END LOOP C
IF (7.GT. 2.0) GO TO 304
C4=C1*(K/V+(C+V)*3/V+JT*(M**2)/V)
C5=(K*C2-(C+V)*.2*C2+JT*Q2+Q2*C2-JT*W2*W2,C2-2.0*JT*Q2*W2*C3+(C+V)
1*.2*C3)/V
C6=(K*C3-(C+V)*.2*C3+JT*Q2+Q2*C3-JT*W2*W2,C3+2.0*JT*Q2*W2*C2-(C+V)
1*.2*C2)/V
PH2T2=C4*EXP(M*T2)+EXP((-W2)*T2)*(C5*COS(W2*T2)+C6*SIN(W2*T2))
1 IF W2/V
GO TO 305
C
304 C4=(JT*C1*L1*B1,(C+V)*C1*B1+K*C1)/V
C5=(JT*C2*L2*B2,(C+V)*C2*B2+K*C2)/V
C6=(JT*C3*L3*B3,(C+V)*C3*B3+K*C3)/V
PH2T2=C4*EXP(R1,T2)+C5*EXP(R2,T2)+C6*EXP(L3,T2)+F*K/V
C
305 TH1T2=(T0/ALFA+H1,X)*EXP(ALFA/(F*T2))-T0/ALFA
TH1FT2=(T0/ALFA+H1,X)*EXP(ALFA/(F*T2))-T0/ALFA
142 FORMAT (1X,'PH1T2=',E13.7,4X,'TH1T2=',E13.7,4X,'PH2T2=',E13.7)
IF (1-ABS(PH1T2-0.2T2).LT. 0.5/V) GO TO 301

```

```

      P=PH+1.0
      IF (PH1T3 .GT. 50.0) GO TO 904
      IF (PH1T3 .GT. PH2T2) PHMAX=PH+MAX
      IF (PH1T3 .LT. PH2T2) PHMAX=PH+MAX
      GO TO 291
C
C END LOOP B
391 IF (PH2T3-(T2-T3)-TAU) .LT. 0.005) GO TO 800
      IF (T2-T3) .GT. TAU) V=V
      IF (T2-T3) .LT. TAU) V=V
      P=PH+1.0
      IF (PH .GT. 12.0) GO TO 903
      GO TO 291
C
C END LOOP A
C CALCULATE POWER AND POWER LOSS
C
800 WORK=V*(T2/20.0)*(W(0.0)+2.0*W(.1*T2)+2.0*W(.2*T2)+2.0*W(.3*T2)+
1 2.0*W(.4*T2)+2.0*W(.5*T2)+2.0*W(.6*T2)+2.0*W(.7*T2)+2.0*W(.8*T2)
2 +2.0*W(.9*T2)+W(T2))
      WORKL=V*(T2/20.0)*(L(0.0)+2.0*L(.1*T2)+2.0*L(.2*T2)+2.0*L(.3*T2)+
1 2.0*L(.4*T2)+2.0*L(.5*T2)+2.0*L(.6*T2)+2.0*L(.7*T2)+2.0*L(.8*T2)
2 +2.0*L(.9*T2)+L(T2))
      PWR=WORK/(6600.0*TAU)
      PWRL=WORKL/(6600.0*TAU)
      WRITE(6,790) PWR, PWRL
790 FORMAT (1X,'PWR=',F9.5,'PWRL=',F9.5)
C
      WRITE (6,105) K
1051 FORMAT (6X,'K=',F4.1)
      WRITE (6,103) N
1031 FORMAT (6X,'N=',F4.1)
      TRQMAX=V*PHMAX
      WRITE (6,791) TRQMAX
791 FORMAT (2X,'TRQMAX=',F12.5)
C
      WRITE (6,110) V
1101 FORMAT (2X,'V=',F12.5)
      WRITE (6,111) P,MAX
1111 FORMAT (2X,'PHMAX=',F12.5)
      WRITE (6,112) P,1T3
1121 FORMAT (2X,'PH1T3=',F12.5)
      WRITE (6,113) T,1MAX
1131 FORMAT (2X,'TH1MAX=',F12.5)
      WRITE (6,780) T,1MIN
7801 FORMAT (2X,'TH1MIN=',F12.5)
      WRITE (6,114) T1
1141 FORMAT (2X,'T1=',F12.5)
      WRITE (6,115) T2
1151 FORMAT (2X,'T2=',F12.5)
      WRITE (6,116) T3
1161 FORMAT (2X,'T3=',F12.5)
      WRITE (6,117) T,113
1171 FORMAT (2X,'TH1T3=',F12.5)
      WRITE (6,368) T,2T2
3681 FORMAT (2X,'TH2T2=',F12.5)
      WRITE (6,118) Q1

```

```

113 FORMAT (2X,'M1= ',F12.5)
   WRITE (6,113) M1
119 FORMAT (2X,'W1= ',F12.5)
C
   IF (7 .GT. 2.0) GO TO 311
   WRITE (6,120) M
120 FORMAT (2X,'M= ',F12.5)
   WRITE (6,121) W
121 FORMAT (2X,'Q2= ',F12.5)
   WRITE (6,122) W
122 FORMAT (2X,'W2= ',F12.5)
   GO TO 310
C
311 WRITE (6,360) B1
360 FORMAT (2X,'B1= ',F12.5)
   WRITE (6,361) B2
361 FORMAT (2X,'B2= ',F12.5)
   WRITE (6,362) B3
362 FORMAT (2X,'B3= ',F12.5)
C
310 WRITE (6,123) F
123 FORMAT (2X,'F= ',F12.5)
   WRITE (6,124) C1
124 FORMAT (2X,'C1= ',F12.5)
   WRITE (6,125) C2
125 FORMAT (2X,'C2= ',F12.5)
   WRITE (6,126) C3
126 FORMAT (2X,'C3= ',F12.5)
   WRITE (6,127) C4
127 FORMAT (2X,'C4= ',F12.5)
   WRITE (6,128) C5
128 FORMAT (2X,'C5= ',F12.5)
   WRITE (6,129) C6
129 FORMAT (2X,'C6= ',F12.5)
C
   IF (7 .GT. 2.0) GO TO 390
   GO TO 392
390 WRITE (6,391)
391 FORMAT (1X,'THE A2 OVERDAMPED')
392 IF (PHIT .LT. BRKOWN) GO TO 901
   GO TO 300
901 WRITE (6,104)
104 FORMAT (1X,'MOTOR BRKOWN')
   GO TO 300
C
903 WRITE (6,106)
106 FORMAT (1X,'NA ,ARGE')
   GO TO 300
904 WRITE (6,107)
107 FORMAT (1X,'NB ,ARGE')
   GO TO 300
905 WRITE (6,108)
108 FORMAT (1X,'NC ,ARGE')
   GO TO 300
C
C DEFINE FUNCTIONS P(T) AND TH(T)
FUNCTION TH(T)

```

```

IF (7 .LT. 2.0)
1 T=C1*EXP(W1*T)+C2*EXP((-W2)*T)*(C3*COS(W2*T)+C3*SIN(W2*T))
2 +EXP((-W2)*T)*(C3+C2*SIN(W2*T)+C3*W2*COS(W2*T))
IF (7 .GT. 2.0) T=C1*EXP(B1*T)+C2*W2*EXP(B2*T)+C3*W3*EXP(B3*T)
RETURN

```

C

```

FUNCTION P(T)
IF (7 .LT. 2.0)
1 P=C4*EXP(W1*T)+XP((-W2)*T)*(C5*COS(W2*T)+C6*SIN(W2*T))+F*K/V
IF (7 .GT. 2.0) P=C4*EXP(B1*T)+C5*EXP(B2*T)+C6*EXP(B3*T)+F*K/V
RETURN
END

```

P.K=	.37919	P.D.L=	.15054
A=	112.0		
n=	16.0		
T _{max} =	1982.45326		
V=	163.25000		
P _{max} =	12.14360		
P _{min} =	11.55369		
T _{max} =	4.40541		
T _{min} =	-30897		
T ₁ =	-.81921		
T ₂ =	.52723		
T ₃ =	-.87582		
T _{1T2} =	4.20208		
T _{12T2} =	7.49020		
Q ₁ =	3.24324		
W ₁ =	3.83492		
B ₁ =	-.81633		
B ₂ =	-2.02242		
B ₃ =	-43.25145		
F=	13.33851		
C ₁ =	-15.32500		
C ₂ =	.93411		
C ₃ =	.24320		
C ₄ =	3.96051		
C ₅ =	-1.47707		
C ₆ =	.16609		
T _{max} Overhead			

APPENDIX B

PLOTS

```

C PLOT OF DISPLACEMENT AND VELOCITIES
  IMPLICIT REAL (A-Z)
  INTEGER I
  DIMENSION THETA(121), THETAF(121), PHIF(121), IBUFF(1200), T(121)
C
  READ (5,201) Z, N, K, V, PHMAX, TH1MIN, T2, T3
201 FORMAT (2F5.0,6F10.0)
  READ (5,202) Q1, W1, M, Q2, W2, F
202 FORMAT (6F10.0)
  READ (5,203) C1, C2, C3, C4, C5, C6
203 FORMAT (6F10.0)
C
  T0=431.0
  ALFA=-2.2
  JF=1.00
  C7=TH1MIN
  CB=(Q1*C7)/W1
C
  IF (Z .LT. 2.0) GO TO 110
  B1=M
  B2=Q2
  B3=W2
C
110 JF=JF*N*N
  T0=T0*N
  ALFA=ALFA*N*N
C
  DO 104 I=1,121
    T(I)=(I-1)*.02
C
    IF (T(I) .GE. T2) GO TO 101
    THETA(I)=TH2(T(I))
    THETAF(I)=TH2F(T(I))
    PHIF(I)=PH2F(T(I))
    GO TO 100
C
101 IF (T(I) .GE. (T2-T3)) GO TO 102
    THETA(I)=TH1(T(I)-(T2-T3))
    THETAF(I)=TH1F(T(I)-(T2-T3))
    PHIF(I)=PH1F(T(I)-(T2-T3))
    GO TO 100
C
102 IF (T(I) .GE. (T2-T3+T2)) GO TO 103
    THETA(I)=TH2(T(I)-(T2-T3))
    THETAF(I)=TH2F(T(I)-(T2-T3))
    PHIF(I)=PH2F(T(I)-(T2-T3))
    GO TO 100
C
103 THETA(I)=TH1(T(I)-2.0*(T2-T3))
    THETAF(I)=TH1F(T(I)-2.0*(T2-T3))
    PHIF(I)=PH1F(T(I)-2.0*(T2-T3))
C
100 T(I)=T(I)*2.5
    THETA(I)=THETA(I)*0.572958
    THETAF(I)=THETAF(I)*(60.0/6.28319)*0.02
104 PHIF(I)=PHIF(I)*(60.0/6.28319)*0.02

```

```

C
CALL PLOTS (IBUFF,1200,3)
CALL PLOT (1.0,5.5,-3)
CALL AXIS (0.0,0.0,'TIME',4,10.0,0.0,0.0,0.4)
CALL AXIS (0.0,-3.0,'THETA (DEG)',11.6,0.0,90.0,-300.0,100.0)
CALL PLOT (T(1),THETA(1),3)
DO 105 I=2,121
105 CALL PLOT (T(I),THETA(I),2)
C
CALL PLOTS (IBUFF,1200,3)
CALL PLOT (12.0,0.0,-3)
CALL AXIS (0.0,0.0,'TIME',4,10.0,0.0,0.0,0.4)
CALL AXIS (0.0,-3.0,'VELOCITY (RPM)',14.6,0.0,90.0,-150.0,50.0)
CALL PLOT (T(1),THETA(I),3)
DO 106 I=2,121
106 CALL PLOT (T(I),THETA(I),2)
CALL PLOT (T(1),PHIF(1),3)
DO 107 I=2,121
107 CALL PLOT (T(I),PHIF(I),2)
CALL PLOT (11.0,0.0,999)
STOP
C
FUNCTION TH1(T)
TH1=EXP((-Q1)*T)*(C7*COS(W1*T)+C8*SIN(W1*T))
RETURN
C
FUNCTION TH2(T)
IF (Z .LT. 2.0) TH2=C1*EXP(M*T)+EXP((-Q2)*T)*(C2*COS(W2*T)
A +C3*SIN(W2*T))+F
IF (Z .GT. 2.0) TH2=C1*EXP(B1*T)+C2*EXP(B2*T)+C3*EXP(B3*T)+F
RETURN
C
FUNCTION TH1F(T)
TH1F=EXP((-Q1)*T)*((W1+C8-Q1*C7)*COS(W1*T)-(W1*C7+Q1*C8)*SIN(W1*T)
A )
RETURN
C
FUNCTION TH2F(T)
IF (Z .LT. 2.0) TH2F=C1*M*EXP(M*T)-Q2*EXP((-Q2)*T)*(C2*COS(W2*T)
A +C3*SIN(W2*T))+EXP((-Q2)*T)*((-C2)*W2*SIN(W2*T)+C3*W2*COS(W2*T))
IF (Z .GT. 2.0) TH2F=C1*B1*EXP(B1*T)+C2*B2*EXP(B2*T)+C3*B3*EXP
A (B3*T)
RETURN
C
FUNCTION PH1F(T)
PH1F=(TQ/ALFA+PHMAX)*EXP(ALFA/JF*T)-TQ/ALFA
RETURN
C
FUNCTION PH2F(T)
IF (Z .LT. 2.0) PH2F=C4*EXP(M*T)+EXP((-Q2)*T)*(C5*COS(W2*T)+C6*
A SIN(W2*T))+F*K/V
IF (Z .GT. 2.0) PH2F=C4*EXP(B1*T)+C5*EXP(B2*T)+C6*EXP(B3*T)+F*K/V
RETURN
END

```


APPENDIX C

TWO MASS WITH DAMPING

```

      DIMENSION TH(50)
      N=8
C
      DO 106 I=1,N
      READ (5,100) TH(I)
100  FORMAT (F6.0)
      WRITE (6,106) I,TH(I)
106  FORMAT (12X,I3,6X,F8.5)
C
      K=0
      C=50.0
      AW=30.0
      AT=3.0
C
      F=66.2
401  CONTINUE
      IF (K .GT. 12) STOP
      E=0.
      EC=0.
      EAW=0.
      EAT=0.
      ECC=0.
      ECAW=0.
      ECAT=0.
      EAWAW=0.
      EAWAT=0.
      EATAT=0.
C
      K=K+1
C
      Q=(-C)*(1./AW+1./AT)
      C1=AW*F/AT/(AT+AW)
C
      DO 500 I=1,N
C
      T=0.1*FLOAT(I-1)
      EX=EXP(Q*T)
C
      D=C1/Q/Q*(EX-1.)-C1*T/Q+F*T/2./(AT+AW)
C
      DQ=(-2.)*C1/(Q**3)*(EX-1.)+C1/Q/Q*T*(EX+1.)
      DC1=(EX-1.)/Q/Q-T/Q
C
      QC=Q/C
      QAT=C/AT/AT
      QAW=C/AW/AW
C
      CIAT=AW*F*(2.*AT+AW)/((AT*AT+AT*AW)**2)
      CIAW=F/((AT+AW)**2)
C
      DC=DQ*QC
      DAW=DQ*QAW+DC1*CIAW-.5*T*T/((AT+AW)**2)
      DAT=DQ*QAT+DC1*CIAT-.5*T*T/((AT+AW)**2)
C
      JQG=.5*C1/(Q**4)*(EX-1.)-2.*C1*T/(Q**3)*(2.*EX+1.)+C1*T*T/Q/Q*EX
      JQC1=T/Q/Q*(EX+1.)-2./(Q**3)*(EX-1.)
      JC1C1=0.0

```

```

QCAW=1./AW/AW
QCAT=1./AT/AT
QATAT=(-2.)*C/(AT**3)
QAWAW=(-2.)*C/(AW**3)

C1ATAT=F*(6.*(AT**4)*AW+12.*(AT**3)*AW*AW+8.*AT*AT*(AW**3)
1 +2.*AT*(AW**4))/(AT*(AT+AW)**4)
C1AWAT=(-2.)*F/(AT+AW)**3)
C1AWAW=C1AWAT

DCC=DQQ*QC*QC
DCAW=DQQ*QAW*QC+DQ*QCAW
DCAT=DQQ*QAT*QC+DQ*QCAT
DAWAW=(DQQ*QAW+DQC1*C1AW)*QAW+(DC1C1*C1AW+DQC1*QAW)*C1AW
1 +DQ*QAWAW+DC1*C1AWAW+F*T/T/((AT+AW)**3)
DAWAT=(DQQ*QAT+DQC1*C1AT)*QAW+(DC1C1*C1AT+DQC1*QAT)*C1AW
1 +DC1*C1AWAT+F*T/T/((AT+AW)**3)
DATAT=(DQQ*QAT+DQC1*C1AT)*QAT+(DC1C1*C1AT+DQC1*QAT)*C1AT
1 +DQ*QATAT+DC1*C1ATAT+F*T/T/((AT+AW)**3)

C
S=TH(I)-D
E=E+S**2

C
EC=EC-2.*S*DC
EAW=EAW-2.*S*DAW
EAT=EAT-2.*S*DAT
ECC=ECC+2.*(DC*DC-S*DCC)
ECAW=ECAW+2.*(DC*DAW-S*DCAW)
ECAT=ECAT+2.*(DC*DAT-S*DCAT)
EAWAW=EAWAW+2.*(DAW*DAW-S*DAWAW)
EAWAT=EAWAT+2.*(DAW*DAT-S*DAWAT)
EATAT=EATAT+2.*(DAT*DAT-S*DATAT)

C
500 CONTINUE
WRITE (6,499) C, AW, AT, E
499 FORMAT(10X,'C=',F12.8,4X,'AW=',F12.8,4X,'AT=',F11.8,8X,'E=',F12.8)

C
EATAW=EAWAT
EATC=ECAT
EAWC=ECAW

C
Z=ECC*(EAWAW*EATAT-EATAW*EAWAT)-EAWC*(ECAW*EATAT-EATAW*ECAT)
1 +EATC*(ECAW*EAWAT-EAWAW*ECAT)

C
DELTC=(EAW*(ECAW*EATAT-EATAW*ECAT)-EC*(EAWAW*EATAT-EATAW*EAWAT)
1 -EAT*(ECAW*EAWAT-EAWAW*ECAT))/Z
DELTAW=(EC*(EAWC*EATAT-EAWAT*EATC)-EAW*(ECC*EATAT-EATC*ECAT)
1 +EAT*(ECC*EAWAT-EAWC*ECAT))/Z
DELTAT=(EAW*(ECC*EATAW-ECAW*EATC)-EC*(EAWC*EATAW-EAWAW*EATC)
1 -EAT*(ECC*EAWAW-ECAW*EAWC))/Z

C
C=C+DELTC
AW=AW+DELTAW
AT=AT+DELTAT

```

```
C      IF (DELTC .GT. 0.001) GO TO 401
      IF (DELTAW .GT. 0.001) GO TO 401
      IF (DELTAT .GT. 0.001) GO TO 401

C      WRITE (6,501) C, AW, AT
501    FORMAT (6X,'FINAL RESULT', C=' ',F12.8,4X,'AW=' ',F12.8,4X,'AT=' ',F11.8)
      STOP
      END
```

APPENDIX D

ONE MASS WITH DAMPING

```

C      DIMENSION TH(50)
C      N=8
C      DO 100 I=1,N
C      READ (5,100) TH(I)
100  FORMAT (F6.0)
C      A=5.0
C      C=30.8
C      F=66.2
C      K=0
C      400 CONTINUE
C      IF (K .GT. 12) STOP
C      E=0.
C      EC=0.
C      EA=0.
C      ECC=0.
C      ECA=0.
C      EAA=0.
C      K=K+1
C      DO 500 I=1,N
C      T=0.1*FLOAT(I-1)
C      EX=EXP((-C)/A*T)
C      D=A*F/C/C*(EX-1.)+F/C*T
C      DC=(-2.)*A*F/(C**3)*(EX-1.)-F*T/C/C*(EX+1.)
C      DA=F/C/C*(EX-1.)+F*T/A/C*EX
C      DCC=6.*A*F/(C**4)*(EX-1.)+2.*F*T/(C**3)*(2.*EX+1.)+T*T*F/A/C/C*EX
C      DCA=(-2.)*F/(C**3)*(EX-1.)-2.*F*T/A/C/C*EX-T*T*F/A/A/C*EX
C      DAA=F*T*T/(A**3)*EX
C      S=TH(I)-D
C      E=E+S**2
C      EC=EC+2.*S*DC
C      EA=EA+2.*S*DA
C      ECC=ECC+2.*(DC*DC-S*DCC)
C      ECA=ECA+2.*(DC*DA-S*DCA)
C      EAA=EAA+2.*(DA*DA-S*DAA)
500  CONTINUE
C      WRITE (6,400) C, A, E
400  FORMAT (10X,'C=',F12.8,4X,'A=',F12.8,4X,'E=',F15.8)
C      EAC=ECA
C      Z=ECC*EAA-ECA*EAC
C      DELTC=(ECC-ECA-EA*EAA)/Z
C      DELTA=(ECC-EAC-EA*ECC)/Z
C      C=C+DELTC
C      A=A+DELTA

```

```
IF (DELTA .GT. 0.001) GO TO 400  
IF (DELTA .GT. 0.001) GO TO 400  
C  
WRITE (6,501) C, A  
501 FORMAT (6X, 'FINAL RESULT; C=', F12.8, 4Y, 'A=', F12.8)  
STOP  
END
```

APPENDIX E

ONE MASS WITHOUT DAMPING

```

      DIMENSION TH(50)
      T(1)=0.1*FLOAT(I-1)
      F=66.2
      N=8
      DEN=0.0
      DO 106 I=1,N
        READ (5,100) TH(I)
100  FORMAT (F5.0)
        THE=TH(I)
        DEN=DEN+THE
        WRITE (6,106) I,THE
106  FORMAT (12X,I3.6X,F8.5)
        NUM=0.0
        DO 700 I=1,N
          NUM=NUM+T(I)*T(I)
700  A=F/2.0*NUM/DEN
          E=0.0
          DO 300 I=1,N
            G=0.5*T(I)*T(I)*F/A
300  E=E+(TH(I)-G)**2
            WRITE (6,701) A,E
701  FORMAT (12X,I3.6X,'A=',F12.7,4X,'E=',F12.6)
          STOP
        END

```

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