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AN APPLICATION OF QUEUING THEORY TO DETERMINE

THE VEHICLE AUTHORIZATION OF A MILITARY UNIT

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AN APPLICATION OF QUEUING THEORY TO DETERMINE
THE VEHICLE AUTHORIZATION OF A MILITARY UNIT

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CHAPTER I

SUMMARY

The proposed method is a new approach to determine the number of vehicles for a military motor pool. It suggests first that there be an appraisal of the average number of requests per unit of time and the average service time of each dispatching, and then a reduction of these two parameters to the requirement of vehicles through the use of a mathematical model. It is believed that the planning staff can make a better appraisal on these parameters than on the number of vehicles required. The answer reached through this method is likely to represent the true transportation requirement of that unit based on standard waiting time.

However, the mathematical model used in this solution is a very theoretical one. The practical case may have some deviations from the model due to various influences. But it will be shown in a later chapter that most of the influences tend to favor the operation of the motor pool. Therefore, if only the satisfactory operation of the motor pool is considered, the number of vehicles determined through the theoretical model can meet the requirement of any influenced case.

Finally, the answer obtained through this method may seem rather high in comparison with the current authorizations of vehicles to the Chinese military units. This is because the answer indicates the true requirement of the unit. For many years, the transportation facilities

assigned to a Chinese military unit have been determined mainly by what is available rather than what is required. This unsatisfactory planning procedure can surely be improved by adoption of this proposed method.

CHAPTER II

INTRODUCTION

The purpose of this research is to provide a new approach for the military planning staff in order to determine the optimum number of vehicles assigned to the motor pool of a military unit. Motor pool service is almost a universally adopted system for military organizations to provide transportation to its officers and men for a field trip of duty. But the adequate number of vehicles for a particular unit is always a subject of argument.

The current practice in determining the vehicle authorization for a unit is based mainly on a rough evaluation of its organizational level and/or its size. This has long been a complaint of the planning staff involved because too many subjective factors are involved. In many cases, even personal relations produce an effect on the determination. However, there have not been any better methods proposed heretofore.

Waiting line theory, or queuing theory as it is called by the British, is an operations research method which can be used to determine the number of facilities a service station should have so as to meet a specified demand for service. This theory is hereby introduced as a solution to the motor pool problem.

The case of a military motor pool is compared to a queuing problem as follows: the motor pool itself can be seen as a service station, and the vehicles in the pool are servers. When a request for transportation comes to the motor pool, a server is assigned to render the service. If there is no empty server, then the request must go to the queue until a server is available.

This represents a several-server queuing model. Based on the assumption that the request arrivals follow a Poisson distribution and the service times follow a negative exponential distribution, the mathematical relations between the number of servers in the station and various queuing properties have been readily derived.

From the military point of view, people pay attention only to the waiting time, so that the activities of a unit will not be adversely affected; the adequacy of the number of servers in the station is thus measured by the average waiting time. However, due to the complicated mathematical expression for the average waiting time of the model and various values of the rate of request arrivals, λ, and the rate of service accomplished, μ, involved in practical application, charts are prepared for use so that the correct number of vehicles resulting in standard waiting time for a motor pool can be determined as soon as the values of λ and μ for that unit are agreed.

CHAPTER III

LITERATURE SURVEY

Queuing theory was developed in order to provide models to predict the behavior of a system that attempted to provide services for randomly arising demands. However, the early phase of this theory was to study problems of telephone-traffic congestion. The pioneer investigator was A. K. Erlang,[1] who, in 1908, published Use of Waiting Line Theory in the Danish Telephone System. In 1930, Pollaczek[2] studied the Poisson-input, arbitrary-holding-time, single-channel case. In 1934, he also tried to study the Poisson-input, arbitrary-holding-time, multiple-channel problem.

In recent years, this theory has gradually become a very useful operations research technique and it has been widely used in almost any problem where there are elements of input and elements of output involving one or more points of service.

In 1956, Russell L. Ackoff[3] published a paper "Waiting Line Theory as a Management Tool," which descriptively introduces the application of this theory to management problems. He explained that a waiting line problem involves changing the behavior of the arriving units, or the service facilities, or both. To effect these changes requires manipulation or control of such factors as rate of arrival, order of service, number of service facilities, and so forth.

He further explained that there were two methods of approach to the solution of a waiting line problem: first, the mathematical approach, and second, the simulated sampling method. This statement is still applicable.

In 1957, Thomas L. Saaty[4] collected all the well-developed mathematical expressions for various queuing models and edited them into Resume of Useful Formulas in Queuing Theory. In this paper, many of the formulas are for single-channel models with various input distributions, holding-time distributions and queuing disciplines, and the formulas concerning multiple-channel models are limited to the following four cases:

a. Poisson input, identical exponential holding times,

b. Poisson input, identical constant holding times,

c. (Priority discipline) different Poisson inputs, a finite number of priorities with the same exponential holding time,

d. (Limited source) exponential holding time.

A later operations research textbook, Introduction to Operations Research (1967) by F. S. Hillier and G. J. Lieberman,[5] states that,

> Multiple-server models are inherently more difficult to analyze than single-server models. Therefore when s (number of channels) > 1, it is almost impossible to obtain useful results, except when the service time distribution is exponential so that the birth-death process is applicable. Certain limited results have been obtained for the constant service times, but the equations involved are too complicated for routine computation.

CHAPTER IV

STATEMENT OF THE PROBLEM

In a military unit, transportation equipment for personnel is always important because it has great effect on staff activities and in turn on the overall combat effectiveness of the unit. For thousands of years, horses had played a very important role in military transportation. But, since World War I, things have changed rapidly and horses in military organizations were replaced by motorized vehicles. Today, new draftees to any branches of an armed force seldom see any horses and the term "cavalry" has long been deleted from the order of combat.

Motor vehicles are indeed indispensable equipment in military units in these days. Even the troops of the most backward countries are equipped with this convenience. However, in most of the Far Eastern countries, wherein their industries are not capable to self-support their troops' requirements, the availability of the vehicles for their military use is subject to many limiting factors such as the financial condition of the government military budget, and source of foreign aid, etc. As the number of this equipment is likely to be limited, military people naturally seek methods to make more effective use of their vehicles in order to meet their transportation requirements.

The concept of a motor pool system is so designed so as to achieve this goal. At present, the military organizations of almost all countries over the world have adopted this system. Under this system, all vehicles for official use* in one unit are pooled together in one station. Officers and enlisted men of the unit may request vehicles when there is an official need.

So far as efficiency is concerned, this system is the most ideal one at this time. But, this system still presents a planning problem to the less industrialized countries such as Thailand, Republic of Korea, and Republic of China in Taiwan--in these countries, because most of their people cannot afford to have his own car. The dependence on government vehicles as a means of transportation is rather vital. This is unlike the case in the well-developed countries where nearly every officer or enlisted man has his own car which can be flexibly used to alleviate any overloaded condition of the motor pool. Hence, an authorization of vehicles in the motor pool of a military unit in these less industrialized countries is more important because when the authorization is too low, it will put adverse effects on its military activities; while it is too high, it will cause unnecessary waste of national resources.

It is noted that the staff work of determining the vehicle authorization for a unit has experienced many changes in the past decades. In the Republic of China, the early phase of this staff work was to determine the vehicle authorization based almost entirely on the organizational level of that unit. In the Air Force, for example, the planning staff merely

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* It is only vehicles for personnel transportation that are to be pooled in the station. These are all jeeps. Staff cars, if there are any, are only for commanding officers' use. The special purpose vehicles such as fire trucks, tractors, and heavy engineering equipment, etc., present no need for pooling.

established the standard as how many vehicles should be assigned for a wing, a group, or a squadron. Other operational units were authorized with a number of vehicles in reference to the established standard for that level.

As time went on, it was experienced that units on the same organizational level did not necessarily represent the same requirement for transportation. A base service squadron might have a strength several times that of a combat squadron, their requirement of transportation apparently would not be the same. Therefore, at a later date, factors such as the total strength of a unit, the nature of its function, its relationship with friendly forces, etc., were all taken into consideration. However, due to the variability of the effect of these factors, the weights of these factors on the vehicle requirement cannot be specifically determined. Hence, although the proposed factors for consideration seemed to be numerous, the equipment assigned to a unit was still determined by the judgment of the planning staff.

Recently, some units stated that they wished to have their vehicle authorization increased based on geographical considerations. The argument was that the geographical condition of the stationed base does have significant effect on the service time of each dispatch. In comparing two units of approximately the same size and similar function, one is likely to draw a conclusion that the two units would have the same requirements for transportation service. But, this only means the same number of requests. If one unit station is in a compact base, while the other station is in a vast area with dispersed installations, the difference in the transportation equipment required is apparent.

The aforementioned facts clearly indicate that the organization and logistic planning personnel are eagerly desirous of a fair and objective measure, with all factors properly considered, in order to determine the proper number of military vehicles for a motor pool. The purpose of this study is to apply queuing theory in order to solve this problem. A new method is suggested for military planning people in order to determine the optimum number of vehicles for a motor pool.

CHAPTER V

PROPOSED METHOD OF SOLUTION

With reference to the above-stated problem in determining the optimum number of vehicles for a military unit, the traditional practice seemed too arbitrary and thus unable to give a satisfactory solution. As more and more factors are suggested for consideration, the problem tends to become even more complicated.

The method of solving this problem, as proposed in this study, is to reduce all these complicated factors, no matter whether they are implicit or explicit, into two parameters: (1) the average number of requests per unit of time, (2) the average number of services which can be completed by one server per unit of time. A queuing model is then introduced to determine the number of vehicles required under this condition. This model is based on an established limit of average waiting time per request.

The reason for reducing all factors into two parameters is not only because these two parameters are required in the queuing model to be used in our problem, but also because the various factors for consideration can generally be sorted into two categories: (1) those factors affecting the rate of request, and (2) those factors affecting service time. When the factors of each category are considered and evaluated separately, the planners are able to make better estimations of each parameter than by making direct estimation of the number of vehicles required. As this is true, the solution obtained from the two better estimated parameters through a mathematical analysis is certainly more reasonable.

Before dealing with the mathematical model to be used in this problem, two important assumptions must be examined in order that the motor pool problem can be approximated by this model. One of the assumptions is that the requests coming to a motor pool follow the Poisson distribution, and the other is that the service times follow the negative exponential distribution. Of course, a realistic situation of the requests coming to a motor pool cannot be expected to follow exactly any theoretical distribution, and the actual distribution will also vary from unit to unit and from time to time. However, due to the following reasoning, the Poisson distribution of request arrivals can be approximated without significant error. This approximation is necessary since this study is intended to suggest a method of solution for cases in general.

A review of the nature of the requests coming to a motor pool shows that it generally agrees with the following conditions:

1. The number of requests coming to a motor pool during a non-overlapping time period can be said to be independent and random. This is particularly true when the requests are coming from a large number of staff personnel.

2. The number of requests during any time interval depends only on the length of that interval and not on the endpoints.

3. If the time interval is sufficiently small, the probability that a request comes during that interval is directly proportional to

the length of that interval.

4. If the time interval is sufficiently small, the probability of two or more requests is negligible.

5. There is no request at the starting hours of the motor pool.

These analyses are sufficient to support the assumption that the request coming to a motor pool tends to follow the Poisson distribution. (The derivation of the Poisson distribution based on the above five conditions can be found in Introductory Probability and Statistical Applications by Paul L. Meyer,[6] p. 152.)

One next examines the service times. Because each trip of a dispatch is independent from the other, the return time of the vehicles can also be assumed as Poisson. This is based on a similar reasoning to that for the request arrivals. Hence, the distribution of the service time can be approximated as a negative exponential. (The relation between the Poisson process and the exponential service time can be found in Introductory Probability and Statistical Applications by Paul L. Meyer,[6] p. 214.)

Since it is permissible to assume the distribution of the requests coming to a motor pool as Poisson and the distribution of service times as negative exponential, one is now ready to discuss the queuing model which will be used in the solution to the stated problem. The following is a brief description of the several-servers queuing model.

Suppose there are k servers in the service station, each server with an exponential service rate of mean μ (μ services completed per unit of time), fed by a queue built up to Poisson arrivals with mean λ (λ requests for service per unit of time). The queuing discipline is that first come, first served, with one request going into service at the moment a server becomes empty. Then, various queuing properties can be expressed in terms of λ, μ, k, and P_0, the probability that no service in the system. P_0 is actually a function of λ, μ, and k. The expression for P_0 in this model is

$$P_0 = \frac{1}{\left[\sum_{n=0}^{k-1} \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n\right] + \frac{1}{k!}\left(\frac{\lambda}{\mu}\right)^k \frac{k\mu}{k\mu-\lambda}} \tag{1}$$

subject to $k\mu > \lambda$. (If $k\mu \leqq \lambda$, the queue will never become stable.)

Among all the queuing properties, the most important one in the problem is the average waiting time of a request because this is a direct indication of the adequacy of a motor pool's service. The equation for the calculation of the average waiting time is

Average waiting time of a request $$E(w) = \frac{\mu\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!\ (k\mu-\lambda)^2} \tag{2}$$

By applying these two equations, for any given combination of λ and μ, the E(w) corresponding to different values of k can be calculated. Hence, the adequacy of a motor pool's service with k vehicles assigned under the specified condition of λ and μ can be measured. After the E(w)'s for various values of k are obtained, their relations can be plotted into a set of charts for use.

In applying these charts to determine the number of vehicles for a motor pool, the military planning staff can first estimate the λ and μ of a unit to be established or evaluate the λ and μ of a unit which already exists, and then determine the correct value of k from the charts based on an approved average waiting time standard.

The other queuing properties of this model are

Average queuing length
$$E(m) = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!\ (k\mu-\lambda)^2} \cdot P_0 \tag{3}$$

Average number of units in system
$$E(n) = \frac{\lambda\mu\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!\ (k\mu-\lambda)^2} \cdot P_0 + \frac{\lambda}{\mu} \tag{4}$$

Average time in arrival spending in system
$$E(v) = \frac{\mu\left(\frac{\lambda}{\mu}\right)^k}{(k-1)!\ (k\mu-\lambda)^2} \cdot P_0 + \frac{1}{\mu} \tag{5}$$

The three properties are not very important in this problem. Therefore, there will be no calculations about these properties in this study. However, they can be calculated without difficulty if the P_0 values for each specific combination of λ and μ are known. For this reason, the P_0 values for various combinations of λ and μ are given in the tables in the next chapter for use when necessary.

CHAPTER VI

NUMERICAL CALCULATION AND CHART PLOTTING

One objective of this thesis is to develop a new method for military planning people to use in determining the number of vehicles for various sized motor pools. The data calculated are to cover a range of λ and μ values.

For practical application to those military units in Far Eastern countries, four charts, each with a mean service time of 30 minutes (μ = 2 services per hour), 40 minutes (μ = 1.5 services per hour), 50 minutes (μ = 1.2 services per hour), and 60 minutes (μ = 1 service per hour), are plotted. The number of requests ranges from λ = 2 requests per hour to λ = 20 requests per hour. P_0, the probability that there are no units in the system, for different combinations of λ and μ, is also shown in a column of the table. Because λ and μ expressed in hours E(w) obtained in equation (2) is also expressed in hours. However, for convenience of planning work, E(w) shown in the following tables has been converted into minutes.

Table 1. E(w) vs. k for λ = 2 to 20, μ = 2

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 2$	k = 2	5	12.0
	k = 3	2.75	1.36
$\lambda = 3$	k = 2	6.9	38.20
	k = 3	4.75	4.74
	k = 4	4.52	0.89
$\lambda = 4$	k = 3	9	13.33
	k = 4	7.67	2.61
	k = 5	7.45	0.59
$\lambda = 5$	k = 3	22.25	42.20
	k = 4	13.57	6.40
	k = 5	12.48	1.56
$\lambda = 6$	k = 4	26.5	15.30
	k = 5	21.44	3.54
	k = 6	20.44	0.94
$\lambda = 7$	k = 4	67.77	44.30
	k = 5	38.57	7.54
	k = 6	34.50	2.13
$\lambda = 8$	k = 5	76.96	16.60
	k = 6	59.91	4.28
	k = 7	56.13	1.35
$\lambda = 9$	k = 5	201.6	45.7
	k = 6	121.0	8.44
	k = 7	95.55	2.60
$\lambda = 10$	k = 6	221.6	17.8
	k = 7	167.3	4.83
	k = 8	155.4	1.66
$\lambda = 11$	k = 6	579.4	46.6
	k = 7	308.8	9.13
	k = 8	264.6	3.10
	k = 9	251.4	1.11

Table 1. (Concluded)

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 12$	k = 7	633.1	18.6
	k = 8	487.6	5.75
	k = 9	445.1	1.87
$\lambda = 13$	k = 7	1,711	47.7
	k = 8	868	9.7
	k = 9	732	3.37
	k = 10	686	1.32
$\lambda = 14$	k = 8	1,796	19.0
	k = 9	1,299	5.78
	k = 10	1,269	2.04
$\lambda = 15$	k = 8	4,849	49.0
	k = 9	2,362	10.5
	k = 10	1,951	3.82
	k = 11	1,817	1.57
$\lambda = 16$	k = 9	5,096	19.6
	k = 10	3,613	6.13
	k = 11	3,220	2.47
$\lambda = 17$	k = 9	14,060	49.4
	k = 10	6,829	10.6
	k = 11	5,591	3.95
	k = 12	5,190	1.68
$\lambda = 18$	k = 10	14,366	20.0
	k = 11	10,040	6.48
	k = 12	8,866	2.66
$\lambda = 19$	k = 10	39,963	49.5
	k = 11	19,070	10.95
	k = 12	15,465	4.2
	k = 13	14,233	1.84
$\lambda = 20$	k = 11	40,391	20.4
	k = 12	26,845	7.0
	k = 13	24,393	2.86

Table 2. E(w) vs. k for λ = 2 to 20, μ = 1.5

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 2$	k = 2	5	32.0
	k = 3	3.93	4.27
	k = 4	3.54	0.83
$\lambda = 3$	k = 3	9	17.8
	k = 4	7.66	3.5
	k = 5	7.44	0.81
$\lambda = 4$	k = 3	35.66	95.8
	k = 4	16.70	11.34
	k = 5	14.90	2.78
$\lambda = 5$	k = 4	46.90	39.5
	k = 5	31.49	7.84
	k = 6	28.90	2.22
$\lambda = 6$	k = 5	77.0	22.2
	k = 6	59.9	5.7
	k = 7	56.1	1.8
$\lambda = 7$	k = 5	329.2	101.0
	k = 6	136.3	14.2
	k = 7	114.7	4.3
	k = 8	110	1.57
$\lambda = 8$	k = 6	433	44.1
	k = 7	252.8	9.7
	k = 8	223.9	3.3
	k = 9	193	1.03
$\lambda = 9$	k = 7	640	23.8
	k = 8	466	7.14
	k = 9	425	2.6
	k = 10	411	1.01
$\lambda = 10$	k = 7	2,833	103.0
	k = 8	1,085	16.0
	k = 9	882	5.36
	k = 10	821	2.10

Table 2. (Continued)

Requests/hr	Number of Servers	$1/P_0$	E(w)
λ = 11	k = 8	3,324	45.0
	k = 9	1,961	11.2
	k = 10	1,682	4.12
	k = 11	1,588	1.43
λ = 12	k = 9	5,086	26.1
	k = 10	3,611	8.2
	k = 11	3,220	3.26
	k = 12	3,076	1.4
λ = 13	k = 9	23,164	105
	k = 10	8,549	17.3
	k = 11	7,323	7.04
	k = 12	6,192	2.6
λ = 14	k = 10	28,847	46.3
	k = 11	15,299	12.2
	k = 12	12,801	4.8
	k = 13	11,933	2.12
λ = 15	k = 11	38,407	28.6
	k = 12	25,723	9.7
	k = 13	22,409	4.15
	k = 14	21,075	1.91
λ = 16	k = 12	67,304	18.1
	k = 13	51,781	6.87
	k = 14	46,691	3.6
	k = 15	44,598	1.44
λ = 17	k = 12	233,810	43.3
	k = 13	118,152	12.9
	k = 14	97,323	5.35
	k = 15	89,648	2.49
λ = 18	k = 13	317,230	28.2
	k = 14	222,984	9.18
	k = 15	184,634	4.27
	k = 16	172,750	2.04
λ = 19	k = 14	523,232	19.0
	k = 15	394,605	7.4
	k = 16	351,929	3.44
	k = 17	333,515	1.7

Table 2. (Concluded)

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 20$	k = 14	1,781,000	45.5
	k = 15	910,000	13.6
	k = 16	738,000	5.8
	k = 17	673,000	2.8

Table 3. E(w) vs. k for λ = 2 to 20, μ = 1.2

Requests/hr	Number of Servers	$1/P_0$	E(w)
λ = 2	k = 2	11	113.5
	k = 3	5.79	11.25
	k = 4	5.38	2.2
λ = 3	k = 3	19.22	81.2
	k = 4	10.97	14.3
	k = 5	9.88	3.3
λ = 4	k = 4	46.66	49.2
	k = 5	31.46	9.8
	k = 6	28.9	2.77
λ = 5	k = 5	101.21	37.3
	k = 6	72.67	8.9
	k = 7	66.79	2.8
λ = 6	k = 6	221.6	29.4
	k = 7	167.4	8.1
	k = 8	154.1	2.79
λ = 7	k = 7	489	24.0
	k = 8	384.2	7.36
	k = 9	356.2	2.72
λ = 8	k = 8	1,089	20.0
	k = 9	882	6.7
	k = 10	821	2.62
λ = 9	k = 9	2,437	17.0
	k = 10	2,024	6.13
	k = 11	1,891	2.5
λ = 10	k = 9	9,472	58.0
	k = 10	5,476	14.6
	k = 11	4,646	5.68
	k = 12	4,353	2.4
λ = 11	k = 10	19,262	43.1
	k = 11	12,333	11.1
	k = 12	10,638	5.15
	k = 13	10,018	2.29

Table 3. (Continued)

Requests/hr	Number of Servers	$1/P_0$	E(w)
λ = 12	k = 11	40,385	33.0
	k = 12	27,845	11.2
	k = 13	24,393	4.7
	k = 14	23,049	2.18
λ = 13	k = 12	86,463	27.9
	k = 13	63,048	9.99
	k = 14	55,893	4.39
	k = 15	53,030	2.07
λ = 14	k = 13	187,700	23.2
	k = 14	143,000	8.9
	k = 15	128,100	4.07
	k = 16	121,900	1.97
λ = 15	k = 13	822,200	83.0
	k = 14	413,700	19.6
	k = 15	339,400	8.5
	k = 16	293,700	3.78
	k = 17	280,400	1.87
λ = 16	k = 14	1,681,000	62.5
	k = 15	909,500	17
	k = 16	738,700	7.25
	k = 17	673,900	3.56
λ = 17	k = 15	3,325,000	45.8
	k = 16	2,253,000	16.05
	k = 17	1,682,000	6.6
	k = 18	1,545,000	3.27
λ = 18	k = 16	5,250,000	36.8
	k = 17	2,355,000	13.5
	k = 18	1,384,000	6.02
	k = 19	815,000	2.92
λ = 19	k = 17	14,490,000	29.9
	k = 18	10,215,000	11.6
	k = 19	8,685,000	5.45
	k = 20	8,129,000	2.71

Table 3. (Concluded)

Requests/hr	Number of Servers	$1/P_0$	E(w)
λ = 20	k = 17	93,889,000	120
	k = 18	37,560,000	20.7
	k = 19	23,167,000	9.98
	k = 20	20,485,000	4.95
	k = 21	19,179,000	2.6

Table 4. E(w) vs. k for λ = 2 to 20, μ = 1

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 2$	k = 3	9	26.7
	k = 4	7.67	5.2
	k = 5	7.45	1.2
$\lambda = 3$	k = 4	26.5	30.6
	k = 5	21.44	7.0
	k = 6	20.43	1.96
$\lambda = 4$	k = 5	76.8	33.2
	k = 6	59.9	8.57
	k = 7	56.1	2.8
$\lambda = 5$	k = 6	221.6	35.3
	k = 7	167.3	9.77
	k = 8	154.4	3.3
	k = 9	150.4	1.2
$\lambda = 6$	k = 7	655.5	36.4
	k = 8	489.2	10.2
	k = 9	447.6	3.72
	k = 10	433.8	1.44
$\lambda = 7$	k = 8	1,798	38.1
	k = 9	1,299	11.5
	k = 10	1,170	4.44
	k = 11	1,124	1.8
$\lambda = 8$	k = 9	5,091	39.2
	k = 10	3,616	11.9
	k = 11	3,222	4.9
	k = 12	3,076	2.1
$\lambda = 9$	k = 10	13,399	39.8
	k = 11	10,040	12.9
	k = 12	8,857	5.32
	k = 13	8,423	2.36
	k = 14	8,238	1.07

Table 4. (Continued)

Requests/hr	Number of Servers	$1/P_0$	E(w)
$\lambda = 10$	k = 11	40,385	41.0
	k = 12	27,867	13.5
	k = 13	24,383	5.71
	k = 14	23,049	2.6
$\lambda = 11$	k = 12	113,380	41.6
	k = 13	77,230	14
	k = 14	67,073	6.05
	k = 15	63,099	2.84
	k = 16	61,362	1.38
$\lambda = 12$	k = 13	317,300	42.3
	k = 14	214,100	14.4
	k = 15	184,700	6.38
	k = 16	172,900	3.07
	k = 17	167,600	1.52
$\lambda = 13$	k = 14	884,900	42.9
	k = 15	591,600	14.8
	k = 16	507,300	6.7
	k = 17	472,600	3.28
	k = 18	457,100	1.66
$\lambda = 14$	k = 15	2.468×10^6	43.4
	k = 16	1.637×10^6	15.3
	k = 17	1.395×10^6	6.96
	k = 18	1.294×10^6	3.47
	k = 19	1.248×10^6	1.79
$\lambda = 15$	k = 16	6.877×10^6	43.7
	k = 17	4.526×10^6	15.6
	k = 18	3.832×10^6	7.22
	k = 19	3.546×10^6	3.67
	k = 20	3.407×10^6	1.92
$\lambda = 16$	k = 17	18.94×10^6	44.5
	k = 18	12.3×10^6	16.1
	k = 19	10.44×10^6	7.51
	k = 20	9.51×10^6	3.91
	k = 21	9.114×10^6	2.10

Table 4. (Concluded)

Requests/hr	Number of Servers	$1/P_0$	E(w)
λ = 17	k = 18	53.22 x 10^6	44.7
	k = 19	34.45 x 10^6	16.3
	k = 20	28.8 x 10^6	7.72
	k = 21	26.55 x 10^6	3.96
	k = 22	25.45 x 10^6	2.16
λ = 18	k = 19	147.4 x 10^6	45.0
	k = 20	95.11 x 10^6	16.5
	k = 21	79.37 x 10^6	7.9
	k = 22	72.66 x 10^6	4.17
	k = 23	69.35 x 10^6	2.28
λ = 19	k = 20	408.8 x 10^6	45.4
	k = 21	261.9 x 10^6	16.8
	k = 22	218.0 x 10^6	8.12
	k = 23	198.8 x 10^6	4.32
	k = 24	189.3 x 10^6	2.4
λ = 20	k = 21	1,133 x 10^6	45.6
	k = 22	722.5 x 10^6	17.05
	k = 23	587.5 x 10^6	8.47
	k = 24	543.9 x 10^6	4.48
	k = 25	517.1 x 10^6	2.51

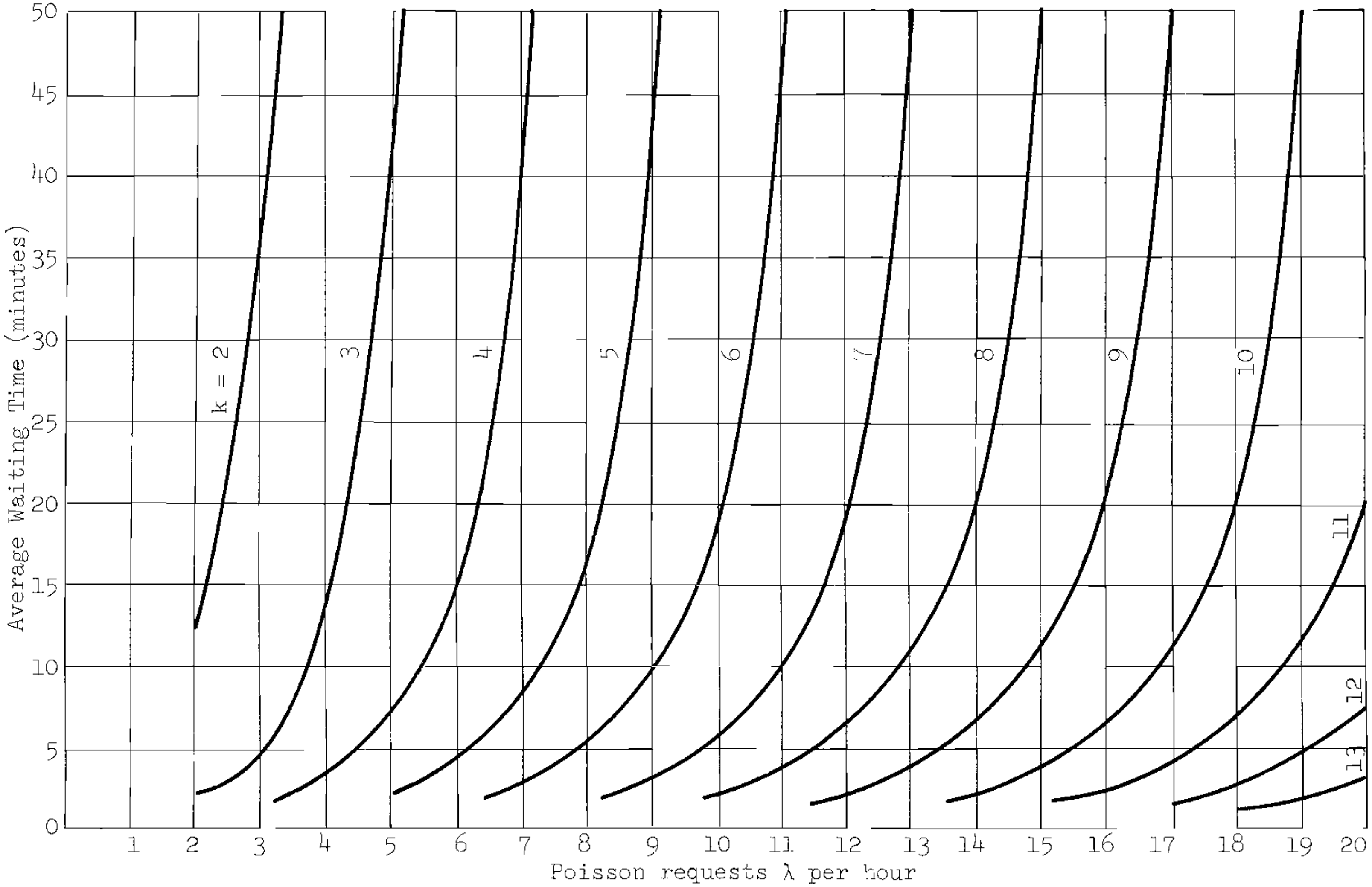

Figure 1. Average Waiting Time vs. Number of Vehicles, $\mu = 2$

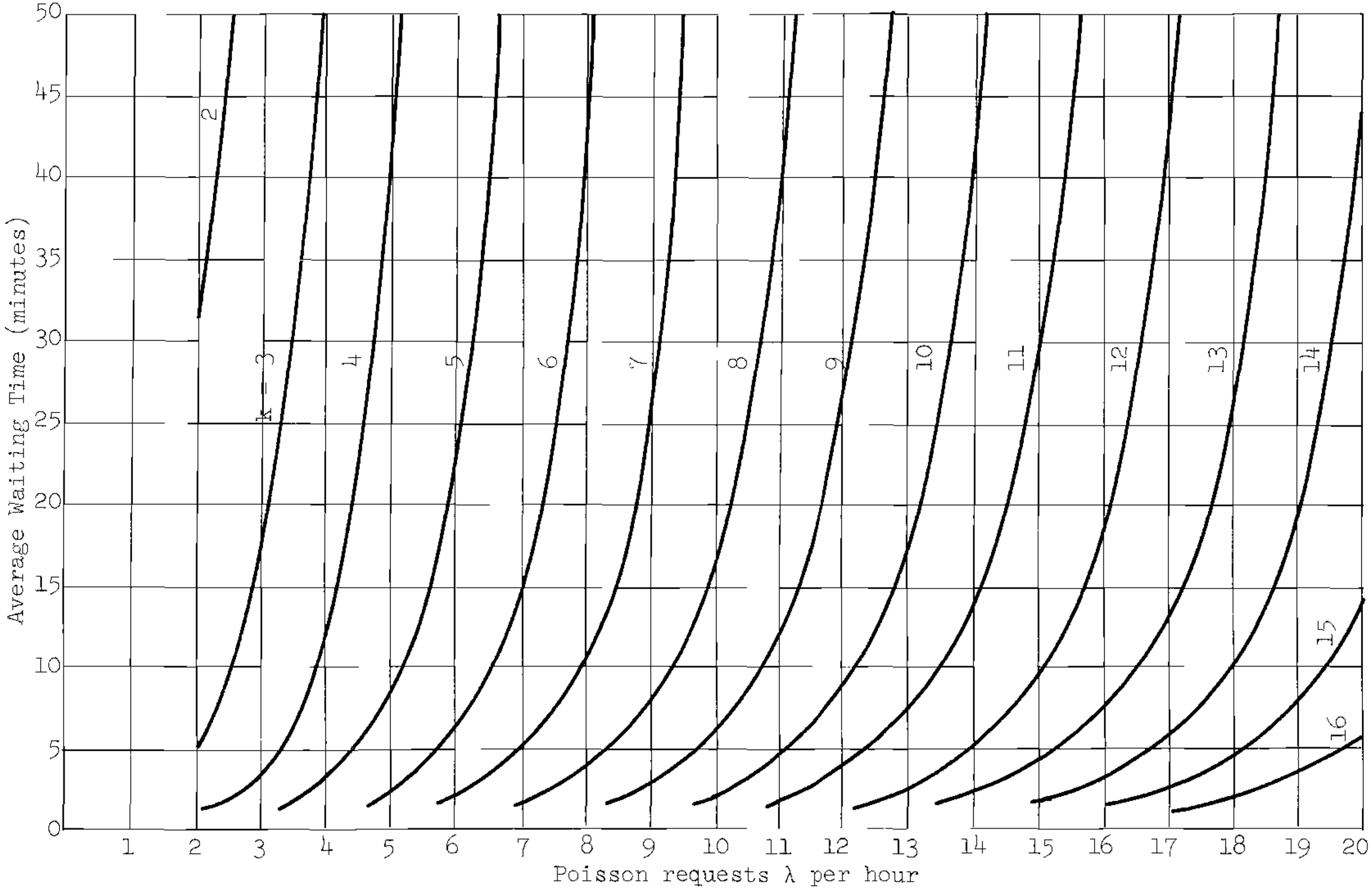

Figure 2. Average Waiting Time vs. Number of Vehicles, $\mu = 1.5$

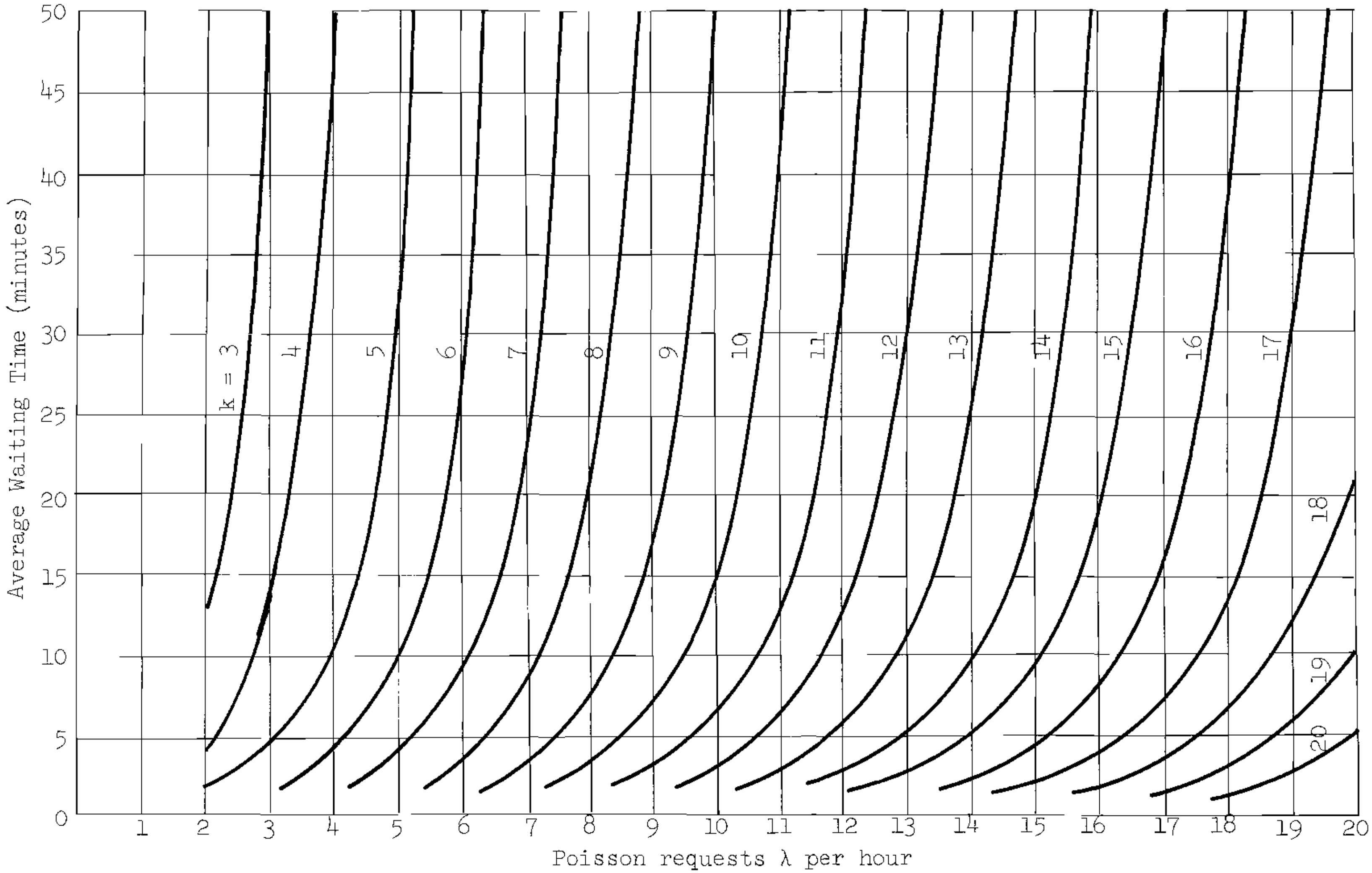

Figure 3. Average Waiting Time vs. Number of Vehicles, $\mu = 1.2$

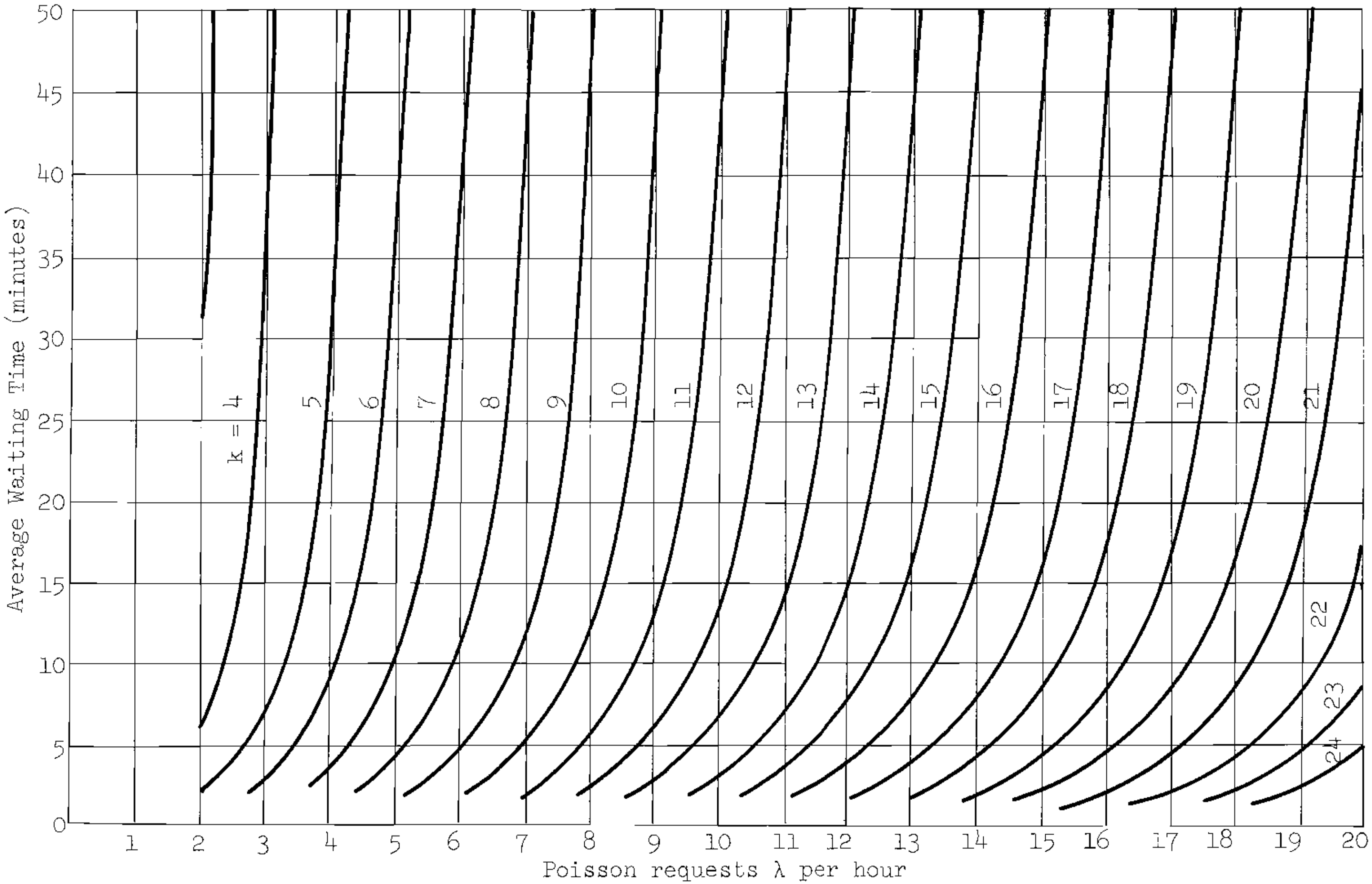

Figure 4. Average Waiting Time vs. Number of Vehicles, $\mu = 1$

CHAPTER VII

DISCUSSION OF RESULTS

Analysis of Charts

In examining the charts plotted in the last chapter, one finds that each chart contains a family of monotonically increasing curves. Though the curve plotted begins from $\lambda = 2$, it is apparent that regardless of the value of μ, when λ approaches zero, the corresponding $E(w)$ approaches zero, also. This is true because when no one asks for transportation, the average waiting time must be zero. As the value of λ increases, the curve representing each value of k has a vertical asymptote when $\lambda = k\mu$. The $E(w)$ corresponding to that value of k tends to go to positive infinity when λ approaches $k\mu$. This is to say that if $\lambda \geqq k\mu$, the queue will build up to an infinite length and the stable condition of the queue can never be achieved.

Next, one examines the slopes of the curves. As one can see in the charts, the slopes of all curves at the higher values of $E(w)$, say 50 minutes or above in the range of μ and λ, become very steep so that this part of the curve is less important. This is because, first, any average waiting time above 50 minutes will be considered as unsatisfactory for a motor pool service, and second, the high slope of this part of the curve means the high sensitivity of $E(w)$ response to the change in k. Hence, we would not consider the k values corresponding to this range of $E(w)$ where a unit increment in k will gain tremendous reduction in the average waiting time.

Neither is one interested in that part of the curve where $E(w)$ is below five minutes, because this represents the inert response of $E(w)$ to the change of k values. It is noted that once k has increased to a value such that $E(w)$ falls into this range, any further addition in k will give only a small reduction in $E(w)$, and thus it is practically unimportant. The increase in the number of vehicles under this condition is likely to increase the idle time of the facilities with little contribution to reduce the average waiting time.

Hence, the above analysis leads one's attention to the $E(w)$ range from five minutes to 50 minutes, where the curves tend to show moderate change in $E(w)$ against k. This range of $E(w)$ is particularly important because it virtually covers most of the acceptable average waiting times for various motor pools. However, because the restriction that the number of vehicles k must be an integer, it is impossible to apply an exact value of $E(w)$ to determine k for all units. One can see in the charts that there are no more than three or four values of k which have $E(w)$ within this range, therefore the planning staff's choice is only those values of k with corresponding $E(w)$ values which are near to the established standard values, rather than equal to them.

One other point that must be mentioned here pertinent to the charts is that they are plotted only in ten-minute intervals of the average service times. In actual applications, it is possible that the estimated average service time for a motor pool is not a value suitable for the available charts. Therefore, one must consider the cases where the estimated μ is an intermediate value.

To deal with these cases, the method of interpolation is to be used. It is suggested that the two k values corresponding to the two neighbor values of μ, indicated on the charts plotted, be determined first, then the correct value of k corresponding to the intermediate value of μ be calculated through interpolation and rounded into integer number. This is not to say that the change in k is directly in proportion to the change in μ, but the non-linearity would be well covered by the integer-rounding procedure.

An example is given below to show the proper way to apply the charts to the problem of determining the vehicle requirement of a motor pool. Suppose that after deliberate evaluation, investigation, and coordination, the planning staff has decided that the estimated average service time of the subject motor pool is 43 minutes, and the average number of requests for a dispatching is 17 per hour. The approved average waiting time standard has been established to be no greater than 20 minutes. Then the procedure of determining the correct value of k is to find first the minimum allowed k values along the $\lambda = 17$ coordinates in both charts of $\mu = 1.5$ (average service time is 40 minutes) and $\mu = 1.2$ (average service time is 50 minutes) such that the corresponding E(w)'s are below the 20 minute limit.

From the charts

for $\mu = 1.5, \lambda = 17;$ k minimum is 13 with $E(w) = 12.9 < 20$

$\mu = 1.2, \lambda = 17;$ k minimum is 16 with $E(w) = 16.6 < 20$

The k value for the motor pool with an average service time = 43 minutes and $\lambda = 17$ can thus be obtained

$$k = 13 + \frac{43 - 40}{50 - 40} \times (16 - 13) = 13 + .9 = 14$$

Hence, k = 14 is the proper number of vehicles required in that motor pool to provide the specified service.

Human Effect on the Distribution of λ

The Poisson distribution of the requests received by a motor pool is based on the assumption that the requests are from a large number of individuals and are completely independent. But in most realistic conditions, the human effect tends to influence this assumption so that the actual distribution of the requests coming to a motor pool varies from the ideal Poisson pattern. This will surely cause certain differences between the actual k - E(w) relations and the k - E(w) relations derived from our model.

Since one is dealing with a problem concerning military motor pools, it would be meaningful to make a discussion on the nature of the human effect on the request arrival distribution in military units. Experience shows that the human effect in military units is usually presented in the form of coordination--the coordination among those people who are going to request transportation. Although the degree of coordination which can be achieved is different from unit to unit, it can be imagined that the coordination which can be achieved is different from unit to unit, it can also be imagined that the coordination is intended to keep the request arrivals at its average rate. This is true because in no cases would people purposely coordinate to send "salvo" requests to the motor pool and waste their time in waiting.

Based on this understanding, it is reasonable to assume that the effort of coordination is to equalize the number of requests for each period of time. Hence, the effect is simply to reduce the variance of the distribution of λ. When there is no coordination, the distribution of request arrivals will follow a Poisson pattern, but when the coordination goes tighter, the distribution will "narrow" up and the probability density will become higher and higher in the vicinity of λ. The extreme condition of the coordination would be the request arrivals becoming equi-interval of λ per unit of time. The tendency of the human effect on the distribution of the requests coming to a military motor pool is shown in Figure 5.

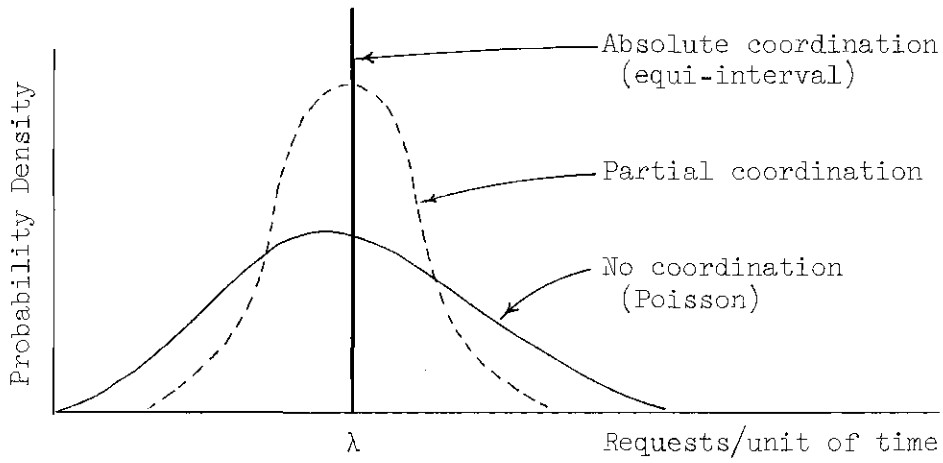

Figure 5. Non-scaled Graph Showing the Tendency of Human Effect on the Distribution of the Requests

In practical cases, a military unit can neither have no coordination nor achieve absolute coordination. A situation which exists between the two extreme conditions would be the result of the human effect. The values of the E(w)'s under various degrees of coordination will also fall between the two extremities.

The E(w) for the Poisson condition can be readily obtained through the model developed in this thesis. The remaining values of E(w) are the equi-interval conditions. There is no easy general formula to obtain the E(w) under the condition of equi-intervaled request arrivals. The Monte Carlo simulation is the only handy method to obtain these values. Unfortunately, this method can only obtain one answer for each time for a particular combination of μ, λ, nad k. Therefore, it is very difficult to prove, on a general basis, that the E(w) values under equi-interval conditions are less than those under Poisson conditions. Table 5 is a Monte Carlo solution of E(w) under equi-interval conditions for a random selected case $\mu = 2$, $\lambda = 10$, and $k = 6$. One can see that it is substantially less than the E(w) under Poisson conditions. Similar results can be obtained for any other combination of μ, λ, and k.

In Table 5, the first 20 requests are used for building up the queue and are not included in the computation. The average waiting time of the last 30 requests is

$$\frac{186}{30} = 6.2 \text{ minutes}$$

Compare this value with the E(w) value obtained through the model developed in this thesis (17.8 minutes). The E(w) value under the equi-

Table 5. Monte Carlo Solution of E(w) for μ = 2 (exponential service time), λ = 10 (equi-interval), and k = 6

Request No.	Time of Arrival	Time Getting a Jeep	Jeep No.	Trip Time	Return Time	Waiting Time
1	6	6	1	23	29	
2	12	12	2	26	38	
3	18	18	3	33	51	
4	24	24	4	13	37	
5	30	30	5	34	64	
6	36	36	6	53	89	
7	42	42	1	8	50	
8	48	48	2	1	54	
9	54	54	3	19	73	
10	60	60	4	21	81	
11	66	66	5	7	73	
12	72	72	1	6	78	
13	78	78	2	78	156	
14	84	84	3	158	242	
15	90	90	4	26	116	
16	96	96	5	62	156	
17	102	102	1	1	103	
18	108	108	6	54	162	
19	114	114	1	24	140	
20	120	116	4	47	163	
21	126*	140	1	25	165	14
22	132*	156	2	10	166	24
23	138*	156	5	1	157	16
24	144*	157	5	8	165	13
25	150*	162	6	19	181	12

Table 5. (Concluded)

Request No.	Time of Arrival	Time Getting a Jeep	Jeep No.	Trip Time	Return Time	Waiting Time
26	156*	163	4	117	280	7
27	162*	165	1	29	194	3
28	168	168	5	106	274	
29	174	174	2	22	196	
30	180*	181	6	39	220	1
31	186*	194	1	3	197	8
32	192*	196	2	50	246	4
33	198	198	1	24	222	
34	204*	220	6	17	237	4
35	210*	222	1	3	225	12
36	216*	225	1	4	229	9
37	222*	229	1	63	292	7
38	228*	237	6	25	262	9
39	234*	242	3	28	270	8
40	240*	246	2	7	253	6
41	246*	253	2	5	258	7
42	252*	258	2	57	315	6
43	258*	262	6	52	314	4
44	264*	270	3	8	278	6
45	270*	274	5	5	279	4
46	276*	278	3	10	288	2
47	282	282	5	15	297	
48	288	288	4	73	361	
49	294	294	3	66	360	
50	300	300	1	30	330	
				Total waiting time		186 min

*Requests are delayed before getting a jeep.

interval condition is much smaller than the E(w) value under the Poisson condition is thus made clear. For other cases with coordination, though one does not know what the degree of coordination is, one thing can be assumed; that the E(w) value must be smaller than that without coordination. Hence, the k value obtained through the model can satisfy a unit with any degree of coordination as the effect of coordination is only to provide a larger margin for the operation of the motor pool.

Effect on the Distribution of μ Due to Practical Conditions

When the destinations of dispatches are of a large variety of distances from the motor pool and the trips of duty require various lengths of time, the return time of the vehicles tends to be Poisson and the service time is described as negative exponential. But, in practical cases the geographical condition may limit the dispatches mainly to a few places, and a number of duties in similar nature may require almost the same period of time. Thus, the actual distribution of the service time of that motor pool is likely to vary from the ideal negative exponential.

To investigate the effect of this variation on the average waiting time is difficult because one does not know the extreme case of the variation. However, there is one indication to show the tendency of the effect. That is a derived relation for a single server model of Poisson input and arbitrary service time. The formula of expected waiting time for that model appears in Resume of Useful Formulas in Queuing Theory, Operation Research, Vol 5, 1957, p. 176.

$$\text{Expected waiting time} \quad W = \frac{\rho}{2\mu(1 - \rho)} \left[1 + (s\mu)^2 \right]$$

where $\rho = \frac{\lambda}{\mu}$ and s is the standard deviation of the service time.

One can see in the formula that in a single server model the average waiting time is tending to reduce if s, the standard deviation of the service time, becomes smaller. Extending this same principle of varying s to a several-server model, one may expect that if a change in service time distribution is to reduce the variance, it will also reduce the average waiting time. Therefore, the answer of k obtained through the model of ideal exponential service time can still satisfy any case with a varying distribution of service time, provided the standard deviation becomes smaller.

The question whether the condition of a unit is to increase or to reduce the standard deviation of its motor pool's service time can only be answered through actual surveying and statistical analysis. The cases with the variance increased are not many.

CHAPTER VIII

RECOMMENDATIONS

Further Study of the Priority Assignment Problem

In the proposed method, it has been assumed that all requests are of equal importance, so that the queuing discipline "first come, first served" can be applied. But in actual military operations, this is not always true because duty trips concerning some special activities such as combat or intelligence, etc., must always be punctual or in time.

To deal with this problem, there is already a competent operations research technique which is the queuing model with priority assignment. Alan Cobham[7] discussed this model in his paper "Priority Assignment in Waiting Line Problems." The formulas for computing the expected waiting times for requests of various levels of priority are shown in Resume of Useful Formulas in Queuing Theory by Thomas L. Saaty.[4] It is recommended that further investigation be made on this problem by studying how many priority levels are adequate, determining the proper portions of λ for different levels, and determining the average waiting time for an arrival of each level. This problem together with other managerial problems in a motor pool are an area worthwhile of further study.

Strengthening the Statistical Work in Military Organizations

Though the proposed method is a new approach to determine the correct number of vehicles required in a military unit, the suitability of the result is still much dependent on the accurate estimate of the values of λ and μ. In this regard, the experiences of the planning staff are important, but the statistical records of the similar units are even more important. Statistical records are virtually "experiences" presented in a much more concrete and mathematical manner.

Statistics have been used by the Chinese military organizations for a long period. More than thirty years ago, immediately after the end of World War II, statistics offices were established in major commands of the Chinese Air Force to handle this kind of work. Similar units were soon established in the Chinese Army and the Chinese Navy. However, the achievement of the statistical work is not remarkable because most of the work of these groups is limited to descriptive statistics. It seems that the statistical work in Chinese military organizations is a show window on the progress of the military strength rather than a means to improve military operations.

In making the above criticism, it seems that I have digressed from the motor pool problem. However, I am concerned that, to insure a better result from the use of the proposed method of determining the proper requirement of vehicles for a unit, improved statistical work in military organizations is absolutely necessary. This is true not only for this particular problem, but also for any other military operations research work.

Using the True Requirement as Logistic Planning Target

It has been mentioned before that the result obtained through the proposed method tends to reflect the true transportation requirement of a unit. When one adopts the proposed method for determining the vehicle requirements for new units to be established, it is mandatory to evaluate the authorizations of the old units with this same method. It is likely that the requirements computed are higher than the original authorizations. However, one appreciates knowing these differences because this will reveal the reason why the activities have long been felt to be inert or slow moving.

It is recommended that even if these requirements cannot be fulfilled under present conditions, these true requirement figures should be used as the target for long range logistic planning so that the correct amount of equipment can be made available when the situation permits.

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