PROJECTION METHODS FOR NONLINEAR BOUNDARY VALUE PROBLEMS

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Finally, I want to dedicate this work to my wife, Bea, and thank her for her patience and understanding during all my years as a graduate student.

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CHAPTER I

INTRODUCTION

The theory of spline functions has undergone rapid development in the 1960's. They have been shown to have wide applicability to eigenvalue and eigenvector problems, initial value problems, optimal quadrature formula, approximation theory and stochastic processes. For extensive bibliographies of papers in spline theory and its applications, see [20], [32], and [33].

One particularly effective application of spline functions has been to the approximation of the solutions to linear and mildly nonlinear boundary value problems using the Galerkin method. Ciarlet, Schultz, and Varga [7], Perrin, Price, and Varga [23], Schultz [32], and Lucas [20] have studied the theoretical rates of convergence for the Galerkin method with splines, while Herbold [11] and Herbold, Schultz, and Varga [12] have investigated the associated computational aspects of the method.

The Galerkin method can be viewed as a specific projection technique in the sense, for example, of Kantorovich and Akilov [13]. See also Vainikko [40], Krasnosel'skii [17], Petryshyn [25], Pol'skii [26], and deBoor [3]. Let X_n be a sequence of finite dimensional subspaces of a normed linear space X and T be a mapping from X to a normed linear space Y. Let P_n be a sequence of projections on Y. A projection method defines an approximation to the solution of the equation Tu=f to be a solution of the equation

$$P_n T u_n = P_n f \tag{1.1}$$

with u_n in X_n. The questions of existence, uniqueness, and convergence of the approximations of course now follow. Projection methods other than the Galerkin method have been used to approximate the solution to linear and nonlinear boundary value problems, see [3], [10], [13], and [38]. Although the use of such methods over subspaces of spline functions apparently offers significant computational advantages, such techniques do not seem to have been as well investigated.

As an example of the equation Tu=f, consider the two-point boundary value problem

$$u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x), \qquad 0<x<1, \tag{1.2}$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Suppose a_0, a_1 and f are continuous. Defining $X = \{u\epsilon C^2[0,1]:u(0)=u(1)=0\}$ and $Y = C[0,1]$, then $Tu = u'' + a_1u' + a_0u$ defines a mapping T from X into Y. Moreover, (1.2) with the boundary conditions may now be written as Tu=f. Let $\{w_i^n\}_{i=1}^n$ be n linearly independent vectors in X and let X_n be their linear span. Let P_n be projections mapping Y onto subspaces Y_n of Y with dimension n. The projections P_n can then be represented by n linear functionals $\{\lambda_i^n\}_{i=1}^n$, i.e. $P_n f=s$ if and only if

$$\lambda_i^n f = \lambda_i^n s, \quad i=1,\ldots,n \text{ and } s\varepsilon Y_n.$$

Then the equation $P_n T u_n = P_n f$ for (1.2) becomes equivalent to the n algebraic equations

$$\lambda_i^n\big(u_n''(x)+a_1(x)u_n'(x)+a_0 u_n(x)\big) = \lambda_i^n f(x), \quad i=1,\ldots,n,$$

where $u_n(x)$ is defined by

$$u_n(x) = \sum_{i=1}^{n} \alpha_i^n w_i^n(x).$$

From a computational standpoint the selection of the basis $\{w_i^n\}$ for X_n and the linear functionals $\{\lambda_i^n\}$ to represent P_n becomes very important. This will be illustrated in Chapters II and III when some specific projections are studied in detail. Selection of norms will also be discussed in Chapter II.

Using the setting of problem (1.2), we now give several examples of projection operators that have been used extensively in the past. For the moment, the approximations u_n can be thought of as being polynomials of degree n+1 that satisfy $u_n(0) = u_n(1) = 0$. Let $\{w_i^n\}_{i=1}^n$ be a basis for this space of polynomials. The linear functionals associated with the Galerkin method are

$$\lambda_i^n f = \int_0^1 f(x)w_i^n(x)dx, \qquad i=1,\ldots,n,$$

that is, orthogonality is required (for $Tu_n - f$) with respect to the subspace of functions X_n. The Galerkin method is closely related to the method of Ritz [14] provided the method of Ritz can be applied to the problem. The Ritz or variational approach was one of the earliest techniques used to establish convergence for the Galerkin method and boundary value problems using splines, see [6].

For the collocation method [10], the λ_i^n take the form

$$\lambda_i^n f = f(x_i^n), \qquad i=1,\ldots,n$$

with $0 \le x_1^n < x_2^n < \ldots < x_n^n \le 1$. Applying this scheme to (1.2), the approximation equations become

$$u_n''(x_i^n) + a_1(x_i^n)u_n'(x_i^n) + a_0(x_i^n)u_n(x_i^n) = f(x_i^n), \qquad i=1,\ldots,n.$$

The method of least squares requires orthogonality with respect to the space of functions $T[X_n]$. Thus the linear functionals defining this projection are

$$\lambda_i f = \int_0^1 f(x)(Tw_i^n)(x)dx, \qquad i=1,\ldots,n.$$

For some recent work using spline functions and the method of least squares, see [2].

Another scheme that has been used is the partition method, see [10]. This technique is also known in the literature as the sub-domain method. The linear functionals associated with it are

$$\lambda_i^n f = \int_{x_{i-1}^n}^{x_i^n} f(x)dx, \qquad i=1,\ldots,n,$$

with $0 = x_0^n < x_1^n < \ldots < x_n^n = 1$.

The last example considered is the method of moments [13], which generalizes both the method of Galerkin and the method of least squares. In this case, a linearly independent set of functions $\{y_i^n\}_{i=1}^n$ is chosen and orthogonality is required with respect to each of the y_i^n, i.e.

$$\lambda_i^n f = \int_0^1 f(x)y_i^n(x)dx, \qquad i=1,\ldots,n.$$

Theoretically, one can generate approximation schemes simply by choosing linear functionals and bases elements. Of course, existence and convergence of the scheme would then need to be shown. The objective of this thesis is threefold. The first is to develop new projection schemes that are easier to apply than the Galerkin method and are applicable to a wider class of problems. The second is to study the Galerkin method itself and extend the class of problems to which it can be applied. And the third is to establish general criteria for the development of new projection schemes with splines and to illustrate several approaches to showing convergence and convergence rates. Emphasis will be on nonlinear problems.

In Chapter II high order methods are developed for general second order boundary value problems of the form

$$D^2u = f(x,u,u'), \quad 0 < x < 1,$$

with boundary conditions

$$u(0) = u(1) = 0.$$

Convergence is established for three methods different from the Galerkin method that achieve the same rate of convergence as the Galerkin method. One method uses cubic splines (see Definition 1.1) and requires orthogonality with respect to linear splines. Apparently this method has been studied in the literature for second order linear problems only. Moreover, convergence there is only third order if the solution is in $C^4[a,b]$, whereas here it is shown to actually be fourth order, see [3]. A new weighted sub-domain is developed. Also, the Galerkin method is studied and the class of problems to which it can be applied are extended. Results of numerical experiments are reported for some of these methods. The results developed in this chapter and Chapter III are based on some developments in the general theory of approximation methods by G. M. Vainikko [40]. For some important work on projection methods applied to second order linear problems, see deBoor [3].

In Chapter III, several new projections of collocation type are introduced. Moreover, the problems studied are generalized to equations of the form

$$D^m u = f(x,u,\dots,u^{m-1}), \qquad a<x<b,$$

with boundary conditions

$$\sum_{j=0}^{m-1} [a_{ij}u^j(a) + b_{ij}u^j(b)] = 0, \qquad i=1,\dots,m.$$

A general result, Theorem 3.2, is proved and it provides criteria for testing if a proposed projection scheme is convergent. Chapter III emphasizes along with the theoretical development the application of these schemes. Several theorems are given describing classes of problems to which the methods of Chapters II and III can be applied. Moreover, numerical implementation of these methods on an electronic computer and the results of numerical experiments are discussed.

In Chapter IV, attention is focused on nonlinear boundary value problems of the form

$$(-1)^m D^{2m} u + f(x,u,\dots,u^j) = 0, \qquad 0<x<1,\ 0\le j\le m,\ m\ge 1$$

with boundary conditions

$$u^k(0) = u^k(1) = 0, \qquad 0\le k\le m-1;$$

and

$$(-1)^m D^{2m-1} u + f(x,u,\dots,u^j) = 0, \qquad 0<x<1,\ 0\le j\le m,\ m\ge 2 \qquad (1.3)$$

with boundary conditions

$$u^i(0) = u^k(1) = 0, \qquad 0 \le i \le m-1,\ 1 \le k \le m-1 \tag{1.4}$$

and the Galerkin method. In contrast to the previous chapters, monotone operator theory is used to establish convergence. A specific regularity hypothesis on f is made and the result is obtained that the rate of convergence is dependent on the order of the highest derivative that appears as an argument in f. This result is applied in Chapter IV to improve the known convergence rate for the Galerkin method applied to a specific third order problem of the form

$$-D^3u = f(x,u,u'), \qquad 0<x<1,$$

with boundary conditions

$$u(0) = Du(0) = Du(1) = 0,$$

see [7].

We conclude this section by recalling some results in the theory of spline functions. The splines used in this thesis are simply piecewise polynomial functions, which is the early characterization of splines. Many generalizations of the notion of spline function have appeared, some involving explicit approaches, others implicit approaches based on some of the extremal properties of splines, see [20] and [33].

Definition 1.1. Let π_n: $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition of the interval $[a,b]$. A real-valued function $s(x)$ satisfying

(1) $s \epsilon C^q[a,b]$ for some integer $q \geq 0$, and

(2) s is a polynomial of degree less than or equal to p, $p \geq q+1$, on each subinterval $[x_i, x_{i+1}]$ of π_n is called a polynomial spline (or simply spline) of degree p. The set of all such functions for some partition π_n will be denoted by $Sp(\pi_n, p, q)$.

Note that $Sp(\pi_n, p, q)$ is a linear subspace of $C^q[a,b]$ of dimension $n(p-q) + q + 1$.

Generalizations of these spline functions include allowing the continuity to vary from one partition point to another. Also, the splines can be chosen to be piecewise functions in the null space of an operator $L^* L$, where $Lu = \sum_{j=0}^{m} a_j(x) u^j(x)$ and L^* denotes the formal adjoint of L. Such functions are called L-splines and include the case of polynomial splines of degree 2m-1 when $L=D^m$, see [33]. These more general splines can be used in many of the methods given in this thesis; however, there appears to be no advantage in doing so. The simpler the spline space used and the less its dimension, the easier it is to set up the algebraic equations defining the approximation and the smaller will be the size of the system. Moreover, convergence depends essentially on the degree of the splines and the number of partition points. Thus, in general, it is advantageous to work with polynomial splines with $q = p-1$, i.e. maximum continuity. In this case the spline space will be denoted simply by $Sp(\pi_n, p)$.

Let $\|f\|_{L^\infty} = \sup_{a \leq x \leq b} |f(x)|$, $\bar{\pi}_n = \max_i (x_{i+1} - x_i)$, and $\underline{\pi}_n = \min(x_{i+1} - x_i)$. A sequence of partitions $\{\pi_n\}$ of $[a,b]$ satisfying $(\bar{\pi}_n / \underline{\pi}_n) \leq \alpha$ for some constant $\alpha > 0$ is said to be *quasi-uniform*. These sequences do not have to be sequences of refinements.

<u>Theorem 1.1</u>. Let f be in $C^t[a,b]$ and $\{\pi_n\}$ be a quasi-uniform sequence of partitions of [a,b]. Let $Sp(\pi_n,p,q)$ be associated spaces of splines of degree $p\geq t$. Then there exist constants $K_j>0$ and independent of f and n so that

$$\inf_{s\in Sp(\pi_n,p,q)} \|D^j(f-s)\|_{L^\infty} \leq K_j\bar{\pi}_n^{t-j}\omega(f^t,\bar{\pi}_n),$$

where $\omega(f^t,\bar{\pi}_n)$ denotes the modulus of continuity of f^t with respect to $\bar{\pi}_n$, i.e. $\sup_{|x-y|\leq\bar{\pi}_n} |f^t(x)-f^t(y)|$, and the L^∞ norm is interpreted over [a,b] less the partition points if D^js is not continuous. Moreover, this result is unchanged if f(a) = f(b) = 0 and the infimum is taken over functions in $Sp(\pi_n,p,q)$ that satisfy the same conditions.

For a proof of this theorem, see [4] and [30]. It is shown in [4] that for the case j=0, the mesh restriction can be removed. Unless results follow for arbitrary partitions without the need to develop additional approximation theory, results are given in this thesis for quasi-uniform partitions (if possible) since from a computational standpoint such partitions represent more than adequate generality. This is particularly true in Chapter II.

<u>Definition 1.2</u>. Given a function $f\varepsilon C^1[a,b]$, let its $Sp(\pi_n,3)$-interpolate be defined by $Q_nf=s$, $s\varepsilon Sp(\pi_n,3)$, where

$$f(x_i) = s(x_i) \quad \text{for all } x_i\varepsilon\pi_n$$

and

$$f'(a) = s'(a), \quad f'(b) = s'(b).$$

It is known that Q_n is well defined, $n \geq 1$, see Schultz and Varga [33]. Moreover, it is also known that given a quasi-uniform sequence of partitions of $[a,b]$, if $f \in C^j[a,b]$, $j=1,2,3$, or 4, then for some constant K independent of n

$$\|f - Q_n f\|_{L^\infty} \leq K\bar{\pi}_n^j.$$

Many authors have studied error bounds for cubic spline interpolation, see [1], [5], and [34]. In particular, deBoor [3] has shown that in the Banach space $C^1[a,b]$ with norm

$$\|f\|_X = |f(a)| + \|f'\|_{L^\infty},$$

that $\|Q_n\|_X \leq M$ for a constant M independent of n. For completeness we develop a special error bound that will be used repeatedly in Chapter II.

<u>Theorem 1.2</u>. Let $\{\pi_n\}$ be a quasi-uniform family of partitions of $[a,b]$. Then there exists a constant $K>0$ and independent of n so that for any $f \in C^1[a,b]$,

$$\|Q_n f - f\|_{L^\infty} \leq K\bar{\pi}_n \omega(f', \bar{\pi}_n).$$

Proof. Let f be in $C^1[a,b]$. There exists a $\bar{y}$ in $[a,b]$ so that

$$\|Q_nf-f\|_{L^\infty} = |(Q_nf)(\bar{y}) - f(\bar{y})|.$$

Let x_i be the closest partition point to $\bar{y}$. Then

$$\|Q_nf-f\|_{L^\infty} = \left|\int_{x_i}^{\bar{y}} \big((Q_nf)(x) - f(x)\big)'dx\right| \leq \bar{\pi}_n\|(Q_nf-f)'\|_{L^\infty}. \tag{1.5}$$

Also,

$$\begin{aligned}\|(Q_nf-f)'\|_{L^\infty} &= \|Q_nf-f\|_X \\ &\leq (1+M)\|f-s_n\|_X \\ &\leq (1+M)\{|f(a)-s_n(a)| + \|f'-s_n'\|_{L^\infty}\}\end{aligned}$$

for any $s_n \varepsilon Sp(\pi_n,3)$. Let s_n be a spline satisfying the conclusion of Theorem 1.1. Then

$$\|(Q_nf-f)'\|_{L^\infty} \leq (1+M)\{K_1\bar{\pi}_n\omega(f',\bar{\pi}_n) + K_2\omega(f',\bar{\pi}_n)\}. \tag{1.6}$$

The result follows by combining (1.5) with (1.6).

CHAPTER II

SOME PROJECTION METHODS FOR SECOND ORDER PROBLEMS

It is convenient theoretically to study projection methods as applied to equations of the form $v=Tv$ with T mapping a normed linear space X into itself rather than $Tu=f$ with T mapping X into some other normed linear space Y as introduced in Chapter I. For boundary value problems, the setting of a single space can be accomplished through a change of variables to be introduced later. Given a normed linear space X, a sequence of projections P_n on X with $P_n[X] = X_n$, and an operator T (nonlinear) on X, then approximations to the solution of $v=Tv$ can theoretically be found by solving $v_n = P_nTv_n$ with $v_n \varepsilon X_n$. The next theorem describes conditions for existence and convergence for the approximations generated by such a scheme. This theorem is a direct consequence of Theorem 3 in [40]. We note that the requirement that X be a Banach space in Theorem 3 is not necessary if approximations to $v=Tv$ are found from $v_n = P_nTv_n$ and each P_n has finite dimensional range. We will assume this is the case and that X is simply a normed linear space.

Theorem 2.1. Let X be a normed linear space with T and P_nT continuous over an open set $V \subset X$. Let the equation $v=Tv$ have a solution $v_0 \varepsilon V$, and let the following conditions be satisfied:

(1) $\|(I-P_n)v_0\| \to 0$ as $n \to \infty$;

(2) The operator T is continuously Fréchet differentiable at

the point v_0 and the linear operator $I - T'(v_0)$ is continuously invertible;

(3) The operator P_nT is Fréchet differentiable at the point v_0 while $(P_nT)'(v_0) = P_nT'(v_0)$ and for each $\epsilon>0$ one can find an integer N and a number $\delta>0$ such that

$$\|(I-P_n)T'(v)\| \leq \epsilon$$

when $n \geq N$ and $\|v-v_0\| \leq \delta$.

Then there exists an integer N_1 and a constant $\delta_1>0$ so that v_0 is unique in the sphere $\|v-v_0\| \leq \delta_1$, and whenever $n \geq N_1$, the equation $v_n = P_nTv_n$ has a unique solution in this same sphere. Moreover, there exists a constant $K>0$ so that

$$\|v_n-v_0\| \leq K\|(I-P_n)v_0\|.$$

Boundary Value Problems

The problems

$$D^2u = f(x,u) + e(x)u'(x) = f(x,u,u'), \qquad 0<x<1, \tag{2.1}$$

and

$$D^2u = f(x,u,u'), \qquad 0<x<1, \tag{2.2}$$

with boundary conditions

$$u(0) = u(1) = 0 \tag{2.3}$$

are considered. The two uses of the notation for the function f in (2.1) has been made so that (2.1) and (2.2) may be referred to later on using the same general notation. A distinction is made here, however, regarding the linearity or nonlinearity of f in u'. The first approximation scheme for these problems seeks approximate solutions over subspaces of cubic splines, $Sp(\pi_n,3)$. The projections P_n considered map $C[0,1]$ into $D^2Sp(\pi_n,3)$ and are defined by the space $D^2Sp(\pi_n,3)$ and the following n+1 linear functionals:

$$\lambda_0 f = \int_0^1 f(x)(x_1-x)_+ dx,$$

$$\lambda_i f = \int_0^1 f(x) g_i(x) dx$$

with

$$g_i(x) = \begin{cases} \dfrac{(x-x_{i-1})_+}{(x_i-x_{i-1})} & x \le x_i \\ \\ \dfrac{(x_{i+1}-x)_+}{(x_{i+1}-x_i)} & x > x_i \end{cases}, \quad i=1,\dots,n-1$$

and

$$\lambda_n f = \int_0^1 f(x)(x-x_{n-1})_+ dx,$$

where

$$(x)_+ = \begin{cases} x & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

Approximations to the solution of either (2.1) or (2.2) with boundary conditions (2.3) are found by solving

$$\lambda_i\left(u_n''(x)\right) = \lambda_i\left(f(x,u_n,u_n')\right), \qquad i=0,1,\ldots,n,$$

with $u_n \varepsilon Sp(\pi_n,3)$ and $u_n(0) = u_n(1) = 0$. We write this as

$$P_n u_n'' = P_n f(x,u_n,u_n') \qquad \text{and} \qquad u_n \varepsilon Sp_0(\pi_n,3).$$

Let G denote the inverse of D^2 with respect to the boundary conditions (2.3), and let $G(x,s)$ be the associated Green's function, that is

$$G(x,s) = \begin{cases} x(s-1) & x \leq s \\ s(x-1) & s < x \end{cases}$$

with $0 \leq x,s \leq 1$. Let $Q_n = GP_nD^2$. Clearly Q_n is a projection from $C^2[0,1]$ onto $Sp_0(\pi_n,3)$. If $GP_nD^2f = GP_nD^2s$, then $P_nf'' = P_ns''$, and so Q_n is defined by the linear functionals $\hat{\lambda}_i f = \lambda_i f''$ and the subspace $Sp_0(\pi_n,3)$. The linear functionals $\lambda_0,\ldots,\lambda_n$ chosen to represent P_n are computationally attractive because the integrals to be evaluated span at most two subintervals. Theoretically, however, the linear functionals could be defined by

$$\lambda_0 f = \int_0^1 (1-x)f(x)dx,$$

$$\lambda_i f = \int_0^1 G(x_i,x)f(x)dx, \qquad i=1,\ldots,n-1,$$

and

$$\lambda_n f = \int_0^1 xf(x)dx$$

since it can be shown that the functions $1-x$, x and $G_i(x_i,x)$, $i=1,\ldots,n-1$, form a basis for $D^2Sp(\pi_n,3)$ as do the functions $(x_1-x)_+$, $(x-x_{n-1})_+$ and $g_i(x)$, $i=1,\ldots,n-1$. It follows then that the Q_n's are represented by the linear functionals giving ordinary cubic spline interpolation, i.e. $Q_n f = s\varepsilon Sp(\pi_n,3)$ with

$$s(x_i) = f(x_i), \qquad i=0,\ldots,n$$

and

$$s'(0) = f'(0), \quad s'(1) = f'(1).$$

In order to establish existence of approximate solutions, convergence, and convergence rates for this method, Theorem 2.1 is applied. Thus equations (2.1), (2.2), and (2.3) must be put in the form $v=Tv$ and the normed space X identified. Let $v(x) = D^2u(x)$. Then $u = Gv$ and $u' = G_1v$, where $G_1v = \int_0^1 G_x(x,s)v(s)ds$. Substituting into (2.1) or (2.2) we have $D^2u = f(x,u,u')$ together with (2.3) becomes

$$v = f(x,Gv,G_1v) = Tv.$$

Let $X = C[0,1]$ and define $\|v\|_X = \|Gv\|_{L^\infty}$. Considering first (2.1) with (2.3), we have the following theorem.

Theorem 2.2. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of $[0,1]$ satisfying $\bar{\pi}_n \to 0$. Suppose u_0 is a solution to (2.1) with boundary conditions (2.3), where f and f_u are continuous on $N = \{0 \le x \le 1, |u(x)-u_0(x)| \le \delta, \delta > 0\}$ and $e \in C^1[0,1]$. If the equation $D^2u - f_u(x,u_0)u - e(x)u' = 0$ with boundary conditions (2.3) has only the zero solution, then the hypotheses of Theorem 2.1 are satisfied for the projections P_n and the associated equation $v = Tv$ in the space $X = C[0,1]$ with norm $\|Gv\|_{L^\infty}$.

Proof. Let $X_1 = \{u \in C^2[0,1]: u(0)=u(1)=0\}$ with norm $\|u\|_{X_1} = \|u\|_{L^\infty}$. Note that G is an isometry between X_1 and X. Let $v_0 = D^2u_0$ and let $V_0 = \{v \in X: \|v-v_0\|_X < \delta\}$ be an open set containing v_0 in X. Defining $Sv = f(x,Gv)$ and $M_1u = eu'$, then $Tv = Sv + M_1Gv$. It will be shown that S and M_1G, hence T are continuous. Let $v_n \to v$ in V_0. Defining $u_n = Gv_n$ and $u = Gv$, it follows that $\|u_n - u\|_{L^\infty} \to 0$. As

$$\|Sv_n - Sv\|_X = \sup_{0 \le s \le 1} \left|\int_0^1 G(s,x)\big(f(x,u_n) - f(x,u)\big)dx\right|$$

$$\le \|G(s,x)\|_{L^\infty} \sup_{0 \le x \le 1} |f(x,u_n) - f(x,u)|,$$

then the continuity of f implies $Sv_n \to Sv$ and so S is continuous on V_0.

Writing $e(x)u'(x) = \{e(x)u(x)\}' - e'(x)u(x)$, let $a_1(x) = e(x)$ and $a_0(x) = -e'(x)$. Then $M_1u = \{a_1(x)u(x)\}' + a_0(x)u(x)$, and

$$GM_1u = \int_0^1 G(s,x)\Big(\{a_1(x)u(x)\}' + a_0(x)u(x)\Big)dx$$

$$= \int_0^1 \{-a_1(x)G_x(s,x) + a_0(x)G(s,x)\}u(x)dx$$

after an integration by parts. Thus,

$$\|M_1u\|_X \leq \|a_1(x)G_x(s,x) + a_0(x)G(s,x)\|_{L^\infty}\|u\|_{L^\infty},$$

and therefore M_1 is a continuous map from X_1 to X. It follows then that M_1G is continuous on X and hence T is continuous.

We next establish P_nT is continuous. Note that as Q_n involves linear functionals with derivative evaluations, Q_n and P_n will not be continuous. Let $v_k \to v$ in V_0. Then

$$\|P_nTv_n - P_nTv\|_X = \|GD^2Q_nGTv_k - GD^2Q_nGTv\|_{L^\infty}$$

$$= \|Q_n \int_0^1 G(s,x)\{f(x,u_k)-f(x,u)+e(x)(u_k'-u')\}dx\|_{L^\infty}.$$

Let $h_k(s) = \int_0^1 G(s,x)\{f(x,u_k)-f(x,u)+e(x)(u_k'-u')\}dx$. Then

$$h_k'(s) = \int_0^1 G_s(s,x)\{f(x,u_k)-f(x,u)+e(x)(u_k'-u')\}dx,$$

and

$$|h_k'(s)| \le \|G_s(s,x)\|_{L^\infty}\|f(x,u_k)-f(x,u)\|_{L^\infty}$$

$$+ \left|D_s \int_0^1 G(s,x)\left[e(x)(u_k-u)\right]'dx\right| + \left|D_s \int_0^1 G(s,x)e'(x)(u-u_k)dx\right|$$

$$\le \|G_s(s,x)\|_{L^\infty}\|f(x,u_k)-f(x,u)\|_{L^\infty} + 2\|e(x)\|_{L^\infty}\|u_k-u\|_{L^\infty}$$

$$+ \|G_s(s,x)\|_{L^\infty}\|e'(x)\|_{L^\infty}\|u_k-u\|_{L^\infty}.$$

Using the continuity of f and the fact that $\|u_k-u\|_{L^\infty}\to 0$, it follows that $h_k'(s)$ converges uniformly to zero. Note that in a similar fashion, it follows that $h_k(s)$ also converges uniformly to zero. Finally, using Theorem 1.2, one has

$$\|Q_n h_k(s)\|_{L^\infty} \le \|Q_n h_k(s)-h_k(s)\|_{L^\infty} + \|h_k(s)\|_{L^\infty}$$

$$\le K\bar{\pi}_n\|h_k'(s)\|_{L^\infty} + \|h_k(s)\|_{L^\infty}.$$

Thus $\|Q_n h_k(s)\|_{L^\infty}\to 0$ as $v_k\to v$, and so P_nT is continuous.

Note that

$$\|(I-P_n)v_0\|_X = \|G(I-D^2Q_nG)v_0\|_{L^\infty} = \|Gv_0-Q_nGv_0\|_{L^\infty}.$$

As v_0 is in $C[0,1]$, then Gv_0 is in $C^2[0,1]$, and using Theorem 1.2 it follows that $\|Gv_0-Q_nGv_0\|_{L^\infty}\to 0$.

If T is Fréchet differentiable at v_1, the Fréchet derivative will equal the weak derivative, $T'(v_1) = f_u(x,Gv_1)G + M_1G$. In order for this to be the Fréchet derivative at v_1, it must be shown that given $\epsilon>0$ there is a $\delta>0$ so that whenever $\Delta v = v_1 - v$ satisfies $\|\Delta u\|_X < \delta$, then $\|T(v_1+\Delta v)-T(v_1)-T'(v_1)\Delta v\|_X \leq \epsilon\|\Delta v\|_X$. Choose v_1 in V_0. Then using the mean value theorem it follows that

$$\|T(v_1+\Delta v)-T(v_1)-T'(v_1)\Delta v\|_X =$$

$$\sup_{0\leq s\leq 1} \Big|\int_0^1 G(s,x)\{f\big(x,G(v_1+\Delta v)\big)+e(x)G_1(v_1+\Delta v)-f(x,Gv_1)$$

$$-e(x)G_1v_1-f_u(x,Gv_1)G\Delta v-e(x)G_1\Delta v\}\,dx\Big|$$

$$\leq \|G(s,x)\|_{L^\infty} \sup_{0\leq x\leq 1} \{|f_u(x,\theta G\Delta v+Gv_1)-f_u(x,Gv_1)|\cdot|G\Delta v|\}$$

with $0\leq\theta\leq 1$. If $\rho>0$ and $\|\Delta v\|_X<\rho$, then $\|G\Delta v\|_{L^\infty}$, $\|\theta G\Delta v\|_{L^\infty}<\rho$. Since f_u is continuous on V_0 and if v_1 is sufficiently close to $D^2u_0=v_0$, it follows that given $\epsilon>0$ there is a $\rho>0$ so that $\|\Delta v\|_X<\rho$ implies

$$\|T(v_1+\Delta v)-T(v_1)-T'(v_1)\Delta v\|_X \leq \epsilon\|\Delta v\|_X,$$

and so T is Fréchet differentiable. In the same manner, it can be shown that $\|T'(v)-T'(v_0)\|_X < \epsilon$ if v is chosen sufficiently close to v_0, and so T is continuously Fréchet differentiable at v_0.

The existence of only the zero solution to $D^2u-f_u(x,u_0)u-e(x)u' = 0$ and the boundary conditions implies the same result for $\big(I-T'(v_0)\big)v = 0$.

Let $M_2u = -f_u(x,u_0)u$, $\tilde{M}_2 = M_2-M_1$, and $M = D^2+\tilde{M}_2$. Then $\left(I-T'(v_0)\right)^{-1}$ and M^{-1} both exist. We will show M^{-1} is continuous as a mapping from X to X_1 and that this implies the continuity of $\left(I-T'(v_0)\right)^{-1}$ on X. If there exists a number $\alpha>0$ so that $\|\tilde{M}_2u\|_X \le \alpha\|Mu\|_X$, then with $u=M^{-1}v$, $GD^2u = G(Mu-\tilde{M}_2u)$, so $\|M^{-1}v\|_{X_1} = \|Mu-\tilde{M}_2u\|_X$, and finally $\|M^{-1}v\|_{X_1} \le (1+\alpha)\|v\|_X$, i.e. M has a continuous inverse. As $D^2 = M-\tilde{M}_2$, after multiplying by the appropriate operators one has

$$G\tilde{M}_2 = G\tilde{M}_2M^{-1}M = G\tilde{M}_2GM - G\tilde{M}_2M^{-1}\tilde{M}_2GM,$$

and

$$\|G\tilde{M}_2\|_{X_1} \le \|G\tilde{M}_2-G\tilde{M}_2M^{-1}\tilde{M}_2\|_{X_1}\|GM\|_{X_1}$$

provided $\alpha = \|G\tilde{M}_2-G\tilde{M}_2M^{-1}\tilde{M}_2\|_{X_1}$ satisfies $\alpha<\infty$. As G and $\tilde{M}_2$ are continuous, the result follows if $M^{-1}\tilde{M}_2$ can be shown to be continuous. Let $H(s,x)$ be the Green's function for M and (2.3). Let $b_0(x) = -f_u(x,u_0)+e'(x)$ and $b_1(x) = -e(x)$. Then

$$\begin{aligned}\|M^{-1}\tilde{M}_2u\|_{X_1} &= \sup_{0\le s\le 1}\left|\int_0^1 H(s,x)\{\left(b_1(x)u(x)\right)' + b_0(x)u(x)\}dx\right| \\ &= \sup_{0\le s\le 1}\left|\int_0^1 \left(-b_1(x)H_x(s,x)+b_0(x)H(s,x)\right)u(x)dx\right|\end{aligned}$$

after an integration by parts. Thus $M^{-1}\tilde{M}_2$ is continuous on X_1 and $\alpha<\infty$ implies M^{-1} is continuous. Now choose v in X. Then $v-T'(v_0)v = D^2u+\tilde{M}_2u$ where D^2u-v. Hence for some positive constant k,

$$\left\|\left(I-T'(v_0)\right)v\right\|_X = \left\|D^2u+\tilde{M}_2u\right\|_X \geq k\|u\|_{X_1}$$

$$= k\|v\|_X,$$

and Condition 2 of Theorem 2.1 is verified.

We next establish P_nT is Fréchet differentiable and $(P_nT)'(x) = P_nT'(x)$. Consider

$$\left\|P_n\left(T(v_1+\Delta v)-T(v_1)-T'(v_1)\Delta v\right)\right\|_X = \left\|Q_nG\left(f(x,u_1+\Delta u)-f(x,u_1)-f_u(x,u_1)\Delta u\right)\right\|_{L^\infty}.$$

Using the mean value theorem, we have

$$h(s) = \int_0^1 G(s,x)\left(f(x,u_1+\Delta u)-f(x,u_1)-f_u(x,u_1)\Delta u\right)dx$$

$$= \int_0^1 G(s,x)\left(f_u(x,u_1+\theta\Delta u)-f_u(x,u_1)\right)\Delta u dx, \quad 0\leq\theta\leq 1.$$

For functions g in $C^1[0,1]$ their projection under Q_n satisfies by Theorem 1.2

$$\|g-Q_ng\|_{L^\infty} \leq K\bar{\pi}_n\|Dg\|_{L^\infty},$$

where K is independent of n,g. Suppose n is large enough so that $K\bar{\pi}_n < 1$. Let $\epsilon>0$ be given. Using the continuity of f_u, there exists a $\delta>0$ so that $\|\Delta u\|_{L^\infty} < \delta$ implies with $K_1 = \max\{\|G(s,x)\|_{L^\infty}, \|G_x(s,x)\|_{L^\infty}\}$ that

$$\sup_{0\leq x\leq 1}\left|f_u(x,u_1+\theta\Delta u)-f_u(x,u_1)\right| \leq \frac{\epsilon}{2K_1}, \qquad 0\leq\theta\leq 1.$$

Proceeding as before, it follows that

$$\|h(s)\|_{L^\infty} \le \frac{\epsilon}{2}\|\Delta u\|_{L^\infty},$$

and

$$\|h'(s)\|_{L^\infty} \le \frac{\epsilon}{2}\|\Delta u\|_{L^\infty}.$$

Thus,

$$\|Q_n h\|_{L^\infty} \le \|Q_n h-h\|_{L^\infty} + \|h\|_{L^\infty} \le K\bar{\pi}_n\|Dh\|_{L^\infty} + \|h\|_{L^\infty} \le \epsilon\|\Delta u\|_{L^\infty},$$

and so P_nT is Fréchet differentiable and $(P_nT)'(v) = P_nT'(v)$.

Now let v in X satisfy $\|v\|_X \le 1$, and suppose v_1 is in V_0. Then with $u_1=Gv_1$, $u=Gv$,

$$\begin{aligned}\|(I-P_n)T'(v_1)v\|_X &= \|G(I-D^2Q_nG)\left(f_u(x,u_1)u+e(x)u'\right)\|_{L^\infty} \\ &= \|(I-Q_n)\int_0^1 G(s,x)\left(f_u(x,u_1)u+e(x)u'(x)\right)dx\|_{L^\infty}.\end{aligned}$$

As $\|v\|_X \le 1$, then $\|u\|_{L^\infty} \le 1$ and

$$\sup_{0\le s\le 1}\left|\int_0^1 G_s(s,x)f_u(x,u_1)u\,dx\right| \le K\|G_s(s,x)\|_{L^\infty},$$

where $K = \sup\{|f_u(s,y)|: 0\le s\le 1, |y-u_0(s)|\le\delta\}$. Also since

$$\int_0^1 G_s(s,x)e(x)u'(x)dx = \int_0^1 G_s(s,x)\left(e(x)u(x)\right)'dx - \int_0^1 G_s(s,x)e'(x)u(x)dx,$$

then

$$\sup_{0\le s\le 1} \left|\int_0^1 G_s(s,x)e(x)u'(x)dx\right| \le 2\cdot\|e\|_{L^\infty} + \|G_s(s,x)\|_{L^\infty}\cdot\|e'\|_{L^\infty}.$$

Thus the set of elements $v\varepsilon X$ satisfying $\|v\|_X \le 1$ gets mapped under $DGT'(v_1)$ into a set bounded in the uniform norm. Then again using Theorem 1.2 we have that for some constant $K>0$,

$$\|(I-P_n)T'(v_1)\|_X \le K\bar{\pi}_n, \tag{2.4}$$

and thus Condition 3 of Theorem 2.1 is verified and the proof of Theorem 2.2 is complete.

The following corollary gives convergence rates for the approximations to the solution. Convergence rates for the derivatives and second derivatives will be given in Corollary 2.3.

Corollary 2.1. If in addition to the hypotheses of Theorem 2.2 the solution u_0 is in $C^j[0,1]$, $j=2,3$, or 4, then

$$\|u_n-u_0\|_{L^\infty} = O(\bar{\pi}_n^j).$$

Proof. Note that

$$\|v_n-v_0\|_X \le K\|(I-P_n)v_0\|_X$$

implies

$$\|u_n-u_0\|_{L^\infty} \le K\|(I-Q_n)u_0\|_{L^\infty}.$$

Additional statements can be made concerning the rate of convergence if u_0 is in $C^2[0,1]$ and something is known about u_0''. For example, suppose u_0'' is in $Lip_M\beta$. Then using an error bound of [34],

$$\begin{aligned}\|u_n-u_0\|_{L^\infty} &\le K\|Q_nu_0-u_0\|_{L^\infty}\\ &\le K'\bar{\pi}_n^2\omega(u_0'',\bar{\pi}_n)\\ &\le K''\bar{\pi}_n^{2+\beta}.\end{aligned}$$

Suppose u_0''' is essentially bounded and u_0'' is absolutely continuous. Then

$$\begin{aligned}\omega(u_0'',\bar{\pi}_n) &= \sup_{|x-y|\le\bar{\pi}_n} |u_0''(x)-u_0''(y)|\\ &= \sup_{|x-y|\le\bar{\pi}_n} \left|\int_y^x u_0'''(t)dt\right| \le \bar{\pi}_n\cdot K.\end{aligned}$$

Thus,

$$\|u_n-u_0\|_{L^\infty} \le K'\bar{\pi}_n^3.$$

A similar set of bounds can be derived if $u_0\epsilon C^3[0,1]$ and something is known about u_0'''. The result needed to do this is $\|Q_nu_0-u_0\|_{L^\infty} \le K\bar{\pi}_n^3\omega(u_0''',\bar{\pi}_n)$. This error bound could not be found in the literature and so we give a proof.

Lemma 2.1. Let $u_0 \varepsilon C^3[0,1]$ satisfy $u_0(0) = u_0(1) = 0$. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of $[0,1]$ satisfying $\bar{\pi}_n \to 0$. Then

$$\|Q_n u_0 - u_0\| = O\left(\bar{\pi}_n^3 \omega(u_0''', \bar{\pi}_n)\right).$$

Proof. Let $X = \{u \varepsilon C^2[0,1]: u(0)=u(1)=0\}$. Let $\|u\|_X = \|u''\|_{L^\infty}$. Then X is a Banach space. Using [34], it is known that if $u \varepsilon X$, then $\|(Q_n u - u)''\|_{L^\infty} \le K\omega(u'', \bar{\pi}_n)$. Thus the Q_n's are continuous and converge pointwise to the identity on X, and so by the Banach-Steinhaus theorem, the Q_n's are uniformly bounded in norm on X. Choose any linear spline r_n. Let s_n be in $Sp(\pi_n, 3)$ with $s_n(0) = s_n(1) = 0$ and $s_n'' = r_n$. Then for some constant M

$$\|(Q_n u_0 - u_0)''\|_{L^\infty} \le (1+M)\|u_0'' - s_n''\|_{L^\infty}.$$

Since s_n'' is any linear spline, Theorem 1.1 implies

$$\|(Q_n u_0 - u_0)''\|_{L^\infty} \le K\bar{\pi}_n \omega(u_0''', \bar{\pi}_n).$$

Arguing as in the proof of Theorem 1.2, the result follows immediately.

We next consider applying the same approximation scheme to the problem (2.2) with (2.3), i.e. $D^2 u = f(x,u,u')$, $0<x<1$, with $u(0) = u(1) = 0$ and f nonlinear in u'. In order to apply Theorem 2.1, we again write the problem in the form $v = f(x, Gv, G_1 v) = Tv$ with $v = D^2 u$. Defining the spaces X_1 and X as before, let their norms be also as before, that is $\|v\|_X = \|Gv\|_{L^\infty}$ and $\|u\|_{X_1} = \|u\|_{L^\infty}$. One of the conditions of

Theorem 2.1 is continuity of T in this setting. Consider the function $f(x,u,u') = (u')^2$. The associated operator T is defined by $Tv = (G_1v)^2$. If T is continuous, then if $v_n \to v_0$, it must be shown that

$$\sup_{0\le x\le 1} \left|\int_0^1 G(x,s)\left[(u_n')^2-(u_0')^2\right]ds\right| \to 0. \tag{2.5}$$

Consider the sequence $u_n(s) = \dfrac{\sin n\pi s}{\sqrt{n}}$ and $u_0(s) = 0$. Let $x = 1/2$ for example. Then direct evaluation of (2.5) shows that $\|Tu_n\|_X \to \infty$. Thus in general, if f is nonlinear in u', the norms cannot be chosen as before. In order to establish continuity of T, convergence to zero of $\|u_n-u_0\|_{L^\infty}$ and $\|u_n'-u_0'\|_{L^\infty}$ when $\|v_n-v_0\|_X \to 0$ would be sufficient. This can be achieved by defining $\|v\|_X = \|DGv\|_{L^\infty}$. For G to remain an isometry between the spaces X_1 and X (defined as before), the norm an X_1 is chosen to be $\|u\|_{X_1} = \|u'\|_{L^\infty}$. In this setting the details of verifying Theorem 2.1 for the problems treated in the next theorem are analogous to the case already done and are omitted.

Theorem 2.3. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of $[0,1]$ satisfying $\bar{\pi}_n \to 0$. Suppose (2.1) with (2.3) has a solution u_0 with f, f_u and $f_{u'}$ continuous on $N = \{0\le x\le 1,\ |u^i-u_0^i|\le\delta,\ \delta>0\}$. If the equation $D^2u - f_u(x,u_0,u_0')u - f_{u'}(x,u_0,u_0')u' = 0$ with boundary conditions $u(0) = u(1) = 0$ has only the zero solution, then the hypotheses of Theorem 2.1 are satisfied for the projections P_n and the associated equation $v=Tv$ in the space $X=C[0,1]$ with norm $\|DGv\|_{L^\infty}$.

Classes of problems meeting the hypotheses of Theorems 2.2 and

2.3 are developed in Chapter III when higher order equations are studied.

Corollary 2.2. If in addition to the hypotheses of Theorem 2.3 the solution u_0 is in $C^j[0,1]$, j=2, 3, or 4, then

$$\|u_n^k-u_0^k\|_{L^\infty} = O(\bar{\pi}_n^{j-1}), \qquad k=0,1.$$

Note that since the problems described in Theorem 2.3 contain those in Theorem 2.2, then Corollary 2.2 applies in both cases and rates of convergence for the derivatives have been obtained. Moreover, an analogous theorem can be obtained using the $\|u\|_{L^\infty}$ norm on X and the norm $\|u''\|_{L^\infty}$ on X_1. The spaces X and X_1 are Banach in this case. Using spline interpolation bounds [34], if f is in $C^2[0,1]$ then

$$\|(Q_nf-f)''\|_{L^\infty} \le K\omega(f'',\bar{\pi}_n),$$

where K is a constant independent of f and n. Thus we can show the P_n's are uniformly bounded on X as in the proof of Lemma 2.1. We delay until Chapter III the proof of a result, Theorem 3.2, enabling us to deduce directly convergence in this setting and hence for the second derivatives. It is possible to proceed also in a manner similar to the preceding. These results are summarized in the next corollary.

Corollary 2.3. If in addition to the hypotheses of Theorem 2.2 the solution u_0 is in $C^j[0,1]$, j=2,3, or 4, then

$$\|u_n^k-u_0^k\|_{L^\infty} = O(\bar{\pi}_n^{j-k}), \qquad k=0,1,2.$$

If in addition to the hypotheses of Theorem 2.3 the solution u_0 is in $C^j[0,1]$, j=2,3, or 4, then the conclusions of Corollary 2.2 hold and

$$\|u_n''-u_0''\|_{L^\infty} = O(\bar{\pi}_n^{j-2}).$$

Other Projection Methods

The details of the proof of Theorem 2.2 remain essentially the same (actually easier) for any sequence of projections P_n with $Q_n = GP_nD^2$ and the range of P_n equal to $Sp(\pi_n,k-2)$, $k\geq 3$, provided the Q_n's are uniformly bounded in the L^∞ norm over X_1, i.e. there exists a constant M independent of n so that $\|Q_nf\|_{L^\infty} \leq M\|f\|_{L^\infty}$ for all $f\varepsilon X_1$. For example, choose any $f\varepsilon X_1$. Then

$$\|Q_nf-f\|_{L^\infty} = \|Q_n(f-s_n)-(f-s_n)\|_{L^\infty}$$

$$\leq (1+M)\|f-s_n\|_{L^\infty}$$

for any $s_n\varepsilon Sp_0(\pi_n,k)$, and so Q_n achieves the same asymptotic rate of convergence as the best approximation given in Theorem 1.1. Thus Condition 1 of Theorem 2.1 can be verified. Note that if Q_n is uniformly bounded, then P_n is also uniformly bounded in the setting of Theorem 2.2 where G is an isometry. With P_n and T continuous, then P_nT will be continuous. Also, if T is Fréchet differentiable, then P_nT will be Fréchet differentiable and $(P_nT)' = P_nT'$, see [17]. Likewise, using Theorem 1.1, (2.4) will hold. Moreover, similar remarks are true for Theorem 2.3. The next two lemmas enable us to deduce that the conclusions of Theorem 2.2 and 2.3 are also true for a sequence of projections uniformly bounded over X_1 with the sup norm.

Lemma 2.2. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of $[0,1]$. Let $Sp_0(\pi_n,k)$, $k\geq 3$, be the associated spaces of spline functions of degree k. Let $\{Q_n\}$ be a sequence of projections from X_1 onto $Sp_0(\pi_n,k)$ satisfying $\|Q_n\|_{L^\infty} \leq M$ for some constant M independent of n. Then there exists a constant M_1 so that $\|(Q_nf)'\|_{L^\infty} \leq M_1\|f'\|_{L^\infty}$ for any $f\varepsilon X_1$ and all n.

Proof. If f is in X_1, then using the best approximation properties of $Sp_0(\pi_n,k)$-splines, it follows that there exists a constant K independent of n so that

$$\|f-Q_nf\|_{L^\infty} \leq K\bar{\pi}_n\omega(f',\bar{\pi}_n). \tag{2.6}$$

There exists some $Sp_0(\pi_n,k)$-spline s_n for each n and constants K_1,K_2 independent of n so that

$$\|f-s_n\|_{L^\infty} \leq K_1\bar{\pi}_n\omega(f',\bar{\pi}_n) \tag{2.7}$$

and

$$\|(f-s_n)'\|_{L^\infty} \leq K_2\omega(f',\bar{\pi}_n). \tag{2.8}$$

Using the Markov inequality [18], (2.6), and (2.7), there exist constants K_3 and K_3' independent of n such that

$$\|(s_n-Q_nf)'\|_{L^\infty} \le \frac{K_3}{\underline{\pi}_n} \|s_n-Q_nf\|_{L^\infty} \tag{2.9}$$

$$\le \frac{K_3}{\underline{\pi}_n} \{\|s_n-f\|_{L^\infty}+\|f-Q_nf\|_{L^\infty}\}$$

$$\le K_3'\omega(f',\bar{\pi}_n).$$

Then using the triangle inequality with (2.8) and (2.9) one has

$$\|(f-Q_nf)'\|_{L^\infty} \le K_4\omega(f',\bar{\pi}_n).$$

Finally,

$$\|(Q_nf)'\|_{L^\infty} \le \|(Q_nf-f)'\|_{L^\infty} + \|f'\|_{L^\infty}$$

$$\le K_4\omega(f',\bar{\pi}_n) + \|f'\|_{L^\infty}$$

$$\le K_5\|f'\|_{L^\infty},$$

where K_5 is a constant independent of n. This completes the proof.

The next lemma follows in a manner similar to Lemma 2.2.

<u>Lemma 2.3</u>. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of [0,1]. Let Q_n be a sequence of projections from X_1 onto $Sp_0(\pi_n,k)$, $k\ge3$,

satisfying $\|(Q_n f)'\|_{L^\infty} \le M\|f'\|_{L^\infty}$ for some constant M and all n. Then there exists a constant M_1 so that $\|(Q_n f)''\|_{L^\infty} \le M_1\|f''\|_{L^\infty}$ for all $f \in X_1$ and all n.

For a sequence of projection operators satisfying the conclusion of Lemma 2.3, Theorem 3.2 of Chapter III can be applied to deduce convergence of the approximation method for the problems treated in Theorems 2.2 and 2.3.

Corollary 2.4. If Q_n is continuous with respect to $\|u\|_{L^\infty}$ ($\|u'\|_{L^\infty}$) for some index n, then it is continuous with respect to $\|u'\|_{L^\infty}$ ($\|u''\|_{L^\infty}$) for that same index.

We next introduce several projections that can be used to approximate solutions to (2.1) or (2.2) with (2.3).

Let π_n be a partition of [0,1]. Let $\tilde{\pi}_n$ be the partition obtained from π_n by removing x_1 and x_{n-1}, i.e. $\tilde{\pi}_n: 0=x_0<x_2 \ldots <x_{n-2}<x_n=1$. Given a function f in C[0,1], define the projection operator R_n mapping C into $Sp(\tilde{\pi}_n,3)$ by $R_n f=s$ with $s(x_i)=f(x_i)$, $i=0,\ldots,n$. DeBoor [3] has shown the following result, see also Schultz [30].

Theorem 2.4. R_n is a well-defined projection of C[0,1] onto $Sp(\tilde{\pi}_n,3)$ and $\|R_n\|_{L^\infty} \le 1 + \frac{5}{2}(\bar{\pi}_n/\underline{\pi}_n)^2$.

Again letting G(x,s) represent the Green's function associated with D^2 and the boundary conditions, the approximation u_n in $Sp_0(\tilde{\pi}_n,3)$ to the solution of $D^2u=f(x,u,u')$, $u(0)=u(1)=0$, is defined as the solution to

$$u_n(x_i) = \int_0^1 G(x_i,s)f\big(s,u_n(s),u_n'(s)\big)ds, \qquad i=0,\ldots,n. \qquad (2.10)$$

This approximation scheme will be called a weighted sub-domain method. Letting $P_n = D^2R_nG$ and writing the boundary value problem in the form $v=Tv$ as before, then $v_n = P_nTv_n$ is equivalent to $u_n = R_nGf(s,u_n,u_n')$ which is in turn equivalent to (2.10).

In order to apply the approximation scheme (2.10) as defined, it would be necessary to calculate integrals over the entire interval $[0,1]$. It is possible to avoid this by applying other linear functionals in the span of those that generate D^2R_nG. In particular, the linear functionals

$$\lambda_i f = \int_0^1 g_i(x)f(x)dx, \qquad i=1,\ldots,n+1,$$

where

$$g_i(x) = \begin{cases} \dfrac{(x-x_{i-1})_+}{x_i - x_{i-1}} & x \leq x_i \\ \\ \dfrac{(x_{i+1}-x)_+}{x_{i+1} - x_i} & x \geq x_i \end{cases}$$

can be applied to both sides of (2.1). The integrals required by each of these linear functionals span only two subintervals in the partition π_n.

Using Theorem 2.4 and Lemmas 2.2 and 2.3, it follows that the projections P_n are uniformly bounded in the norms $\|Gv\|_{L^\infty}$, $\|DGv\|_{L^\infty}$, and $\|v\|_{L^\infty}$ on $X=C[0,1]$. Thus the analysis in the previous section will go over for this projection.

We digress for a moment to take advantage of the uniform boundedness of the projections (2.10) to demonstrate a different technique for verifying Condition 3 of Theorem 2.1 in this case. Suppose X and X_1 are defined as before with $\|v\|_X = \|Gv\|_{L^\infty}$ and $\|u\|_{X_1} = \|u\|_{L^\infty}$. Let v be in some neighborhood $\|v-v_0\|_X < \delta$ of the point $v_0 = D^2u_0$, and suppose for any z in X the following inequality is satisfied:

$$\gamma = \inf\{\|T'(x)z-s_n\|_X : s_n \epsilon P_n X\} \le r_n\|z\|_X \tag{2.11}$$

with $r_n\|P_n\|_X \to 0$ as $n\to\infty$. Then following Vainikko [40],

$$\begin{aligned}\|(I-P_n)T'(x)z\|_X &\le \|I-P_n\|_X \cdot \gamma \\ &\le (1+\|P_n\|_X)r_n\|z\|_X,\end{aligned}$$

and so $\|(I-P_n)T'(x)\|_X \to 0$ which is Condition 3 of Theorem 2.1. Note that we have actually only required the P_n's to be continuous and $r_n\|P_n\|_X \to 0$, not that $\|P_n\|_X$ be uniformly bounded, for the verification of (2.11). However as they are uniformly bounded for the projection (2.10), then only $r_n \to 0$ need be shown. If $f(x,u,u') = f(x,u) + e(x)u'(x)$, then as before,

$$T'(v_1) = f_u(x,u_1)G + e(x)DG$$

and

$$T'(v_1)z = f_u(x,u_1)u + e(x)u'$$

where $u = Gz$. Let $T'(v) = S'(v) + e(x)DG$. Condition (2.11) will be applied to $S'(v)$ and Condition 3 of Theorem 2.1 will be shown directly for $e(x)DG$. Note that

$$\gamma = \inf\{\|S'(v_1)v-s_n\|_X : s_n \varepsilon P_n[X]\}$$

$$= \inf\{\|GS'(v_1)v-u_n\|_{L^\infty} : u_n \varepsilon Sp_0(\pi_n,3)\}.$$

Since $GS'(v_1)v\varepsilon C^2[0,1]$, Theorem 1.1 implies that for some constant $K>0$ and independent of v_1, v, and n,

$$\gamma \leq K\bar{\pi}_n^2\|f_u(x,Gv_1)\|_{L^\infty}\|Gv\|_{L^\infty}.$$

As f_u is continuous, then if v_1 is sufficiently close to v_0, there exists a constant $K'>0$ so that

$$\gamma \leq K'\bar{\pi}_n^2\|v\|_X.$$

Thus (2.11) is verified for $S'(v_1)$.

Let $eDGv = eu' = \hat{S}u$ with $u = Gv$. Following [3], write

$$G\hat{S}u = \int_0^s \int_0^t \Big[\big(e(x)u(x)\big)'-e'(x)u(x)\Big]dxdt$$

$$- s\int_0^1 \int_0^t \Big\{\big(e(x)u(x)\big)'-e'(x)u(x)\Big\}dxdt.$$

Then

$$(G\hat{S}u)' = e(s)u(s) - \int_0^s e'(x)u(x)dx - \int_0^1 e(x)u(x)dx$$

$$- \int_0^1 \int_0^t \Big[\big(e(x)u(x)\big)' - e'(x)u(x)\Big]dxdt.$$

Thus

$$\|(G\hat{S}u)'\|_{L^\infty} \le k\|u\|_{L^\infty} \tag{2.12}$$

for some constant k. Letting $V = \{G\hat{S}u : u \in X_1 \text{ and } \|u\|_{L^\infty} \le 1\}$, it follows from (2.12) that

$$\omega(v,\bar{\pi}_n) \le k\bar{\pi}_n, \quad \text{all } \bar{\pi}_n > 0, \quad v \in V.$$

This implies the set V is totally bounded in X_1 and since D^2 is an isometry from X_1 to X, then $\hat{S}$ is a totally bounded operator, see [16, p.70]. Since the P_n's are uniformly bounded in norm and thus converge pointwise to the identity on X, the fact that for an equi-continuous family of linear maps from one topological vector space to another the topologies of pointwise and uniform convergence coincide on totally bounded sets (see [16]) may be used to deduce $P_n\hat{S}$ converges uniformly to $\hat{S}$. This implies Condition 3 of Theorem 2.1. An application of the triangle inequality shows that the condition also holds for T'.

Results for the method (2.10) as applied to the problem (2.1) with (2.3) are summarized in the next theorem.

Theorem 2.5. Let $\{\pi_n\}$ be a quasi-uniform sequence of partitions of [0,1] satisfying $\bar{\pi}_n \to 0$. Suppose u_0 is a solution to (2.1) with boundary conditions (2.3) with f and f_u continuous on $N = \{0\le x\le 1,\ |u(x)-u_0(x)|\le\delta,\ \delta>0\}$ and $e\varepsilon C^1[0,1]$. If the equation $D^2u - f_u(x,u_0)u - e(x)u' = 0$ with boundary conditions $u(0) = u(1) = 0$ has only the zero solution, then the hypotheses of Theorem 2.1 are satisfied for the projections defined by (2.10) and the associated equation $v=Tv$ in the space $X=C[0,1]$ with norm $\|Gv\|_{L^\infty}$. Moreover,

$$\|u_0^k-u_n^k\|_{L^\infty} = O\left(\inf\{\|u^k-u_0^k\|_{L^\infty} : u^k\varepsilon D^k Sp_0(\pi_n,3)\}\right),\quad k=0,1,2,$$

where $Sp_0(\pi_n,3)$ represents those functions in $Sp(\pi_n,3)$ satisfying the boundary conditions (2.3).

We next investigate a projection method with quintic splines. Let $\pi_n : 0 = x_0<x_1<x_2<\ldots<x_{n+2}<x_{n+3}<x_{n+4} = 1$ be a partition of $[0,1]$ with $x_1 = 1/4n$, $x_2 = 1/2n$, $x_{n+2} = 1-1/2n$, $x_{n+3} = 1-1/4n$, and $x_i = (i-2)/n$, $i=3,\ldots,n+1$. Let $\tilde{\pi}_n$ be the partition obtained from π_n by deleting x_1, x_2, x_{n+2}, and x_{n+3}. Let $Sp(\tilde{\pi}_n,5)$ be the associated space of quintic splines. Define the projection U_n from $C[0,1]$ into $Sp(\tilde{\pi}_n,5)$ by $U_nf = s$ where

$$s(x_i) = f(x_i),\qquad i=0,\ldots,n+4. \tag{2.13}$$

<u>Theorem 2.6</u>. U_n is a well-defined projection of $C[0,1]$ onto $Sp(\tilde{\pi}_n,5)$ for $n\ge 2$. Moreover, $\|U_n\|_{L^\infty}$ are uniformly bounded.

Proof. Define a basis for $Sp(\tilde{\pi}_n,5)$ using the $\{s_i\}_{i=-2}^{n+2}$ defined by (3.26) after dividing by $\bar{\tilde{\pi}}_n^5$. It is known (or may be seen by direct computation) that $\sum s_i(x) = 120$ for x in [0,1], therefore

$$\left\| \sum_{i=-2}^{n+2} \alpha_i s_i(x) \right\|_{L^\infty} \le 120 \max|\alpha_i|. \tag{2.14}$$

If f is in C[0,1], then $U_n f = s = \sum \alpha_i s_i$ which from (2.13) gives (to three places) the matrix system

$$\left.\begin{aligned}
&\alpha_{-2} + 26\alpha_{-1} + 66\alpha_0 + 26\alpha_1 + \alpha_2 = f(x_0),\\
&.237\alpha_{-2} + 14.989\alpha_{-1} + 62.357\alpha_0 + 39.369\alpha_1 + 3.046\alpha_2 + .001\alpha_3 = f(x_1)\\
&.031\alpha_{-2} + 7.406\alpha_{-1} + 52.563\alpha_0 + 52.563\alpha_1 + 7.406\alpha_2 + .031\alpha_3 = f(x_2)\\
&\alpha_{i-2} + 26\alpha_{i-1} + 66\alpha_i + 26\alpha_{i+1} + \alpha_{i+2} = f(x_i), \quad 3 \le i \le n+1,\\
&.031\alpha_{n-1} + 7.406\alpha_n + 52.563\alpha_{n+1} + 52.563\alpha_{n+2}\\
&\qquad + 7.406\alpha_{n+3} + .031\alpha_{n+4} = f(x_{n+2})\\
&.001\alpha_{n-1} + 3.046\alpha_n + 39.369\alpha_{n+1} + 62.357\alpha_{n+2}\\
&\qquad + 14.989\alpha_{n+3} + .237\alpha_{n+4} = f(x_{n+3})\\
&\alpha_n + 26\alpha_{n+1} + 66\alpha_{n+2} + 26\alpha_{n+3} + \alpha_{n+4} = f(x_{n+4})
\end{aligned}\right\} \tag{2.15}$$

Note that except for the first three and last three rows, the system (2.15) is strictly diagonally dominant. If suitable linear combinations of the first (last) few rows are used to replace the first (last) three rows of (2.15), the entire system can be made diagonally dominant. For example, row one may be multiplied by .5740, row two by -1.0, and row three by .46472 and the sum used to replace row two. Row one requires use of a linear combination of the next seven rows. This argument shows that the system (2.15) after the indicated modifications is strictly diagonally dominant and hence invertible, so U_n is well defined for $n \geq 4$. Let $|\alpha_i| = \max_j |\alpha_j|$. If $3 \leq i \leq n+1$, then from (2.15) one has

$$\|f\|_{L^\infty} \geq 66|\alpha_i| - (1+26+26+1)|\alpha_i|$$

$$= 12|\alpha_i|.$$

A similar computation for the first and last three rows shows existence of a constant $K>0$ so that $|\alpha_i| \leq K\|f\|_{L^\infty}$ for $n \geq 4$. A similar result follows for $n=2,3$ by direct use of (2.15). Combining this result with (2.14) gives

$$\|U_n f\|_{L^\infty} \leq 120K\|f\|_{L^\infty},$$

concluding the proof.

Using Lemmas 2.2 and 2.3, the projections U_n are also uniformly bounded in the norms $\|u'\|_{L^\infty}$ and $\|u''\|_{L^\infty}$ on the space of functions in $C^2[0,1]$ satisfying the boundary conditions (2.3), i.e. X_1. Defining $P_n = D^2U_nG$, approximations to the solution of $D^2u = f(x,u,u')$, $u(0) = u(1) = 0$, can be found by solving

$$U_n u_n = U_n \int_0^1 G(s,x)f(x,u_n,u_n')dx \tag{2.16}$$

with u_n in $Sp(\pi_n,5)$ and $u_n(0) = u_n(1) = 0$. The span of the integrals in (2.16) can be reduced by applying to the original differential equation the linear functionals

$$\lambda_i f = \int_0^1 f(x)g_i(x)dx, \qquad i=1,\ldots,n+3,$$

with

$$g_i(x) = \begin{cases} \dfrac{(x-x_{i-1})_+}{x_i-x_{i-1}} & x\le x_i \\[2ex] \dfrac{(x_{i+1}-x)_+}{x_{i+1}-x_i} & x>x_i \end{cases}$$

We summarize results for this method applied to the problem (2.1) with (2.3).

<u>Theorem 2.7</u>. Suppose u_0 is a solution to (2.1) with boundary conditions (2.3) with f and f_u continuous on $N = \{0\le x\le 1,\ |u_0(x)-u(x)|\le\delta,\ \delta>0\}$ and $e\varepsilon C^1[0,1]$. Let the equation $D^2u - f_u(x,u_0)u - e(x)u' = 0$ with boundary conditions (2.3) have only the zero solution. If the projection scheme is defined by (2.16) and the associated sequence of partitions $\{\pi_n\}$ satisfies $\bar{\pi}_n\to 0$, then the hypotheses of Theorem 2.1 are satisfied for this method and the associated equation v=Tv in the space X=C[0,1] with norm $\|Gv\|_{L^\infty}$. Moreover,

$$\|u_0^k-u_n^k\|_{L^\infty} = O\left(\inf\{\|u_0^k-u^k\|_{L^\infty}:u^k\varepsilon D^k Sp_0(\tilde{\pi}_n,5)\}\right) \quad k=0,1,2,$$

where $Sp_0(\tilde{\pi}_n,5)$ represents those functions in $Sp(\tilde{\pi}_n,5)$ satisfying the boundary conditions (2.3).

Note that as the space of splines used in this method is quintic as opposed to the cubic splines of the previous two methods, better rates of convergence can be achieved if the solution possesses additional regularity. For example, if $u_0 \varepsilon C^6$, then

$$\|u_0 - u_n\|_{L^\infty} = O(\bar{\pi}_n^6).$$

DeBoor has shown in [3] that the Galerkin projection is uniformly bounded in the norm $\|Gv\|_{L^\infty}$ on $X=C[0,1]$ provided the partitions $\{\pi_n\}$ are uniform, i.e. $(\bar{\pi}_n/\underline{\pi}_n)=1$, and cubic splines are used. Thus an identical theorem to Theorem 2.5 holds for this method.

We complete this chapter by describing results from numerical experiments using the projections described in Theorem 2.2 and Theorem 2.5 as applied to two problems. Results of some other numerical experiments are contained in Chapter III. Consider first the problem $D^2u(x) = 4u(x) + 4\cosh 1$, $0<x<1$, with boundary conditions $u(0) = u(1) = 0$. It can be verified that $u_0(x) = \cosh(2x-1) - \cosh 1$ is the unique solution to this problem and that the hypotheses of Theorems 2.2 and 2.5 are satisfied. Using first the method of Theorem 2.2, we have the following results.

Table 1. Application of Theorem 2.2

$\bar{\pi}_n$	$\|u_n-u_0\|_{L^\infty}$	$\|(u_n-u_0)'\|_{L^\infty}$	$\|(u_u-u_0)''\|_{L^\infty}$
$\frac{1}{5}$	$.906\cdot 10^{-4}$	$.145\cdot 10^{-2}$	$.764\cdot 10^{-1}$
$\frac{1}{7}$	$.245\cdot 10^{-4}$	$.541\cdot 10^{-3}$	$.398\cdot 10^{-1}$
$\frac{1}{10}$	$.606\cdot 10^{-5}$	$.188\cdot 10^{-3}$	$.198\cdot 10^{-1}$

Then using Theorem 2.5 we have the next table.

Table 2. Application of Theorem 2.5

$\bar{\pi}_n$	$\|u_n-u_0\|_{L^\infty}$	$\|(u_n-u_0)'\|_{L^\infty}$	$\|(u_n-u_0)''\|_{L^\infty}$
$\frac{1}{10}$	$.632\cdot 10^{-5}$	$.396\cdot 10^{-3}$	$.335\cdot 10^{-1}$
$\frac{1}{50}$	$.124\cdot 10^{-7}$	$.323\cdot 10^{-5}$	$.138\cdot 10^{-2}$

As the solution is in $C^4[0,1]$, then convergence for $\|(u_n-u_0)^j\|_{L^\infty}$ is $O(\bar{\pi}_n^{\,4-j})$, j=0,1,2, which can be verified for the results in the previous tables. Both of these methods give essentially give five diagonal matrices defining the approximations and these matrix problems were solved using Gaussian elimination. The two methods are similar computationally.

The second problem considered is $D^2u = e^u$, $0<x<1$, with boundary conditions $u(0) = u(1) = 0$. This problem has a unique solution

$u_0(x) = \ln 2 + 2\ln[c \sec(c(x-.5)/2)]$, where $c \doteq 1.3360556949$. This problem meets the hypotheses of Theorem 2.2 and 2.5, see Theorem 3.4. Using first the method of Theorem 2.2, we have the following results.

Table 3. A Second Application of Theorem 2.2

$\bar{\pi}_n$	$\|u_n-u_0\|_{L^\infty}$	$\|(u_n-u_0)'\|_{L^\infty}$	$\|(u_n-u_0)''\|_{L^\infty}$
$\frac{1}{5}$	$.448\cdot10^{-5}$	$.148\cdot10^{-3}$	$.628\cdot10^{-2}$
$\frac{1}{7}$	$.125\cdot10^{-5}$	$.555\cdot10^{-4}$	$.329\cdot10^{-2}$
$\frac{1}{9}$	$.485\cdot10^{-6}$	$.265\cdot10^{-4}$	$.202\cdot10^{-2}$

Then using Theorem 2.5, we have the next table.

Table 4. A Second Application of Theorem 2.5

$\bar{\pi}_n$	$\|u_n-u_0\|_{L^\infty}$	$\|(u_n-u_0)'\|_{L^\infty}$	$\|(u_n-u_0)''\|_{L^\infty}$
$\frac{1}{5}$	$.462\cdot10^{-5}$	$.700\cdot10^{-4}$	$.374\cdot10^{-2}$
$\frac{1}{7}$	$.124\cdot10^{-5}$	$.261\cdot10^{-4}$	$.195\cdot10^{-2}$
$\frac{1}{9}$	$.461\cdot10^{-6}$	$.124\cdot10^{-4}$	$.120\cdot10^{-2}$

Newton's method was used to solve the nonlinear systems defining the approximations with a relative error check of 10^{-11}. Convergence was obtained in four iterations in all cases. An initial guess of zero was made. The integrals required by the linear functionals of these methods were calculated using the four-point Gaussian quadrature formula over each subinterval.

CHAPTER III

GENERAL PROJECTION METHODS FOR HIGH ORDER PROBLEMS

The next theorem is a direct consequence of Theorem 5 in [42]. It concerns approximation methods that can be applied to the equation $v=Tv$ where T is a mapping (nonlinear) from a Banach space X into itself. In this setting, projection methods that can be applied to very general differential equations are developed.

Theorem 3.1. Let X be a Banach space, $\{P_k\}$ a sequence of continuous projections converging pointwise to the identity operator on X, and T an operator (nonlinear) on X. Let v_0 be a solution to the equation $v=Tv$ with T completely continuous on an open set containing v_0, T continuously Fréchet differentiable at v_0, and the equation $v-T'(v_0)v = 0$ having only the trivial solution in X. Then v_0 is unique in some sphere $\{v\varepsilon X:\|v-v_0\|\leq\delta,\ \delta>0\}$, and there exists an integer N so that for $k\geq N$ the equation $v=P_kTv$ has a unique solution v_k in the same sphere. Moreover, there exists a constant $K>0$ and independent of k so that

$$\|v_0-v_k\| \leq K\|P_kv_0-v_0\|.$$

Corollary 3.1. There exists a constant $M>0$ so that

$$\|v_0-v_k\| \leq M \inf\{\|v_0-v\|:v\epsilon X_k\}.$$

Proof. This follows directly from the Banach-Steinhaus Theorem and the fact that for any $v \in X_k$,

$$\|P_k v_0 - v_0\| = \|P_k(v_0 - v) - (v_0 - v)\| \leq \|I - P_k\| \|v_0 - v\|.$$

The next lemma gives a criterion for establishing that the P_k's converge pointwise to the identity.

Lemma 3.1. If $\{P_k\}$, $k \geq 1$, are uniformly bounded in norm on X and

$$\lim_{k\to\infty}\{\inf\|x - x_k\| : x_k \in P_k[X]\} = 0$$

for all x in X, then $P_k \to I$ pointwise on X.

Boundary Value Problems

The class of problems to be considered in this chapter are general nonlinear boundary value problems of the form

$$D^m u = f(x, u, \ldots, u^{m-1}), \qquad a<x<b, \tag{3.1}$$

with boundary conditions

$$\sum_{j=0}^{m-1} [a_{ij} u^j(a) + b_{ij} u^j(b)] = 0, \qquad 1 \leq i \leq m, \tag{3.2}$$

with the a_{ij}, b_{ij} real constants.

Assume the only solution to $D^m u=0$ with boundary conditions (3.2) is the trivial solution. We may then let $G(x,s)$ denote the Green's function associated with D^m and (3.2), i.e. if $D^m u=v$, then

$$u(x) = \int_a^b G(x,s)v(s)ds \equiv G_0[v(x)].$$

Approximations to the solution of (3.1)-(3.2) are sought in the space $G_0 Sp(\pi_k,p,q) = S_k$. This is the space of polynomial splines in $Sp(\pi_k,p+m,q+m)$ that satisfy the boundary conditions (3.2). Let P_k project $C[a,b]$ onto $Sp(\pi_k,p,q)$. An approximation u_k to the solution of (3.1)-(3.2) is defined by (1.1), i.e. the function in S_k satisfying the projection equation

$$D^m u_k(x) = P_k f\big(x,u_k(x),\ldots,u_k^{m-1}(x)\big). \tag{3.3}$$

Theorem 3.2. Let $\{\pi_k\}$ be a sequence of partitions of I so that $\bar{\pi}_k \to 0$ and let $\{P_k\}$ be a sequence of uniformly bounded projections from $C(I)$ onto $Sp(\pi_k,p,q)$. Let u_0 be a solution to (3.1)-(3.2). Let $f = f(x,x_0,\ldots,x_{m-1})$ and suppose f, $\partial f/\partial x_i$, $0\le i\le m-1$, are defined and continuous on $S = \{a\le x\le b,\ |x_i-u_0^i(x)|\le\delta,\ 0\le i\le m-1,\ \delta>0\}$, a neighborhood of u_0. Finally, assume both $D^m u=0$ and

$$D^m u(x) - \sum_{j=0}^{m-1} \frac{\partial f}{\partial x_j}(x,u_0,\ldots,u_0^{m-1})u^j(x) = 0 \tag{3.4}$$

with boundary conditions (3.2) have only the trivial solution. Then a constant $\rho>0$ can be found so that u_0 is unique in the sphere $\{u\varepsilon C^m[a,b]: \|u_0^m(x)-u^m(x)\|_{L^\infty}\leq\rho\}$. Moreover, there exists an integer N so that for $k\geq N$ equations (3.3) have a solution u_k in S_k which is unique in the same sphere and there exists a positive constant K, independent of k, such that for $k\geq N$

$$\|u_k^j(x)-u_0^j(x)\|_{L^\infty} \leq KE_k(u_0^m), \qquad 0\leq j\leq m,$$

where $E_k(u_0^m)$ represents the error of the best approximation to $u_0^m(x)$ in $Sp(\pi_k,p,q)$.

Proof. Rewrite (3.1)-(3.2) as $v = f(x,G_0v,\ldots,G_{m-1}v) = Tv$ where $D^mu=v$ and

$$G_jv = \int_a^b \frac{\partial^j G(x,s)}{\partial x^j} v(s)ds, \qquad 0\leq j\leq m-1.$$

Note that each G_j is a continuous mapping from C[a,b] to C[a,b] and that $\frac{\partial^j G(x,s)}{\partial x^j}$ is uniformly bounded on $[a,b] \times [a,b]$ for $j=0,\ldots,m-1$, [9]. It is a consequence of the Arzela-Ascoli theorem that the G_j are completely continuous mappings in the setting. As f is continuous, then it follows that T is completely continuous relative to the uniform norm on a sufficiently small neighborhood of $v_0 = D^mu_0$. Viewing $v=Tv$ as an equation in C[a,b], conditions (3.3) may be written as $v_k = P_kTv_k$, an equation in C[a,b] or more specifically in $Sp(\pi_k,p,q)$.

The continuity of the partials of f imply the continuous Fréchet differentiability of the operator T about v_0. The existence of only the trivial solution to (3.4)-(3.2) implies the same result for $v-T'(v_0)v = 0$. Finally, Lemma 3.1 implies the pointwise convergence of the P_n's to the identity on $C[a,b]$. Hence the hypotheses of Theorem 3.1 are satisfied. Thus the solution u_0 is unique in some sphere. There exists an integer N so that for all $k \geq N$, u_k exists and is unique in the same sphere. Moreover, using Corollary 3.1, there exists a constant $K>0$ and independent of k so that with $v_k = D^m u_k$,

$$\|v_k - v_0\|_{L^\infty} \leq K \inf\{\|v_0 - v\|_{L^\infty} : v \in Sp(\pi_k, p, q)\}. \tag{3.5}$$

This and (3.5) complete the proof.

The next corollary follows by applying Theorem 1.1 to (3.5) and using some of the arguments presented in Chapter II.

<u>Corollary 3.2</u>. If in addition to the hypotheses of Theorem 3.2 the solution u_0 satisfies for $m+p \geq r \geq m$

(i) $u_0 \varepsilon C^r[a,b]$, then

$$\|u_k^j - u_0^j\|_{L^\infty} = O\left(\bar{\pi}_k^{r-m}\omega(f^r, \bar{\pi}_k)\right), \qquad 0 \leq j \leq m,$$

(ii) $u_0 \varepsilon C^r[a,b]$ and $u_0^r \varepsilon Lip_M \beta$, then

$$\|u_k^j - u_0^j\|_{L^\infty} = O(\bar{\pi}_k^{r-m+\beta}), \qquad 0 \leq j \leq m,$$

or

(iii) $u_0 \varepsilon C^r[a,b]$, u_0^r absolutely continuous, and u_0^{r+1} essentially bounded, then

$$\|u_k^j - u_0^j\|_{L^\infty} = O(\bar{\pi}_k^{-r-m+1}), \qquad 0 \le j \le m.$$

We next give several theorems describing classes of problems that meet the hypotheses of Theorem 3.2 (and also Theorems 2.2 and 2.3). The most difficult hypothesis to verify in all of these theorems is that the equation (3.4) with boundary conditions (3.2) has only the zero solution. The linear case is treated first.

Theorem 3.3. For the problem $D^m u + \sum_{j=0}^{m-1} a_j(x) u^j(x) = f(x)$ with boundary conditions (3.2), assume $a_j(x)$, $0 \le j \le m-1$, and $f(x)$ are continuous, a unique solution exists to the problem, and that the equation $D^m u = 0$ with boundary conditions (3.2) has only the trivial solution. Then the hypotheses of Theorem 3.2 concerning the differential equation are satisfied.

Proof. It only needs to be shown that equation (3.4) with (3.2) has only the zero solution. But if it had a non-zero solution, the uniqueness of the solution to the original problem would be contradicted. This completes the proof.

The next theorem essentially contains the mildly nonlinear problem treated by Ciarlet, Schultz, and Varga [6], Schultz [32], and Lucas [20] using the Galerkin method.

Theorem 3.4. Consider the problem

$$L[u(x)] \equiv \sum_{0 \le i,j \le m} (-1)^j D^j\left(\sigma_{ij}(x) D^i u(x)\right) = f\left(x, u(x)\right), \qquad a<x<b, \quad (3.6)$$

subject to the homogeneous boundary conditions

$$D^k u(a) = D^k u(b) = 0, \qquad 0 \le k \le m-1,$$

where $\sigma_{mm}(x) \ge \omega > 0$; $\sigma_{ij} \in C^j[a,b]$, $0 \le i, j \le m$; $f(x,u)$ and $f_u(x,u)$ are continuous functions on $[a,b] \times R$;

$$\int_a^b u(x)L[u(x)]dx \ge c\|u\|_m^2 \tag{3.8}$$

for some $c>0$, all $u \in C_0^{2m}[a,b]$, where $\|u\|_m^2 = \int_a^b \sum_{i=0}^m [D^i u(x)]^2 dx$ and $C_0^{2m}[a,b]$ the subspace of $C^{2m}[a,b]$ satisfying (3.7); and

$$f_u(x,u) \le \gamma < cc_1 \tag{3.9}$$

for all $(x,u) \in [a,b] \times R$ where

$$c_1 = \inf\{\|u\|_m^2 / \int_a^b [u(x)]^2 dx : u \in C_0^{2m}[a,b]\}. \tag{3.10}$$

If $u_0 \in C_0^{2m}[a,b]$ is a solution to (3.6)-(3.7), then the hypotheses of Theorem 3.2 concerning the differential equation (after dividing through by $\sigma_{mm}(x)$) are satisfied.

Proof. Suppose u is a solution to

$$L[u(x)] - f_u\big(x,u_0(x)\big)u(x) = 0 \tag{3.11}$$

subject to the boundary conditions (3.7). Then by (3.8), (3.9), (3.10) and (3.11)

$$0 \geq c\|u\|_m^2 - \gamma \int_a^b [u(x)]^2 dx$$

$$\geq (cc_1 - \gamma) \int_a^b [u(x)]^2 dx$$

and since

$$cc_1 - \gamma > 0, \; u(x) \equiv 0.$$

This completes the proof as the other verifications are immediate.

Theorem 3.5. For the problem $-D^2u=f(x,u)$, $0<x<1$, $u(0)=u(1)=0$, suppose a classical solution u_0 exists and $f, \frac{\partial f}{\partial u}$ are continuous on a neighborhood of u_0. In addition, suppose $n^2\pi^2 < c_1 \leq \frac{\partial f}{\partial u} \leq c_2 < (n+1)^2\pi^2$ for constants c_1, c_2 and some positive integer n on a neighborhood of u_0. Then the hypotheses of Theorem 3.2 concerning the differential equation are satisfied.

Proof. It is a direct consequence of the Sturm comparison theorem [9, p. 208] that the only solution to $-D^2u = \frac{\partial f}{\partial u}(x,u_0)u$, $u(0) = u(1) = 0$, is the trivial solution. This and the other assumptions above imply the hypotheses of Theorem 3.2 are satisfied.

Note that this problem also meets the hypotheses of Theorem 2.2. Thus the Galerkin method as described in Chapter II can be applied and best order convergence obtained. This class of problem is not contained in those of Ciarlet, Schultz, Varga [7].

The next theorem concerns a class of problems that will be treated in more detail in Chapter IV.

Theorem 3.6. For the problem $(-1)^m D^{2m} u = f(x,u,\ldots,u^m)$ with boundary conditions $u^j(0) = u^j(1) = 0$, $0 \le j \le m-1$, suppose a classical solution u_0 exists and f, $\frac{\partial f}{\partial u^j}$, $0 \le j \le m$, are continuous on a neighborhood of u_0. In addition, suppose $\frac{\partial f}{\partial u} \le a_0$, $\left|\frac{\partial f}{\partial u^j}\right| \le a_j$ for $1 \le j \le m$, on a neighborhood of u_0, and

$$e = \left(\frac{\max(+a_0,0)}{\pi^{2m}} + \sum_{j=1}^{m} a_j/\pi^{2m-j}\right) < 1.$$

Then the hypotheses of Theorem 3.2 concerning the differential equation are satisfied.

Proof. First note that it follows from the Rayleigh-Ritz [6] and Cauchy-Schwarz inequalities that

$$\int_0^1 |D^j u(x) u(x)|\,dx \le \frac{1}{\pi^{2m-j}} \int_0^1 \left(D^m u(x)\right)^2 dx. \qquad (3.12)$$

Now let u be a solution to

$$(-1)^m D^{2m} u = \sum_{J=0}^{m} \frac{\partial f}{\partial u^j}(x,u_0,\ldots,u_0^m) u^j(x) \qquad (3.13)$$

and the boundary conditions. Taking the L^2 inner product of both sides of (3.13) with u and integrating by parts on the left, it follows that

$$(D^m u, D^m u) = \sum_{j=0}^{m} \left(\frac{\partial f}{\partial u^j}(x,u_0,\ldots,u_0^m) u^j, u\right). \qquad (3.14)$$

Then using the bounds on $\frac{\partial f}{\partial u^j}$ and the Rayleigh-Ritz and Cauchy-Schwarz inequalities as in (3.12) on the right-hand side of (3.14), one finds $(D^m u, D^m u) \le e(D^m u, D^m u)$ and so $u(x) \equiv 0$. This and the other assumptions imply the hypotheses of Theorem 3.2 are satisfied.

Before introducing several specific projections, we make several remarks. A result analogous to Theorem 3.6 can be established for problems of the type

$$(-1)^{m+1} D^{2m-1} u = f(x, u, \ldots, u^m)$$

with boundary conditions

$$u^j(0) = u^k(1) = 0, \qquad 0 \le j \le m-1,\ 1 \le k \le m-1.$$

This problem is studied in detail in Chapter IV. A slight improvement can be achieved in the denominator of a_1 in Theorem 3.6 (one π can be replaced by a four) using Opial's inequality [22]. This is also discussed in Chapter IV. These problems are related to some specific problems considered in [7, Section 7].

As a final remark, we note that if the hypotheses of any of Theorems 3.2, 3.3, 3.4, 3.5, 3.6 hold on all of $[a,b]\times R\times \ldots \times R$, then approximations will exist and be unique in any given neighborhood of the solution provided k is taken sufficiently large. For the linear case, Theorem 3.3, global uniqueness of the approximation for k large follows directly from [38].

Collocation Methods

This section is begun by recalling an important result by Swartz and Varga [37].

Theorem 3.7. For any $n\geq 1$, $k\geq 2n-1$, let $\pi_k:\{x_i=a+(b-a)i/k:\ 0\leq i\leq k\}$ be the uniform partition of $[a,b]$ containing k intervals. For any f in the Banach space $C[a,b]$, let s be the unique element in $Sp(\pi_k,2n-1)$ such that

$$\begin{aligned} s(x_i) &= f(x_i), && 0\leq i\leq k, \\ D^j s(a) &= D^j(L_{2n-1,0}f)(a), && 1\leq j\leq n-1 \\ D^j s(b) &= D^j(L_{2n-1,1}f)(b), && 1\leq j\leq n-1 \end{aligned} \tag{3.15}$$

where $L_{2n-1,0}f(L_{2n-1,1}f)$ is the Lagrange polynomial interpolation of f at the knots $x_0,x_1,\ldots,x_{2n-1}(x_{k-2n-1},x_{k-2n},\ldots,x_k)$. Then there exists a constant K independent of k such that

$$\|f-s_k\|_{L^\infty} \leq K\omega(f,\tilde{\pi}_k).$$

Corollary 3.3. The projections P_k from $C[a,b]$ onto $Sp(\pi_k,2n-1)$ defined by (3.15) converge pointwise to the identity. Thus the $\|P_k\|_{L^\infty}$ are uniformly bounded, $k\geq 2n-1$.

The next two theorems may be deduced from results of Swartz and Varga [37] (Theorems 4.1 and 6.6) and provide a means of modifying the above projection.

Theorem 3.8. Given $f\varepsilon C^t[a,b]$, $t\geq 0$, and given a quasi-uniform family of partitions of $[a,b]$ containing at least $n+1$ mesh points, let $L_{n,i}f$, $n\geq 1$ fixed, denote the Lagrange polynomial interpolation of f at the points $x_i, x_{i+1}, \ldots, x_{i+n}$ where $0\leq i\leq N-n$, i.e.,

$$(L_{n,i}f)(x_j) = f(x_j), \qquad i\leq j\leq i+n.$$

Then for $r \equiv \min(t,n)$

$$K\bar{\pi}^{r-j}\omega(D^r f,\bar{\pi}) \geq \begin{cases} \|D^j(f-L_{n,i}f)\|_{L^\infty[x_i,x_{i+n}]}, & 0\leq j\leq r \\ \|D^j(L_{n,i}f)\|_{L^\infty[x_i,x_{i+n}]}, & j>r. \end{cases}$$

The next theorem is a stability result.

Theorem 3.9. Given $f\varepsilon C^t[a,b]$ with $0\leq t<2m$ and given a uniform partition of $[a,b]$, let s be the unique element in $Sp(\pi,2m-1)$ such that

$$\begin{cases} s(x_i) = \alpha_{i,0}, & 0\leq 1\leq n \\ D^j s(a) = \alpha_{0,j},\ D^j s(b) = \alpha_{N,j}, & \text{if } 1\leq j\leq m-1, \end{cases}$$

where it is assumed that a function $F_i(f,\bar{\pi})$ exists such that

$$\left.\begin{aligned}
K\bar{\pi}^{t}F_{i}(f,\bar{\pi}) &\geq |f(x_i)-\alpha_{i,0}|, && 0\leq i\leq k \\
K\bar{\pi}^{t-j}F_{0}(f,\bar{\pi}) &\geq |D^{j}f(a)-\alpha_{0,j}|, && \text{if } 1\leq j\leq \min(t,m-1) \\
K\bar{\pi}^{t-j}F_{0}(f,\bar{\pi}) &\geq |\alpha_{0,j}|, && \text{if } \min(t,m-1)<j\leq m-1
\end{aligned}\right\} \tag{3.16}$$

with similar inequalities holding at x=b. Then, with

$$\|F\|_{\infty} \equiv \max_{i}|F_{i}(f,\pi)|$$

$$K\bar{\pi}^{k-j}\left(\omega(D^{k}f,\bar{\pi}) + \|F\|_{\infty}\right) \geq \begin{cases} \|D^{j}(f-s)\|_{L^{\infty}[a,b]}, & 0\leq j\leq k \\ \\ \|D^{j}s\|_{L^{\infty}[a,b]}, & \text{if } k<j\leq 2m-1. \end{cases} \tag{3.17}$$

In particular, if the partition has at least 2m knots and if s and its first (m-1) derivatives are defined by

$$D^{j}s(a) = D^{j}(L_{2m-1,0}f)(a)$$

in terms of Lagrange polynomial interpolation of $\{\alpha_{i,0}\}_{i=0}^{2m-1}$ where the $\alpha_{i,0}$ satisfy the first inequality of (3.16), then the bounds of (3.17) are valid.

The following extension of Theorem 3.7 is a straightforward application of Theorems 3.8 and 3.9.

<u>Theorem 3.10</u>. Let π_k be the uniform partition of $[a,b]$ used in Theorem 3.7. Define the projection Q_k from $C[a,b]$ onto $Sp(\pi_k,2n-1)$ by (3.15) where $L_{2n-1,0}$ is redefined to be the Lagrange polynomial interpolation of f at the 2n uniformly spaced points $x_j = a + \left(\bar{\pi}_k/(2n-1)\right)j$, $0\le j\le 2n-1$ between x_0 and x_1, and $L_{2n-1,1}$ is redefined in a similar manner to interpolate at points between x_{k-1} and x_k. Then there exists a constant K independent of k such that

$$\|f-Q_kf\|_{L^\infty} \le K\omega(f,\bar{\pi}_k)$$

for any $k\ge 1$. Hence the projection Q_k converges pointwise to the identity and the $\|Q_k\|_{L^\infty}$ are uniformly bounded.

<u>Theorem 3.11</u>. Suppose the boundary value problem (3.1)-(3.2) and a solution $u_0(x)$ satisfy the hypotheses of Theorem 3.2. Then for any $n\ge 1$, a $\delta>0$ can be found such that u_0 is unique in the sphere $\{u\in C^m[a,b]: \|u_0^m(x)-u^m(x)\|_{L^\infty}\le\delta\}$, and there exists an integer N such that for $k>n$, and uniform partitions $\{\pi_k\}$, the equations

$$s_k^m(x) = P_kf\left(x,s_k(x),\ldots,s_k^{m-1}(x)\right)$$

and (3.18)

$$\tilde{s}_k^m(s) = Q_kf\left(x,\tilde{s}_k(x),\ldots,\tilde{s}_k^{m-1}(x)\right)$$

each have a unique solution in $Sp_0(\pi_k,2n+m-1)$ and the above sphere for all $k\ge N$, where Sp_0 is the subspace of $Sp(\pi_k,2n+m-1)$ satisfying the boundary conditions (3.2). Moreover

$$\|u_0^j - s_k^j\|_{L^\infty} \le K \inf\{\|u_0^m - s\|_{L^\infty} : s \in Sp(\pi_k, 2n-1, 2n-2)\}, \qquad 0 \le j \le m.$$

and a similar bound holds for $\tilde{s}$.

Corollary 3.4. Under the hypotheses of Theorem 3.11 both s_k^j and $\tilde{s}_k^j$ converge to u_0^j in the uniform norm for $0 \le j \le m$, as $\bar{\pi}_k \to 0$. If in addition the solution u_0 satisfies for some r, $m \le r \le m+2n-1$

(i) $u_0 \varepsilon C^r[a,b]$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O\left(\pi_k^{r-m} \omega(u_0^r, \bar{\pi}_k)\right), \qquad 0 \le j \le m, \tag{3.19}$$

(ii) $u_0 \varepsilon C^r[a,b]$ and $u_0^r \varepsilon \mathrm{Lip}_M \beta$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O(\bar{\pi}_k^{r-m+\beta}), \qquad 0 \le j \le m, \tag{3.20}$$

or

(iii) $u_0 \varepsilon C^r[a,b]$, u_0^r absolutely continuous, and u_0^{r+1} essentially bounded, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O(\bar{\pi}_k^{r-m+1}), \qquad 0 \le j \le m. \tag{3.21}$$

A similar set of bounds hold for $\tilde{s}$.

As an example of Theorem 3.11, suppose equations (3.1)-(3.2) give the second order problem $-u'' = f(x,u,u')$, $u(0) = u(1) = 0$, where f, $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial u'}$ are continuous in $[a,b] \times R \times R$, with $\frac{\partial f}{\partial u}(x,\theta_1,\theta_2) \le a_0$, $\left|\frac{\partial f}{\partial u'}(x,\theta_1,\theta_2)\right| \le a_1$ for all $(x,\theta_1,\theta_2) \varepsilon [a,b] \times R \times R$, and $\left(\max(+a_0,0)/\pi^2 + a_1/4\right) < 1$. We shall also assume that there exists a solution to this problem. Then by Theorem 3.6 and one of the remarks following it, all of the hypotheses of Theorem 3.2 concerning the differential equations are satisfied. Now consider the application of the projections (3.15) to approximating the solution u_0 of problem. If in Theorem 3.16 n is chosen to be one, both P_k and Q_k coincide and the conditions (3.15) become ordinary collocation on the uniform partition π_k of [0,1] over the space $Sp_0(\pi_k,3)$ of cubic polynomial splines satisfying the boundary conditions. This gives the equations

$$s''(x_i) = f\big(x_i, s(x_i), s'(x_i)\big), \qquad 0 \le i \le k, \tag{3.22}$$

$$s(0) = s(1) = 0, \tag{3.23}$$

over $Sp(\pi_k,3)$, which will require solving a (nonlinear) tridiagonal matrix system. By the preceding corollary if $u_0 \varepsilon C^4[0,1]$,

$$\| u_0^j - s_k^j \|_{L^\infty} = O(\bar{\pi}_k^2), \qquad 0 \le j \le 2.$$

If n is taken to be two, P_k and Q_k will be slightly different projections into the space of quintic splines over the uniform partition $\pi_k, Sp_0(\pi_k,5)$, given by (3.22), (3.23), and

$$s'''(a) = \frac{1}{6\bar{\pi}_k}\{-11f([x]) + 18f([x_1 2]) - 9f([x_2]) + 2f([x_3])\}, \tag{3.24}$$

$$s'''(b) = \frac{1}{6\bar{\pi}_k}\{11f([b]) - 18f([x_{k-1}]) + 9f([x_{k-2}]) - 2f([x_{k-3}])\} \tag{3.25}$$

for P_k and

$$s'''(a) = \frac{1}{18\bar{\pi}_k}\{-11f([a]) + 18f([a+\bar{\pi}_k/3]) \tag{3.24}'$$

$$- 9f([a+2\bar{\pi}_k/3]) + 2f([x_1])\},$$

$$s'''(b) = \frac{1}{18\bar{\pi}_k}\{11f([b]) - 18f([b-\bar{\pi}_k/3]) \tag{3.25}'$$

$$+ 9f([b-2\bar{\pi}_k/3]) - 2f([x_{k-1}])\}$$

for Q_k where the notation "f([x])" means "$f(x,s(x),s'(x))$". By Corollary 3.4, if $u_0 \varepsilon C^6[0,1])$, $\|u_0^j - s_k^j\|_{L^\infty} = O(\bar{\pi}_k^4)$, $0 \le j \le 2$.

There are computational advantages to both of the above methods. For either method suppose a basis for $Sp(\pi_k,5)$ with minimum span is used and the first and last equation are given by the boundary conditions (3.23). Then the contribution of equations (3.22) will be of the form of a five-diagonal band matrix. With P_k, equations (3.24)-(3.25) will introduce an extra set of four elements in the second and next to last rows of the matrix outside the band structure which will thus require special treatment. On the other hand, equations (3.24) and (3.25) can be transformed through use of (3.22) into two identities involving relationships among the basis elements alone, and these can be coded for a computer program independently of the problem. If Q_k is used, only one additional element is introduced outside the band structure in the same two rows, and this continues to be the case for n>2, unlike P_k, where the number of additional elements continue to increase along with the difficulty of computing the earlier mentioned identity. Further details and a convenient basis for quintic splines will be given in the last section in this chapter.

The next theorem will lead to a third projection method for the general mth order problem. This method preserves the advantages of the Q_k projection in giving a band matrix of width m+3 (except in two rows where there is one additional element) for splines of degree m+3, and allowing for rates of convergence up to $(\bar{\pi}_k)^4$. Moreover, the mesh requirement is weakened to be quasi-uniform.

Let π_k: $a = x_0<x_1<\ldots x_{k-1}<x_k = b$ be a partition of [a,b] with $k\geq 3$, and $\hat{\pi}_k$: $a = x_0<x_2<\ldots x_{k-2}<x_k = b$ be the associated partition formed by omitting the points x_1 and x_{k-1} from π_k. Let R_k be the projection from C[a,b] onto $Sp(\hat{\pi}_k,3)$ given by

$$R_kf(x_i) = f(x_i), \qquad 0\leq i\leq k,$$

that is, interpolation is required at all partition points of the space of cubic splines over $\hat{\pi}_k$ and in addition at x_1 and x_{k-1}. Then from Theorem 2.4, we have the next result.

Theorem 3.12. The projection R_k is a well-defined linear mapping of C[a,b] onto $Sp(\hat{\pi}_k,3)$. If $\{\pi_k\}$ is a quasi-uniform sequence of partitions of [a,b], then the R_k's are uniformly bounded in norm.

The following theorem is an immediate consequence of Theorems 3.2 and 3.12.

Theorem 3.13. Suppose the boundary value problem (3.1)-(3.2) and a solution u_0 satisfy the hypotheses of Theorem 3.2. Then letting $S_k = Sp_0(\hat{\pi}_k, m+3)$ be the subspace of $Sp(\hat{\pi}_k, m+3)$ satisfying (3.2), and $\{\pi_k\}$ a quasi-uniform set of partitions with $\bar{\pi}_k \to 0$, a $\delta > 0$ can be found such that u_0 is unique in the sphere $\{u \epsilon C^m[a,b]: \|u_0^m(x) - u^m(x)\|_{L^\infty} \le \delta\}$ and there exists an integer N such that for $k \ge N$ the equations

$$s_k^{(m)}(x_i) = f\left(x_i, s_k(x_i), \ldots, s_k^{m-1}(x_i)\right), \qquad 0 \le i \le k,$$

have a unique solution in S_k and the above sphere. Moreover

$$\|u_0^j - s_k^j\|_{L^\infty} \le K \inf\{\|u_0^m - s\|_{L^\infty}: s \epsilon Sp(\hat{\pi}_k, 3)\}, \quad 0 \le j \le m.$$

Corollary 3.5. Under the hypotheses of Theorem 3.11, s_k converges to u_0 in the uniform norm for $0 \le j \le m$. If in addition the solution u_0 satisfies for some r, $m \le r \le m+3$

(i) $u_0 \epsilon C^r[a,b]$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O\left(\bar{\pi}_k^{r-m} \omega(u_0^r, \bar{\pi}_k)\right) \qquad 0 \le j \le m,$$

(ii) $u_0 \epsilon C^r[a,b]$ and $u_0^r \epsilon Lip_M \beta$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O(\bar{\pi}_k^{r-m+\beta}), \qquad 0 \le j \le m,$$

or

(iii) $u_0 \epsilon C^r[a,b]$, u_0^r absolutely continuous and u_0^{r+1} essentially bounded, then

$$\|u_0^j - s_k^j\|_{L^\infty} = O(\bar{\pi}_k^{r-m+1}), \qquad 0 \le j \le m.$$

Next let π_k: $0 = x_0 < x_1 < \ldots < x_{k+2} < x_{k+3} < x_{k+4} = 1$ be a partition of $[0,1]$ with $x_1 = 1/4k$, $x_2 = 1/2k$, $x_{k+2} = 1 - 1/2k$, $x_{k+3} = 1 - 1/4k$, and $x_i = (i-2)/k$, $i=3,\ldots,k+1$. Let $\tilde{\pi}_k$ be the partition obtained from π_k by deleting x_1, x_2, x_{k+2}, and x_{k+3}. Let $Sp(\tilde{\pi}_k, 5)$ be the space of quintic splines over $\tilde{\pi}_k$. It was shown in Theorem 2.6 that the projections U_k defined by $U_k f = s$ where

$$s(x_i) = f(x_i), \qquad i=0,\ldots,n+4$$

are well defined projections for $k \ge 2$ and uniformly bounded in the L^∞ norm.

Theorem 3.14. Suppose the boundary value problem (3.1) with (3.2) and a solution $u_0(x)$ satisfy the hypotheses of Theorem 3.2. Then letting $S_k = Sp_0(\tilde{\pi}_k, m+5)$ be the subspace of $Sp(\tilde{\pi}_k, m+5)$ satisfying (3.2) and $\{\pi_k\}$ be a sequence of partitions defined as above and satisfying $\bar{\pi}_k \to 0$, a constant $\delta > 0$ can be found such that u_0 is unique in the sphere $\{u \in C^m[a,b]: \|u^m - u_0^m\|_\infty < \delta\}$ and there exists an integer N such that for $k \ge N$ the equations

$$s_i^m(x_i) = f\left(x_i, s_k(x_i), \ldots, s_k^{m-1}(x_i)\right), \qquad i=0,\ldots,k+4,$$

have a unique solution in S_k and the above sphere.

Moreover,

$$\|u_0^j - s_k^j\|_{L^\infty} \le K \inf\{\|u_0^m - s\|_{L^\infty} : s \text{ is in } Sp(\tilde{\pi}_k, 5)\}, \qquad 0 \le j \le m.$$

Corollary 3.6. Under the hypotheses of Theorem 3.14, s_k converges to u_0 in the uniform norm for $0 \le j \le m$. If in addition the solution u_0 satisfies for some r, $m \le r \le m+5$.

(i) $u_0 \varepsilon C^r[a,b]$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = 0\left(\bar{\pi}^{r-m}\omega(u_0^r, \bar{\pi}_k)\right), \qquad 0 \le j \le m,$$

(ii) $u_0 \varepsilon C^r[a,b]$ and $u_0^r \varepsilon Lip_M \beta$, then

$$\|u_0^j - s_k^j\|_{L^\infty} = 0(\bar{\pi}^{r-m+\beta}), \qquad 0 \le j \le m,$$

or

(iii) $u_0 \varepsilon C^r[a,b]$, u_0^r absolutely continuous and u_0^{r+1} essentially bounded, then

$$\|u_0^j - s_k^j\|_{L^\infty} = 0(\bar{\pi}_k^{r-m+1}), \qquad 0 \le j \le m.$$

Numerical Results

In this section, a convenient basis for quintic splines and numerical results for some of the methods presented in this chapter are given. Equations simplifying the use of (3.24) and (3.25) are also given. For a convenient basis for cubic splines, see [6, p.418].

For the actual application of Theorems 3.11, 3.13 and 3.14 to mth order problems over $Sp_0(\pi_k, 2n+m-1)$ one can keep the band width of the resulting matrix at a minimum by choosing splines having a minimum span. Thus for second order problems with n=2 or fourth order problems with n=1 such a basis is required for quintic splines. For uniform partitions, i.e. $(\bar{\pi}_k/\underline{\pi}_k) = 1$, with $h = \bar{\pi}_k$ such a basis is given by

$$s_i(x) = \begin{cases} (x_{i+3}-x)^5 & x \text{ in } [x_{i+2}, x_{i+3}] \\ h^5 + 5h^4(x_{i+2}-x) + 10h^3(x_{i+2}-x)^2 & \\ + 10h^2(x_{i+2}-x)^3 + 5h(x_{i+2}-x)^4 & x \text{ in } [x_{i+1}, x_{i+2}] \\ - 5(x_{i+2}-x)^5 & \\ 26h^5 + 50h^4(x_{i+1}-x) + 20h^3(x_{i+1}-x)^2 & \\ -20h^2(x_{i+1}-x)^3 - 20h(x_{i+1}-x)^4 & x \text{ in } [x_i, x_{i+1}] \\ + 10(x_{i+1}-x)^5 & \\ s_i(2x_i-x) & x \text{ in } [x_{i-3}, x_i] \\ 0 & x \text{ not in } [x_{i-3}, x_{i+3}] \end{cases} \tag{3.26}$$

where $x_i = a+hi$, and $-2 \le i \le k+2$, with the elements $\{s_i\}_{-2}^{k+2}$ being restricted by the boundary conditions. For computational purposes, it is convenient to divide $s_i(x)$ as defined in (3.26) by h^5 and then calculate the values of any one of the basis functions and its first few derivatives once and for all at the collocation points which are affected by it. For second order problems and the projections P_k, this would be the points x_{i-2}, x_{i-1}, x_i, x_{i+1}, and x_{i+2} for s_i. In this case, if s satisfies (3.2) and is given by $\sum_{i=-2}^{k+2} \alpha_i s_i(x)$, and if the first and last equations are given by the boundary conditions, the second and next to last by (3.24) and (3.25) and the rest by (3.22), the resulting matrix will be five-diagonal except for the first and last rows. Moreover, if (3.24) and (3.25) are combined with (3.22), the following new equations, which can be coded in the second and next to last rows independently of the problem, result:

$$-7\alpha_{-2} + 40\alpha_{-1} - 93\alpha_0 + 110\alpha_1 - 65\alpha_2 + 12\alpha_3 + 5\alpha_4 - 2\alpha_5 = 0, \qquad (3.27)$$

$$-2\alpha_{k-5} + 5\alpha_{k-4} + 12\alpha_{k-3} - 65\alpha_{k-2} + 110\alpha_{k-1} - 93\alpha_k + 40\alpha_{k+1} - 7\alpha_{k+2} = 0.$$

If instead of the P_k, the projections Q_k or R_k are used, only the equations in the second and next to last row will be changed. For Q_k, since evaluations one-third and two-thirds of the way between the node points of the basis functions $s_i(x)$ are required along with evaluations at x_0, x_1, x_{k-1}, and x_k, only a partial simplification of the form (3.27) is possible. If instead R_k is used, evaluations of the basis functions at the midpoints is required. Note that for both Q_k and R_k the second and next to the last row will involve relations among only six bases elements instead of eight as in (3.27), coming close to the five diagonal form of the equations in between. Thus for any of these three projection schemes, after a few initial evaluations of the basis functions are made, no further explicit use of (3.26) is required except if desired when writing the solution.

If the original problem (3.1)-(3.2) is linear, Gaussian elimination is an effective way to solve the resulting matrix system. The matrices developed are with the exception of a few rows diagonally dominant and so Gaussian elimination is generally stable with respect to round-off error. If the problem (3.1)-(3.2) is nonlinear, we have found that Newton's method has been successful in all of the problems which have been used for numerical experiments. A starting value of zero was used and convergence obtained within four to five iterations in all cases. Each iteration essentially involves going through the preceding linear loop once, and thus relatively little coding is required to modify a computer program that handles the linear case to treat the nonlinear case.

Some specific examples of (3.1)-(3.2) are discussed next. First, consider the linear problem

$$D^2u(x) = 4u(x) + 4\cosh 1, \tag{3.28}$$

with boundary conditions $u(0) = u(1) = 0$. It is easy to verify that $u_0(x) = \cosh(2x-1) - \cosh 1$ is the unique solution to (3.28) so by Theorem 3.4 the hypotheses of Theorem 3.2 concerning (3.28) are satisfied.

Table 5 lists some numerical results of applying the projections over subspaces of quintic splines. Since u_0 is analytic, in particular it is in $C^6[0,1]$ and so convergence of the approximates and the first two derivatives will be fourth order.

Table 5. Three Collocation Methods Over $Sp_0(\pi_k,5)$

$h=1/k$	$\dim(Sp_0(\pi_k,5))$	$\|s_k^P-u_0\|_{L^\infty}$	$\|s_k^Q-u_0\|_{L^\infty}$	$\|s_k^R-u_0\|_{L^\infty}$
1/7	10	$3.35\cdot10^{-6}$	$3.42\cdot10^{-6}$	$3.35\cdot10^{-6}$
1/9	12	$8.01\cdot10^{-7}$	$1.27\cdot10^{-6}$	$1.25\cdot10^{-6}$
1/11	14	$4.28\cdot10^{-7}$	$5.71\cdot10^{-7}$	$5.66\cdot10^{-7}$

For comparison, the results of two other fourth order methods, the method of Galerkin over cubic splines and the collocation method of [29] are given in Tables 6 and 7.

Table 6. Galerkin Method Over $Sp_0(\pi_k,3)$

$h=1/k$	$\dim Sp_0(\pi_k,3)$	$\|s_k^G-u_0\|_{L^\infty}$
1/5	6	$3.98\cdot10^{-5}$
1/7	8	$1.12\cdot10^{-5}$
1/9	10	$4.36\cdot10^{-6}$

We note that the Galerkin method yields seven diagonal matrices for this problem.

Table 7. Collocation Over $Sp_0(\pi_k,5,2)$

$h=1/k$	$\dim Sp_0(\pi_k,5,2)$	$\lVert s_k-u_0\rVert_{L^\infty}$
1/5	16	$1.49\cdot 10^{-6}$
1/7	22	$3.90\cdot 10^{-7}$
1/9	28	$1.43\cdot 10^{-7}$

As a second example, consider the nonlinear problem

$$D^2u(x) = e^{u(x)}, \qquad 0<x<1, \tag{3.29}$$

with $u(0) = u(1) = 0$. The unique solution to (3.29) is $u_0(x) = \ln 2 + 2\ln[c\sec(c(x-.5)/2)]$, where $c \doteq 1.3360556949$. Letting $a_0=a_1=0$, Theorem 3.4 applies since $-\exp(u(x))\leq 0$ and so the hypotheses of Theorem 3.2 regarding (3.29) are satisfied. Convergence will be fourth order. Table 8 contains the results of applying P_k, Q_k, R_k to this problem.

Table 8. A Second Application of Three Collocation Methods Over $Sp_0(\pi_k,5)$

$h=1/k$	$\|s_k^P-u_0\|_{L^\infty}$	$\|s_k^Q-u_0\|_{L^\infty}$	$\|s_k^R-u_0\|_{L^\infty}$
1/7	$1.78\cdot 10^{-7}$	$2.09\cdot 10^{-7}$	$2.02\cdot 10^{-7}$
1/9	$4.23\cdot 10^{-8}$	$7.79\cdot 10^{-8}$	$7.65\cdot 10^{-8}$
1/11	$2.41\cdot 10^{-8}$	$3.53\cdot 10^{-8}$	$3.49\cdot 10^{-8}$

The error bounds for the first and second derivatives are summarized in Table 9 on the following page. For comparison, we list results for the Galerkin method over cubic splines, which will also be a fourth order method for this problem. The Galerkin method yields a seven diagonal matrix.

Table 10. A Second Application of Galerkin Method Over $Sp_0(\pi_k,3)$

$h=1/k$	$\|s_k-u_0\|_{L^\infty}$
1/5	$2.39\cdot 10^{-6}$
1/7	$6.44\cdot 10^{-7}$
1/9	$2.50\cdot 10^{-7}$

As a final example, consider the nonlinear problem

$$xu'' = u' - (u')^3, \qquad 1<x<2, \tag{3.30}$$

Table 9. Results for First and Second Derivatives

$h=1/k$	$\Vert(s_k^P-u_0)'\Vert_{L^\infty}$	$\Vert(s_k^Q-u_0)'\Vert_{L^\infty}$	$\Vert(s_k^R-u_0)'\Vert_{L^\infty}$	$\Vert(s_k^P-u_0)''\Vert_{L^\infty}$	$\Vert(s_k^Q-u_0)''\Vert_{L^\infty}$	$\Vert(s_k^R-u_0)''\Vert_{L^\infty}$
1/7	$4.06\ 10^{-6}$	$8.26\ 10^{-7}$	$7.18\ 10^{-7}$	$6.93\ 10^{-5}$	$4.46\ 10^{-6}$	$5.27\ 10^{-6}$
1/9	$1.21\ 10^{-6}$	$3.19\ 10^{-7}$	$2.88\ 10^{-7}$	$2.75\ 10^{-5}$	$1.84\ 10^{-6}$	$2.14\ 10^{-6}$
1/11	$4.47\ 10^{-7}$	$1.48\ 10^{-7}$	$1.37\ 10^{-7}$	$1.30\ 10^{-5}$	$8.95\ 10^{-7}$	$1.03\ 10^{-6}$

with nonhomogeneous boundary conditions

$$u(1) = \sqrt{2} \quad \text{and} \quad u(2) = \sqrt{5}.$$

One solution to (3.30) is $u_0(x) = \sqrt{1+x^2}$. In order to apply the techniques of this chapter to (3.30), the problem must be modified to the form (3.1)-(3.2). This is in general easy to do, and in this case we have

$$\hat{u}'' = f(x,\hat{u},\hat{u}') = [\hat{u}' + \sqrt{5} - \sqrt{2} - (\hat{u}'+\sqrt{5}-\sqrt{2})^3]/x, \qquad 1<x<2, \quad (3.31)$$

with $\hat{u}(1) = \hat{u}(2) = 0$, where $u_0(x) = \hat{u}(x) + \sqrt{2} + (\sqrt{5}-\sqrt{2})(x-1)$. Since $\hat{u}'(x) = (x/\sqrt{1+x^2}) - \sqrt{5} + \sqrt{2}$, $f_{\hat{u}} = 0$, and $f_{\hat{u}'} = [1 - 3(\hat{u}'+\sqrt{5}-\sqrt{2})^2]/x$, then equation (3.4) becomes

$$u'' - \left[\frac{1}{x} - \frac{3x}{1+x^2}\right]u' = 0, \qquad 1<x<2, \qquad (3.32)$$

with

$$u(1) = u(2) = 0. \qquad (3.33)$$

Integrating (3.32) it follows that $u(x) = c_1(1+x^2)^{-1/2} + c_2$ and then evaluating the boundary conditions shows $c_1=c_2=0$, and so (3.32) has only the trivial solution. Thus it follows that the hypotheses of Theorem 3.2 are satisfied. Since u_0 is in $C^6[1,2]$, convergence achieved by applying the projections P_k, Q_k, and R_k will be fourth order. Tables

11 and 12 contain results of applying the projections P_k, Q_k, and R_k to the modified problem (3.31) with (3.33).

Table 11. A Third Application of Three Collocation Methods Over $Sp_0(\pi_k,5)$

h=1/k	$\|s_k^P-u_0\|_{L^\infty}$	$\|s_k^Q-u_0\|_{L^\infty}$	$\|s_k^R-u_0\|_{L^\infty}$
1/7	$5.40\cdot10^{-8}$	$5.78\cdot10^{-8}$	$5.87\cdot10^{-8}$
1/9	$1.94\cdot10^{-8}$	$2.05\cdot10^{-8}$	$2.07\cdot10^{-8}$
1/11	$9.06\cdot10^{-9}$	$9.03\cdot10^{-9}$	$9.11\cdot10^{-9}$

The error bounds for the derivatives and second derivatives are summarized in Table 12 on the following page.

Table 12. Results for Derivatives

$h=1/k$	$\|(s_k^P-u_0)'\|_{L^\infty}$	$\|(s_k^Q-u_0)'\|_{L^\infty}$	$\|(s_k^R-u_0)'\|_{L^\infty}$	$\|(s_k^P-u_0)''\|_{L^\infty}$	$\|(s_k^Q-u_0)''\|_{L^\infty}$	$\|(s_k^R-u_0)''\|_{L^\infty}$
1/7	$3.74\cdot10^{-7}$	$1.98\cdot10^{-7}$	$2.11\cdot10^{-7}$	$5.51\cdot10^{-6}$	$1.23\cdot10^{-6}$	$1.22\cdot10^{-6}$
1/9	$2.38\cdot10^{-7}$	$6.31\cdot10^{-8}$	$6.78\cdot10^{-8}$	$4.07\cdot10^{-6}$	$4.80\cdot10^{-7}$	$4.81\cdot10^{-7}$
1/11	$1.27\cdot10^{-7}$	$2.85\cdot10^{-8}$	$2.91\cdot10^{-8}$	$2.48\cdot10^{-6}$	$2.22\cdot10^{-7}$	$2.24\cdot10^{-7}$

CHAPTER IV

HIGH ORDER PROBLEMS AND THE GALERKIN METHOD

In an important series of papers, Ciarlet, Schultz, and Varga [6], [7], Perrin, Price, and Varga [23], Schultz [31], [32], and Lucas [20] among others have made a systematic study of the application of the Galerkin method using spline functions to approximate the solution of a class of linear and mildly nonlinear boundary value problems. In this chapter, some of the results of these papers are extended to problems with more general nonlinearities. The proofs will use monotone operator theory [7] and provide a contrast to the methods of Chapters II, III. In particular, a result that the rate of convergence depends on the order of the second highest derivative in the equation will be obtained simply by algebraic manipulations. This phenomenon occurred in Chapter II for second order problems, nonlinear in the derivative term, due to the selection of a particular norm.

The following nonlinear boundary value problems are considered:

$$(-1)^m D^{2m} u + f(x,u,\ldots,u^j) = 0, \qquad 0<x<1,\ 0\le j\le m,\ m\ge 1 \tag{4.1}$$

with boundary conditions

$$u^k(0) = u^k(1) = 0, \qquad 0\le k\le m-1; \tag{4.2}$$

$$(-1)^m D^{2m-1}u + f(x,u,\ldots,u^j) = 0, \qquad 0<x<1,\ 0\le j\le m,\ m\ge 2 \qquad (4.3)$$

with boundary conditions

$$u^i(0) = u^k(1) = 0, \qquad 0\le i\le m-1,\ 1\le k\le m-1. \qquad (4.4)$$

The even order problem is considered first. The Galerkin form [7] associated with (4.1)-(4.2) is

$$a(u,v) = \int_0^1 D^m u D^m v\,dx + \int_0^1 f(x,u,\ldots,u^j)v\,dx.$$

Generalized solutions and approximations to (4.1)-(4.2) with respect to $a(u,v)$ are sought over the space of functions $W_0^{m,2}[0,1]$, i.e. those that are in the Sobolev space [42] $W^{m,2}[0,1]$ and satisfy the boundary conditions (4.2). The usual Sobolev norm, $\{\sum_{k=1}^{m} \|u^k\|^2_{L^2}\}^{1/2}$, will be denoted as $\|u\|_m$. A generalized solution to (4.1)-(4.2) is a function $u_0 \varepsilon W_0^{m,2}[0,1]$ satisfying $a(u_0,v) = 0$ for all $v\varepsilon W_0^{m,2}[0,1]$.

Let $f(x,u,\ldots,u^j)$ and $\frac{\partial f}{\partial u^k}$ be continuous on $[0,1] \times R \times \ldots \times R$ for $0\le k\le j$. This set of functions will be denoted by $C^1[A_j]$. Let

$$\frac{\partial f}{\partial u} \ge a_0, \quad \left|\frac{\partial f}{\partial u^k}\right| \le a_k \qquad \text{for } 1\le k\le j,$$

and

$$\alpha_j = \left(\frac{\max(-a_0,0)}{\pi^{2m}} + \frac{a_1}{4\pi^{2m-2}} + \sum_{k=2}^{j} a_K/\pi^{2m-k}\right) < 1. \qquad (4.5)$$

Condition (4.5) is needed to develop a higher-order convergence result. Other regularity conditions assuring convergence of the Galerkin method are possible, see [7] and [31], but convergence will be of a lower order than the rate developed here. Moreover, in the event only low order convergence can be achieved, application of a collocation technique as in Chapter III may be preferred.

The cases $j=m$ and $j<m$ in (4.1) are markedly different due to the fact that using the boundary conditions (4.2), it follows that bounds on $\|u\|_n$ yield uniform bounds for $D^j u$, $0\le j\le m-1$, but not for $D^m u$. Uniform bounds on the arguments of f are needed in the proofs to follow. For the case $j=m$, a classical solution u to (4.1)-(4.2) is assumed to exist, a priori bounds for $D^j u$, $0\le j\le m$, are obtained, and f is modified to obtain an equivalent problem with sufficient regularity to permit application of the Galerkin method. If $j\le m-1$, it will be shown that the Galerkin method can be applied to the problem directly.

If u is a generalized solution to (4.1)-(4.2) with $j=m$, then it follows using the mean value theorem that

$$\begin{aligned}\int_0^1 (D^m u)^2 dx &= -\int_0^1 f(x,u,\ldots,D^m u)u\,dx \qquad (4.6)\\ &= -\int_0^1 f(x,0,\ldots,0)u\,dx - \int_0^1 \frac{\partial f}{\partial u}(x,\theta u,\ldots,\theta D^m u)u^2 dx\\ &\quad - \ldots - \int_0^1 \frac{\partial f}{\partial D^m u}(x,\theta u,\ldots,\theta D^m u)D^m u\,u\,dx\end{aligned}$$

where $0\le\theta(x)\le 1$. The Cauchy-Schwarz and Rayleigh-Ritz inequalities imply

$$\int_0^1 |D^k uu|dx \le \frac{1}{\pi^{2m-k}} \int_0^1 (D^m u)^2 dx. \tag{4.7}$$

Applying (4.7) to (4.6), it follows then that

$$\int_0^1 (D^m u)^2 dx \le \frac{A_1}{\pi^m} \|D^m u\|_{L^2} + \alpha_m \|D^m u\|^2_{L^2} \tag{4.8}$$

where $A_1 = \sup_{0\le x\le 1} |f(x,0,0,\ldots,0)|$. Note that the Opial inequality [22] was also used in finding α_m, i.e. if $u(0) = u(1) = 0$ and u is in $W^{1,2}[0,1]$, then

$$\int_0^1 |Duu|dx \le 1/4 \int_0^1 (Du)^2 dx.$$

From (4.8) one has

$$\|D^m u\|_{L^2} \le \frac{A_1}{\pi^m(1-\alpha_m)} = B_1.$$

Using the boundary conditions it is easy to see that $\|D^j u\|_{L^\infty} \le \|D^{j+1}u\|_{L^2}$ for $0\le j\le m-1$, and so $\|D^j u\|_{L^\infty} \le B_1$ for $0\le j\le m-1$.

Now assuming u is a classical solution to (4.1)-(4.2), the boundary conditions and Rolle's theorem can be used repeatedly to write

$$D^m u(x) = D^m u(y) + \int_y^x \int_{c_{m-1}}^{t_{m-1}} \cdots \int_{c_1}^{t_1} D^{2m} u(t)dtdt_1\ldots dt_{m-1}$$

where $0<c_i<1$ and $D^{2m-i}u(c_i) = 0$, $1\le i\le m-1$, $0\le x,y\le 1$. Then

$$|D^m u(x)|^2 \le 2|D^m u(y)|^2 + 2\Big(\int_0^1 |f(t,u,\dots,D^m u)|dt\Big)^2 \tag{4.9}$$

$$\le 2|D^m u(y)|^2 + 2\Big(\int_0^1 |f(t,u,\dots,D^{m-1}u,0)|dt$$

$$+ \int_0^1 \Big|\frac{\partial f}{\partial D^m u}(t,u,\dots,D^{m-1}u,\theta D^m u)D^m u\Big|dt\Big)^2$$

$$\le 2|D^m u(y)|^2 + 2A_2^2 + 4A_2 a_m B_1 + 2a_m^2 B_1^2$$

where $A_2 = \sup\{|f(x,u,\dots,D^{m-1}u,0)| : |D^j u| \le B_1, \quad 0\le j\le m-1,\ 0<x<1\}$. Integrating both sides of (4.9) with respect to y over [0,1], the following uniform bound for $D^m u(x)$ is obtained:

$$|D^m u(x)|^2 \le 2B_1^2 + 2A_2^2 + 4A_2 a_m B_1 + 2a_m^2 B_1^2 = B_2^2.$$

The problem (4.1)-(4.2) can be transformed as follows. Using the real-valued continuously differentiable function defined for all real u by

$$h_M(u) = \begin{cases} M+1-\exp(M-u) & M<u \\ u & |u|\le M \\ -M-1-\exp(M+u) & u<-M, \end{cases}$$

consider the problem

$$(-1)^m D^{2m}u + \hat{f}(x,u,\dots,D^m u) = 0, \qquad 0<x<1, \tag{4.10}$$

with boundary conditions

$$D^j u(0) = D^j u(1) = 0, \qquad 0 \le j \le m-1, \tag{4.11}$$

where $\hat{f}(x,u,\ldots,D^m u) = f\left(x,h_{B_1}(u),\ldots,h_{B_1}(D^{m-1}u),\ h_{B_2}(D^m u)\right)$. Noting that $0<h_M'(u) \le 1$, the bounds established for the solution to (4.1)-(4.2) with j=m are valid for (4.10)-(4.11). Consequently, from the definition of $\hat{f}$, a classical solution to (4.1)-(4.2) is a classical solution to (4.10)-(4.11) and conversely. Note that $\hat{f}$ satisfies the same regularity hypotheses as f, but in addition, $\hat{f}$ as well as its partials with respect to $D^j u$, $0 \le j \le m$, are uniformly bounded. That is, in addition to (4.5) there exist constants B_3, B_4 so that

$$\left|\hat{f}(x,u,\ldots,D^m u)\right| \le B_3, \quad \frac{\partial \hat{f}}{\partial u} \le B_4. \tag{4.12}$$

We now state a result from the theory of monotone operators which can be found in Ciarlet, Schultz, and Varga [7].

Lemma 4.1. Let X be a real Hilbert space, and T be an operator (nonlinear) mapping X into X. If T is strongly monotone, i.e. there exists an $\alpha>0$ such that $(Tu-Tv,\ u-v)_X \ge \alpha\|u-v\|_X^2$ for all u,v in X; and T is Lipschitz continuous for bounded arguments, i.e. given K>0 there exists a constant C(K) so that $\|Tu-Tv\|_X \le C(K)\|u-v\|_X$ for all u,v in X with $\|u\|_X$, $\|v\|_X \le K$, then the problem of determining a u in X such that

$$(Tu,v)_X = 0 \qquad \text{for all } v \text{ in } X$$

and for any finite dimensional subspace of S_k of X, the problem of determining a u_k in S_k such that

$$(Tu_k,v)_X = 0 \qquad \text{for all } v \text{ in } S_k,$$

each have a unique solution. Moreover, there exists a constant K' such that the following error bound is valid:

$$\|u_k-u\|_X \le K' \inf\{\|w-u\|_X\colon w\varepsilon S_k\}.$$

Theorem 4.1. Let $f\varepsilon C^1[A_j]$ satisfy (4.5). If j=m, suppose in addition (4.12) is satisfied. Then the problem (4.1)-(4.2) has a unique generalized solution over $W_0^{m,2}[0,1]$. If S is any finite dimensional subspace of $W_0^{m,2}[0,1]$, then there exists a unique Galerkin approximation $u_S \varepsilon S$. Furthermore, there exists a constant K independent of S such that

$$\|u-u_S\|_M \le K \inf\{\|w-u\|_m\colon w\varepsilon S\}.$$

Proof. Suppose first that j=m. Applying Lemma 4.1, the result will follow once the Galerkin form a(u,v) is shown to be Lipschitz continuous in u and strongly monotone. It follows from the definition of a(u,v) that for fixed u it will be a continuous linear functional in v over $W_0^{m,2}[0,1]$. Hence by the Riesz representation theorem, there exists an operator T from $W_0^{m,2}[a,b]$ into itself such that $(Tu,v)_m = a(u,v)$. In what follows, Lemma 4.1 is applied to T, but only as it is defined through a(u,v). Lipschitz continuity is established first.

Using the triangle inequality and the mean value theorem, one has

$$|a(u_1-u_2,v)| \le \int_0^1 |D^m(u_1-u_2)D^m v|\,dx + \Big|\int_0^1 \big(f(x,u_1,\ldots,D^m u_1) - f(x,u_2,\ldots,D^m u_2)\big) v\,dx\Big|$$

$$\le \int_0^1 |D^m(u_1-u_2)D^m v|\,dx + \Big|\int_0^1 \frac{\partial f}{\partial u}(u_1-u_2)v\,dx + \ldots + \int_0^1 \frac{\partial f}{\partial D^m u} D^m(u_1-u_2)v\,dx\Big|,$$

where the partials are evaluated at $\theta(x,u_1,\ldots,D^m u_1) + (1-\theta)(x,u_2,\ldots,D^m u_2)$, $0\le\theta(x)\le 1$, and u_1, u_2, and v are in $W_0^{m,2}[a,b]$. Using the bounds on the partials of f and the Cauchy-Schwarz and Rayleigh-Ritz inequalities on the right-hand side of (4.14), then it follows that there exists a constant $K>0$ so that

$$|a(u_1-u_2,v)| \le K\|u_1-u_2\|_m\|v\|_m.$$

This implies Lipschitz continuity.

The mean value theorem and the bounds on the partials of f imply

$$a(u-v,u-v) = \int_0^1 \big(D^m(u-v)\big)^2 dx + \int_0^1 \big(f(x,u,\ldots,D^m u) - f(x,v,\ldots,D^m v)\big)(u-v)\,dx \tag{4.15}$$

$$= \int_0^1 \big(D^m(u-v)\big)^2 dx + \int_0^1 \frac{\partial f}{\partial u}(u-v)^2 dx + \ldots$$

$$+ \int_0^1 \frac{\partial f}{\partial D^m u} D^m(u-v)(u-v)dx$$

$$\geq \int_0^1 \left(D^m(u-v)\right)^2 dx - \max(-a_0,0)\int_0^1 (u-v)^2 dx - \ldots - a_m \int_0^1 |D^m(u-v)(u-v)|dx.$$

Applying (4.5) and (4.7) to (4.15), then

$$a(u-v,u-v) \geq (1-\alpha_m) \int_0^1 \left(D^m(u-v)\right)^2 dx. \tag{4.16}$$

Finally, applying the Rayleigh-Ritz inequality repeatedly to the right-hand side of (4.16) and adding the resulting inequalities, we find

$$a(u-v,u-v) \geq \left(\frac{1-\alpha_m}{m}\right)\|u-v\|_m^2.$$

This completes the proof for the case $j=m$. If $j<m$, the only change required in the above proof is in establishing Lipschitz continuity. If $\|u\|_m \leq K$, it follows that $\|u^j\|_{L^\infty} \leq K$ for $0\leq j\leq m-1$. Using this fact and the continuity of the partials of f, Lipschitz continuity can be established as in (4.14).

We can now immediately apply known error bounds for various classes of splines and deduce convergence rates. We choose to state a generalization to Theorem 7.10 of [7] using polynomial splines.

Corollary 4.1. Let $\{\pi_n\}$ be any sequence of partitions satisfying $\bar{\pi}_n \to 0$. Let S_n be those functions in $Sp(\pi_n,2k-1)$, $2k-1\geq 2m$, that satisfy the boundary conditions (4.2). Then there exists a unique Galerkin approximation u_n over S_n, and if the generalized solution u is of class

$W^{t,2}[a,b]$, $k \le t \le 2k$, then

$$\|(u-u_n)^j\|_{L^\infty} = O(\bar{\pi}_n^{t-m}), \qquad 0 \le j \le m-1.$$

We note that many results stating error bounds are possible here as the error depends on the best approximation properties of splines.

For the odd order problem (4.3)-(4.4), the theory of K-positive definite operators [24, p.128] and [7] is used with K=D. Let $A = (-1)^m D^{2m-1}$ and let the domain of A be those functions in $C^{2m-1}[0,1]$ satisfying the boundary conditions (4.4). Then it follows from [24] that the Galerkin form associated with this problem is

$$a(u,v) = \int_0^1 D^m u D^m v dx - \int_0^1 f(x_1 u,\ldots,D^j u) D v dx \tag{4.17}$$

for u,v in the domain of A. Denoting H_D as the completion of the domain of A with respect to

$$\int_0^1 (D^m u)^2 dx = \|u\|_{H_D}^2,$$

generalized solutions to (4.3)-(4.4) are sought over H_D relative to (4.17). A description of this procedure for the third order case is contained in [7]. Conditions can now be put on f analogous to the even order case to deduce a(u,v) is Lipschitz continuous and strongly monotone over H_D, and thus the Galerkin method converges. If f depends on $D^m u$, then before applying the Galerkin method, it will be necessary to establish bounds for the solution and its derivatives up to order m and then modify the odd order problem in the manner as the even order case. We simply state here the analogous to Theorem 4.1 and Corollary 4.1

Theorem 4.2. Let $f \varepsilon C^1[A_j]$. If $j=m$, assume $\left|\frac{\partial f}{\partial D^j u}\right| \le a_j$ for $j=0,2,\ldots,m$, $A_1 \le \frac{\partial f}{\partial Du} \le a_1$, and

$$\beta_m = \left(\frac{2a_0}{\pi^{2n-1}} + \frac{\max(a_1,0)}{\pi^{2n-2}} + \sum_{j=3}^{m} a_j/\pi^{2n-j-1}\right) < 1.$$

If $j<m$, assume $\frac{\partial f}{\partial Du} \le a_1$, $\left|\frac{\partial f}{\partial D^k u}\right| \le a_k$ for $k=0,2,\ldots,j$, and $\beta_j<1$. Then the problem (4.3)-(4.4) has a unique generalized solution relative to (4.17) over H_D. If S is any finite dimensional subspace of H_D, then there exists a unique Galerkin approximate u_S over S and a constant K>0 and independent of S so that

$$\|u-u_S\|_{H_D} \le K \inf\{\|u-w\|_{H_D} : w \varepsilon S\}.$$

Corollary 4.2. Let $\{\pi_n\}$ be any sequence of partitions of [0,1] satisfying $\bar{\pi}_n \to 0$. Let S_n be those functions in $Sp(\pi_n, 2k-1)$, $2k-1 \ge 2m$, satisfying (4.4). Then there exists a unique Galerkin approximation u_n over S_n, and if the generalized solution u is of class $W^{t,2}[a,b]$, $k \le t \le 2k$, then

$$\|(u-u_n)^j\|_{L^\infty} = O(\bar{\pi}_n^{t-m}), \qquad 0 \le j \le m-1.$$

The previous convergence rates can be improved. We treat the odd order problem in detail. Let S be those polynomial splines in $Sp(\pi,2m-1)$ satisfying (4.4). Here $2m-1$ is the order of the equation (4.3). Let f satisfy the hypotheses of Theorem 4.2 with $j<m$. If $j=m$, convergence is given by Corollary 4.2. Let u be the unique generalized solution to (4.3)-(4.4), u_S the Galerkin approximation over S, and w the S-interpolate of u defined by $w(x_i) = u(x_i)$ for $x_i \varepsilon \pi$, and $w^j(0) = u^j(0)$, $w^j(1) = u^j(1)$, for $i \le j \le m-1$. A comparison of u_S and w will be made extending a technique used for mildly nonlinear problems in [23].

The orthogonality relation of odd-order polynomial spline interpolation [33] implies

$$\int_0^1 D^m(w-u)D^m(w-u_S)dx = 0.$$

The definitions of u and u_S imply

$$\int_0^1 \left(D^m u D^m(w-u_S) - f(x,u,\ldots,D^j u)D(w-u_S)\right)dx = 0, \qquad (4.19)$$

and

$$\int_0^1 \left(D^m u, D^m(w-u_S) - f(x,u_S,\ldots,D^j u_S)D(w-u_S)\right)dx = 0. \qquad (4.20)$$

Now define

$$\hat{K} = \int_0^1 \left(D^m u D^m(w-u_S) - f(x,w,\ldots,D^j w)D(w-u_S)\right)dx. \qquad (4.21)$$

Subtracting (4.19) from (4.21) and using (4.18), one has

$$\hat{K} = -\int_0^1 \big(f(x,w,\ldots,D^j w) - f(x,u,\ldots,D^j u)\big) D(w-u_S)dx. \qquad (4.22)$$

Subtracting (4.20) from (4.21), then

$$\hat{K} = \int_0^1 \big[(D^m(w-u_S)\big)^2 - \big(f(x,u_S,\ldots,D^j u_S) \qquad (4.23)$$

$$- f(x,w,\ldots,D^j w)\big) D(w-u_S)\big] dx$$

From the proof of Theorem 4.2 as in the proof of Theorem 4.1, a(u,v) is strongly monotone and so (4.23) implies that for some constant $K>0$,

$$\hat{K} \geq K \int_0^1 \big(D^m(w-u_S)\big)^2 dx. \qquad (4.24)$$

Note that as w is the S-interpolate of u, $\|D^j u - D^j w\|_{L^\infty}$ can be bounded for $0\leq j\leq m-1$ independently of w and S using L-spline convergence theorems. For example, Theorem 6 of [33] implies $\|D^j u - D^j w\|_{L^\infty} = O(\bar{\pi}^{m-j-1/2}\|D^m u\|_{L^2})$. Thus having uniform bounds for $D^j u$, $0\leq j\leq m-1$, one can obtain through the triangle inequality uniform bounds for $D^j w$, $0\leq j\leq m-1$. Furthermore, using the boundary conditions (4.4), it follows that

$$\|D^j u\|_{L^\infty} \leq \|D^{j+1} u\|_{L^\infty} \leq \|u\|_{H_D} \quad \text{for } 0\leq j\leq m-1. \qquad (4.25)$$

The above remarks can be used to bound the arguments of f in (4.22).

Then using the mean value theorem and the Rayleigh-Ritz inequality, there will be a constant $K_1>0$ so that

$$\hat{K} \le K_1 \| D^j(u-w) \|_{L} \; \| D(w-u_S) \|_{L^2}. \tag{4.26}$$

Applying the Rayleigh-Ritz inequality to the right-hand sides of (4.24), (4.26) and combining the two, it follows that

$$\| D^k(w-u_S) \|_{L^2} \le K_2 \| D^j(u-w) \|_{L^2}, \qquad 1 \le k \le m. \tag{4.27}$$

Now from the triangle inequality, (4.25), and (4.27), it follows finally that

$$\begin{aligned} \| D^i(u-u_S) \|_{L^\infty} &\le \| D^i(u-w) \|_{L^\infty} + \| D^i(w-u_S) \|_{L^\infty} \\ &\le \| D^i(u-w) \|_{L^\infty} + K_2 \| D^j(w-u_S) \|_{L^2} \end{aligned}$$

for $0 \le i \le m-1$. Combining (4.28) with known error bounds for splines gives the next theorem.

<u>Theorem 4.3</u>. Let the hypotheses of Theorem 4.2 hold. Let $\{\pi_n\}$ be a sequence of partitions of [0,1] satisfying $\bar{\pi}_n \to 0$. Let S_n be those splines in $Sp(\pi_n, 2m-1)$ satisfying (4.4). Let u_n represent the Galerkin approximation over S_n, and let u, the generalized solution to (4.3)-(4.4), be of class $W^{2m,2}[0,1]$. Then if j is the highest derivative that f depends on where $1 \le j \le m$, then

$$\|D^k(u-u_n)\|_{L^\infty} = O(\pi_n^{-2m-j}), \qquad 0<k\le j-1,$$

and

$$\|D^k(u-u_n)\|_{L^\infty} = O(\pi_n^{-2m-k-1/2}), \qquad j\le k\le m-1.$$

If j=0,

$$\|D^k(u-u_n)\|_{L^\infty} = O(\pi_n^{-2m-j-1/2}), \qquad 0\le j\le m-1$$

Theorem 3 can be used to improve a result of Ciarlet, Schultz, Varga [Theorem 7.5, 7] for the specific problem $-D^3u = f(x,u,Du)$, $0<x<1$, with $u(0) = Du(0) = Du(1) = 0$; and where f is measurable in x, Lipschitz continuous in u, D_u and satisfies

$$\pi^2 > \alpha \ge \frac{f(x,\theta,\phi) - f(x,\theta',\phi')}{\phi - \phi'} \tag{4.29}$$

for all $0\le x\le 1$, $-\infty<\theta,\phi,\theta',\phi'<\infty$. Using (4.29), it is established in [7] that the associated Galerkin form is strongly monotone. Using this fact and the Lipschitz continuity of f in u,Du, inequality (4.28) follows as before, although the constant K_2 cannot be determined a priori. Assuming the generalized solution is in $W^{4,2}[0,1]$ and using cubic splines, Theorem 4.3 implies convergence of order 3 in the uniform norm, whereas the result of [7] for the same conditions states convergence of order 2 for general spline subspaces. Suppose in addition to the above hypotheses, $f(x,0,0)$ is in $L^2[0,1]$. If u is the generalized solution, then proceeding as before and using (4.29),

$$\int_0^1 (D^2u)^2 dx = \int_0^1 f(x,u,Du)Du dx \tag{4.30}$$

$$= \int_0^1 f(x,0,0)Du dx + \int_0^1 \big(f(x,u,Du) - f(x,0,0)\big)Du dx$$

$$\leq \|f(x,0,0)\|_{L^2}\|Du\|_{L^2} + \alpha\|Du\|^2_{L^2}$$

Then using the Rayleigh-Ritz inequality and solving (4.30) for $\|D^2u\|_{L^2}$ it follows that

$$\|D^2u\|_{L^2} \leq \frac{\|f(x,0,0)\|_{L^2}}{\pi^2(1-\alpha/\pi^2)} .$$

Using this a priori uniform bound for u,Du, the constant K_2' can be specified.

Note from (4.22) that if f is a function of x alone then $\hat{K}=0$. This implies $w=u_S$ on [0,1]. This result extends a theorem of Rose [27] to the class of odd order problems (4.3)-(4.4). Proceeding in a manner analogous to the proof of Theorem 4.3, a similar theorem can be established for problems (4.1)-(4.2).

<u>Theorem 4.4</u>. Let f satisfy the hypotheses of Theorem 4.1, $\{\pi_n\}$ be a sequence of partitions of [0,1] satisfying $\bar{\pi}_n \to 0$, and S_n be those functions in $Sp(\pi_n, 2m-1)$ satisfying (4.2). Let u_n be the Galerkin approximation over S_n and suppose u, the generalized solution to (4.3)-(4.4) is in $W^{2m,2}[0,1]$. Then the conclusion to Theorem 4.3 holds.

It is possible to extend the preceding results to establish higher-order convergence if the solution possesses additional smoothness. The technique necessary to do this is known and is described for example in [23]. We close with several observations. Note that monotone operator theory enabled us to relax the mesh restriction and consider problems of general order. The results in Chapter II depended on special spline error bounds, the arguments for which appear difficult to generalize except for special cases. Monotone operator theory also allowed the introduction of the idea of a generalized solution and hence permits application when a classical solution might not necessarily exist. However, monotone operator theory requires rather strong hypotheses on the equation and for special problems it seems that application of the techniques in either Chapters II or III are best to develop an appropriate approximation scheme.

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