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CAPITAL BUDGETING AND MIXED ZERO-ONE

INTEGER QUADRATIC PROGRAMMING

A THESIS

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Singaravelu Ramaswamy

S. R. Radhakrishnan

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SUMMARY

The object of investment analysis is to select, from the set of available proposals, a feasible package to be undertaken, which is preferred above all others. Most of the work done in the area of "Capital Budgeting" assumes linearity, independency of projects, and deterministic cashflows. A quadratic objective function is justified in the following cases to build realism into the models developed:

1. Where the objective is to maximize the utility and the utility function is quadratic instead of linear.
2. Where the interrelationship between projects are considered explicitly in the models.
3. Where risk factor is taken into consideration in project selection.

The objective of this research is to analyze capital budgeting problem as a mixed-zero one integer quadratic programming problem by relaxing the independency assumption and linear utility assumption in the models developed by Weingartner, Bernhard and Baumol and Quandt. The duality concepts developed by Balas for the quadratic case, are used to analyze the properties of optimal solutions to the models. Solution techniques for a class of capital budgeting problems are also discussed. Specifically, the problem involves the allocation of limited amounts of capital among a specified set of investment opportunities in such a way as to maximize the utility of a discounted sum of cash distributions made to the firm's shareholders.

CHAPTER I

INTRODUCTION

The area of capital budgeting is vast and diversified. Discussions on the proper criterion to use to evaluate investment proposals and discussions on the proper horizon, proper interest rates, the problem of optimal dividends, optimal renewal, etc., have been dealt with in the literature from various points of view. A variety of formal techniques have been used in these areas. In principle, the object of the investment analysis is to select from the set of available proposals a feasible package to be undertaken, which is preferred above all others.

Analysis for investment decisions by firms has become more sophisticated largely through use of mathematical models and computers to solve them. A major contribution for the theoretical formulation and treatment of constrained capital budgeting problems through the use of mathematical programming was made in 1962 by Weingartner (23). Since that time, several additional contributions by that author, Baumol and Quandt (5), Naslund (18), Byrne et al. (8) and others have discussed further the use of programming techniques, especially the choice of suitable objective function and the development of methods for handling uncertainty.

Most of the work done in this area assume linearity of objective function, independency of projects, and deterministic cash flows. The desirability of one prospective investment is not always a question that can be answered independently of others. It is common practice to reduce all problems to the case of independent proposals by amalgamating those projects among which interdependencies seem strong, and ignoring any remaining interdependencies. The extent to which this practice leads to good decisions has been obscured by lack of a technique of analysis capable of finding good decisions in specific instances.

The interrelationships between projects may take on many forms. Some investments may be competitive, such as developing new products, which would in part compete for the same market. Some investments may be complementary, such as developing and installing new production systems, which would share common facilities or which would mutually benefit from common research and development work. These factors may be internal to the firm or they may be exogoneous to the firm, such as the general state of the economy, or relevant technological factors.

This research has been directed towards relaxing the linearity assumptions made in the capital budgeting models. A quadratic objective function is justified in the following cases:

i) Where the objective is to maximize the utility and the utility function is quadratic instead of linear.

ii) Where the interrelationships between the projects under consideration are explicitly considered in the model.

iii) Where risk factor is taken into consideration in project selection. If we could calculate the risk associated with an alternative or a set of alternatives, in terms of the variance of returns, or that alternative, and the covariance between alternatives, then the problem of allocating a limited amount of capital to a group of interrelated investment alternatives under risk can be formulated as a quadratic binary programming problem.

Balas (4) has studied a pair of problems, in which the min-max (max-min) of a nonlinear function is to be found over a domain defined by linear inequalities, and the variables are constrained to belong to arbitrary sets of real numbers, i.e. some or all of the variables may be discrete. Mixed-integer and all integer quadratic problems are special cases of these problems, and the duality construction is symmetric, i. e. the dual of the dual is primal.

The objective of this research is to analyze capital budgeting, as a mixed-zero one integer quadratic programming problem, by relaxing the independency assumption and linear utility assumption in the models developed by Weingartner, Bernhard and Baumol and Quandt. The duality concepts developed by Balas (4), for the nonlinear discrete case, in particular the quadratic case, are used to analyze the properties of optimal solutions to the models formulated.

In Chapter II, the duality concepts of Balas for discrete programming are discussed. Particular emphasis has been placed on the quadratic discrete programming problem as they have direct bearings on the models to be formulated in Chapter III.

In Chapter III, three quadratic mixed zero-one integer programming models are formulated and analyzed. The results derived from an analysis of optimal solutions to the primal and dual models are stated as lemmas.

Chapter IV deals with discussions on the solution techniques for a class of capital budgeting problems. Specifically, the problem we will be concerned with involves the allocation of limited amounts of capital among a specified set of investment opportunities in such a way as to maximize the utility of a discounted sum of cash distributions made to the firm's shareholders.

CHAPTER II

BALAS' CONCEPTS OF DUALITY IN DISCRETE PROGRAMMING

Introduction

Duality theory plays a crucial role in the theory and computational algorithms of linear programming. The inception of duality theory in linear programming may be traced to the classical "min-max" theorem of Von Neuman, and was first explicitly given by Gale, Kuhn and Tucker (13). The main result of linear programming duality theory is that the primal has a finite optimal solution if an only if the dual has one, in which case the values of both the objective functions are equal at optimality.

The duality theory of quadratic programming has been studied by Dennis (11) and principally by Dorn (12). Wolfe (25) has specialized his results in nonlinear programming to the case of quadratic programming. In nonlinear programming, in order to establish a symmetric duality, conditions have to be imposed on the problem, but under less stringent conditions, duality can be established. Under certain conditions, a solution to the primal problem provides solution to the dual problem and vice-versa.

Balas (2) has studied a pair of dual problems in which the min-max (max-min) of a linear function is to be found over a domain, defined by linear inequalities, and some of the variables are to be constrained to belong to arbitrary sets of real numbers. For example, some or all of the variables may be discrete. Balas shows mixed-integer and all-integer programming problems are special cases of these symmetric dual problems. Balas extends the results of linear cases to the case of quadratic objective function. In this, a quadratic objective function is to be optimized subject to linear constraints. A pair of symmetric dual quadratic programs is studied, where some or all of the variables belong to arbitrary set of real numbers. Quadratic all-integer and quadratic mixed-integer programs are shown to be special cases of the general problem. Further, Balas extends the results of linear and quadratic case to mixed-integer and pure integer nonlinear programs, with convex objective functions and constraints (4).

The duality concepts studied by Balas is symmetric, i. e. the dual is the primal. Subject to qualification, the primal problem has an optimal solution if and only if the dual has one, and in this case, the values of their respective objective functions are equal.

In this chapter, the duality concepts developed by Balas are discussed and analyzed for the quadratic case. A brief summary of the results of the linear case is given before considering the quadratic case.

Duality in Linear Programming

Consider a pair of dual linear programs:

Primal: Max cx	Dual: Min ub
s.t. $Ax + y = b$	$uA - v = c$
$x, y \geqq 0$	$u, v, \geqq 0$

where,

$A = (a_{ij})$ is an 'm x n' matrix $i = (1,2,\ldots .m)$ and

$j = (1,2,\ldots .n)$

c, x = 'n' dimensional vectors

u, v, b = 'm' dimensional vectors.

The duality theorem states that if the primal problem has an optimal solution, then the dual has also an optimal solution, and indeed

$$\text{Max } cx = \text{Min } ub$$

If $(\bar{\bar{x}}, \bar{y})$ and $(\bar{u}, \bar{\bar{b}})$ are the two optimal solutions to the primal and the dual respectively, we have

$$\bar{c}\bar{x} = \bar{u}\bar{b} \quad \text{and}$$

$$\bar{u}\bar{y} = \bar{v}\bar{x} = 0$$

These relations play a central role in linear programming.

Duality in Discrete Programming--Linear Case

Consider a mixed-integer linear programming problem:

Given

$$c = (c_j)$$

$$A = (a_{ij}) \qquad j \in N = (1,2 \ldots\ldots m)$$

$$b = (b_i) \qquad i \in M = (1,2 \ldots\ldots n)$$

Find vectors,

$$x = (x_j),\ y = (y_i) \qquad i \in M, \quad j \in N$$

and Max cx

$$\text{s.t. } Ax + y = b \tag{I}$$

$$x, y \geqq 0$$

$$x_j \text{ integer} \qquad j \in N_1 \subset N$$

The above problem is different from the linear programming problem discussed earlier, since in this case some or all of the variables are constrained to belong to an arbitrary set, for instance the set of integers.

The basic feature of Balas' duality concept for linear discrete programming can be summarized as follows. Whenever a primal variable 'x_j' is constrained to be integer, or arbitrarily constrained, the associated dual constraint is relaxed by the introduction of an unconstrained slack variable 'v_j'. By this, the 'j'th constraint disappears from the dual set, but its slack appears in the objective function, multiplied by its complementary primal variable 'x_j'.

Essentially, in a linear programming problem, we are looking for a feasible solution to the primal and dual to solve the problem. In the discrete case, we are looking for a feasible solution to the primal with the property that the associated solution to the dual satisfies all the dual constraints, corresponding to the continuous primal variables, and comes "as close as possible," to satisfying the dual constraints corresponding to the integer constrained primal variables. This "coming as close as possible" is to be interpreted in the sense that the "gap" the amount '$-(v_j)$' by which each dual constraint corresponding to an integer constrained primal variable 'x_j' is potentially (i.e. if $v_j < 0$) violated, is weighted with 'x_j' and the weighted sum of these "gaps" is being minimized with respect to 'v' at the same time as the objective function is maximized with respect to 'x' (1).

Consider the following problem (P) in which,

$$A = (a_{ij})$$

$$b = (b_i)$$

$$c = (c_j)$$

Find $x = (x_j)$, $y = (y_i)$ $\quad i \in M = (1,2,\ldots .m)$,

$j \in N = (1,2,\ldots .n)$.

Let $x = \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$, where x_j is a component of x^1 if $j \in N_1$

and a component of x^2, if $j \in N - N_1$.

Further, let us introduce the variables u_i, $i \in M$, and let $u = (u^1, u^2)$,

where u_i is a component of u^1, if $i \in M_1 \subset M$, and a component of u^2, if $i \in M - M_1$.

Accordingly, let $c = (c^1, c^2)$, $y = \begin{pmatrix} y^1 \\ y^2 \end{pmatrix}$, $b = \begin{pmatrix} b^1 \\ b^2 \end{pmatrix}$, and

$$A = \begin{pmatrix} A^{11} & A^{12} \\ A^{21} & A^{22} \end{pmatrix}.$$

Consider now the problem:

$$\min_{u^1} \max_{X} \quad cx + u^1 y^1 + u^1 A^{11} x^1$$

S.T.

$$
\begin{aligned}
Ax + y &= b \\
x,\ u^1 &\geqq 0 \\
x_j &\text{ integer, } j \in N_1 \\
u_i &\text{ integer, } i \in M_1 \\
y_i &\begin{cases} \text{unconstrained,} & i \in M_1 \\ = 0 & i \in M - M_1 \end{cases}
\end{aligned}
\qquad \text{(P)}
$$

Problem 'I' is a special case of 'P' where $M_{\cdot} = \emptyset$.

Balas defines the dual (D) to the problem (P) as finding,

$$u = (u_i) \qquad i \in M$$

$$v = (v_j) \qquad j \in N \; ; \quad v = (v^1, v^2), \; v^1 \in N_1$$

$$x^1 = (x_j) \qquad j \in N_1 \qquad\qquad \text{that}$$

$$\underset{x^1}{\text{Max}} \quad \underset{u}{\text{Min}} \qquad ub - v^1 x^1 + u^1 A^{11} x^1$$

$$\text{S.T.} \qquad uA - v = c \tag{D}$$

$$u_i = \text{integer } i \in M_1$$

$$x_j = \text{integer } j \in N_1$$

$$v_j \begin{cases} \text{unconstrained} & j \in N_1 \\ \geqq 0 & j \in N - N_1 \end{cases}$$

The above pair of dual problems have certain complementary properties between the primal variable 'x_j' and the dual slack 'v_j', and between the primal slack 'y_i' and the dual variable 'u_i'. The relationships are

$$x_j \text{ integer} \longleftrightarrow v_j \text{ unconstrained}$$

$$x_j \text{ continuous} \longleftrightarrow v_j \geqq 0$$

$$y_i \text{ unconstrained} \longleftrightarrow u_i \text{ integer}$$

$$y_i \geqq 0 \longleftrightarrow u_i \text{ continuous}$$

Duality in Quadratic Programming

A quadratic programming problem is a nonlinear programming problem, having linear constraints, and an objective function which is the sum of linear and a quadratic forms. The general quadratic programming problem can be stated as follows:

$$\begin{array}{llcl} & \text{Max} \quad f & = & cx + \tfrac{1}{2}x'Cx \\ \text{S.T.} & Ax & = & b \\ & x & \geq & 0 \end{array} \qquad \ldots (QP)$$

Where,

'A' is a 'm x n' matrix

'C' is a 'n x n' matrix

'b' is a 'm' component vector

'c', and 'x' are 'n' component vectors.

The matrix 'C' can be taken to be symmetric matrix without loss of generality by letting,

$$C = (c_{ij}) = (c_{ji}) = \tfrac{1}{2}(c'_{ij} + c'_{ji})$$

The set of feasible solutions to the constraints is a convex set. If the objective function is a concave function, then any relative maximum is a global maximum. The objective function 'f' is the sum of a linear form (which is concave) and a quadratic form. Since the sum of two concave functions is concave, the objective function will be concave if 'x'Cx' is a concave function.

Hadley (14) shows that a dual problem to the quadratic programming problem can be stated as

$$\begin{array}{llcl} \text{Min} & F & = & u'b - \tfrac{1}{2}x'Cx \\ \text{S.T.} & Au - Cx & \geq & c' \\ & x & \geq & 0 \end{array} \qquad \ldots (QD)$$

If the problem (QP) has an optimal solution, then so does the problem (QD), its dual, and furthermore,

$$\max f = \min F.$$

In fact, an optimal solution 'x*' to primal forms an optimal solution to the dual.

Duality in Discrete Programming--Quadratic Case

The symmetric dual nonlinear programs studied by Dantzig, Eisenberg and Cottle (10), as well as the dual nonlinear programs formulated by Wolfe (25), Mangasarian (16), and Huard (15), are generalized by Balas (4), by allowing some of the variables to be constrained to belong to arbitrary sets of real numbers, and dropping the requirement that the objective function and the constraints be concave (convex) in these variables.

Consider the following symmetric dual nonlinear programs with arbitrary constraints:

Let,

$x \in R^n$, $u \in R^n$ (rows or columns, according to context)

$(1,\ldots .n) = N$, $(1,\ldots .m) = M$

$(1,\ldots .n_1) = N_1 \subset N$, i.e. $0 \leqq n_1 \leqq n$

$(1,\ldots .m_1) = M_1 \subset M$ i.e. $0 \leqq m_1 \leqq m$

$x = (x^1, x^2)$, $u = (u^1, u^2)$

$x_1 = (x_1 \ldots .x_{n_1})$, $x^2 = (x_{n_1} + 1, \ldots .x_n)$

$u_1 = (u_1 \ldots .u_m)$, $u^2 = (u_{m_1} + 1, \ldots .u_m)$

Consider a function K (x,u) differentiable in x^2 and u^2, and denoting

$$\nabla_{x^2} K(x,u) = \left(\frac{\delta K(x,u)}{\delta x_{n_1} + 1}, \ldots . \frac{\delta K(x,u)}{\delta x_n} \right)$$

$$\nabla_{u^2} K(x,u) = \left(\frac{\delta K(x,u)}{\delta x_{n_1+1}} \cdots\cdots \frac{\delta K(x,u)}{\delta x_n} \right)$$

$$\nabla_{u^2} K(x,u) = \left(\frac{\delta K(x,u)}{\delta u_{m_1+1}} \cdots\cdots \frac{\delta K(x,u)}{\delta u_m} \right)$$

Let 'P' be the problem of finding $x \in R^n$ and $u \in R^m$ and

(Primal) $$\min_{u^1} \max_{x,u^2} \quad K(x,u) - u^2 \nabla_{u^2} K(x,u)$$

Subject to $$\nabla_{u^2} K(x,u) \geqq 0$$

$$x^1 \in X^1, \; u^1 \in U^1$$

$$x^2, \; u^2 \geqq 0$$

where, $x^1 \in R^{n1}$ and $U^1 \in R^{m1}$ are <u>arbitrary sets</u>

Balas defines the dual of (Primal) as the problem of finding $x \subset R^n$, $u \in R^m$ and

(Dual) $$\operatorname{Max}_{x^1} \min_{x^2,u} \quad K(x,u) - x^2 \nabla_{x^2} K(x,u)$$

Subject to

$$\nabla_{x^2} K(x,u) \leqq 0$$

$$x^1 \in X^1, \quad u^1 \in U^1$$

$$x^2, \; u^2 \geqq 0$$

If we set, in the nonlinear programming problem,

$$K(x,u) = cx + ub - uAx + \tfrac{1}{2}(xCx - uEu) + u^1 A^{11} x^1$$

we have a quadratic programming problem, where some or all of the variables are constrained to belong to arbitrary set of real numbers. Then the objective function of primal and dual become,

$$\min_{u^1} \; \max_{x,\, u^2} \quad cx + \tfrac{1}{2} xCx + \tfrac{1}{2} uEu + u^1 y^1 + u^1 A^{11} x^1 \quad \text{and}$$

$$\max_{x^1} \; \min_{x^2,\, u} \quad ub - \tfrac{1}{2} xCx - \tfrac{1}{2} uEu - v^1 x^1 + u^1 A^{11} x^1$$

where 'E' and 'C' are symmetric matrices with 'C^{22}' and 'E^{22}' negative semidefinite; y^1, v^1 are the slack and surplus variables.

Consider now the problem (P) in which,

i) 'C' and 'E' are symmetric matrices with 'C^{22}' and 'E^{22}' negative semidefinite

ii) X^1 and U^1 are arbitrary sets of 'n_1' vectors and 'm_1' vectors respectively such that X^1 is independent of x^2, u; i.e. neither X^1 nor U^1 is defined in terms of other variables of the problem.

Find,

$$x = (x_j)$$

$$y = (y_i) \qquad\qquad j \in N$$

$$u = (u_i) \qquad\qquad i \in M$$

$$\min_{u^1} \; \max_{x,u^2} \quad f = cx + \tfrac{1}{2} xCx + \tfrac{1}{2} uEu + u^1 y^1 + u^1 A^{11} x^1$$

subject to

$$Ax + Eu + y = b$$

$$x^1 \in X^1 \qquad u^1 \in U^1$$

$$x^2,\ y^2 \geqq 0 \qquad \qquad \text{(P)}$$
$$u^2,\ y^1 \quad \text{unconstrained in sign.}$$

A solution to 'P' will be written as (x,u) as the vector 'y' is uniquely determined by 'x' and 'u'. This solution will be called a feasible solution if it satisfies the constraint set.

The dual problem to (P) as defined by Balas is as (D) of finding,

$$u = (u_i) \qquad\qquad i \in M$$
$$x = (x_j) \qquad\qquad j \in N$$
$$v = (v_j)$$

$$\underset{x^1}{\text{Max}} \quad \underset{x^2,\, u}{\text{Min}} \quad g = ub - \tfrac{1}{2}\, uEu - \tfrac{1}{2}\, xCx - v^1\, x^1 + u^1\, A^{11}\, x^1$$

$$\text{Subject to} \qquad uA - xC - v = c$$
$$u^1 \in U^1 \qquad\qquad x^1 \in X^1$$
$$v^2,\ u^2 \geqq 0 \qquad\qquad \text{(D)}$$
$$v^1,\ x^2 \quad \text{unconstrained}$$

If we write the dual (D) in the form (P), we obtain by changing the signs of the objective function and in the equation set

$$-\underset{x^1}{\min} \quad \underset{x^2,u}{\max} \quad u\,(-b) + \tfrac{1}{2}\, u\,(E)\,u + \tfrac{1}{2}\, x\,(C)\,x + v^1\, x^1 + u^1(-A^{11})x^1$$

$$\text{Subject to} \qquad u\,(-A) + x\,C + v = (-c)$$
$$u^1 \in U^1 \qquad\qquad x^1 \in X^1$$
$$u^2,\ v^2 \geqq 0$$
$$x^2,\ v^1 \quad \text{unconstrained}$$

It can be observed that the dual of the above problem is the Primal, thus proving that the dual is involutory.

Properties of Primal and Dual

Let 'Z' and 'W' denote the constraint set of (P) and (D) respectively, and let

$$z = \min_{u^1} \max_{x,u^2} \left\{ f \;\middle|\; (x,u) \in Z \right\}$$

$$w = \max_{x^1} \min_{x^2,u} \left\{ g \;\middle|\; (x,u) \in W \right\} \text{ if it exists.}$$

Definition

Let s^1, s^2 . . . . .s^p be elements of arbitrary finite dimensional vector spaces.

A vector function G (s^1, s^2, ...s^p) is called separable with respect to 's^1' if there exist vector functions H (s^1) (independent of s^2, ...s^p) and K (s^2. . . . .s^p) (independent of s^1), such that

$$G\,(s^1, \ldots\ldots s^p) \equiv H\,(s^1) + K\,(s^1, \ldots\ldots s^p)$$

G (s^1 . . . . .s^p) is called Componentwise separable with respect to s^1, if each component, g_i of G can be written either as g_i (s^1), or as g_i (s^2. . . . .s^p).

Lemma 1

Let r, s, t, be elements of arbitrary vector spaces. Let f (r, s, t) be a scalar function and G (r, s, t) be a vector function.

If f (r, s, t) is separable and G (r, s, t) is componentwise separable with respect to 'r' and 's', then

$$\operatorname*{Inf}_{s} \ \operatorname*{Supr}_{r,\, t} \left\{ f\,(r, s, t) \;\middle|\; G\,(r, s, t) \leqq 0 \right\}$$

$$= \operatorname*{Supr}_{r} \ \operatorname*{inf}_{s} \left\{ \operatorname*{sup}_{t} \left\{ f\,(r, s, t) \;\middle|\; G\,(r, s, t) \leqq 0 \right\} \right\}$$

In this lemma, it can be noted that the definitions infimum and supremum were used instead of minimum and maximum to enable us to consider open sets of feasible solutions, which do not contain the boundary points or some of the boundary points.

<u>Theorem 1</u>

Assume $E^{21} = 0$ and v^2 is componentwise separable with respect to 'u^1' (or $C^{12} = 0$ and y^2 is componentwise separable with respect to 'x^1'). Then if (P) has an optimal solution, $(\bar{x}, \bar{u})$, there exists u^2 such that $(\bar{x}, \hat{u})$ where, $u = (\bar{u}^1, \hat{u}^2)$ is an optimal solution to (P) and (D) with $E\,\bar{u} = E\,\hat{u}$,

$$\operatorname*{min}_{u^1} \ \operatorname*{max}_{x,\, u^2} \left\{ f \ \left\{ (x, u) \in Z \right\} \right. = \operatorname*{max}_{x^1} \ \operatorname*{min}_{x^2,\, u} \left\{ g \;\middle|\; (x, u) \in W \right\}$$

and the function

$$F\,(x, u) = cx + \tfrac{1}{2}\, x\, C\, x + ub - \tfrac{1}{2}\, u\, E\, u - u\, A\, x + u^1\, A^{11}\, x^1$$

has a saddle-point at $(\bar{x}, u)$:

$$F\,(x, \hat{u}) \leqq F\,(\bar{x}, \hat{u}) \leqq F\,(\bar{x}, u)$$

for all $x \in X\,(u, \bar{y}^2)$ and all $u \in U\,(\bar{x}, \bar{v}^2)$, where $\bar{y}^2$, and $\bar{v}^2$ are defined by $(\bar{x}, \hat{u})$.

Corollary 1-A

If (x, u) is an optimal solution to (P) and (D), then

$$\bar{u}^2 \bar{y}^2 = 0 \qquad \bar{v}^2 \bar{x}^2 = 0$$

$$\bar{u}^2 (b^2 - A^{21} \bar{x}^1 - E^{21} \bar{u}^1 - E^{22} \bar{u}^2) - (c^2 - \bar{u}^1 A^{12} + \bar{x}^1 C^{12} + \bar{x}^2 C^{22}) \bar{x}^2 = 0$$

Consider the problem (D'),

$$\underset{s}{\text{Max}} \quad \underset{x,u}{\text{Min}} \qquad g' = ub - \tfrac{1}{2} uEu - \tfrac{1}{2} xCx - v^1 s + u^1 A^{11} s$$

Subject to

$$\begin{aligned} & uA - xC - v = c \qquad\qquad (D') \\ & u^1 \in U^1 \qquad s \in X^1 \\ & u^2, v^2 \geqq 0 \\ & x, v^1 \text{ unconstrained.} \end{aligned}$$

where 's' is an 'n_1' vector and all the other symbols have the same meaning as in (D). Let 'W' and "W'" denote the constraint sets of (D) and (D').

Theorem 2

If the matrix 'C' is negative semidefinite, (D') is equivalent to (D)

(a) If $(\bar{x}, \bar{u})$ solves (D), then $(\bar{s}, \bar{x}, \bar{u})$ where $\bar{s} = \bar{x}^1$ solves (D').

(b) If $(\hat{s}, \hat{x}, \hat{u})$ solves (D'), there exists $\bar{x}^2$ such that $(\hat{s}, \bar{x}, \hat{u})$, where $\bar{x} = (\bar{s}, \bar{x}^2)$ also solves (D'), and $(\bar{x}, \hat{u})$ solves (D).

In both cases,

$$\underset{s}{\max} \ \underset{x,\, u}{\min} \left\{ g' \,\middle|\, (s, x, u) \in W' \right\} = \underset{x^1}{\max} \ \underset{x^2, u}{\min} \left\{ g \,\middle|\, (x, u) \in W \right\}$$

Special Case

Consider the following mixed-integer programming problem:

$$\begin{aligned} \text{Max} \quad & f = cx + \tfrac{1}{2}\, x\, C\, x \\ & Ax \leqq b \\ & x^1 \in X^1 \quad x^2 \geqq 0 \end{aligned} \qquad \text{(I)}$$

where $x = (x^1, x^2)$ is an 'n' vector, X^1 is the set of all non-negative 'n_1' vectors ($n_1 \leqq n$) with integer components.

The problem (I) can be written as

$$\text{Max } f = c^1\, x^1 + c^2\, x^2 + \tfrac{1}{2}\, (x^1, x^2) \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}$$

subject to

$$\begin{aligned} A^{21}\, x^1 + A^{22}\, x^2 & \leqq b^2 \\ x^1 \in X^1 \quad x^2 & \geqq 0. \end{aligned}$$

The problem (I) is a special case of the general problem (P) where, we have

$$E = 0 \text{ and } M_1 = \emptyset.$$

The dual of (I) can be written as (D)

$$\max_{x^1} \quad \min_{x^2,\, u} \quad g = u^2\, b^2 - \tfrac{1}{2}\, xCx - v^1\, x^1$$

Subject to

$$u^2\, A^{21} - (x^1\, C^{11} + x^2\, C^{21}) - v^1 = c^1$$

$$u^2 A^{22} - (x^1 C^{12} + x^2 C^{22}) - v^2 = c^2$$

$$x^1 \epsilon X^1$$

$$u^2, v^2 \geqq 0$$

$$x^2, v^1 \text{ unconstrained.}$$

For this special case, Theorem 1 of Balas reduces to

$$\underset{x^1, x^2, u^2}{\text{Max}} \left\{ f \;\middle|\; (x^1, x^2, u^2) \epsilon Z \right\} = \underset{x^1}{\text{Max}} \; \underset{x^2, u^2}{\text{min}} \left\{ g \;\middle|\; (x, u) \epsilon W \right\}$$

and the function

$$F (x^1, x^2, u^2) = (c^1 x^1 + c^1 x^2 + c^1 x^2 + \tfrac{1}{2} x C x + u^2 b^2 - u^2 A^{21} x^1 - u^2 A^{22} x^2)$$

has a saddle point: $(\bar{x}^1, \bar{x}^2, \bar{u}^2)$:

$$F (x^1, x^2 \bar{u}^2) \leqq F (\bar{x}^1, \bar{x}^2, \bar{u}^2) \leqq F (\bar{x}^1, \bar{x}^2, u^2)$$

and the corollary to theorem 1 for this special case reads as,

$$\bar{u}^2 \bar{y}^2 = 0$$

$$\bar{v}^2 \bar{x}^2 = 0$$

and

$$\bar{u}^2 (b^2 - A^{21} \bar{x}^1) - (c^2 + \bar{x}^1 C^{12} + \bar{x}^2 C^{22}) \bar{x}^2 = 0$$

where $(\bar{x}^1, \bar{x}^2, \bar{u}^2)$ is an optimal solution to (P) and (D).

CHAPTER III

QUADRATIC DISCRETE MODELS

Introduction

The capital budgeting models developed by Weingartner (23), Baumol and Quandt (5), and Bernhard (7) invariably assume independence of projects. However, the desirability of one prospective investment is not always a question that can be answered independently of others, since their performances often would be interrelated. These interrelations may take many different forms. Some of the investments may be competitive, such as the development of new products, which would in part compete for the same market. Some investments may be complementary, such as developing and installing a new production system which would share common facilities or which would mutually benefit from common research and development work. The amount of income resulting from each of a group of investments may be correlated because these incomes are affected by common factors. These common factors may be internal to the firm or external factors such as the general state of economy or relevant technological factors. So, in general, the independency assumption does not hold.

In this chapter, we will consider the interrelationships between projects explicitly in the model. Relaxation of any assumption of independency give rise to a quadratic objective function. Quadratic discrete programming models are discussed as an extension to the cases of the following models:

i) Weingartner's Linear Programming Model

ii) Weingartner's Basic Horizon Model

iii)Bernhard's general model

The duality concepts of Balas (3) for the quadratic discrete case are applied to the quadratic models developed, and the optimal solution to primal and dual are analyzed, to obtain certain results.

Interrelationships Between Projects

We are concerned here discrete indivisible projects. The decision variable 'x_j' $j = 1, 2, \ldots .n$, is a binary vector with values either '0' or '1'. This makes the decision of undertaking the project fully or rejected fully depending upon the value of x_j. If the 'j' the project is accepted, $x_j = 1$, and $x_j = 0$, otherwise.

When the projects are interrelated, these interrelations must be taken into consideration in the mathematical programming models. Weingartner discusses interrelationships like mutually exclusive projects, contingent projects, and compound projects, and suggests methods to take care of them as part of the constraint set. A mixture of independent and mutually exclusive projects presents neither conceptual nor computational difficulties in the formulation. To make an extension, it is only necessary to add one inequality for the set of mutually exclusive alternatives. These interrelations are therefore handled in the linear models discussed by Weingartner (23).

Quadratic Objective Function

We are concerned here with investment decisions which are not independent and also are such that the interdependencies cannot be handled by a set of constraints. Consider a set of investment proposals that are not mutually independent. That is the pay off, whether measured in profit, revenue, cost productivity or the like to any one project may depend upon the other projects undertaken with it. For instance, if among the available projects there are

i) that of building a concrete road in a certain distant and inaccessible part of the world and

ii) that of building a cement plant in the same region.

The cost of the road might depend upon whether the cement plant were or were not built, due to an assumed difference in the cost of transporting cement under the two conditions, while the return to the cement plant might depend on whether or not the road were built, because of assumed absence of other demand for cement in that region. Problems of this kind are to be found confronting a single firm in its investment planning as well as an entire economy.

If we assume that it is sufficient to consider pairwise interreactions between the projects, it is possible to represent the interdependencies by means of a quadratic term in the objective function. Reiter (21) developed a model to include all pairwise second order effects, i. e. involving interreaction terms between pairs of projects. Nemhauser and Ullmann (19) solves such a quadratic model by a dynamic programming algorithm.

Suppose a decision maker is confronted with 'n' investment projects. The performance of any project may be measured in various ways, e. g. total discounted net profitability over the estimated life-time of the project. A triangular pay-off matrix

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots\cdots & b_{1n} \\ 0 & b_{22} & \cdots\cdots & b_{2n} \\ 0 & 0 & \cdots\cdots & b_{3n} \\ 0 & & \cdots\cdots & b_{4n} \\ 0 & & \cdots\cdots & \\ 0 & 0 & 0 \ \cdots & b_{nn} \end{bmatrix}$$

is defined for the set of 'n' investment alternatives such that the pay-off (e. g. net present value) from the acceptance of project 'r' alone is 'b_{rr}' and the additional pay-off from the acceptance of both the project 'r' and 's' is 'b_{rs}', in addition to the pay-off from the acceptance of project 'r', 'b_{rr}' and project 's', 'b_{ss}'.

The problem of interrelationships can therefore be handled as a quadratic term

$$X\,B\,X' = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}\, x_i\, x_j$$

$$X = (x_1, x_2, \ldots\ldots x_n)$$

$$X' = \text{Transpose of 'X'}$$

The pay-off is realized only if $x_i^* = 1$; $x_j^* = 1$. Otherwise the product, $b_{ij}\, x_i^*\, x_j^* = 0$. Since the '$x_i$'s are restricted to zero and unity, it is unnecessary to distinguish between 'x_i' and 'x_i^2'. Therefore, if the objective is to maximize the present worth, it is possible to formulate the objective function as

$$\text{Max,} \sum_j b_j\, x_j + \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j$$

separating the interraction terms as quadratic term. Without loss of generality, the matrix (b_{ij}) can be made to an equicalent symmetric matrix, with the diogonal elements zeros.

Extension of Weingartner's Basic Model to the Quadratic Case

Introduction

The basic model of Weingartner involves the allocation of limited amounts of capital among a specified set of 'n' investment opportunities with the goal of selecting those projects whose total present value is maximum, but whose total outlay in each period falls within the budget limitation.

Consider, a planning horizon 'T' divided into a finite number of periods, and during this period, the exogenous and internal conditions remain constant. The model considered is deterministic, i.e., all information about cash flows and budget limitations up to the planning horizon is assumed to be known with certainty. The discount rates are assumed to be known for the purpose of obtaining the present values. The projects are not independent and the interractions between them are taken into consideration explicitly in the model.

The Model

Let,

C_t = the budget ceiling in time 't'

c_{tj} = the cost of project 'j' in time period 't'

b_j = the present value of all cash flows (revenues and costs)

$(\frac{1}{2}b_{ij})$= the Matrix containing the present value of all cash-flows due to interraction of projects only.

x_j = the decision variable to be either "go" or "no-go"

i.e. x_j takes on values '0' or '1'.

The model can be stated as,

Maximize

$$\sum_{j=1}^{n} b_j\ x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\ x_i\ x_j$$

Subject to

$$\sum_{j=1}^{n} c_{tj}\ x_j \leqq C_t$$

$$t = 1, 2, \ldots\ldots T$$

$$j = 1, 2, \ldots\ldots n$$

$$x_j \geqq 1$$

$$x_j \text{ integer.}$$

This is a pure 0-1 integer programming problem. The above model differs from Weingartner's Basic Model (13) by having a quadratic function in the objective function to explicitly take care of the project interractions. This model is a special case of the general model (P) discussed in Chapter II, where E = 0 and $M_1 = \emptyset$, with X^1 a set of all real 'n' vectors with values either '0' or '1'.

The problems (P) and (D) read in Balas notation for the quadratic case as

$$\text{Max} \quad c^1\ x^1 + \tfrac{1}{2}\ x^1\ C^{11}\ (x^1)'$$

S.T.

$$A^{21}\ x^1 + y^2 = b^2 \qquad (P')$$

$$x^1 \in X^1 \text{ (either '0' or '1')}$$

$$y^2 \geqq 0,$$

and the dual to (P') as,

$$\underset{x^1}{\text{Max}} \ \underset{u^2}{\text{Min}} \quad u^2 \, b^2 - \tfrac{1}{2} \, x^1 \, C^{11} \, x^1 - v^1 \, x^1$$

S. T. $$u^2 \, A^{21} - x^1 \, C^{11} - v^1 = c^1$$

$$x^1 \in X^1 \text{ (either '0' or '1')}$$

$$u^2 \geqq 0$$

$$v^1 \text{ unconstrained.}$$

Using the notational correspondence, between (P') and (D'), it could be stated as:

<u>Weingartner</u>	<u>Balas</u>
$(x_1, x_2 \, . \, . \, . \, . \, . x_n)$	x^1
$(C_1, C_2 \, . \, . \, . \, . \, . C_T)$	b^2
$(u_1, u_2 \, . \, . \, . \, . \, . u_T)$	u^2
$(b_1, b_2 \, . \, . \, . \, . \, . b_n)$	c^1
(b_{ij})	C^{11}
(c_{tj})	A^{21}

$(i = 1, 2, ... n)$

$(j = 1, 2, ... n)$

$(t = 1, 2, ... T)$

The Dual Problem

Based on the above notational correspondence, the dual problem to the model developed can be formulated as

$$\underset{x_j}{\text{Max}} \quad \underset{u_t}{\text{Min}} \quad \sum_{t=1}^{T} u_t\, C_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j - \sum_{j=1}^{n} v_j\, x_j$$

$$\text{S.T.} \qquad \sum_{t=1}^{n} u_t\, c_{tj} - \sum_{i=1}^{n} x_i\, b_{ij} - v_j = b_j$$

$$(j = 1, 2, \ldots .n)$$

$$x_j = \text{'0' or '1'} \qquad (j = 1, 2, \ldots .n)$$

$$u_t \geqq 0 \qquad (t = 1, 2, \ldots .T)$$

$$v_j \text{ unconstrained.}$$

The dual is a (max-min) type optimization of a quadratic mixed integer programming problem. The dual variables 'u_t' and the surplus variables 'v_j' for all 't' and 'j' are continuous, the variables 'v_j' being unconstrained in sign, as they correspond to the integer constrained primal variables 'x_j' (partial relaxation of constraints).

Properties of Solution to Primal and Dual

From the theorems and lemmas stated in Chapter II, the properties of the primal and dual are derived.

Since the vector of primal slack variable $(y_1, y_2, \ldots .y_T)$ is componentwise separable with respect to the primal decision vector $(x_1, x_2 \ldots .x_n)$, the saddle point theorem states the "if (P) has an optimal solution $(\bar{x}_1, \bar{x}_2 \ldots .\bar{x}_n)$, there exists a set of dual variables $(\bar{u}_1, \bar{u}_2 \ldots .\bar{u}_T)$ such that $(\bar{x}_1, \bar{x}_2 \ldots .\bar{x}_n; \bar{u}_1 \ldots .\bar{u}_T)$

is an optimal solution to the dual.

Further we have

$$\text{Max} \sum_{j=1}^{n} b_j\, x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j$$

$$= \underset{x_j}{\text{Max}} \quad \underset{u_t}{\text{Min}} \sum_{t=1}^{T} u_t\, C_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j - \sum_{j=1}^{n} v_j\, x_j \tag{3.1}$$

From the dual constraint set we have

$$\sum_{t=1}^{n} u_t\, c_{tj} - \sum_{i=1}^{n} x_i\, b_{ij} - b_j = v_j \tag{3.2}$$

$$(j = 1, 2, \ldots\ldots n)$$

Substituing (3.2 in (3.1)

$$\text{Max} \sum_{j=1}^{n} b_j\, x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j = \underset{x_j}{\text{Max}} \quad \underset{u_t}{\text{Min}} \sum_{t=1}^{T} u_t\, C_t$$

$$- \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, x_i\, x_j - \sum_{j=1}^{n} \left(\sum_{t=1}^{T} u_t\, c_{tj} - \sum_{i=1}^{n} x_i\, b_{ij} - b_j \right) x_j \tag{3.3}$$

If $(\bar{x}_j, \bar{u}_t)$ is an optimal solution to both primal and dual, then we have

$$\sum_{j=1}^{n} b_j\, \bar{x}_j + \tfrac{1}{2} \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij}\, \bar{x}_i\, \bar{x}_j = \sum_{t=1}^{T} \bar{u}_t\, C_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, \bar{x}_i\, \bar{x}_j$$

$$- \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t\, c_{tj}\, \bar{x}_j + \sum_{j=1}^{n} \sum_{i=1}^{n} \bar{x}_i\, b_{ij}\, \bar{x}_j + \sum_{j=1} b_j\, \bar{x}_j \tag{3.4}$$

which reduces to

$$\sum_{t=1}^{T} \bar{u}_t\, C_t = \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t\, c_{tj}\, \bar{x}_j \tag{3.5}$$

The relation (3.5) relates the optimal dual and primal variables with the corresponding budget and cash flows.

According to the Theorem 1 of Chapter II, the function

$$F\ (x, u) = cx + \tfrac{1}{2}\ xCx + ub - \tfrac{1}{2}\ uEu - uAx + + u^1\ A^{11}\ x^1$$

has a saddle point at $(\bar{x}, \bar{u})$; for the quadratic model under consideration, we have

$$F\ (x,u) = c^1\ x^1 + \tfrac{1}{2}\ x^1\ C^{11}\ x^1 + u^2\ b^2 - u^2\ A^{21}\ x^1 \qquad \text{or}$$

$$F\ (x_j, u_t) = \sum_{j=1}^{n} b_j\ x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\ x_i\ x_j + \sum_{t=1}^{T} u_t\ C_t - \sum_{j=1}^{n} \sum_{t=1}^{T} u_t\ c_{tj}\ x_j \tag{3.6}$$

$F\ (x_j, u_t)$ has a saddle point at $(\bar{x}_1, \dots \bar{x}_n; \bar{u}_1, \dots \bar{u}_t)$ and

$$F\ (x_j, \bar{u}_t) \leqq F\ (\bar{x}_j, \bar{u}_t) \leqq F\ (\bar{x}_j, u_t) \tag{3.7}$$

for all $x_j \in X\ (\bar{u}_t, \bar{y}_t)$ and, all $u \in U\ (\bar{x}_j, \bar{v}_j)$, where $\bar{y}_t$, and $\bar{v}_j$ are defined by $(\bar{x}_j, \bar{u}_t)$.

From the corollary stated in Chapter II, we have for the quadratic model under consideration

$$\bar{u}^2\ \bar{y}^2 = 0 \qquad \bar{v}^2 = 0 \qquad \bar{x}^2 = 0$$

$$\bar{u}^2\ (b^2 - A^{21}\ \bar{x}^1) = 0 \tag{3.8}$$

If $(\bar{x}_1, \dots \bar{x}_n)$ and $(\bar{u}_1, \bar{u}_2, \dots \bar{u}_T; \bar{x}_1, \bar{x}_2, \dots \bar{x}_n)$ are the optimal solutions to (P) and (D) respective, then

$$\sum_{t=1}^{T} \bar{u}_t\ \bar{y}_t = 0 \tag{3.9}$$

Since $u_t \geqq 0$ for all t and $y_t \geqq 0$ for all t,

$$\bar{u}_t \; \bar{y}_t = 0 \qquad \text{for all 't'}. \tag{3.10}$$

Further, from (3.8), we have

$$\sum_{t=1}^{T} \bar{u}_t \; c_t = \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t \; c_{tj} \; \bar{x}_j \tag{3.11}$$

which is the same result as obtained in (4.5). These complementary slack conditions differ from those in linear programming in that, optimality is a sufficient, but not a necessary condition for them to hold.

Extension of Weingartner's Terminal Wealth Model to the Quadratic Case

Introduction

In some respects, capital budgeting represents the central problem of the firm. Most of the projects have considerable period of duration time. Any set of decision taken today has consequences at later times, during the period of execution of project. Similarly, the opportunities available at later dates are related to the decisions being implemented today. To quote from Weingartner,

> While the only decisions that need to be made today are those which require action today. Specifically, the decision to utilize resources for the acquisition of assets which yield streams of revenue, but which cannot be turned back into cash or liquid assets without some cost call for careful analysis.(23)

The objective of Weingartner's Terminal Wealth Model is to maximize the sum of all discounted values due to flows subsequent to the horizon 'T' plus the difference between the lending and borrowing at the horizon, subject to the condition that the outflows of money in each period must not be larger than the inflows. Furthermore, the investment, borrowing and lending must be non-negative and one cannot invest in more than one complete project of each kind.

The model to be developed here differs from Weingartner's terminal wealth model in two distinct respects.

i) Here the projects' interactions are considered explicitly, by having a quadratic term in the objective function, and

ii) The firm is not operating in a "perfect capital market." This means that the borrowing and lending rates are not equal; further, they are different from different time periods.

The above assumptions bring the model under consideration closer to the real world situation, where we often have dependent projects and imperfect capital markets.

The Model

Consider a firm operating in an imperfect capital market. Let the horizon time be 'T', divided into a finite number of periods. It is assumed that the information about cash flows and budget limitations up to and beyond the horizon time is assumed to be known with certainty. The discount rates are known for the purpose of calculating the present values. The projects are not independent and the interactions between them are taken into consideration explicitly in the model. The financial transactions are introduced into the model by lending and borrowing without limit, at some stated rate of interest 'l_t' for lending and 'b_t' for borrowing; where, t = 1, 2, ...T. Both lending and borrowing are accomplished by means of renewable one year contract, where by convention, all interests are payable at the end of the year.

Letting,

D_t = the amount of money generated by outside sources.

a_{tj} = the flow of money associated with project 'j' in period 't'. Negative signs are used to associate with out-flows and positive signs with inflows.

v_t = the amount lent in period 't' to (t+1), at the lending rate 'l_t'.

w_t = the amount borrowed in period 't' to (t+1) at the rate 'b_t'.

l_t = $(1+r_{lt})$ where, 'r_{lt}' is the lending rate of interest from time 't' to 't+1'.

b_t = $(1+r_{bt})$, where, 'r_{bt}' is the borrowing rate of interest from 'r' to 't+1'.

x_j = the decision variable associated with prospect 'j'.

$\hat{a}_j$ = the horizon value of all flows subsequent to the horizon, associated with project 'j'.

$(\frac{1}{2}\hat{p}_{ij})$ = the 'horizon value' matrix of all flows subsequent to the horizon, due to pairwise interactions of projects under consideration.

Maximize,

$$\sum_{j=1}^{n} \hat{a}_j x_j + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{p}_{ij} x_i x_j + v_T - w_T$$

Subject to

$$- \sum_{j=1}^{n} a_{tj} x_j - l_{t-1} v_{t-1} + v_t + b_{t-1} w_{t-1} - w_t \leqq D_t$$

$$(t = 1, 2, \ldots\ldots T)$$

$$v_t,\ w_t \geqq 0 \qquad v_o,\ w_o = 0$$

$$x_j \leqq 1$$

$$x_j \text{ integer.}$$

$$(j = 1, 2, \ldots\ldots n)$$

The 'T' cash balance restrictions in this model can be interpreted as "the net cash outflow from time 't' loans plus the cash outflow for time (t-1) borrowing minus the cash inflow from time 't' borrowing must be less than or equal to the cash available from outside sources at time 't'. The objective function contains basically two components:

i) the net amount of financial assets accumulated at the horizon $(v_T - w_T)$; and

ii) the post-horizon cash flows along with flows due to interaction of projects discounted back to the horizon time.

Without loss of generality, the matrix $(\frac{1}{2}p_{ij})$ can be converted into a symmetric matrix $(\frac{1}{2}\hat{p}_{ij})$,

The above model is a mixed-zero-one integer quadratic programming problem. This is the special case of Bala's general model (P), discussed in Chapter II, where, $E = 0$, $M_1 = \emptyset$ and $x_j \in X^1$, a set of all vectors with '0' or '1' as values; v_t, w_t, $\in x^2$. The general problem (P) reduces to this special case as

$$\text{Max} \quad f = (c^1\ x^1 + c^2\ x^2) + \tfrac{1}{2}\ x^1\ C^{11}\ x^1$$

Subject to

$$A^{21} x^1 + A^{22} x^2 \leqq b^2 \qquad (P')$$

$x^1 \epsilon X^1$ where x^1 is the set of all binary variables for all 'j'

$$x^2 \geqq 0$$

The dual of (P') can be stated as (D'),

$$\underset{x^1}{\text{Max}} \quad \underset{x^2, u^2}{\text{Min}} \quad g = u^2 b^2 - \tfrac{1}{2} x^1 C^{11} x^1 - v^1 x^1$$

S.T.

$$\begin{aligned} u^2 A^{21} - x^1 C^{11} - v^1 &= c^1 \\ u^2 A^{22} \qquad\qquad - v^2 &= c^2 \end{aligned} \qquad (D')$$

$$x^1 \epsilon X^1$$

$$u^2 \geqq 0$$

$$v^2 \geqq 0$$

$$v^1 \text{ unconstrained}$$

The problems (P') and (D') are compared with the quadratic model under discussion, and using the notational correspondence, we have

Weingartner	Balas
$(x_1, x_2, \ldots .x_n)$	x^1
$(v_1, v_2, \ldots .v_T; w_1, w_2, \ldots .w_T)$	x^2
$(u_1, u_2 \ldots .u_T)$	u^2
$(D_1, D_2 \ldots . D_T)$	b^2

Weingartner	Balas
$(\hat{a}_1, \hat{a}_2, \ldots \hat{a}_n)$	c^1
$(0,0. \ldots .0,1\!: 0,0,. \ldots .0,-1)$	c^2
$(\hat{p}_{ij})$	C^{11}
$(-a_{tj})$	A^{21}

$$\begin{bmatrix} 1 & 0 & 0 & . & . & 0 & -1 & 0 & 0 & 0 & 0 & 0 & . & . & 0 \\ -1_1 & 1 & 0 & . & . & 0 & b_1 & -1 & 0 & 0 & 0 & 0 & . & . & 0 \\ 0 & -1_2 & 1 & . & . & 0 & 0 & b_2 & -1 & 0 & 0 & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 1_{t-1} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_{t-1} & 1 \end{bmatrix} \qquad A^{22}$$

$(i = 1, 2, \ldots n)$

$(j = 1, 2, \ldots n)$

$(t = 1, 2, \ldots T)$

The Dual

Based on the above notational correspondence, the dual problem to the model under consideration can be formulated as (D). This dual is a (max-min) type optimization of quadratic mixed-integer programming problem.

$$\max_{x_j} \min_{u_t} \sum_{t=1}^{T} u_t D_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j - \sum_{j=1}^{n} k_j x_j \tag{3.12}$$

Subject to

$$- \sum_{t=1}^{T} u_t a_{tj} - \sum_{i=1}^{n} x_i \hat{p}_{ij} - k_j = \hat{a}_j \tag{3.13}$$

$(j = 1, 2, \ldots .n)$

$$u_t - l_t\, u_{t+1} \geqq 0$$

$$(t = 1,2, \ldots . T\text{-}1) \qquad (3.14)$$

$$- u_t + b_t\, u_{t+1} \geqq 0$$

(D)

$$(t = 1, 2, \ldots . T\text{-}1) \qquad (3.15)$$

$$u_T \geqq 1 \qquad (3.16)$$

$$- u_T \geqq -1 \qquad (3.17)$$

$$u_t \geqq 0 \text{ for all 't'} \qquad (3.18)$$

$$x_j = \text{'0' or '1'}.$$

k_j unconstrained.

The dual variable 'u_t' and the surplus variable 'k_j' are continuous, 'k_j' being unconstrained in sign, since they correspond to the integer constrained primal variables 'x_j',(partial relaxation of constraints). The surplus variables in constraints (4.14) to (4.18) are greater than or equal to zero. In the next section, the optimal solution to the primal and dual are analyzed to derive certain relationships.

Analysis of Primal and Dual Problems

Based on the theorems and lemma stated in Chapter II, various properties of the primal and dual problems formulated in the precious section can be derived. As in the case of the basic model, the vector of primal slack variables (y_1, y_2 . . . .y_T) is component-wise separable with respect to the vector of primal variables (x_1, x_2 . . . . .x_n) and thus the Theorem 1 becomes, "If (P) has an optimal solution $(\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n, \bar{v}_1, \bar{v}_2, \ldots \bar{v}_T, \bar{w}_1, \bar{w}_2 \ldots \bar{w}_T)$ then there exists a set of dual variables $(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_T)$ such that $(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_T; \bar{x}_1, \bar{x}_2, \ldots \bar{x}_n)$ is an optimal solution to the dual with

$$\text{Max} \sum_{j=1}^{n} \hat{a}_j x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j + v_T - w_T =$$
$$\max_{x_j} \min_{u_t} \sum_{t=1}^{T} u_t D_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j - \sum_{j=1}^{n} k_j x_j \tag{3.19}$$

and, at optimality,

$$\sum_{j=1}^{n} \hat{a}_j \bar{x}_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{x}_i \hat{p}_{ij} \bar{x}_j + \bar{v}_T - \bar{w}_T =$$
$$\sum_{t=1}^{T} \bar{u}_t D_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{x}_i \hat{p}_{ij} \bar{x}_j - \sum_{j=1}^{n} k_j \bar{x}_j \tag{3.20}$$

Substituting the value of 'k_j' from (3.13), we have

$$+ \sum_{t=1}^{T} \bar{u}_t D_t - \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t a_{tj} \bar{x}_j = (v_T - w_T) \tag{3.21}$$

The relation (3.21) relates the optimal dual and primal variables with the corresponding cash-flows and the net amount of financial assets accumulated at horizon $(v_T - w_T)$.

Further, the function

$$F(x, u) = cx + \tfrac{1}{2} xCx + ub - uEu - uAx + u^1 A^{11} x^1$$

has a saddle point at $(\bar{x}, \bar{u})$:

$$F(x, \bar{u}) \leqq F(\bar{x}, \bar{u}) \leqq F(\bar{x}, u)$$

for all $x \in X(\bar{u}, \bar{y}^2)$ and all $u \in U(\bar{x}, \bar{v}^2)$ where $\bar{y}^2$ and $\bar{v}^2$ are defined by $(\bar{x}, \bar{u})$.

For the model under consideration, the above inequalities become

$$F(x_j, v_t, w_t; u_t) = c_1^1 x_1^1 + c_2^2 x_2^2 + \tfrac{1}{2} x_1^1 C^{11} x^1 + u^2 b^2 - u^2 A^{21} x^1 - u^2 A^{22} x^2 \tag{3.22}$$

or

$$F(x_j, v_t\ w_t; u_t) = \sum_{j=1}^{n} \hat{a}_j x_j + v_T - w_T + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j + \sum_{t=1}^{T} u_t D_t + \sum_{j=1}^{n} \sum_{t=1}^{T} u_t a_{tj} x_j - \sum_{t=1}^{T} u_t v_t + \sum_{t=1}^{T} u_t w_t + \sum_{t=1}^{T-1} l_t u_{t+1} v_t - \sum_{t=1}^{T-1} b_t u_{t+1} w_t \tag{3.23}$$

has a saddle point at:

$$F(x_j, v_t, w_t; \bar{u}_t) \leqq F(\bar{x}_j, \bar{v}_t, \bar{w}_t; \bar{u}_t) \leqq F(\bar{x}_j, \bar{v}_t, \bar{w}_t; u_t) \tag{3.24}$$

From the corollary to the theorem 1, in Chapter II, we have the following relationships:

$$\sum_{t=1}^{T} \bar{u}_t \bar{y}_t = 0$$

$$\bar{v}^2 \bar{x}^2 = 0$$

and

$$\bar{u}^2 b^2 - \bar{u}^2 A^{21} \bar{x}^1 - c^2 \bar{x}^2 = 0$$

<u>Lemma 1</u>. Under the explicit and implicit assumptions made, i. e., $l_t, b_t > 0$, the following relationships hold:

(a) $u_t > 0$ for all 't'

(b) $u_T = 1$

(c) $b_t \geqq l_t$ $t = 1, 2, \ldots\ldots T-1$ for a feasible dual solution.

(d) $\bar{y}_t = 0$ for all 't'

Proof: We have from (3.16) and (3.17),

$$1 \leqq u_T \leqq 1$$

Therefore,

$$u_T = 1$$

and also

$$\bar{u}_T = 1.$$

From (3.14)

$$u_t \geqq l_t \ (u_{t+1})$$

we find

$$u_t \geqq \left(\prod_{r=t}^{T-1} l_r \right) u_T$$

But we have shown,

$$u_T = 1$$

Therefore,

$$u_t \geqq \left(\prod_{r=t}^{T-1} \right) l_r$$

By assumption, $l_r > 0$ for all 't', and so $u_t > 0$ for all 't'.

From (3.14) and (3.15),

$$u_t \geqq l_t \, u_{t+1}$$

$$-u_t \geqq -b_t \, u_{t+1}$$

$$u_t \leqq b_t \, u_{t+1}$$

We have from the above inequalities,

$$l_t \leqq b_t \text{ for all 't'} = 1, 2, \ldots . T-1$$

necessarily follows for a feasible dual solution.

From the complementary slackness conditions, we have

$$\sum_{t=1}^{T} \bar{u}_t \, \bar{y}_t = 0$$

Since $u_t \geqq 0$ and $y_t \geqq 0$

$$\bar{u}_t \, \bar{y}_t = 0 \text{ for all 't'}.$$

and since it has been shown that $u_t > 0$ for all 't', it is necessary that '$\bar{y}_t$' = 0, which proves part (d) of Lemma 1.

<u>Lemma 2</u>. If $(\bar{x}_j, \bar{v}_t, \bar{w}_t, \bar{u}_t)$ is an optimal solution to the model then, the following equalities hold:

(a) $(\bar{u}_t - 1_t \bar{u}_{t+1}) \bar{v}_t = 0$

$$t = 1, 2, \ldots\ldots (T-1)$$

(b) $(-\bar{u}_t + b_t \bar{u}_{t+1}) \bar{w}_t = 0$

(c) $\sum_{t=1}^{T} \bar{u}_t D_t - \sum_{t=1}^{T} \sum_{j=1}^{n} \bar{u}_t a_{tj} \bar{x}_j = v_T - w_T$

Proof: from (3.14), we have

$$u_t - 1_t u_{t+1} = K_t'$$

where K_t' is the surplus variable for the constraint set. At optimality,

$$\bar{u}_t - 1_t \bar{u}_{t+1} = \bar{K}_t'$$

From the complementary slackness conditions, we have

$$\bar{K}_t' \bar{v}_t = 0$$

Therefore,

$$(\bar{u}_t - 1_t \bar{u}_{t+1}) \bar{v}_t = 0 \qquad \text{Q.E.D.}$$

(b) From (4.14), we have

$$(- u_t + b_t u_{t+1}) = K_t''$$

where K_t'' is the surplus variable for the constraint set. At optimality, we have

$$- \bar{u}_t + b_t \bar{u}_{t+1} = \bar{K}_t''$$

From complementary slackness conditions, we have

$$\bar{K}_t'' \bar{w}_t = 0$$

which gives the equality

$$(- \bar{u}_t + b_t \bar{u}_{t+1}) \bar{w}_t = 0 \qquad \text{Q.E.D.}$$

(c) Please see (3.21) for proof.

Lemma 3. If lending rate at any period of time 't' is strictly less than the borrowing rate, then for optimal conditions

$$\bar{v}_t \bar{w}_t = 0$$

Proof:

Let us assume $\bar{v}_t \not> 0$. From part (a) of lemma 2,

$$(\bar{u}_t - l_t \bar{u}_{t+1}) \bar{v}_t = 0 \qquad \text{and}$$

therefore,

$$\bar{u}_t = l_t \bar{u}_{t+1}$$

or

$$\frac{\bar{u}_t}{\bar{u}_{t+1}} = l_t$$

But from part (b) of lemma 2

$$(- \bar{u}_t + b_t \bar{u}_{t+1}) \bar{w}_t = 0$$

Consider the case that $\bar{w}_t \neq 0$

then

$$- u_t + b_t \, u_{t+1} = 0$$

$$b_t = \frac{u_t}{u_{t+1}} = l_t$$

The above relationship contradicts the assumption that the lending rate is strictly less than the borrowing rate. Therefore, if $\bar{v}_t > 0$, then $\bar{w}_t$ must be equal to zero. Similarly, it could be shown that if we assume $\bar{w}_t > 0$ then $\bar{v}_t$ must be equal to zero. This proves that the product

$$\bar{v}_t \, \bar{w}_t = 0 \qquad \text{Q.E.D.}$$

The above result has an important interpretation. According to this lemma, it will never be optimal to borrow and lend during the same time period, i.e. if $l_t < b_t$, the optimal policy would be only to borrow or lend, not both.

Extension of Bernhard's Model to the Quadratic Integer Case

Introduction

In the capital budgeting literature, an appropriate objective in the planning of productive investment and financing policy is the maximization of some function, usually a discounted sum of all anticipated dividend payments to the owners of the firm's present shares. If the function's argument, the stream of dividend is truncated at some finite horizon 'T', as required in a programming formulation, then it seems reasonable to include also in the argument, the time 'T' terminal wealth, 'G' as a proxy for the post 'T' stream of dividends (7).

The approach taken by Weingartner is to maximize the net value of assets, financial and physical as of the horizon, where the former are expressed in terms of the funds available for 'lending' at that time and the latter are represented by the discounted streams of net revenues past the horizon.

A different approach to that taken in the basic horizon model is to introduce a utility function that places preferences over alternative dividend streams and terminal wealth valuations. Baumol and Quandt (5) formulate a model of this type which is linear in dividend payments (meaning that the constant marginal utility weights serve as discount factors).

Bernhard (7) combines the objective goals of Weingartner and Baumol and Quandt, and assumes a general objective function to be maximized, a finite stream of dividend payments $W_1, W_2 \dots\dots W_T$ and the terminal wealth 'G'. This objective could therefore be stated as

$$\text{Maximize} \quad f\,(W_1, W_2, W_3, \dots\dots W_T; G)$$

In this section, we will consider a special version of this objective function. Following the paths of Baumol and Quandt, a linear utility function is assumed and the project interactions are considered explicitly in the model.

The Model

Consider a firm having a decision problem of choosing projects among 'n' interrelated proposals. The firm is operating in a capital market, where it can lend and borrow money at a pre-determined discount rate. The firm's objective is to maximize the discounted streams of dividend payments (which are assumed to be linear), and the terminal wealth of the company as of the horizon.

Letting

D_t = the amount of money generated by other activities of the firm, than the investment projects, we are considering.

a_{tj} = the flow of money associated with project 'j' in period 't'. Negative signs are associated with outflows and positive signs are for inflows, as a rule.

v_t = the amount lent in period (t) to (t+1), at the lending rate 'l_t'.

w_t = the amount borrowed in period 't' to 't+1' at the borrowing rate 'b_t'.

l_t = $(1+r_{lt})$, where 'r_{lt}' is the lending rate of interest from time 't' to 't+1'.

b_t = $(1+r_{bt})$, where 'r_{bt}' is the borrowing rate of interest from time 't' to 't+1'.

x_j = the decision variable either '0' or '1' as values.

$\hat{a}_j$ = the horizon value of all flows subsequent to the horizon associated with project 'j'.

$(\frac{1}{2}\hat{p}_{ij})$ = the horizon value matrix of all flows subsequent to the horizon due to pair-wise interaction of projects.

W_t = the dividend to be paid to the shareholders at time 't'.

p_t = the rate at which dividend paid in the 't'th period are discounted.

v_0 = w_0 = 0 by assumption.

Maximize

$$\sum_{j=1}^{n} \hat{a}_j \, x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{p}_{ij} \, x_i \, x_j + v_T - w_T + \sum_{t=1}^{T} p_t \, W_t$$

Subject to

$$W_t - \sum_{j=1}^{n} a_{tj} \, x_j - l_{t-1} \, v_{t-1} + v_t + b_{t-1} \, w_{t-1} - w_t + y_t = D_t$$

$$t = 1, 2, \ldots\ldots T$$

$$W_t, w_t, v_t, \geqq 0$$

$$x_j = '\theta' \text{ or } '1'$$

$$j = 1, 2, \ldots\ldots n$$

The 'T' cash balance restrictions in the model can be interpreted as "the net cash outflow from time 't' loans plus cash outflow from 't-1' borrowing plus the dividends payment minus the cash inflow from time 't' borrowing must be less than or equal to the cash available from outside sources at time 't'.

The objective function contains basically three components. (1) The net amount of financial assets accumulated at the horizon, $(v_T - w_T)$, (2) the post horizon cash outflows along with flows due to interaction of projects, discounted back to the horizon time, and (3) the discounted sum of dividend payments up to the horizon.

Without loss of generality, the matrix $(\frac{1}{2}\hat{p}_{ij})$ can be taken into a symmetric matrix (n x n). The above model is a mixed-zero-one integer quadratic programming model, and is a special case of Balas' general model (P), for the quadratic case, where

$$E = 0$$

$M_1 = \emptyset$ and

$x_j \epsilon X^1$, a set of all vectors with values either '0' or '1'.

$$w_t, v_t W_t \epsilon X^2$$

For this special case, the primal and dual problem of Balas general model reduces to (p') and (D') respectively.

$$\text{Max} \quad f = c^1 x^1 + c^2 x^2 + \tfrac{1}{2} x^1 C^{11} x^1$$

Subject to

$$A^{21} x^1 + A^{22} x^2 \leqq b^2 \qquad (P')$$

$$x_j \epsilon X^1 \quad \text{(either (0) or (1))}$$

$$x^2 \geqq 0$$

and the dual to (P') can be stated as (D')

$$\max_{x^1} \quad \min_{x^2, u} \quad g = u^2 b^2 - \tfrac{1}{2} x^1 c^{11} x^1 - v^1 x^1$$

S.T.

$$u^2 A^{21} - x^1 C^{11} - v^1 = c^1$$

$$u^2 A^{22} - v^2 = c^2$$

$$x^1 \epsilon X^1$$

$$v^2, u^2 \geqq 0$$

v^1 unconstrained

The following notational correspondence are observed in the case of the model under consideration and the special case given above.

The Model	Balas
$(x_1, x_2, \ldots\ldots x_n)$	x^1
$(v_1, v_2, \ldots\ldots v_T, w_1, w_2 \ldots w_T; W_1, W_2 \ldots W_T)$	x^2
$(u_1, u_2 \ldots\ldots u_T)$	u^2
$(D_1, D_2 \ldots\ldots D_T)$	b^2
$(\hat{a}_1, \hat{a}_2 \ldots\ldots \hat{a}_n)$	c^1
$(0, 0 \ldots 1; 0, 0, \ldots - 1; p_1, p_2 \ldots p_T)$	c^2
$(\hat{p}_{ij})$	C^{11}
$(-a_{tj})$	A^{21}
$\begin{bmatrix} 1 & 0 & 0 & 0 & . & . & 0 & -1 & 0 & 0 & 0 & 1 & 0 & . & 0 \\ -1_1 & 1 & 0 & 0 & . & . & 0 & b_1 & -1 & 0 & 0 & 0 & 1 & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 1_{t-1} & 1 & 0 & 0 & b_{r-1} & -1 & 0 & 0 & . & 1 \end{bmatrix}$	A^{22}

$$(i = 1, 2, \ldots\ldots n)$$

$$(j = 1, 2, \ldots\ldots n)$$

$$(t = 1, 2, \ldots\ldots T)$$

The Dual Problem

Based on the notational correspondence, the dual problem (D) to the model can be formulated as

$$\underset{x_j}{\text{Max}} \quad \underset{u_t}{\text{Min}} \quad \sum_{t=1}^{T} u_t D_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j - \sum_{j=1}^{n} k_j x_j \qquad (3.25)$$

S.T.

$$- \sum_{t=1}^{T} u_t a_{tj} - \sum_{i=1}^{n} x_i \hat{p}_{ij} - k_j = \hat{a}_j \qquad (3.26)$$

$$j = 1, 2, \ldots \ldots n.$$

$$u_t - l_t u_{t+1} - k_t' = 0 \qquad (3.27)$$

$$t = 1, 2, \ldots \ldots T-1$$

$$u_T \geqq 1 \qquad (3.28)$$

$$- u_t + b_t u_{t+1} - k_t'' = 0 \qquad (3.29)$$

$$- u_T \geqq - 1 \qquad (3.30)$$

$$u_t \geqq p_t \quad \text{for all 't'} \qquad (3.31)$$

and

$$u_t \geqq 0$$

$$k_j \text{ unrestricted.}$$

It could be observed that the dual problem contains binary variable 'x_j' and continuous variables 'u_t'; the surplus variable 'k' is unconstrained in sign as it corresponds to the integer constrained primal variable 'x_j' (partial relaxation of constraints). All other surplus variables, k_t', k_t'' corresponding to the primal variables v_t, and w_t are non-negative.

Analysis of Primal and Dual Problems

Let the primal slack vector be $(Y_1, Y_2, \ldots \ldots Y_T)$. Since this vector is componentwise separable with respect to the decision vector $(x_1, x_2, \ldots \ldots x_n)$, according to the theorem stated in Chapter II, we have

"If (P) has an optimal solution $(\bar{x}_1, \bar{x}_2, \ldots \bar{x}_n; \bar{v}_1, \bar{v}_2, \ldots \bar{v}_T; \bar{w}_1, \bar{w}_2, \ldots \bar{w}_T; \bar{W}_1, \bar{W}_2, \ldots \bar{W}_T)$, then there exists a set of dual variables, $(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_T)$ such that $(\bar{u}_1, \ldots \bar{u}_T; \bar{x}_1 \ldots \bar{x}_n; \bar{v}_1 \ldots \bar{v}_T; \bar{w}_1 \ldots \bar{w}_T; \bar{W}_1 \ldots \bar{W}_T)$ is an optimal solution to the dual, and the following equality holds:

$$\begin{aligned} &\text{Max} \sum_{j=1}^{n} \hat{a}_j x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j + v_T - w_T + \sum_{t=1}^{T} p_t W_t \\ &= \underset{x_j}{\text{Max}} \ \underset{u_t}{\text{Min}} \sum_{t=1}^{T} u_t D_t - \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \hat{p}_{ij} x_j - \sum_{j=1}^{n} k_j x_j \end{aligned} \tag{3.32}$$

But from (3.26) we have

$$- \sum_{t=1}^{T} u_t a_{tj} - \sum_{i=1}^{n} x_i p_{ij} - \hat{a}_j = k_j \tag{3.33}$$

$$j = 1, 2, \ldots n.$$

Substituting this in (3.32), and at optimality

$$\sum_{t=1}^{T} \bar{u}_t D_t + \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t a_{tj} \bar{x}_j = \bar{v}_T - \bar{w}_T + \sum_{t=1}^{T} p_t \bar{W}_t \tag{3.34}$$

According to the theorem 1 of Chapter II, we have

$$F(x, u) = cx + \tfrac{1}{2} xCx + ub - \tfrac{1}{2} uEu - uAx + u^1 A^{11} x^1$$

has a saddle point $(\bar{x}, \bar{u})$:

$$F(x, \bar{u}) \leqq F(\bar{x}, \bar{u}) \leqq F(\bar{x}, u)$$

for all $x \in X(\bar{u}, \bar{y}^2)$ and for all $u \in U(\bar{x}, \bar{v}^2)$ where $\bar{y}^2$ and $\bar{v}^2$ are defined by $(\bar{x}, \bar{u})$.

For the model now under consideration,

$$F\,(x_j,\, v_t,\, w_t,\, W_t;\, u_t) = \sum_{j=1}^{n} \hat{a}_j\, x_j + v_T - w_T + \sum_{t=1}^{T} p_t\, W_t +$$

$$\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} x_i\, \hat{p}_{ij}\, x_j + \sum_{t=1}^{T} u_t\, D_t + \sum_{j=1}^{n} \sum_{t=1}^{T} u_t\, a_{tj}\, x_j - \qquad (3.35)$$

$$\sum_{t=1}^{T} u_t\, v_t + \sum_{t=1}^{T} u_t\, w_t - \sum_{t=1}^{T} u_t\, W_t + \sum_{t=1}^{T-1} 1_t\, u_{t+1}\, v_t - \sum_{t=1}^{T-1} b_t\, u_{t+1}\, w_t$$

The function (3.35) has a saddle point, $(\bar{x}_1, \ldots \bar{x}_n;\ \bar{v}_1, \ldots \bar{v}_T;\ \bar{w}_1, \ldots \bar{w}_T;\ \bar{W}_1, \ldots \bar{W}_T;\ \bar{u}_1, \ldots \bar{u}_T)$:

$$F\,(x_j,\, v_t,\, w_t,\, W_t;\, \bar{u}_t) \leq F\,(\bar{x}_j,\, \bar{v}_t,\, \bar{w}_t,\, \bar{W}_t;\, \bar{u}_t) \leq F(\bar{x}_j, \bar{v}_t, \bar{w}_t, \bar{W}_t; u_t) \qquad (3.36)$$

The complementary slackness conditions follow from corollary to theorem 1,

$$\bar{u}_t\, \bar{Y}_t = 0 \qquad t = 1, 2, \ldots T \qquad (3.37)$$

$$(\bar{u}_t - 1_t\, \bar{u}_{t+1})\, \bar{v}_t = 0 \quad t = 1, 2, \ldots T\text{-}1 \qquad (3.38)$$

$$(-\bar{u}_t + \bar{b}_t\, \bar{u}_{t+1})\, \bar{w}_t = 0$$

$$t = 1, 2, \ldots T - 1 \qquad (3.39)$$

$$(\bar{u}_t - p_t)\, \bar{W}_t = 0$$

$$t = 1, 2, \ldots T \qquad (3.40)$$

$$\sum_{t=1}^{T} \bar{u}_t\, D_t + \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t\, a_{tj}\, \bar{x}_j = \bar{v}_T - \bar{w}_T + \sum_{t=1}^{T} p_t\, \bar{W}_t \qquad (3.41)$$

Lemma 4. Under optimality, the following relations hold:

i) $\bar{u}_t > 0$ for all 't'

ii) $\bar{u}_T = 1$

iii) $\bar{Y}_t = 0$ for all 't'

iv) $b_t \geqq 1_t$ $t = 1, 2, 3, \ldots T - 1$, for a feasible solution.

Proof.

i) From (3.28) and (3.30)

$$1 \leqq \bar{u}_T \leqq 1 \quad \text{Therefore } u_T = 1 \text{ and } \bar{u}_T = 1.$$

ii) From (3.27), we have

$$\begin{aligned} u_t &\geqq 1_t \, u_{t+1} \\ &\geqq \left(\prod_{r=t}^{T-1} 1_r \right) u_t \\ &\geqq \left(\prod_{r=t}^{T-1} \right) 1_r \end{aligned}$$

But $1_r > 0$, by assumption, therefore $u_t > 0$ for all 't'.

iii) From complementary slackness (3.37)

$$\bar{u}_t \, \bar{Y}_t = 0 \quad \text{for all 't'}$$

Since it has been shown $\bar{u}_t > 0$, $Y_t = 0$ for all 't'.

iv) From equations (3.27) and (3.29)

$$u_t \geqq l_t \; u_{t+1}$$

$$u_t \leqq b_t \; u_{t+1}$$

$$l_t \; u_{t+1} \leqq u_t \leqq b_t \; u_{t+1}$$

which proves

$$l_t \leqq b_t \quad \text{for all } t = 1, 2, 3, \ldots T.$$

If this condition is not satisfied at optimality, then the primal constraint set can be violated, which would mean that we can borrow without limits and lend it at a lower rate to satisfy the constraints. For optimality, the lending rate must be less than or equal to the borrowing rate.

<u>Lemma 5.</u> If lending rate at any time period 't' is strictly less than the borrowing rate, then, at any time period 't', the firm either borrows or lends and not both, for optimality.

<u>Proof:</u> From (3.38) and (3.39) we have

$$(\bar{u}_t - l_t \; \bar{u}_{t+1}) \; \bar{v}_t = 0$$

$$(- \bar{u}_t + b_t \; \bar{u}_{t+1}) \; \bar{w}_t = 0$$

Let us assume that $\bar{v}_t \neq 0$ then

$$(\bar{u}_t - l_t \; \bar{u}_{t+1}) = 0$$

$$\bar{u}_t = l_t \; \bar{u}_{t+1}$$

If we take that $w_t \neq 0$, then

$$b_t \bar{u}_{t+1} = \bar{u}_t = l_t \bar{u}_{t+1}$$

which means the $l_t = b_t$. This contradicts the assumption made that the lending rate is strictly less than the borrowing rate. Therefore, if $\bar{v}_t \neq 0$, then $\bar{\bar{w}}_t$ must be equal to zero. Similarly, by the same argument, it can be shown that if $\bar{w}_t \neq 0$, then $\bar{v}_t$ must be equal to zero, i. e.

$$\bar{v}_t \bar{w}_t = 0.$$

According to this lemma, it will never be optimal to borrow or lend money during the same time period. If the lending rate is strictly less than the borrowing rate for a particular time period 't', then the optimal policy would be to lend or borrow, not both.

Lemma 6. For an optimal strategy, dividends are declared in any time period only when the dual evaluator for the period is equal to the rate at which the dividend paid, in the 't'th period, are discounted.

Proof: From complementary slackness conditions (3.40)

$$(\bar{u}_t - p_t) \bar{W}_t = 0$$

If $\bar{W}_t > 0$, then

$$\bar{u}_t - p_t = 0 \quad \text{i.e.,} \quad u_t = p_t$$

which proves the lemma.

In the section above entitled "The Model", it has been assumed that the interaction among projects are only at post horizon, which may not be realistic. To obviate this, the model is considered, where the interactions among projects are taken into consideration for all periods, by modifying the constraint set as follows:

Maximize

$$\sum_{j=1}^{n} \hat{a}_j x_j + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \hat{p}_{ij} x_i x_j + v_T - w_T \sum_{t=1}^{T} p_t W_t$$

Subject to

$$W_t - \sum_{j=1}^{n} a_{tj} x_j - \sum_{i=1}^{n} \sum_{j=1}^{n} x_i g_{tij} x_j - 1_{t-1} v_{t-1} + v_t$$

$$+ b_{t-1} w_{t-1} - w_t \leqq D_t \tag{3.42}$$

$$t = 1, 2, \ldots T$$

$$W_t, w_t, v_t \geqq 0$$

$$x_j = \text{'0' or '1'}$$

$$j = 1, 2, \ldots n.$$

where the term (g_{tij}) represents the pay-off due to the interaction among projects 'i' and 'j' at the time period 't'. All other symbols are the same as in the above model.

The 'T' cash balance restrictions in the model can be interpreted as "the net cash outflow from time 't' loans plus cash outflow from 't-1' borrowing plus the dividend payment minus the cashflow from 't' borrowings must be less than or equal to the cash available from outside sources at time 't'.

In Balas' notation, the model can be stated as:

Maximize

$$f = c^1\, x^1 + c^2\, x^2 + \tfrac{1}{2}\, x^1\, C^{11}\, x^1$$

S.T.

$$(X^1, X^1 .. X^1)\begin{bmatrix} G^1 & 0 & 0 & 0 & 0 \\ 0 & G^2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & G^T \end{bmatrix}\begin{bmatrix} X^1 \\ X^1 \\ \cdot \\ \cdot \\ X^1 \end{bmatrix} + A^{21}\, X^1 + A^{22}\, X^2 \leqq b^2$$

$x_j \in X^1$ the set of binary variables

'0' or '1'.

$$X^2 \geqq 0$$

and

$$G^t = (g_{tij})$$

For any given X^1, the model is a linear programming model in X^2. If we solve the linear programming model for all possible X^1, the optimal solution to the model,will be the one set of X^1, which maximizes the objective function of the linear programming problem. The linear programming problem can be stated as follows:

For the optimal $X^1 = \bar{x}_j$

Maximize

$$\sum_{t=1}^{T} p_t W_t + v_T - w_T$$

S.T.

$$W_t - l_{t-1} v_{t-1} + v_t + b_{t-1} w_{t-1} - w_t + y_t =$$

$$D_t^* = (D_t + \sum_{j=1}^{n} a_{tj} \bar{x}_j + \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{x}_i g_{tij} \bar{x}_j) \tag{3.43}$$

$$t = 1, 2, . . . T.$$

$$W_t, v_t, w_t \geqq 0$$

The dual of the above model can be stated as:

Minimize

$$\sum_{t=1}^{T} u_t D_t^*$$

S.T.

$$u_t \geqq p_t \tag{3.44}$$

$$u_t - l_t u_{t+1} \geqq 0 \tag{3.45}$$

$$- u_t + b_t u_{t+1} \geqq 0 \tag{3.46}$$

$$u_T \geqq 1 \tag{3.47}$$

$$- u_T \geqq - 1 \tag{3.48}$$

From complementary slackness conditions of the linear programming problem, we have

$$\bar{u} \bar{y} = \bar{v} \bar{x} = 0$$

It could be seen that the results (3.37) to (3.40) follow from complementary slackness conditions. At optimality, the results of lemmas stated in section entitled, "Analysis of Primal and Dual Problems," hold even if we were to consider the interaction of projects during all time periods.

CHAPTER IV

SOLUTION TECHNIQUES

Introduction

If under capital rationing, external interest rates are irrelevant, and if one cannot insist on a present value formulation of the objective function, the relevant discount rates cannot be determined internally by the problem. To resolve this dilemma, one approach is to introduce a utility function that places preferences over alternative dividend streams and terminal wealth valuations. Charnes, Cooper and Miller (9) and Baumol and Quandt (5) have considered this problem via analysis of the dual. The objective function of Baumol and Quandt is to maximize a function of the dividend payments, and their main model is a linear function.

$$\text{Max} \quad f\,(W_1, W_2, \ldots W_T) = \sum_{t=1}^{T} u_t\, W_t$$

where u_t for $t = 1, 2, \ldots T$ is a set of 'T' constants with $u_t \geqq 0$.

Unger (22) developed special solution techniques based on Bala's zero-one algorithm and Bender's (6) partitioning procedure for a mixed zero-one integer programming model, which explicitly considers the firm's dividend policy. This is similar to the utility function formulation of Baumol and Quandt, and Manne (17).

In this chapter, it is assumed that the utility function is quadratic to take care of any risk averse behavior of the shareholders, and a quadratic mixed zero-one integer programming model is developed.

Based on the duality concepts developed by Balas for the quadratic integer programming case, special solution techniques for the model are discussed.

Quadratic Utility Function

In this chapter, the objective function to be optimized is a utility function. The capital budgeting problem is therefore treated as a part of the general theory of choice, where utility is to be maximized subject to the opportunities and constraints. The utility function, to be considered, can be construed as the management's perception of the utility to the owners of consumption alternatives, available in different periods. One cannot rule out the possibility that in ascertaining the owner's time preferences, management may err, consciously or unconsciously. However, based on past performances, and by taking into consideration the relevant factors of the firm, it is assumed that a proper utility function which represents the firm's behavior in the market can be derived.

Let y_t for $t = 1, 2, \ldots T$ be the sums available for the period 't' for withdrawal from the firm, as dividend to its owners (shareholders). The utility function to be maximized is

$$U = U\ (y_1, y_2, \ldots y_T).$$

If we assume that the shareholders are risk averse, then a linear function of utility does not hold. Therefore, it is necessary to consider non-linear utility functions.

The requirement of positive marginal utility can be met by a linear function, but the risk aversion axiom requires at least a second degree. K. J. Arrow (1), and J. W. Pratt (20), show that the axiom of decreasing risk aversion cannot be represented by a second degree and it requires at least a third degree polynomial.

The following assumptions are made for the model under consideration:

i) The marginal rate of substitution between dollars in any two periods is not a constant.

ii) The shareholders are risk averse; the utility function is a continuous, monotonically increasing.

iii)The firm is operating in an imperfect market condition. In the presence of imperfect market, the discount rates cannot be determined <u>ex-ante</u>. Therefore, the objective function chosen, to be maxim zed is a utility function, instead of maximizing the present value.

iv) The effect of decreasing risk aversion of the shareholders is ignored and it is assumed that its effect on the utility function is constant.

Under these assumptions, the utility function can be represented by a second degree polynomial, i.e. a quadratic function as shown below:

$$U\ (y_1, y_2, \ldots y_t) = \sum_{t=1}^{T} p_t\, y_t + \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i$$

A concave utility function will take care of the risk averse behavior of the owners. The above function will be concave if the term,

$$\sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i$$

is concave, which implies that the matrix $C = (c_{ti})$, must be negative semi-definite or negative definite. We assume here that the matrix C is negative semi-definite.

The Model

Consider a firm which has under consideration 'n' different indivisible investment projects. The firm operates without recourse to outside financing, but it may reinvest the returns from any adopted project as well as the returns invested in prior to the first period under consideration. Further, any funds not used in one period may be carried over to the next period. The firms' objective is to maximize the discounted flow of dividends paid to its shareholders.

$$U\,(y_1, y_2, \ldots y_T) = \sum_{t=1}^{T} p_t\, y_t + \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i$$

Letting,

a_{tj} = the net cash flow received from the 'j' the project in the (t)th period.

E_t = the net cash flow in the 't'th period from all projects adopted prior to the first period.

y_t = the dividend paid to the shareholders at time period 't'.

s_t = the amount of cash carried over from the 't'th to the (t+1)st period; and assuming a planning horizon of 'T' years, the mathematical statement of the problem is

Maximize

$$\sum_{t=1}^{T} p_t\, y_t + \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i$$

Subject to

$$y_t - s_{t-1} + s_t - \sum_{j=1}^{n} a_{tj}\ x_j \leqq E_t$$

$$t = 1, 2, \ldots T \qquad \text{(P)}$$

$$s_o = 0$$

$$x_j = \text{'0' or '1'}$$

$$j = 1, 2, \ldots n$$

$$y_t,\ s_t \geqq 0$$

where $x_j = 1$ if the project 'j' is accepted, and $x_j = 0$ if the 'j'th project is rejected.

The model is a mixed-zero one quadratic programming model. If we assume that $p_t > 0$, for all 't', the coefficient of y_t in the objective function will be greater than zero. Thus, we may treat the constraints as equality constraints, since any slack would clearly be paid out as a dividend. If, however, one were to impose the restriction that funds could not be carried over from one period to the next, i.e., $s_t = 0$ for all 't', then

$$y_t = E_t + \sum_{j=1}^{n} a_{tj}\ x_j$$

$$t = 1, 2, \ldots T$$

and the model can be reformulated as

Maximize

$$\sum_{t=1}^{T} p_t \left(E_t + \sum_{j=1}^{n} a_{tj}\, x_j\right) +$$

$$\tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} \left(E_t + \sum_{j=1}^{n} a_{tj}\, x_j\right) c_{ti} \left(E_t + \sum_{j=1}^{n} a_{tj}\, x_j\right)$$

subject to

$$\left(E_t + \sum_{j=1}^{n} a_{tj}\, x_j\right) \geqq 0$$

$$t = 1, 2, \ldots T$$

$$x_j = \text{'0' or '1'}$$

$$j = 1, 2, \ldots n$$

which is a pure zero-one integer quadratic programming model.

The model under consideration is a special case of general quadratic model (P) discussed in Chapter II, wherein we have

$$E = \emptyset \quad \text{and}$$

$$M_1 = \emptyset$$

The primal problem in Balas notation reads as (P')

$$\text{Max} \quad c^2\, x^2 + \tfrac{1}{2}\, x^2\, C^{22}\, x^2$$

S.T.

$$A^{21}\, x^1 + A^{22}\, x^2 \leqq b^2 \qquad \text{(P)}$$

$$x^1 \in X^1\, x^2,\ b^2 \geqq 0$$

C^{22} is negative semi-definite.

The dual D' of (P') can be stated as

$$\max_{x^1} \; \min_{x^2, u^2} \quad g = u^2 \, b^2 - \tfrac{1}{2} \, x^2 \, C^{22} \, x^2 - v^1 \, x^1$$

Subject to

$$u^2 \, A^{21} - v^1 = 0$$

$$u^2 \, A^{22} - x^2 \, C^{22} - v^2 = c^2$$

$$x^1 \in X^1$$

$$u^2, \; v^2 \geqq 0$$

$$v^1 \text{ unconstrained}$$

The following are the notational correspondence between the model under consideration and Balas' general model for the quadratic case.

The Model	Balas
$(x_1, x_2 \; . \; . \; . \; . x_n)$	x^1
$(y_1, y_2, \; . \; . \; . y_T : s_1, s_2 \; . \; . \; . \; . s_T)$	x^2
$(u_1, u_2 \; . \; . \; . \; . u_T)$	u^2
$(E_1, E_2 \; . \; . \; . \; . E_T)$	b^2
$(p_1, p_2 \; . \; . \; . \; . p_T)$	c^2
(c_{ti})	C^{22}
$(-a_{tj})$	A^{21}

$$\begin{bmatrix} 1 & 0 & 0 & . & . & . & 0 & 1 & 0 & 0 & 0 & 0 & . & 0 \\ 0 & 1 & 0 & . & . & . & 0 & -1 & 1 & 0 & 0 & 0 & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & . & . & . & . & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \qquad A^{22}$$

(T X 2T)

$$t = 1, 2, \ldots T$$

$$i = 1, 2, \ldots T$$

$$j = 1, 2, \ldots n$$

Using the notational correspondence, the dual of the model can be stated below.

<u>The Dual</u>

$$\underset{x_j}{\text{Max}} \quad \underset{y_t, u_t}{\text{Min}} \quad \sum_{t=1}^{T} u_t \, E_t - \tfrac{1}{2} \sum_{i=1}^{T} \sum_{i=1}^{T} y_t \, c_{ti} \, y_i - \sum_{j=1}^{n} v_j \, x_j$$

subject to

$$- \sum_{t=1}^{T} u_t \, a_{tj} - v_j = 0 \qquad \text{(D)}$$

$$j = 1, 2, \ldots n$$

$$u_t - y_t \sum_{i=1}^{T} c_{ti} - v_t{}' = p_t$$

$$t = 1, 2, \ldots T$$

$$u_t - u_{t+1} - v_t{}'' = 0$$

$$t = 1, 2, \ldots T\text{-}1$$

$$u_T \geqq 0$$

$$u_t, \, y_t \geqq 0 \quad \text{for all 't'}$$

$$v_j \text{ unconstrained}$$

The dual problem is a max-min type optimization of the quadratic mixed integer programming problem. The dual variables 'u_t' for all 't', and the surplus variables 'v_j' for all 'j' are continuous; 'v_j' is unconstrained in sign, as the vector corresponding to the integer constrained primal variable 'x_j' (partial relaxation of constraints).

Based on the theorems and lemmas stated in Chapter II, various properties of the primal and dual problems formulated can be derived.

Let the vector of slack variables of Primal Problem (P) be $(K_1, K_2, \ldots K_T)$. Since this vector is componentwise separable with respect to the vector $(x_1, x_2, \ldots x_n)$, we have the following results:

'If (P) has an optimal solution $(\bar{y}_1, \bar{y}_2, \ldots \bar{y}_T, \bar{s}_1, \bar{s}_2, \ldots \bar{s}_T, \bar{x}_1, \bar{x}_2, \ldots \bar{x}_n)$, then there exists a set of dual variables $(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_T)$ such that $(\bar{u}_1, \bar{u}_2, \ldots \bar{u}_T; \bar{x}_1, \bar{x}_2, \ldots \bar{x}_n; \bar{y}_1, \bar{y}_2, \ldots \bar{y}_T; \bar{s}_1, \bar{s}_2, \ldots \bar{s}_T)$ is an optimal solution to the dual with

$$\text{Max} \quad \sum_{t=1}^{T} p_t\, y_t + \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i = \underset{x_j}{\text{Max}} \quad \underset{u_t, y_t}{\text{Min}} \quad \sum_{t=1}^{T} u_t\, E_t$$

$$- \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i - \sum_{j=1}^{n} v_j\, x_j$$

and at optimality, we have

$$\sum_{t=1}^{T} p_t\, \bar{y}_t + \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} \bar{y}_t\, c_{ti}\, \bar{y}_i =$$

$$\sum_{t=1}^{T} \bar{u}_t\, E_t - \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} \bar{y}_t\, c_{ti}\, \bar{y}_i - \sum_{j=1}^{n} \bar{v}_j\, \bar{x}_j$$

From the dual constraint set

$$\bar{v}_j = - \sum_{t=1}^{T} \bar{u}_t \, a_{tj}$$

Substituting, we have

$$\sum_{t=1}^{T} \bar{u}_t \, E_t - \sum_{t=1}^{T} p_t \, \bar{y}_t - \sum_{t=1}^{T} \sum_{i=1}^{T} \bar{y}_t \, c_{ti} \, \bar{y}_i$$

$$+ \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t \, a_{tj} \, \bar{x}_j = 0$$

The above relation relates the optimal dual and primal variables with the corresponding cash-flows and dividend payments.

Further, the function

$$F\,(x,\,u) = cx + \tfrac{1}{2}\, xCx + ub - \tfrac{1}{2}\, uEu - uAx + u^1\, A^{11}\, x^1$$

has a saddle point at $(\bar{x},\,\bar{u})$:

$$F\,(x,\,\bar{u}) \leqq F\,(\bar{x},\,\bar{u}) \leqq F\,(\bar{x},\,u)$$

for all $x \in X\,(\bar{u},\,\bar{y}^2)$, and for all $u \in U\,(\bar{x},\,\bar{v}^2)$, where $\bar{y}^2$, $\bar{v}^2$ are defined by $(\bar{x},\,\bar{u})$.

For the model under consideration

$$F\,(x,\,u) = c^2\, x^2 + \tfrac{1}{2}\, x^2\, C^{22}\, x^2 + u^2\, b^2 - u^2\, A^{21}\, x^1 - u^2\, A^{22}\, x^2$$

or

$$F\,(x_j,\, y_t,\, s_t;\, u_t) = \sum_{t=1}^{T} p_t\, y_t + \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t\, c_{ti}\, y_i + \sum_{t=1}^{T} u_t\, E_t$$

$$+ \sum_{j=1}^{n} \sum_{t=1}^{T} u_t\, a_{tj}\, x_j - \sum_{t=1}^{T} u_t\, s_t \sum_{t=1}^{T-1} u_{t+1}\, s_t$$

has a saddle point at

$$F\ (x_j,\ y_t,\ s_t;\ \bar{u}_t) \leqq F\ (\bar{x}_j,\ \bar{y}_t,\ \bar{s}_t;\ \bar{u}_t) \leqq F\ (\bar{x}_j,\ \bar{y}_t,\ \bar{s}_t;\ u_t)$$

Further, from corollary to the theorem 1, of Chapter II, the following relationships hold:

$$\sum_{t=1}^{T} \bar{u}_t\ \bar{K}_t = 0$$

or

$$\bar{u}_t\ \bar{K}_t = 0$$

$$t = 1,\ 2,\ \ldots\ T \qquad \text{I}$$

$$(\bar{u}_t - \bar{u}_{t+1})\ \bar{s}_t = 0$$

$$t = 1,\ 2,\ \ldots\ T\text{-}1 \qquad \text{II}$$

$$u_T\ s_T = 0$$

$$\sum_{t=1}^{T} \bar{u}_t\ E_t + \sum_{j=1}^{n} \sum_{t=1}^{T} \bar{u}_t\ a_{tj}\ \bar{x}_j - \sum_{t=1}^{T} p_t\ \bar{y}_t - \sum_{t=1}^{T} \sum_{i=1}^{T} \bar{y}_t\ c_{ti}\ \bar{y}_i = 0 \qquad \text{III}$$

The complementary property of result (III) is the same as we have shown earlier.

It could be seen from result (II) that if at optimality, $\bar{u}_t \neq \bar{u}_{t+1}$, then $\bar{s}_t$ must be equal to zero and vice-versa. Similarly, at the 'T'th period, for optimal conditions, if u_T is strictly greater than zero, then s_T must be equal to zero and vice-versa.

Partitioning the Problem

The model (P) under consideration can be stated in matrix notation as

$$\text{Maximize} \quad P' Y + \tfrac{1}{2} Y' C Y$$

Subject to

$$I Y + D S - A X = E \tag{4.1}$$

$$Y, S \geqq 0$$

$$X = '0' \text{ or } '1'$$

where P, Y, S and E are 'T' component column vectors with elements p_t, y_t, s_t, and E_t respectively; 'X' an 'n' component binary column vector; A, a 'T' by 'n' matrix with elements 'a_{tj}'; I, a T by T identity matrix; 'D' a 'T' by 'T' matrix with ones on the diagonal, minus ones immediately below the diagonal and zeros elsewhere; 'C' a negative semi-definite matrix with elements as 'c_{ti}'.

Letting A^t represent the 't'th row of the matrix, A, and defining $s_o = 0$, the 't'th constraint may be written as

$$y_t - s_{t-1} + s_t - A^t X = E_t$$

Adding the first 't' constraints, we obtain

$$y_1 \ldots y_t + s_t = E_1 + A^1 X + \ldots E_t + A^t X$$

and since s_t, $y_t \geqq 0$ for all 't' obtain

$$E_t + A^t X \ldots + E_1 + A^1 X = 0.$$

Since this holds for any $t = 1, 2, \ldots T$, any feasible solution to the model must satisfy the constraint set,

$$\sum_{k=1}^{t} (E_k + A^k X) \geqq 0$$

$$t = 1, 2, \ldots T$$

Conversely, any feasible solution to the preceding constraint set, we can obtain a feasible solution to the model. One such solution will be

$$s_t = \sum_{k=1}^{t} (E_k + A^k X) \text{ for all 't'}$$

$$y_t = \text{'0' for all 't'}.$$

Now consider the problem:

$$\text{Maximize } z_o$$

Subject to

$$z_o \leqq u^{k'} (E + AX) - \tfrac{1}{2} Y^{k'} C Y^k \qquad U^k, y^k \in K$$

$$\sum_{k=1}^{t} (E_k + A^k X) \geqq 0 \qquad (4.2)$$

$$t = 1, 2, \ldots T$$

$$\text{'X'} = \text{'0' or '1'}$$

where 'K' is a set of feasible solutions to the convex polyhedral set:

$$R = \left\{ U \middle| \begin{pmatrix} I \\ D \end{pmatrix} U - \begin{pmatrix} C \\ \emptyset \end{pmatrix} \begin{pmatrix} Y \\ S \end{pmatrix} \geqq \begin{matrix} P \\ \emptyset \end{matrix} \right\}$$

'U' a 'T' component column vector with elements 'u_t' and '$\emptyset$' a 'T' component null vector.

<u>Theorem 1</u>

If (z_o^*, X^*, Y^*) is an optimal solution to (4.2), there exists a vector (S^*) such that (Y^*, S^*, X^*) is an optimal solution to (4.1) with value

$$P'\, Y^* + \tfrac{1}{2}\, Y^{*\prime}\, C\, Y^* = z_o^*$$

<u>Proof:</u> Consider the following pair of quadratic programming problems:

For any given vector 'X' say X^+ we obtain from (4.1)

Maximize

$$P'\, Y + \tfrac{1}{2}\, Y'\, C\, Y$$

Subject to

$$\begin{aligned} I\, Y + D\, S &= E + A\, X^+ \\ Y,\ S &\geqq 0 \end{aligned} \tag{4.3}$$

with dual

Minimize

$$U'\, (E + A\, X^+) - \tfrac{1}{2}\, Y'\, C\, Y$$

Subject to

$$\begin{aligned} I\, U - C\, Y &\geqq P \\ D\, U &\geqq 0 \\ Y,\ S &\geqq 0 \end{aligned} \tag{4.4}$$

'U' unrestricted

From quadratic programming duality theory

$$U^{k'} (E + A X^{+}) - \tfrac{1}{2} Y^{k'} C Y^{k} \geqq P' Y^{k} + \tfrac{1}{2} Y^{k'} C Y^{k}$$

for all $U^k \varepsilon K$, where 'K' is the set of dual feasible solutions. Now for $X^{+} = X^{*}$, we have a feasible solution to (4.1) and hence to (4.3). Further, for $p_t > p_{t+1} > 0$ for all 't', one may verify that $U = P$; $Y = 0$ is a feasible solution to (4.4). Therefore, since (4.3) and (4.4) have feasible solutions, they must have finite optimum solutions.

Let Y^*, S^*, be the optimal solution to (4.3); for $X^{+} = X^{*}$, then we must have

$$P' Y^{*} + \tfrac{1}{2} Y'^{*} C Y^{*} = \min_{U^k \varepsilon K} U^{k'} (E + A X^{*}) - \tfrac{1}{2} Y^{k'} C Y^{k}$$

and for problem (4.2) we have

$$z_o^{*} = \min_{U^k \varepsilon K} U^{k'} (E \pm A X^{*}) - \tfrac{1}{2} Y^{k'} C Y^{k}$$

and hence

$$z_o^{*} = P' Y^{*} + \tfrac{1}{2} Y'^{*} C Y^{*}$$

Since Y^*, S^* is a feasible solution to (4.3), for $X^{+} = X^{*}$, (Y^*, S^*, X^*) is a feasible solution to (4.1). To show that it is optimal to (4.1), assume the contrary, i.e., assume (z_o^*, X^*, Y^*) is optimal to (4.2), but there exists a solution to (4.1), say (X^{**}, Y^{**}, S^{**}) with $P' Y^{**} + \tfrac{1}{2} Y'^{**} C Y^{**} > z_o^*$. Clearly, for

$X^{+} = X^{**}$, we obtain Y^{**}, an optimal solution to (4.3) for, if not, Y^{**} could not be optimal to (4.1). Thus for $X^{+} = X^{**}$, we obtain

$$P' \, Y^{**} + \tfrac{1}{2} \, Y'^{**} \, C \, Y^{**} = \min_{U^k \in K} U^{k'} \, (E + A \, X^{**}) - \tfrac{1}{2} \, Y'^{**} \, C \, Y^{**}$$

Now since, X^{**}, is a feasible 'X' vector for (4.1), it is also feasible for (4.2) and thus we obtain a solution to (4.2) with value z_o^{**}

$$z_o^{**} = \min_{U^k \in K} \quad U^{k'} \, (E + A \, X^{**}) - \tfrac{1}{2} \, Y^{k'} \, C \, Y^{k} =$$

$$P'Y^{**} + \tfrac{1}{2} \, Y'^{**} \, C \, Y^{**} > z_o^{*}$$

This contradicts our assumption that (z_o^*, X^*, Y^*) is the optimal solution to (4.2), and thus given an optimal solution to (4.2), there exists an optimal solution to (4.1) with value z_o^*.

The problem (4.1) can be solved by solving first the problem (4.2) and then given X^*, the optimal (X) vector, for (4.2), set $X^{+} = X^{*}$, and solve the quadratic programming problem (4.3) for Y^*, S^*. (X^*, Y^*, S^*) will be the optimal solution to (4.1). A direct solution of problem (4.2) would require the enumeration of all feasible solution to the set 'R'. This, of course, is not possible, since the set is infinite. One may therefore use a procedure identical to the one developed by Unger (22), which is based on the partitioning algorithm developed by Benders (6). The salient features of the solution procedure are discussed in the next section.

A Solution Procedure

Benders (6) suggests the following iterative procedure to solve the mixed integer programming problem.

Step 1: Start with some subset K' and K and solve the integer programming problem.

Step 2: Given X^*, the optimal integer solution for the subset, solve the quadratic programming problem. Check whether $z_o^* = U^{*'}\ (E + A\ X^*) - \frac{1}{2}\ y^{k'}\ C\ y^{k'}\ C\ y^k$; if 'yet' stop. If not, we have found a U^*, contained in K, but not in K'; add U^*, Y^* to K' and go to step 1.

Application of the above procedure may require the solution of many integer programming problems. Using Balas' concept of implicit enumeration, we may avoid this by solving more quadratic programming problems.

The solution procedure is essentially similar to the procedure developed by Unger (22) for the linear case for solving problem (4.2).

$$\text{Max } z_o$$

Subject to

$$z_o \leqq U^{k'}\ (E + A\ X) - \tfrac{1}{2}\ Y^{k'}\ C\ Y^k$$

$$\Sigma_{k=1}^{t}\ (E_k + A^k\ X) \geqq 0$$

$$t = 1, 2, \ldots T \tag{4.5}$$

$$X = \text{'0' or '1'}$$

Letting

$$c_j^k = \sum_{t=1}^{T} u_t^k\, a_{tj}$$

$$b^k = \sum_{t=1}^{T} u_t^k\, E_t - \tfrac{1}{2} \sum_{t=1}^{T} \sum_{i=1}^{T} y_t^k\, c_{ti}\, y_i^k$$

$$b_{tj} = \sum_{k=1}^{t} a_{kj}$$

$$f_t = \sum_{k=1}^{t} E_k$$

We may restate the problem (4.5) as

$$\text{Maximize} \quad z_o$$

subject to

$$
\begin{aligned}
& z_o \leqq b^k + \sum_{j=1}^{n} c_j^k\, x_j \\
& k = 1, 2, \ldots m \\
& F + B\, X \geqq 0 \\
& x_j = \text{'0' or '1'} \\
& j = 1, 2, \ldots n
\end{aligned}
\qquad (4.6)
$$

where F is the T component column vector with elements 'f_t', B the 'T' by 'n' matrix with components 'b_{tj}' and 'm' the number of feasible solutions in the set K'.

If K' = K, we shall refer to (4.6) as the complete problem. Otherwise, we shall refer to (4.6) as the restricted problem.

Any binary vector 'X' will be called a solution to (4.6).

A solution satisfying the constraint set $F + B\ X \geqq 0$ will be called a feasible solution, and a feasible solution that maximizes z_o over all feasible solutions for the complete problem will be called an optimal feasible solution. We shall refer to constraints of the form

$$z_o \leqq b^k + \sum_{j=1}^{n} c_j^k\ x_j$$

as objective function constraints since they limit the maximum value of z_o, but do not affect feasibility. A partial solution S will be an assignment of binary values to a subject of the 'n' binary variables. The variables not assigned values by S can take on either the value '0' or '1' and are called free. We define a completion of a partial solution S as the binary vector X^S determined by S and an assignment of binary values to the free variables.

As a bookkeeping procedure, we adopt the convention that symbol 'j' denotes $x_j=1$ and the symbol '-j' denotes $x_j=0$. z^+ will denote the current best feasible solution. We employ the backtracking scheme discussed by Geoffrion, and, given any partial solution S, we shall attempt to fathom that solution by either of the following:

1. Finding the feasible completion of S that maximizes z_o for all $U^k \in K$;
2. Showing that there exists no feasible completion of S with value greater than the current best feasible solution z^+.

To accomplish this fathoming, we use the simple computational algorithm developed by Unger (22). Steps (1) through (10) of the algorithm are the same except step (9), where in the present case, having obtained the optimal solution (z_o^*, X^*), a quadratic programming problem has to be solved instead of a linear programming problem.

For the given vector X^s, the solution of quadrating programming problem defined by Primal (4.3) and dual (4.4) consists of finding a feasible solution to the following constraint sets:

$$y_t^k - s_{t-1}^k - s_t^k + K_t = E_t + \sum_{j=1}^{n} a_{tj}\, x_j^s$$

$$t = 1, 2, \ldots T \tag{4.7}$$

$$u_t^k - y_t^k \sum_{i=1}^{T} c_{ti} - v_t^{'k} = p_t$$

$$t = 1, 2, \ldots T$$

$$u_t^k - u_{t+1}^k - v_t^{''k} = 0 \tag{4.8}$$

$$t = 1, 2, \ldots T-1$$

$$u_T^k - v_T^{''k} = 0$$

$$y_t^k,\ s_t^k,\ u_t^k,\ K_t^k,\ v_t^{'k},\ v_t^{''k} \geqq 0$$

$$u_t^k\, K_t^k = 0$$

$$y_t^k\, v_t^{'k} = 0$$

$$s_t^k\, v_t^{''k} = 0 \tag{4.9}$$

$$t = 1, 2, \ldots . T$$

To show that the above procedure is finite for a given X^+, we get the solution for the quadratic programming problem. If the solution is not optimal, a constraint is generated. Since there are only finite number of X^+ s, we can only generate finite number of constraints. If we have considered all possible constraints, any further constraint that will be generated will be the one that has already been considered. The stopping rule has to be met. The procedure therefore is finite.

To show that the solution is optimal consider that given $k' \subset k$, X^* is the best possible solution for that subset. Given X^*, and if we solve the quadratic programming problem and the stopping rule is met at this stage, then the constraint generated by the problem has already been satisfied by X^*. That is, we have found X^* such that

$$z_o^* = \min_{U^k \in k} \quad U^{k'} \; (E + A\;X^*) - \tfrac{1}{2}\;Y^{k'}\;C\;Y^k \quad \text{and}$$

$$z_o = U^{*'} \; (E + A\;X^*) - \tfrac{1}{2}\;y^{k'}\;C\;y^k$$

$$\text{all } U^k \in K$$

Therefore adding any additional constraint will not result in an increase in z_o. Therefore the solution is optimal.

We start the algorithm with $S = \emptyset$ $Y = \emptyset$ and as the initial set K', we may use $U = P$, since this is a feasible solution to the set 'R', under the assumption $p_t > p_{t+1} > 0$. Further under the assumption $E_t' \geq 0$ for all 't', $X = \emptyset$ is a feasible solution to the complete problem, and thus we may use the initial best solution $z^+ = \sum_{t=1}^{T} p_t \; E_t$. After termination, we calculate the optimal Y, and S, by solving the problem (4.3) with $X^+ = X^*$, the optimal integer solution.

The above solution procedure is illustrated by solving a numerical example, a two period three projects case, in the following section.

<u>An Example</u>

To illustrate the solution procedure discussed in (22), consider the following problem:

Maximize

$$0.9\ y_1 + 0.8\ y_2 - y_1^{\ 2}$$

S.T.

$$y_1 + s_1 + 100\ x_1 + 100\ x_2 + 100\ x_3 \leqq 200$$

$$y_2 - s_1 + s_2 - 100\ x_1 + 300\ x_2 + 100\ x_3 \leqq 300$$

$$x_j = \text{'0' or '1'}$$

$$j = 1, 2, 3$$

$$y_t,\ s_t \geqq 0$$

$$t = 1, 2$$

In Balas' notation, we have

$$c^1 = (0, 0, 0)$$
$$c^2 = (0.9, 0.8, 0, 0)$$
$$x^1 = (x_1, x_2, x_3)$$
$$x^2 = (y_1, y_2, s_1, s_2)$$

$$C^{11} = \emptyset \quad C^{12} = \emptyset \quad C^{21} = \emptyset \quad C^{22} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^{21} = \begin{bmatrix} -100 & -100 & -100 \\ 100 & -300 & -100 \end{bmatrix}$$

$$A^{22} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$b^2 = \begin{bmatrix} 200 \\ 300 \end{bmatrix}$$

<u>Start</u>

$$S = \emptyset \quad K' = (U') \quad \text{where } U' = (0.9,\ 0.8)$$

$$z^+ = \sum_{t=1}^{2} P_t\ E_t = 420$$

and the best known solution is $X = \emptyset$.

Given $K' = (U')$ the integer programming problem may be formulated as

$$\text{Max} \quad z_o$$

S.T.
$$z_o \leqq 420 - 10\ x_1 - 330\ x_2 - 170\ x_3$$

$$200 - 100\ x_1 - 100\ x_2 - 100\ x_3 \geqq 0$$

$$500 - 0\ x_1 - 400\ x_2 - 200\ x_3 \geqq 0$$

$$x_1,\ x_2,\ x_3 = \text{'0' or '1'}$$

Steps of the Algorithm

$$X^{s1} = (-1, -2, -3)$$

$$z_1 = 420 \qquad (1)$$

$$Z = z_1 \qquad X^s = X^{s1} \qquad (2)$$

$$z'_1 = z_1 \text{ thus } z' = z_1 \qquad (4)$$

$$Y^s = (200, 500) \qquad (5)$$

and is feasible. So as per step 9, the quadratic programming problem to be solved is

$$y_1 + s_1 + K_1 = 200$$

$$y_2 - s_1 + s_2 + K_2 = 300$$

$$2y_1 + u_1 - v_1 = 0.9$$

$$u_2 - v_2 = 0.8$$

$$u_1 - u_2 - v_3 = 0$$

$$u_2 - v_4 = 0$$

$$u_1 K_1 = 0 \qquad u_2 K_2 = 0$$

$$y_1 v_1 = 0 \qquad y_2 v_2 = 0$$

$$s_1 v_3 = 0 \qquad s_2 v_4 = 0$$

$$y_1, y_2, s_1, s_2, u_1, u_2, v_1, v_2, v_3, v_4 \geq 0$$

The optimal solution is

$$s_1^* = 199.95 \qquad y_1^* = 0.05 \qquad K_1^* = 0$$

$$s_2^* = 499.95 \qquad y_1^* = 0 \qquad K_2^* = 0$$

$$u_1^* = 0.8 \qquad v_1^* = v_2^* = v_3^* = 0$$

$$u_2^* = 0.8 \qquad v_4^* = 0.8$$

The new objective function constraint is

$$z_o \leqq 400.0025 + 20\ x_1 - 320\ x_2 - 160\ x_3$$

The integer programming problem to be solved is

$$\text{Max } z_o$$

$$z_o \leqq 420 - 10\ x_1 - 330\ x_2 - 170\ x_3$$

$$z_o \leqq 400.0025 + 20\ x_1 - 320\ x_2 - 160\ x_3$$

$$200 - 100\ x_1 - 100\ x_2 - 100\ x_3 \geqq 0$$

$$500 - 0\ x_1 - 400\ x_2 - 200\ x_3 \geqq 0$$

$$x_1,\ x_2,\ x_3 = \text{'0' or '1'}.$$

$$X^{s_1} = (-1,\ -2,\ -3)$$

$$X^{s_1} = (1,\ -2,\ -3) \qquad (1)$$

$$z_1 = 420$$

$$z_2 = 400.0025$$

$$z = z_1 \qquad X^{s_1} = X^{s} \tag{2}$$

$$z_1' = 420 \qquad z_2' = 400.0025$$

$$z' = z_2' \tag{4}$$

$$z' < z$$

$$Y^s = (200,\ 500) \geqq 0 \tag{5}$$

The solution is feasible. Since $X = (-1, -2, -3)$ is the same as the one obtained earlier. Therefore, backtracking

$$X^s = (1, -2, -3)$$

is the optimal solution to the problem and we have values as:

$$z_o^* = 400.0025$$

$$y_1^* = 0.05 \qquad K_1^* = 0$$

$$y_2^* = 499.95 \qquad K_2^* = 0$$

$$s_1^* = 99.95 \qquad u_1^* = 0.8 \qquad v_1^*,\ v_2^*,\ v_3^*, = 0$$

$$s_2^* = 0 \qquad u_2^* = 0.8 \qquad v_4^* = 0.8.$$

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