SPECTRAL ANALYSIS OF TWO-VARIATE STOCHASTIC PROCESSES

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SUMMARY

The paper is concerned with the determination of autocorrelation functions for zero-one stochastic processes. The length of time the process persists in either one of its two states is assumed to be constant or distributed either exponentially, normally, uniformly or according to some combination thereof. Repeated reentries into any state are expected and are assumed to be governed by the same probability laws and parameters which characterized previous occupations.

The process has the form of a random wave for which a power density spectrum has been established. The inverse transform of the spectral density function constitutes the autocorrelation function of the process. Because of the unattainability of the inverse transform by analytical means, methods of numerical integration were employed for its determination. Autocorrelation functions were obtained for a total of 60 parameter combinations. Attempts to draw conclusions from this experimental evidence were made. Only qualitative rules could be given pertaining to the "damping" characteristics. With respect to the periods of the autocorrelation functions a quantitative rule of high likelihood was derived experimentally.

CHAPTER I

INTRODUCTION

Objective of Study

The objective of this study is to determine quantitatively the correlation between two or more occurrences of a sampled, two-valued, process activity. In the analysis of defectives, the evaluation of machine breakdowns as well as in various other industrial systems an activity can frequently be defined and subsequently sampled which has bernoullian characteristics: it occurs or it does not occur, it will occupy only one of two possible states at a time. The length of time which an activity spends in either one of the two states is a random variable whose distribution depends on the particular system under consideration. Under the assumption of multiple occurrences of the activity, the particular objective is then to determine the autocorrelation functions associated with the process giving rise to realizations.

Purpose of Study

In recent years the theory underlying the field of activity sampling has been extended considerably. In particular, attempts have been made to justify the use of systematic sampling procedures for the occurrence of non-periodic activities. These attempts were based upon a comparison of the variances of estimates obtained through random, stratified random, and systematic sampling, the objective being to demonstrate that the precision associated with systematic sampling is higher than for the other two sampling procedures. A publication by Davis[1] stated the mathematical models for the computation of the estimate variances. A decisive component in each of these models is the process serial correlation. Davis points out that the proof of superiority of systematic sampling as opposed to random sampling hinges upon the availability of the correlation function.

It is therefore the purpose of this study to supply this possible basis for a successful comparison of the known sampling procedures.

Scope of Study

This research is concerned with the determination of the autocorrelation functions associated with various distributions of the span lengths of the two states. Its results are intended to broaden the basis for future research in the field of activity sampling. It does not, however, concern itself with the problem of drawing conclusions about the superiority of either of the sampling procedures on the basis of its results.

Method of Procedure and Assumptions

The given situation is best to be described by a process whose sample space consists of two possible states. A covariance stationary zero-one stochastic process X(t) is therefore assumed, where a particular realization of the process at time t is given by x(t), with

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1. Davis, H., "A Mathematical Evaluation of a Work Sampling Technique," *Naval Research Logistics Quarterly*, pp. 111-117.

$$x(t) = 1; \text{ for those instants, } t, \text{ at which the activity occurs,} \tag{1.1}$$

$$x(t) = 0; \text{ for those instants, } t, \text{ at which the activity does not occur.}$$

The method of procedure rests upon a harmonic analysis of the process under the following assumptions:

(a) The times between value changes are independent random variables;

(b) The time it takes to change from 0 to 1 is distributed as a random variable U, and the time it takes to change from 1 to 0 is distributed as a random variable V;

(c) The distribution functions of U and V are $F_o(u)$ and $F_1(v)$, respectively.

CHAPTER II

LITERATURE SURVEY

The basis for the general treatment of stochastic or random processes and for the determination of their properties and characteristics has been established by Norbert Wiener. Between the years from 1925 to 1930, Wiener published a series of papers on Harmonic Analysis of Irregular Motion which culminated in a paper on "Generalized Harmonic Analysis."[2] Y. W. Lee[3] extended Wiener's work and gave a comprehensive account on the statistical theory underlying the field of communication. Lee applied the generalized harmonic analysis to periodic, aperiodic or transient, and random or stochastic processes.

Given a periodic function f(t) of the independent variable t, which in general represents time, and expressing f(t) as a Fourier series, the Fourier transform of f(t) is found to be

$$F(n) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f(t)\, e^{-jn\omega_1 t}\, dt, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where ω_1 (radians per second) is the fundamental angular frequency,

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2. Wiener, N., "Generalized Harmonic Analysis," *Acta Mathematica,* 1930.

3. Lee, Y. W., *Statistical Theory of Communication,* pp. 5-14.

which is related to the period T_1 (seconds) of the function by the formula $T_1 = \frac{2\pi}{\omega_1}$. This Fourier transform is a representation of the periodic time function in the frequency domain. Since F(n) is, in general, complex, it is called the complex spectrum of f(t). Since the harmonic order n assumes only discrete values, the spectrum is a line spectrum.

In the general theory of harmonic analysis, an expression of considerable importance is the correlation which, in the case of periodic functions, takes on the form

$$\frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f_1(t)\cdot f_2(t+\tau)dt = \sum_{n=-\infty}^{\infty}[\bar{F}_1(n)\ F_2(n)]e^{jn\omega_1\tau}. \tag{2.2}$$

$f_1(t)$ and $f_2(t)$ are two periodic functions having the same fundamental angular frequency ω_1 and τ is a continuous time of displacement in the range $[-\infty,\infty]$, independent of t. If $f_1(t)$ has the complex spectrum $F_1(n)$, and $f_2(t)$, $F_2(n)$ then the Fourier transform of Equation (2.2) is $\bar{F}_1(n)\cdot F_2(n)$. In fact, Lee shows that

$$\frac{1}{T_1}\int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f_1(t)\cdot f_2(t+\tau)dt \quad \text{and} \quad \bar{F}_1(n)\cdot F_2(n) \tag{2.3}$$

are Fourier transforms of each other and he calls this relation the correlation theorem for periodic functions.

The integral (2.2) involves a combination of three operations:

1. One of the periodic functions concerned, $f_2(t)$, is given a time displacement τ.

2. The displaced function is multiplied by the other periodic function of the same angular frequency.

3. The product is averaged by integration over a complete period.

These steps are repeated for every value of τ in the interval $[-\infty,\infty]$ so that a function is generated. This combination of the three operations, namely, DISPLACEMENT, MULTIPLICATION, and INTEGRATION, is termed CORRELATION.

If a function is correlated with itself, that is, if $f_1(t) = f_2(t)$, then the autocorrelation function $\rho(\tau)$ is obtained: in that case the integral (2.2) takes on the form

$$\rho(\tau) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f_1(t)\, f_1(t+\tau)dt = \sum_{n=-\infty}^{\infty} |F_1(n)|^2\, e^{jn\omega_1\tau}, \qquad (2.4)$$

by definition. Let the Fourier transform of (2.4), the *power spectrum,* be denoted by $S(n)$. Then

$$S(n) = \bar{F}_1(n)\, F_1(n) = |F_1(n)|^2. \qquad (2.5)$$

Since the spectrum of the autocorrelation function is the square of the absolute value of the complex spectrum of the given periodic function, its phase spectrum is always zero for all harmonics. In other words, all periodic functions having the same harmonic amplitudes but differing in their initial phase angles have the same autocorrelation function.

This property of the autocorrelation function holds according to Lee for transient and random functions, and particularly with respect to the harmonic analysis of random processes (ensembles of random functions[4]), where an infinite variety in waveform is an inherent attribute, this property is of utmost importance.

The relation between the autocorrelation functions of random processes and their spectra has been derived by Wiener and establishes the *Wiener Theorem for Autocorrelation.* The derivation starts with the autocorrelation theorem for periodic functions which follows from (2.4) and (2.5) to

$$\rho(\tau) = \sum_{n=-\infty}^{\infty} S(n)\, e^{jn\omega_1\tau}, \tag{2.6}$$

and inversely

$$S(n) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \rho(\tau)\, e^{-jn\omega_1\tau}\, d\tau. \tag{2.7}$$

Combining (2.6) and (2.7) one obtains the single expression

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4. Lee defines the infinite aggregate of messages or noise, or of their combination, as "Ensemble," and a specific function in the ensemble as a "Member Function."

$$\rho(\tau) = \sum_{n=-\infty}^{\infty} e^{jn\omega_1\tau} \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \rho(\eta)\, e^{-jn\omega_1\eta}\, d\eta. \tag{2.8}$$

Associated with this Fourier expression for $\rho(\tau)$ of a periodic function $f_1(t)$ is the definition of $\rho(\tau)$ which is restated (2.4):

$$\rho(\tau) = \frac{1}{T_1} \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} f_1(t)\, f_1(t+\tau)\, dt. \tag{2.9}$$

On this basis Wiener applies the following limiting process[5]: As we allow the period T_1 to approach infinity, $f_1(t)$ is made to approach the random function. Since $1/T_1 = \omega_1/2\pi$, expression (2.8) becomes

$$\rho(\tau) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{jn\omega_1\tau} \cdot \omega_1 \int_{-\frac{T_1}{2}}^{\frac{T_1}{2}} \rho(\eta)\, e^{-jn\omega_1\eta}\, d\eta. \tag{2.10}$$

As T_1 tends to infinity, the fundamental angular frequency ω_1 becomes a differential of angular frequency $d\omega$, and $n\omega_1$, which is the nth harmonic angular frequency, becomes the continuous angular frequency ω, and the summation over all harmonics becomes an integration over the entire continuous frequency range $[-\infty,\infty]$. The result of this limiting process is that $\rho(\tau)$ loses its periodic form and becomes an aperiodic function. We have thus obtained formally and heuristically the autocorrelation

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5. Lee, Y. W., *Op. Cit.*, pp. 56-58.

function of a random function in the form of the Fourier integral

$$\rho(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega\tau}\, d\omega \int_{-\infty}^{\infty} \rho(\eta)\, e^{-j\omega\eta}\, d\eta. \tag{2.11}$$

Here we assume that

$$\int_{-\infty}^{\infty} |\rho(\tau)|\, d\tau \tag{2.12}$$

is finite. Furthermore, with the substitution $T = \frac{T_1}{2}$, the limiting form of (2.9) is

$$\rho(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t)\, f_1(t+\tau)\, dt, \tag{2.13}$$

which is the autocorrelation function of the random function $f_1(t)$. If a function $S(\omega)$ is defined, such that

$$S(\omega) = \int_{-\infty}^{\infty} \rho(\tau)\, e^{-j\omega\tau}\, d\tau, \tag{2.14}$$

then the autocorrelation function (2.11) assumes the form

$$\rho(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, e^{j\omega\tau}\, d\omega. \tag{2.15}$$

By (2.14) $S(\omega)$ is defined as the Fourier transform of $\rho(\tau)$. Since $\rho(\tau)$ is an aperiodic function, $S(\omega)$ is the continuous spectrum of $\rho(\tau)$. In fact, Lee asserts[6] and proves[7] that $S(\omega)$ is the *Power Density Spectrum or Spectral Density Function* of the random function $f_1(t)$.

The autocorrelation function $\rho(\tau)$ is a real and even function and so is its spectrum. This fact permits the rewriting of Equations (2.14) and (2.15) as cosine transforms:

$$\rho(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \cos \omega\tau \, d\omega; \tag{2.16}$$

$$S(\omega) = \int_{-\infty}^{\infty} \rho(\tau) \cos \omega\tau \, d\tau. \tag{2.17}$$

The Wiener Theorem for autocorrelation states that

> The autocorrelation function of a random function and the power density spectrum of the random function are related to each other by a Fourier Cosine Transformation as given by (2.16) and (2.17).

Lee as well as Papoulis[8] state and prove the properties of both the autocorrelation function and the power spectrum. For convenience they are restated here:

a. $\rho(\tau)$ is a real and even function and so is its spectrum $S(\omega)$, that is

$$\rho(-\tau) = \rho(\tau), \quad \text{and} \quad S(-\omega) = S(\omega). \tag{2.18}$$

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6. Lee, Y. W., *Op. Cit.*, p. 58.

7. *Ibid.*, pp. 93-96.

8. Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, pp. 336-343.

b. For a single process X(t) [Lee's notation: $f_1(t)$], with $\tau = 0$, Equation (2.16) becomes

$$\rho(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)\, d\omega = E\{|X(t)|^2\} \geq 0. \qquad (2.19)$$

Thus: The total area of $S(\omega)/2\pi$ is nonnegative and equals the "Average Power" of the process X(t). In fact, Papoulis[8] proves that $S(\omega)$ is nonnegative for all ω.

c. $\rho(\tau)$ attains its maximum value for $\tau = 0$.

d. $\rho(\tau)$ is zero if τ approaches infinity, that is

$$\lim_{\tau \to \infty} \rho(\tau) = 0, \qquad (2.20)$$

if the process X(t) is truly random (i.e., contains no periodic components).

Papoulis[8] illustrates the relation between the autocorrelation function and its transform for a number of processes. Table 1 is reproduced at this place so as to visualize some of the properties of $\rho(\tau)$ and $S(\omega)$ as stated above. Few publications are known which are related specifically to that type of stochastic process with which

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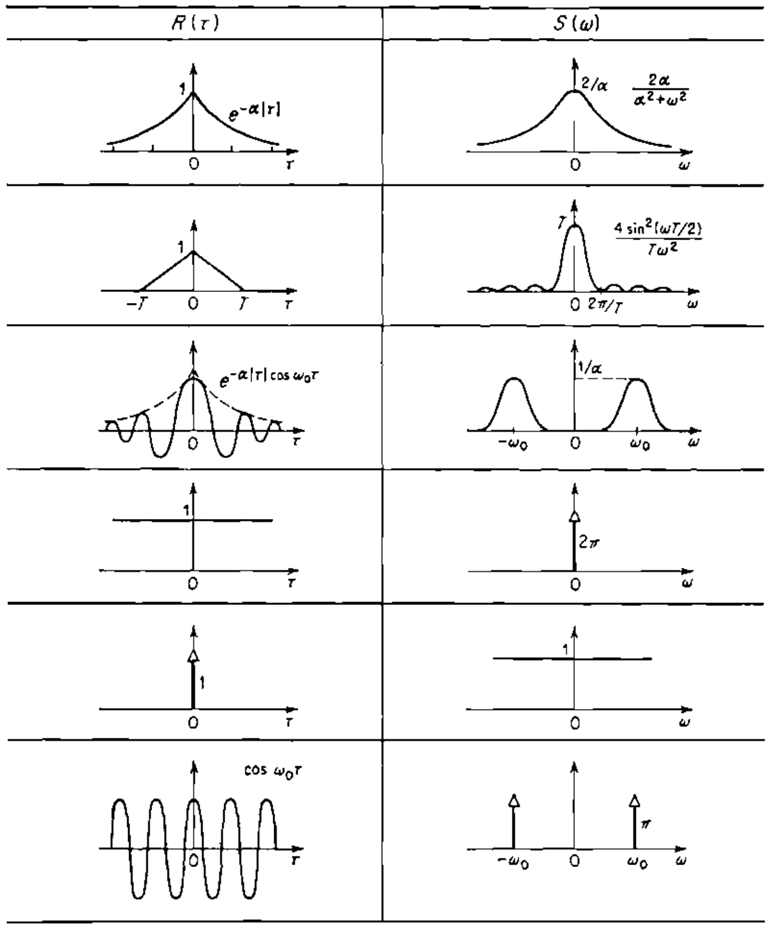
8. *Ibid.*, pp. 336-343.

Table 1. A Selection of Autocorrelations and Their Spectra[9]

Table 10-2



Comment. The power spectrum $S(\omega)$ of a process $\mathbf{x}(t)$ can be expressed directly in terms of its second-order density $f(x_1,x_2;\tau)$. To this end we introduce the Fourier transform of $f(x_1,x_2;\tau)$ with respect to τ:

$$G(x_1,x_2;\omega) \triangleq \int_{-\infty}^{\infty} f(x_1,x_2;\tau)e^{-j\omega\tau}\,d\tau$$

Since

$$R(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2f(x_1,x_2;\tau)\,dx_1\,dx_2$$

9. Papoulis, A., *Op. Cit.*, p. 340.

this research is concerned. Parzen[10] devoted a section in his book to two-valued processes of which the zero-one process is a member. A typical sample function of a zero-one process is graphed in Figure 1.

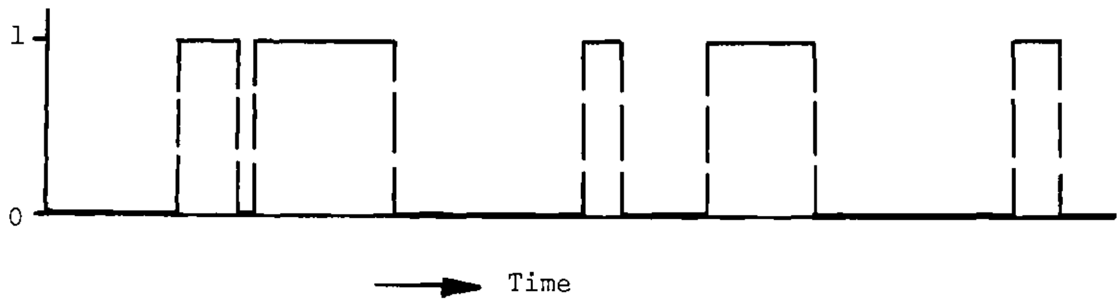


Figure 1. A Typical Sample Function of a Zero-One Process

The defined process can be represented by

$$\begin{cases} X(t), & t \geq 0 \\ 0, & \text{elsewhere,} \end{cases} \tag{2.21}$$

and it is a zero-one process whose properties can be studied if two assumptions are satisfied.

(i) The times between value changes are independent random variables;

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10. Parzen, E., "Two-Valued Processes," *Stochastic Processes*, pp. 35-40.

(ii) the time it takes to change from 0 to 1 is distributed as a random variable U, and the time it takes to change from 1 to 0 is distributed as a random variable V.

On the basis of these assumptions Parzen elaborates briefly on the following two properties:

(a) The probabilities $P\{X(t) = 0\}$ and $P\{X(t) = 1\}$ that $X(t)$ will at time t be in each of its two possible states,

(b) The probability law of the fraction of time during the interval $[0,t]$ that the process has the value 1.

Parzen uses the limit theorems of probability theory to derive his answers, assuming that it suffices to know the behavior of the process after it has been operating for a long time ($t \gg 0$). This assumption of a stationary process, in fact, can be validated since it is applicable to many real world situations. Parzen's treatment of zero-one processes appears rather brief and its value for the object of this research is therefore limited. The assumptions, however, will be maintained throughout this study and Parzen's notation is used for the sake of consistency.

In an effort to contribute to the theoretical basis of the field of activity sampling, W. W. Hines[11] made the first serious attempt to acquire a knowledge about the autocorrelation functions associated with

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11. Hines, W. W., "The Relationship Between the Properties of Certain Sample Statistics and the Structure of Activity in Systematic Activity Sampling," Ph.D. Dissertation, pp. 93-138.

zero-one processes. In this work the term span length is used to denote the time the process spends in either one of its two states. Hines considered a time period [0,T] during which the activity of interest occurs more than once and assumes no initial knowledge pertaining to a possible truncation of the distribution of span length at some value greater than zero. The autocorrelation function given has the following form:

$$\rho(\tau) = E\{X(t)\ X(t+\tau)\}, \tag{2.22}$$

$$= P\{X(t) = 1,\ X(t+\tau) = 1\},$$

$$= P\{X(t) = 1\} \cdot P\{X(t+\tau) = 1 | X(t) = 1\},$$

where

$$P\{X(t+\tau) = 1 | X(t) = 1\} = \int_0^\infty \left[\sum_{k=0}^\infty P\{X(t+\tau) = 1,\ \text{no.changes} = 2k | X(t) = 1, y\} \right] f_1(y)\ dy,$$

where the dummy variable y denotes the time that X(t) has been equal to 1 at time t, and where $f_1(\cdot)$ is the density of the span length for the process being in state 1. Evidently an analytical evaluation of expression (2.22), in general, incurs severe difficulties which are exclusively mathematical in nature, and since mathematicians consulted did not have any knowledge of the existence of appropriate literature, Hines employed Monte Carlo Simulation Techniques to obtain his results. For two different distributions of span length--gamma and truncated normal--a total of 18 autocorrelation functions were obtained for various parameter combinations. For easy reference Tables 2 and 3 provide a survey of the parameters considered.

Table 2.[12] Values of Parameters Employed in Simulation with Truncated Normal Distributions for Span Lengths of U (Subscript 0) and V (Subscript 1)

μ_1	μ_0	$\frac{\mu_1}{\mu_0}$	σ_1^2	σ_0^2	Approximate Coefficient of Variation
5.0	5.0	1.0	6.0	6.0	0.347
10.0	10.0	1.0	6.0	6.0	0.173
10.0	10.0	1.0	4.0	4.0	0.141
10.0	10.0	1.0	2.0	2.0	0.100
4.0	20.0	0.2	1.0	4.0	0.093
30.0	5.0	6.0	4.0	4.0	0.081
10.0	10.0	1.0	2.0	0.0	0.071
5.0	30.0	0.167	1.0	4.0	0.064
4.0	20.0	0.2	0.25	0.25	0.003

A brief discussion of the autocorrelation functions obtained by Hines on the basis of the results of this research is given in Appendix A. Considering the feasibility of the chosen approach, simulation is an adequate procedure from an engineering standpoint. A wide range of application, however, appears prohibitive in light of the large amount of computer time needed. The figures given were 25 to 104 minutes for

12. Hines, W. W., *op. cit.*, p. 105.

each parameter set of the gamma distribution and 109 minutes per parameter combination if the span lengths are distributed normally.

Table 3.[13] Values of Parameters Employed in Simulation with Gamma Distributions for Length of Span U (Subscript 0) and V (Subscript 1)

λ_1	r_1	Mean r_1/λ_1	λ_0	r_0	Mean r_0/λ_0	$\frac{r_1/\lambda_1}{r_0/\lambda_0}$	Coefficient of Variation
0.40	2	5	0.08	2	25	0.2	1.667
0.10	1	10	0.10	1	10	1.0	0.707
1.00	10	10	1.00	10	10	1.0	0.224
3.00	9	3	2.00	14	7	0.43	0.212
4.00	16	4	4.00	16	4	1.0	0.177
2.00	20	10	2.00	20	10	1.0	0.150
5.00	30	6	4.00	16	4	1.5	0.149
4.00	64	16	4.00	16	4	4.0	0.112
10.00	50	5	10.00	30	3	1.67	0.112
5.00	50	10	5.00	50	10	1.0	0.100

A recent publication by Kume[14] originated in pursuit of the same objectives that motivated Hines' research. It underlies this study. Starting with the assumptions given by Parzen, Kume applies Wiener's Theorem of autocorrelation to zero-one stochastic processes: If $X(t)$ is the process with mean μ and variance σ^2, a Fourier transformation yields an expression for the spectral density function of the form

13. *Ibid.*, p. 103.

14. Kume, Hitoshi, "On the Spectral Analysis of 0-1 Processes."

$$S(\omega) = \lim_{T\to\infty} E\{ \frac{1}{T} \cdot | \int_0^T X(t)\, e^{-j\omega t}\, dt|^2\}. \tag{2.23}$$

Considering now a realization of the process X(t), the sequence $\{U_1,V_1,U_2,V_2,...\}$ is a sequence of independent nonnegative random variables where the U's have a common distribution with distribution function $F_0(u)$ and the V's have a common distribution function $F_1(v)$. Kume defines (Figure 2)

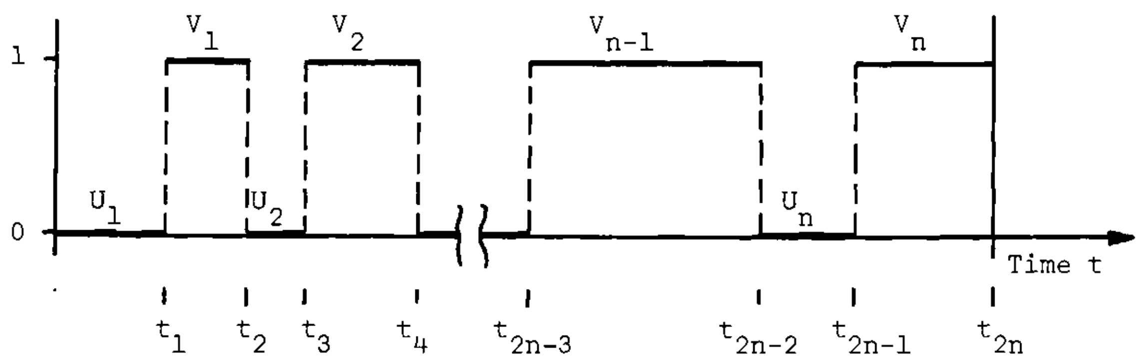

Figure 2. A Zero-One Sample Function

$$t_{2j} = \sum_{i=1}^{j} (U_i+V_i), \tag{2.24}$$

$$t_{2j+1} = \sum_{i=1}^{j} (U_i+V_i) + U_{j+1} = t_{2j} + U_{j+1},$$

and obtains a realization of X(t), x(t), given by

$$x(t) = \begin{cases} 1, & t_{2j-1} < t \le t_{2j}, \quad j \ge 1, \\ \\ 0, & t_{2j} < t \le t_{2j+1}, \quad j \ge 0. \end{cases} \tag{2.25}$$

If $X_{2n}(j\omega)$ is the Fourier transform of $x(t)$ in the interval $[0,t_{2n}]$, then

$$X_{2n}(j\omega) = \int_0^{t_{2n}} x(t)\, e^{-j\omega t}\, dt \tag{2.26}$$

becomes

$$X_{2n}(j\omega) = -\frac{1}{j\omega} \sum_{r=1}^{2n} (-1)^r\, e^{-j\omega t_r}. \tag{2.27}$$

Defining the spectrum of $X_{2n}(j\omega)$ as

$$S_{2n}(\omega) = \frac{|X_{2n}(j\omega)|^2}{t_{2n}} = \frac{1}{t_{2n}} X_{2n}(j\omega) \cdot X_{2n}(-j\omega), \tag{2.28}$$

substitution of (2.27) into (2.28) yields

$$S_{2n}(\omega) = \frac{1 + \frac{1}{2n} \sum_{k=1}^{2n-1} \sum_{r=1}^{2n-k} (-1)^k \{e^{j\omega y_{r,k}} + e^{-j\omega y_{r,k}}\}}{\omega^2 \cdot \frac{t_{2n}}{2n}} \tag{2.29}$$

with

$$y_{r,k} = t_{r+k} - t_r.$$

At this stage Kume applies a limiting process and obtains:

$$\lim_{n\to\infty} \frac{t_{2n}}{2n} = \frac{\mu_0 + \mu_1}{2}, \qquad (2.30)$$

The spectral density function

$$S(\omega) = \lim_{n\to\infty} E[S_{2n}(\omega)] \qquad (2.31)$$

$$= \frac{2}{\omega^2(\mu_0+\mu_1)} \lim_{n\to\infty} \left[1 + \frac{1}{2n} \sum_{k=1}^{2n-1} (-1)^k \sum_{r=1}^{2n-k} \{E(e^{j\omega y_{r,k}}) + E(e^{-j\omega y_{r,k}})\}\right]$$

where (2.30) follows from the strong law of large numbers, and μ_0 and μ_1 are the expected values of U and V, respectively. Noting that

$$\phi_{r,k}(\omega) = E\{e^{j\omega y_{r,k}}\}$$

is the characteristic function of the distribution of $y_{r,k}$, and defining the characteristic functions of the distributions of U and V as $\phi_0(\omega)$ and $\phi_1(\omega)$, respectively, Kume shows expression (2.31) to be equivalent to

$$S(\omega) = \frac{2}{\omega^2 \cdot (\mu_0+\mu_1)} \left[1 + \mathrm{Re}\{ \frac{2\phi_0\phi_1 - (\phi_0+\phi_1)}{1 - \phi_0\phi_1} \}\right], \qquad (2.32)$$

if

$$|\phi_0\ \phi_1| < 1. \tag{2.33}$$

The "Re" denotes the real part of the expression in parentheses. When the distributions of the U's and V's are the same

$$S(\omega) = \frac{1}{\mu\omega^2} \{ \frac{1 - \phi\ \bar{\phi}}{(1+\phi)(1+\bar{\phi})} \}, \tag{2.34}$$

where $\phi = \phi_0 = \phi_1$ and $\mu = \mu_0 = \mu_1$.

As an example Kume determines the autocorrelation function of the process if the U's and V's are distributed exponentially. Since this case will not be treated in the text the example is stated here.

Let

$$f_0(u) = \lambda_0\ e^{-\lambda_0 u}, \quad f_1(v) = \lambda_1\ e^{-\lambda_1 v}, \tag{2.35}$$

Then the means and characteristic functions are given by

$$\mu_0 = \frac{1}{\lambda_0}, \quad \mu_1 = \frac{1}{\lambda_1}, \tag{2.36}$$

$$\phi_0(\omega) = (1-j\ \frac{\omega}{\lambda_0})^{-1},\ \phi_1(\omega) = (1-j\ \frac{\omega}{\lambda_1})^{-1}. \tag{2.37}$$

Applying (2.35) and (2.36) to (2.32), the spectral density function becomes

$$S(\omega) = \frac{2\lambda_0 \cdot \lambda_1}{(\lambda_0+\lambda_1)} \cdot \frac{1}{\omega^2 + (\lambda_0+\lambda_1)^2} \tag{2.38}$$

and the inverse transform, the covariance function, is given by

$$R(\tau) = \frac{2\lambda_0 \cdot \lambda_1}{(\lambda_0+\lambda_1)\pi} \int_0^\infty \frac{\cos \omega\tau}{\omega^2 + (\lambda_0+\lambda_1)^2} d\omega \tag{2.39}$$

$$= \frac{\lambda_0 \lambda_1}{(\lambda_0+\lambda_1)^2} \cdot e^{-(\lambda_0+\lambda_1) \tau} . \tag{2.40}$$

As the process variance is $\lambda_0\lambda_1/(\lambda_0+\lambda_1)^2$, the autocorrelation function is expressed by

$$\rho(\tau) = e^{-(\lambda_0+\lambda_1)\tau} . \tag{2.41}$$

For the parameter combination $\lambda_0 = \lambda_1 = 1$, Equation (2.37) has been graphed in Figure 3 and Equation (2.40) in Figure 4.

Kume states yet another example, namely, the case where the U's and V's are distributed normally.[15] Here, after some computational work, the spectral density function, $S(\omega)$, too, is relatively easily to be obtained, but Kume fails to elaborate on the method of finding the inverse transform which involves an integration over an infinite interval

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15. Kume does not state the limitations associated with this case. It is, however, necessary to consider the remarks on p. 33 to be able to validate his findings.

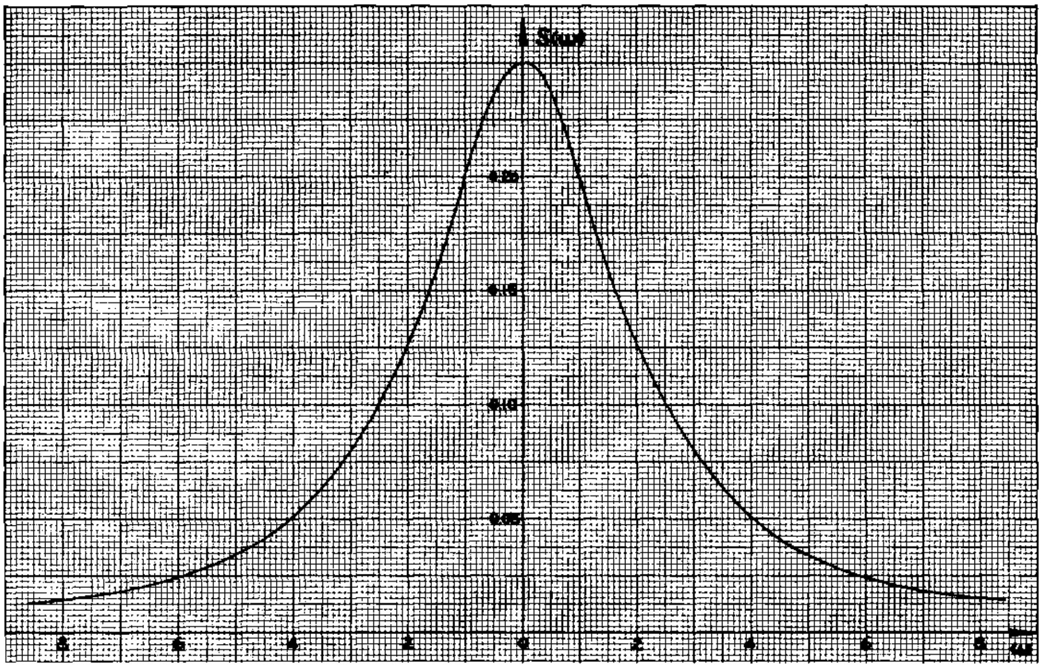

Figure 3. Spectral Density Function $S(\omega) = \dfrac{1}{\omega^2 + 2^2}$.

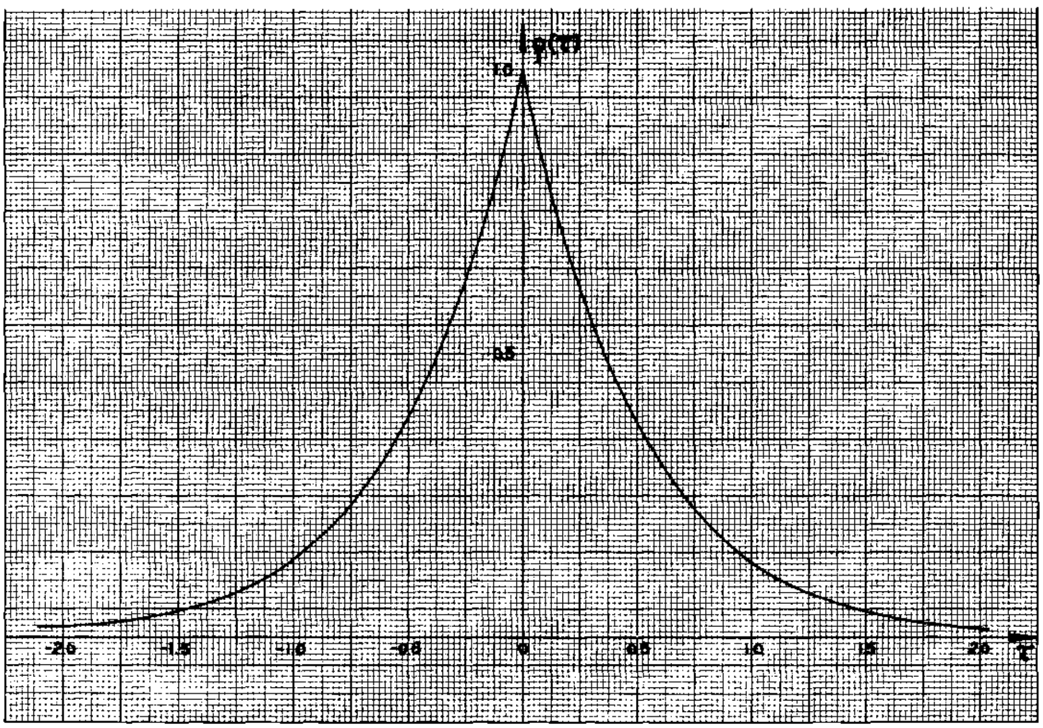

Figure 4. Autocorrelation Function $\rho(\tau) = e^{-2\varepsilon}$.

of a rather complicated integrand. This difficulty seems to be intrinsically associated with the procedure and it is therefore necessary to develop means for the determination of the inverse transform for a large variety of spectral density functions.

CHAPTER III

MODEL DEVELOPMENT

It has been stated before that the random variables U and V are assumed to denote the time which the process X(t) spends in states 0 and 1, respectively. U and V can be distributed according to a variety of probability laws for the process to be an appropriate representation of certain real world systems. In other words, there exists in almost all cases an industrial situation to which the process with particular distributions of the random variables is applicable. Considering the fact that the case of both U and V being distributed exponentially is treated in the existing literature already, this study depicts the following five combinations for further treatment:

Case I: $U = \text{Constant}; \; V \approx EXP(\lambda)$;

Case II: $U = \text{Constant}; \; V \approx U(t)$;

Case III: $U = \text{Constant}; \; V \approx N(\mu,\sigma^2)$;

Case IV: $U \approx EXP(\lambda)$; $V \approx N(\mu,\sigma^2)$;

Case V: $U \approx N(\mu_0,\sigma_0^2)$; $V \approx N(\mu_1,\sigma_1^2)$.

Model building, as understood here, then consists of determining the spectral density function $S(\omega)$ in each of the above five cases.

The expression for $S(\omega)$ was given in (2.32) as:

$$S(\omega) = \frac{2}{\omega^2(\mu_0+\mu_1)} \left[1 + \mathrm{Re} \left\{ \frac{2\phi_0\phi_1 - (\phi_0+\phi_1)}{1 - \phi_0 \phi_1} \right\} \right] \tag{3.1}$$

with

$$|\phi_0\phi_1| < 1. \tag{3.2}$$

The quantities μ_0 and μ_1 are the expected values of the random variables U and V; that is

$$\mu_0 = E(U), \qquad \mu_1 = E(V). \tag{3.3}$$

If the density functions of U and V are given by $f_0(u)$ and $f_1(v)$ and the distribution functions by $F_0(u)$ and $F_1(v)$, respectively, then the expected values are equally well to be expressed by

$$E(U) = \int_0^\infty u \cdot f_0(u)\, du = \int_0^\infty u \, dF_0(u), \tag{3.4}$$

$$E(V) = \int_0^\infty v \cdot f_1(v)\, dv = \int_0^\infty v \, dF_1(v).$$

The most significant entries in (3.1) are the characteristic functions associated with both random variables, namely, $\phi_0(\omega)$ and $\phi_1(\omega)$ for U and V, respectively. In general, the characteristic function $\phi_Y(\omega)$ of a random variable Y, with density function f(y) and distribution function F(y), is given by

$$\phi_Y(\omega) = E(e^{j\omega Y}) \tag{3.5}$$

$$= \int_{-\infty}^{\infty} e^{j\omega y} \, dF(y)$$

$$= \int_{-\infty}^{\infty} e^{j\omega y} \, f(y) \, dy,$$

and it is a complex quantity of a real variable ω. The development of the model requires the determination of $\phi(\omega)$ for both U and V. Furthermore, since the validity of expression (3.1) depends on (3.2), the model building process must include a check on the condition

$$|\phi_0(\omega) \cdot \phi_1(\omega)| < 1. \tag{3.2}$$

The absolute value of a product, however, equals the product of the absolute values, so that

$$|\phi_0(\omega) \cdot \phi_1(\omega)| = |\phi_0(\omega)| \cdot |\phi_1(\omega)| < 1.$$

Since both $\phi_0(\omega)$ and $\phi_1(\omega)$ are complex quantities, their absolute values are given by their respective magnitudes.

The real part of the complex expression

$$\text{Re} \left\{ \frac{2\phi_0(\omega)\ \phi_1(\omega) - [\phi_0(\omega) + \phi_1(\omega)]}{1 - [\phi_0(\omega) \cdot \phi_1(\omega)]} \right\}, \tag{3.6}$$

added to one, constitutes a multiplicand of the product which equals $S(\omega)$. Because of the very nature of expression (3.6) and of the whole product (3.1) the distributions of U and V can be interchanged. In Case IV, for instance, the same spectral density function will be obtained for $U \approx EXP(\lambda)$, $V \approx N(\mu,\sigma^2)$, as indicated, as will for $U \approx N(\mu,\sigma^2)$, $V \approx EXP(\lambda)$. This property extends the range of possible applications of the results.

The real purpose of the development of the model $S(\omega)$ is to transform the process from the time domain into the frequency domain, bearing in mind that those transformations are an integral part of general system's theory and frequently yield easier obtainable answers. The final answer, i.e. the resulting autocorrelation function, however, is expected to be represented in the time domain again so that an inverse transformation has to be performed. The practical aspects of this inverse transformation will be discussed in Chapter V.

<u>Case I: U = Constant, V $\approx$ EXP(λ)</u>

Let u = constant and

$$f_1(v) = \lambda e^{-\lambda v}, \quad v \geq 0, \tag{3.7}$$

$$= 0, \quad \text{elsewhere,}$$

then the means and variances are given by

$$E(U) = \mu_0 = u, \quad E(V) = \mu_1 = \frac{1}{\lambda}, \tag{3.8}$$

$$\mathrm{Var}(U) = \sigma_0^2 = 0; \quad \mathrm{Var}(V) = \sigma_1^2 = \frac{1}{\lambda^2} \tag{3.9}$$

The characteristic functions are

$$\phi_0(\omega) = E(e^{j\omega U}) = e^{j\omega u} \tag{3.10}$$

for constant U, and

$$\begin{aligned} \phi_1(\omega) = E(e^{j\omega V}) &= \int_{-\infty}^{\infty} e^{j\omega v} f_1(v)\, dv \\ &= \lambda \int_0^{\infty} e^{-(\lambda - j\omega)v}\, dv \\ &= \frac{\lambda}{\lambda - j\omega} = (1 - j\,\frac{\omega}{\lambda})^{-1} \end{aligned} \tag{3.11}$$

for V exponentially distributed. Applying (3.2), (3.4) and (3.5) to Equation (3.1), the spectral density function is found to

$$S(\omega) = \frac{2\lambda}{1 + \lambda u} \cdot \frac{1 - \cos \omega u}{\lambda^2(1 - \cos \omega u)^2 + (\omega + \lambda \sin \omega u)^2} . \tag{3.12}$$

For this expression to be valid the condition (3.2) has to be satisfied. It is

$$|\phi_0(\omega)| = |e^{j\omega u}| = 1 ,$$

and

$$|\phi_1(\omega)| = \left|\frac{1}{1 - j\,\frac{\omega}{\lambda}}\right| = \frac{1}{\left|1 - j\,\frac{\omega}{\lambda}\right|} ,$$

with

$$\left|1 - j\,\frac{\omega}{\lambda}\right| = 1 + \frac{\omega^2}{\lambda^2} > 1.$$

Hence $|\phi_1(\omega)| < 1$, and the product

$$|\phi_0(\omega)|\ |\phi_1(\omega)| < 1, \text{ Q.E.D.}$$

Equation (3.12) attains an indeterminate form for $\omega = 0$. Repeated application of L'Hospital's Rule yields

$$\lim_{\omega \to 0} S(\omega) = \frac{\lambda u^2}{(1+\lambda u)^3} . \qquad (3.13)$$

<u>Case II: U = Constant, V $\approx$ U(t)</u>

Let u = constant and

$$f_1(v) = \frac{1}{t}, \quad 0 \le v \le t, \qquad (3.14)$$

$$= 0, \quad \text{elsewhere},$$

then the means and variances are given by

$$E(U) = \mu_0 = u, \qquad E(V) = \mu_1 = \frac{t}{2}, \tag{3.15}$$

$$Var(U) = \sigma_0^2 = 0, \quad Var(V) = \sigma_1^2 = \frac{t^2}{12}. \tag{3.16}$$

The characteristic functions are

$$\phi_0(\omega) = E(e^{j\omega U}) = e^{j\omega u} \tag{3.17}$$

for constant U, and

$$\begin{aligned} \phi_1(\omega) = E(e^{j\omega V}) &= \int_0^\infty e^{j\omega v} f_1(v)\, dv \\ &= \frac{1}{t} \int_0^t e^{j\omega v}\, dv \\ &= \frac{1}{j\omega t} (e^{j\omega t} - 1) \end{aligned} \tag{3.18}$$

for V uniformly distributed.

Applying (3.15), (3.17), and (3.18) to Equation (3.1), the spectral density function is obtained to

$$\begin{aligned} S(\omega) = &\frac{4}{(2u+t)\,\omega^2} \\ &\cdot \frac{(1 - \cos \omega u)[\omega^2 t^2 + 2 \cos \omega t - 2]}{\{2(1 - \cos \omega t) + 2\omega t[\sin \omega u - \sin \omega(u+t)] + \omega^2 t^2\}} \end{aligned} \tag{3.19}$$

It is

$$|\phi_0(\omega)| = |e^{j\omega u}| = 1,$$

and

$$|\phi_1(\omega)| = \left|\frac{e^{j\omega t} - 1}{j\omega t}\right| < 1, \tag{3.20}$$

since

$$\frac{e^{j\omega t} - 1}{j\omega t} = \frac{1 + j\omega t + \frac{(j\omega t)^2}{2!} + \frac{(j\omega t)^3}{3!} + \cdots - 1}{j\omega t}$$

$$= 1 + \frac{j\omega t}{2!} + \frac{(j\omega t)^2}{3!} + \frac{(j\omega t)^3}{4!} + \cdots$$

$$= (1 - \frac{\omega^2 t^2}{3!} + \frac{\omega^4 t^4}{5!} - + \cdots) + j(\frac{\omega t}{2!} + \frac{\omega^3 t^3}{4!} + \frac{\omega^5 t^5}{y!} - + \cdots),$$

$$\left|\frac{e^{j\omega t} - 1}{j\omega t}\right| = \sqrt{1 - \frac{\omega^2 t^2}{12} + \frac{\omega^4 t^4}{360} - \frac{\omega^6 t^6}{20160} + - \cdots} \tag{3.21}$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{(\omega t)^{2n}}{(n+1)(2n+1)!}\right]^{1/2}; \tag{3.21}$$

application of the ratio test to the infinite series in (3.21) proves absolute convergence for all values of ωt. Furthermore, the sum of the series is less than one which proves relation (3.20). It is thus established that condition (3.2) is satisfied and that (3.19) is the spectrum of the process.

Through repeated application of L'Hospital's Rule (six times) the limiting value of $S(\omega)$ (3.19) for $\omega=0$ is determined to

$$\lim_{\omega \to 0} S(\omega) = \frac{2u^2t^2}{3(2u+t)^3} . \tag{3.22}$$

Case III: U = Constant, $V \approx N(\mu,\sigma^2)$[16]

Let u = constant and

$$f_1(v) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(v-\mu)^2}{2\,\sigma^2}} \tag{3.23}$$

then the means and variances are given by

$$E(U) = \mu_0 = u, \quad E(V) = \mu_1 = \mu, \tag{3.24}$$

$$\mathrm{Var}(U) = \sigma_0^2 = 0, \quad \mathrm{Var}(V) = \sigma_1^2 = \sigma^2. \tag{3.25}$$

The characteristic function for constant U is given by

$$\phi_0(\omega) = E(e^{j\omega U}) = e^{j\omega u}. \tag{3.26}$$

The determination of the characteristic function of $V \approx N(\mu,\sigma^2)$ requires

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16. It is, of course, understood that the nonnegativity assumption for the random variables U and V generally does not permit the assignment of a normal distribution. For all practical purposes, however, this and all other combinations of distributions involving normal densities can be considered feasible for proper choices of the parameters.

a derivation procedure which, as a sketch, was given by Parzen[17] and which, for completeness, will be outlined here:

Let

$$y = \frac{v - \mu}{\sigma} . \tag{3.27}$$

Then Y is distributed normally with mean 0 and variance 1, that is $Y \approx N(0,1)$. The density function of Y, f(y), is

$$f(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}, \tag{3.28}$$

and the characteristic function follows as

$$\phi_Y(\omega) = \int_{-\infty}^{\infty} e^{j\omega y} f(y)\, dy \tag{3.29}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{j\omega y} e^{-\frac{1}{2}y^2} dy.$$

Expansion of $e^{j\omega y}$ into a Taylor series yields

$$\phi_Y(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{(j\omega y)^n}{n!} e^{-\frac{1}{2}y^2} dy. \tag{3.30}$$

The following expression is obtained upon interchange of the order of integration and summation

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17. Parzen, E., *Modern Probability Theory and Its Application,* p. 398.

$$\phi_Y(\omega) = \sum_{n=0}^{\infty} \frac{(j\omega)^n}{n!} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^n e^{-\frac{1}{2}y^2} dy. \tag{3.31}$$

The integrand in expression (3.31) is an odd function for odd values of n, so that only the even values of n are retained and

$$\int_{-\infty}^{\infty} y^n e^{-\frac{1}{2}y^2} dy = 2 \int_0^{\infty} y^n e^{-\frac{1}{2}y^2} dy, \quad n \text{ even}. \tag{3.32}$$

Thus, a substitution $n = 2m$ is valid, and

$$\phi_Y(\omega) = \sum_{m=0}^{\infty} \frac{(j\omega)^{2m}}{(2m)!} \frac{2}{\sqrt{2\pi}} \int_0^{\infty} y^{2m} e^{-\frac{1}{2}y^2} dy. \tag{3.33}$$

An integral $\int_0^{\infty} x^r e^{-ax^2} dx$ has the general solution

$$\int_0^{\infty} x^r e^{-ax^2} dx = \frac{\Gamma\left(\frac{r+1}{2}\right)}{2a^{\left(\frac{r+1}{2}\right)}}, \quad \text{for } a > 0,\ r > -1. \tag{3.34}$$

Applied to the corresponding expression in Equation (3.33) one obtains

$$\int_0^{\infty} y^{2m} e^{-\frac{1}{2}y^2} dy = \frac{\Gamma(m + 1/2)}{2(1/2)^{(m + 1/2)}} = \frac{2^m}{\sqrt{2}} \Gamma(m + 1/2). \tag{3.35}$$

Noting that

$$\Gamma(x) \cdot \Gamma(x + 1/2) = \frac{\sqrt{\pi}}{2^{(2x-1)}} \cdot \Gamma(2x), \tag{3.36}$$

it is

$$\Gamma(m + 1/2) = \frac{\sqrt{\pi}}{2^{(2m-1)}} \cdot \frac{\Gamma(2m)}{\Gamma(m)} = \frac{\sqrt{\pi}\ (2m-1)!}{2^{(2m-1)}\ (m-1)!} . \tag{3.37}$$

Applying Equation (3.35) and (3.37) to (3.33), $\phi_Y(\omega)$ results in

$$\phi_Y(\omega) = \sum_{m=0}^{\infty} \frac{[(j\omega)^2]^m}{(2m)!} \cdot \frac{2}{\sqrt{2\pi}} \cdot \frac{2^m}{\sqrt{2}} \cdot \frac{\sqrt{\pi}\ (2m)!}{2^{2m-1}\ 2m(m-1)!} \tag{3.38}$$

$$= \sum_{m=0}^{\infty} \frac{(-\omega^2)^m}{2^m \cdot m!} = \sum_{m=0}^{\infty} \frac{\left(-\frac{1}{2}\omega^2\right)^m}{m!} = e^{-\frac{1}{2}\omega^2} .$$

In order to find the characteristic function of $f_1(v)$, $\phi_1(\omega)$, one observes that from relation (3.21)

$$v = \mu + \sigma y \tag{3.39}$$

and

$$\phi_1(\omega) = \phi_v(\omega) = \phi_\mu(\omega) \cdot \phi_{\sigma Y}(\omega) \tag{3.40}$$

$$= e^{j\omega\mu} \cdot \phi_Y(\sigma\omega).$$

Combining (3.38) and (3.40), it is

$$\phi_1(\omega) = e^{(j\omega\mu - \frac{1}{2}\sigma^2\omega^2)} . \tag{3.41}$$

Applying (3.24), (3.26) and (3.41) to Equation (3.1), the spectral density function is obtained.

$$S(\omega) = \frac{2}{\omega^2(\mu+u)} \cdot \frac{(1 - e^{-\omega^2\sigma^2})(1 - \cos \omega u)}{1 - 2e^{-\frac{1}{2}\omega^2\sigma^2} \cos \omega(\mu+u) + e^{-\omega^2\sigma^2}} . \qquad (3.42)$$

To validate $S(\omega)$, one observes that

$$|\phi_0(\omega)| = |e^{j\omega\mu}| = 1$$

and

$$|\phi_1(\omega)| = |e^{j\omega\mu - \frac{1}{2}\omega^2\sigma^2}| = e^{-\frac{1}{2}\omega^2\sigma^2} < 1, \quad \text{for } |\omega| > 0. \qquad (3.43)$$

Hence condition (3.2) is satisfied. The limiting value of $S(\omega)$ (3.42) for $\omega = 0$ is determined through repeated application of L'Hospital's rule (4 times). It is

$$\lim_{\omega \to 0} S(\omega) = \frac{\sigma^2 u^2}{(\mu+u)^3} .$$

<u>Case IV: $U \approx EXP(\lambda)$, $V \approx N(\mu,\sigma^2)$</u>

Let

$$f_0(u) = \lambda e^{-\lambda u}, \quad u \geq 0,$$

$$= 0, \quad \text{elsewhere},$$

$$f_1(v) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(v-\mu)^2}{2\sigma^2}}, \tag{3.45}$$

then the means and variances are given by

$$E(U) = \mu_0 = \frac{1}{\lambda},\ E(V) = \mu_1 = \mu, \tag{3.46}$$

$$\mathrm{Var}(U) = \sigma_0^2 = \frac{1}{\lambda^2},\ \mathrm{Var}(V) = \sigma_1^2 = \sigma^2. \tag{3.47}$$

The characteristic functions for both distributions were derived in Sections 3.1 and 3.3, respectively. On the basis of Equations (3.11) and (3.41) one obtains

$$\phi_0(\omega) = (1 - j\,\frac{\omega}{\lambda})^{-1} \tag{3.48}$$

$$\phi_1(\omega) = e^{(j\omega\mu - \frac{1}{2}\sigma^2\omega^2)}. \tag{3.49}$$

Since the absolute values of the characteristic functions have already been determined in the respective sections it can be shown that condition (3.2) is satisfied.

The spectral density function is therefore valid and under application of expressions (3.46), (3.48) and (3.49) to Equation (3.1) $S(\omega)$ is obtained to

$$S(\omega) = \frac{2\lambda}{(1+\mu\lambda)} \tag{3.50}$$

$$\cdot \frac{1 - e^{-\frac{1}{2}\omega^2\sigma^2} \cos \mu\omega}{\lambda^2(1 - e^{-\frac{1}{2}\omega^2\sigma^2} \cdot \cos \mu\omega)^2 + (\omega + \lambda e^{-\frac{1}{2}\omega^2\sigma^2} \cdot \sin \mu\omega)^2}$$

Applying L'Hospital's rule twice one obtains

$$\lim_{\omega \to 0} S(\omega) = \frac{\lambda(\sigma^2+\mu^2)}{(1+\mu\lambda)^3} . \tag{3.51}$$

Case V: $U \simeq N(\mu_0,\sigma_0^2)$, $V \simeq N(\mu_1,\sigma_1^2)$

Let

$$f_0(u) = \frac{1}{\sqrt{2\pi}\,\sigma_0} e^{-(u-\mu_0)^2/2\sigma_0^2} , \tag{3.52}$$

$$f_1(v) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-(v-\mu_1)^2/2\sigma_1^2} ,$$

then the means and variances are given by

$$E(U) = \mu_0, \quad E(V) = \mu_1, \tag{3.53}$$

$$\mathrm{Var}(U) = \sigma_0^2, \quad \mathrm{Var}(V) = \sigma_1^2 \tag{3.54}$$

Following the derivation in Case III the characteristic functions are given by

$$\phi_0(\omega) = e^{\left(j\omega\mu_0 - \frac{1}{2}\omega^2\sigma_0^2\right)}, \tag{3.55}$$

$$\phi_1(\omega) = e^{\left(j\omega\mu_1 - \frac{1}{2}\omega^2\sigma_1^2\right)}. \tag{3.56}$$

Following the evaluation in the same section, condition (3.2) is found to be satisfied. Applying expressions (3.53), (3.55) and (3.56) to Equation (3.1), the spectral density function is obtained to

$$S(\omega) = \frac{2}{(\mu_0+\mu_1)\omega^2} \cdot \tag{3.57}$$

$$\cdot \frac{1-e^{-\omega^2(\sigma_0^2+\sigma_1^2)} - e^{-\frac{1}{2}\omega^2\sigma_0^2}(1-e^{-\omega^2\sigma_1^2})\cos\mu_0\omega - e^{-\frac{1}{2}\omega^2\sigma_1^2}(1-e^{-\omega^2\sigma_0^2})\cos\mu_1\omega}{1 - 2e^{-\frac{1}{2}\omega^2(\sigma_0^2+\sigma_1^2)}\cos(\mu_0+\mu_1)\omega + e^{-\omega^2(\sigma_0^2+\sigma_1^2)}},$$

or

$$S(\omega) = \frac{2}{(\mu_0+\mu_1)\omega^2} \cdot \tag{3.58}$$

$$\cdot \frac{\sinh\frac{\omega^2}{2}(\sigma_0^2+\sigma_1^2) - \sinh\left(\frac{\omega^2}{2}\sigma_1^2\right)\cdot\cos\mu_0\omega - \sinh\left(\frac{\omega^2}{2}\sigma_0^2\right)\cos\mu_1\omega}{\cosh\frac{\omega^2}{2}(\sigma_0^2+\sigma_1^2) - \cos(\mu_0+\mu_1)\omega}.$$

Applying L'Hospital's rule four times, the indeterminate form of $S(\omega)$

for $\omega = 0$ is resolved to

$$\lim_{\omega \to 0} S(\omega) = \frac{\sigma_0^2 \mu_1^2 + \sigma_1^2 \mu_0^2}{(\mu_0 + \mu_1)^3} \,. \tag{3.59}$$

CHAPTER IV

THE EVALUATION OF THE SPECTRAL DENSITY FUNCTION

The spectral density function for each of the five cases considered has been determined in Chapter III. It is, as mentioned before, the representation of the given real world system--the zero-one process--in the frequency domain. The objective is to utilize this model representation for the solution of the given problem in the time domain. It will be accomplished by means of an inverse transformation. For the latter to be performable, however, a detailed knowledge of the characteristics of the model--the spectral density function $S(\omega)$--has to be obtained, and it is to this end, that this chapter has been included. The evaluation of $S(\omega)$ progresses in three steps:

1. For each of the five cases a table is provided indicating the parameter combinations considered. These tables contain in addition the ratio of the means of the two contributing distributions, namely,

$$\text{mean ratio} = \frac{E(V)}{E(U)} = \frac{\mu_1}{\mu_0} , \tag{4.1}$$

and the coefficient of variation, defined by

$$CV = \frac{\sqrt{Var(U+V)}}{E(U+V)} = \frac{\sqrt{Var(U+V)}}{E(U) + E(V)} \quad . \tag{4.2}$$

Since the random variables U and V are assumed to be independent, the coefficient of variation becomes

$$CV = \frac{\sqrt{Var(U) + Var(V)}}{E(U) + E(V)} = \frac{\sqrt{\sigma_0^2 + \sigma_1^2}}{\mu_0 + \mu_1} . \qquad (4.3)$$

2. A tabulation of the spectral density function $S(\omega)$ is obtained numerically and is evaluated with respect to the maxima and minima of $S(\omega)$. These data enter Tables 5, 7, 9, 11, and 13. The tables provide a possibility to quantitatively check on the periods and amplitudes of the spectrum. Since $S(\omega)$ is an even function its numerical values are obtained for various positive values of ω only, starting with $\omega = 0$ and proceeding in increments of 0.01 up to a certain ω, say ω_1, and in increments of 0.1 thereafter. The choice of two different increments is due to the fact that certain spectra show very high amplitudes with periods of 0.05 or less for smaller values of ω. The chosen value of ω_1 is not unique for all cases. Since the tables do not indicate ω_1, information about its value can be derived by considering that entering ω--values are given with one or two decimals for ω above or below ω_1, respectively.

In the tables the absolute maxima and minima are specifically marked for easy inspection.

For a more qualitatively oriented evaluation of $S(\omega)$ Appendix B gives plots of the spectra obtained as output of an Algol program.

3. In the final step of the evaluation procedure a summary of pertinent properties of the spectra is accumulated on the basis of both tabulations and plots.

Case I: U = Constant, $V \approx EXP(\lambda)$

Table 4 indicates the parameter combinations considered. The general expressions for the means and variances are given in (3.8) and (3.9), for the mean ratios and the coefficients of variation in (4.1) and (4.3).

Table 4. Values of Parameters Employed in Evaluation of Spectrum for Case I--Distributions of Span Lengths

$\mu_0 = u$	λ	$\mu_1 = \frac{1}{\lambda}$	$\sigma_0^2 = 0$	$\sigma_1^2 = \frac{1}{\lambda^2}$	Mean Ratio $\frac{\mu_1}{\mu_0} = \frac{1}{\lambda u}$	Coefficient of Variation $\frac{1}{(\lambda u+1)}$
1	1	1	0	1	1	1/2
2	1/2	2		4		
3	1/3	3		9		
4	1/4	4		16		
1	1/2	2	0	4	2	2/3
2	1/4	4		16		
3	1/6	6		36		
4	1/8	8		64		
1	1/3	3	0	9	3	3/4
2	1/6	6		36		
3	1/9	9		81		
4	1/12	12		144		
1	1/4	4	0	16	4	4/5
2	1/8	8		64		
3	1/12	12		144		
4	1/16	16		256		

Table 5 gives a listing of consecutive extrema of the spectral density function as extracted from the numerical tabulation. The

Table 5. Tabulated Maxima and Minima of the Spectral Density Function S(ω) for U = Constant and V ≈ EXP(λ)

μ	1				2				3				4			
λ	1	1/2	1/3	1/4	1/2	1/4	1/6	1/8	1/3	1/6	1/9	1/12	1/4	1/8	1/12	1/16
S(ω=0)	0.1280	0.1482	0.1406	0.1280	0.2500	0.2963	0.2814	0.2560	0.3752	0.4442	0.4218	0.3834	0.5000	0.5926	0.5619	0.5136
S(ω) MAX ω		0.148 / 0	0.141 / 0	0.128 / 0		0.296 / 0	0.281 / 0	0.256 / 0		0.444 / 0	0.422 / 0	0.383 / 0		0.592 / 0	0.562 / 0	0.514 / 0
S(ω) MIN ω	0.125 / 0				0.250 / 0				0.375 / 0				0.500 / 0			
S(ω) MAX ω									0.433 / 1.04				0.577 / 0.78			
FIRST ABSOLUTE MIN MINIMUM													≈0 / 1.58	≈0 / 1.58	≈0 / 1.58	≈0 / 1.58
S(ω) MAX ω					0.288 / 1.57											
S(ω) MIN ω									≈0 / 2.1	≈0 / 2.1	≈0 / 2.1	≈0 / 2.1				
S(ω) MAX ω	0.144 / 3.14								0.064 / 3.1	0.045 / 3.1	0.034 / 3.0	0.028 / 3.0	0.086 / 2.4	0.060 / 2.3	0.046 / 2.3	0.037 / 2.3
SECOND ABSOLUTE MIN MINIMUM					≈0 / 3.14	≈0 / 3.14	≈0 / 3.14	≈0 / 3.14					≈0 / 3.1	≈0 / 3.1	≈0 / 3.1	≈0 / 3.1
S(ω) MAX ω													0.032 / 3.9	0.022 / 3.9	0.016 / 3.9	0.013 / 3.9
S(ω) MIN ω									≈0 / 4.2	≈0 / 4.2	≈0 / 4.2	≈0 / 4.2				
S(ω) MAX ω					0.043 / 4.7	0.030 / 4.6	0.025 / 4.6	0.018 / 4.5								
THIRD ABSOLUTE MIN MINIMUM													≈0 / 4.7	≈0 / 4.7	≈0 / 4.7	≈0 / 4.7
S(ω) MAX ω									0.024 / 5.2	0.016 / 5.2	0.012 / 5.2	0.010 / 5.2				
S(ω) MIN ω	≈0 / 6.28	≈0 / 6.28	≈0 / 6.28	≈0 / 6.28	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3	≈0 / 6.3				
S(ω) MAX ω					0.016 / 7.9	0.011 / 7.8	0.008 / 7.8	0.007 / 7.8								
S(ω) MIN ω					≈0 / 9.4	≈0 / 9.4	≈0 / 9.4	≈0 / 9.4								
S(ω) MAX ω	0.022 / 9.4	0.015 / 9.2	0.011 / 9.2	0.009 / 9.1												
S(ω) MIN ω	≈0 / 12.6	≈0 / 12.6	≈0 / 12.6	≈0 / 12.6												
S(ω) MAX ω	0.008 / 15.7	0.005 / 15.6	0.004 / 15.5	0.003 / 15.5												
S(ω) MIN ω	≈0 / 18.8	≈0 / 18.8	≈0 / 18.8	≈0 / 18.8												
HIGHEST ω-LIMIT FOR WHICH S(ω) HAS BEEN DETERMINED	20	20	20	20	10	10	10	10	6.6	6.6	6.6	6.6	4.9	4.9	4.9	4.9

ABSOLUTE MAXIMUM

ABSOLUTE MINIMUM

following summary of general properties of the spectra $S(\omega)$ can be derived from Table 5:

$S(\omega)$ is a positive and even function.

The first absolute minimum is attained at $\omega = 2\pi/u$, the value of $S(\omega)$ at that point being zero.

Consecutive absolute minima with $S(\omega) = 0$ are attained $2\pi/u$ units apart.

$S(\omega)$ is a smooth function throughout with no superposed variations.

$S(\omega)$ is heavily damped.

$S(\omega)$ is small ($\leq u \cdot 10^{-2}$) after two periods.

An upper limit for $S(\omega)$ depends almost exclusively on u. It can for all practical purposes be approximated by

$$M = S_{max}(\omega) \simeq \frac{u \cdot \pi}{20} .$$

<u>Case II: U = Constant, V ≃ U(τ)</u>

Table 6 indicates the parameter combinations considered. The general expressions for the means and variances are given in (3.15) and (3.16), for the mean ratios and the coefficients of variation in (4.1) and (4.3). The coefficient of variation for this case

$$CV = \frac{\sqrt{\sigma_0^2 + \sigma_1^2}}{\mu_0 + \mu_1} = \frac{\sqrt{3}}{3 + 6\,\frac{u}{t}}$$

has an upper bound which is attained if t approaches infinity.

Table 6. Values of Parameters Employed in Evaluation of Spectra for Case II--Distribution of Span Lengths

$\mu_0 = u$	t	$\mu_1 = \frac{t}{2}$	$\sigma_0^2 = 0$	$\sigma_1^2 = \frac{t^2}{12}$	Mean Ratio $\frac{\mu_1}{\mu_0} = \frac{t}{2u}$	Coefficient of Variation $\frac{\sqrt{3}}{\left(\frac{6u}{t} + 3\right)}$
1 2 3	2 4 6	1 2 3	0	0.333 1.333 3.000	1 1 1	0.289
1 2 3	4 8 12	2 4 6	0	1.333 5.333 12.000	2 2 2	0.385
1 2 3	6 12 18	3 6 9	0	3.000 12.000 27.000	3 3 3	0.433
1 2 3	8 16 24	4 8 12	0	5.333 21.333 48.000	4 4 4	0.462
1 2 3	10 20 30	5 10 15	0	8.333 33.333 75.000	5 5 5	0.481
1 2 3	20 40 60	10 20 30	0	33.333 133.333 300.000	10 10 10	0.525

The maximum value is given by

$$\lim_{t\to\infty} \frac{\sqrt{3}}{3 + 6\,\frac{u}{t}} = \frac{\sqrt{3}}{3} = 0.57735.$$

The lower bound clearly is zero.

An extract from the tabulation of the spectral density function is given in Table 7. It includes a listing of consecutive optima. For easy inspection of their magnitude the differences between maxima and minima (or vice versa) of the spectra are indicated by the values of $\Delta S(\omega)$. Plots of the functions are given in Appendix B.

The data contained in Table 7 give rise to the following summary of general properties of the spectral density functions associated with this case:

$S(\omega)$ is a positive function.

The first absolute minimum is attained at $\omega = 2\pi/u$, the value of that minimum being zero.

Consecutive absolute minima with $S(\omega) = 0$ are attained $2\pi/u$ units apart.

Between $\omega = 0$ and the first absolute minimum the function oscillates between various relative maxima and minima with irregular periods. The number of those superposed variations increases with an increasing coefficient of variation whereas their amplitude decreases with increasing coefficient of variation.

These variations cease rapidly for $S(\omega)$ beyond the first absolute minimum, that is for $\omega > 2\pi/u$. $S(\omega)$ attains a high degree of smoothness for higher values of ω.

$S(\omega)$ is heavily damped.

$S(\omega)$ is small ($\leq 10^{-2}$) after three periods.

Table 7. Tabulated Maxima and Minima of the Spectral Density Function $S(\omega)$ for U = Constant and $V \simeq U(t)$.

MAX												
THIRD [illegible]	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0
MAX												
MAX												
					$(8.7/8.8)^{***}$				$(8.7/8.8)^{***}$			
MAX	9.0	0.025	9.1	0.016	9.2	0.012	9.2	0.009	9.2	0.008	9.0	0.004
											$(9.2/9.3)^{****}$	
MAX												
	12.6	≈0	12.6	≈0	12.6	≈0	12.6	≈0	12.6	≈0	12.6	≈0
MAX	15.4	0.008	15.4	0.006	15.5	0.004	15.5	0.003	15.5	0.003	15.6	0.001
	18.8	≈0	18.8	≈0	18.8	≈0	18.8	≈0	18.8	≈0	18.8	≈0
MAX	21.7	0.004	21.8	0.003	21.8	0.002	21.8	0.002	21.8	0.001	21.9	[illegible]
	25.1	≈0	25.1	≈0	25.1	≈0	25.1	≈0	25.1	≈0	25.1	≈0
*****	25.1		25.1		25.1		25.1		25.1		25.1	

MAX												
THIRD [illegible]	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0
MAX												
MAX	7.7	0.017	7.7	0.011	7.8	0.008	7.8	0.007	7.8	0.006	7.8	0.003
MAX												
	9.4	≈0	9.4	≈0	9.4	≈0	9.4	≈0	9.4	≈0	9.4	≈0
MAX	10.9	0.008	10.9	0.006	10.9	0.004	10.9	0.003	11.0	0.003	10.9	0.002
	12.6	$\approx 0^{**}$	12.6	$\approx 0^{**}$	12.6	$\approx 0^{**}$	12.6	$\approx 0^{**}$	12.6	$\approx 0^{**}$	12.6	$\approx 0^{**}$
MAX												
MAX												
*****	12.5		12.5		12.5		12.5		12.5		12.5	

MAX	6.5	0.018	5.1	0.017	5.3	0.012	5.2	0.010	5.2	0.008	5.2	0.005
THIRD [illegible]	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0	6.3	≈0
MAX	7.2	0.013	7.3	0.008	7.3	0.006	7.3	0.005	7.4	0.004	7.3	0.002
MAX												
	8.5	$\approx 0^{*}$	8.5	$\approx 0^{*}$	8.5	$\approx 0^{*}$	8.5	$\approx 0^{*}$	8.5	$\approx 0^{*}$	8.5	$\approx 0^{*}$
MAX												
MAX												
MAX												
MAX												
*****	8.3		8.3		8.3		8.3		8.3		8.3	

* : VALUE INTERPOLATED

*** : RELATIVE MAXIMA AT 8.7 (T = 8)
8.8 (T = 20)
RELATIVE MINIMA AT 8.9 (FOR BOTH T)
(DIFFERENCE NEGLIGIBLE)

**** : RELATIVE MAXIMUM AT 9.2
RELATIVE MINIMUM AT 9.3
(DIFFERENCE NEGLIGIBLE)

***** : HIGHEST W-VALUE FOR WHICH S(W) HAS BEEN DETERMINED

An upper limit for $S(\omega)$ depends more heavily on u than on t and can for all practical purposes be approximated by

$$M = S_{max}(\omega) \approx \frac{\pi \cdot u}{10} .$$

Case III: U = Constant, $V \approx N(\mu,\sigma^2)$

Table 8 indicates the parameter combinations considered for this case. The general expressions for the means and variances are given in (3.24) and (3.25), for the mean ratios and the coefficients of variation in (4.1) and (4.3).

Table 8. Values of Parameters Employed in Evaluation of Spectra for Case III--Distribution of Span Lengths

$\mu_o=u$	$\mu_1=\mu$	$\sigma_0^2=0$	$\sigma_1^2 = \sigma^2$	$\frac{\mu_1}{\mu_0} = \frac{\mu}{u}$	Coefficient of Variation $\frac{\sigma}{(u+\mu)}$
1	0		$(1.0)^2$	0	1.00
1	1		$(1.0)^2$	1	0.50
1	2		$(0.3)^2$	2	0.10
5	5	0	$(1.0)^2$	1	0.10
5	10		$(1.5)^2$	2	0.10
1	1		$(0.1)^2$	1	0.05
5	15		$(1.0)^2$	3	0.05
5	20		$(1.25)^2$	4	0.05

From Table 9 and the plots in Appendix B the following general properties of the spectral density function can be summarized:

Table 9. Tabulated Maxima and Minima of the Spectral Density Function $S(\omega)$ for U = Constant and $V \simeq N(\mu,\sigma^2)$

1								5								U
0		1		1		2		5		10		15		20		μ
1		1		0.1		0.3		1		1.5		1		1.25		σ
w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	
0	1.0	0	0.125	0	0.00125	0	0.0033	0	0.025	0	0.0166	0	0.003125	0	0.0026	S (W=0)
0	1.0															
		0	0.125	0	0.001	0	0.003	0	0.025	0	0.017	0	0.003	0	0.003	MIN
											11.403		6.535		1.843	ΔS(w)
										0.42	11.42	0.32	6.336	0.26	1.846	MAX
									9.124				6.294		1.806	ΔS(w)
												0.46	0.042	0.36	0.040	MIN
											11.272		3.068		5.27	ΔS(w)
								0.62	9.149			0.62	3.11	0.5	5.31	MAX
															5.247	ΔS(w)
										0.62	0.148			0.62	0.063	MIN
											0.626		3.068		0.945	ΔS(w)
										0.82	0.774			0.76	1.008	MAX
			0.182				2.281								0.869	ΔS(w)
												0.80	0.042	0.90	0.038	MIN
					8.195								0.479		0.111	ΔS(w)
												0.94	0.821	1.00	0.150	MAX
								1.26	≈0	1.26	≈0	1.26	≈0	1.26	≈0	MIN FIRST ABSOLUTE MINIMUM
										1.7	0.078	1.6	0.067	1.5	0.033	MAX
													0.051		0.004	ΔS(w)
												1.7	0.036	1.6	0.029	MIN
													0.040		0.022	ΔS(w)
		2.26	0.307			2.10	2.284	1.9	0.153			1.9	0.076	1.8	0.051	MAX
							2.254									ΔS(w)
						3.14	0.030	2.5	≈0	2.5	≈0	2.5	≈0	2.5	≈0	MIN SECOND ABSOLUTE MINIMUM
				3.14	8.196			3.1	0.042	3.1	0.027	3.1	0.021	3.1	0.016	MAX
							0.126									ΔS(w)
								3.8	≈0	3.8	≈0	3.8	≈0	3.8	≈0	MIN THIRD ABSOLUTE MINIMUM
						4.14	0.158									MAX
6.28	≈0	6.28	≈0	6.28	≈0	6.28	≈0									MIN
9.0	0.047	9.0	0.024	9.4	0.103	8.6	0.016									MAX
12.6	≈0	12.6	≈0	12.6	≈0	12.6	≈0									MIN
15.5	0.016	15.5	0.008	15.7	0.015	16.5	0.005									MAX
18.8	≈0	18.8	≈0	18.8	≈0	18.8	≈0									MIN
20.0		20.0		20.0		20.0		3.9		3.9		3.9		3.9		HIGHEST W-VALUE FOR WHICH S(W) HAS BEEN DETERMINED

ABSOLUTE MAXIMUM OF S(W)

ABSOLUTE MINIMUM

$S(\omega)$ is a positive and even function.

$S(\omega)$ attains its first absolute minimum at $\omega = 2\pi/u$, the value of that minimum being zero.

Consecutive absolute minima are placed $2\pi/u$ units apart.

In the first period $(0 \leq \omega \leq \frac{2\pi}{u})$, $S(\omega)$ varies greatly with amplitudes of up to 1000 times the base value and with periods of less than 10^{-1}. This characteristic is more pronounced for higher values of u and μ, and in those cases, too, a higher frequency of these irregular variations is to be observed. Although it appears to be certain that there is a definite quantitative dependency between the height of the peaks of those vibrations and the parameters considered, attempts have not been made to establish such a relationship.

Case IV: $U \approx EXP(\lambda)$, $V \approx N(\mu,\sigma^2)$

Table 10 surveys the parameter combinations considered in this case. The general expressions for the means and variances are given by (3.46) and (3.47), for the mean ratios and the coefficients of variation in (4.1) and (4.3). For the above parameter combinations $S(\omega)$ has been tabulated and an extract of this tabulation is given in Table 11. The data contained in that table give rise to the following summary of general properties of $S(\omega)$:

For a mean ratio of 1 and below the spectra oscillate roughly with period $2\pi/\mu$ between two consecutive minima. These are relative minima in that the spectrum is unequal zero at ω_{min}. However, in about the same way as the amplitudes are decreased through damping, the numerical value of each consecutive minimum is decreasing for increasing ω, that is

$$S\left(\frac{2\pi i}{\mu}\right) > S\left(\frac{2\pi(i+1)}{\mu}\right)$$

For values of the mean ratio greater than 1, $S(\omega)$ does not oscillate at all but decreases monotonically after attaining a low maximum at some value of ω close to zero.

Table 10. Values of Parameters Employed in Evaluation of Spectra for Case IV--Distributions of Span Lengths

λ	$\mu_0 = \frac{1}{\lambda}$	$\mu_1 = \mu$	$\sigma_0^2 = \frac{1}{\lambda^2}$	$\sigma_1^2 = \sigma^2$	Mean Ratio $\frac{\mu_1}{\mu_0} = \mu\lambda$	Coefficient of Variation $\frac{\sqrt{\lambda^2\sigma^2 + 1}}{(1 + \lambda\mu)}$
1/8	8	1	64	0.01	1/8	0.889
1/16	16	4	256	0.01	1/4	0.8
1/9	9	3	81	0.01	1/3	0.75
1/4	4	2	16	0.01	1/2	0.6669
1	1	1	1	0.01	1	0.5025
1	1	2	1	1.00	2	0.4714
2	1/2	2	1/4	1.00	4	0.4472
1	1	4	1	1.00	4	0.2828
2	1/2	4	1/4	1.00	8	0.2485
1	1	9	1	1.00	9	0.1414

It is most apparent that there exist similarities between Cases I and IV. In fact, the mean of the normal distribution has about assumed the effect of the constant span length u. The characteristics are not that pronounced as in the constant-exponential case, but the general properties of the spectra are maintained. It is to be noted that inferences from Case I to this case can only be drawn for similar mean ratios, that is

$$\frac{E(V_I)}{E(U_I)} \simeq \frac{E(V_{IV})}{E(U_{IV})}$$

if

Table 11. Tabulated Maxima and Minima of the Spectral Density Function $S(\omega)$ for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu,\sigma^2)$

1		2		3		4		2		4		2		4		1		9		μ
0.1		0.1		0.1		0.1		1.0		1.0		1.0		1.0		0.1		1.0		σ
1		1/4		1/9		1/16		1		1		2		2		1/8		1		λ
ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	ω	$S(\omega)$	
0	0.1243	0	0.287	0	0.4222	0	0.5123	0	0.1852	0	0.136	0	0.080	0	0.0466	0	0.0687	0	0.082	$S(\omega=0)$
		$\boxed{0}$	$\boxed{0.297}$	$\boxed{0}$	$\boxed{0.422}$	$\boxed{0}$	$\boxed{0.512}$									$\boxed{0}$	$\boxed{0.069}$			MAX
0	0.126							0.	0.185	0	0.136	0	0.080	0	0.047			0	0.082	MIN
																		$\boxed{0.62}$	$\boxed{0.511}$	MAX
																		0.77	0.048	MIN
								$\boxed{1.31}$	$\boxed{0.218}$	$\boxed{1.14}$	$\boxed{0.278}$			$\boxed{1.28}$	$\boxed{0.126}$			1.22	0.128	MAX
						1.57	0.0005											1.50	0.043	MIN
						2.27	0.036					$\boxed{1.75}$	$\boxed{0.115}$					1.80	0.055	MAX
		3.15	0.002	2.10	0.001	3.14	0.00048													MIN
$\boxed{3.00}$	$\boxed{0.143}$			3.05	0.034	3.87	0.013													MAX
				4.19	0.0009	4.71	0.00047													MIN
		4.57	0.029	5.2	0.011	5.5	0.006													MAX
6.3	0.005	6.3	0.002	6.3	0.0008	6.3	0.00046									6.3	0.001			MIN
		7.7	0.009	7.3	0.006	7.0	0.004													MAX
				8.4	0.0007	7.9	0.00045													MIN
9.1	0.018			8.4	0.003	8.6	0.002									8.8	0.004			MAX
		9.4	0.001			9.4	0.00041													MIN
						10.2	0.0018													MAX
				10.5	0.00062	11.0	0.00038													MIN
		10.9	0.004	11.5	0.002	11.7	0.001													MAX
12.7	0.003	12.6	0.001	12.6	0.00058	12.6	0.00035									12.6	0.00077			MIN
		14.0	0.002	12.5	0.001	13.3	0.0008													MAX
				14.7	0.0005	14.1	0.00032													MIN
15.2	0.005			15.6	0.0009	14.9	0.0006									15.0	0.001			MAX
		15.8	0.0009			15.7	0.00029													MIN
		17.1	0.001			16.4	0.00044													MAX
				16.8	0.00045	17.3	0.00026													MIN
				17.7	0.00064	18.0	0.00037													MAX
18.3	0.002	18.9	0.0008	18.9	0.00039	18.5	0.00022									18.3	0.0005			MIN
				18.8	0.00048	18.6	0.0003													MAX
20	0.0023	20	0.0009	20	0.00047	20	0.00026	20	0.00017	20	0.001	20	0.002	20	0.0011	20	0.00082	20	0.0005	*

$\boxed{}$ ABSOLUTE MAXIMUM OF $S(\omega)$

* HIGHEST ω-VALUE FOR WHICH $S(\omega)$ HAS BEEN DETERMINED

$$V_I \simeq EXP(\lambda), \qquad V_{IV} \simeq EXP(\lambda),$$

$$U_I = CONSTANT, \qquad U_{IV} \simeq N(\mu,\sigma^2).$$

Case V: $U \simeq N(\mu_0,\sigma_0^2)$, $V \simeq N(\mu_1,\sigma_1^2)$

Eight parameter combinations were considered in this case with a coefficient of variation (4.03) ranging from 0.07 to 0.6, and a mean ratio (4.01) between 0.1 and 5.0. Table 12 provides an account.

Table 12. Values of Parameters Employed in Evaluation of Spectra for Case V--Distributions of Span Lengths

μ_0	μ_1	σ_0	σ_1	Mean Ratio $\frac{\mu_1}{\mu_0}$	Coefficient of Variation $\frac{\sqrt{\sigma_0^2 + \sigma_1^2}}{(\mu_0 + \mu_1)}$
4	1	3.0	0.5	0.25	0.6083
10	10	5.0	5.0	1.00	0.3536
10	10	1.0	5.0	1.00	0.255
2	10	0.4	3.0	5.00	0.2522
3	10	0.6	3.0	3.33	0.2353
5	10	1.0	3.0	2.00	0.2108
9	1	0.9	0.1	0.11	0.0906
10	10	1.0	1.0	1.00	0.0707

An extract from the tabulation of $S(\omega)$ is again provided in Table 13. The contents of this table in addition to the plots permit the following summary of general properties of the spectral density function $S(\omega)$:

Table 13. Tabulated Maxima and Minima of the Spectral Density Function $S(\omega)$ for $U \simeq N(\mu_0,\sigma_0^2)$ and $V \simeq N(\mu_1,\sigma_1^2)$.

4		9		10		10		10		5		3		2		μ_0
1		1		10		10		10		10		10		10		μ_1
3.0		0.9		1.0		1.0		5.0		1.0		0.6		0.4		σ_0
0.5		0.1		1.0		5.0		5.0		3.0		3.0		3.0		σ_1
w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	w	S(w)	
0	0.104	0	0.0016	0	0.025	0	0.326	0	0.625	0	0.0963	0	0.0533	0	0.0301	S(0)
0	0.104	0	0.002	0	0.025	0	0.325	0	0.625	0	0.096	0	0.053	0	0.030	MIN
			1.177		24.52		3.645		2.066		2.748		1.159		0.645	ΔS(w)
		$\boxed{0.63}$	$\boxed{1.179}$	$\boxed{0.31}$	$\boxed{24.77}$	$\boxed{0.23}$	$\boxed{3.97}$	$\boxed{0.26}$	$\boxed{2.691}$	$\boxed{0.41}$	$\boxed{2.844}$	$\boxed{0.47}$	$\boxed{1.212}$	$\boxed{0.50}$	$\boxed{0.673}$	MAX
	0.112		1.162		24.52		3.925						0.840		0.334	ΔS(w)
		0.9	0.017	0.63	0.025	0.63	0.045					0.73	0.372	0.76	0.239	MIN
			0.264		0.494		0.155				2.798		0.004		0.009	ΔS(w)
$\boxed{0.88}$	$\boxed{0.216}$	1.25	0.281	0.94	0.519	0.88	0.200					0.80	0.376	0.89	0.248	MAX
			0.244		0.495		0.165									ΔS(w)
		1.55	0.037	1.26	0.024	1.26	0.035			1.29	0.046					MIN
			0.082		0.051		0.020				0.008		0.357			ΔS(w)
		1.87	0.119	1.55	0.075	1.50	0.055			1.60	0.054					MAX
			0.070		0.056		0.032								0.239	ΔS(w)
		2.19	0.049	1.90	0.020	1.92	0.023					2.15	0.019			MIN
			0.018		0.006		0.001						0.004			ΔS(w)
		2.47	0.067	2.14	0.026	2.08	0.024					2.67	0.023			MAX
			0.066													ΔS(w)
		6.29	0.001											3.22	0.009	MIN
															0.002	ΔS(w)
		8.8	0.004											4.00	0.011	MAX
		12.6	0.0007													MIN
		15.0	0.001													MAX
50	$1.6\cdot10^{-4}$	50	$8\cdot10^{-5}$	50	$4\cdot10^{-5}$	50	$4\cdot10^{-5}$	50	$4\cdot10^{-5}$	50	$5.3\bar{3}\cdot10^{-5}$	50	$6.15\cdot10^{-5}$	50	$6.6\bar{6}\cdot10^{-5}$	HIGHEST W-VALUE FOR WHICH S(W) HAS BEEN DETERMINED

$\boxed{}$ ABSOLUTE MAXIMUM OF S(W)

$S(0)$ is a relative minimum of the spectra.

No regular fixed period oscillations are to be observed.

After the relative minimum at $\omega = 0$, $S(\omega)$ attains its absolute maximum at some value $0 < \omega \leq 1$. The one oscillation leading to the absolute maximum has very small period ($\leq 10^{-1}$) but high amplitude. The plots in Appendix B support this finding.

Once the maximum is passed, $S(\omega)$ either decreases monotonically towards zero, or runs through a finite number of damped oscillations before it is for all practical purposes close enough to zero.

Hence, $S(\omega)$ is very irregularly shaped and becomes a smooth function only for higher values of ω.

CHAPTER V

THE INVERSE TRANSFORMATION: DETERMINATION OF THE AUTOCORRELATION FUNCTIONS

The inverse transformation which transforms the spectral density function $S(\omega)$ in the frequency domain into the autocovariance $R(\tau)$ in the time domain rests on the following relation:

$$R(\tau) = \frac{1}{\pi}\int_0^\infty S(\omega)\cos\omega\tau\, d\omega, \tag{5.1}$$

$$= \int_0^\infty f(\omega,\tau)\, d\omega,$$

with

$$f(\omega,\tau) = \frac{1}{\pi} S(\omega)\cos\omega\tau. \tag{5.2}$$

Once the autocovariance, $R(\tau)$, is obtained, the autocorrelation function, $\rho(\tau)$, follows from the fact that both $R(\tau)$ and $\rho(\tau)$ are proportional. Proportionality factor is the process variance which is a constant because of the stationarity assumption. The relation is

$$\rho(\tau) = \frac{R(\tau)}{\sigma^2} = \frac{R(\tau)}{R(0)}, \tag{5.3}$$

or in terms of the inverse transform (5.1)

$$\rho(\tau) = \frac{\frac{1}{\pi}\int_0^\infty S(\omega)\cos\omega\tau\, d\omega}{\frac{1}{\pi}\int_0^\infty S(\omega)\, d\omega} . \tag{5.4}$$

In Chapter III the model development included the determination of $S(\omega)$ for each of the five cases considered. Since $S(\omega)$ is therefore readily available, $f(\omega,\tau)$ can be determined with the help of relation (5.2). A glance at the expressions for $S(\omega)$ (3.12), (3.19), (3.42), (3.50), and (3.57) convinces that the most elegant way of determining the autocorrelation function $\rho(\tau)$ (5.4), namely by means of analytical integration appears to be prohibitively difficult. In fact, a substantial part of the time allotted for this study has been spent in an attempt to evaluate the integral (3.57) analytically. The only known case in which analytical integration has been performed is the one discussed in some detail by Kume.[18]

This study, therefore, contents itself with determining the inverse transform by means of numerical integration. From an engineering standpoint this method is capable of yielding adequate results.

It is an inherent property of the spectral density function that

$$\lim_{\omega\to\infty} S(\omega) = 0. \tag{5.5}$$

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18. Kume, H., "On the Spectral Analysis of Zero-One Processes."

From the contents of this study relation (5.5) can be validated intuitively by means of the plots in Appendix B and the tables in Chapter IV. Both show $S(\omega)$ to be basically decreasing. Quantitatively, the relation (5.5) can be proved for each of the five considered cases by simple substitution of ω as $\omega \to \infty$, or, in case that substitution yields one of the two indeterminate forms $[0/0]$, $[\infty/\infty]$, by application of L'Hospital's rule. Relation (5.5) is taken to be valid throughout.

A numerical evaluation of the integral on the interval $[0,\infty]$ is possible when the upper limit is replaced by some estimated value, UL. The compound form of Simpson's Rule is then utilized:

Let

$$0 = \omega_0 < \omega_1 < \omega_2 < \cdots < \omega_{2n-1} < \omega_{2n} = UL$$

be a sequence of equally-spaced points in [0,UL]:

$$\omega_{i+1} - \omega_i = h; \qquad i = 0,1,\ldots,2n-1.$$

Set $f_i = f(\omega_i,\tau)$. Then the compound form of Simpson's Rule is given by

$$\int_0^{UL} f(\omega,\tau)d\omega = \frac{h}{3}\left[f_0+4(f_1+f_3+\ldots+f_{2n-1}) + 2(f_2+f_4+\ldots+f_{2n-2})+f_{2n}\right] + E_n. \tag{5.6}$$

Noting that $N = 2n$ is the (even) number of subdivisions of [0,UL] the length of the subintervals, h, becomes

$$h = \frac{UL}{N} .$$

The remainder E_n is given by

$$E_n = -\frac{(UL)^5}{180\ N^4} f^{(4)}\ (\xi,\tau), \quad 0 < \xi < UL.$$

Since the spectral density functions $S(\omega)$ do have four continuous derivatives the compound Simpson's Rule converges to the true value of the integral with rapidity N^{-4} at worst. In practice, therefore, one might expect that the use of ten subintervals would secure about four decimal places. For any reasonable choice of N the magnitude of the error E_n is therefore negligible in comparison to the error incurred by cutting off the tail end of the function $f(\omega,\tau)$ in (5.2) through the estimation of UL.

Rather than dividing the interval [0,UL] by a predetermined number of subdivisions N and thus arriving at the subinterval length h, in this case h has been fixed first so as to account for specific irregularities of the function $f(\omega,\tau)$.

The basis for the determination of h is given in Chapter IV. In each of the five cases h will be chosen according to the characteristics of the function to be integrated. Since $S(\omega)$ is known to some detail, and most of its properties will prevail after the multiplication with a regular cosine function (except for the nonnegativity property), sufficient knowledge about $f(\omega,\tau)$ is at hand.

Estimation of the Upper Limit, UL

Given the relation (5.1) and separating it by splitting up the interval of integration, one obtains

$$R(\tau) = \int_0^{UL} f(\omega,\tau)\, d\omega + \int_{UL}^{\infty} f(\omega,\tau)\, d\omega. \qquad (5.7)$$

UL has to be determined so as to reduce the numerical value of the integral on the right of the plus sign to a quantity less than some desired accuracy a_ω, that is

$$\int_{UL}^{\infty} f(\omega,\tau)\, d\omega = \frac{1}{\pi} \int_{UL}^{\infty} S(\omega) \cos \omega\tau \, d\omega \le a_\omega . \qquad (5.8)$$

If $S(\omega) \cos \omega\tau$ is a bounded Riemann integrable function on $[UL,\infty]$, then so is $|S(\omega) \cos \omega\tau|$ and it is

$$\left|\frac{1}{\pi} \int_{UL}^{\infty} S(\omega) \cos \omega\tau \, d\omega\right| \le \frac{1}{\pi} \int_{UL}^{\infty} \left|S(\omega) \cos \omega\tau\right| d\omega \qquad (5.9)$$

$$\le \frac{1}{\pi} \int_{UL}^{\infty} S(\omega)\, d\omega \le a_\omega ,$$

since $S(\omega)$ is a positive function. For each particular spectral density function $S(\omega)$, the expression (5.9) has to be approximated, so as to arrive at a lower limit for UL. This value is used in a way that the actual integration procedure is terminated when an even multiple of the subinterval length h has surpassed that limit.

The accuracy value a_ω has been fixed to 0.004. The value was obtained as a compromise between the effects of different considerations which tended to increase or decrease the error magnitude. Contributing factors were:

The amount of additional computer time required for a decrease in error magnitude.

The requirement of a high reliability of the results (the autocorrelation functions)--particularly in those cases where the amplitude of the first oscillation already dropped below a_ω--necessitated a decrease in error magnitude.

It is to be noted that a_ω accounts exclusively for the error incurred by cutting off the "tail" of the integrand $f(\omega,\tau)$. The error incurred through application of Simpson's Rule is assumed to be negligible, although no quantitative proof for the validity of this assumption will be provided.

The integrand $f(\omega,\tau)$ is a function of both ω and τ. The numerical integration procedure requires

a τ to be fixed, say $\tau = c$,

the integral,

$$\int_0^{UL} f(\omega,c)\, d\omega,$$

to be integrated with respect to ω,

a new τ to be fixed and the process to be repeated until sufficient data are at hand.

The question to be asked at this point certainly is: What is the maximum value of τ, τ_{max}, for which the autocovariance $R(\tau)$ has to be determined and in what size increments is τ_{max} to be approached? The answer to both parts of the question requires more knowledge on autocorrelation functions than was initially available. Therefore, the practical tests rather than mathematical theory provided most of the answers at a later stage. The foundation regarding this question, however, was laid in the following manner:

According to (5.7) and (5.2) the autocovariance $R(\tau)$ can be approximated by

$$R(\tau) \approx \frac{1}{\pi} \int_0^{UL} S(\omega) \cos \omega\tau \; d\omega. \tag{5.10}$$

If $S(\omega)$ is a Riemann-integrable function on $[0,UL]$, then the theorem of Lebesque can be applied to (5.10) and one obtains

$$\lim_{\tau\to\infty} R(\tau) = \lim_{\tau\to\infty} \frac{1}{\pi} \int_0^{UL} S(\omega) \cos \omega\tau \, d\omega = 0. \tag{5.11}$$

Hence it can be concluded that there is a τ, say τ_{max}, for which Lebesgue's theorem is closely enough satisfied.

To compute τ_{max}, it is

$$\left|\frac{1}{\pi} \int_0^{UL} S(\omega) \cos \omega\tau \, d\omega\right| \le \frac{1}{\pi} \int_0^{UL} \left|S(\omega) \cos \omega\tau\right| d\omega \tag{5.12}$$

In Chapter IV, in some cases an upper bound M of $S(\omega)$ was determined, where M is greater than or equal the value of the absolute maximum of $S(\omega)$. Using this notation, expression (5.12) becomes

$$\left|\frac{1}{\pi} \int_0^{UL} S(\omega) \cos \omega\tau \, d\omega\right| \le \frac{M}{\pi} \int_0^{UL} \left|\cos \omega\tau\right| d\omega \tag{5.13}$$

$$\le \frac{M}{\pi\tau} \left|\sin[\omega(UL)]\right|$$

$$\leq \frac{M}{\pi \tau_{max}} \leq a_{\tau} \ .$$

Thus

$$\tau_{max} \geq \frac{M}{\pi a_{\tau}} \ . \tag{5.14}$$

The determination of τ_{max} requires thus the fixing of a significance level a_{τ}.

It depends exclusively on the autocorrelation functions whether or not a knowledge of τ_{max} is of advantage. For functions with a high damping[19] factor and therefore a small τ_{max}, a knowledge of the latter may lead to a meaningful determination of the incremental interval length in steps of which τ is to be used for the evaluation of $\rho(\tau)$. For autocorrelation functions with low damping characteristics a knowledge of τ_{max} is of little value, because the function is well enough described after, say, four completed oscillations, whereas τ_{max} may be attained only after a multiple of those oscillations are passed. In this latter case the determination of the increments depends much more on a knowledge of the periods of oscillation.

In general, the question of in which increments of τ to determine $\rho(\tau)$ and where to stop has been answered for each particular case separately, based on some acquired experience.

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19. The term "damping" in this context is deprived of its mechanical impact. It is used to denote the property of some autocorrelation functions, which oscillate with unchanged frequency, but with decreasing amplitude.

Case I: U = Constant, V ≃ EXP(λ)

The spectral density function $S(\omega)$ is given in (3.12). Considering (5.4) the autocorrelation function is obtained as the result of the evaluation of the following expression.

$$\rho(\tau) = \frac{R(\tau)}{R(0)} \simeq \frac{\dfrac{2\lambda}{\pi(1+\lambda u)} \displaystyle\int_0^{UL} \frac{(1-\cos\omega u)\cos\omega\tau}{\lambda^2(1-\cos\omega u)^2 + (\omega+\lambda\sin\omega u)^2}\, d\omega}{\dfrac{2\lambda}{\pi(1+\lambda u)} \displaystyle\int_0^{UL} \frac{(1-\cos\omega u)}{\lambda^2(1-\cos\omega u)^2 + (\omega+\lambda\sin\omega u)^2}\, d\omega} . \tag{5.15}$$

For the determination of the upper limit UL expression (5.9) has to be approximated. Given (3.12), (5.9) attains the form

$$\frac{1}{\pi}\int_{UL}^{\infty} S(\omega)\, d\omega \le \frac{2\lambda}{\pi(1+\lambda u)} \int_{UL}^{\infty} \frac{2}{\omega^2}\, d\omega$$

$$\le \frac{4\lambda}{\pi(1+\lambda u)UL} \le a_\omega = 4\cdot 10^{-3} .$$

Hence

$$UL \le \frac{1000\lambda}{\pi(1+\lambda u)} . \tag{5.16}$$

The integrand in the numerator of expression (5.15) is a superposition of $S(\omega)$ and $\cos\omega\tau$. $S(\omega)$ is a smooth function for all parameter combinations used. It is considered sufficient to divide a complete period of $\cos\omega\tau$ into 24 equidistant subintervals. For values of $\tau \ge 1.0$ the subinterval length was therefore fixed to be $h = \pi/12\tau$. For values of

$\tau < 1.0$, h was determined to $h = \pi/24$ for $\tau = 0.1$, and $h = \pi/12$ for $0.1 < \tau < 1.0$.

The evaluation procedure of (5.15) computed the denominator first, thus obtaining a numerical value for R(0) or the process variance. Here the numerical integration was performed in interval steps of $h = 0.1$, a value which was proved to be sufficient by test evaluation of R(0) with $h = 0.01$. The autocorrelation function $\rho(\tau)$ (5.15) takes on the value 1 for $\tau = 0$. Thereafter $\rho(\tau)$ has been computed for values of τ starting with $(0.1 \div 0.25) \cdot u$ and proceeding in increments of the same size. The upper limit of τ was estimated with $\tau_{max} = 10 \cdot u$, the results show, however, that the amplitudes of the autocorrelation functions are far below the error magnitude of 0.004 at values of τ beyond 4u. It was therefore decided to terminate τ at those values where the amplitude of $\rho(\tau)$ drops below the error limit. For the application of Simpson's Rule the value of the integrand

$$f(\omega,\tau) = \frac{1}{\pi} S(\omega) \cos \omega\tau$$

has to be computed at $\omega = 0$. As in the case of S(0), $f(0,\tau)$ is an indeterminate form which resolves after repeated application of L'Hospital's Rule to

$$\lim_{\omega \to 0} f(\omega,\tau) = \frac{\lambda u^2}{\pi(1+\lambda u)^3} .$$

For each of the parameter combinations listed in Table 4 the autocovariance $R(\tau)$ and the autocorrelation function $\rho(\tau)$ have been tabulated for various values of τ. This output is accumulated in Appendix C. Graphs of the autocorrelation function are given in Figures 5 through 20.

<u>Case II: U = Constant, V ≈ U(τ).</u>

For this case the spectral density function is given by [Equation (3.19)]

$$S(\omega) = \frac{4}{(2u+t)} \cdot \frac{(1 - \cos \omega u)\,[\omega^2 t^2 + 2\cos \omega t - 2]}{\omega^2\{2(1-\cos \omega t) + 2\omega t[\sin \omega u - \sin \omega(u+t)] + \omega^2 t^2\}} .$$

Substitution of $S(\omega)$ into a modified expression (5.4),

$$\rho(\tau) = \frac{R(\tau)}{R(0)} \simeq \frac{\frac{1}{\pi} \int_0^{UL} S(\omega) \cos \omega\tau \, d\omega}{\frac{1}{\pi} \int_0^{UL} S(\omega) \, d\omega} , \qquad (5.18)$$

yields the basic relation for the autocorrelation function which is to be evaluated.

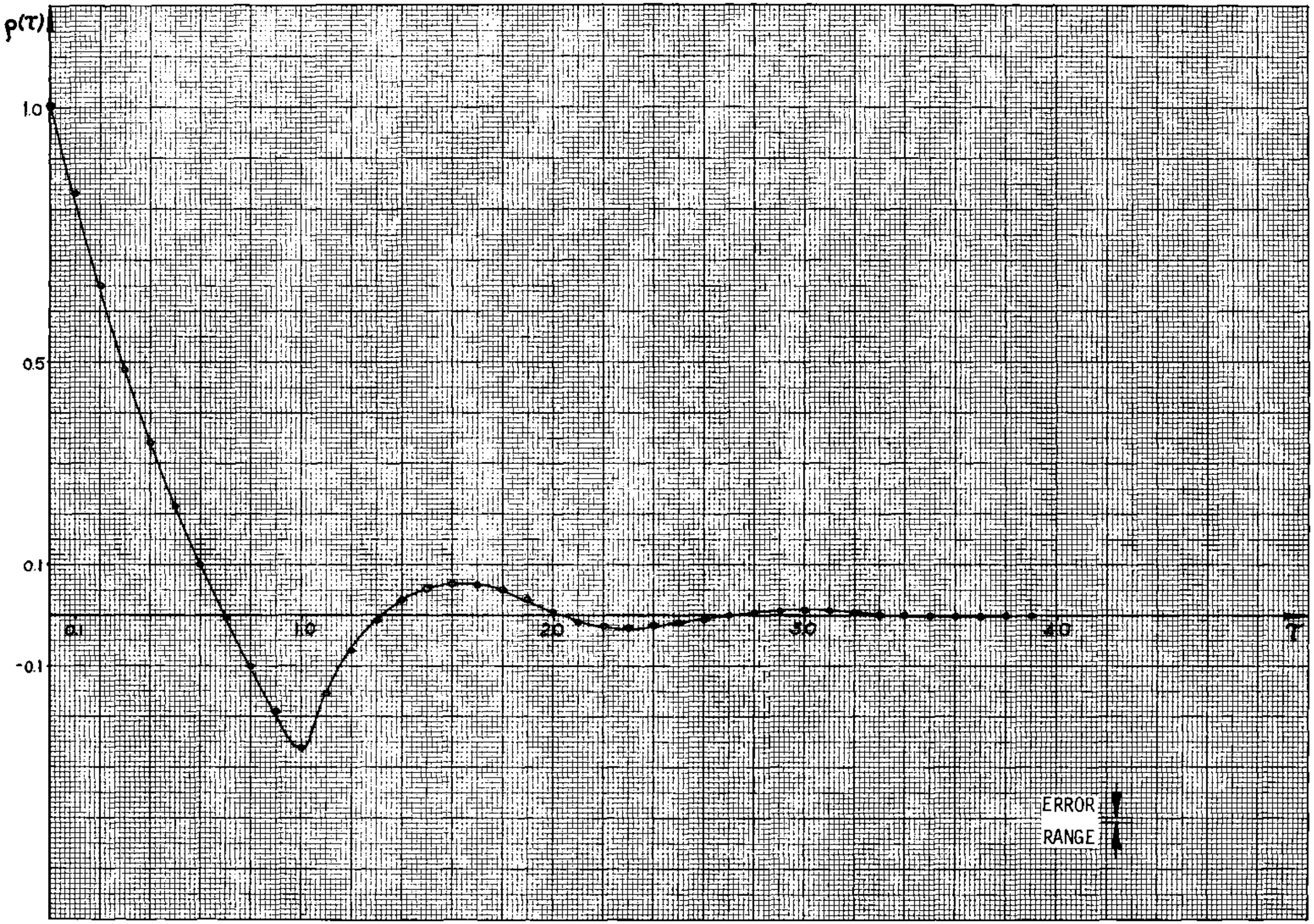

Figure 5. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ). Parameters: $u = 1$, $\lambda = 1$.

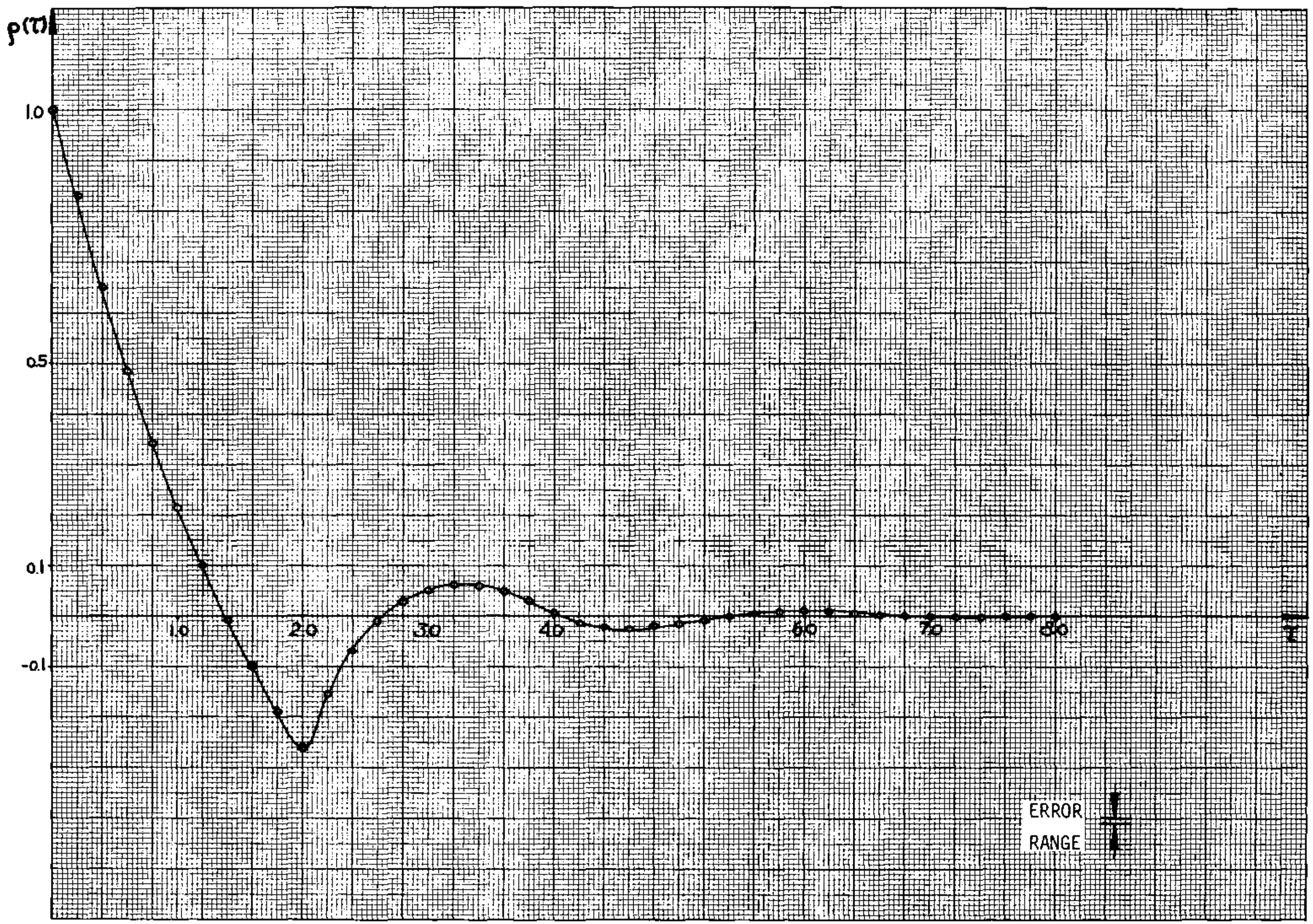

Figure 6. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq EXP(\lambda)$.
Parameters: $u = 2$, $\lambda = 1/2$.

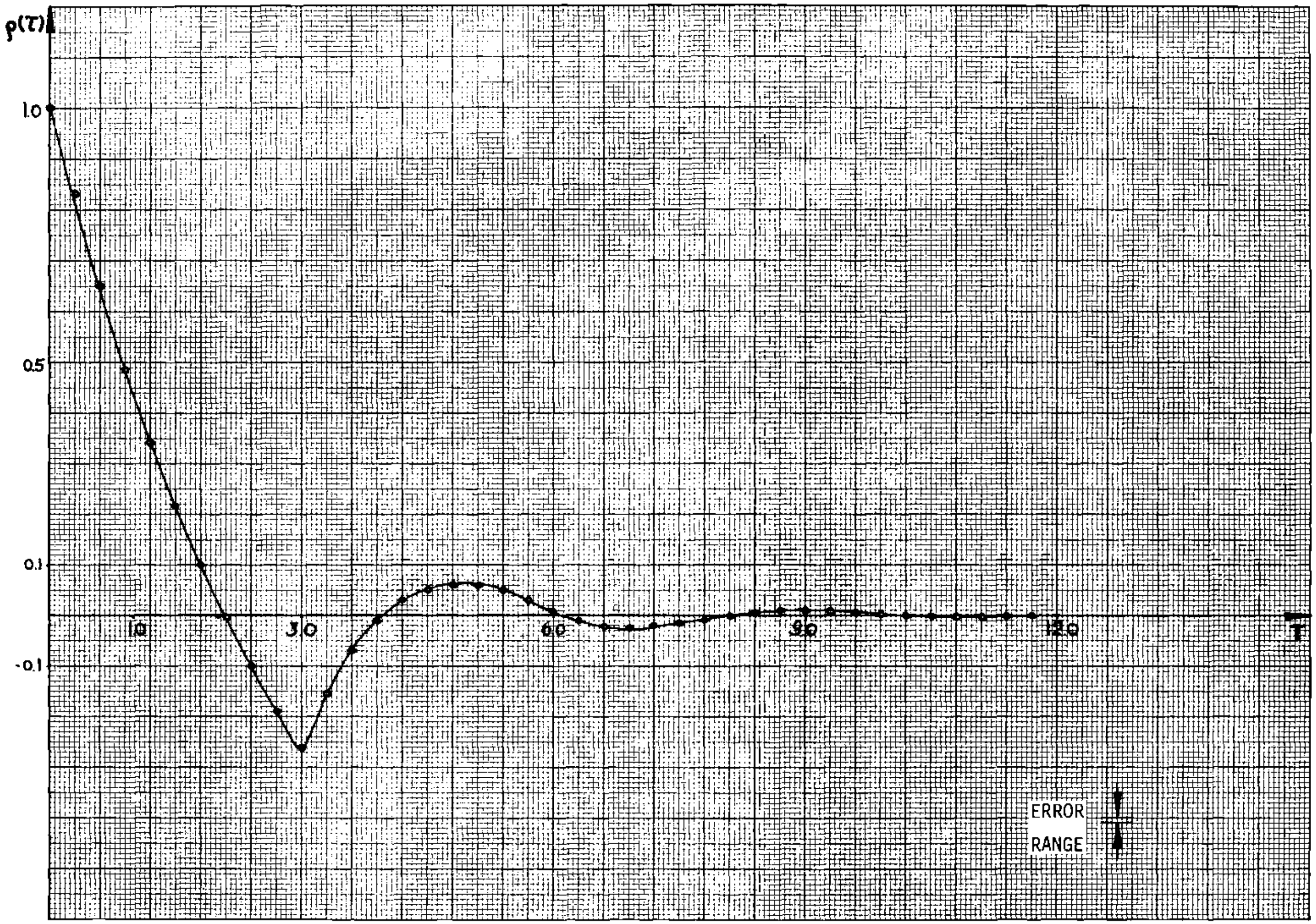

Figure 7. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ EXP(λ).
Parameters: $u = 3$, $\lambda = 1/3$.

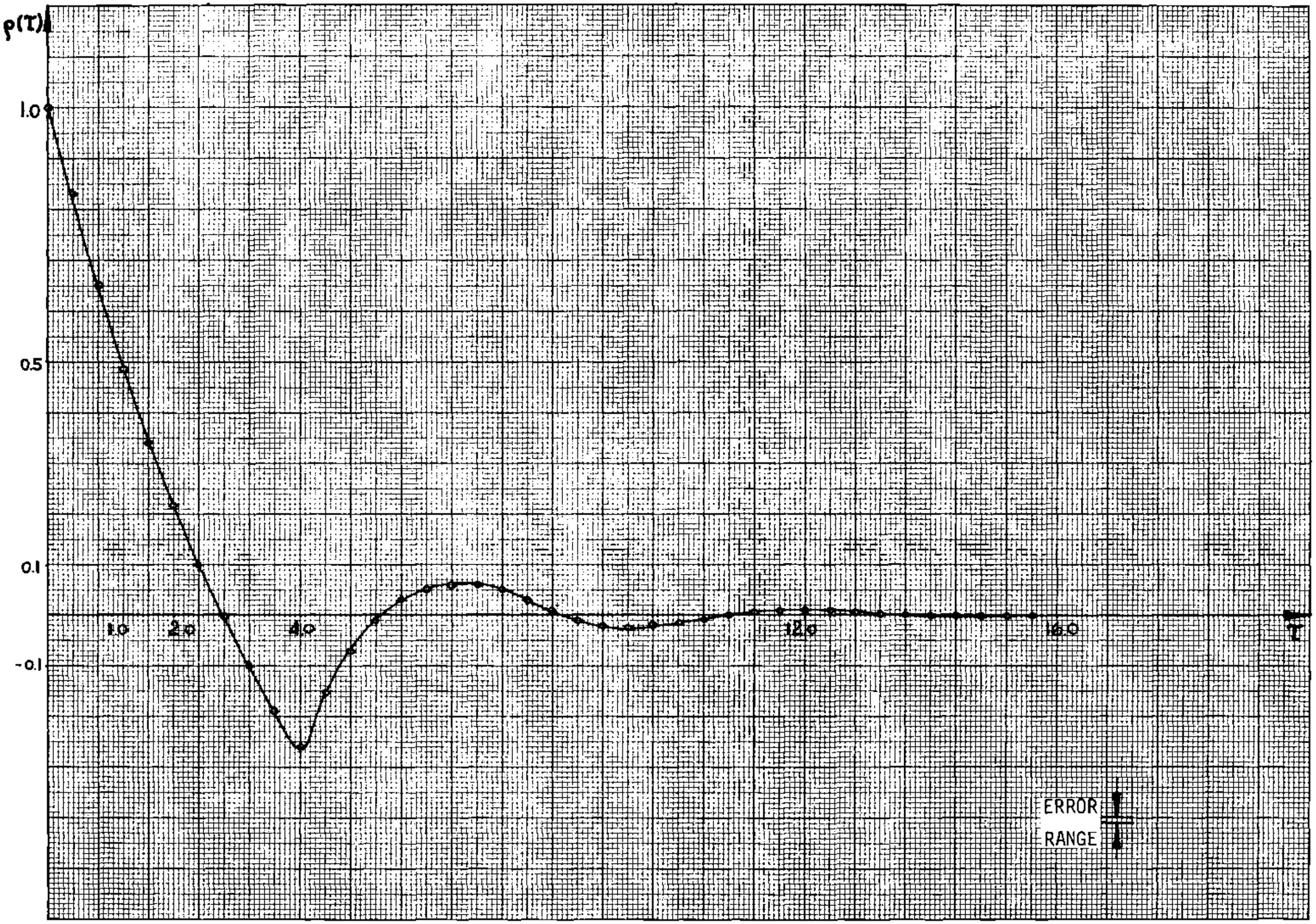

Figure 8. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ EXP(λ).
Parameters: $u = 4$, $\lambda = 1/4$.

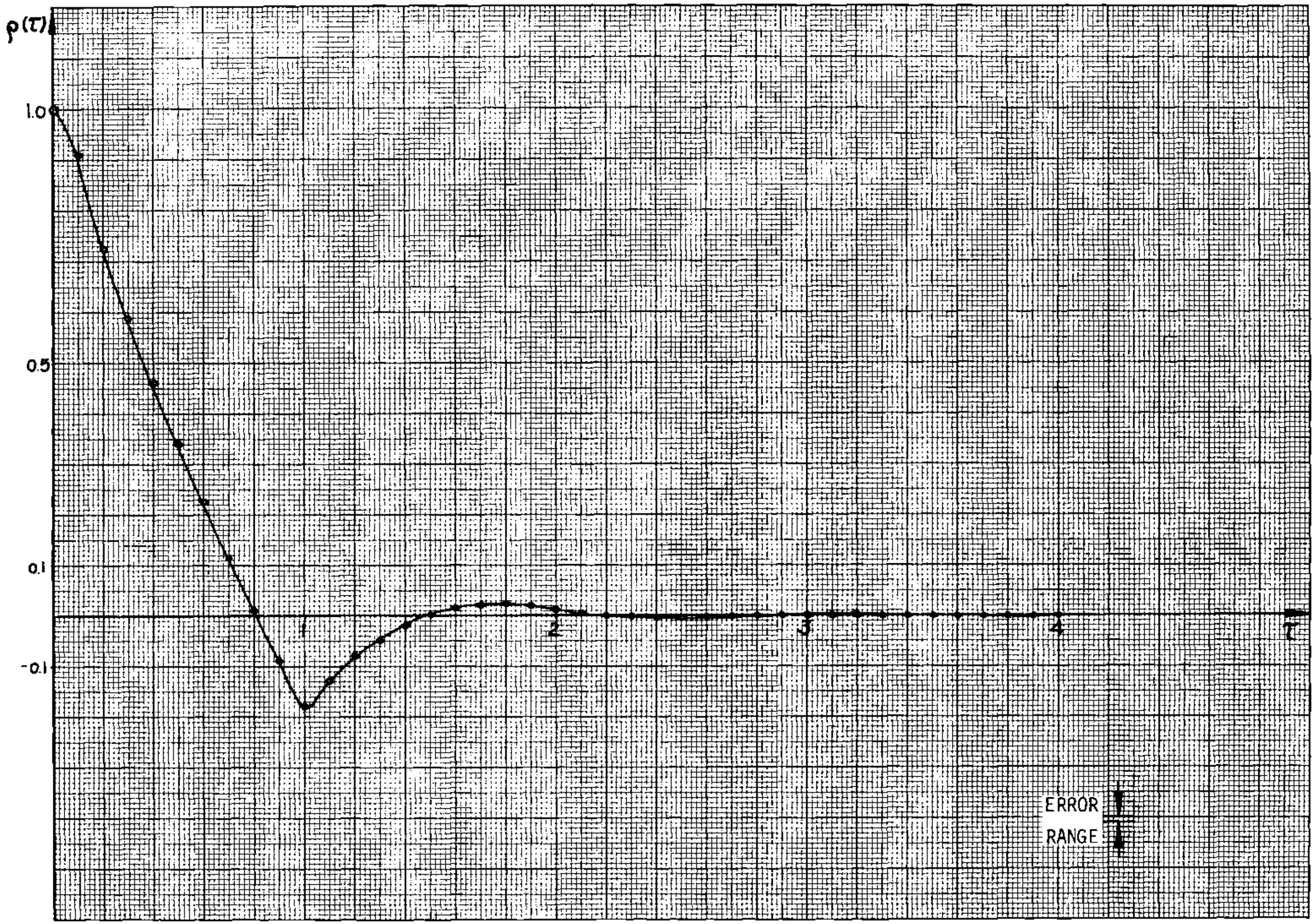

Figure 9. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ EXP(λ).
Parameters: $u = 1$, $\lambda = 1/2$.

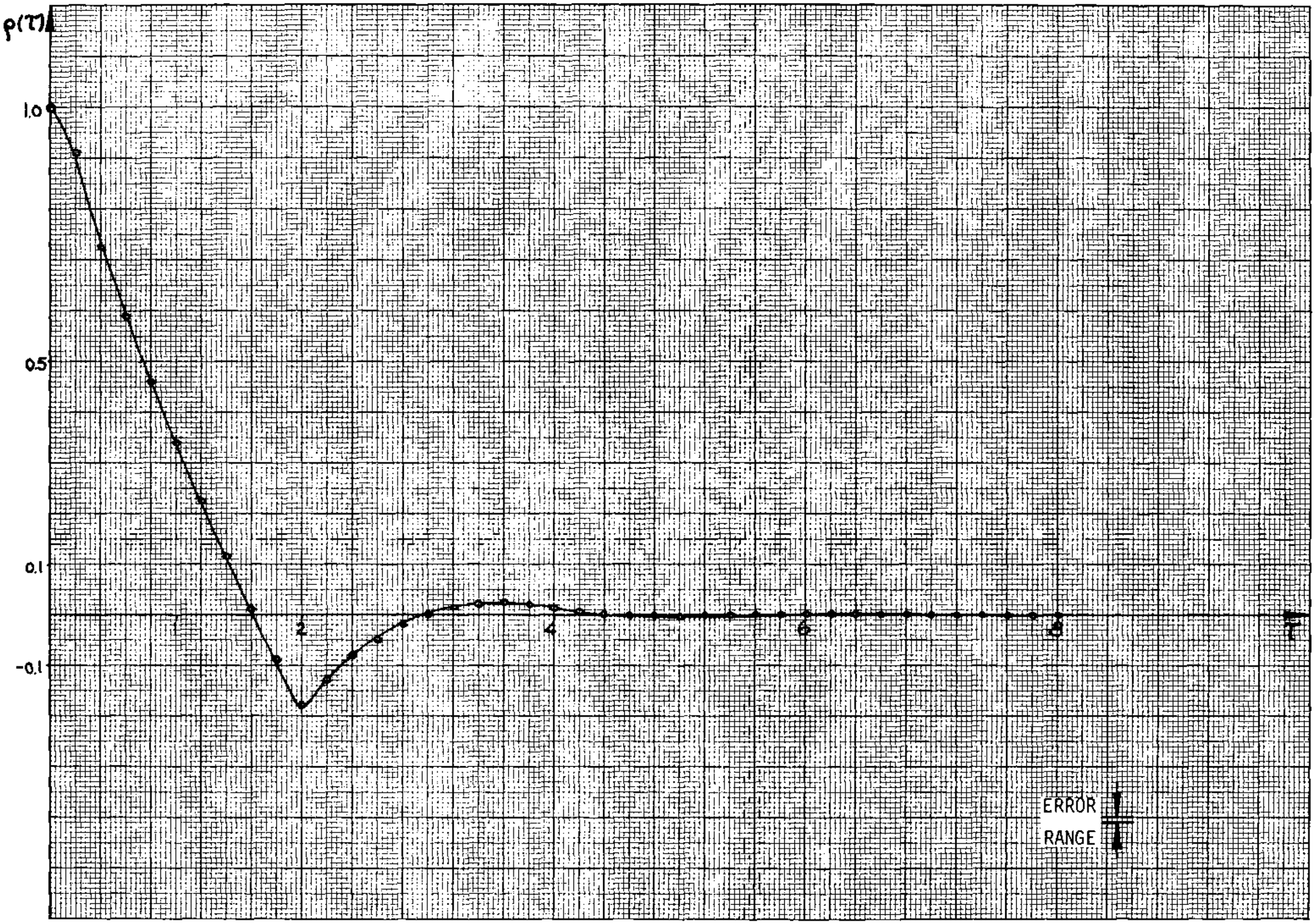

Figure 10. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\approx$ EXP(λ).
Parameters: $u = 2$, $\lambda = 1/4$.

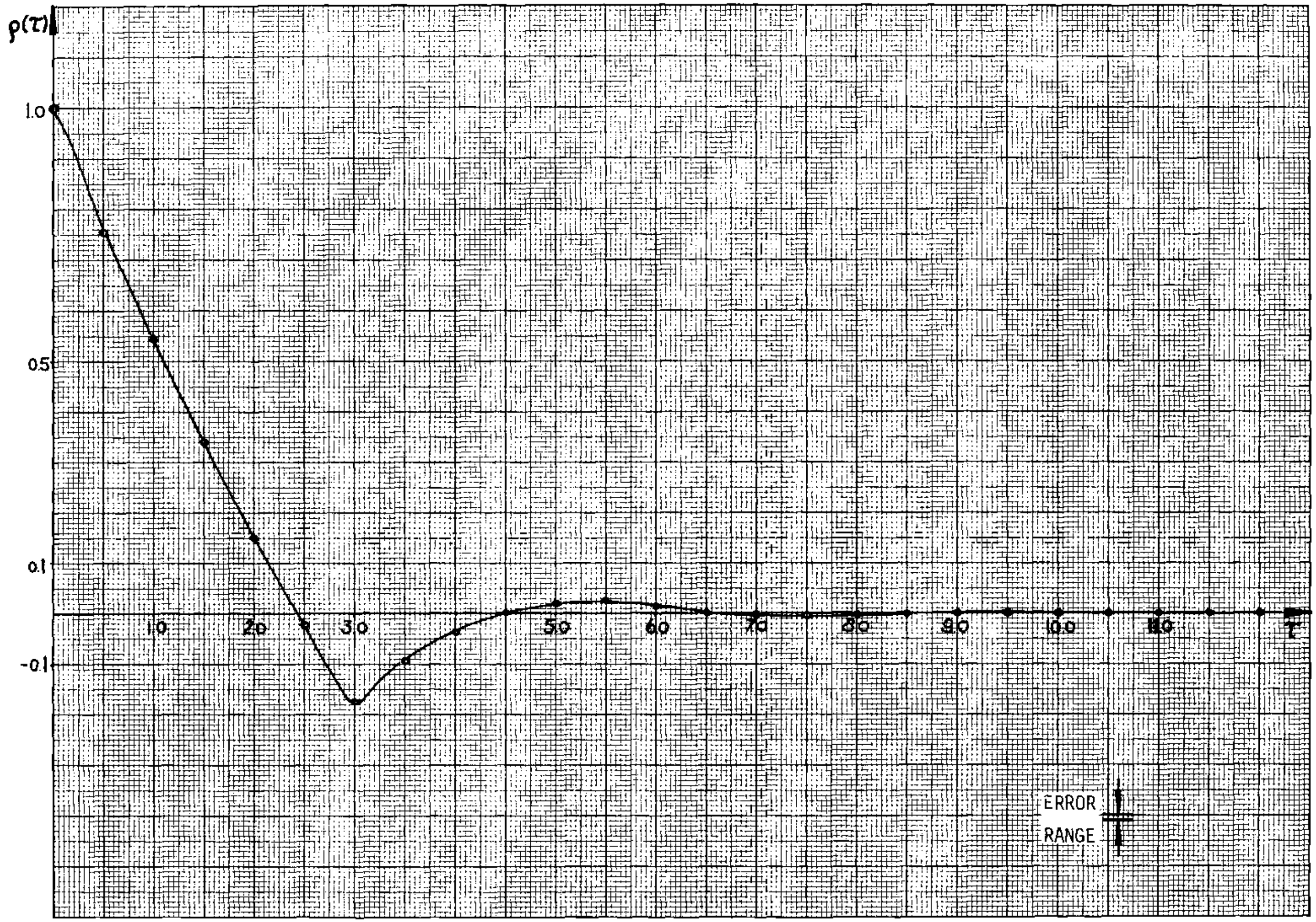

Figure 11. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq EXP(\lambda)$.
Parameters: $u = 3$, $\lambda = 1/6$.

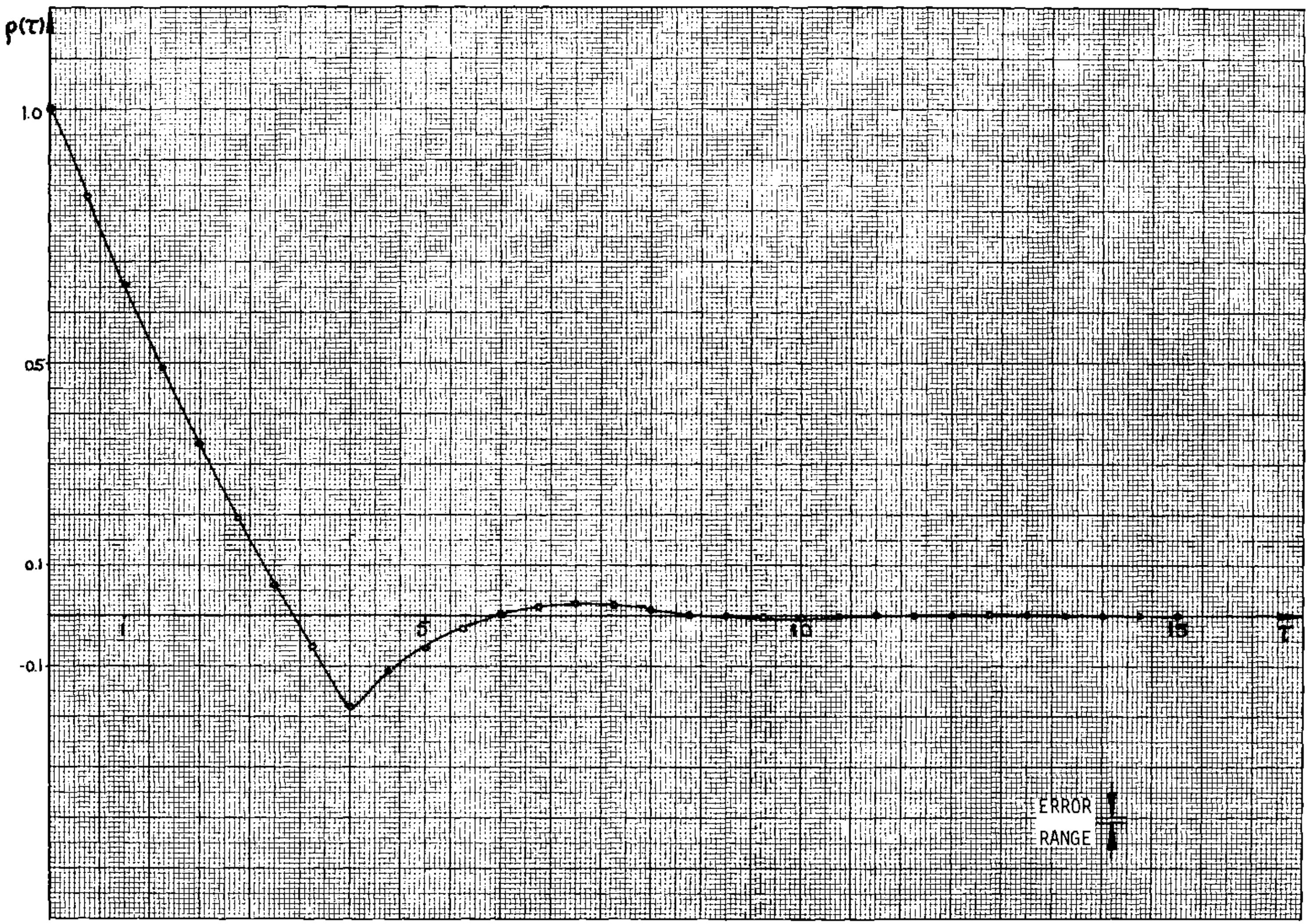

Figure 12. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ).
Parameters: $u = 4$, $\lambda = 1/8$.

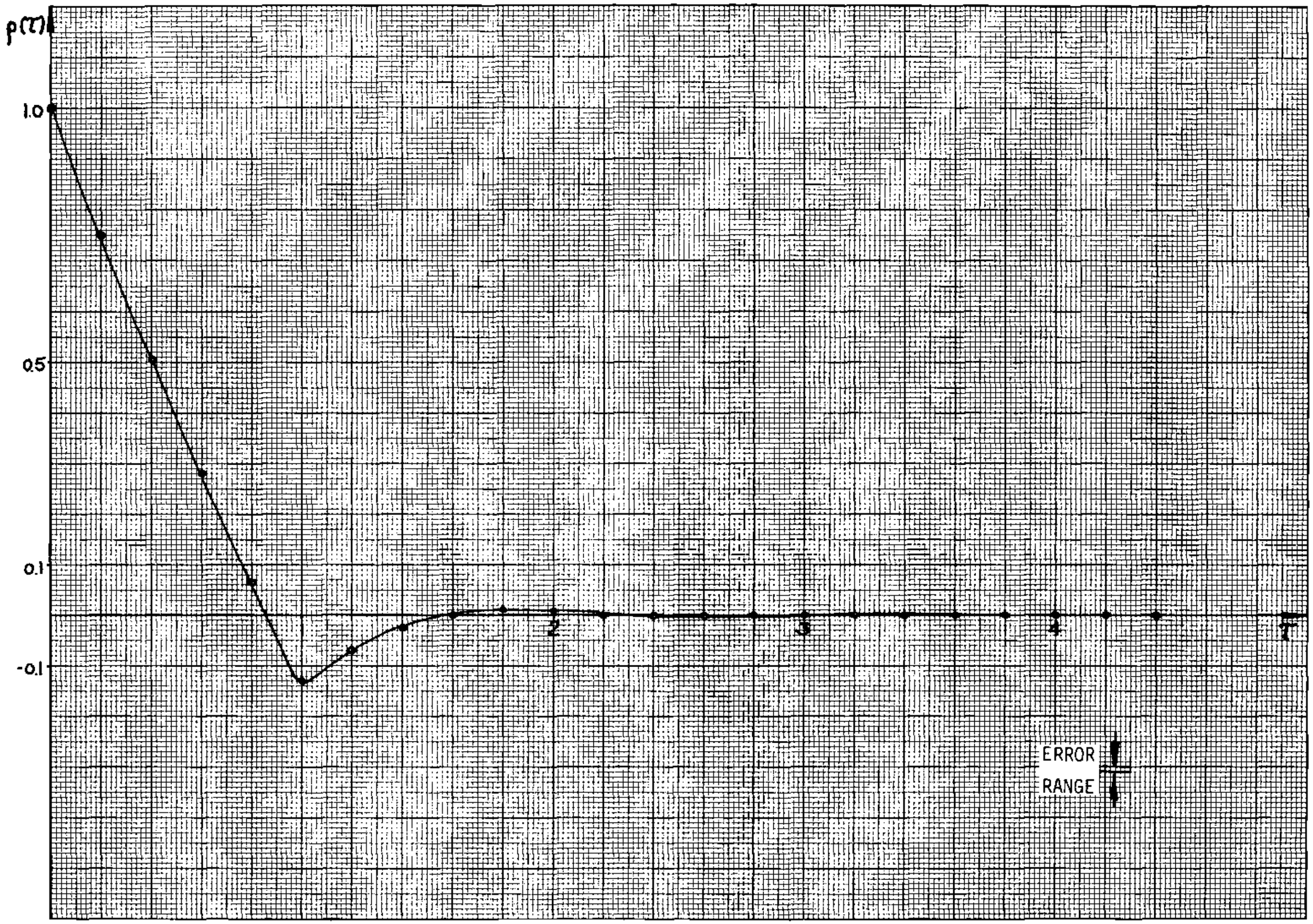

Figure 13. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ).
Parameters: $u = 1$, $\lambda = 1/3$.

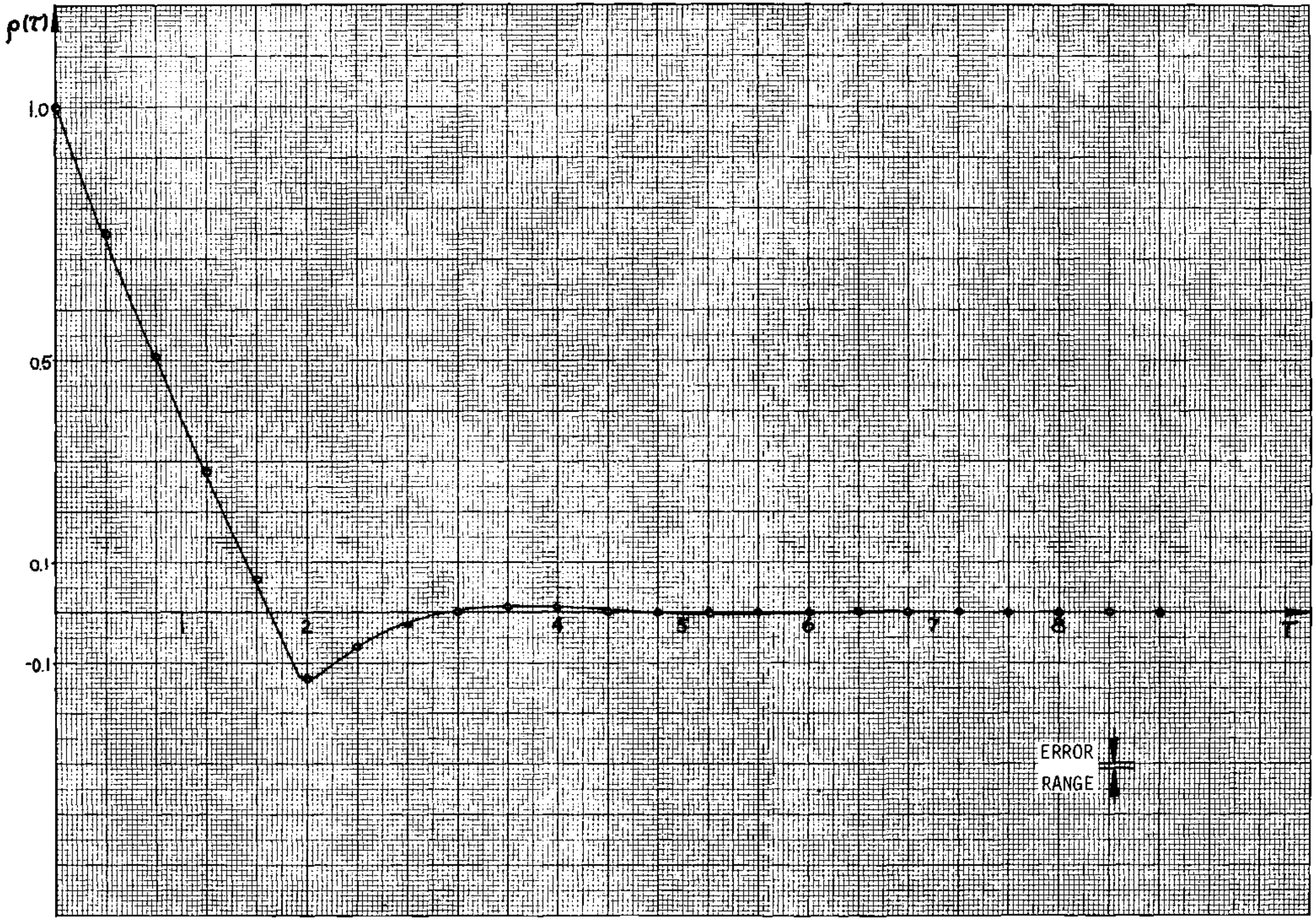

Figure 14. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ).
Parameters: $u = 2$, $\lambda = 1/6$.

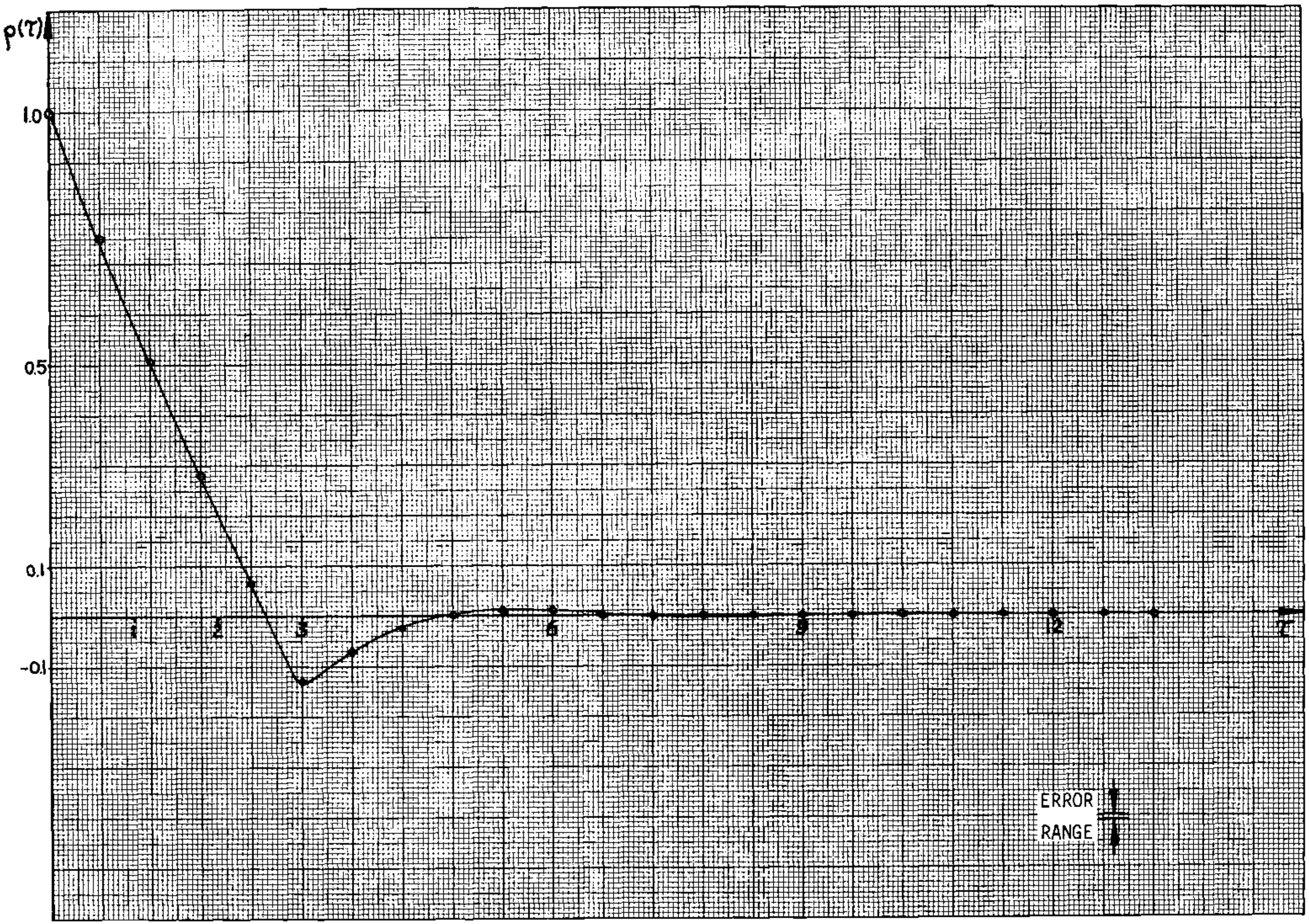

Figure 15. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ).
Parameters: $u = 3$, $\lambda = 1/9$.

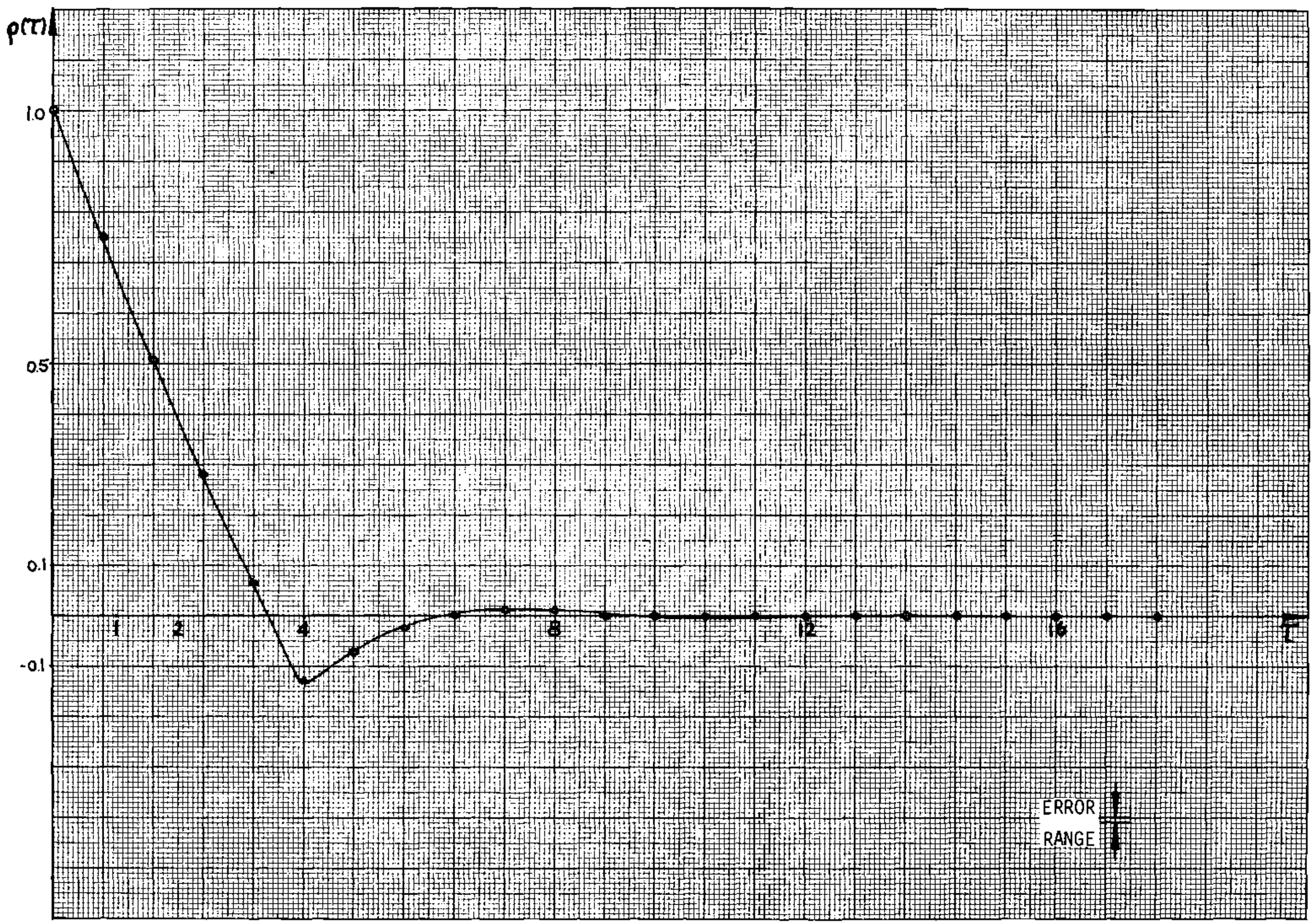

Figure 16. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ EXP(λ).
Parameters: $u = 4$, $\lambda = 1/12$.

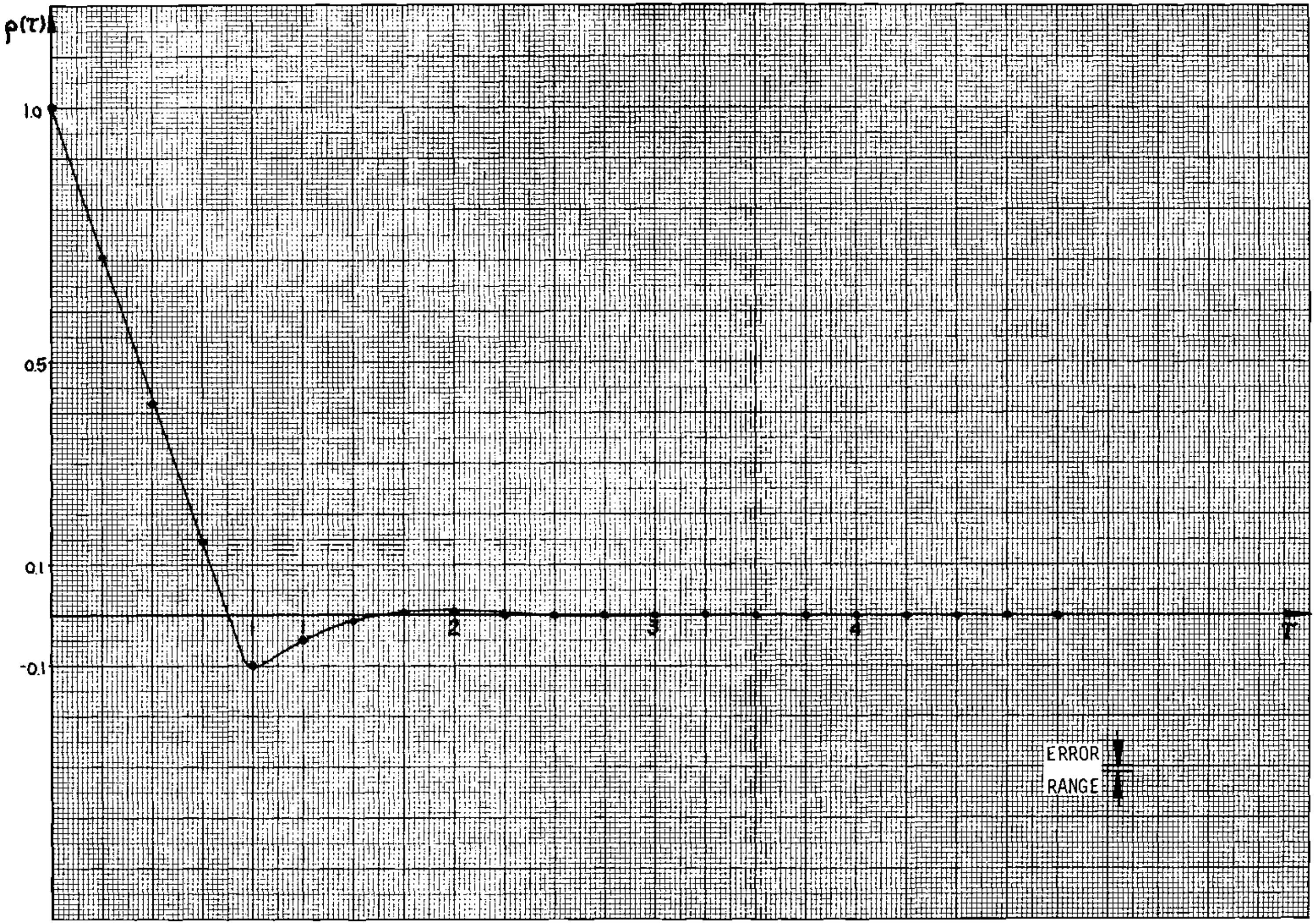

Figure 17. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ EXP(λ).
Parameters: $u = 1$, $\lambda = 1/4$.

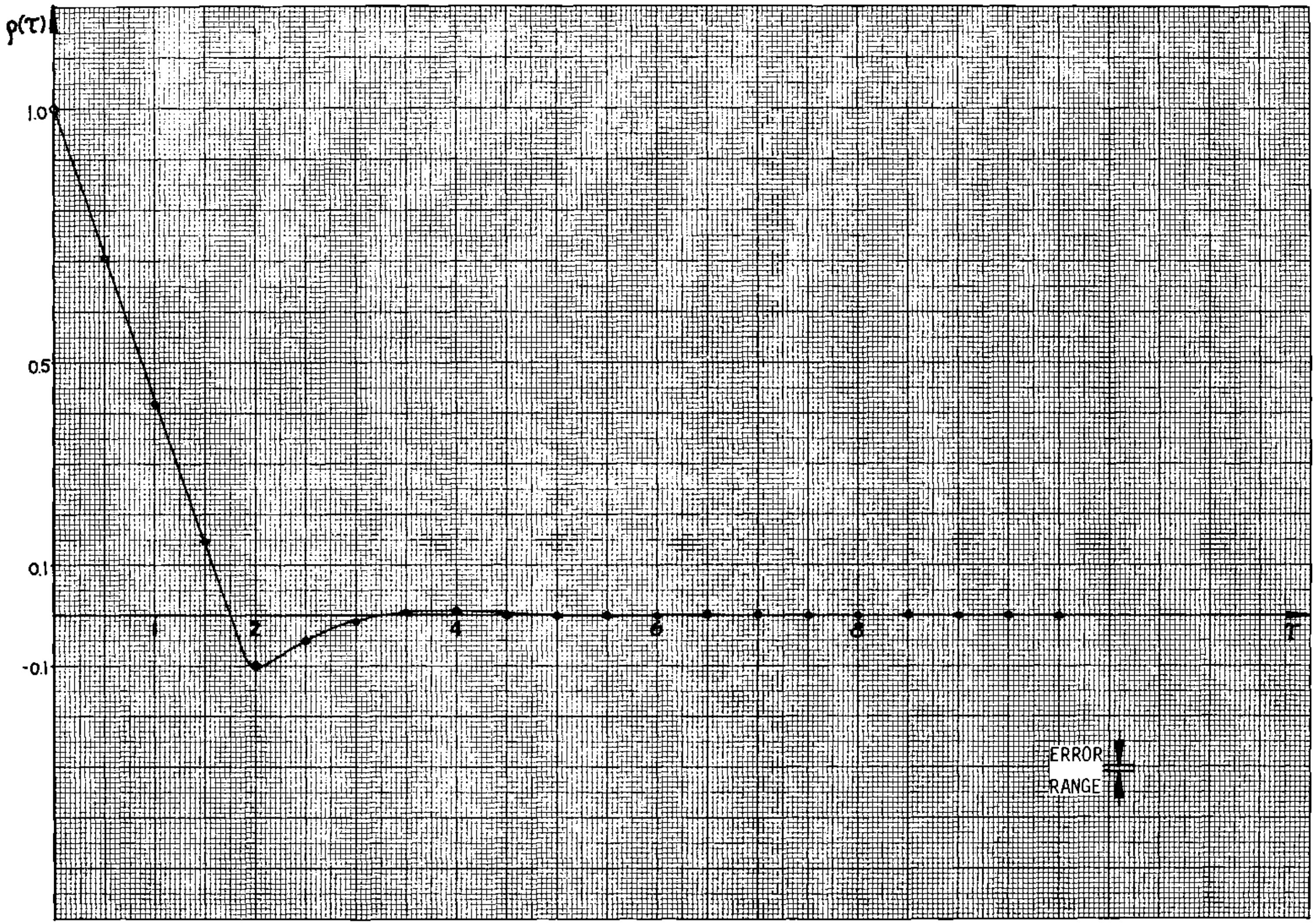

Figure 18. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ EXP(λ).
Parameters: $u = 2$, $\lambda = 1/8$.

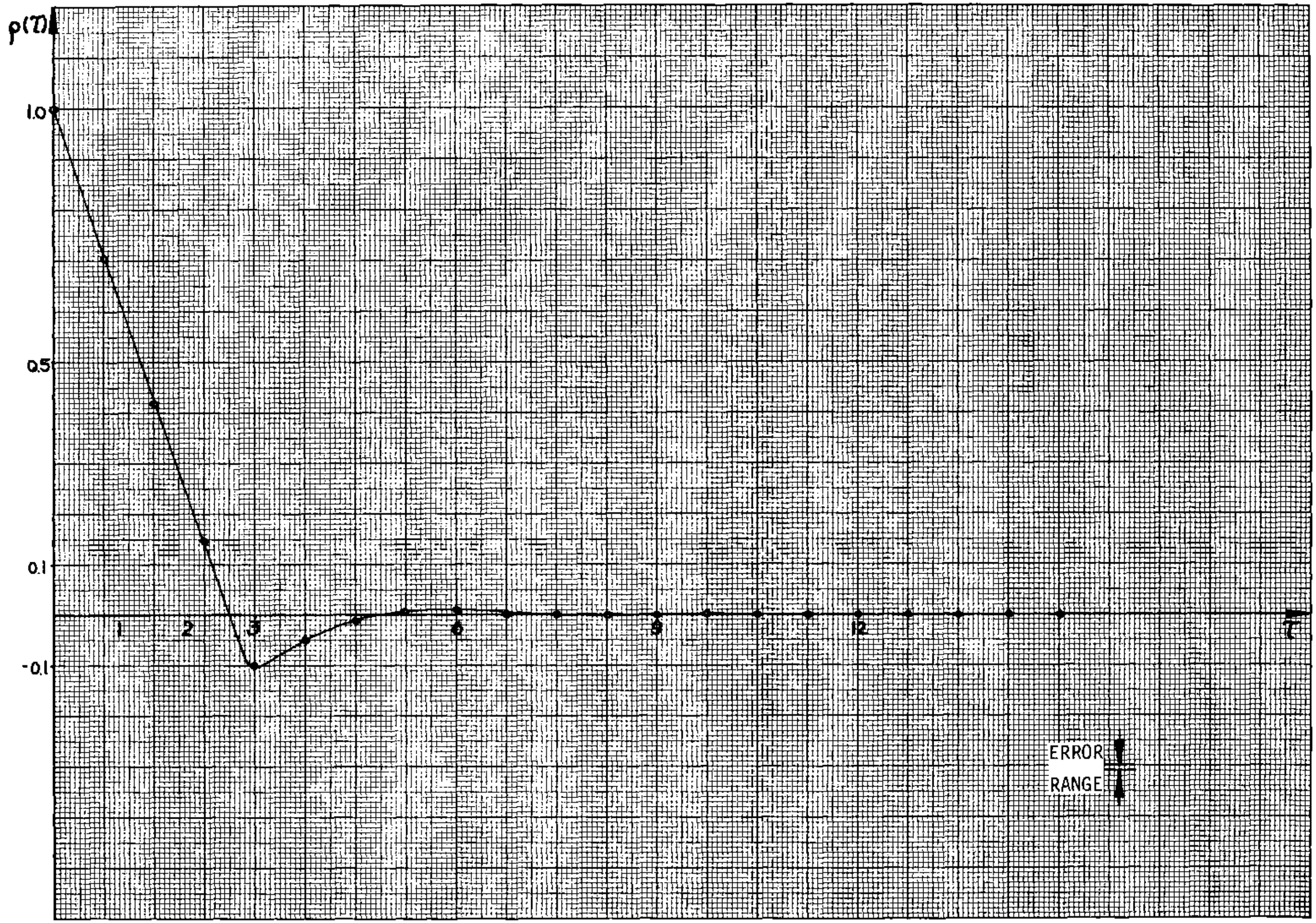

Figure 19. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≃ EXP(λ).
Parameters: $u = 3$, $\lambda = 1/12$.

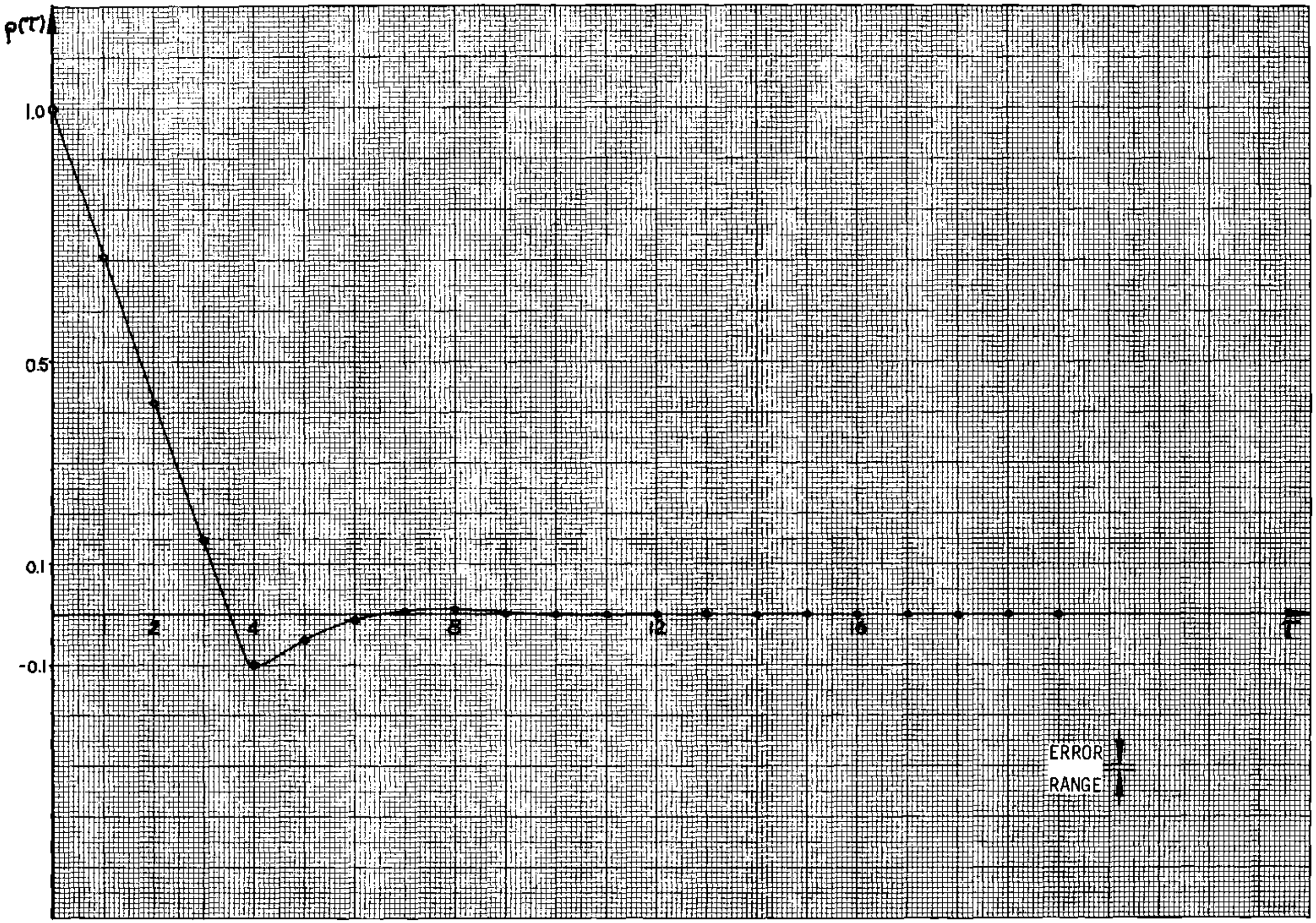

Figure 20. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq EXP(\lambda)$.
Parameters: $u = 4$, $\lambda = 1/16$.

For the determination of the upper limit UL, expression (5.9) has to be approximated. Given (3.19), (5.9) becomes

$$\frac{1}{\pi} \int_{UL}^{\infty} S(\omega)\, d\omega \le \frac{8t^2}{\pi(2u+t)} \cdot \int_{UL}^{\infty} \frac{d\omega}{(\omega t+2)^2}$$

$$\le \frac{8t}{\pi(2u+t)\,[(UL)t + 2]} \le a_\omega = 4 \cdot 10^{-3}.$$

Hence

$$UL \ge \frac{2000}{\pi(2u+t)} - \frac{2}{t}$$

or

$$UL \ge \frac{2000}{\pi(2u+t)}\ . \qquad (5.19)$$

With regard to the numerical integration leading to the determination of R(0), it is to be observed that the spectrum oscillates heavily in the range up to the first absolute minimum. The frequency of these oscillations, however, is not so high that an interval length of $h = 0.1$ cannot sufficiently seize the amplitudes. For the evaluation of $R(\tau)$ an interval length of $h = \pi/12$ was maintained throughout for all chosen values of τ. $R(\tau)$ was determined for τ starting with $\tau/10$ or less and proceeding in steps of $\tau/10$ until some value τ_{max}. τ_{max} was reached whenever either the amplitude of $\rho(\tau)$ dropped below the error range or the number of oscillations completed gave sufficient information for the analysis of $\rho(\tau)$.

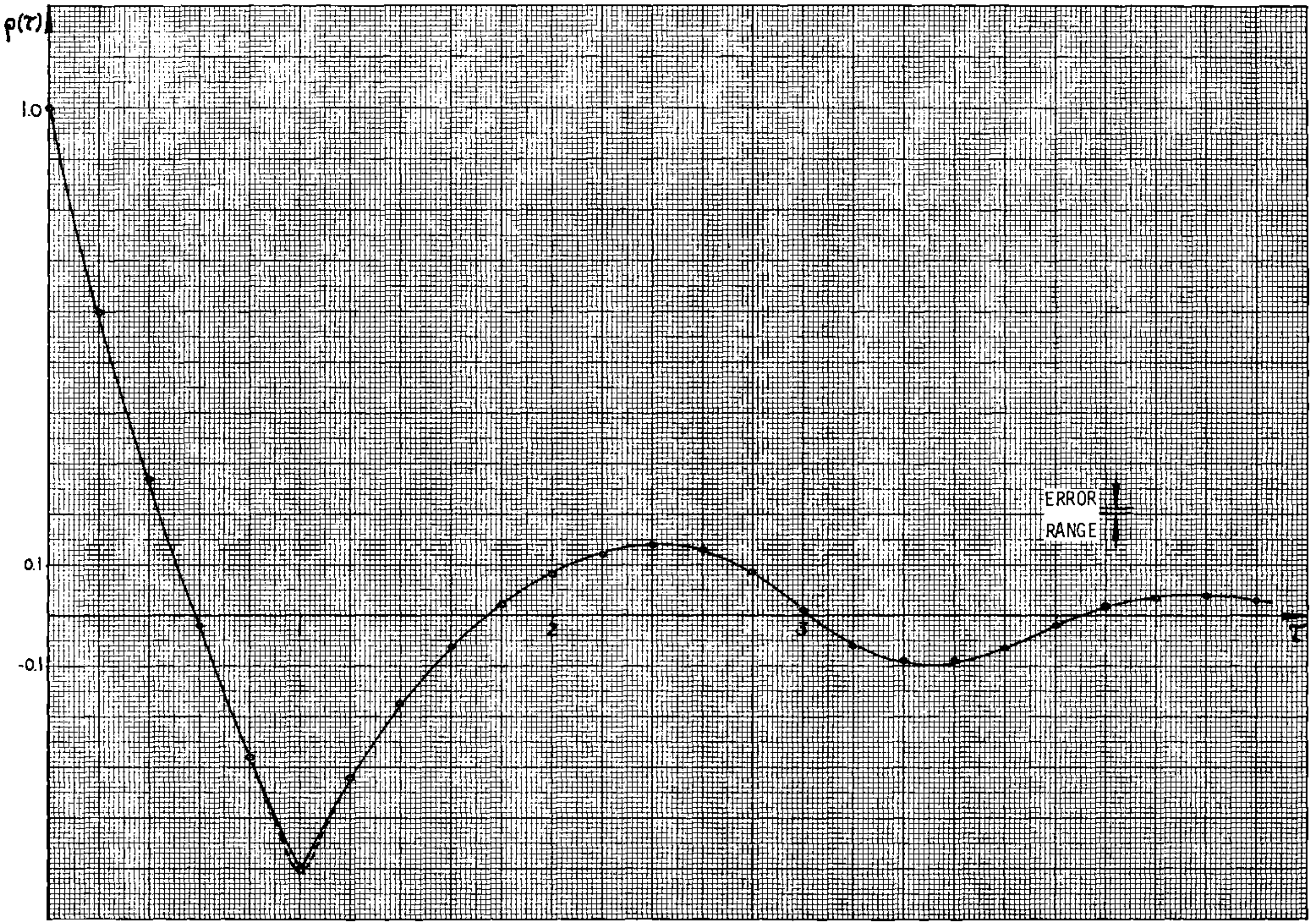

Figure 21. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ U(τ).
Parameters: $u = 1$, $\tau = 2$.

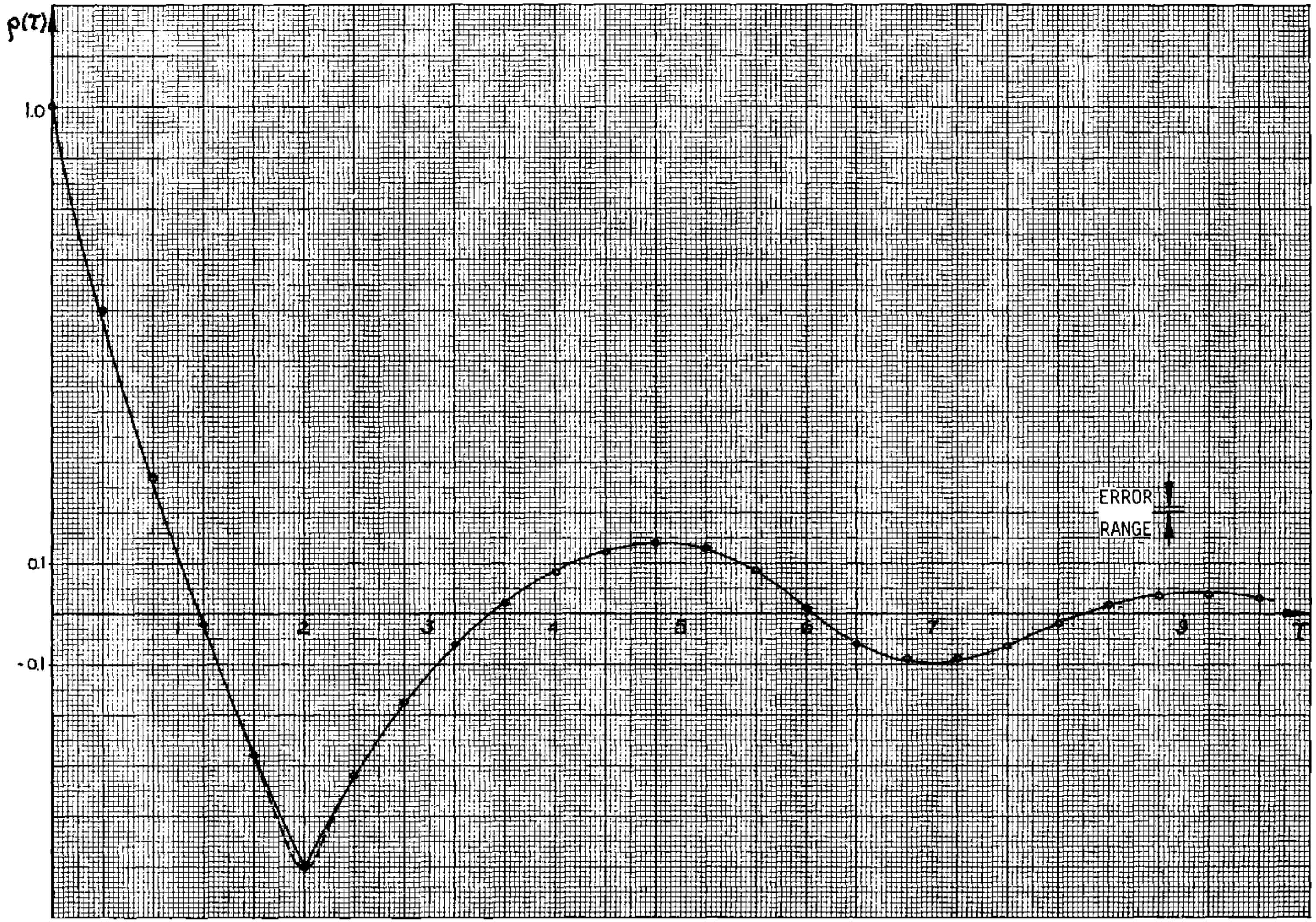

Figure 22. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ U(τ).
Parameters: u = 2, τ = 4.

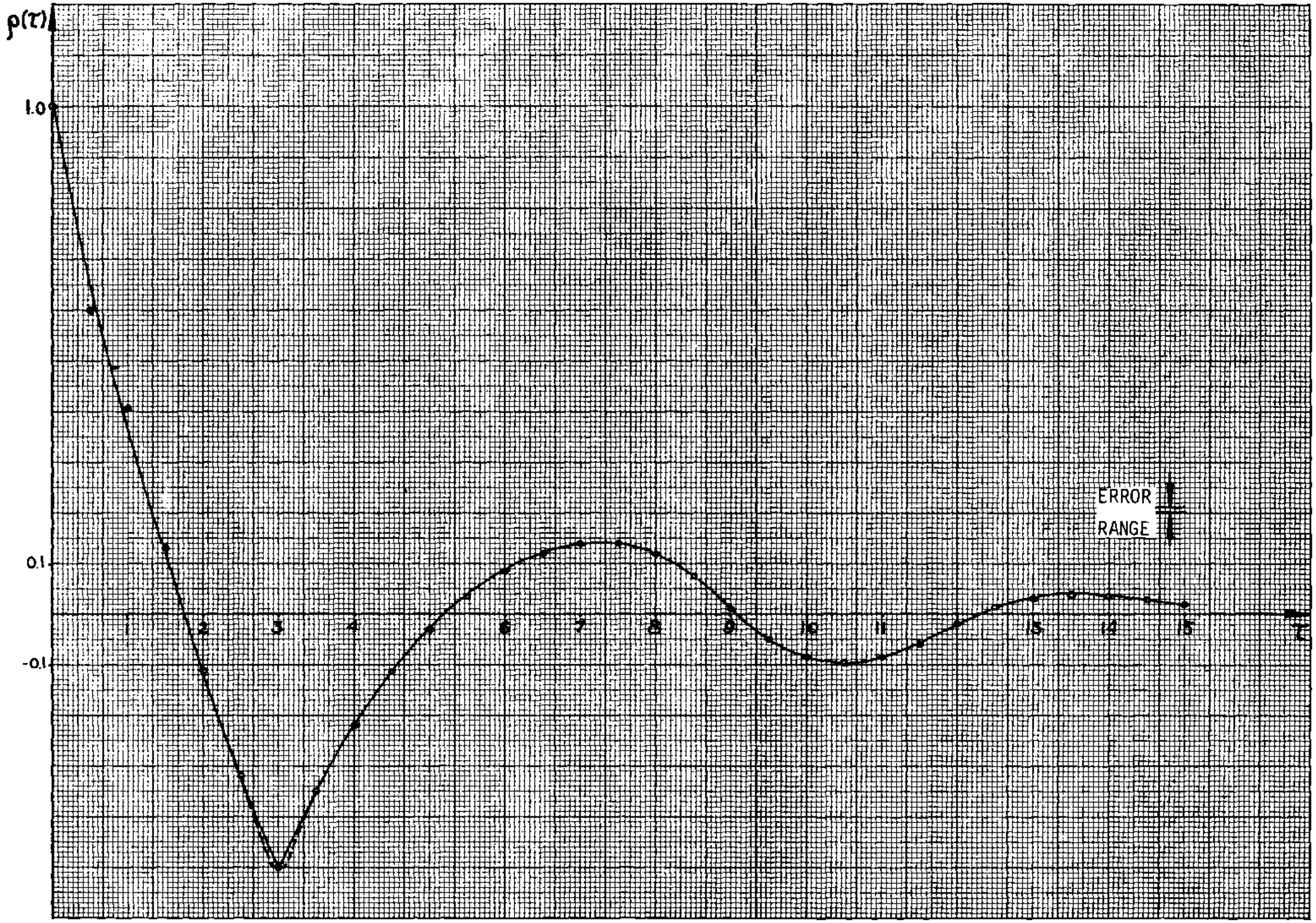

Figure 23. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ U(τ).
Parameters: u = 3, τ = 6.

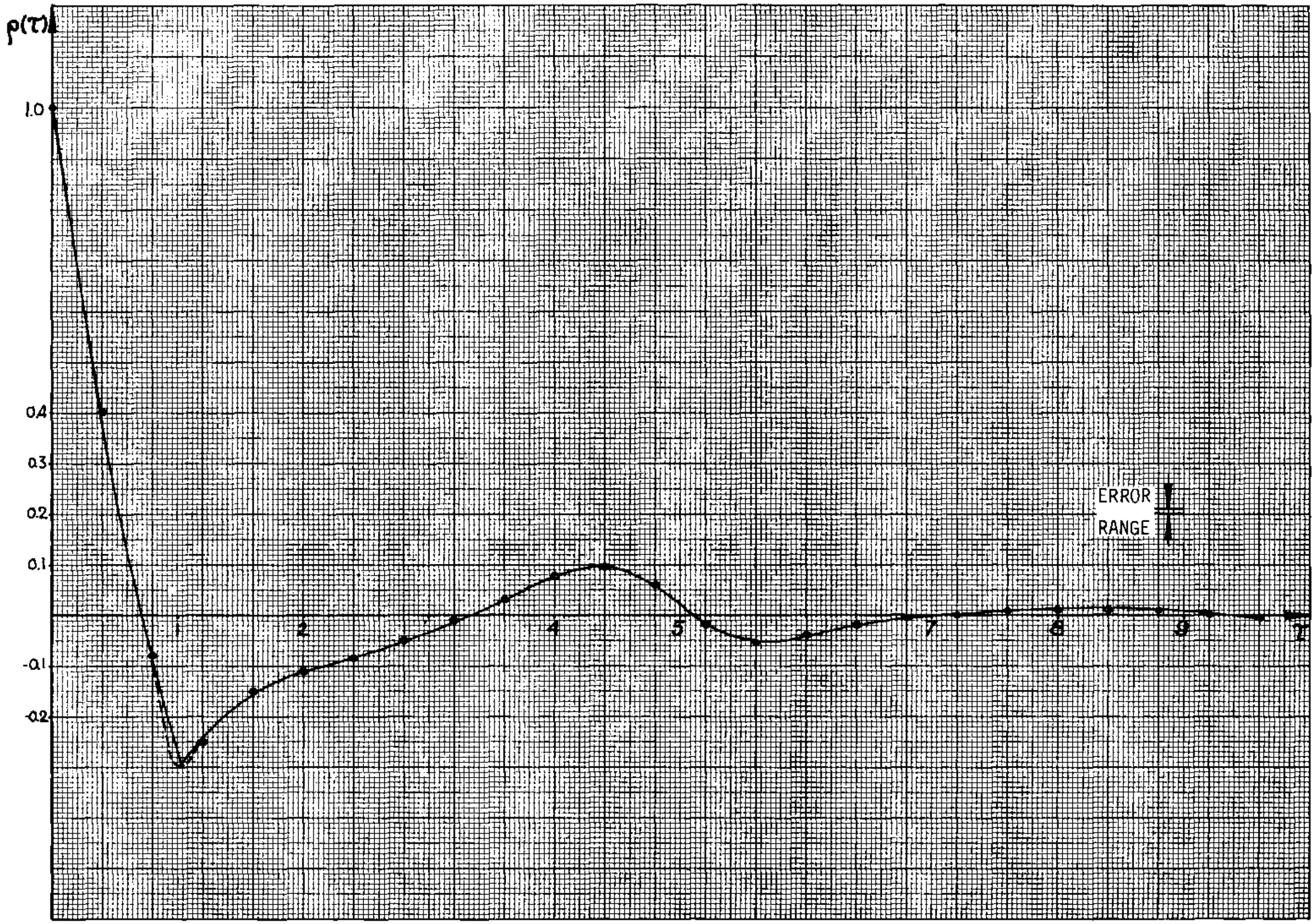

Figure 24. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 1$, $\tau = 4$.

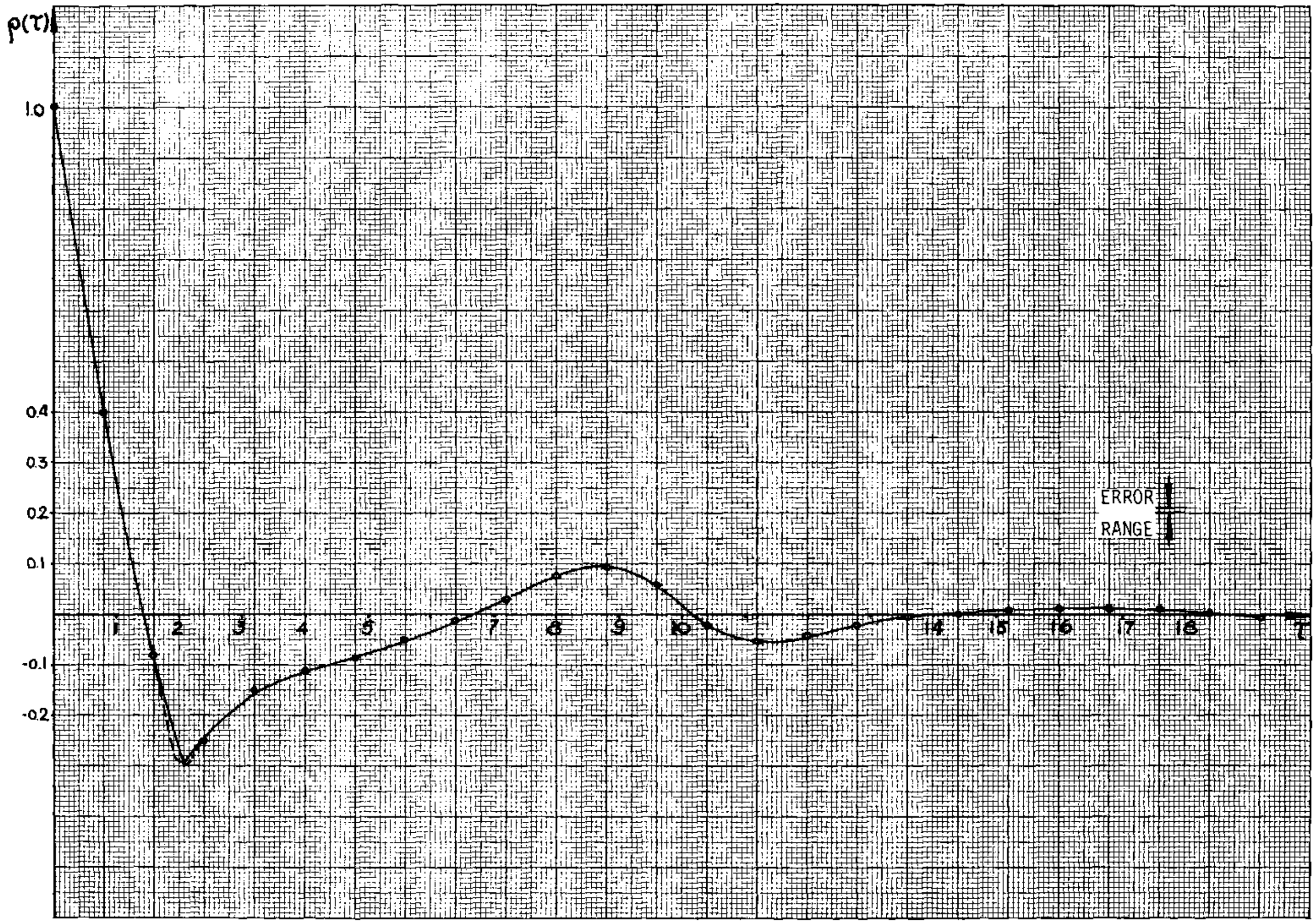

Figure 25. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ U(τ).
Parameters: $u = 2$, $\tau = 8$.

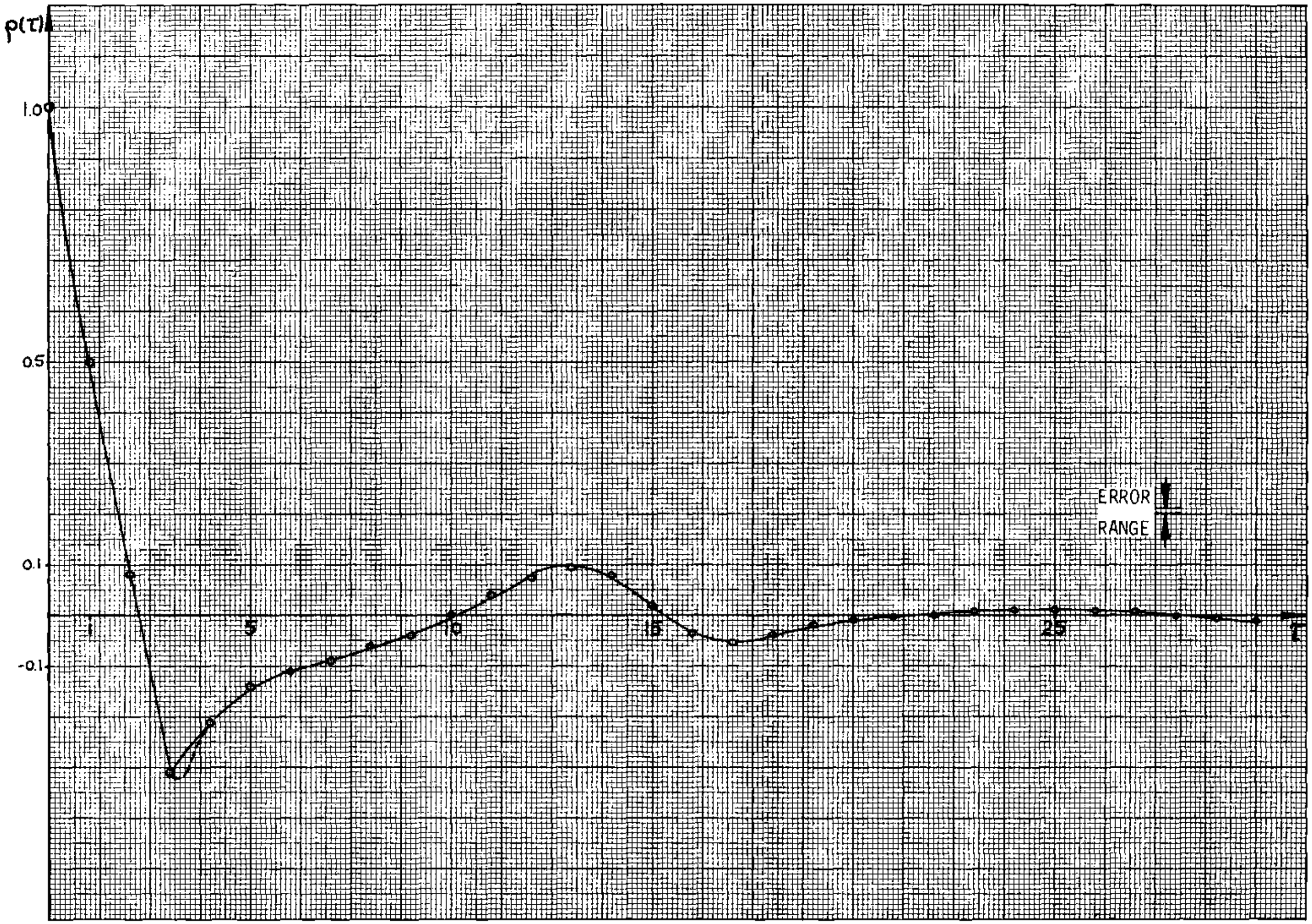

Figure 26. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ U(τ).
Parameters: $u = 3$, $\tau = 12$.

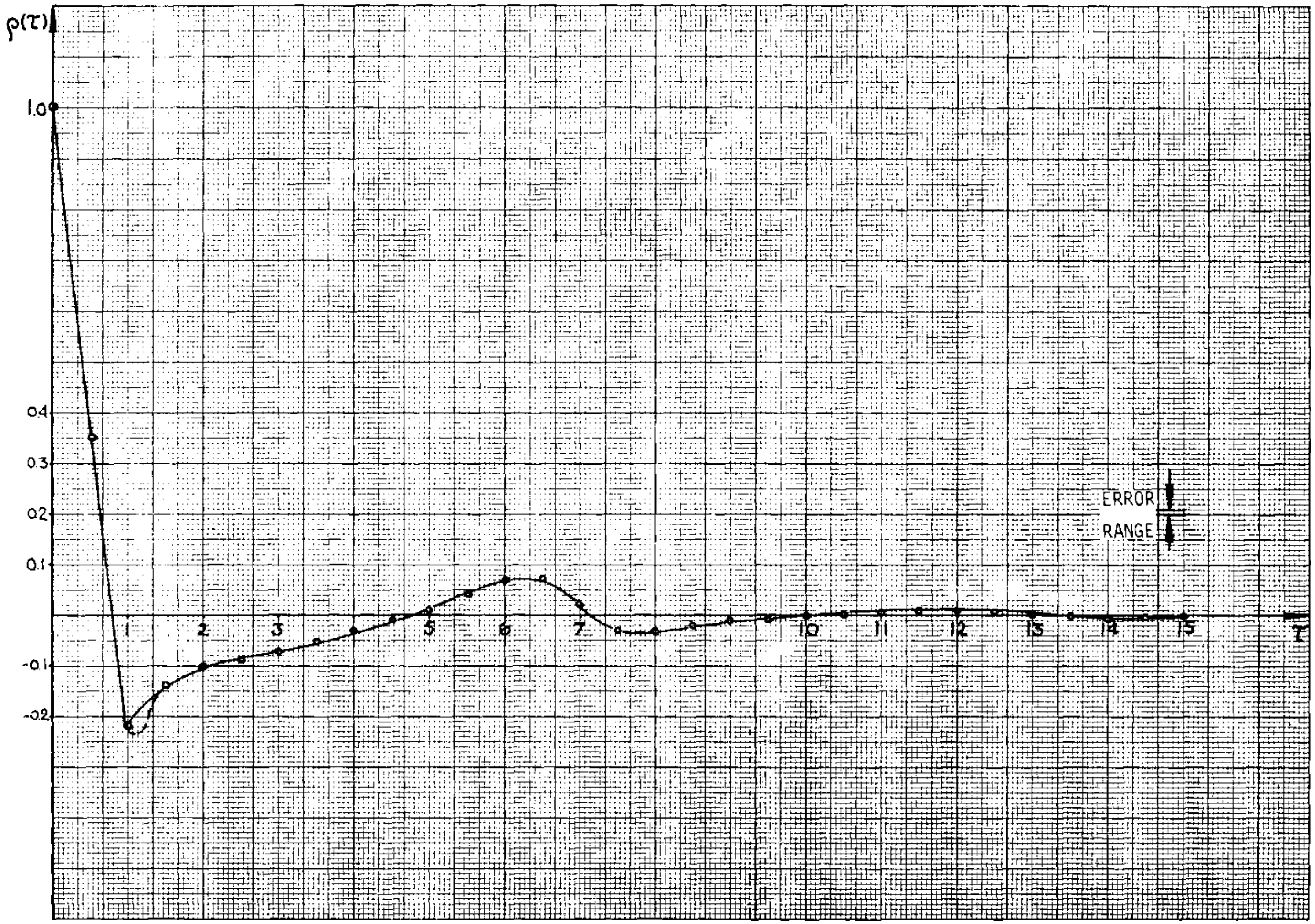

Figure 27. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 1$, $\tau = 6$.

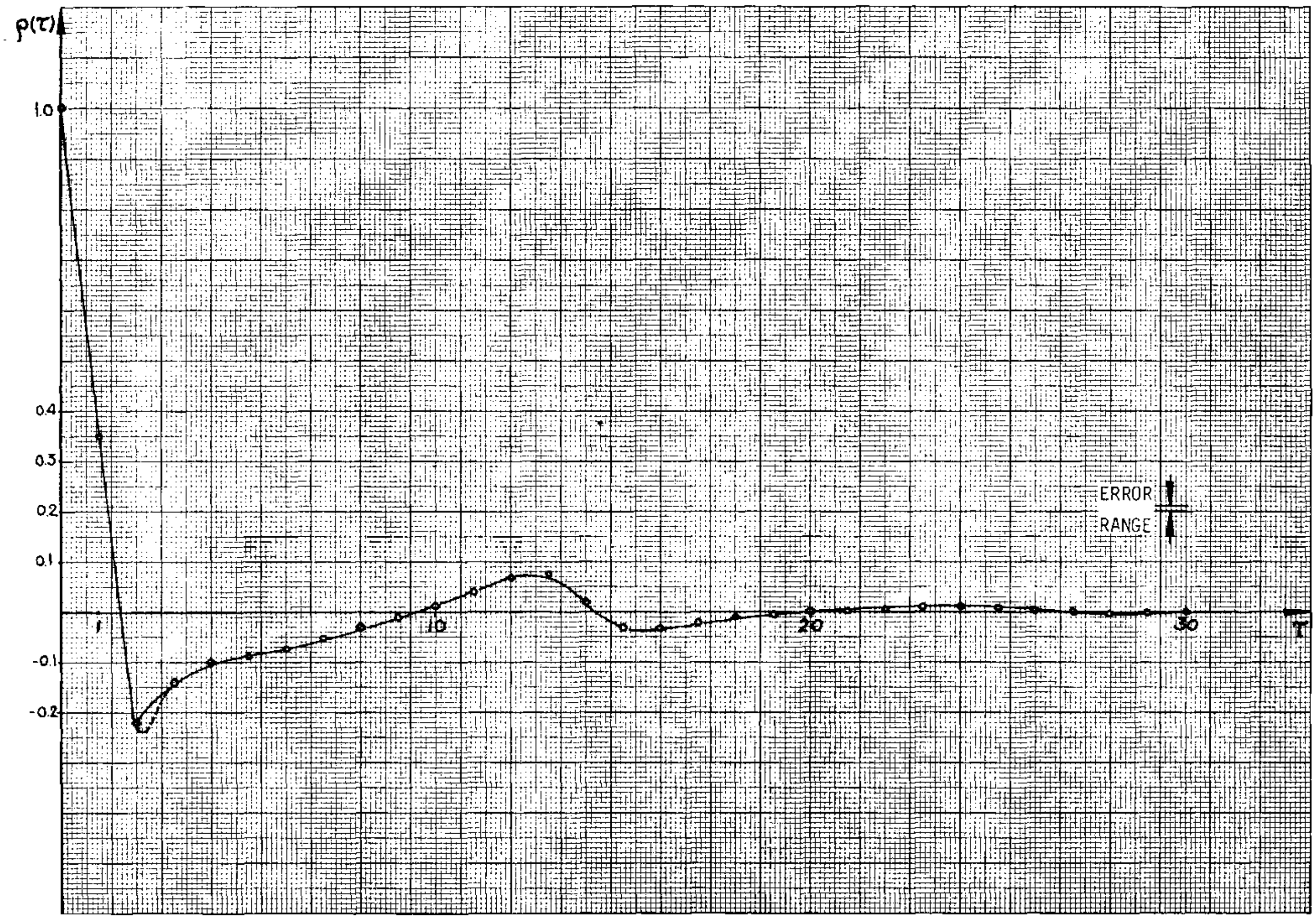

Figure 28. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 2$, $\tau = 12$.

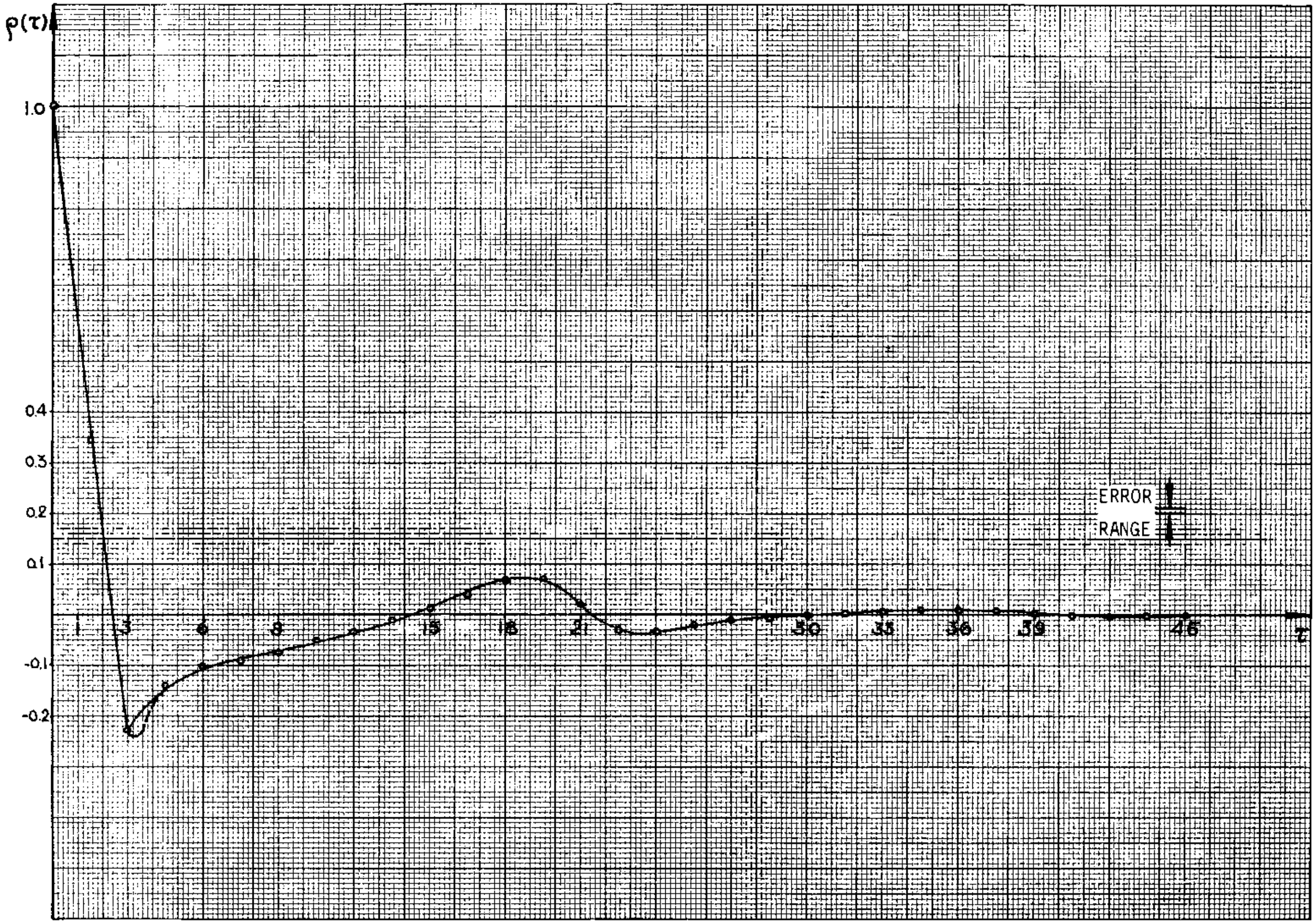

Figure 29. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 3$, $\tau = 18$.

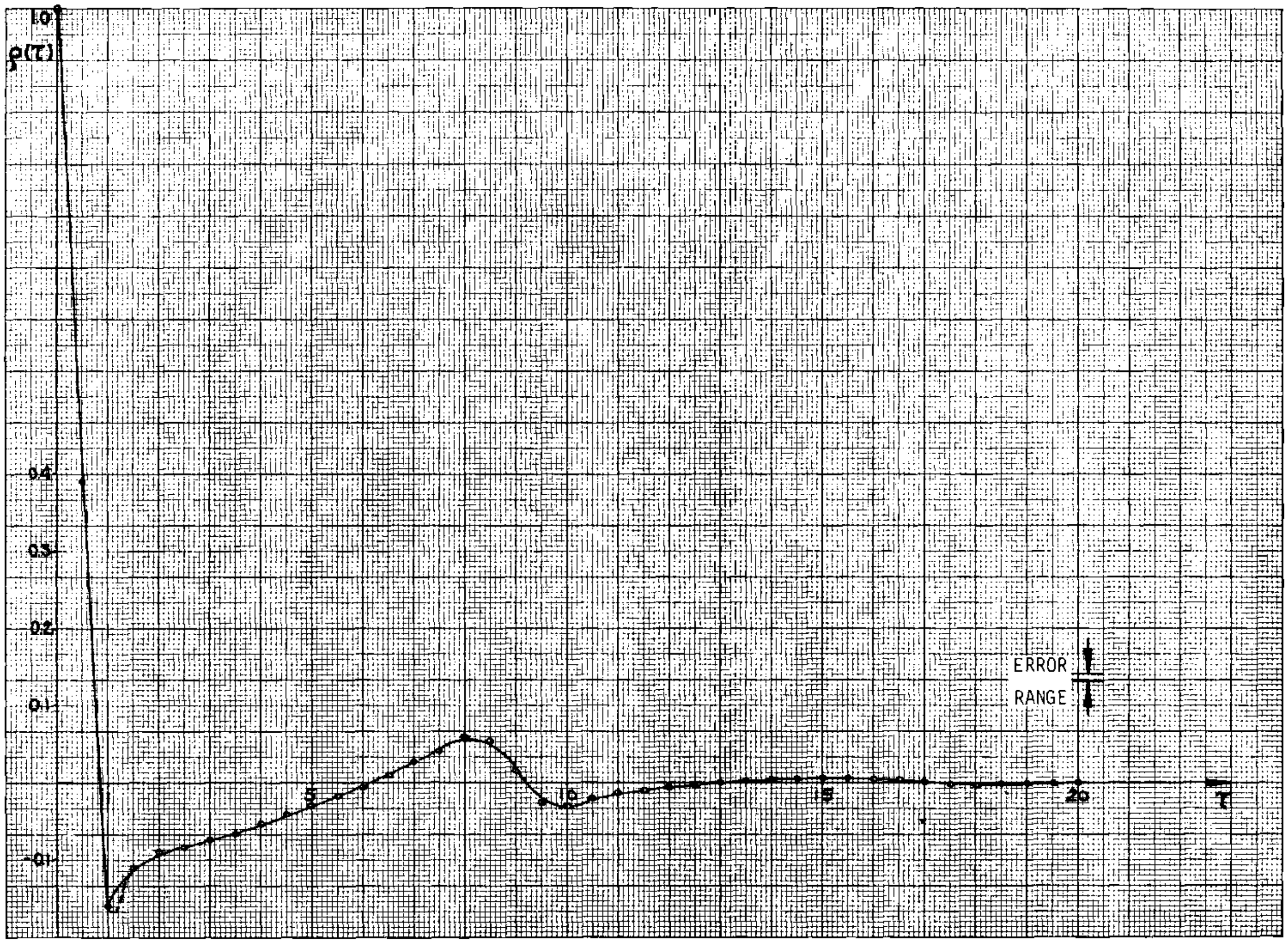

Figure 30. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≃ U(τ).
Parameters: $u = 1$, $\tau = 8$.

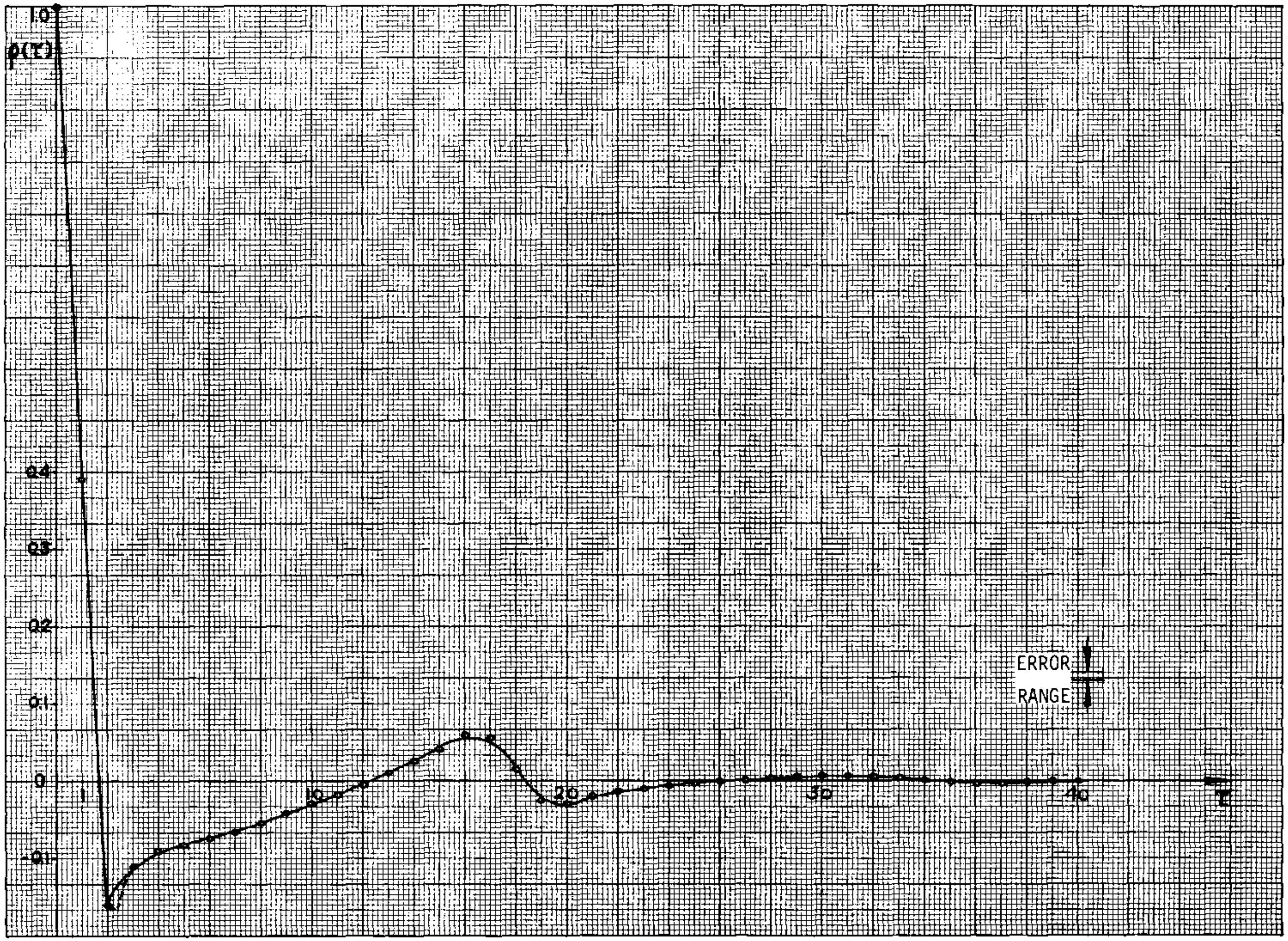

Figure 31. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ $U(\tau)$.
Parameters: $u = 2$, $\tau = 16$.

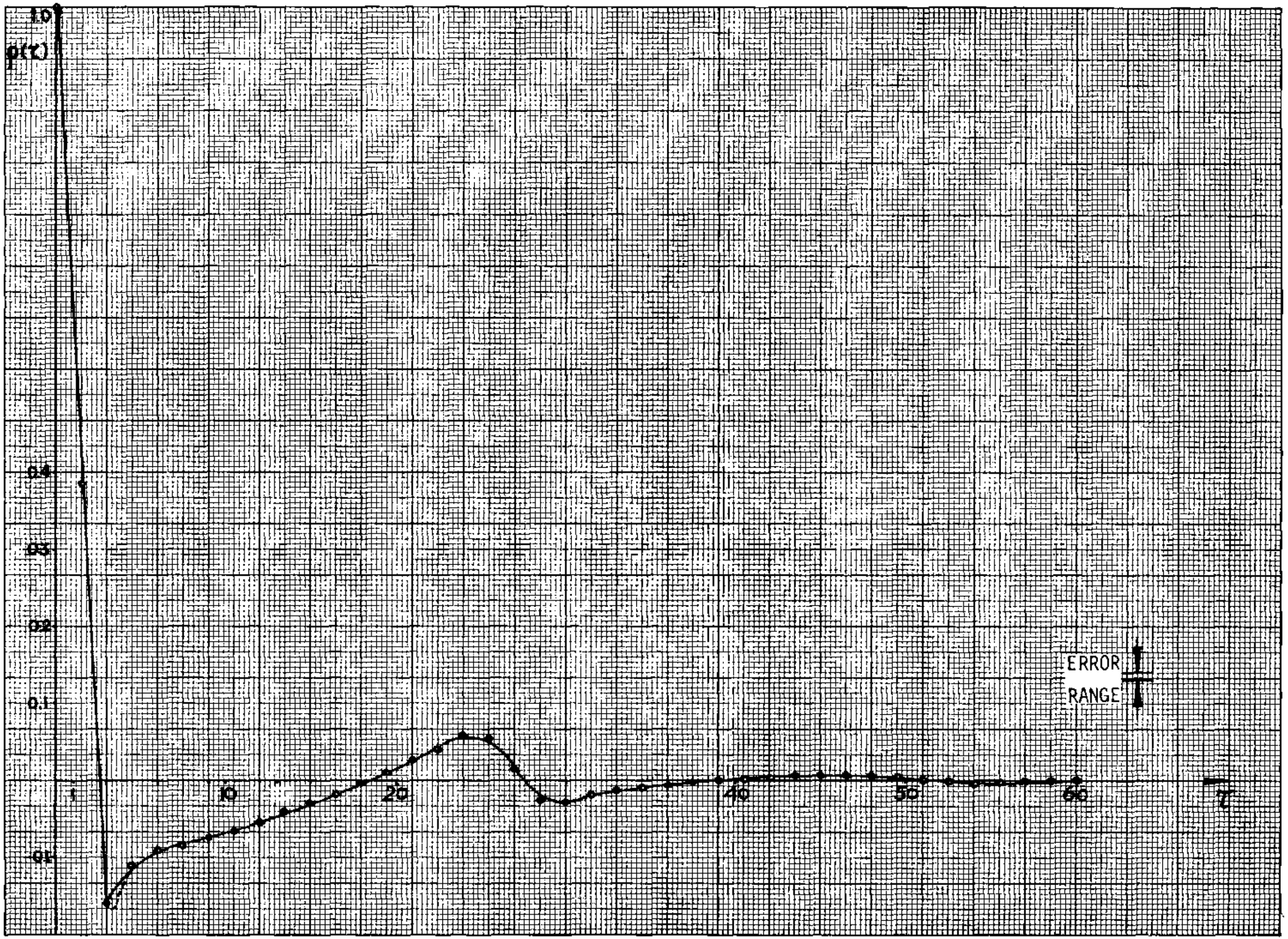

Figure 32. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \approx U(\tau)$.
Parameters: $u = 3$, $\tau = 24$.

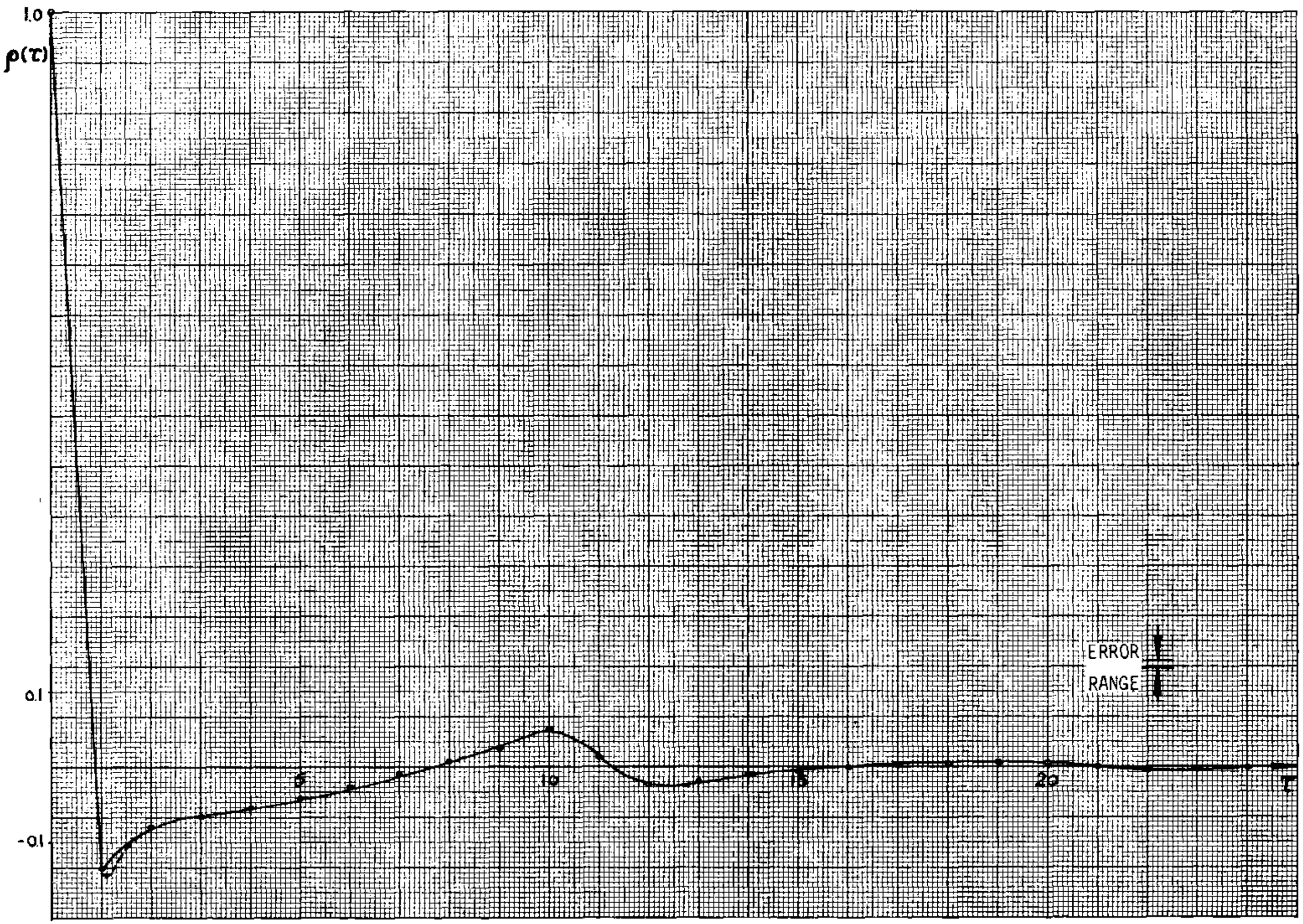

Figure 33. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≃ U(τ).
Parameters: u = 1, τ = 10.

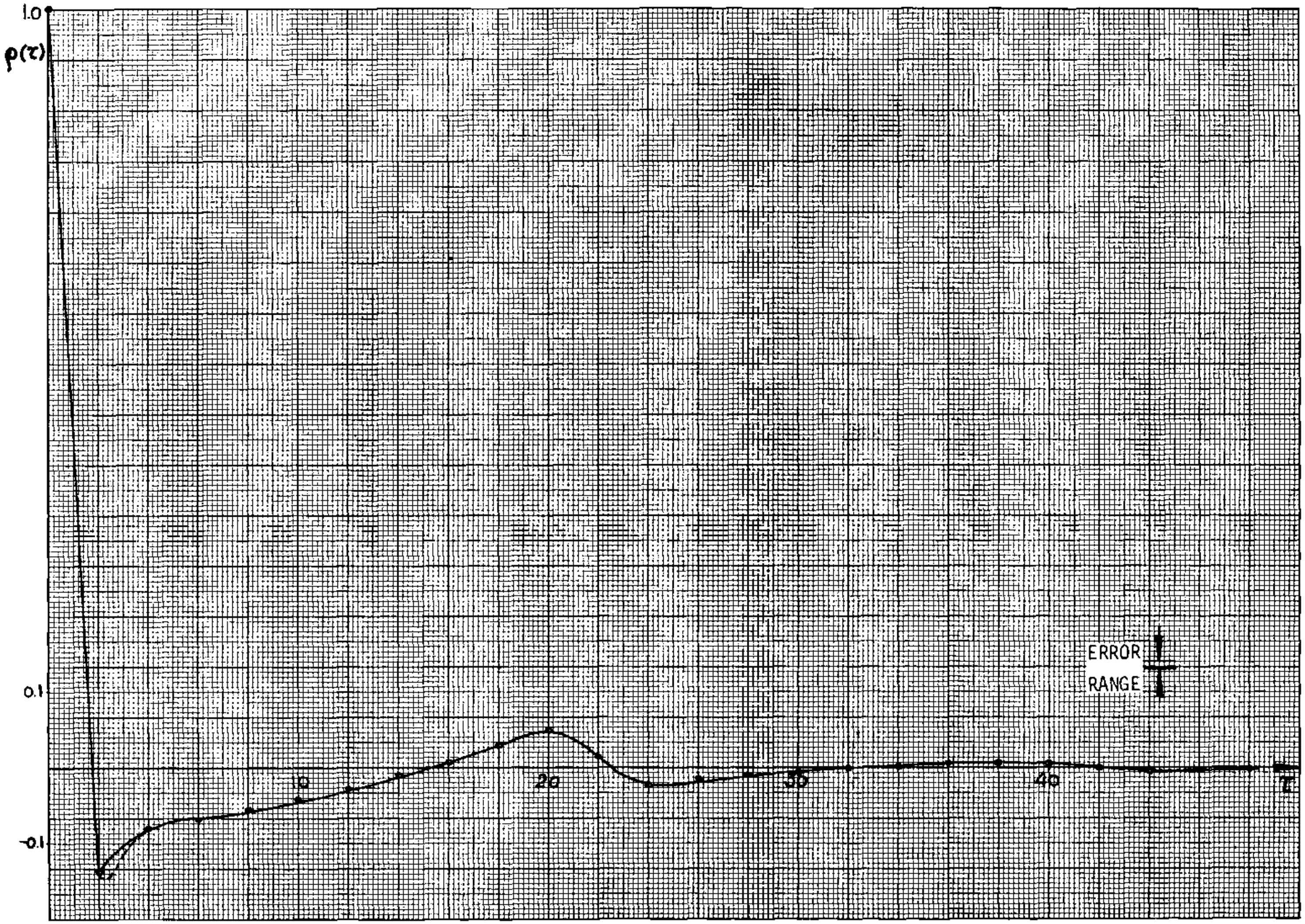

Figure 34. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 2$, $\tau = 20$.

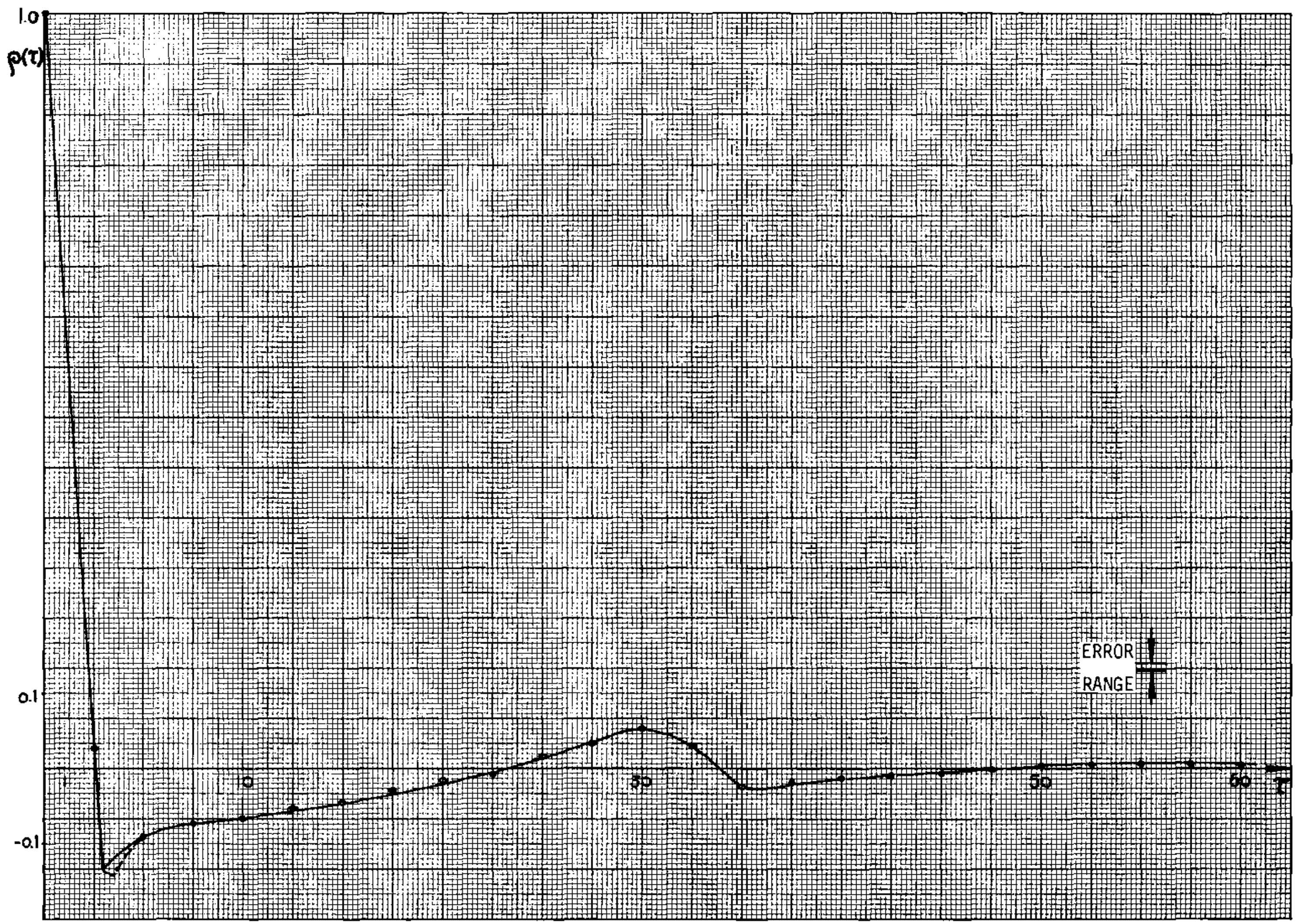

Figure 35. Autocorrelation Function $\rho(\tau)$ for U = Constant and V $\simeq$ U(τ).
Parameters: $u = 3$, $\tau = 30$.

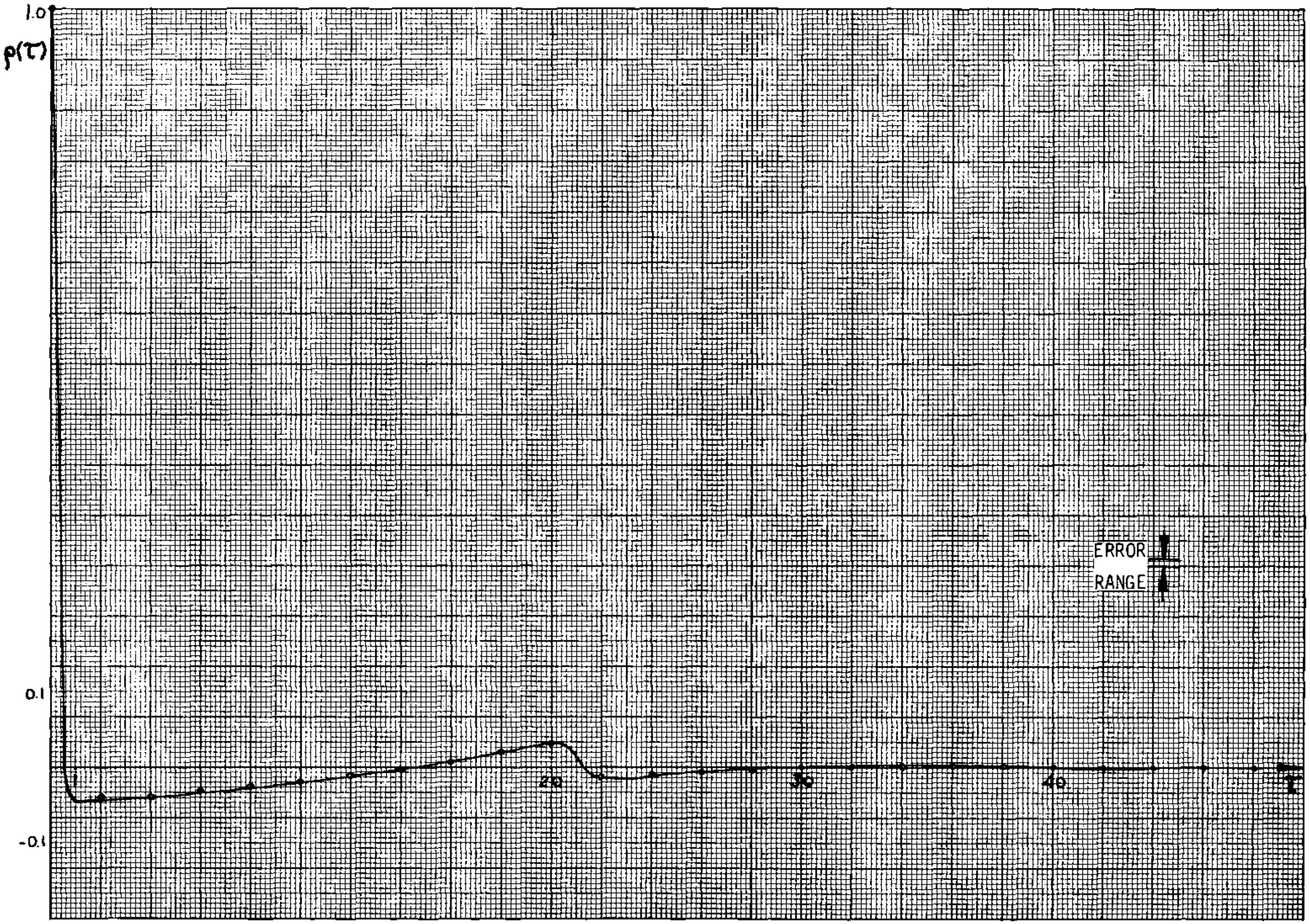

Figure 36. Autocorrelation Function $\rho(\tau)$ for U = Constant and V ≈ U(τ).
Parameters: u = 1, τ = 20.

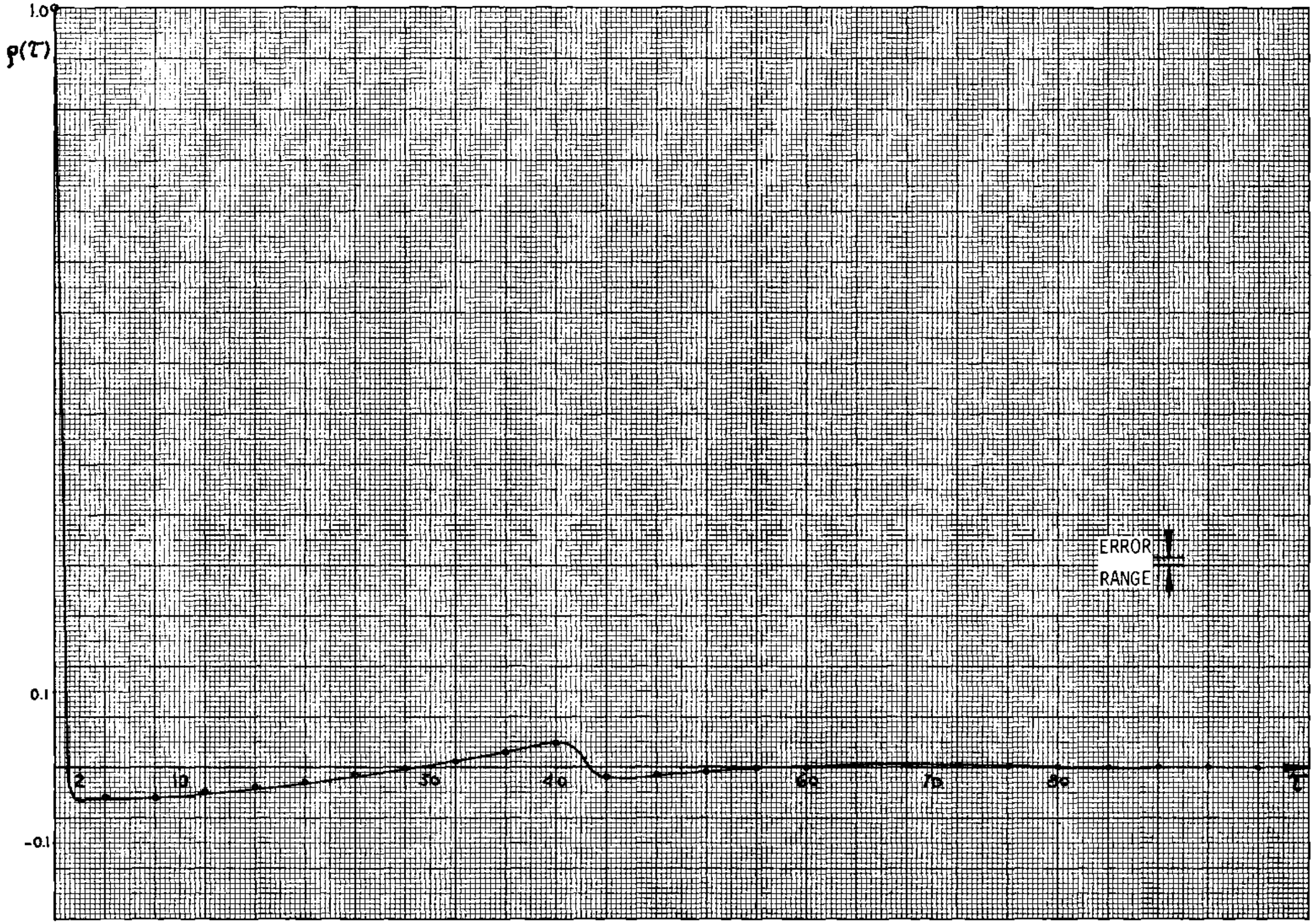

Figure 37. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 2$, $\tau = 40$.

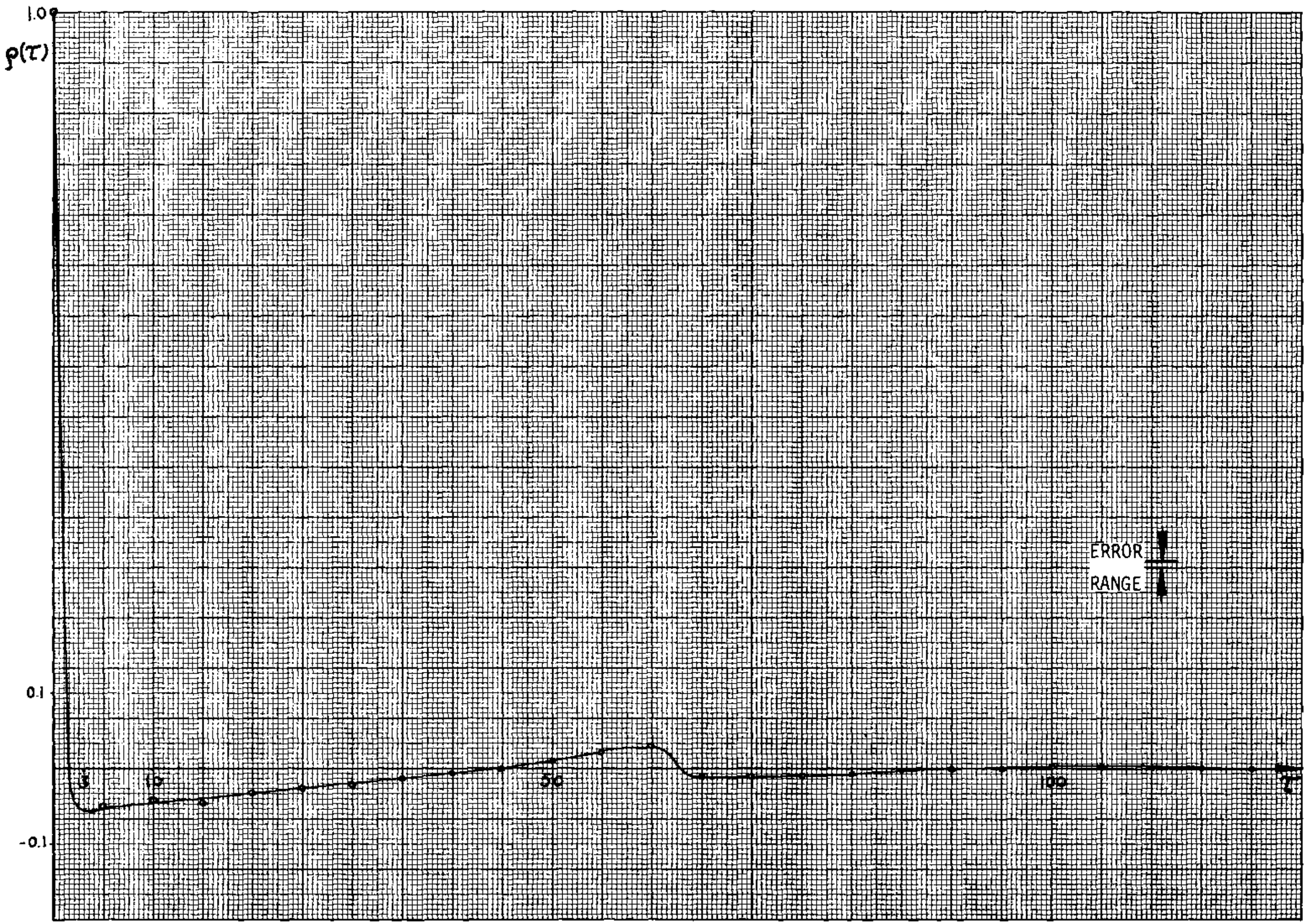

Figure 38. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq U(\tau)$.
Parameters: $u = 3$, $\tau = 60$.

In order to successfully apply Simpson's Rule the indeterminate form of the function

$$f(\omega,\tau) = \frac{1}{\pi} S(\omega) \cos \omega\tau$$

at $\omega = 0$ has to be resolved. Following the methods which lead to Equation (3.20) one obtains

$$\lim_{\omega\to 0} f(\omega,\tau) = \frac{2u^2t^2}{3\pi(2u+t)^3} \ .$$

The parameter combinations considered are listed in Table 6. For each set of parameters autocovariance and autocorrelation function were tabulated. Graphs of the latter appear in Figures 21 through 38.

Case III: U = Constant, $V \simeq N(\mu,\sigma^2)$

Application of Equation (3.42) to formula (5.1) yields the following expression for the autocovariance

$$R(\tau) \simeq \frac{2}{\pi(u+\mu)} \int_0^{UL} \frac{(1 - e^{-\omega^2\sigma^2})(1 - \cos \omega u) \cos \omega\tau}{\omega^2\{1-2e^{-1/2\ \omega^2\sigma^2} \cos \omega(u+\mu) + e^{-\omega^2\sigma^2}\}} d\omega. \tag{5.21}$$

Clearly, the process variance is given by

$$R(0) \simeq \frac{2}{\pi(u+\mu)} \int_0^{UL} \frac{(1 - e^{-\omega^2\sigma^2})(1 - \cos \omega u)}{\omega^2\{1-2e^{-1/2\ \omega^2\sigma^2} \cos \omega(u+\mu) + e^{-\omega^2\sigma^2}\}} d\omega. \tag{5.22}$$

In order to determine the upper limit UL expression (5.9) can be approximated on the basis of (3.42). It is

$$\frac{1}{\pi} \int_{UL}^{\infty} S(\omega)\, d\omega \le \frac{2}{\pi(u+\mu)} \int_{UL}^{\infty} \frac{d\omega}{\omega^2}$$

$$\le \frac{2}{\pi(u+\mu)UL} \le a_\omega = 4 \cdot 10^{-3}.$$

Hence UL is determined to

$$UL \ge \frac{500}{\pi(\mu+u)} \quad . \tag{5.23}$$

With the limits determined, it only remains to fix the interval length h before both expressions (5.21) and (5.22) can be evaluated. In Chapter IV under this respective case it was stated that the spectrum shows oscillations with very high amplitude but with period less than 10^{-1}. This characteristic, it was stated, vanishes after completion of one main period at $2\pi/u$. It is this very behavior of the spectrum that advises a double application of Simpson's Rule. It amounts to a split-up of the integral

$$R(\tau) \approx \frac{1}{\pi} \int_{0}^{UL} S(\omega) \cos \omega\tau \, d\omega$$

into two integrals

$$R(\tau) \approx \frac{1}{\pi} \int_{0}^{2\pi/u} S(\omega) \cos \omega\tau \, d\omega + \frac{1}{\pi} \int_{2\pi/u}^{UL} S(\omega) \cos \omega\tau \, d\omega, \tag{5.24}$$

with the same method holding for the determination of $R(0)$. The first integral in (5.24) is then evaluated for a fixed τ using an interval length of $h = 0.01$ which is considered sufficient to seize the particular irregularities of this spectrum within the given interval length of ω. Following, the second integral in (5.24) is determined for the same fixed τ, using the usual subinterval length $h = \pi/12\tau$. Both results are added, the sum is then divided by $R(0)$ and one obtains the numerical value of $\rho(\tau)$ for a particular τ. The results show that $\rho(\tau)$ oscillates smoothly with period equal to the sum of the means of the two distributions. This characteristic suggested the evaluation in increments of an even fraction of the sum of the means and it was chosen

$$\Delta\tau = \frac{1}{10}(u+\mu) \ . \tag{5.25}$$

τ_{max} was placed at a point where $\rho(\tau)$ has completed five oscillations.

The relation

$$f(\omega,\tau) = \frac{1}{\pi} S(\omega) \cos \omega\tau$$

has again to be determined for $\omega = 0$. Through repeated application of L'Hospital's Rule the resulting indeterminate form is resolved to

$$\lim_{\omega\to 0} f(\omega,\tau) = \frac{\sigma^2 u^2}{\pi(u+\mu)^3} \ . \tag{5.26}$$

Table 8 provides a listing of those parameter combinations for which both autocovariance and autocorrelation function were tabulated. The following graphs on Figures 39 through 46 picture the latter for one parameter set at a time.

Case IV: $U \simeq EXP(\lambda)$; $V \simeq N(\mu,\sigma^2)$

The spectral density function $S(\omega)$ for this case is given in (3.50). Applying $S(\omega)$ to (5.4) the autocorrelation function is expressed by

$$\rho(\tau) = \frac{R(\tau)}{R(0)} = \tag{5.27}$$

$$\frac{\dfrac{2\lambda}{\pi(1+\lambda\mu)} \displaystyle\int_0^{UL} \frac{(1 - e^{-1/2\ \omega^2\sigma^2} \cos \omega\mu) \cos \omega\tau}{\lambda^2(1-e^{-1/2\ \omega^2\sigma^2} \cos \omega\mu)^2 + (\omega + \lambda e^{-1/2\ \omega^2\sigma^2} \sin \omega\mu)^2}\, d\omega}{\dfrac{2\lambda}{\pi(1+\lambda\mu)} \displaystyle\int_0^{UL} \frac{1 - e^{-1/2\ \omega^2\sigma^2} \cos \omega\mu}{\lambda^2(1-e^{-1/2\ \omega^2\sigma^2} \cos \omega\mu)^2 + (\omega + \lambda e^{-1/2\ \omega^2\sigma^2} \sin \omega\mu)^2}\, d\omega}$$

An approximation of expression (5.9) on the basis of (3.50) provides an estimate for the upper limit UL. It is

$$\frac{1}{\pi} \int_{UL}^{\infty} S(\omega)\, d\omega \le \frac{2\lambda}{\pi(1+\lambda\mu)} \int_{UL}^{\infty} \frac{d\omega}{\omega^2}$$

$$\le \frac{2\lambda}{\pi(1+\lambda\mu)UL} \le a_\omega = 4 \cdot 10^{-3}.$$

UL follows to

$$UL \ge \frac{500\lambda}{\pi(1+\lambda\mu)} . \tag{5.28}$$

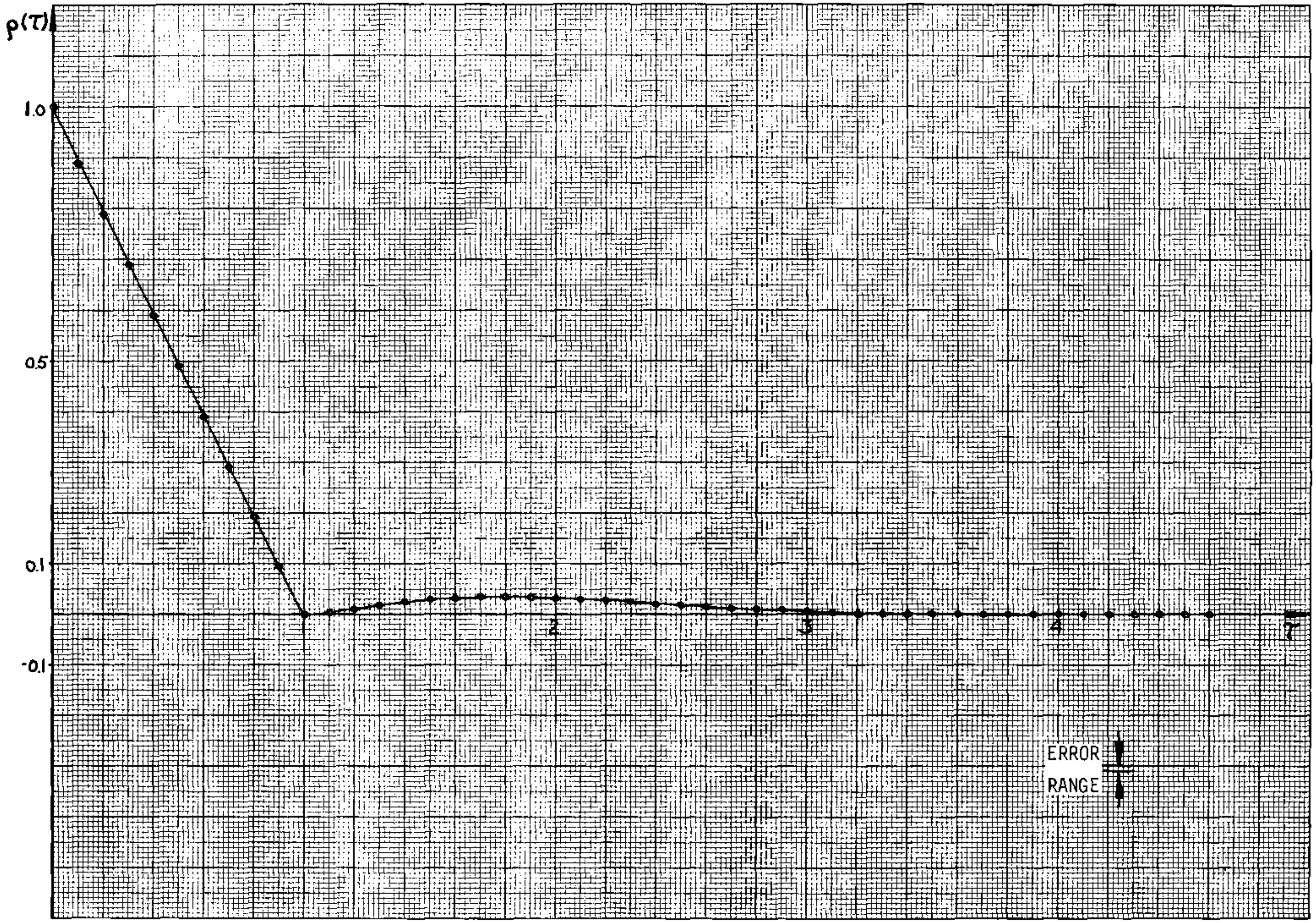

Figure 39. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $u = 1$, $\mu = 0$, $\sigma = 1.0$.

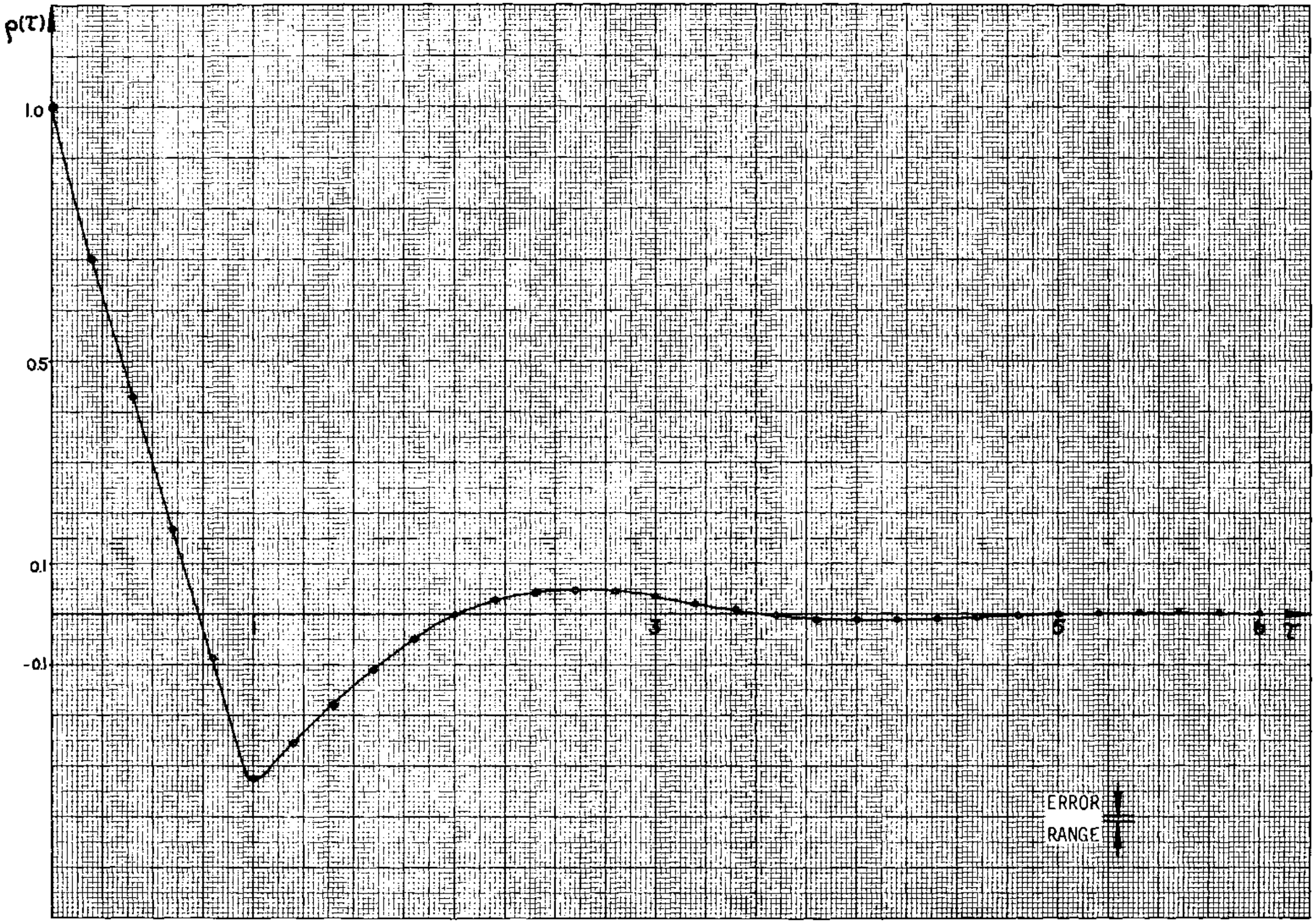

Figure 40. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $u = 1$, $\mu = 1$, $\sigma = 1.0$.

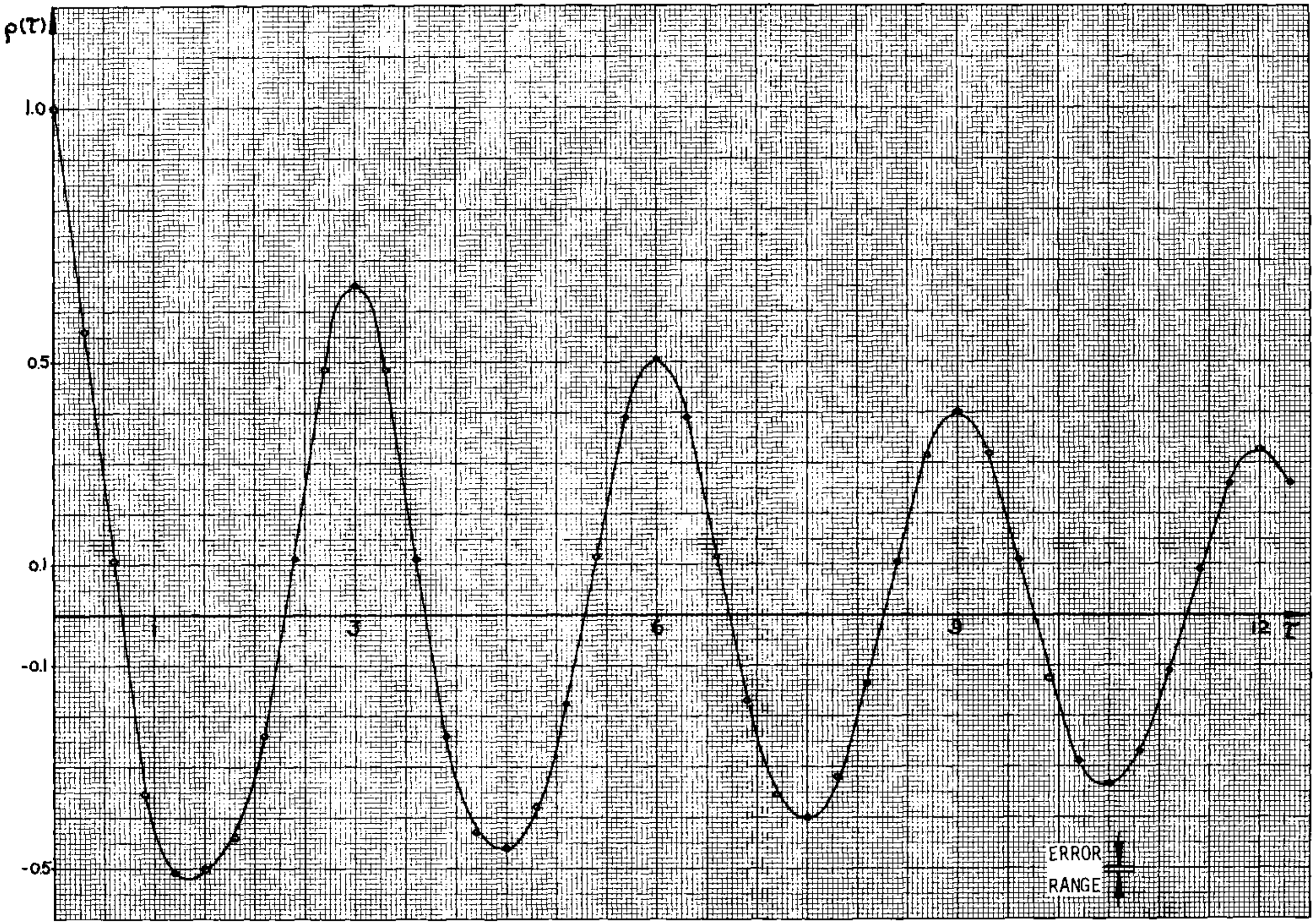

Figure 41. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $u = 1$, $\mu = 2$, $\sigma = 0.3$.

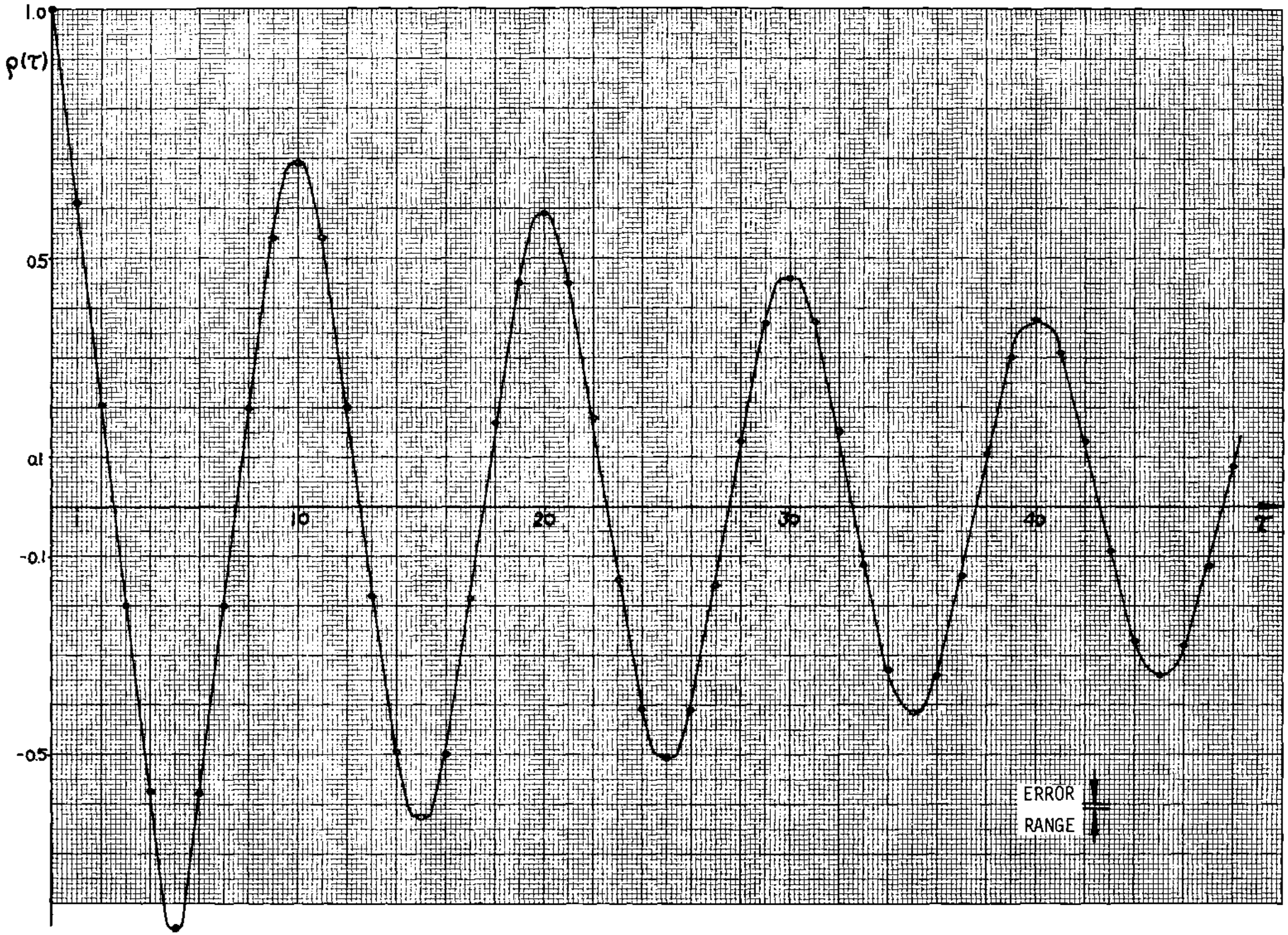

Figure 42. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $u = 5$, $\mu = 5$, $\sigma = 1.0$.

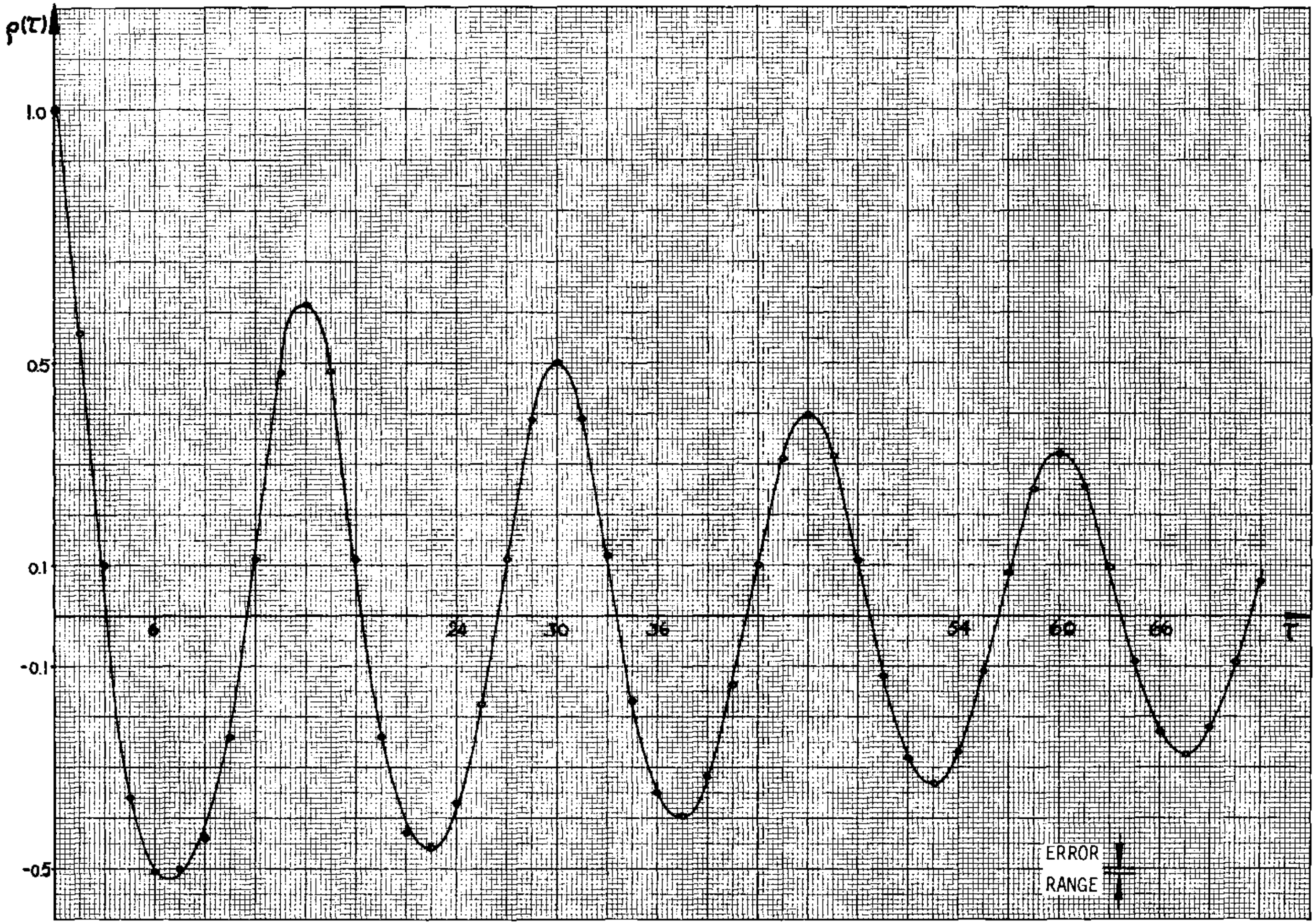

Figure 43. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $u = 5$, $\mu = 10$, $\sigma = 1.5$.

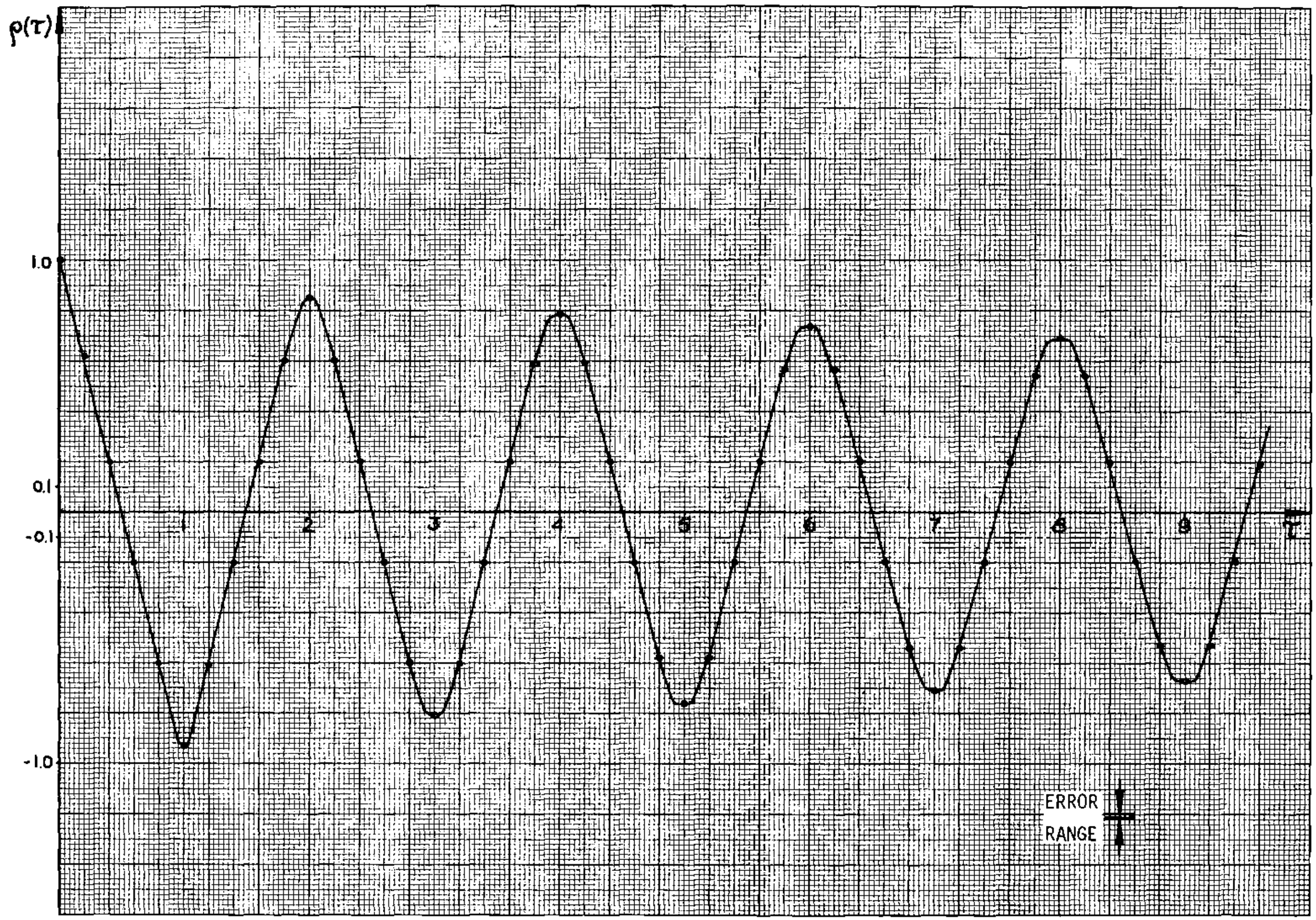

Figure 44. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $u = 1$, $\mu = 1$, $\sigma = 0.1$.

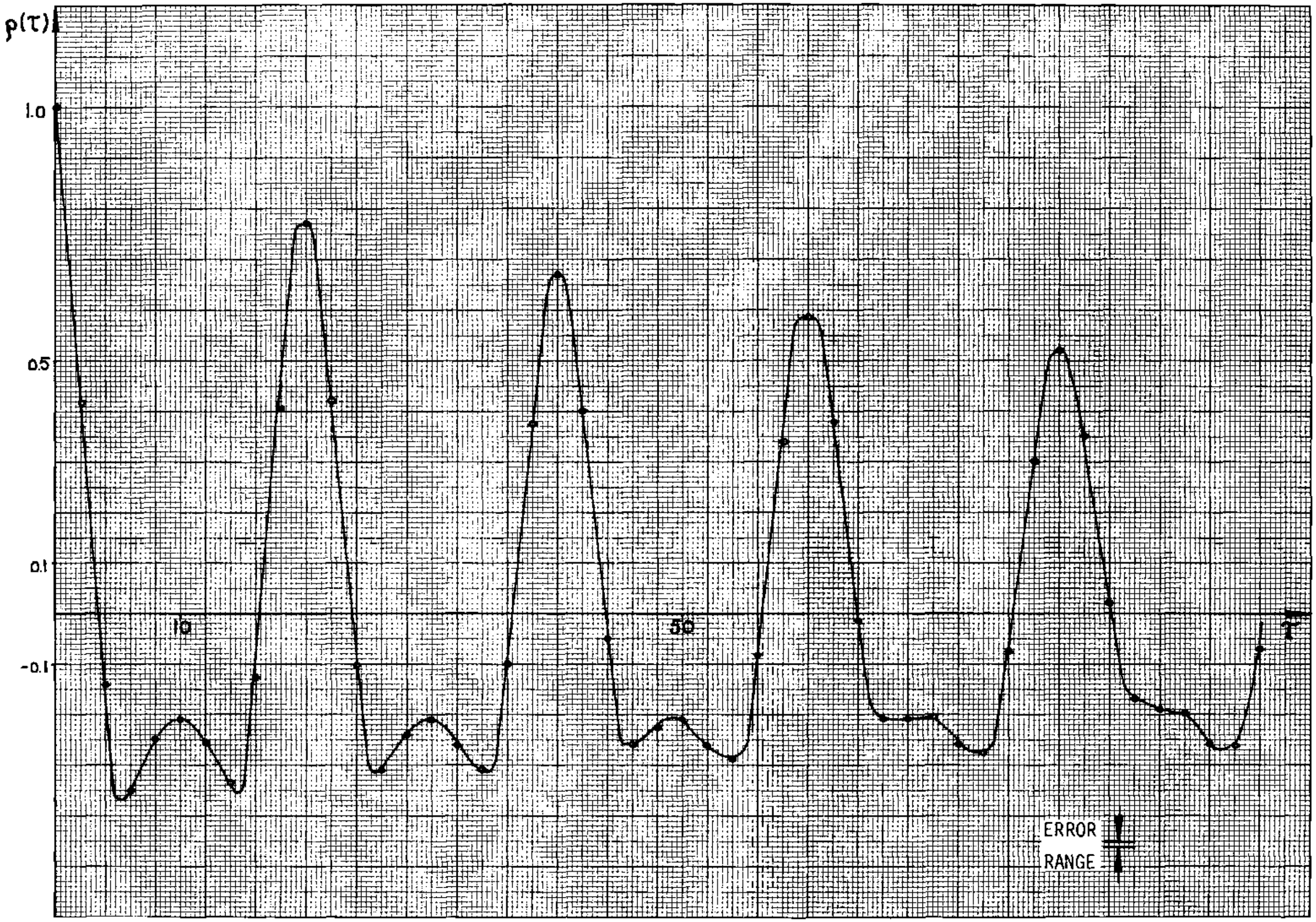

Figure 45. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $u = 5$, $\mu = 15$, $\sigma = 1.0$.

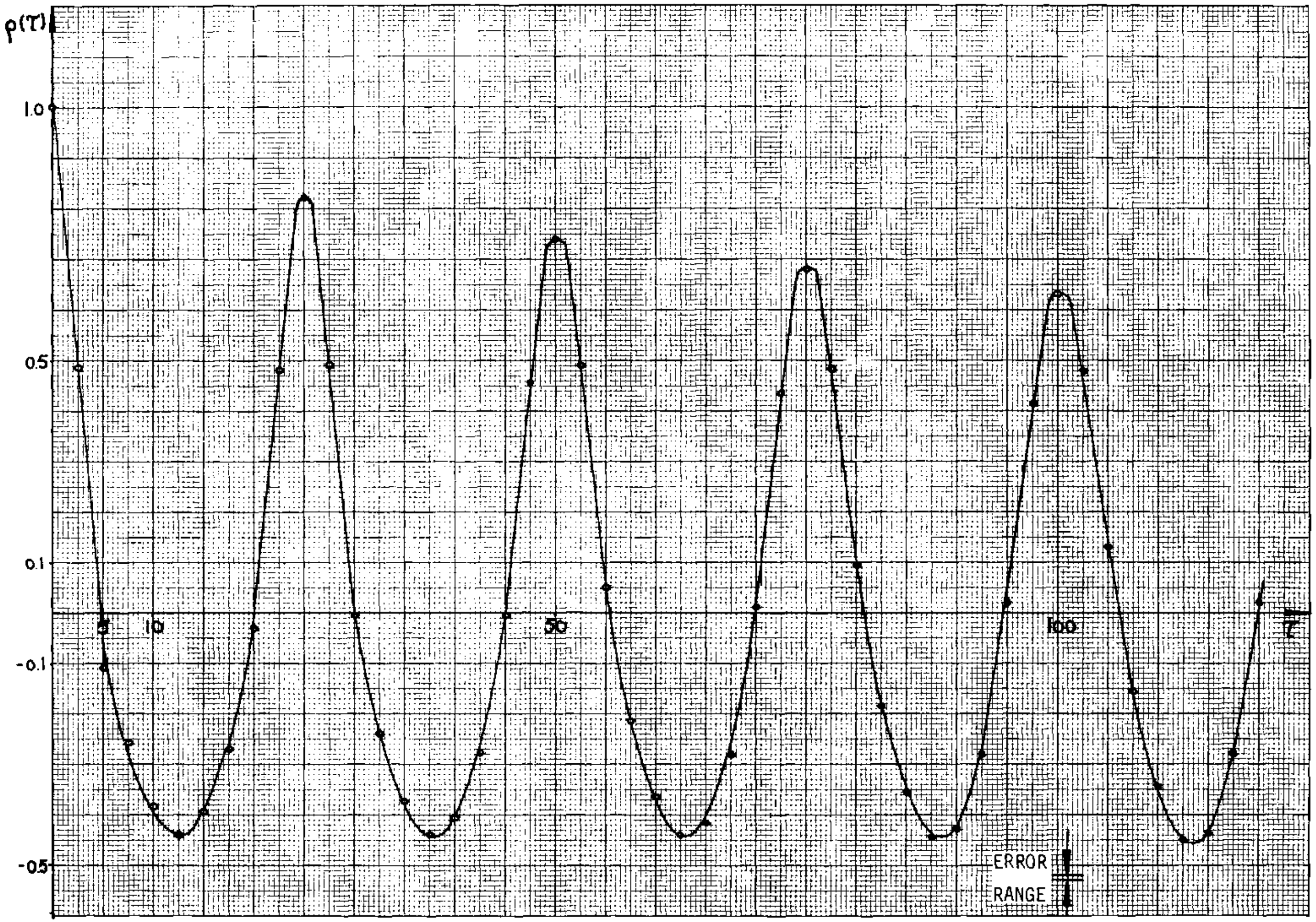

Figure 46. Autocorrelation Function $\rho(\tau)$ for U = Constant and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $u = 5$, $\mu = 20$, $\sigma = 1.25$.

It is apparent that Cases I and IV have certain similarities. If the mean of the normal distribution, μ, takes the place of the constant u, then the only difference between expressions (5.27) and (5.15) lies in the term EXP$(-\frac{1}{2}\,\omega^2\sigma^2)$. The similarities are discernible also through inspection of Tables 5 and 11 and of the respective plots in Appendix B.

For the numerical evaluation of (5.27) the same integration data will therefore be employed, as were used in Case I.

The indeterminate form of

$$f(\omega,\tau) = \frac{1}{\pi}\, S(\omega) \cos \omega\tau$$

at $\omega = 0$ can be resolved through repeated application of L'Hospital's rule to

$$\lim_{\omega\to 0} f(\omega,\tau) = \frac{\lambda(\sigma^2+\mu^2)}{\pi(1+\lambda\mu)^3}\,. \tag{5.29}$$

The autocorrelation function for each of the parameters combinations listed in Table 10 has been determined and graphed in Figures 47 through 56.

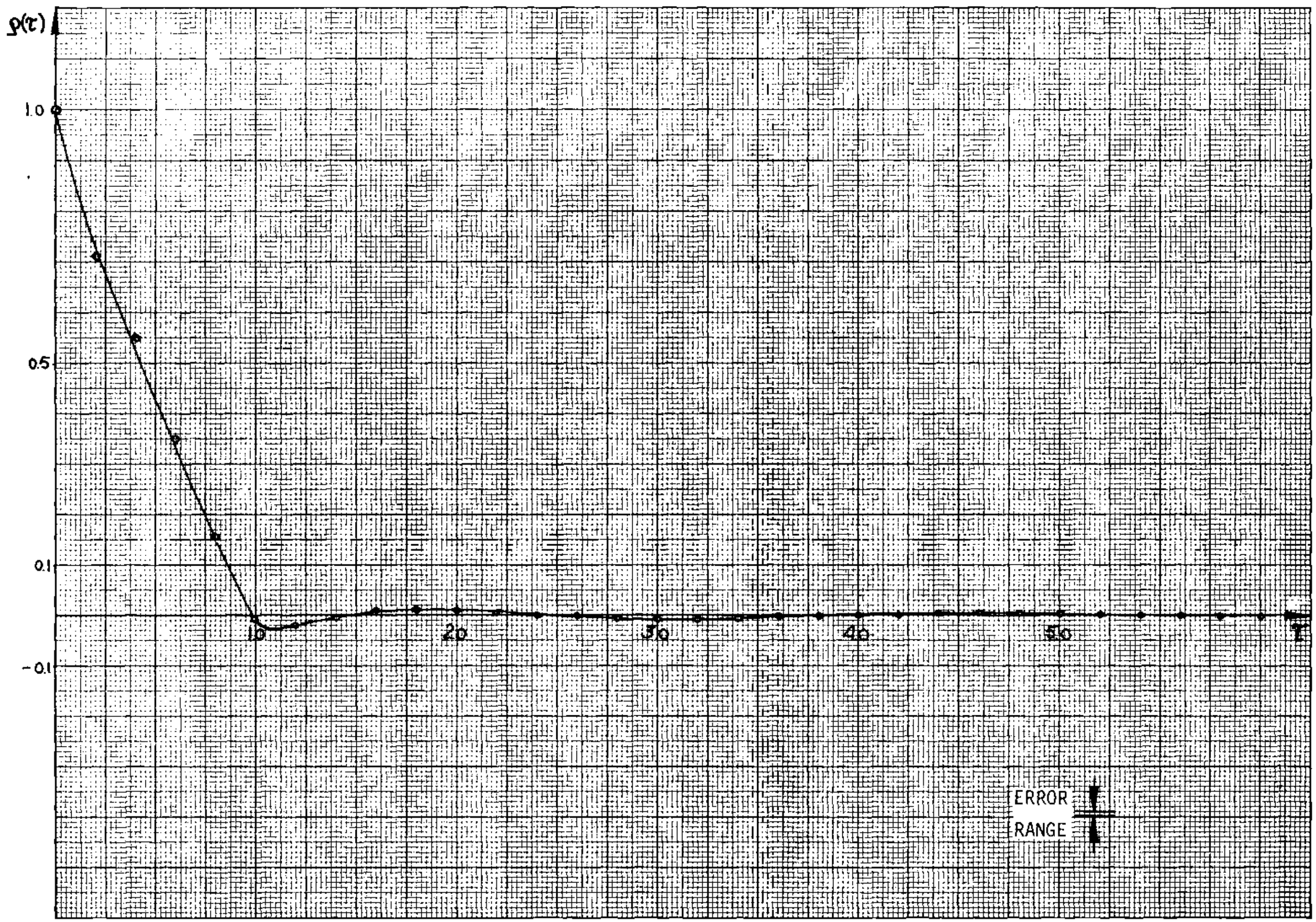

Figure 47. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1/8$, $\mu = 1$, $\sigma = 0.1$.

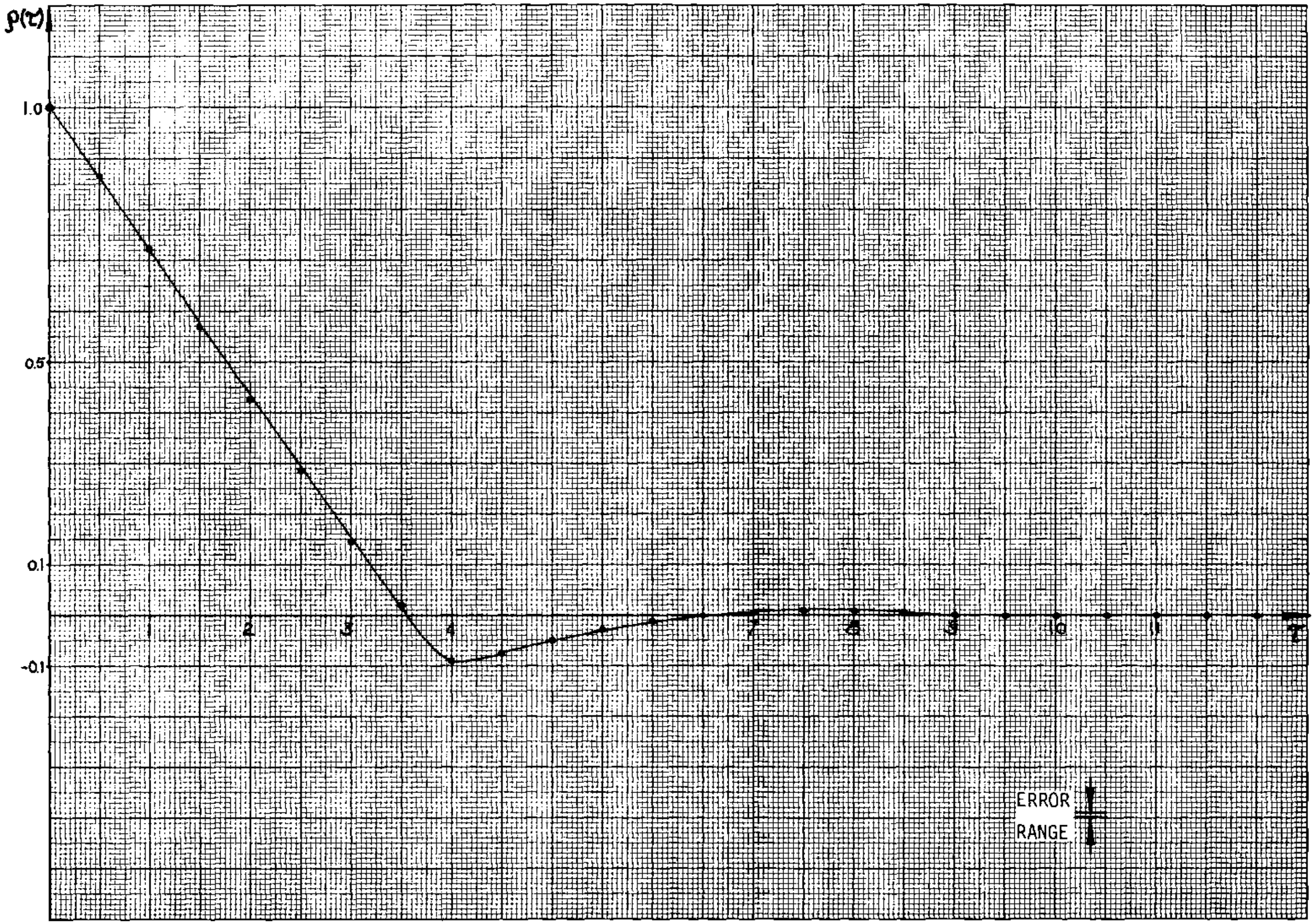

Figure 48. Autocorrelation Function $\rho S(\tau)$ for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1/16$, $\mu = 4$, $\sigma = 0.1$.

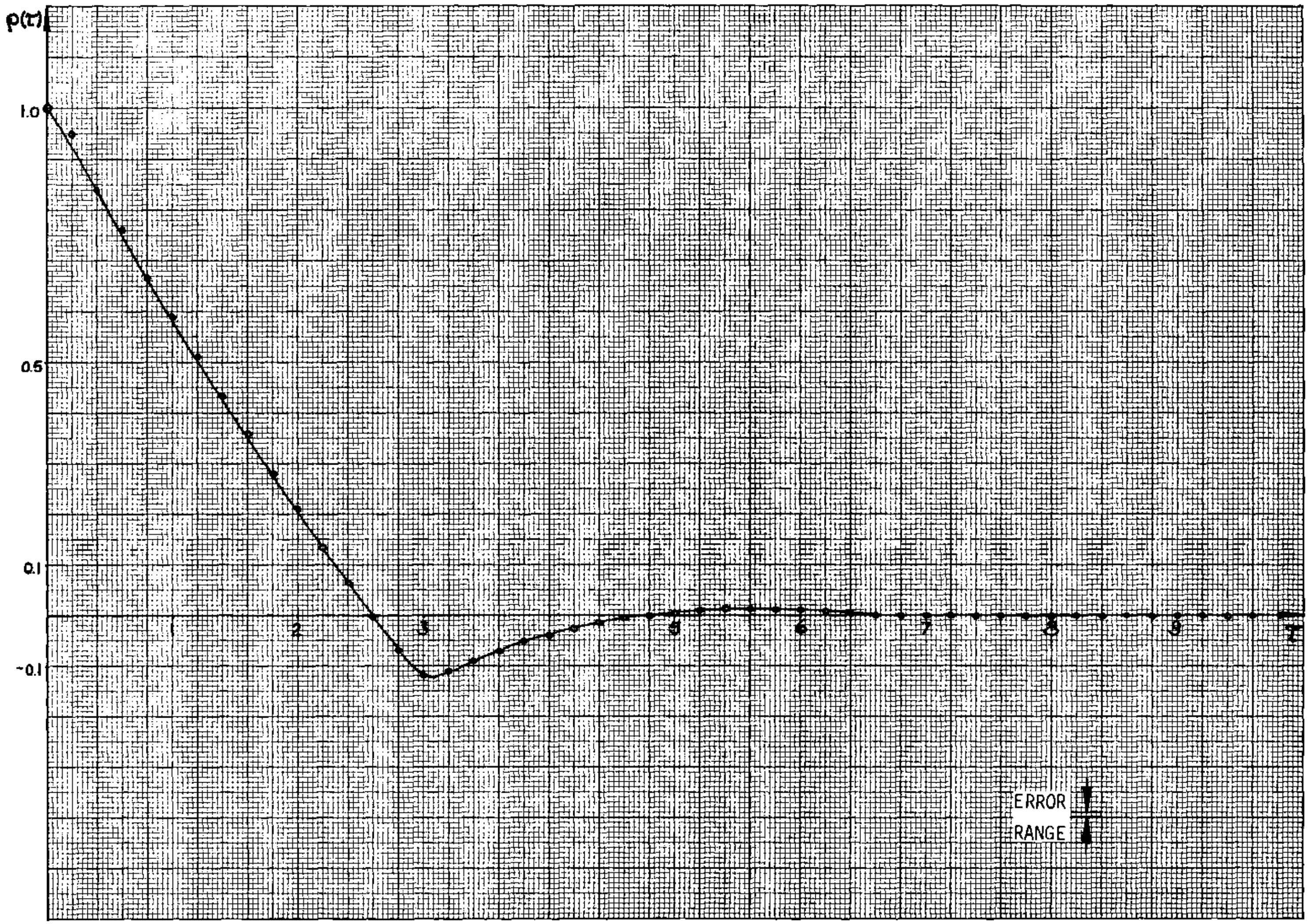

Figure 49. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1/9$, $\mu = 3$, $\sigma = 0.1$.

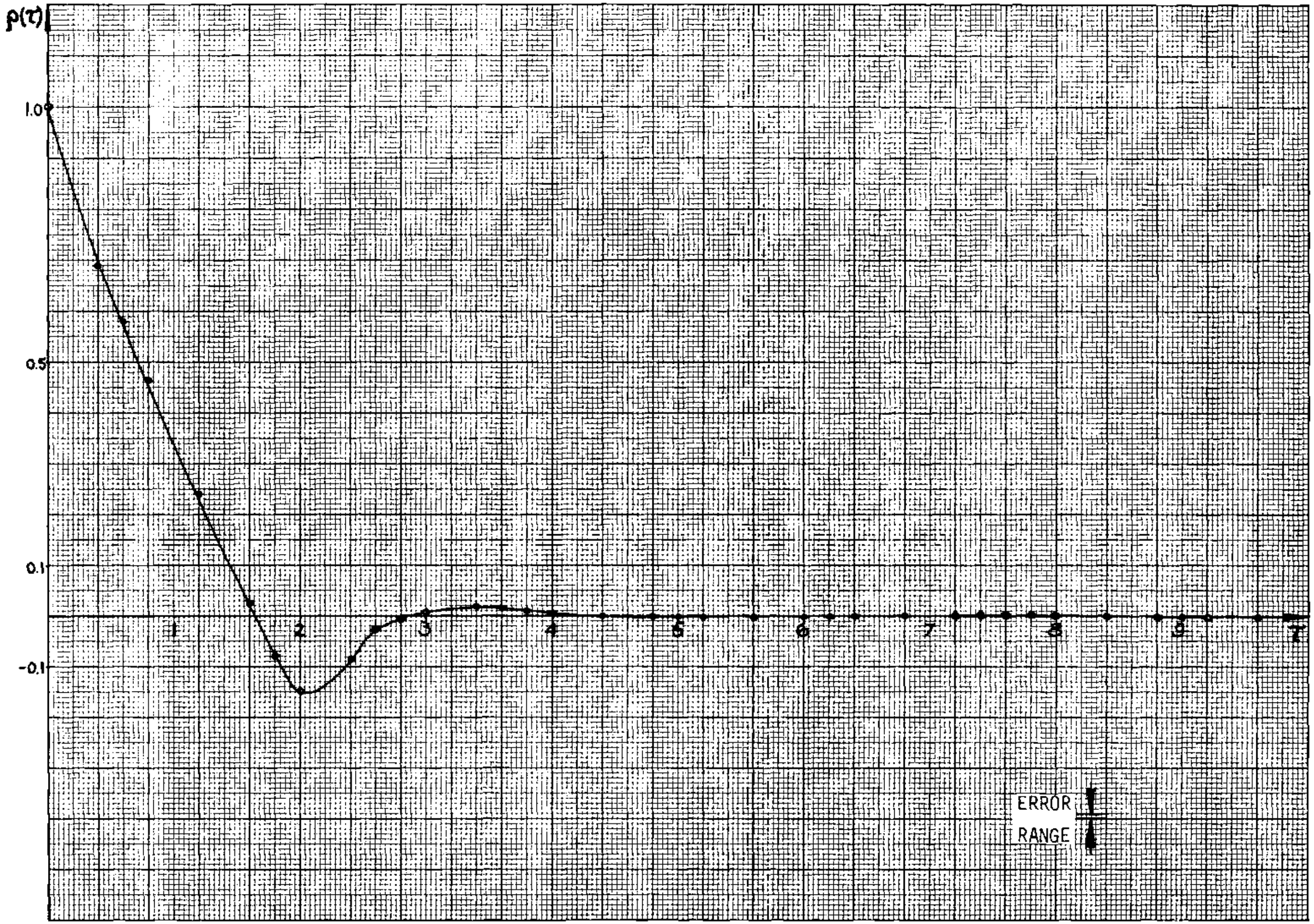

Figure 50. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1/4$, $\mu = 2$, $\sigma = 0.1$.

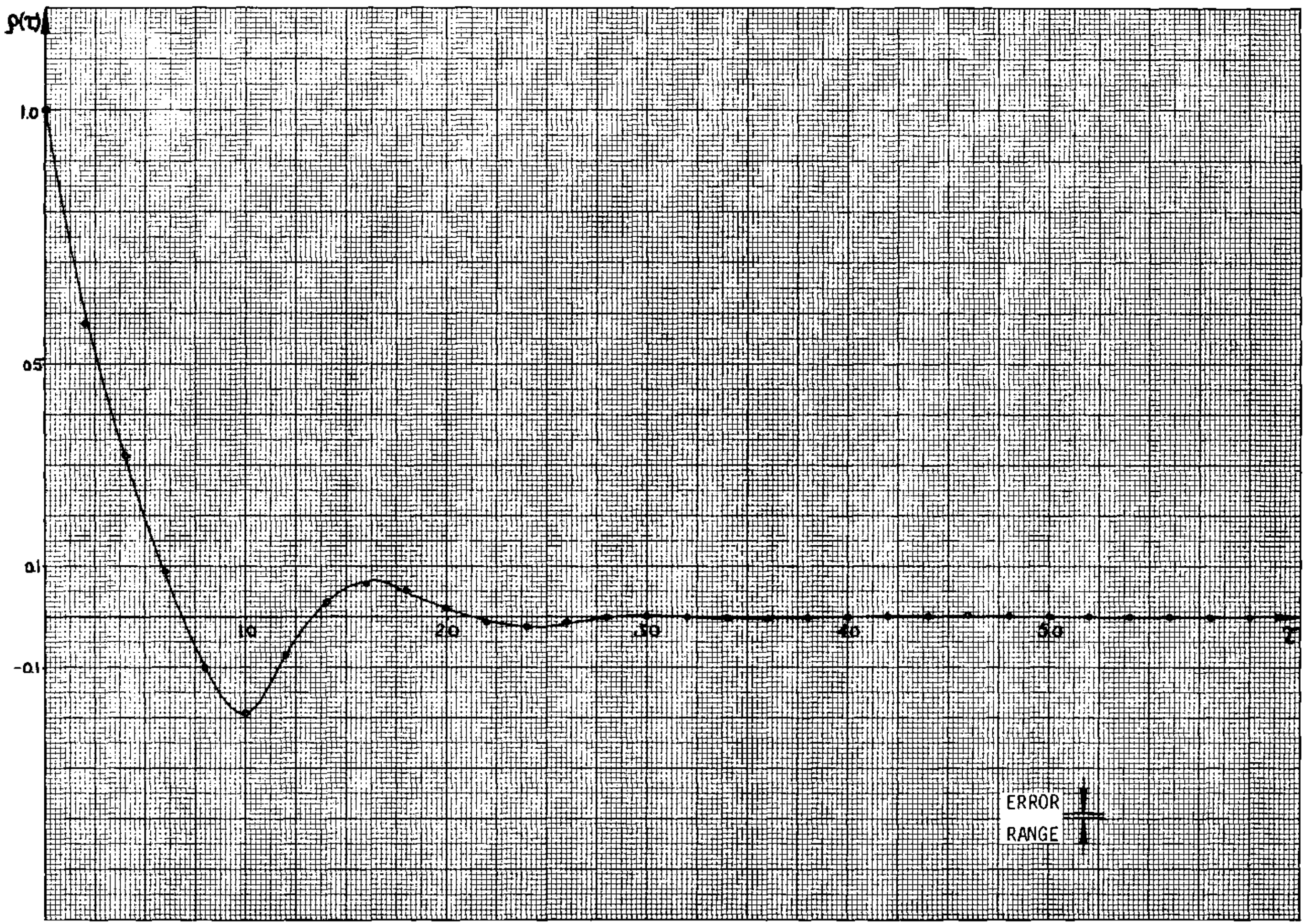

Figure 51. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1$, $\mu = 1$, $\sigma = 0.1$.

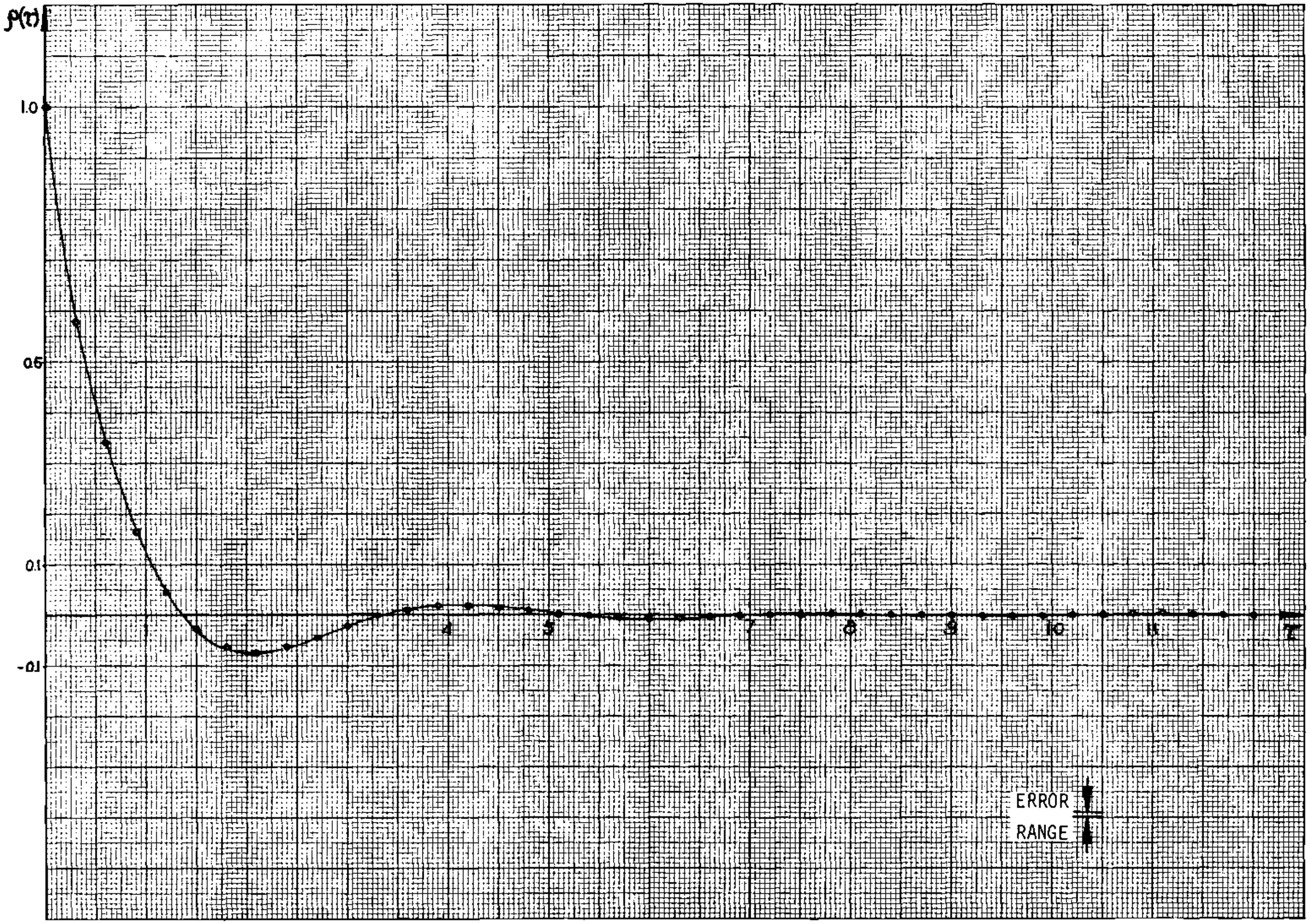

Figure 52. Autocorrelation Function $\rho S(\tau)$ for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1$, $\mu = 2$, $\sigma = 1.0$.

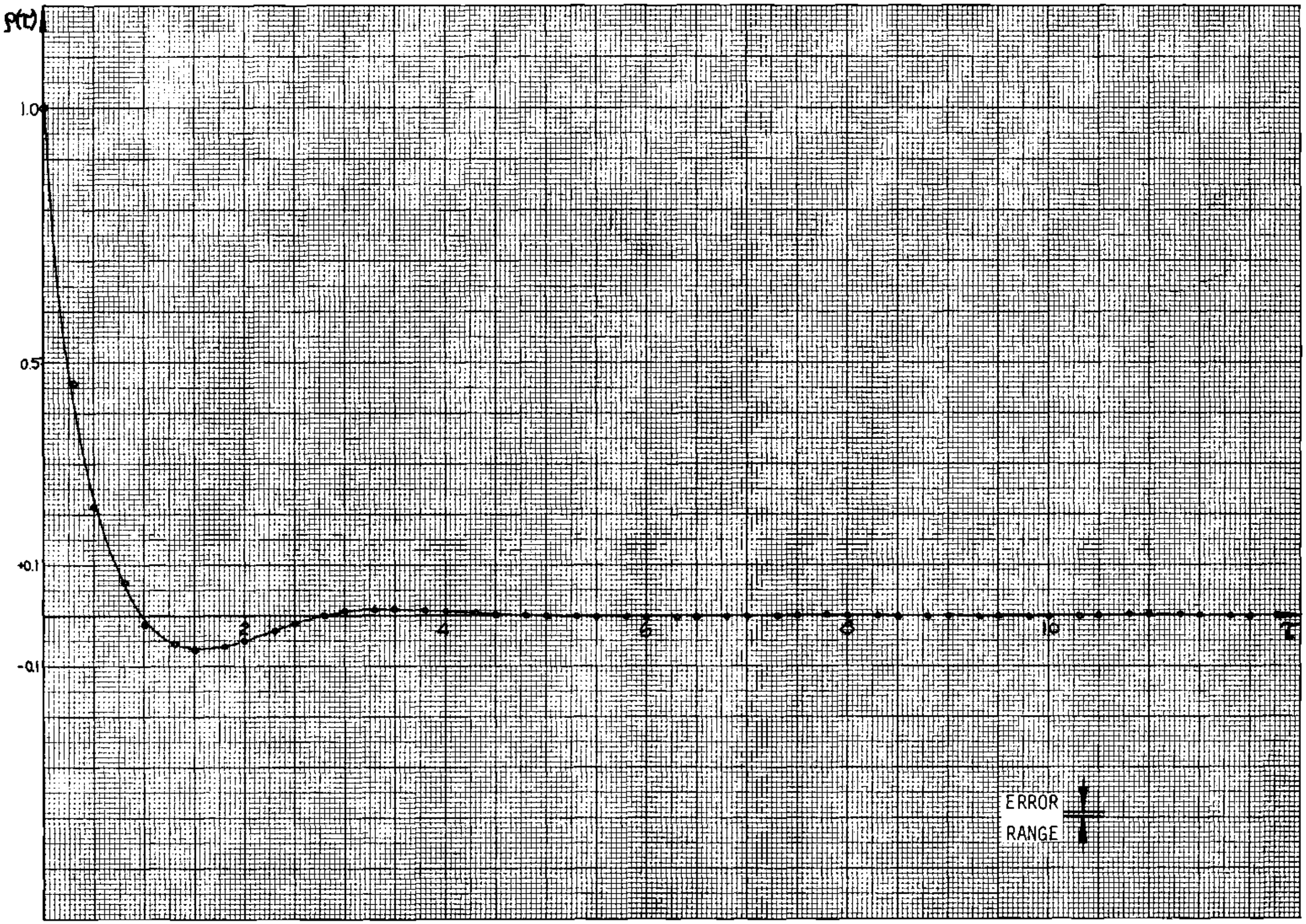

Figure 53. Autocorrelation Function $\rho S(\tau)$ for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 2$, $\mu = 2$, $\sigma = 1.0$.

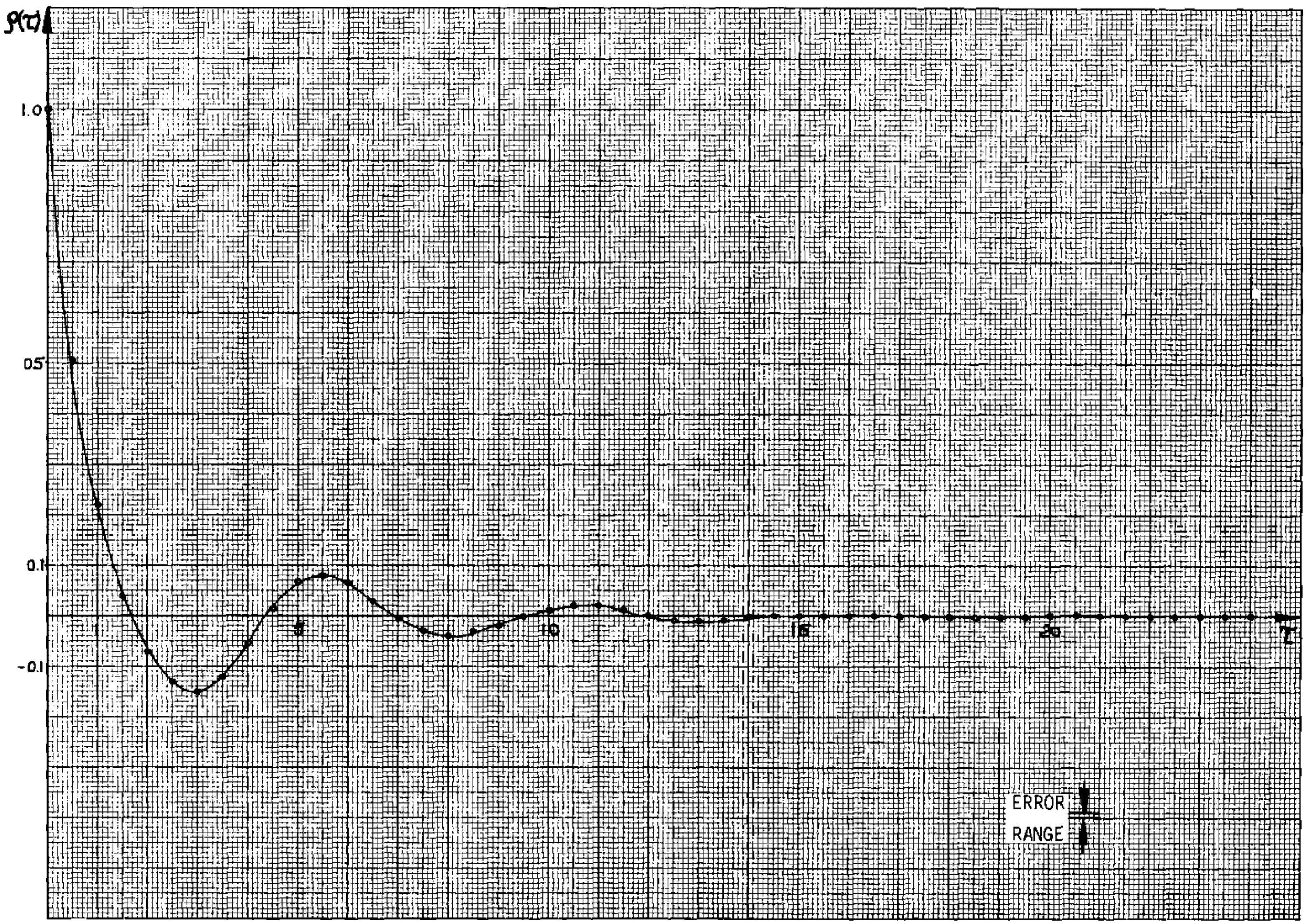

Figure 54. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1$, $\mu = 4$, $\sigma = 1.0$.

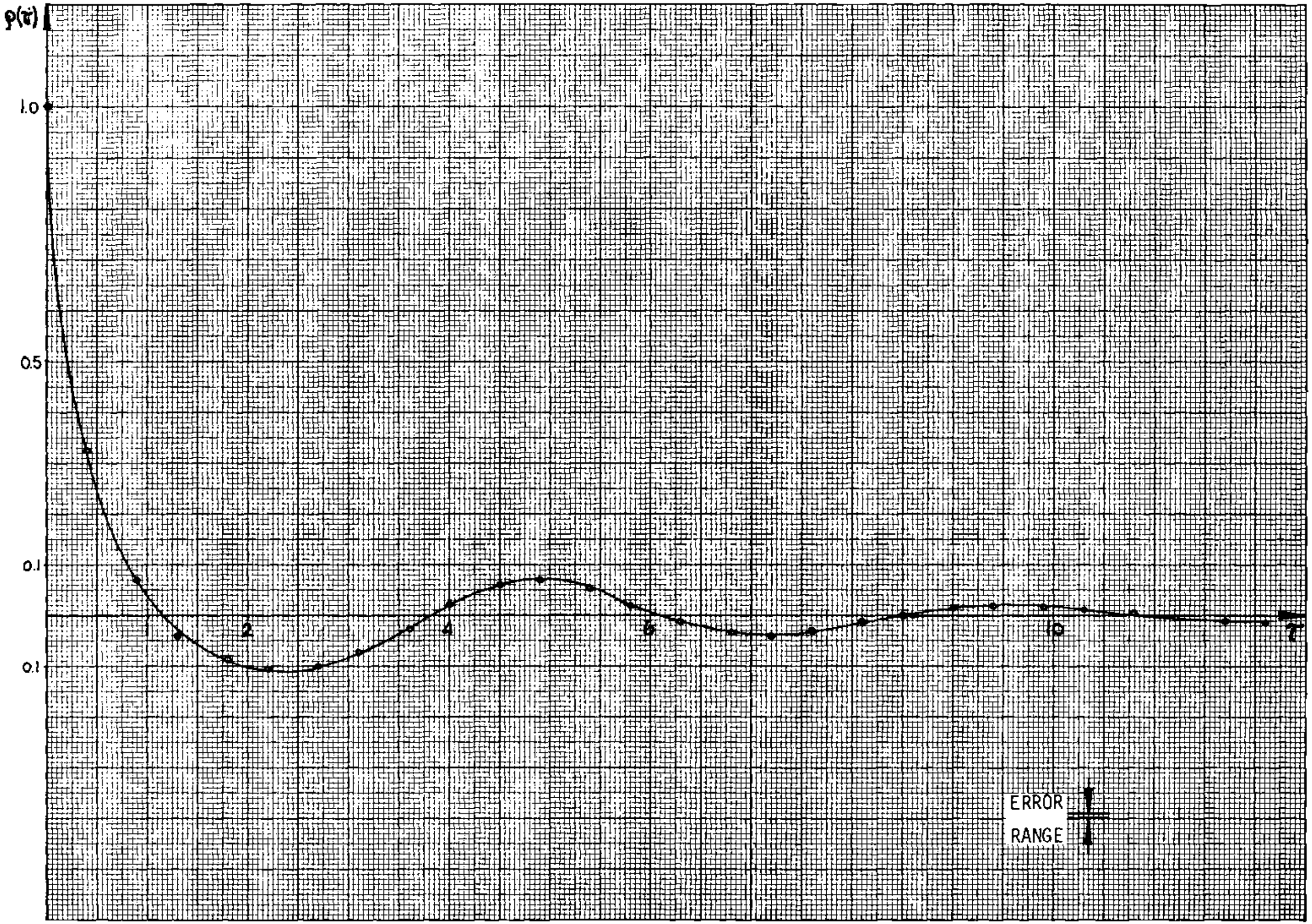

Figure 55. Autocorrelation Function $\rho S(\tau)$ for $U \approx EXP(\lambda)$ and $V \approx N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 2$, $\mu = 4$, $\sigma = 1.0$.

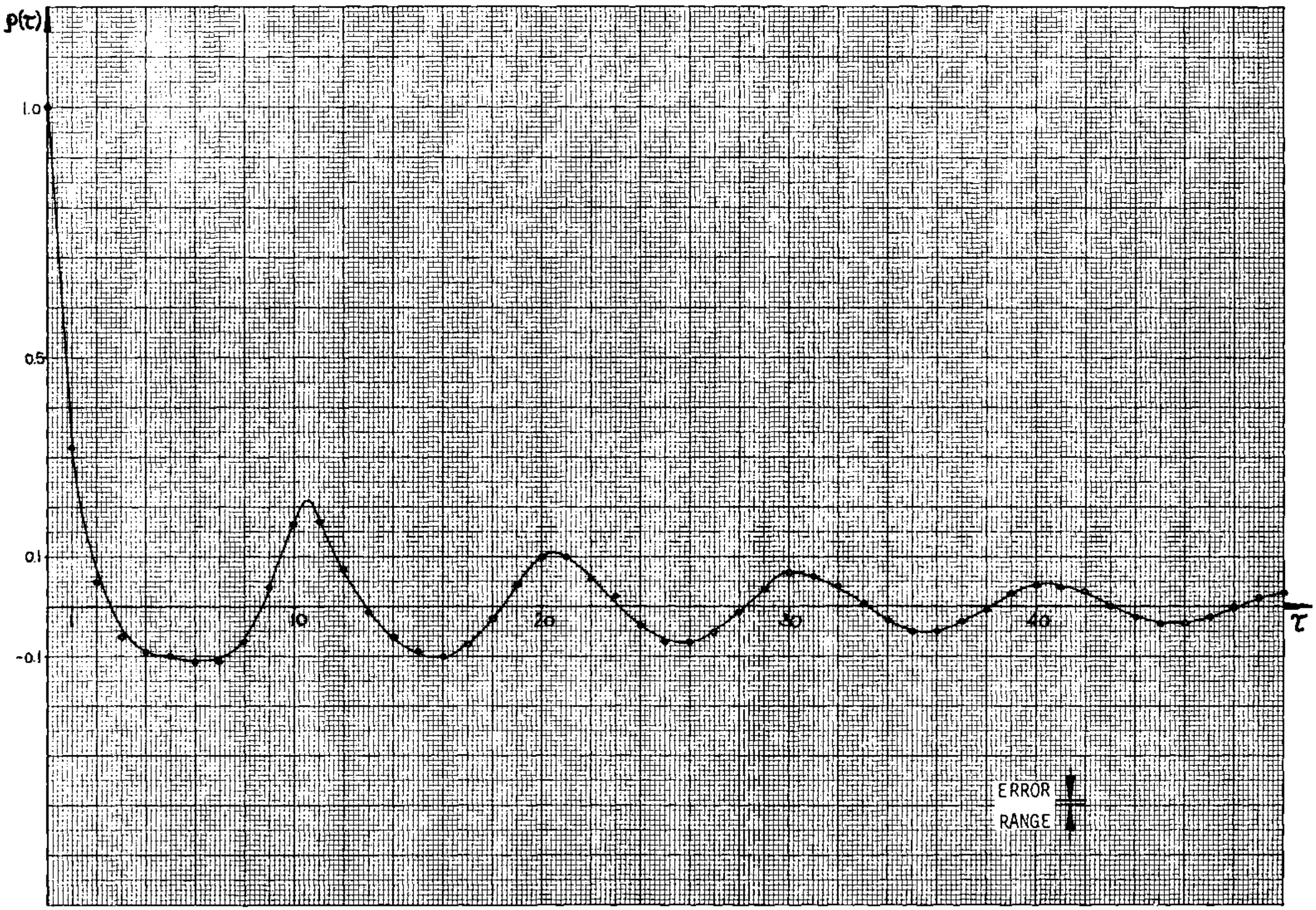

Figure 56. Autocorrelation Function $\rho S(\tau)$ for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu_1, \sigma^2)$.
Parameters: $\lambda = 1$, $\mu = 3$, $\sigma = 1.0$.

Case V: $U \simeq N(\mu_0,\sigma_0^2)$, $V \simeq N(\mu_1,\sigma_1^2)$

The expression to be evaluated is

$$\rho(\tau) = \frac{R(\tau)}{R(0)} = \frac{\frac{1}{\pi}\int_0^{UL} f(\omega,\tau)\, d\omega}{\frac{1}{\pi}\int_0^{UL} S(\omega)\, d\omega} = \frac{\frac{1}{\pi}\int_0^{UL} S(\omega)\cos\omega\tau\, d\omega}{\frac{1}{\pi}\int_0^{UL} S(\omega)\, d\omega}, \quad (5.30)$$

where $S(\omega)$ is given in (3.57). The upper limit UL can be estimated, by approximation of (5.9) with $S(\omega)$ mentioned above. It is

$$\frac{1}{\pi}\int_{UL}^{\infty} S(\omega)\, d\omega \le \frac{2}{\pi(\mu_0+\mu_1)} \int_{UL}^{\infty} \frac{d\omega}{\omega^2}$$

$$\le \frac{2}{\pi(\mu_0+\mu_1)UL} \le a_\omega = 4 \cdot 10^{-3}.$$

Thus

$$UL \ge \frac{500}{\pi(\mu_0+\mu_1)} . \quad (5.31)$$

$S(\omega)$ for this case shows the same characteristics as in Case III, namely superposed variations with very high amplitudes but small periods. A specified double application of Simpson's Rule was employed in that case to cope with the specific irregularities of $S(\omega)$. The same method is used to provide an answer to (5.30) with the same pertinent data as in Case III. The indeterminate form of $f(\omega,\tau)$ in (5.30) for $\omega = 0$ is resolved through repeated application of L'Hospital's rule and the limit is found to

$$\lim_{\omega \to 0} f(\omega,\tau) = \frac{\sigma_0^2\mu_1^2 + \sigma_1^2\mu_0^2}{\pi(\mu_0+\mu_1)^3} \quad . \tag{5.32}$$

The combinations of different parameters for which both autocovariance and autocorrelation function are tabulated in Appendix C and listed in Table 12. The following Figures 57 through 64 are graphs of the autocorrelation function for the parameters specified.

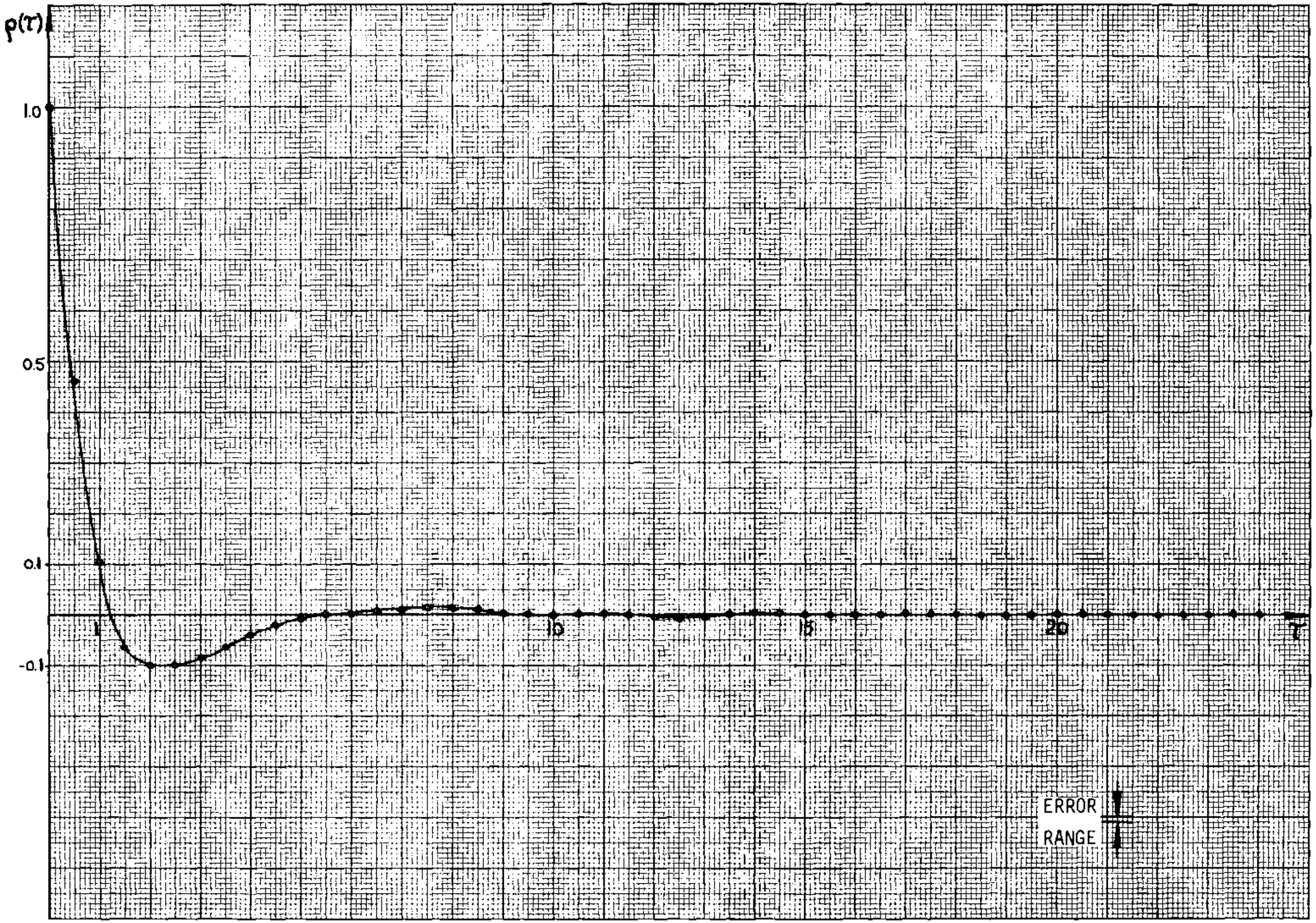

Figure 57. Autocorrelation Function $\rho(\tau)$ for $U \approx N(\mu_0, \sigma_0^2)$ and $V \approx N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 4$, $\mu_1 = 1$, $\sigma_0 = 3.0$, $\sigma_1 = 0.5$.

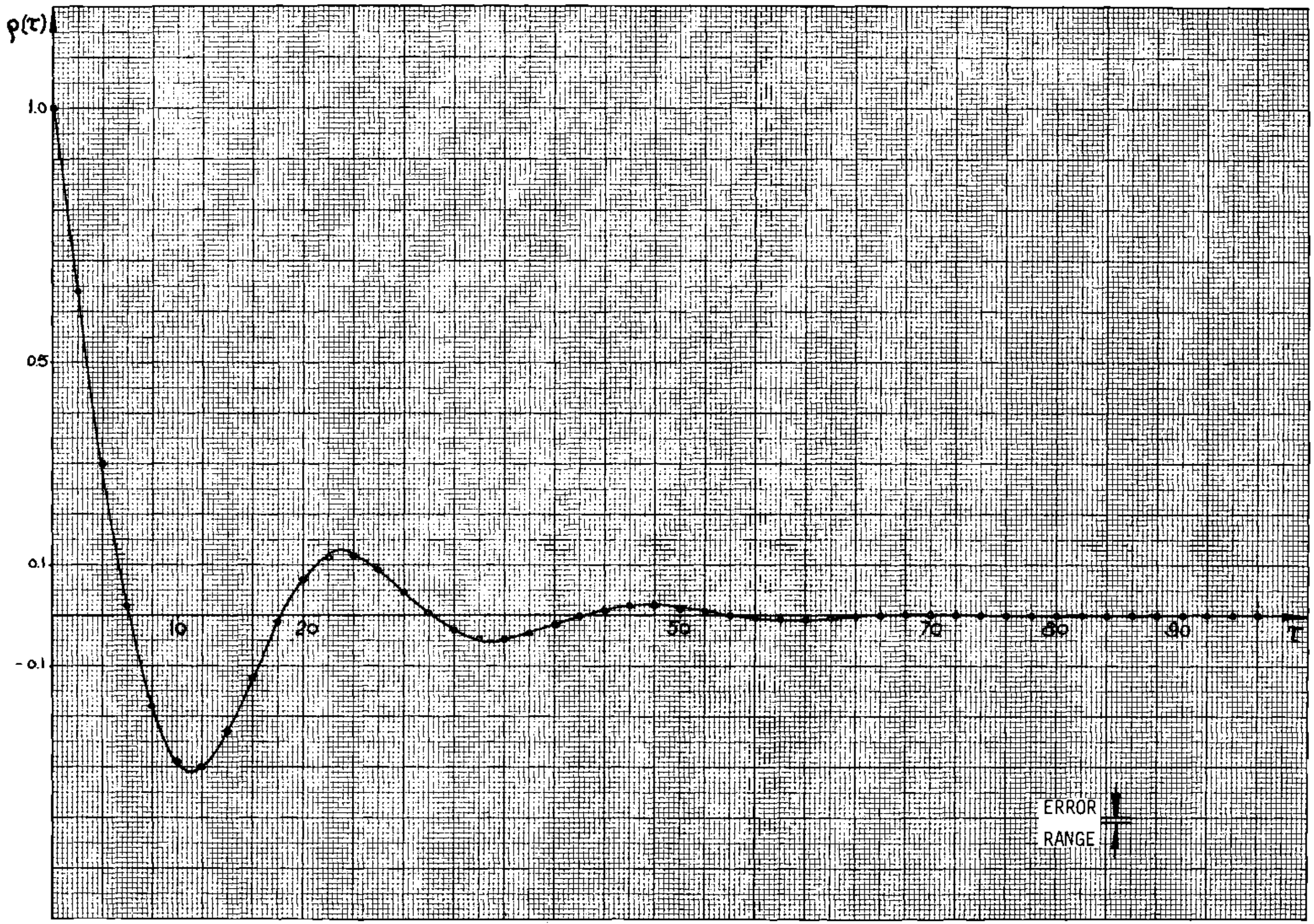

Figure 58. Autocorrelation Function $\rho(\tau)$ for $U \approx N(\mu_0, \sigma_0^2)$ and $V \approx N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 10$, $\mu_1 = 10$, $\sigma_0 = 5.0$, $\sigma_1 = 5.0$.

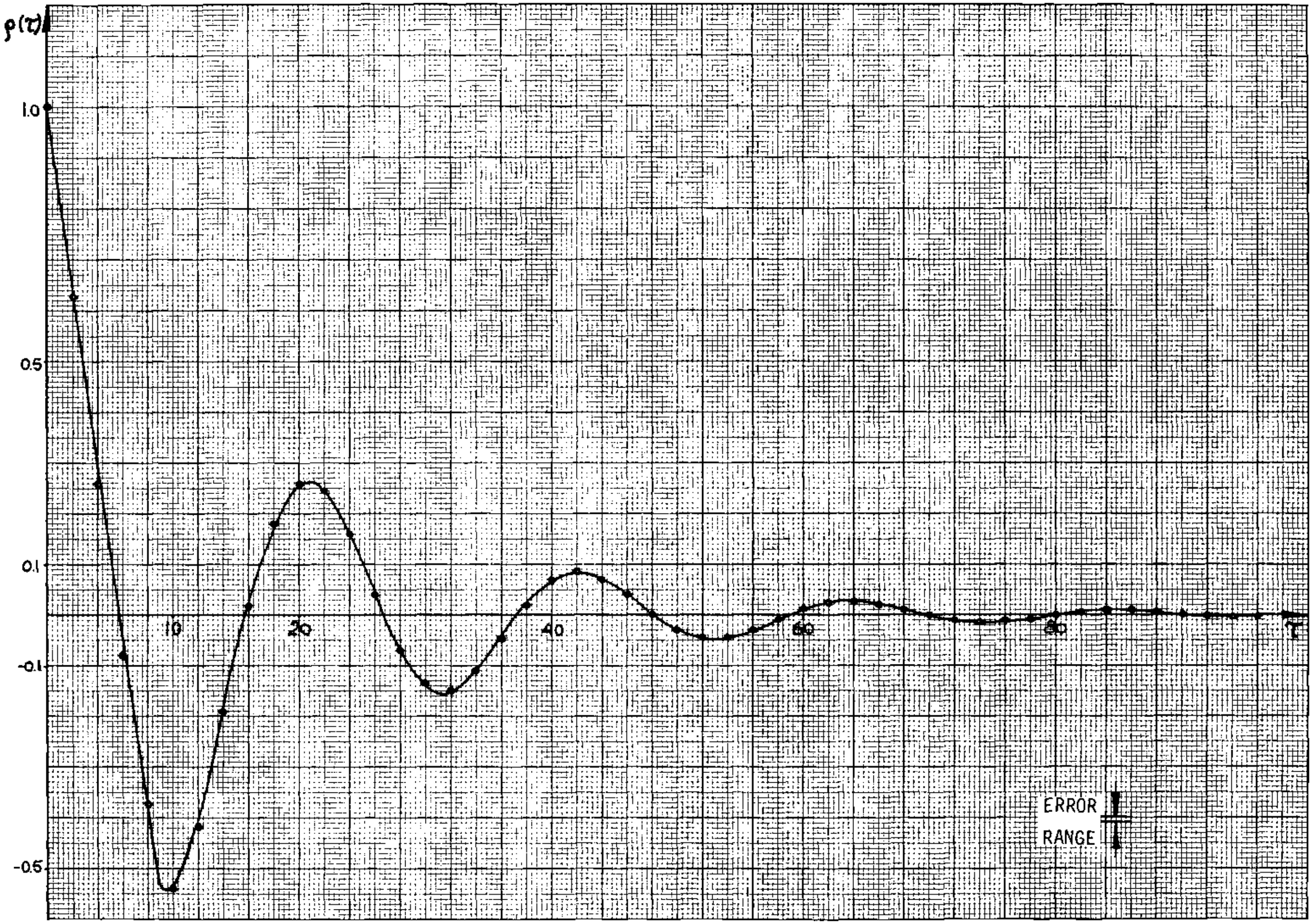

Figure 59. Autocorrelation Function $\rho(\tau)$ for $U \simeq N(\mu_0, \sigma_0^2)$ and V $N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 10$, $\mu_1 = 10$, $\sigma_0 = 1.0$, $\sigma_1 = 5.0$.

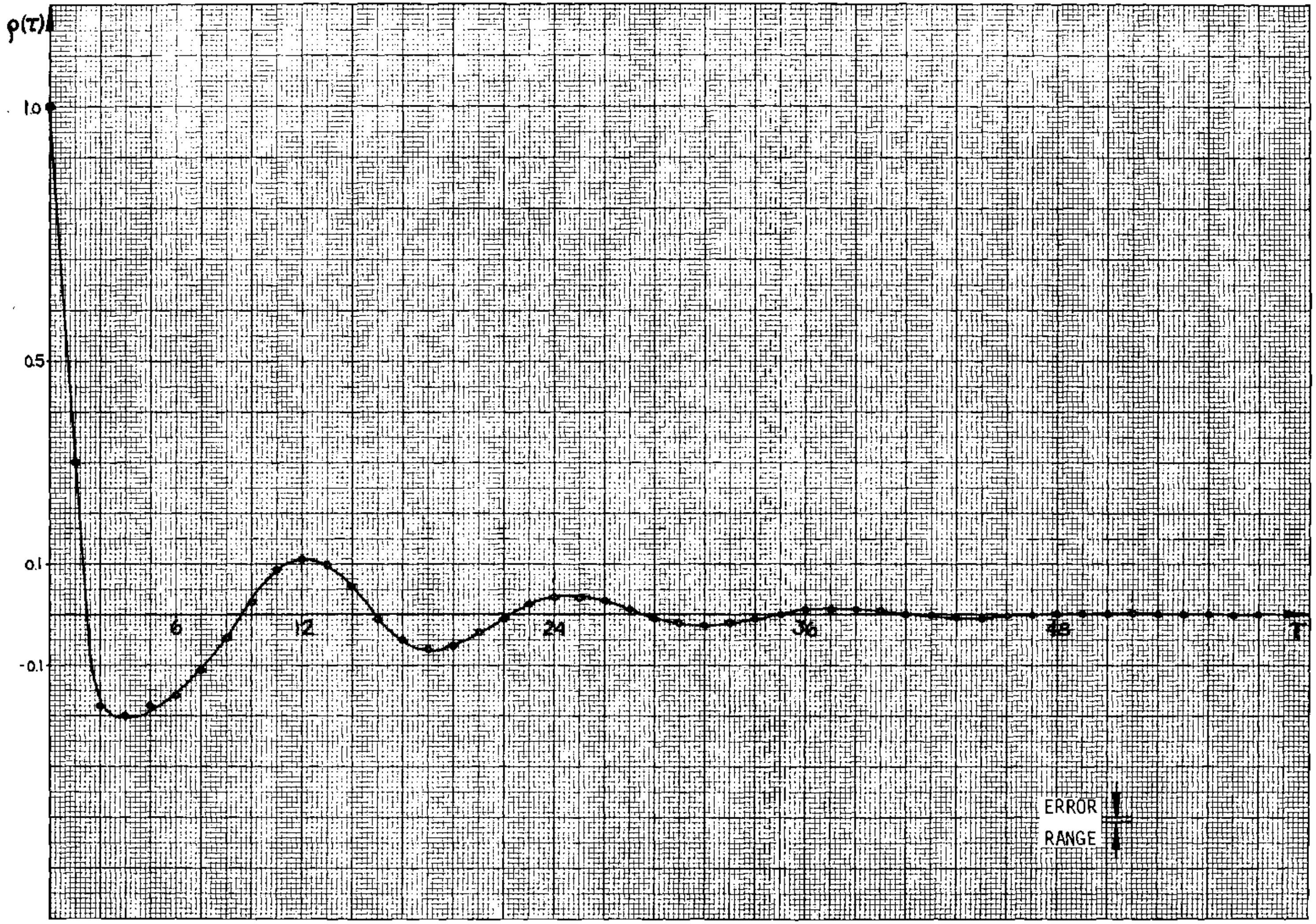

Figure 60. Autocorrelation Function $\rho(\tau)$ for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 2$, $\mu_1 = 10$, $\sigma_0 = 0.4$, $\sigma_1 = 3.0$.

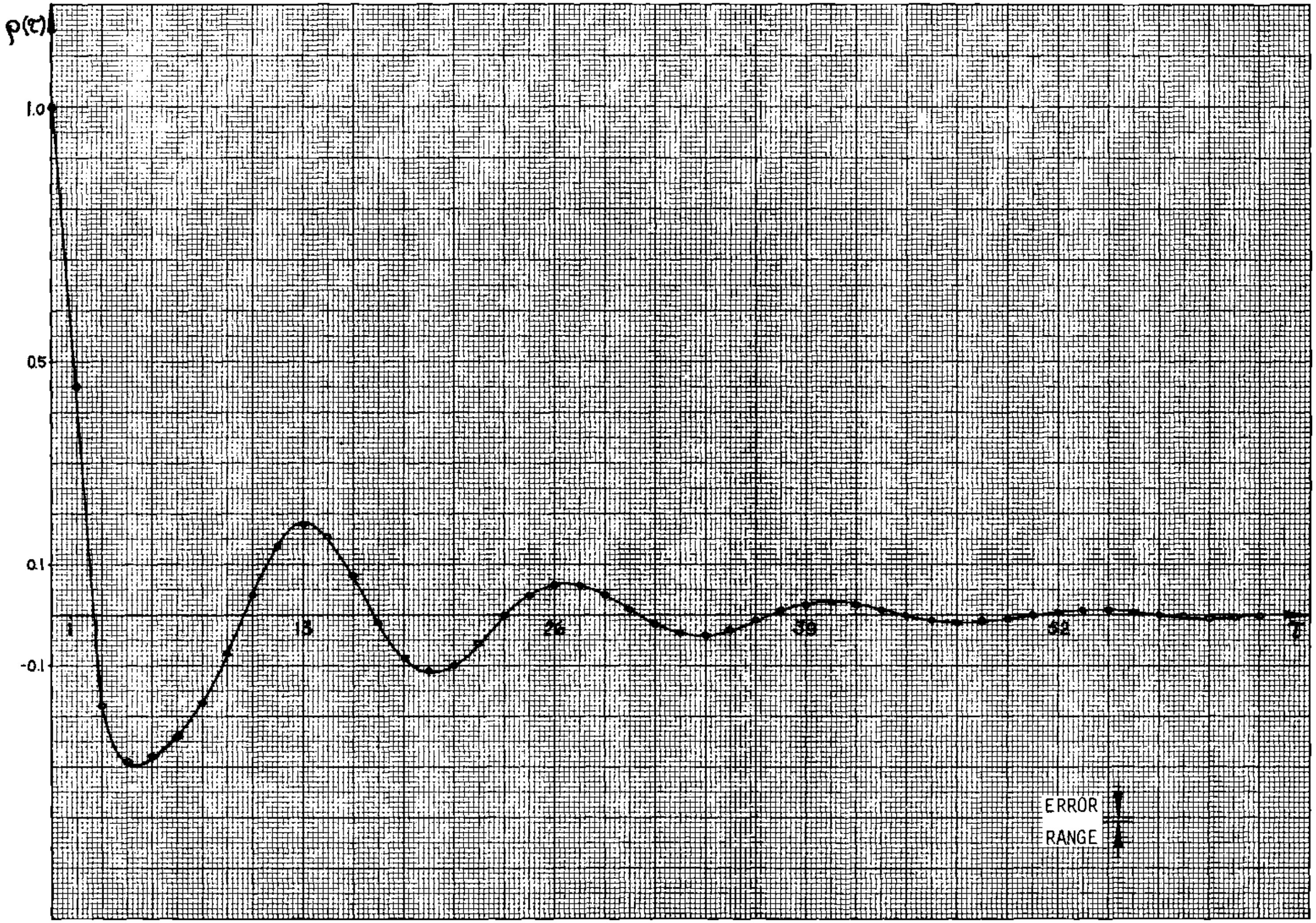

Figure 61. Autocorrelation Function $\rho(\tau)$ for $U \approx N(\mu_0, \sigma_0^2)$ and $V \approx N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 3$, $\mu_1 = 10$, $\sigma_0 = 0.6$, $\sigma_1 = 3.0$.

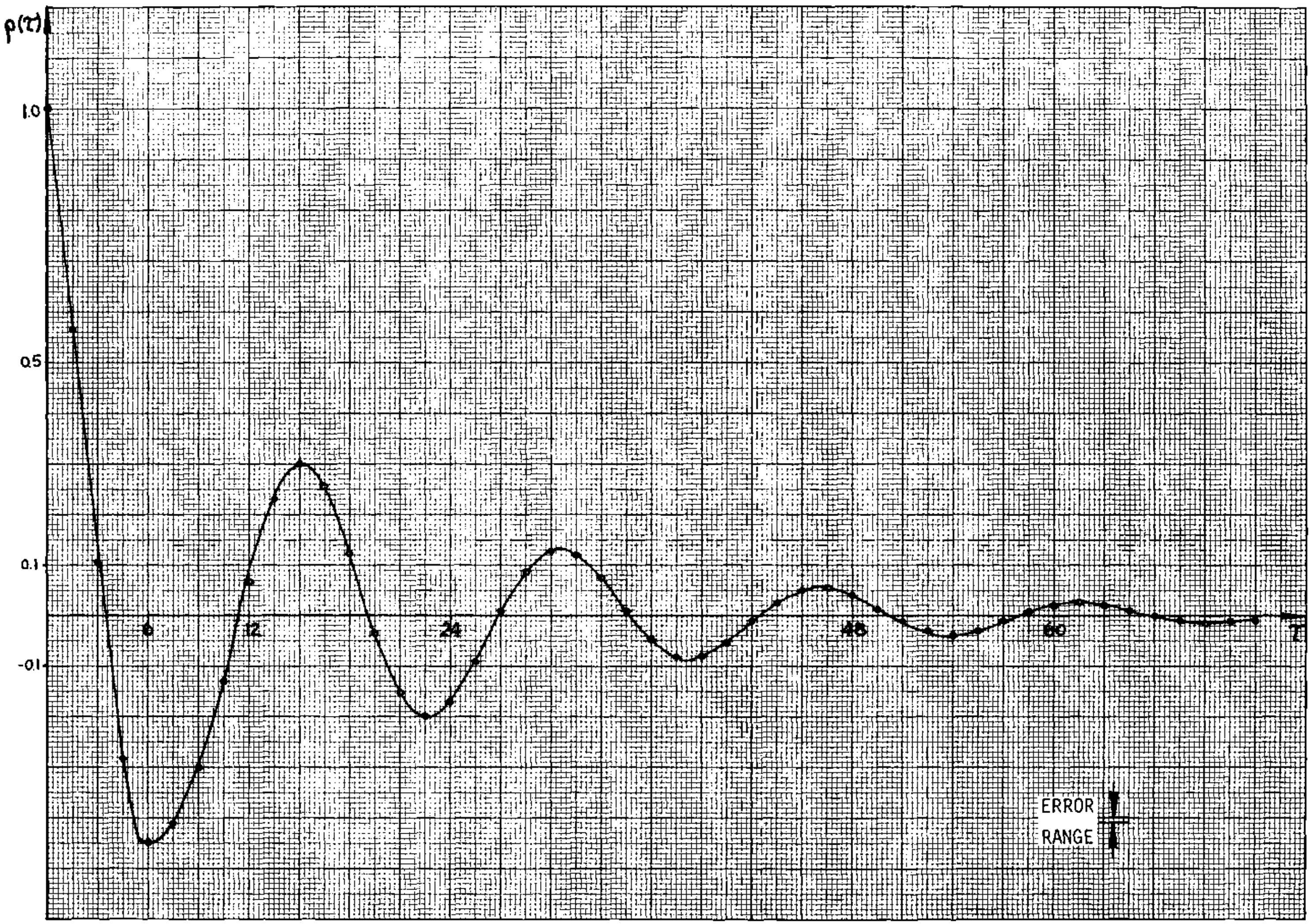

Figure 62. Autocorrelation Function $\rho(\tau)$ for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 5$, $\mu_1 = 10$, $\sigma_0 = 1.0$, $\sigma_1 = 3.0$.

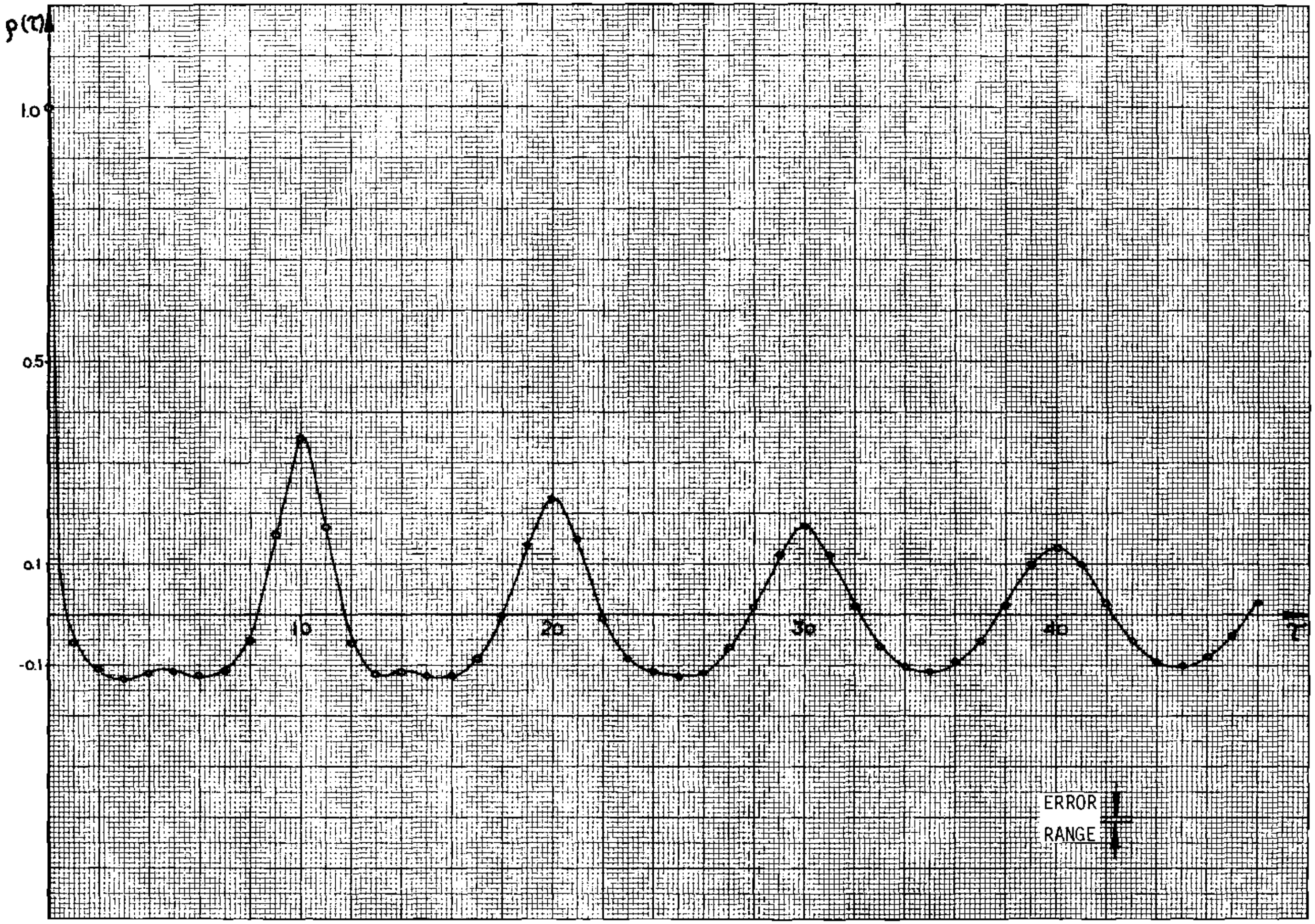

Figure 63. Autocorrelation Function $\rho(\tau)$ for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 9$, $\mu_1 = 1$, $\sigma_0 = 0.9$, $\sigma_1 = 0.1$.

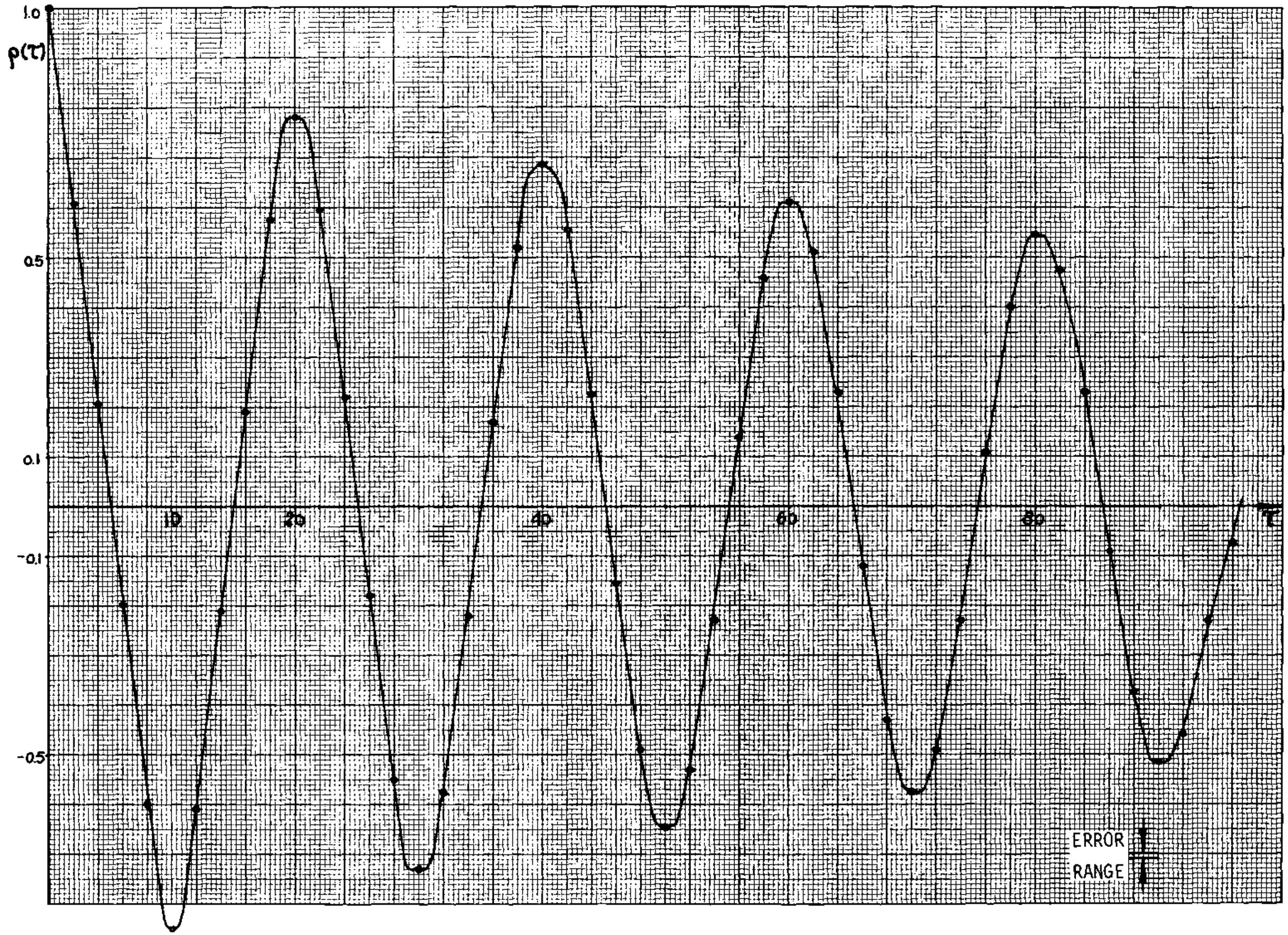

Figure 64. Autocorrelation Function $\rho(\tau)$ for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$.
Parameters: $\mu_0 = 10$, $\mu_1 = 10$, $\sigma_0 = 1.0$, $\sigma_1 = 1.0$.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

In the two cases where the U's are constant and the V's are distributed either exponentially or normally the sets of parameters can be grouped into classes with constant coefficients of variation. In the constant-exponential case four such classes are created with the coefficient or variation (CV) ranging from 0.5 to 0.8 (see Table 4). Each class consists of four parameter combinations, with $u = 1,2,3,4$, respectively, and $\lambda = 1/nu$, n being a constant. In the constant-uniform case six classes are created with the coefficient of variation ranging from 0.29 to 0.53 (see Table 6). Each class here consists of three parameter combinations, with $u = 1,2,3$, respectively, and $t = 2nu$ with n being constant for the class. Figures 5 through 38 reveal, by inspection, that the autocorrelation functions of all members of a class are congruent in a $\rho(\tau)/\tau^1$ coordinate system where

$$\tau^1 = \frac{\tau}{u} \quad . \tag{6.1}$$

This fact leads to the conclusion that there can be derived on the basis of relation (6.1) a number of autocorrelation functions of processes with the above distributions of span lengths, having one of the mentioned coefficients of variation.

The difference between classes is a difference in coefficients of variation. It seems therefore advisable to study the properties of the autocorrelation functions, $\rho(\tau)$, on the basis of the varying coefficients of variation, CV. $\rho(\tau)$ starts at a value of 1 for $\tau = 0$ and decreases for increasing values of τ until it attains its absolute minimum. From there it oscillates around $\rho(\tau) = 0$ with decreasing amplitudes. For both cases considered here the amplitudes of the oscillations, which are the magnitudes of the absolute minimum and subsequent relative optima, are graphed over the coefficient of variation in Figure 65. The points indicating the magnitude of the absolute minimum can be connected by a straight line which passes through CV = 1 in the constant-exponential case and through CV = $\sqrt{3}/3$ in the constant-uniform case. The latter value is familiar from the evaluation of CV_{max} for this case in Chapter IV. The values indicate that in the constant exponential case, for instance, a cycle with coefficient of variation of one leads to an autocorrelation function with absolute minimum on the τ-axis, as to be verified in Figure 39. Points indicating subsequent relative optima can be connected by a straight line, too, although with a slightly smaller margin of exactness.

In general, the amplitudes of the autocorrelation functions for these two distributions of span length decrease rather rapidly. This property is shown in Figure 65 by the straight lines--indicating relative optima--entering the shaded $\rho(\tau) \le 0.05$ region. It follows from the graph that for a higher coefficient of variation the oscillations damp out faster. In none of the considered situations, however,

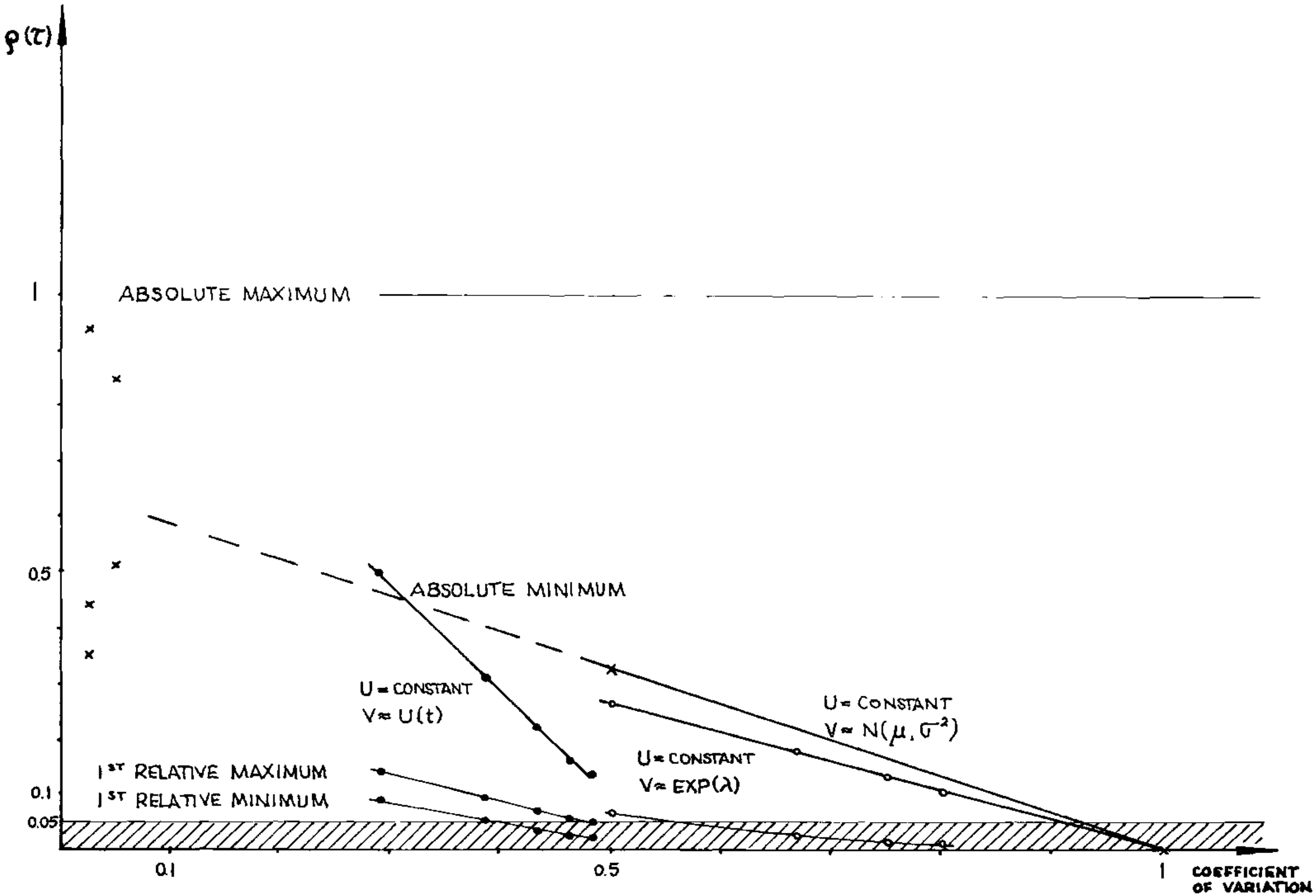

Figure 65. The Amplitudes of the Oscillations of the Autocorrelation Functions for U = Constant and V Distributed Exponentially, Uniformly, and Normally.

does the autocorrelation function drop below the above-mentioned significance level after more than 1.5 periods. The fact that the magnitudes of the optima lie on a straight line provides a possibility to extrapolate those values for autocorrelation functions of processes with different coefficients of variation. For the range of CV from 0.5 to 1.0 in the constant-exponential case and from 0.25 to 0.5 in the constant-uniform case the straight line assumption seems to be sufficiently correct. No knowledge, however, can be assumed whether or not this trend prevails for smaller values of CV. In fact, inspection of Graphs 39 through 46, pertaining to the case where the U's are constant and the V's are distributed normally, gives a somewhat different impression. The coefficients of variation in this case were CV = 1, 0.5, then three times each CV = 0.1 and CV = 0.05. However, the simultaneous occurrence of the same coefficient of variation in this case does not indicate that the parameter sets are again linked by a clear relation as it was previously observable. The parameter combination yielding the same value of CV were not related at all and the resulting autocorrelation functions had three different optimal magnitudes. Figure 65 includes these values for comparison purposes.

The Periods of $\rho(\tau)$

In both the constant-exponential and the constant-uniform case the autocorrelation function does not oscillate with a clearly defined and regular period. In the former case an approximation can be given by the relation

$$\tau_p = u(1.6 + \frac{1}{10\lambda}) \tag{6.2}$$

for the period between two consecutive maxima. It is to be noted that the absolute minimum always occurs at a value of $\tau = u$.

In the constant-uniform case the period between two consecutive maxima is roughly equal

$$\tau_p = (0.4 + \tau)u \ , \tag{6.3}$$

a value which is found to be between the mean cycle length and twice the mean cycle length. In this case, too, it is to be noted that the absolute minimum always occurs at an abscissa value of $\tau = u$. In general, the autocorrelation functions for the zero and uniform densities are rather irregular functions. The oscillations are not symmetric about a vertical axis through an optimum. This property is easily discernible by inspection of Figures 21 through 38. In the case with zero and normal densities for values of $CV \leq 0.1$ the oscillations of the autocorrelation functions are very pronounced, they are symmetric throughout and attain a period between two consecutive maxima which is exactly equal the mean cycle length

$$\tau_p = u + \mu \ . \tag{6.4}$$

The property that the period of the autocorrelation function is equal to the mean cycle length is approached in the exponential-normal case for coefficients of variation below 0.25 (Figures 55 and 56) and is repeated in the case of normal densities for coefficients of variation less than

0.26 (Figures 59 through 64). It is to be mentioned at this point that the lowest CV attained in Case I was CV = 0.5, and CV = 0.29 in Case II. In both cases the choice of parameter combination did not yield coefficients of variation low enough to permit the conclusion that the above property of "period equal mean cycle length" holds also in these cases. There is a strong likelihood for such a conclusion to be reasonable, however. It is mainly supported by the fact that in the constant-uniform case the autocorrelation functions of the process with the lowest coefficient of variation (CV = 0.289) are close to satisfy the property (Figures 21 to 23).

With regard to the period of oscillation the following property can be formulated:

> The autocorrelation functions associated with processes with normal, exponential, uniform or constant distribution of span length (or any combination thereof) oscillate with periods between two consecutive maxima equal the mean cycle length of the process, provided the coefficient of variation does not exceed CV $\leq$ 0.25.

This formulation clearly is a consequence of experimental results and lacks a rigorous mathematical proof. It is given as an experimental guideline, which, for instance, would let it appear permissible to apply its contents to other case distributions.

The Damping of $\rho(\tau)$

It has been pointed out before that the rapidity with which the amplitudes of the autocorrelation functions decrease seems to depend on the coefficient of variation. The higher the value of CV the fewer oscillations it takes for the amplitudes to fall below $|\rho(\tau)|\leq 0.05$. In the case with normal densities for both U and V the autocorrelation functions for values of $CV > 0.2$ damped out after less than three full periods (Figure 66). The amplitudes of the autocorrelation function for $CV = 0.09$ did not decrease below $|\rho(\tau)| = 0.05$ within the five observed periods but showed the tendency to reach this limit within a few oscillations. In the exponential-normal case the lowest attained coefficient of variation was $CV = 0.1414$. The graphs in Figures 47 through 56 revealed that the autocorrelation functions for all considered parameter combinations did damp out after a finite number of periods. Figure 66 shows the number of oscillations performed before falling below the $|\rho(\tau)| = 0.05$ limit in dependence on the coefficients of variation. The changing slopes of the two plots seem to indicate the absence of a persistent narrow trend. The general trend, however, is maintained and it appears certain that an increasing coefficient of variation accelerates the damping of the oscillations.

This conclusion may appear rather vague and it definitely lacks a rigorous mathematical foundation. In order to obtain the latter, attempts should be made to derive envelopes to the autocorrelation functions. Those envelopes would contain the optimal points of each oscillation and would limit the function towards higher and lower values of $\rho(\tau)$. The spectral density functions seems to be the basis for the derivation of the envelopes which are most likely exponential or reciprocal functions. It is recommended to emphasize this problem since the envelope function constitutes the damping factor. Knowledge about the envelopes would determine the amplitudes of the autocorrelation function and their behavior with respect to damping. For small

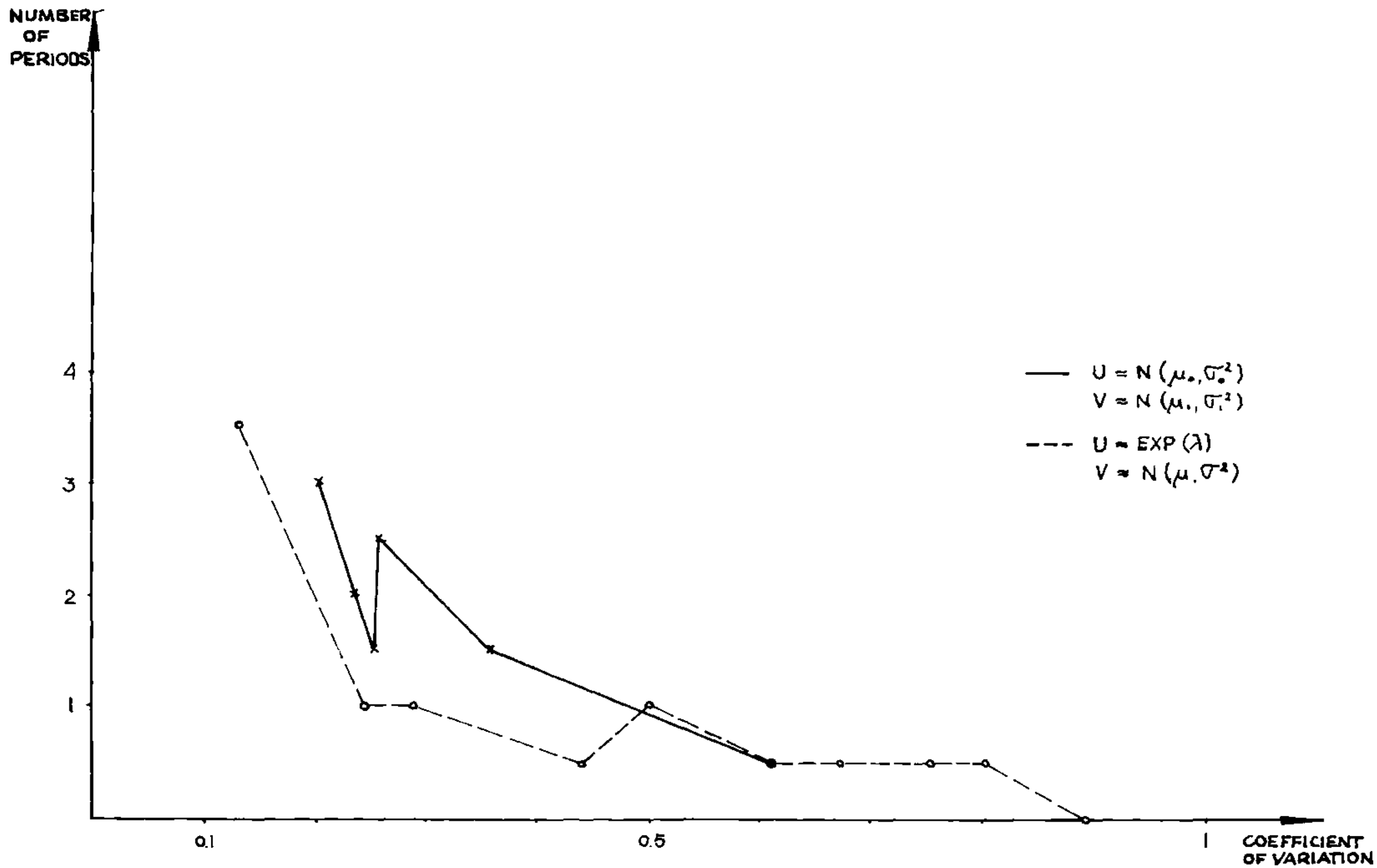

Figure 66. Number of Completed Periods of Autocorrelation Function Before Amplitude Falls Below Significance Level ($\rho(\tau) \leq 0.05$).

coefficients of variation the period can accurately be calculated and it is known that the autocorrelation functions are symmetric about verticals through the optima. With the above information on hand it would be possible to determine an infinite number of autocorrelation functions for processes with certain span lengths distributions.

Some natural extensions of this research are to be recommended. At first it has to be acknowledged that there exist real world systems which are described by a zero-one process with distributions of span lengths different from those with which this study was concerned. For instance, a gamma distribution may be an accurate representation, that is, let

$$f_0(u) = \frac{\lambda_0}{\Gamma(r_0)} (\lambda_0 u)^{r_0 - 1} e^{-\lambda_0 u}, \quad u \geq 0,$$

$$= 0, \text{ elsewhere},$$

and

$$f_1(v) = \frac{\lambda_1}{\Gamma(r_1)} (\lambda_1 v)^{r_1 - 1} e^{-\lambda_1 v}, \quad v \geq 0 \tag{6.5}$$

$$= 0, \text{ elsewhere},$$

where r_0 and r_1 are integers and greater than zero $(r > 0)$. The means and variances are given by

$$E(U) = \mu_0 = \frac{r_0}{\lambda_0}, \quad E(V) = \mu_1 = \frac{r_1}{\lambda_1}, \tag{6.6}$$

$$\mathrm{Var}(U) = \sigma_0^2 = \frac{r_0}{{\lambda_0}^2}\,, \quad \mathrm{Var}(V) = \sigma_1^2 = \frac{r_1}{{\lambda_1}^2}\,. \tag{6.7}$$

The characteristic functions are derived on the basis of expression (3.5), that is,

$$\phi_0(\omega) = E(e^{j\omega U}) = \int_0^\infty e^{j\omega u}\, f_0(u)\, du$$

$$= \frac{\lambda_0^{\,r_0}}{\Gamma(r_0)} \int_0^\infty u^{(r_0-1)}\, e^{-(\lambda - j\omega)u}\, du.$$

Successive integration-by-part, (r_0-1) times, yields

$$\phi_0(\omega) = \left(1 - j\,\frac{\omega}{\lambda_0}\right)^{-r_0}\,. \tag{6.8}$$

Likewise,

$$\phi_1(\omega) = \left(1 - j\,\frac{\omega}{\lambda_1}\right)^{-r_1}\,. \tag{6.9}$$

Applying (6.6), (6.8) and (6.9) to (3.1) the spectral density function for this case is found to

$$S(\omega) = \frac{2\lambda_0\lambda_1}{\omega^2(r_0\lambda_1+r_1\lambda_0)} \cdot \frac{1 - a_0^2a_1^2 - a_0(1-a_1^2)\cos r_0\theta_0 - a_1(1-a_0^2)\cos r_1\theta_1}{1 - 2a_0a_1\cos(r_0\theta_0+r_1\theta_1) + a_0^2a_1^2}\,, \tag{6.10}$$

with

$$a_0 = \left[1 + \frac{\omega^2}{\lambda_0^2}\right]^{-\frac{r_0}{2}}, \qquad a_1 = \left[1 + \frac{\omega^2}{\lambda_1^2}\right]^{-\frac{r_1}{2}},$$

$$\theta_0 = \text{arc tan}\,\frac{\omega}{\lambda_0}, \qquad \theta_1 = \text{arc tan}\,\frac{\omega}{\lambda_1} .$$

On the basis of (6.10) autocorrelation functions can be determined following the procedure outlined in Chapters IV and V.

There are other distributions which accurately enough represent real world situation and which can therefore be considered in this context. The Weibull distribution, for instance, has a wide applicability to a variety of problems which are not necessarily confined to engineering or any particular subject. It has the density function[20]

$$f(x) = \frac{k}{\ell - \varepsilon} \left(\frac{x-\varepsilon}{\ell-\varepsilon}\right)^{k-1} e^{-\left(\frac{x-\varepsilon}{\ell-\varepsilon}\right)^k} \qquad x > \varepsilon, \tag{6.11}$$

$$= 0 \qquad x \le \varepsilon,$$

where ℓ and k are real valued parameters with $\ell > \varepsilon$ and $k > 1$. For this particular application ε would certainly be chosen as $\varepsilon = 0$. The difficulties associated with this case start already at the point where the characteristic function is to be determined.

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20. Parzen, E., *Stochastic Processes*, p. 169.

There is a second natural extension to this study: It is imaginable that each occurrence of the activity is characterized by a set of parameters different from the previous one or by an altogether different distribution. In this case a process is created (Figure 67), in which the activity occurs for the first time with span length equal to a random

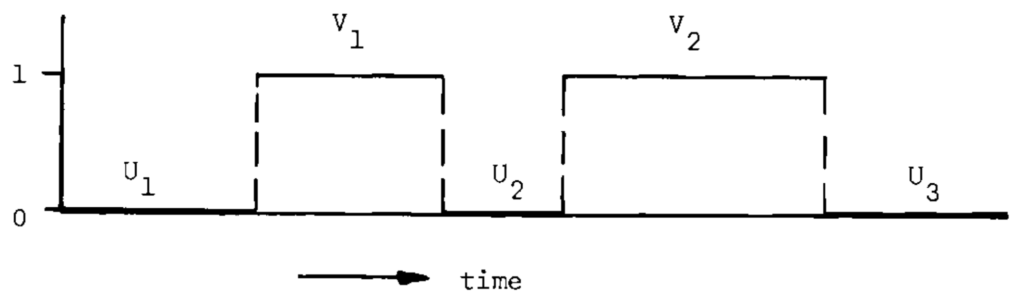

Figure 67. A Different Sample Function of a Zero-One Process

variable V_1, it occurs for the second time with span length equal to a random variable V_2, with $V_1 \neq V_2$, and so forth. If the same property holds for the activity being in state 0, that is $U_1 \neq U_2 \neq U_3$ ..., then the procedure starts with two sums of random variables rather than with two single random variables as before. The sums are

$$S_{0n} = U_1 + U_2 + U_3 + \cdots + U_n, \tag{6.12}$$

$$S_{1n} = V_1 + V_2 + V_3 + \cdots + V_n. \tag{6.13}$$

Rather than attempting to find the probability law of S_{in} and to determine the characteristic function from there, it is to be preferred to proceed the opposite way. It is

$$\phi_0(\omega) = E(e^{j\omega S_{0n}}) = E(e^{j\omega[U_1+U_2+U_3+\cdots+U_n]}) , \qquad (6.14)$$

$$= E(e^{j\omega U_1}) \cdot E(e^{j\omega U_2}) \ldots E(e^{j\omega U_n}) ,$$

$$= \phi_{0_1}(\omega) \cdot \phi_{0_2}(\omega) \ldots \phi_{0_n}(\omega) .$$

Hence, the characteristic function of the sum of independent random variables is equal to the product of the characteristic functions of each of the random variables separately.

This property facilitates the task of determining the characteristic function. The expression for the spectral density function (3.1) upon substitution of those products, however, will be further complicated.

APPENDIX

APPENDIX A

It was indicated earlier that a brief comparison would be conducted between the findings of W. W. Hines[21] and the results of this study. For the case where both the U's and the V's are normally distributed two sets of parameters were singled out. Figure 68 represents the case where $U \approx N(10,6)$ and $V \approx N(10,6)$ and Figure 69 depicts the combination $U \approx N(20,4)$ and $V \approx N(4,1)$. The correlogram from simulation for both cases was taken from Hines' results.[22] Figures 68 and 69 show that both methods, simulation and spectral analysis, lead to about the same result. The differences are marginal. The method of spectral analysis, however, provides more precise values for the various optima of the initial oscillations. Since these two comparisons did not show any differences and since Hines did not change his method of attack, it can readily be assumed that an equally high degree of correspondence will be factual for all 18 correlograms from simulation. Thus, the results of this study and Hines' findings supplement each other favorably.

There is, however, a vital difference between the methods employed. Whereas for the simulation experiments each autocorrelation function required 109 minutes to be determined, the respective figure for the method of spectral analysis was 6-10 minutes.

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21. Hines, W. W., Ph.D. Dissertation.

22. *Ibid.*, pp. 118, 120.

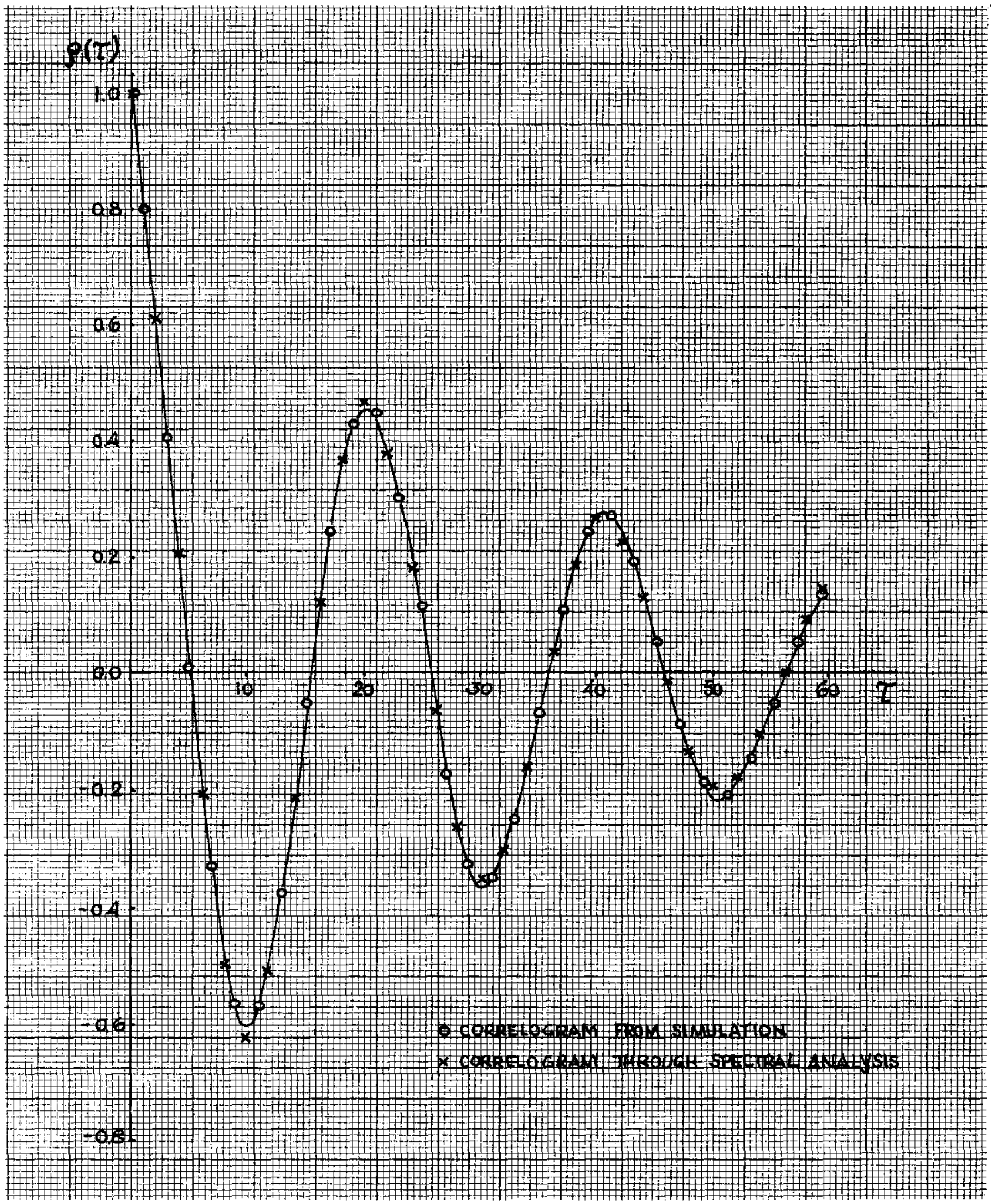

Figure 68. Comparison of Autocorrelation Functions Determined Through Simulation and Through Spectral Analysis for U ≃ N(10, 6) and V ≃ N(10, 6).

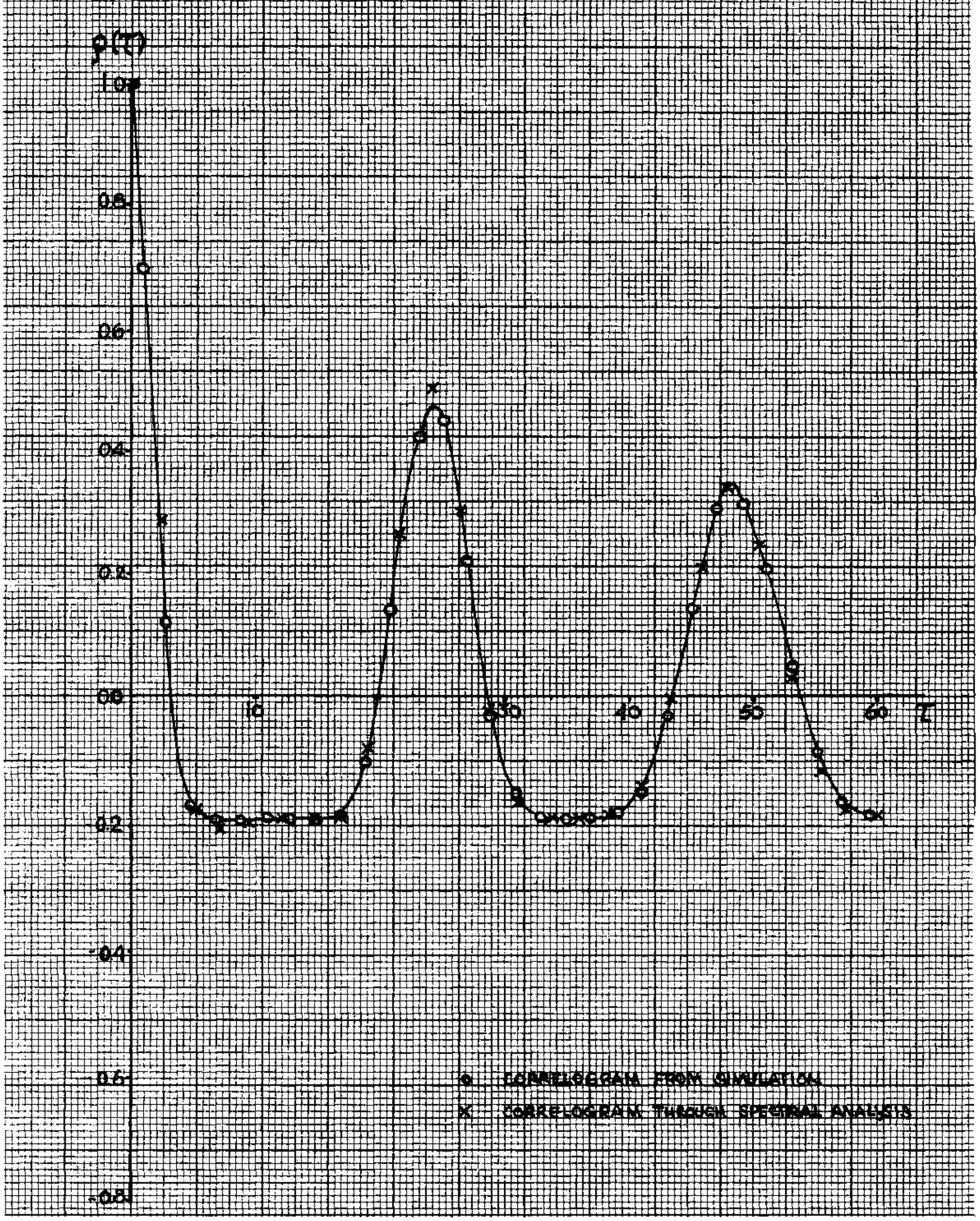

Figure 69. Comparison of Autocorrelation Functions Determined Through Simulation and Through Spectral Analysis for U ≈ N(20, 4) and V ≈ N(4, 1).

Table 14. Numerical Values of Autocovariance and Autocorrelation Function Obtained Through Spectral Analysis

AUTOCOVARIANCE R(TAU) AND AUTOCORRELATION FUNCTION RHO(TAU)
FOR U=N(MUO,SIGO) AND V=N(MU1,SIG1) [ERROR ≤ 0.004]

PARAMETERS: MU[0] = 20 SIGMA[0] = 2.0
MU[1] = 4 SIGMA[1] = 1.0

le 14

TAU	R(TAU)	RHO(TAU)
0.0	1.320E-01	1.000E+00
2.4	3.768E-02	2.855E-01
4.8	-2.335E-02	-1.769E-01
7.2	-2.703E-02	-2.049E-01
9.6	-2.577E-02	-1.952E-01
12.0	-2.531E-02	-1.918E-01
14.4	-2.545E-02	-1.928E-01
16.8	-2.405E-02	-1.901E-01
19.2	-1.035E-02	-7.844E-02
21.6	3.349E-02	2.537E-01
24.0	6.493E-02	4.920E-01
26.4	3.908E-02	2.962E-01
28.8	-3.627E-03	-2.748E-02
31.2	-2.265E-02	-1.716E-01
33.6	-2.534E-02	-1.921E-01
36.0	-2.492E-02	-1.888E-01
38.4	-2.411E-02	-1.827E-01
40.8	-1.863E-02	-1.412E-01
43.2	-6.467E-04	-4.900E-03
45.6	2.779E-02	2.106E-01
48.0	4.402E-02	3.335E-01
50.4	3.128E-02	2.370E-01
52.8	3.778E-03	2.863E-02
55.2	-1.588E-02	-1.204E-01
57.6	-2.309E-02	-1.750E-01
60.0	-2.386E-02	-1.808E-01
62.4	-2.138E-02	-1.620E-01
64.8	-1.335E-02	-1.012E-01
67.2	2.841E-03	2.153E-02
69.6	2.242E-02	1.699E-01
72.0	3.246E-02	2.460E-01
74.4	2.475E-02	1.876E-01
76.8	5.893E-03	4.466E-02
79.2	-1.111E-02	-8.415E-02
81.6	-1.981E-02	-1.501E-01
84.0	-2.150E-02	-1.629E-01
86.4	-1.824E-02	-1.382E-01
88.8	-9.791E-03	-7.420E-02
91.2	3.857E-03	2.923E-02
93.6	1.812E-02	1.373E-01
96.0	2.489E-02	1.886E-01
98.4	1.953E-02	1.480E-01
100.8	5.841E-03	4.426E-02
103.2	-7.941E-03	-6.018E-02
105.6	-1.632E-02	-1.236E-01
108.0	-1.845E-02	-1.398E-01
110.4	-1.526E-02	-1.157E-01
112.8	-7.283E-03	-5.519E-02
115.2	4.020E-03	3.046E-02
117.6	1.462E-02	1.108E-01
120.0	1.918E-02	1.454E-01
122.4	1.522E-02	1.153E-01
124.8	5.115E-03	3.876E-02
127.2	-5.747E-03	-4.355E-02
129.6	-1.317E-02	-9.981E-02

PROCESS = 445.93333 SECS.
I/O = 12.43333 SECS.
RUN TIME = 458.36667 SECS.

PARAMETERS: MU[0] = 10 SIGMA[0] = 2.4 = √6
MU[1] = 10 SIGMA[1] = 2.4 = √6

TAU	R(TAU)	RHO(TAU)
0.0	2.455E-01	1.000E+00
2.0	1.503E-01	6.121E-01
4.0	5.069E-02	2.065E-01
6.0	-4.492E-02	-1.830E-01
8.0	-1.213E-01	-4.939E-01
10.0	-1.526E-01	-6.216E-01
12.0	-1.227E-01	-4.996E-01
14.0	-5.046E-02	-2.055E-01
16.0	2.926E-02	1.192E-01
18.0	8.975E-02	3.655E-01
20.0	1.132E-01	4.612E-01
22.0	9.416E-02	3.835E-01
24.0	4.378E-02	1.783E-01
26.0	-1.677E-02	-6.832E-02
28.0	-6.474E-02	-2.637E-01
30.0	-8.457E-02	-3.445E-01
32.0	-7.281E-02	-2.966E-01
34.0	-3.650E-02	-1.486E-01
36.0	8.872E-03	3.614E-02
38.0	4.608E-02	1.877E-01
40.0	6.316E-02	2.573E-01
42.0	5.619E-02	2.289E-01
44.0	3.002E-02	1.223E-01
46.0	-3.729E-03	-1.519E-02
48.0	-3.267E-02	-1.331E-01
50.0	-4.701E-02	-1.915E-01
52.0	-4.311E-02	-1.756E-01
54.0	-2.459E-02	-1.002E-01
56.0	6.747E-04	2.748E-03
58.0	2.307E-02	9.396E-02
60.0	3.479E-02	1.417E-01
62.0	3.311E-02	1.349E-01
64.0	1.995E-02	8.125E-02
66.0	1.043E-03	4.248E-03
68.0	-1.604E-02	-6.534E-02
70.0	-2.575E-02	-1.049E-01
72.0	-2.535E-02	-1.033E-01
74.0	-1.596E-02	-6.500E-02
76.0	-2.048E-03	-8.342E-03
78.0	1.113E-02	4.533E-02
80.0	1.906E-02	7.763E-02
82.0	1.929E-02	7.855E-02
84.0	1.280E-02	5.212E-02
86.0	2.479E-03	1.010E-02
88.0	-7.713E-03	-3.142E-02
90.0	-1.397E-02	-5.690E-02
92.0	-1.472E-02	-5.995E-02
94.0	-1.021E-02	-4.159E-02
96.0	-2.479E-03	-1.010E-02
98.0	5.218E-03	2.125E-02
100.0	1.029E-02	4.190E-02
102.0	1.127E-02	4.591E-02
104.0	8.041E-03	3.275E-02
106.0	2.412E-03	9.822E-03
108.0	-3.496E-03	-1.424E-02
110.0	-7.636E-03	-3.110E-02

PROCESS = 387.30000 SECS.
I/O = 10.90000 SECS.
RUN TIME = 398.20000 SECS.

APPENDIX B

This appendix contains the plots of the various spectral density functions obtained as output of an algol program. The program utilized is reproduced in Table 15. It consists of two parts, the actual plot routine and the part taking account of the particular spectrum involved. The former has been created by Richard Rosenbaum of the Rich Electronic Computer Center, Georgia Institute of Technology. As for Table 15, the latter pertains to the constant-normal case of span length distributions. This latter part was subject to change and had to be reformulated for each of the five considered cases.

The plots of the spectral density functions appear in Figures 70 through 99. There are two functions on each print distinguished by characters A and B. The respective parameter combinations pertaining to each of the curves are printed in the upper right-hand corner of each graph. The output is based on an XY-coordinate system. The X-axis, the abscissa, is the ω-axis in this context, whereas the Y-axis as ordinate stands for the spectral density function $S(\omega)$.

The plots serve as supporting evidence for the findings in Chapter IV. In the context of the material presented there, the graphs are widely self-explanatory. Consideration, however, has to be given to the fact that apparent discontinuities of the function $S(\omega)$ in the graphical form are explainable as round-off errors associated with the plotted numerical values of $S(\omega)$.

Table 15. Algol Program: Plotting of the Spectral Density Functions.

```
BEGIN
INTEGER U,MU ;
REAL SIG ;
FILE IN INPIN (2,10) ;
FILE OUT INPOUT 16(2,15) ;
FORMAT HEAD (X3,"SPECTRAL DENSITY S(W) FOR U=CONST AND V=N(MU,SIG)"),
       PARM1 (/,X3,"PARAMETERS:",X3,"U = ",I1,X5,"MU = ",I2,X5,
                               "SIGMA = ",F4.2,      "(A)"),
       PARM2 (/,X3,"PARAMETERS:",X3,"U = ",I1,X5,"MU = ",I2,X5,
                               "SIGMA = ",F4.2,      "(B)");
REAL PROCEDURE S(W) ;
                  VALUE W;
                  REAL  W;
   IF W=0 THEN S←((SIG*2)×(U*2))/((U+MU)*4)) ELSE
              S←((2×((1-EXP(-(W*2)×(SIG*2)))×(1-COS(W×U)))/((MU+U)×
                (W*2)×(1-(2×EXP(-((W*2)×(SIG*2)/2))×COS(W×(MU+U)))
                +(EXP(-(W*2)×(SIG*2)))))) ;
BOOLEAN PREFILE;
SAVE FILE PLOTSYS DISK SERIAL [20:1000] (2,20,SAVE 2);
PROCEDURE PLOTTER (FUNCT,XMIN,XMAX,XINC,YMAX,YMIN,FILE1,CHAR,PRINT);
VALUE CHAR;
FILE OUT FILE1; ALPHA CHAR; BOOLEAN PRINT;
REAL PROCEDURE FUNCT ;
REAL XMIN,XMAX,YMIN,YMAX,XINC;
        BEGIN
INTEGER       NHL,NSBH,NVL,NSBV,TNHL,TNVL,NDATA,M,N,MI;

REAL I,XI,TYMIN,TYMAX;
FORMAT FMT1(X3,*(F6.2,X*5));

FORMAT FMT2(F6.2,*(19A6));

FORMAT FMT3(20A6);

FORMAT FMT4(F12.5,*,*);

FORMAT
FMTMN ("THE Y AXIS HAS A SCALE FACTOR OF 10 TO THE",I4,/
        "THE X AXIS HAS A SCALE FACTOR OF 10 TO THE",I4,///,X57,
              "Y AXIS");

IF PREFILE THEN BEGIN ;
                READ (PLOTSYS,/,YMIN,YMAX,XMIN,XMAX,NDATA,N,M);
XINC←ABS(XMAX-XMIN)/(NDATA-1); END ELSE
NDATA←ABS((XMAX-XMIN)/XINC)+1;
                NSBH←10; NVL←11; NSBV←10;
NHL←(NDATA-1)/10;

TNHL←NHL×NSBH  ; TNVL←(NVL-1)×NSBV;
       BEGIN
ALPHA ARRAY IMAGE [0:TNHL,0:20];

REAL ARRAY     ORD[0:TNHL],ABSCIS[0:TNVL],X,Y[0:NDATA]  ;
ALPHA ARRAY    TSIX[0:6]  ;
INTEGER        I,J,JU1,I1,J1,K;
REAL EACH;
FORMAT OUT     FO ("THE POINT (X,Y) = (",2F10.5,") IS OUT OF RANGE"/)  ;

REAL          TEMX,TEMY  ;
LABEL TEND1;
I←1; XI←XMIN;
WHILE XI≤XMAX DO BEGIN ;
Y[I]←FUNCT(XI); X[I]←XI;
XI←XI+XINC; I←I+1; END;
IF PREFILE THEN BEGIN
  FOR I←0 STEP 1 UNTIL TNHL DO
  READ (PLOTSYS,20,IMAGE[I,*]); REWIND (PLOTSYS) END ELSE BEGIN
                FOR I ← 0 STEP 1 UNTIL TNHL DO
                FILL IMAGE[I,*] WITH "      ","      ","      ","      ",
                "      ","      ","      ","      ","      ","      ",
                "      ","      ","      ","      ","      ","      ",
                "      ","      ","      ","      " ;
TYMIN←10*40; TYMAX←-10*40;
FOR I←1 STEP 1 UNTIL NDATA DO BEGIN ;
IF TYMIN>Y[I] THEN TYMIN←Y[I];
IF TYMAX <Y[I] THEN TYMAX←Y[I];
END;
IF YMAX>TYMAX THEN YMAX←TYMAX;
IF YMIN<TYMIN THEN YMIN←TYMIN;
IF ((YMAX-YMIN) MOD TNVL) MOD 0.01>0.1 THEN
DO YMAX←YMAX+0.01  UNTIL ((YMAX-YMIN) MOD TNVL) MOD 0.01<0.1;
IF YMIN<-99.99 OR YMAX>99.99 THEN BEGIN
IF ABS(YMIN)>ABS(YMAX) THEN TYMAX←ABS(YMIN) ELSE TYMAX←ABS(YMAX); N←0;
DO N←N+1 UNTIL (TYMAX←TYMAX/10)≤99.99;
END;
IF XMIN<-99.99 OR XMAX>99.99 THEN BEGIN
IF ABS(XMIN)>ABS(XMAX) THEN TYMAX←ABS(XMIN) ELSE TYMAX←ABS(XMAX); M←0;
DO N←N+1 UNTIL (TYMAX←TYMAX/10)<99.99;
END;
END;
ABSCIS[0]←YMIN; ABSCIS[TNVL]←YMAX;
ORD[0]←XMIN; ORD[TNHL]←XMAX;
IF NOT PREFILE THEN ;
        BEGIN
JU1← TNVL DIV 6; IF TNHL MOD 6 ≠ 0 THEN JU1←JU1+1;
            FILL TSIX[*] WITH "      ","      ","      ","      ",
                "      ","      ","      " ;

                FOR I ← 1 STEP 1 UNTIL TNHL DO
        BEGIN
                FOR J ← 1  STEP NSBV UNTIL TNVL+1 DO ;
                IMAGE[I,(J DIV 6)+1] ← TSIX[(J MOD 6)] ;
                IF I MOD NSBH = 0 THEN
        BEGIN
                J ← 1  ;  DO
        BEGIN
                IMAGE[I,J] ← "------"
        END
                UNTIL (J ← J+1) > JU1
        END
        END
        END  ;
COMMENT         THE "ABSCIS" AND "ORD" VECTORS ARE FILLED WITH UNIFORMLY
                INCREASING INCREMENTS OF X AND Y FORM ONE LIMIT TO THE
                OTHER  E.G. ABSCIS[0] = YMIN  ,  ORD[0] = XMIN  .  ;
EACH←ABS((YMAX-YMIN))/TNVL;
                JU1 ← TNVL-1  ;
                FOR J ← 1 STEP 1 UNTIL JU1 DO
                ABSCIS[J] ← J×EACH + YMIN  ;
FOR J←1 STEP 1 UNTIL TNHL-1 DO ;
ORD[J]←J×XINC+XMIN;
                I ← 1  ;  DO
        BEGIN
COMMENT         TEMX AND TEMY ARE THE COORDINATES OF THE POINT CURRENTLY
                BEING TESTED FOR PLOTTING INTO THE IMAGE MATRIX  .  ;
                TEMX ← Y[I]  ;  TEMY ← X[I]  ;  IF (TEMX < ABSCIS[0]) OR
                (ABSCIS[TNVL] < TEMX) OR (TEMY > ORD[TNHL]) OR (ORD[0] >
                TEMY) THEN
        BEGIN
                WRITE (FILE1,FO,TEMX,TEMY)  ;  GO TO TEND1
        END  ;
                I1 ← -1  ;  DO I1 ← I1+1 UNTIL (ABSCIS[I1] ≤ TEMX AND TEMX
                ≤ ABSCIS[I1+1])  ;  IF ABS(TEMX - ABSCIS[I1]) >
                ABS(ABSCIS[I1+1] - TEMX) THEN I1 ← I1+1  ;
                J1 ← -1 ;  DO J1 ← J1+1 UNTIL (ORD[J1] ≤ TEMY AND TEMY ≤
                ORD[J1+1])  ;  IF ABS(ORD[J1] - TEMY) >ABS(TEMY
                - ORD[J1+1]) THEN J1 ← J1+1  ;
                K ← (I1 DIV 6) + 1  ;  MI← I1 MOD 6  ;
                TEMX ← IMAGE[J1  ,K]  ;
CASE MI OF BEGIN ;
TEMX←TEMX&CHAR[12:42:6];
TEMX←TEMX&CHAR[18:42:6];
TEMX←TEMX&CHAR[24:42:6];
TEMX←TEMX&CHAR[30:42:6];
TEMX←TEMX&CHAR[36:42:6];
TEMX←TEMX&CHAR[42:42:6];
END;

                IMAGE[J1  ,K] ← TEMX  ;
TEND1:    END
                UNTIL (I ← I+1) > NDATA  ;
IF NOT PREFILE THEN BEGIN ;
FOR I←0 STEP 1 UNTIL NVL-1 DO ;
IMAGE[0,I]←ABSCIS[I×NSBV];
IMAGE[I,0]←ORD[0];
FOR I←NSBH STEP NSBH UNTIL TNHL DO ;
IMAGE[I,0]←ORD[I]; END;
IF PRINT THEN BEGIN ;
WRITE (FILE1[PAGE]);
WRITE (FILE1,FMTMN,M,N);
WRITE (FILE1,FMT1,NVL+1,FOR I←0 STEP 1 UNTIL NVL-1 DO
[IMAGE[0,I],NSBV*6]);
FOR I←1 STEP 1 UNTIL TNHL DO ;
IF IMAGE[I,0]="      " THEN WRITE (FILE1,FMT3,
FOR J←0 STEP 1 UNTIL 19 DO IMAGE[I,J]) ELSE ;
WRITE (FILE1,FMT2,FOR J←0 STEP 1 UNTIL 19 DO IMAGE[I,J]);
PREFILE←FALSE;

END ELSE BEGIN ;
IF NOT PREFILE THEN BEGIN ;
WRITE(PLOTSYS,FMT4,YMIN,YMAX,XMIN,XMAX,NDATA,N,M); ;
WRITE (PLOTSYS,20,IMAGE[0,*]); END ELSE SPACE(PLOTSYS,1); ;
FOR I←1 STEP 1 UNTIL TNHL DO WRITE (PLOTSYS,20,IMAGE[I,*]);
REWIND (PLOTSYS); PREFILE←TRUE;
END;
END;

END;
    U ← 1 ;
    MU ← 0 ; SIG ← 1 ;
      WRITE (INPOUT[PAGE]) ;
      WRITE (INPOUT,HEAD) ;
      WRITE (INPOUT,PARM1,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,1,0,INPOUT,"A",FALSE) ;
    U ← 5 ;
    MU ← 5 ; SIG ← 1 ;
      WRITE (INPOUT,PARM2,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,1,0,INPOUT,"B",TRUE) ;
    U ← 1 ;
    MU ← 1 ; SIG ← 1 ;
      WRITE (INPOUT[PAGE]) ;
      WRITE (INPOUT,HEAD) ;
      WRITE (INPOUT,PARM1,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,0.5,0,INPOUT,"A",FALSE) ;
    U ← 5 ;
    MU ← 10 ; SIG ← 1.5 ;
      WRITE (INPOUT,PARM2,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,0.5,0,INPOUT,"B",TRUE) ;
    U ← 1 ;
    MU ← 1 ; SIG ← 0.1 ;
      WRITE (INPOUT[PAGE]) ;
      WRITE (INPOUT,HEAD) ;
      WRITE (INPOUT,PARM1,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,3.5,0,INPOUT,"A",FALSE) ;
    U ← 5 ;
    MU ← 15 ; SIG ← 1 ;
      WRITE (INPOUT,PARM2,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,3.5,0,INPOUT,"B",TRUE) ;
    U ← 1 ;
    MU ← 2 ; SIG ← 0.5 ;
      WRITE (INPOUT[PAGE]) ;
      WRITE (INPOUT,HEAD) ;
      WRITE (INPOUT,PARM1,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,2.5,0,INPOUT,"A",FALSE) ;
    U ← 5 ;
    MU ← 20 ; SIG ← 1.25 ;
      WRITE (INPOUT,PARM2,U,MU,SIG) ;
      PLOTTER (S,0,20,0.1,2.5,0,INPOUT,"B",TRUE) ;
END.
```

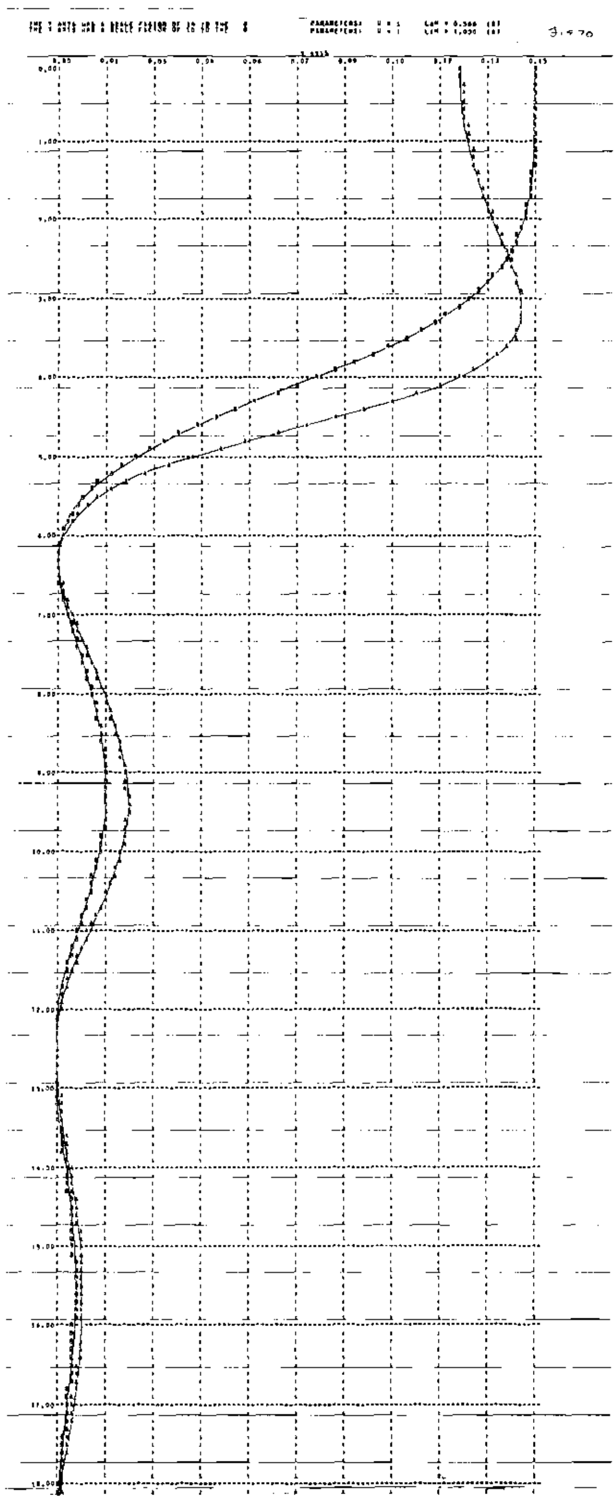

Figure 70. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ EXP(λ).

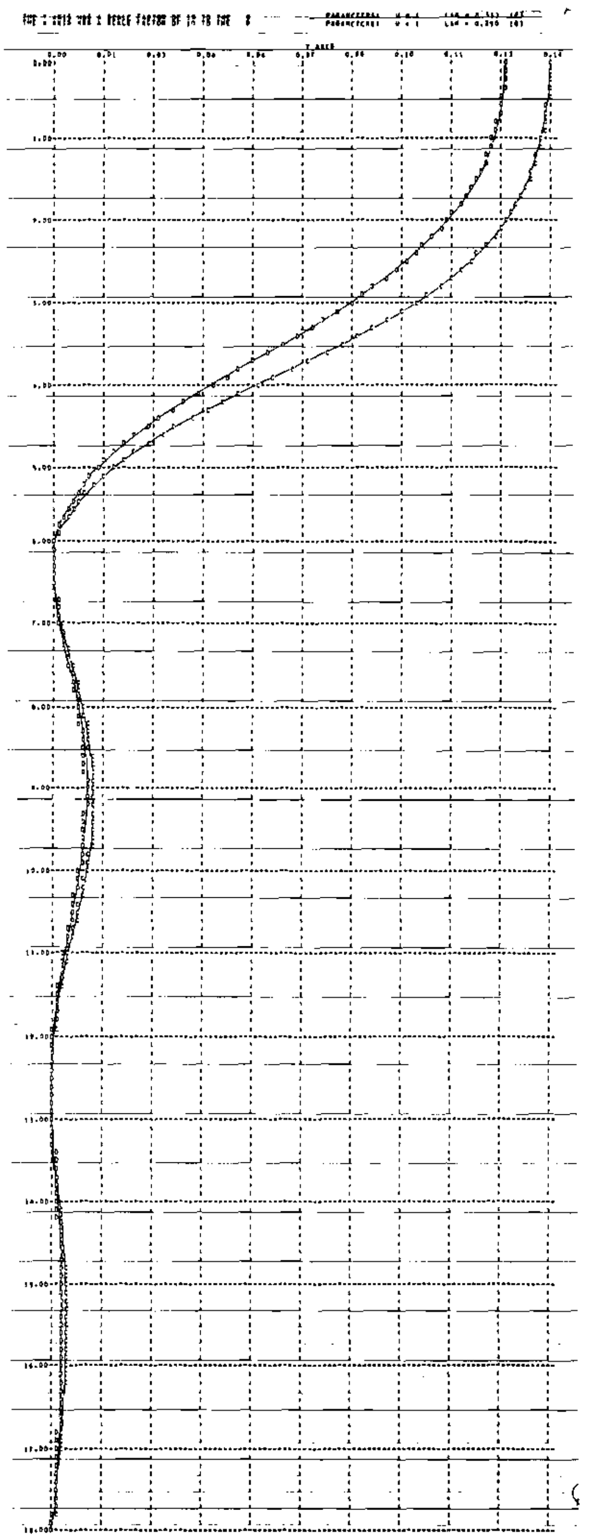

Figure 71. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for U = Constant and $V \approx EXP(\lambda)$.

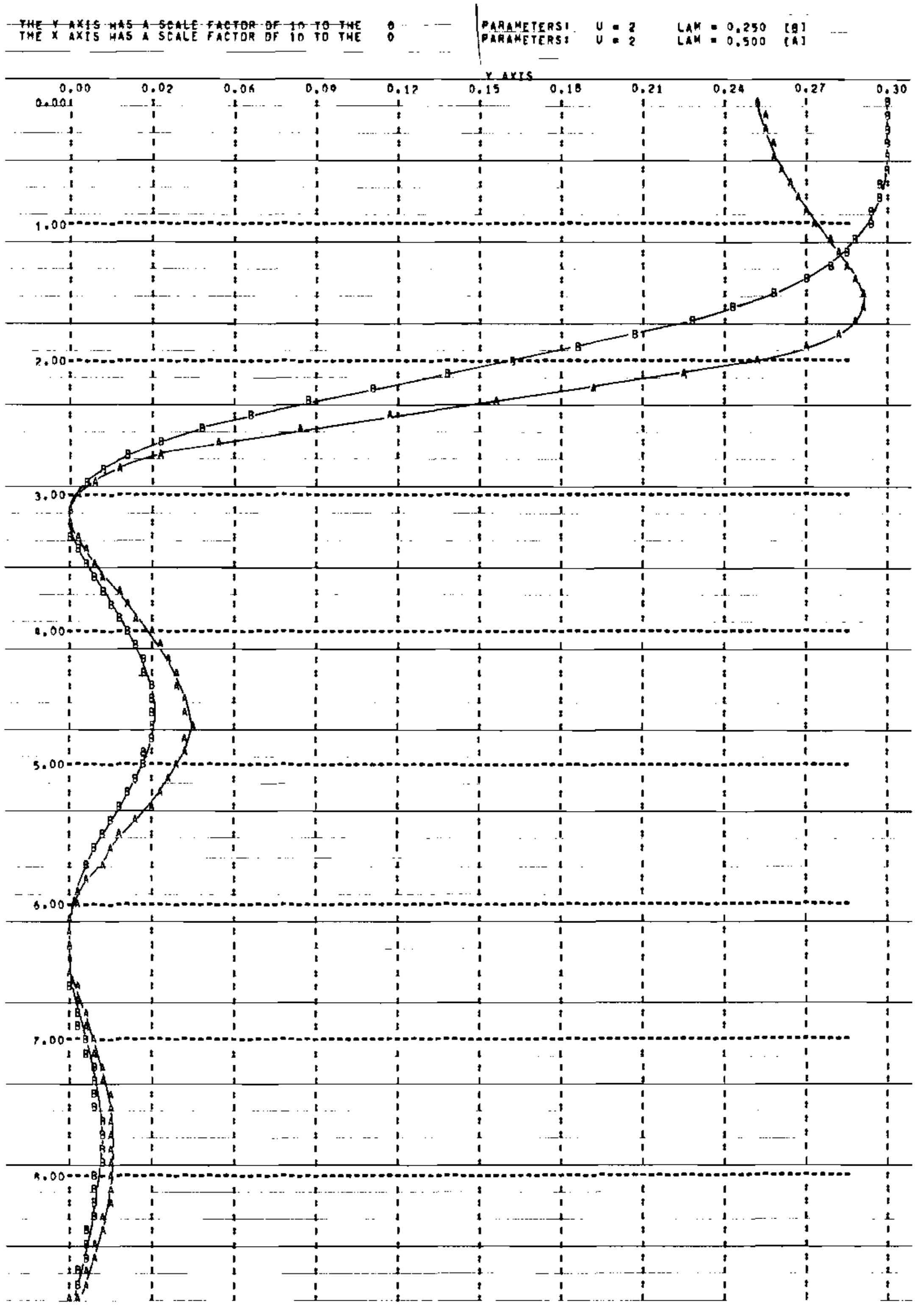

Figure 72. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ EXP(λ).

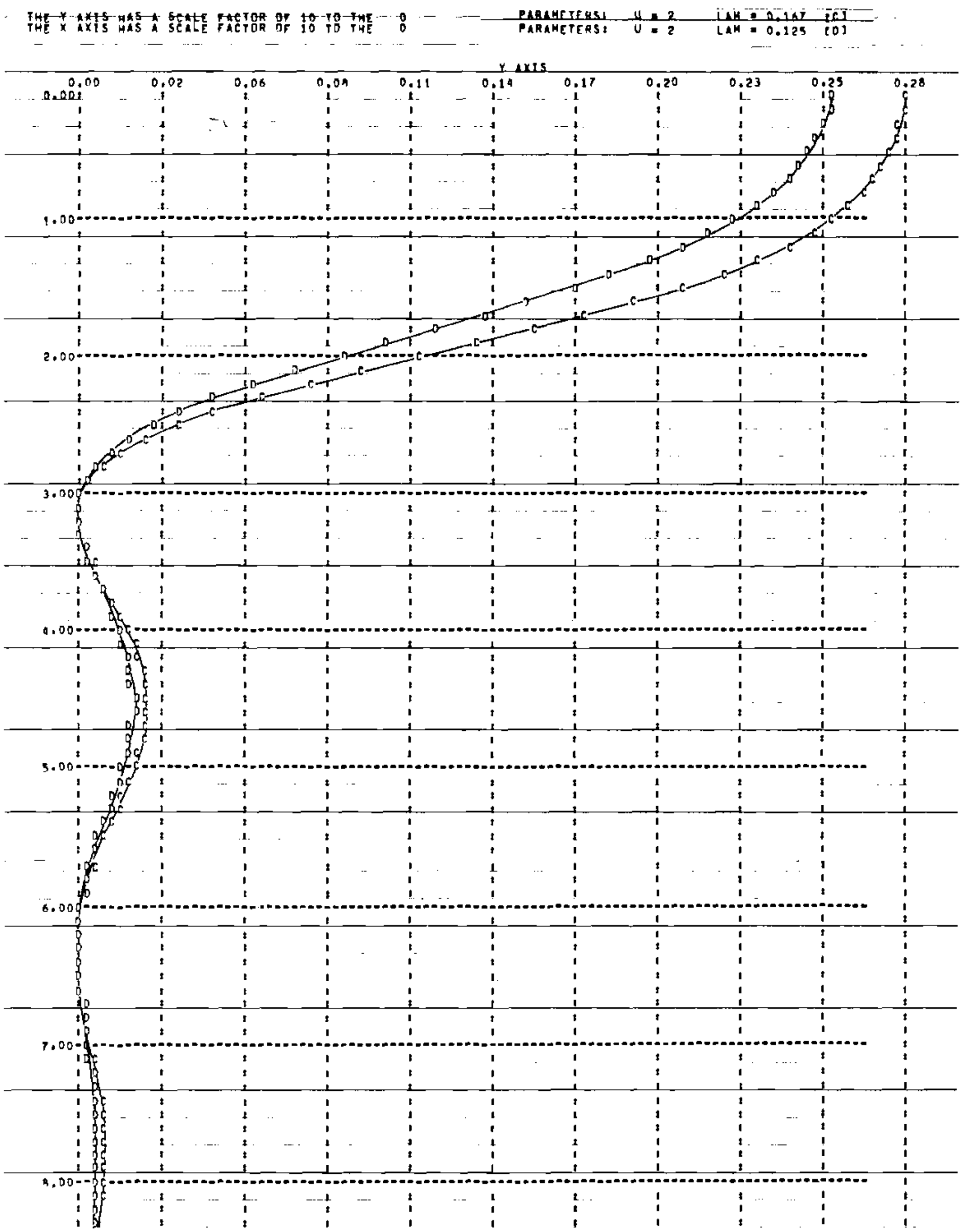

Figure 73. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for U = Constant and $V \approx EXP(\lambda)$.

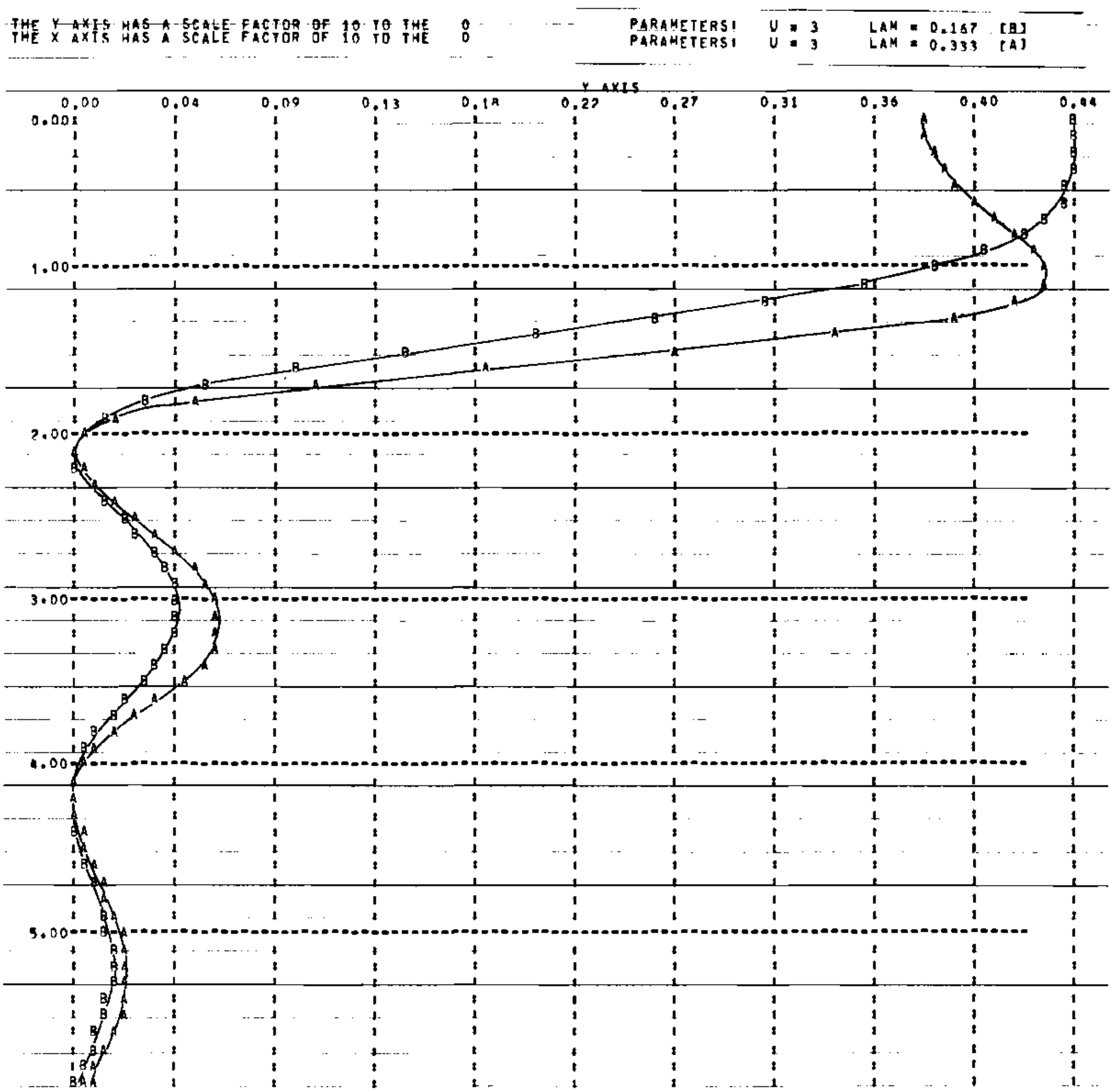

Figure 74. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ EXP(λ).

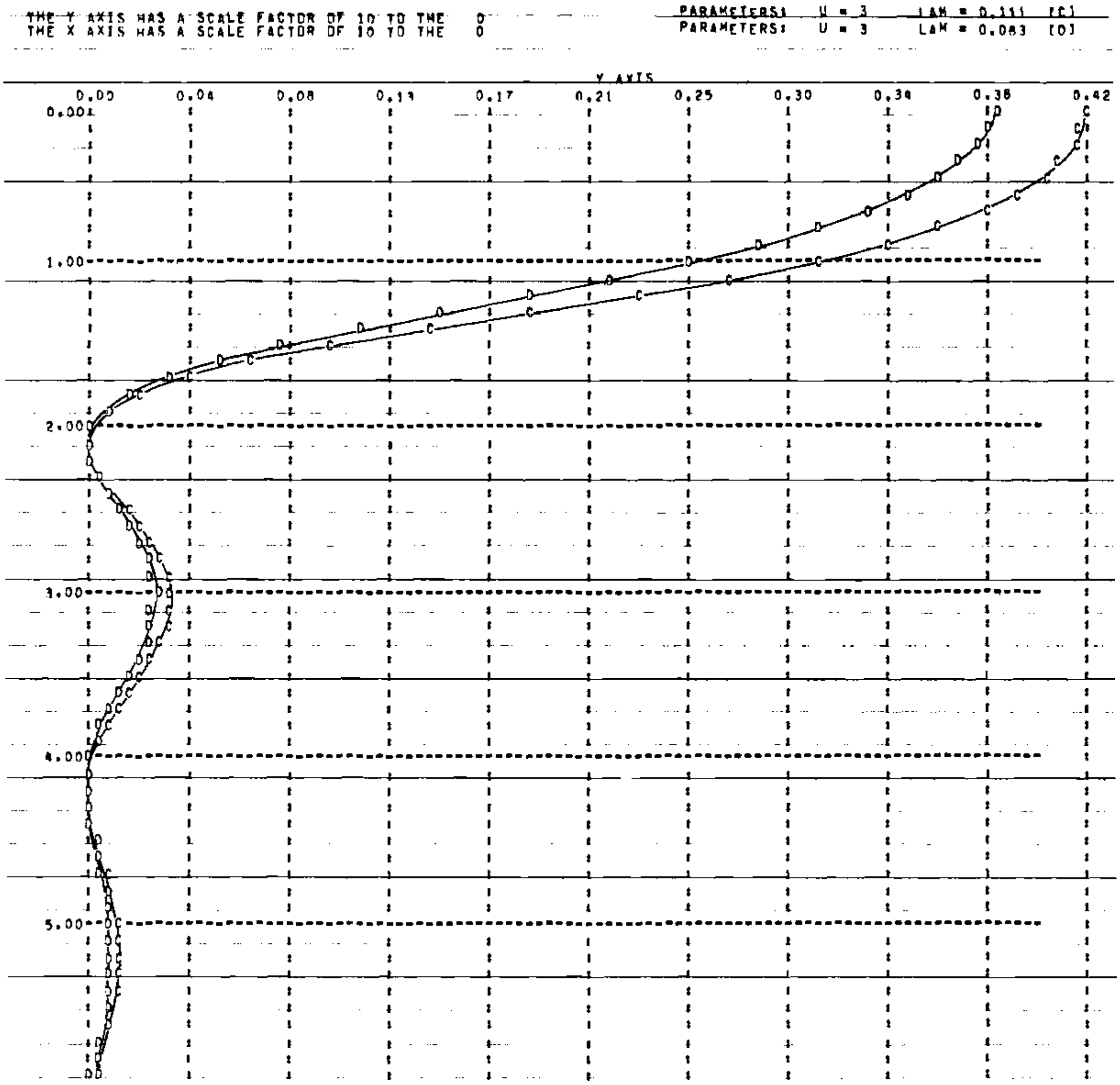

Figure 75. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ EXP(λ).

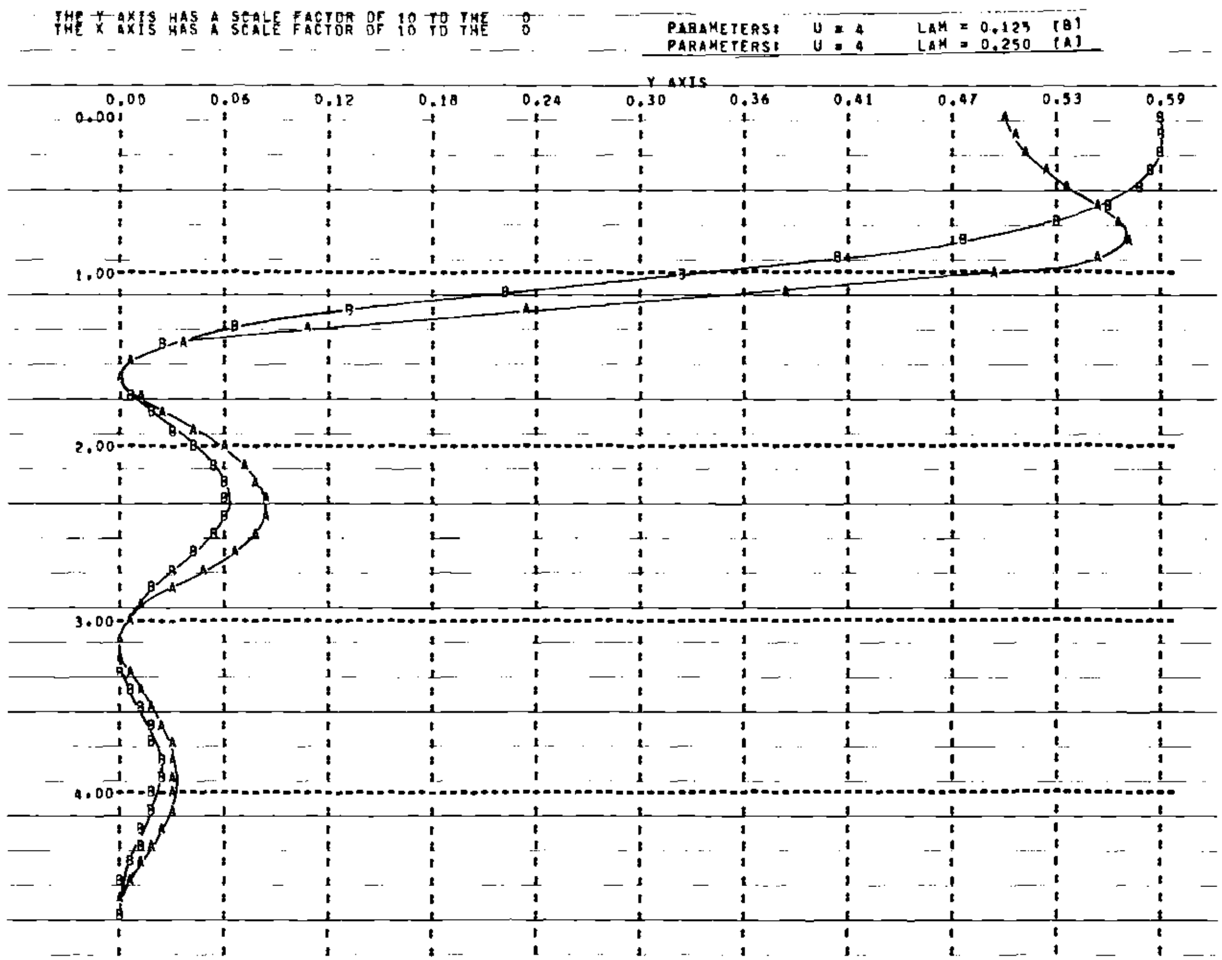

Figure 76. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ EXP(λ).

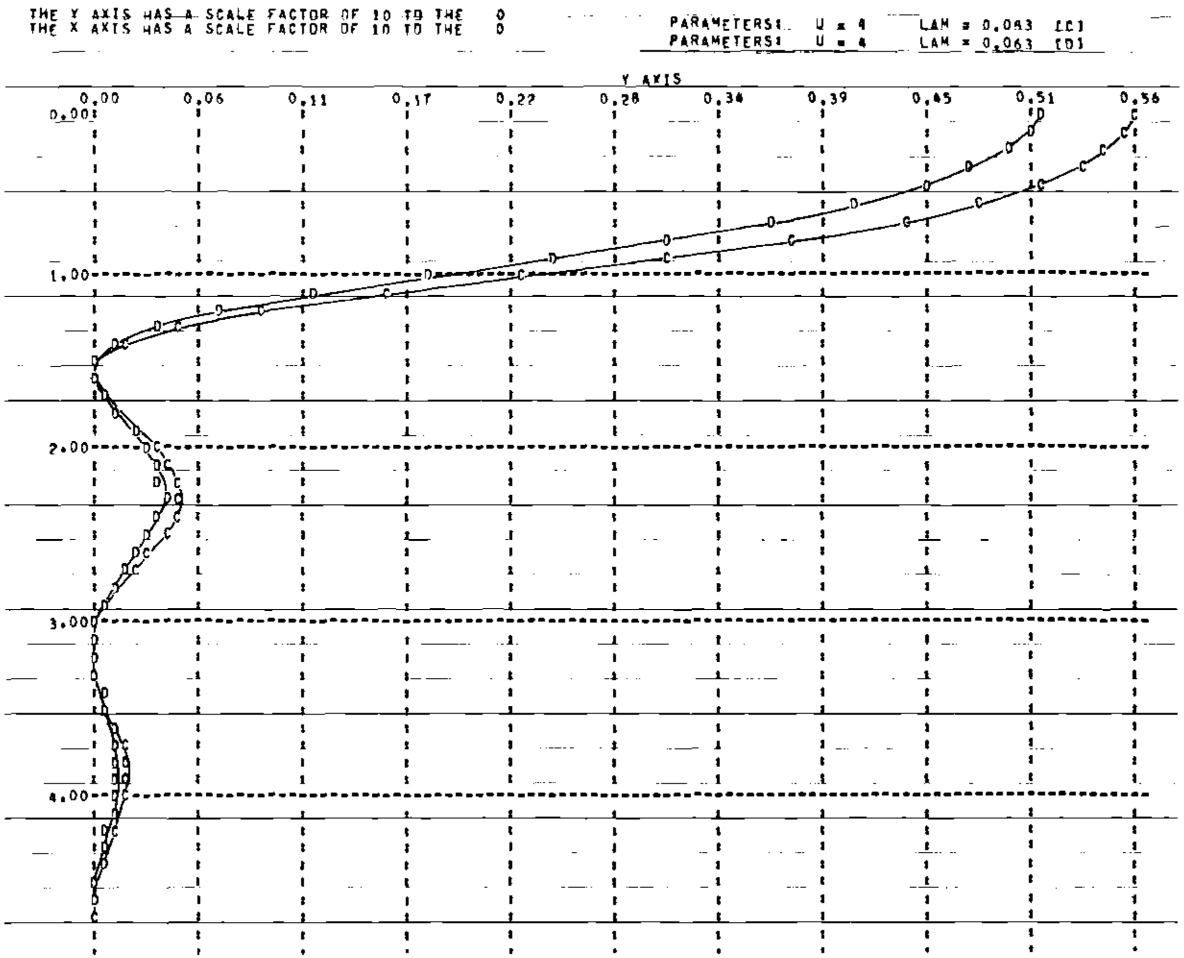

Figure 77. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ EXP(λ).

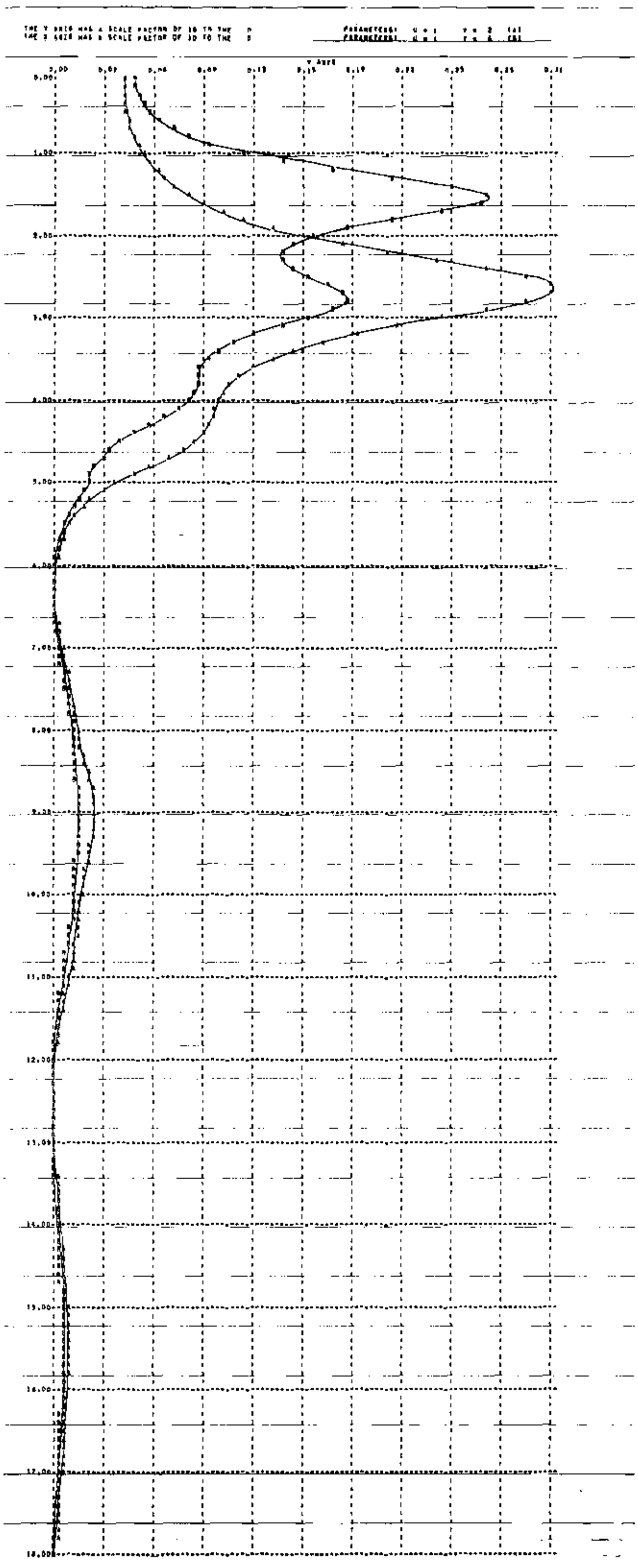

Figure 78. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

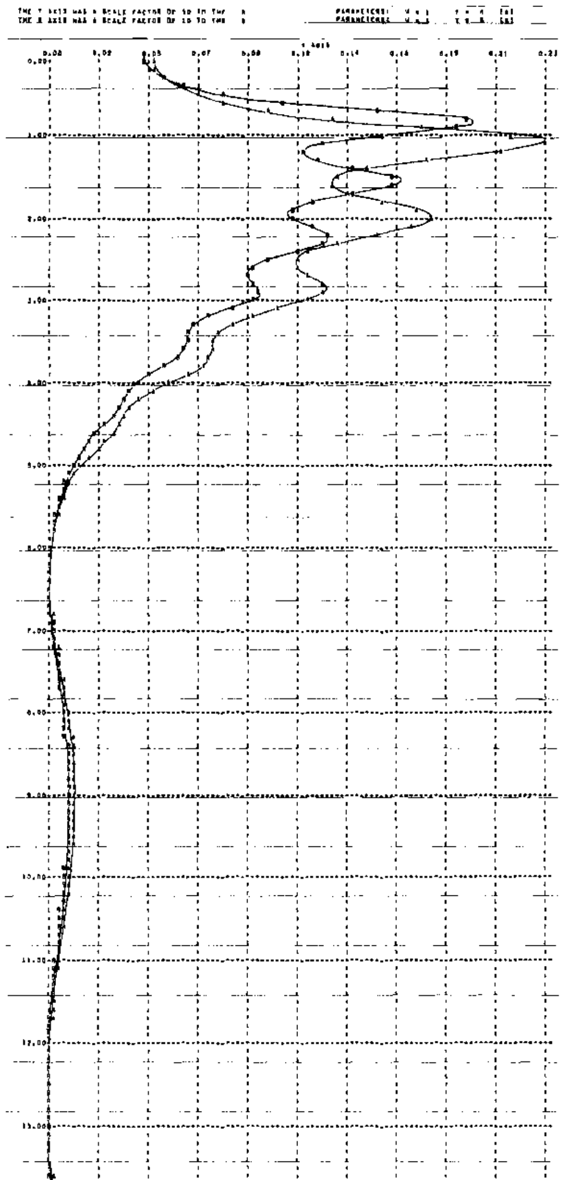

Figure 79. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

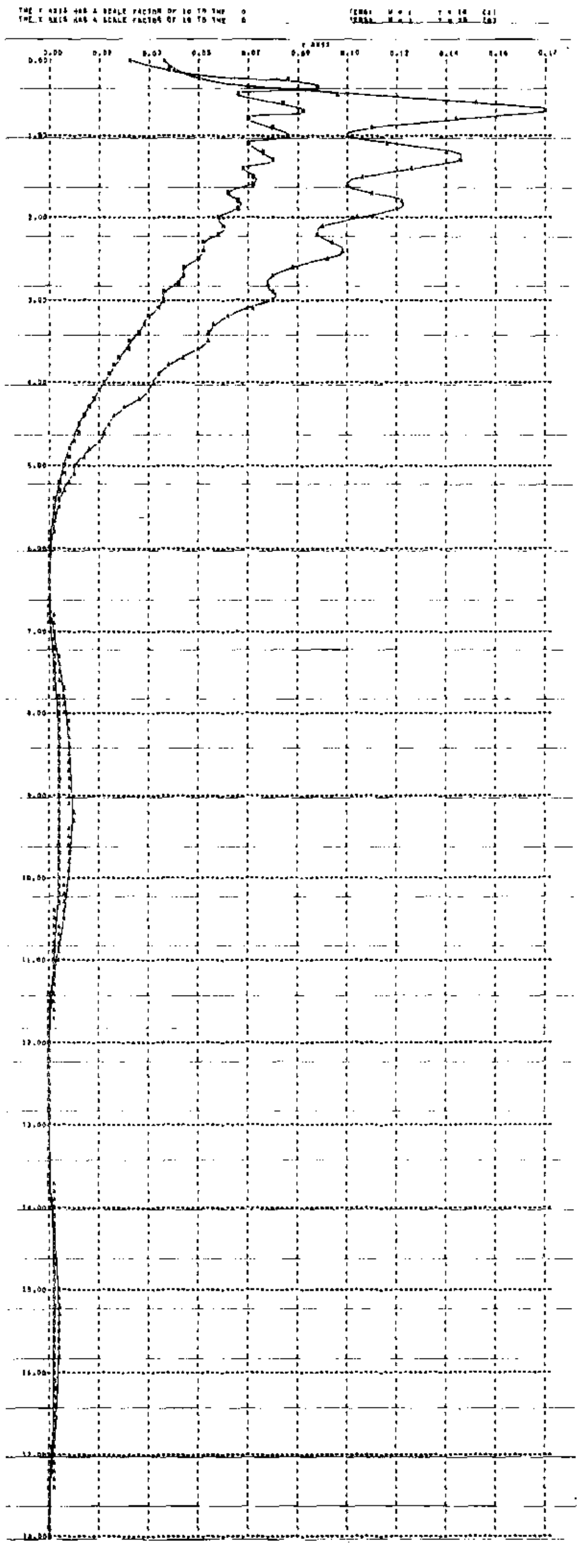

Figure 80. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ U(t).

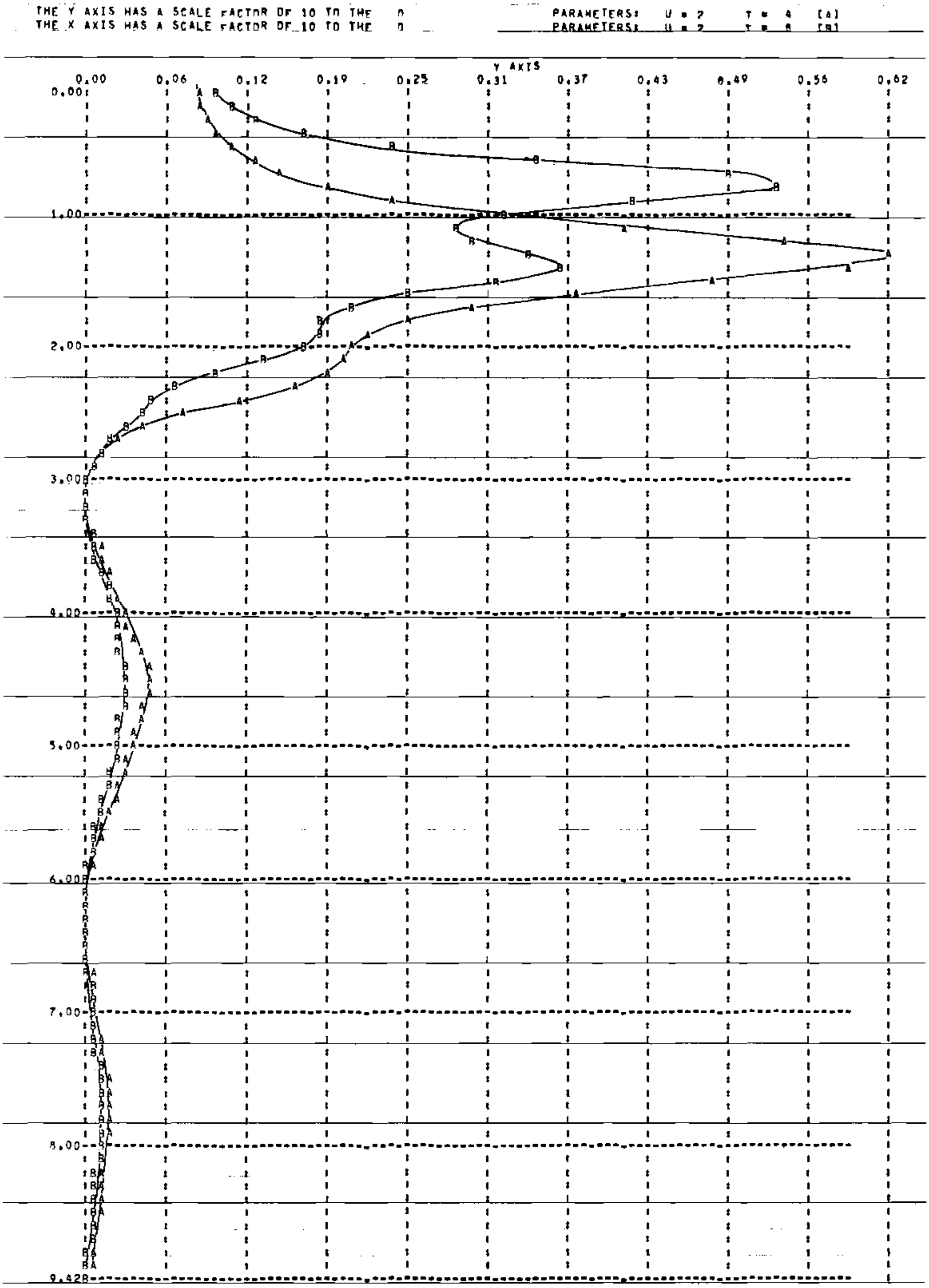

Figure 81. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

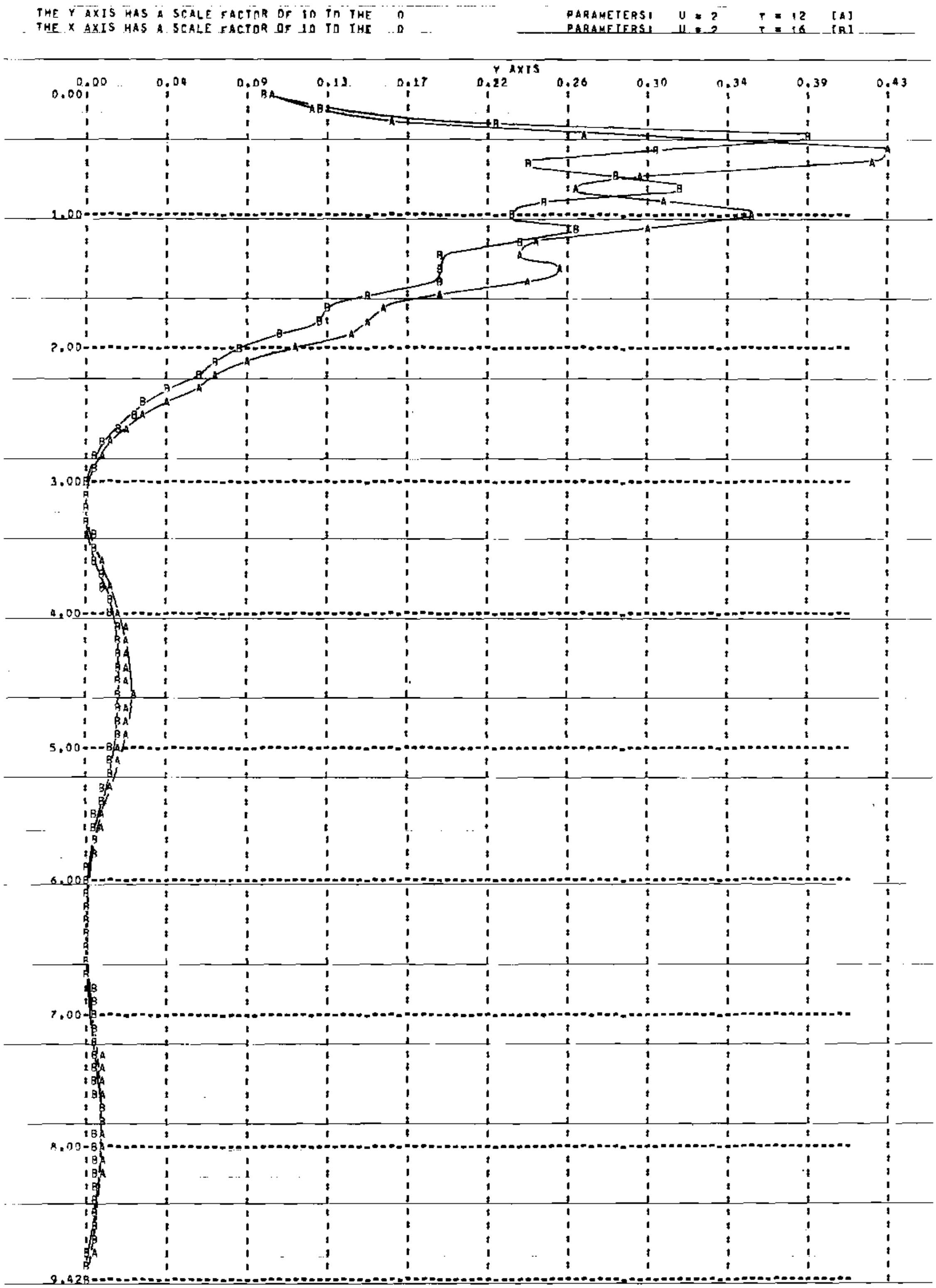

Figure 82. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

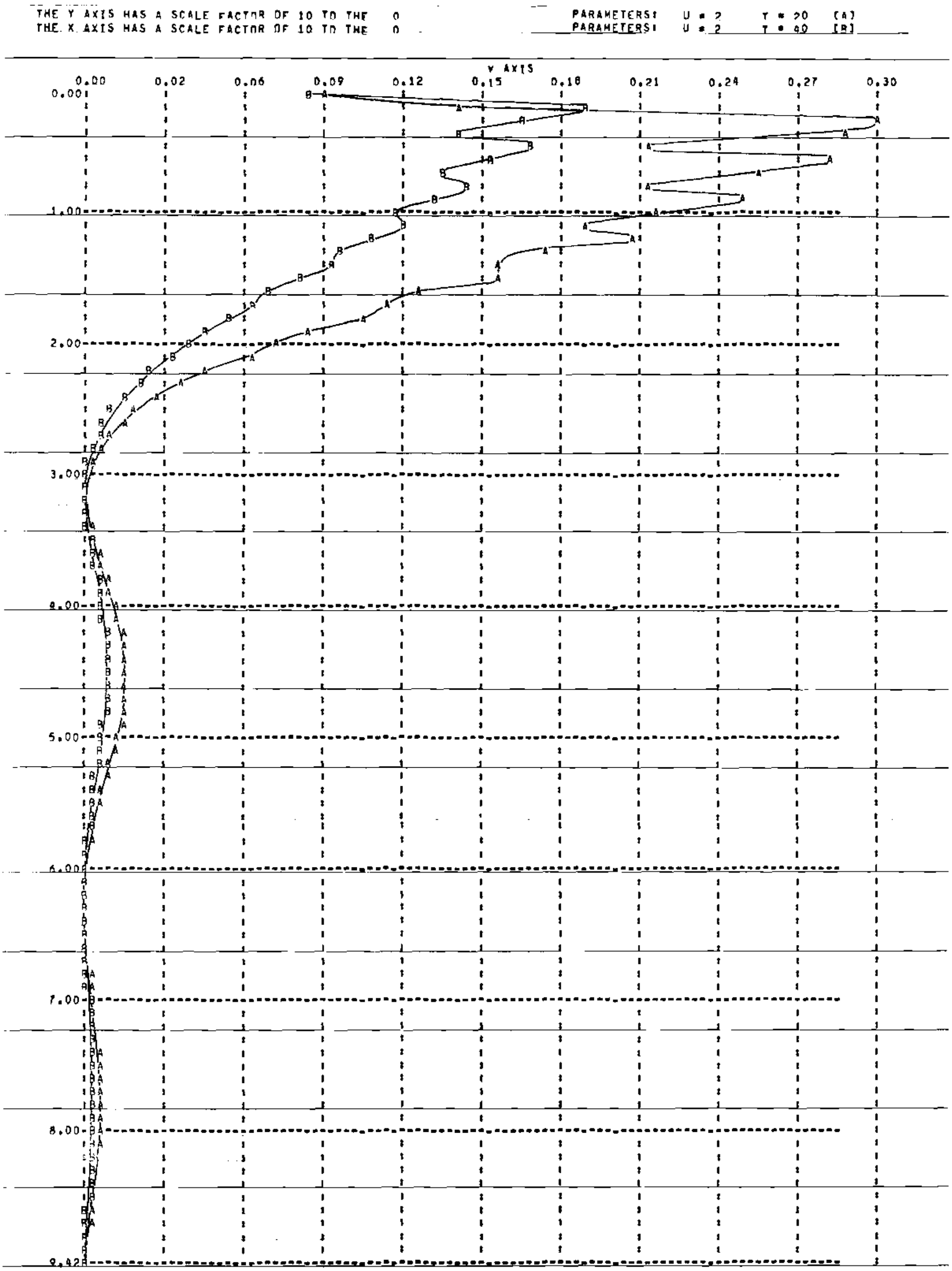

Figure 83. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

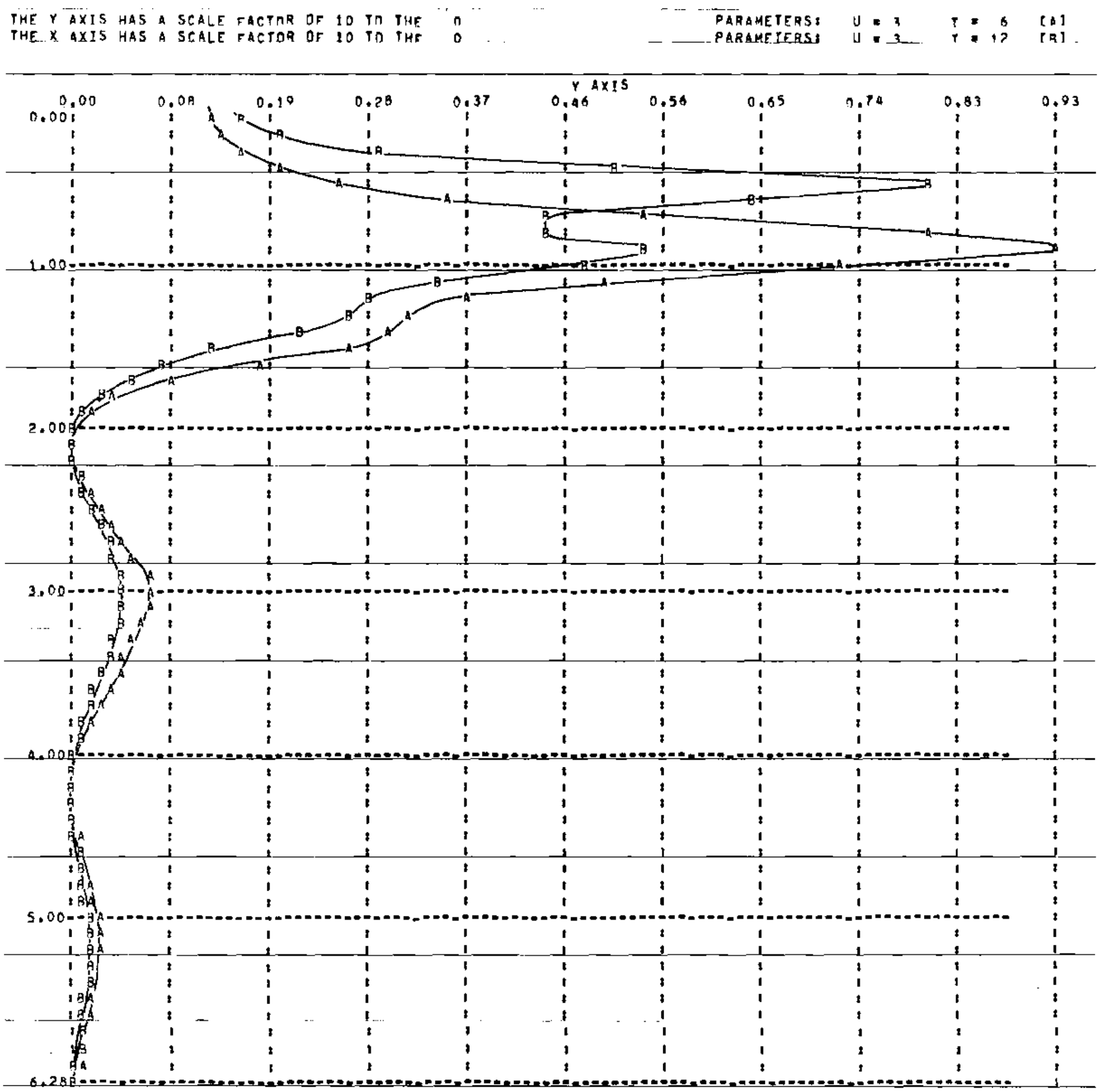

Figure 84. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

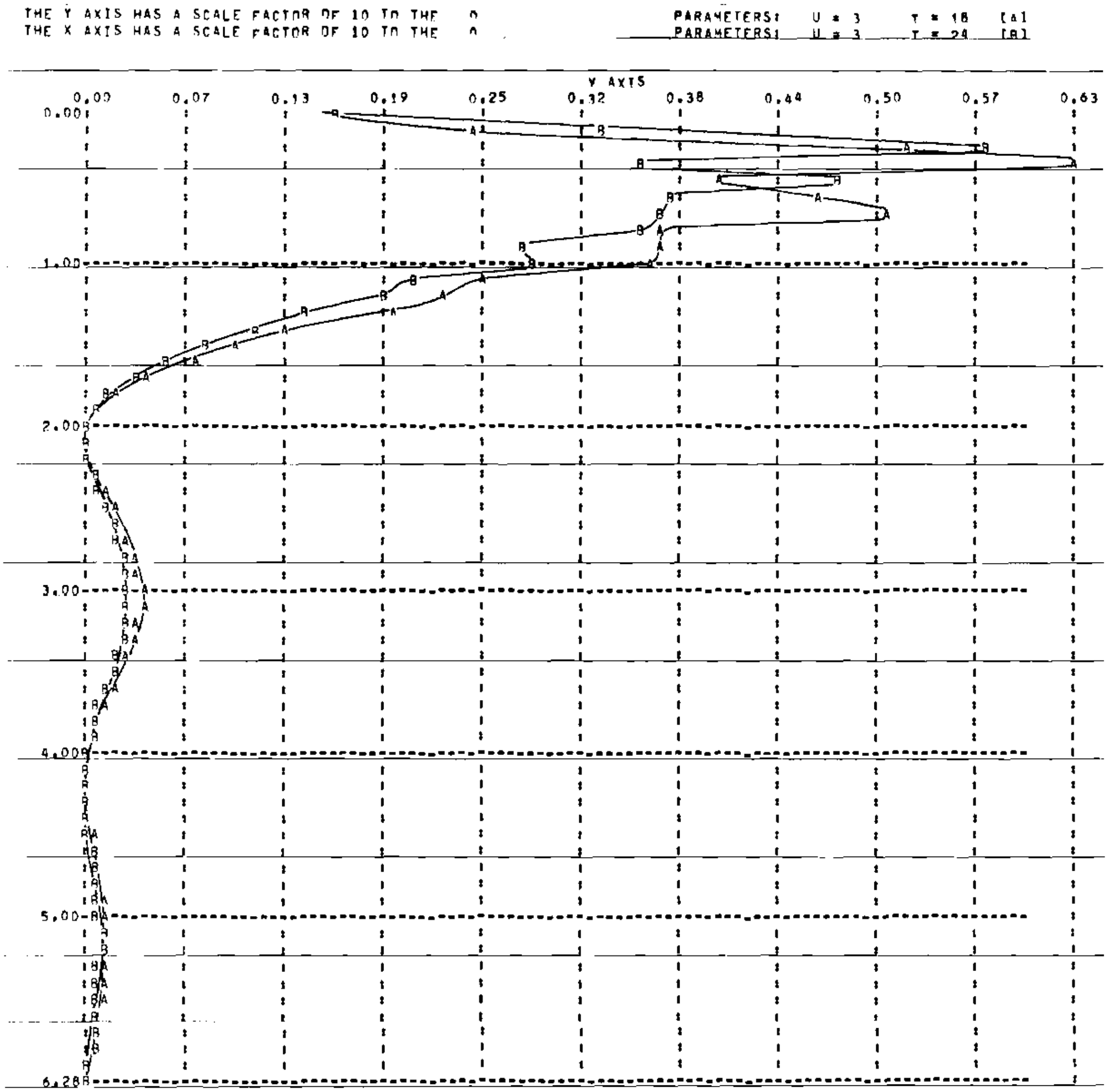

Figure 85. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≈ U(t).

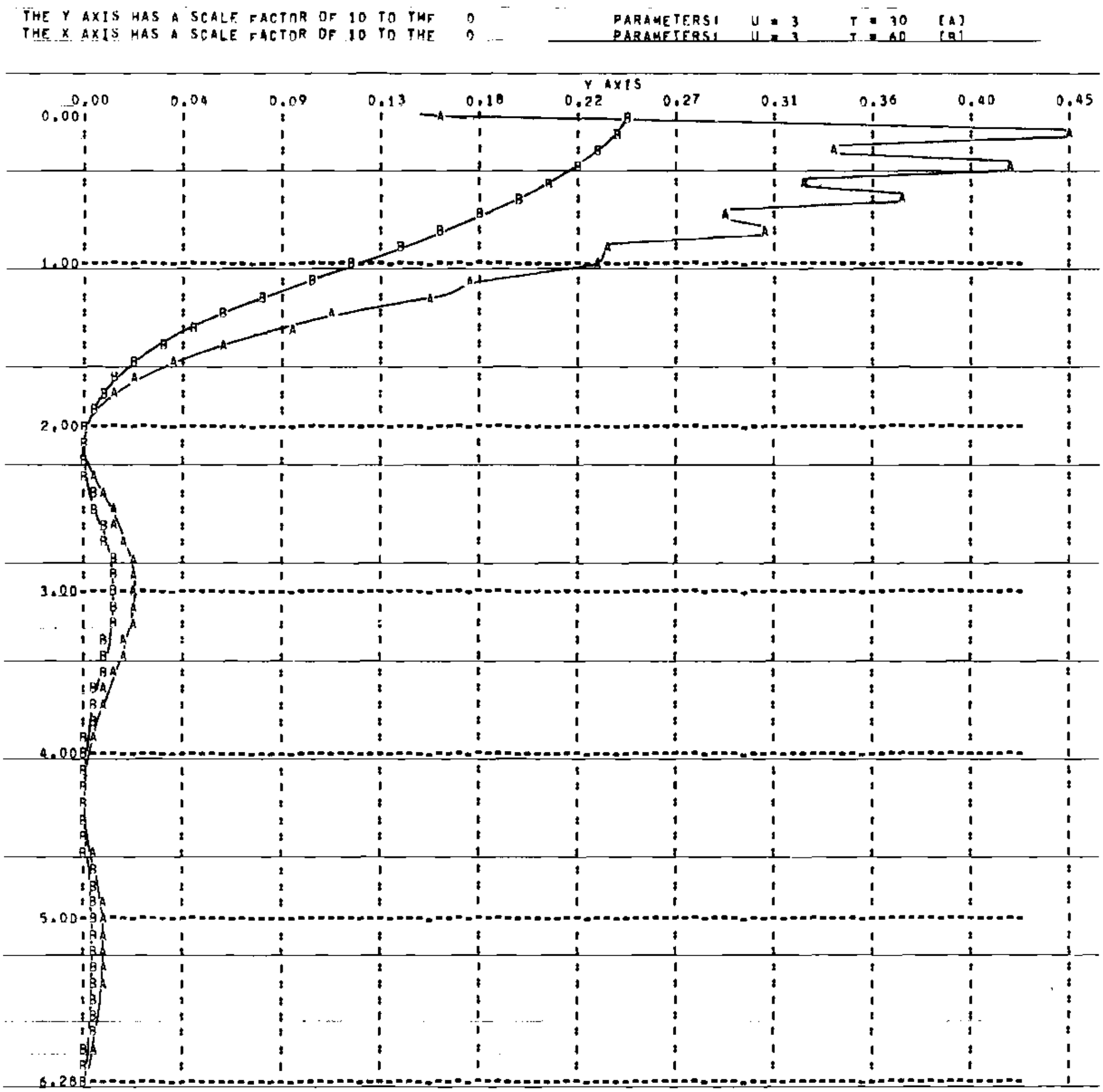

Figure 86. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ U(t).

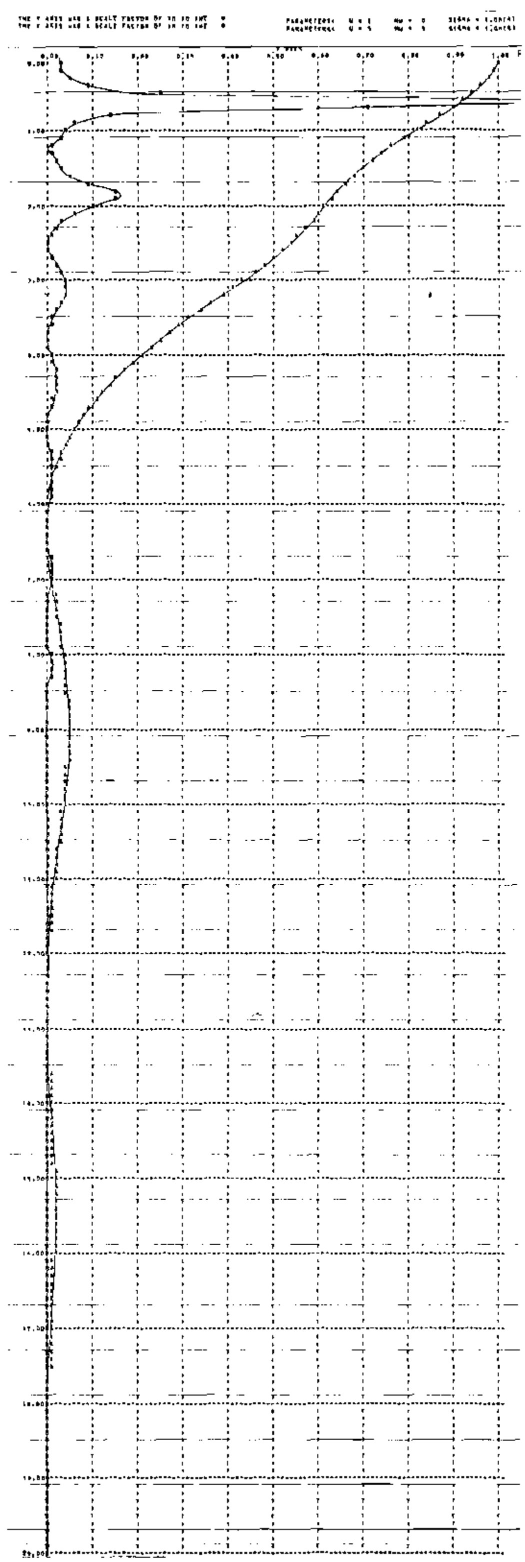

Figure 87. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for U = Constant and $V \simeq N(\mu, \sigma^2)$.

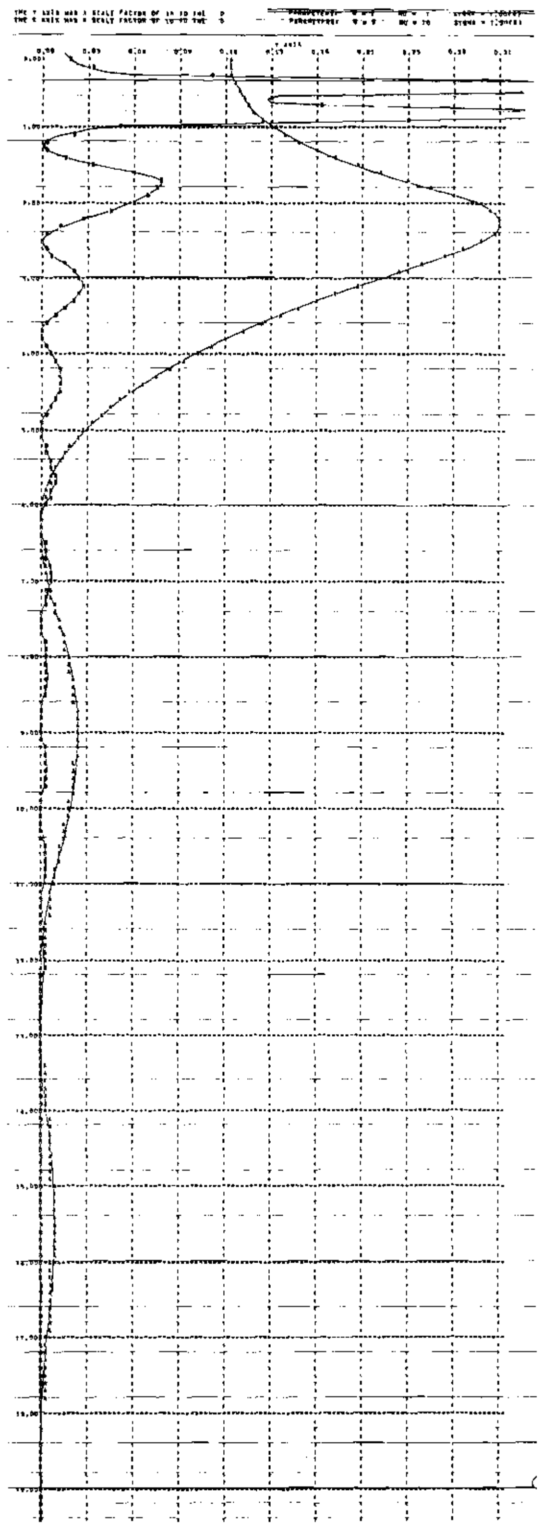

Figure 88. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ N(μ, σ²).

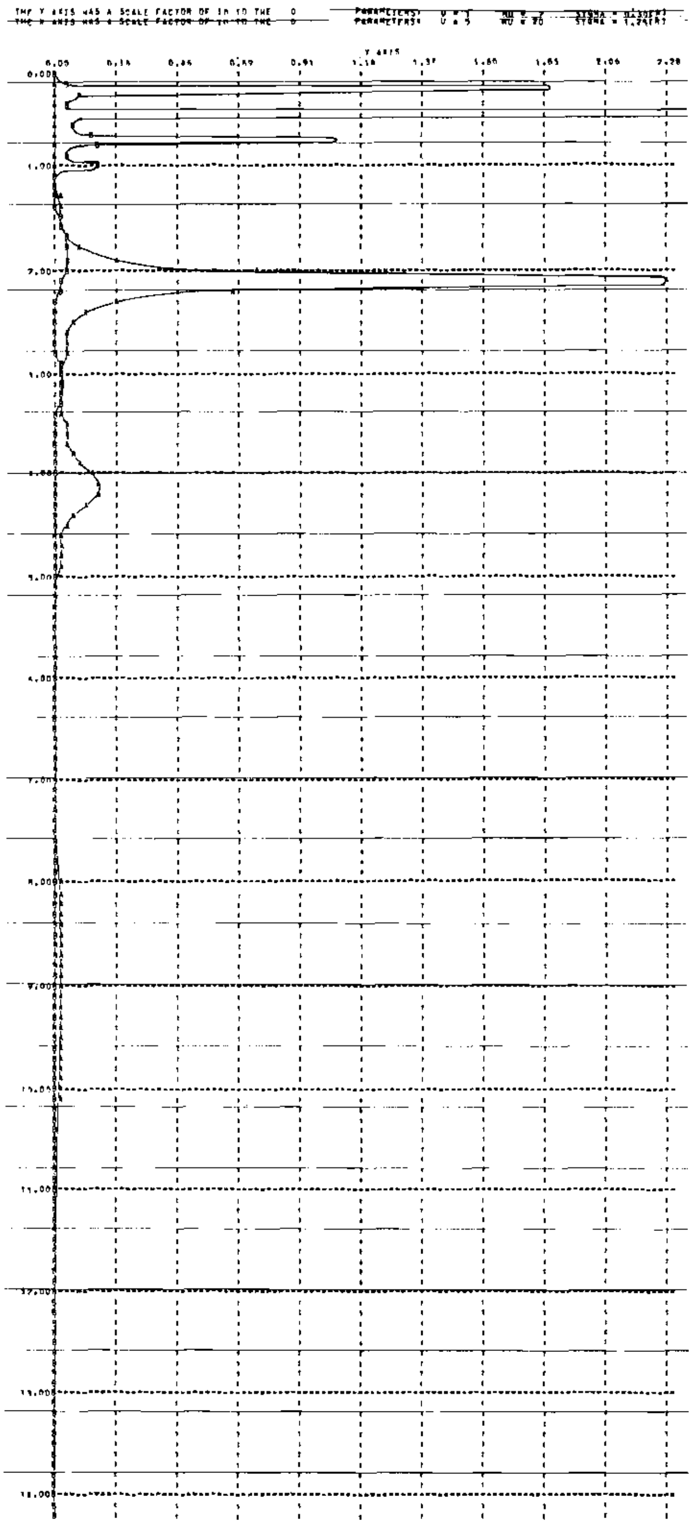

Figure 89. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ N(μ, σ²).

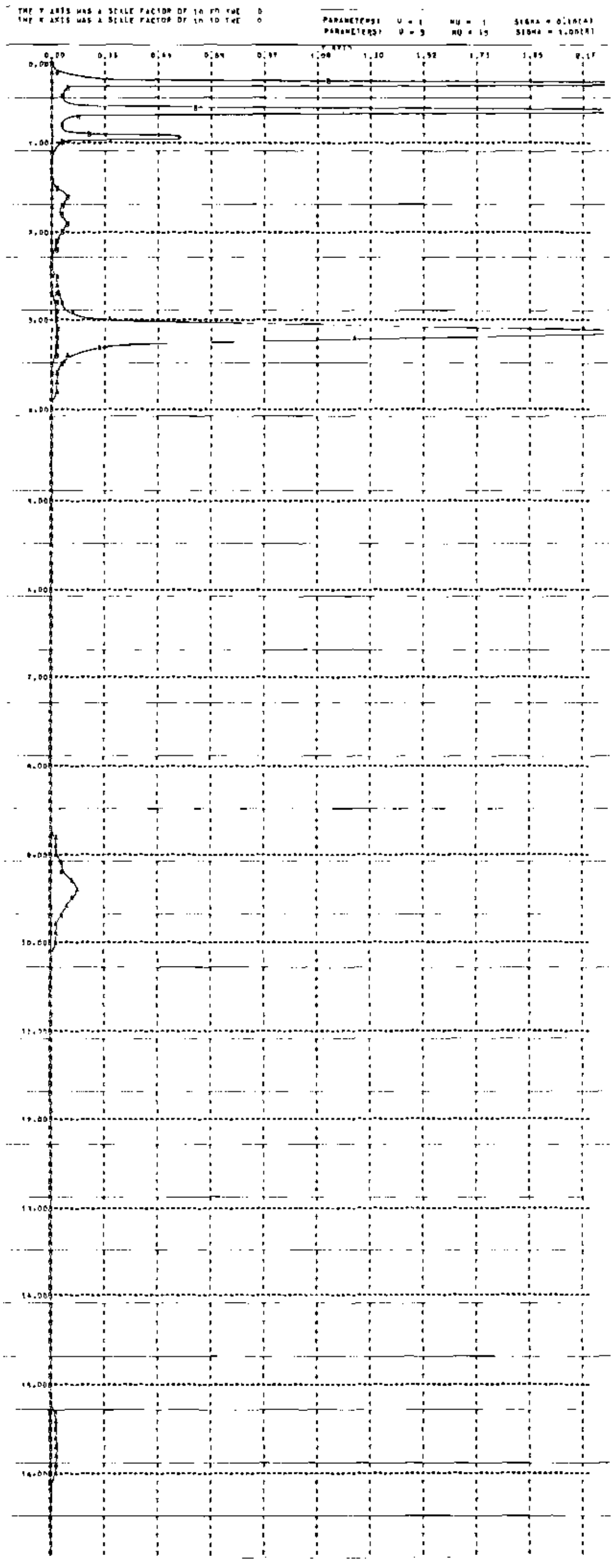

Figure 90. Spectral Density Function S(ω) [Y-Axis] over ω[X-Axis] for U = Constant and V ≃ N(μ, σ^2).

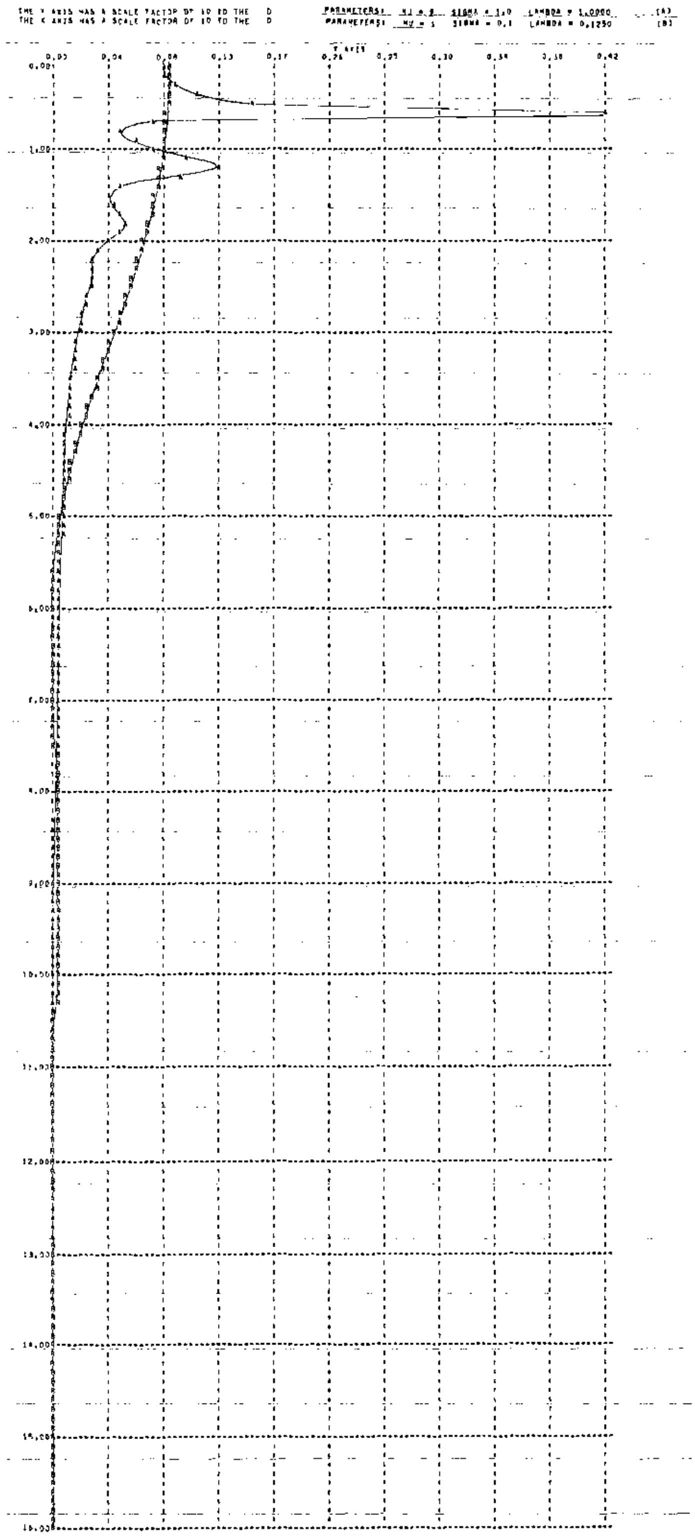

Figure 91. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \approx EXP\ (\lambda)$ and $V \approx N(\mu,\sigma^2)$.

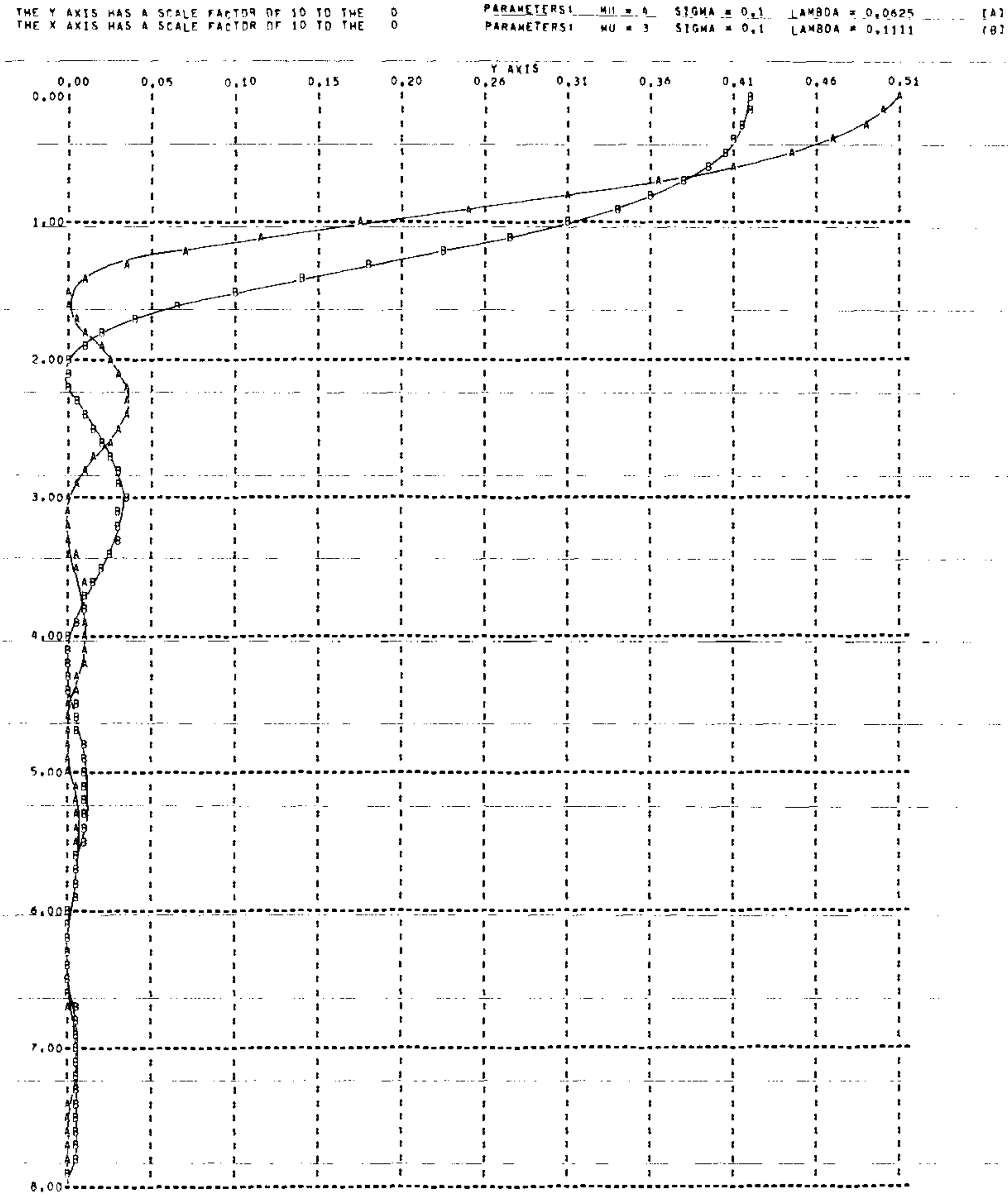

Figure 92. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu,\sigma^2)$.

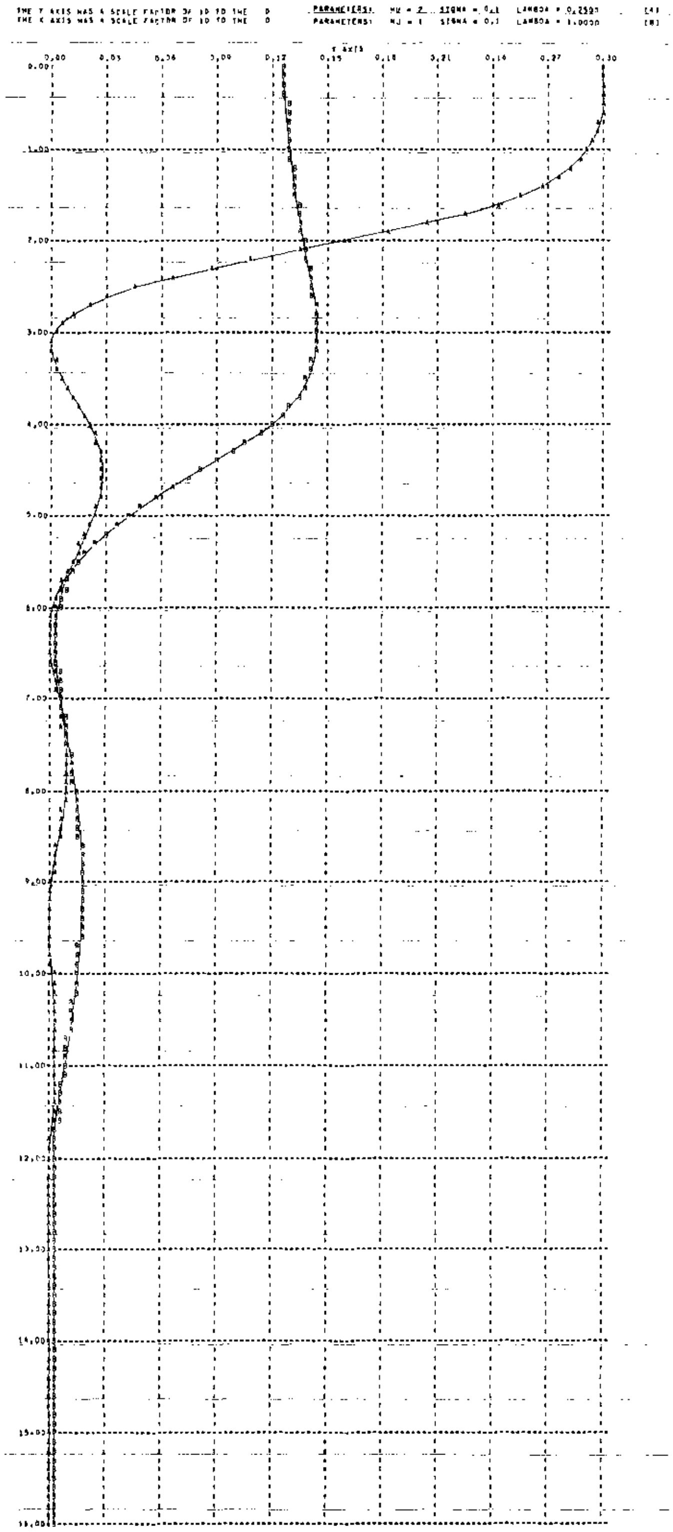

Figure 93. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq EXP\ (\lambda)$ and $V \simeq N(\mu,\sigma^2)$.

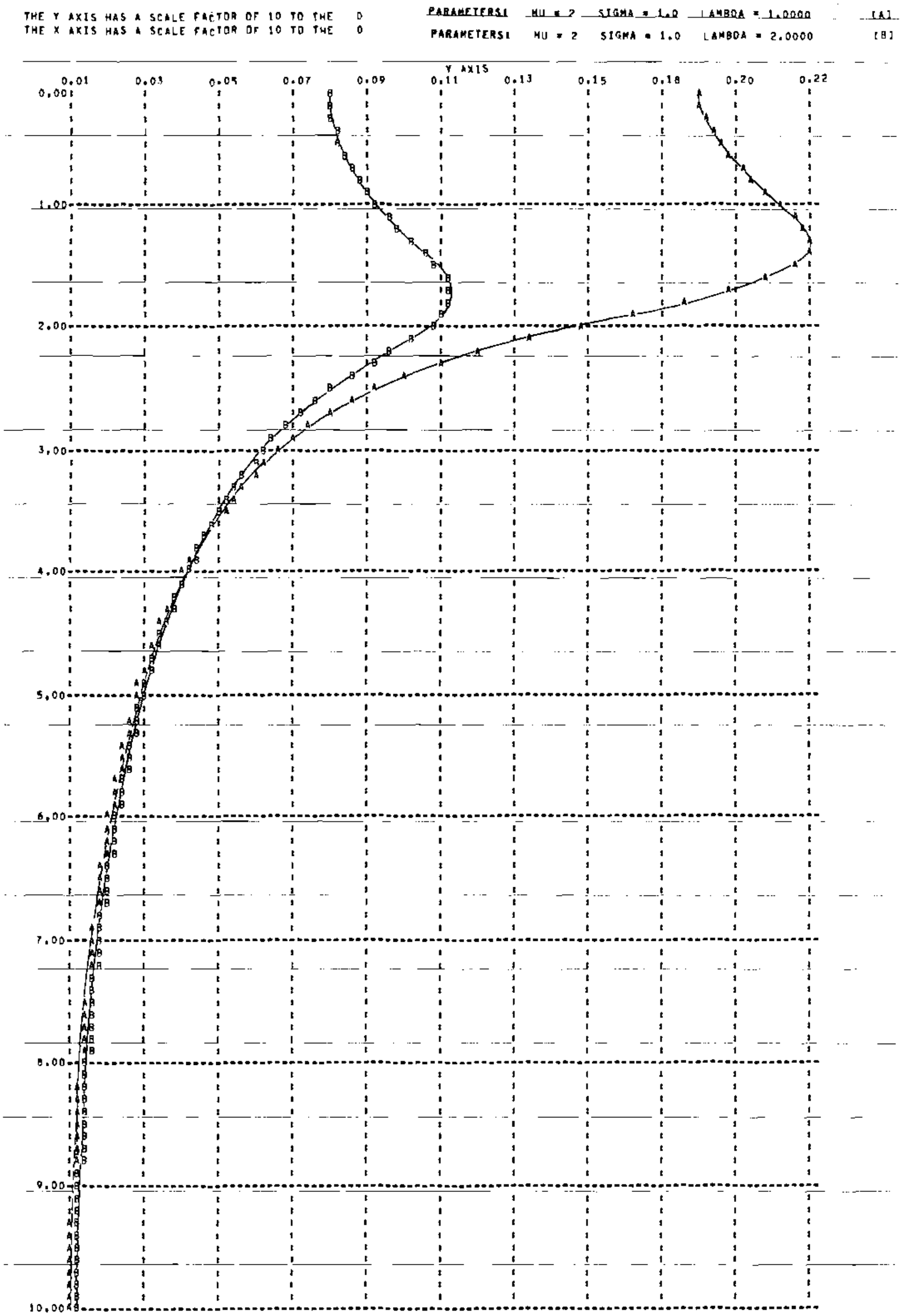

Figure 94. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq EXP(\lambda)$ and $V \simeq N(\mu,\sigma^2)$.

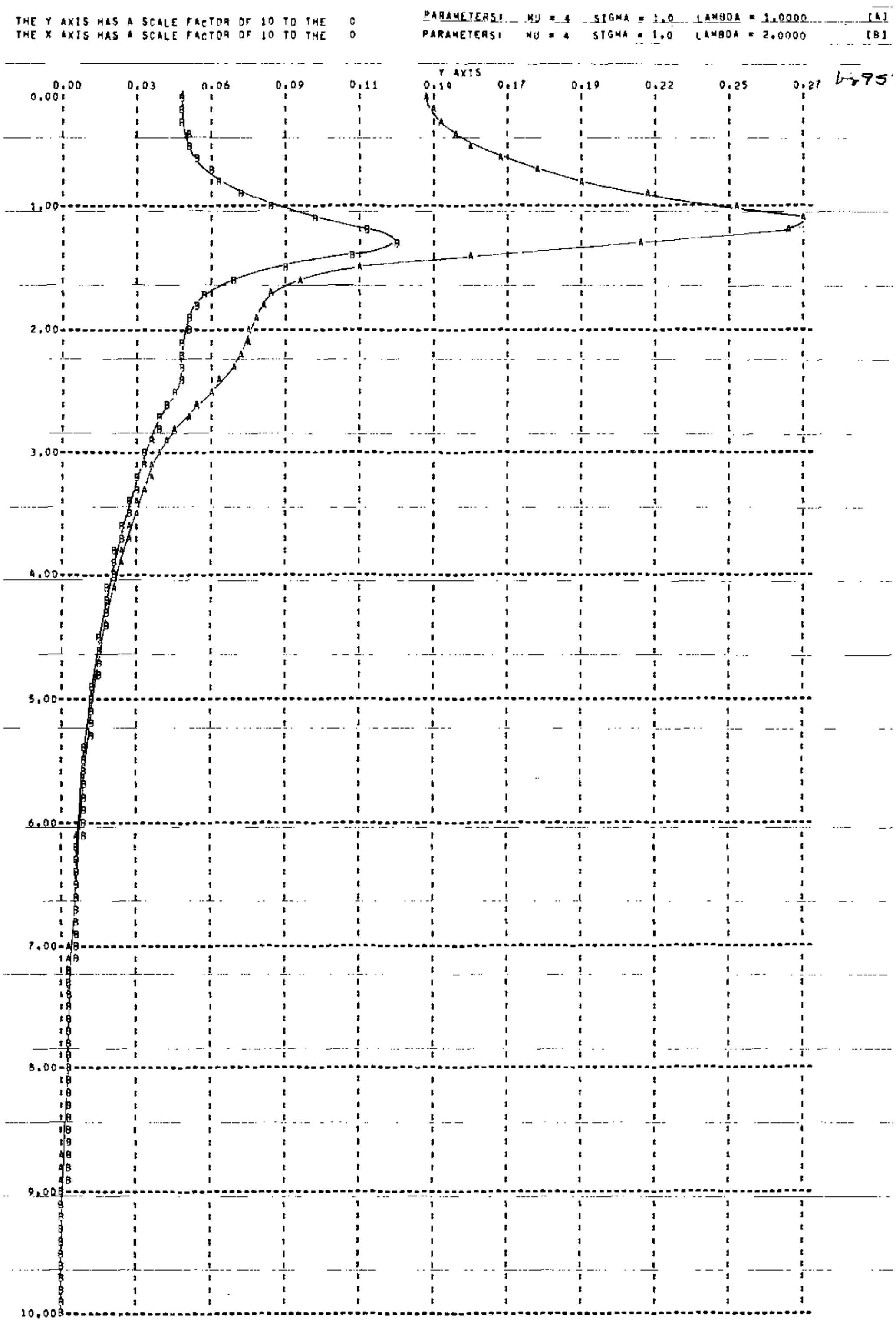

Figure 95. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq EXP\ (\lambda)$ and $V \simeq N(\mu,\sigma^2)$.

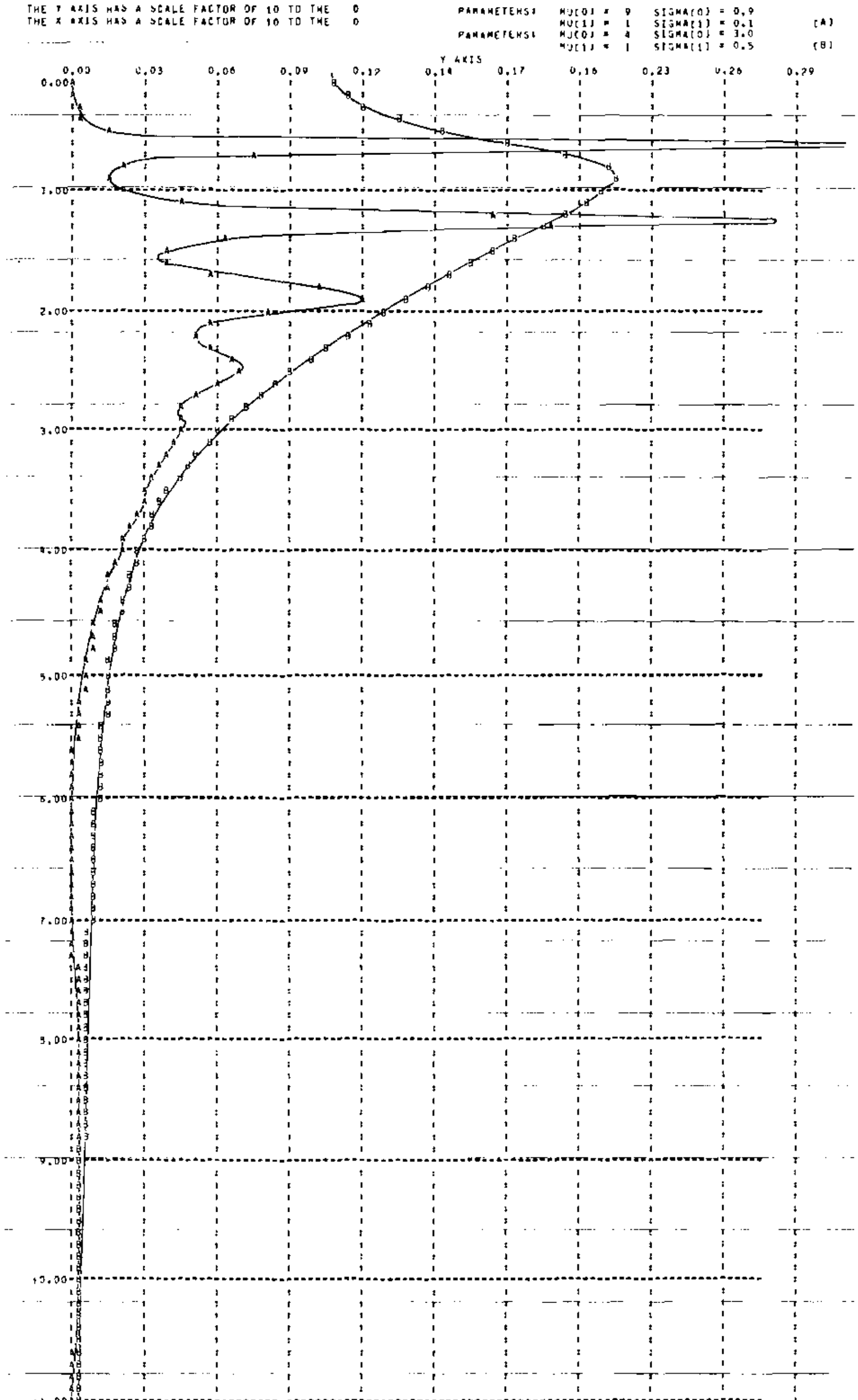

Figure 96. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq N(\mu_0,\sigma_0^2)$ and $V \simeq N(\mu_1,\sigma_1^2)$.

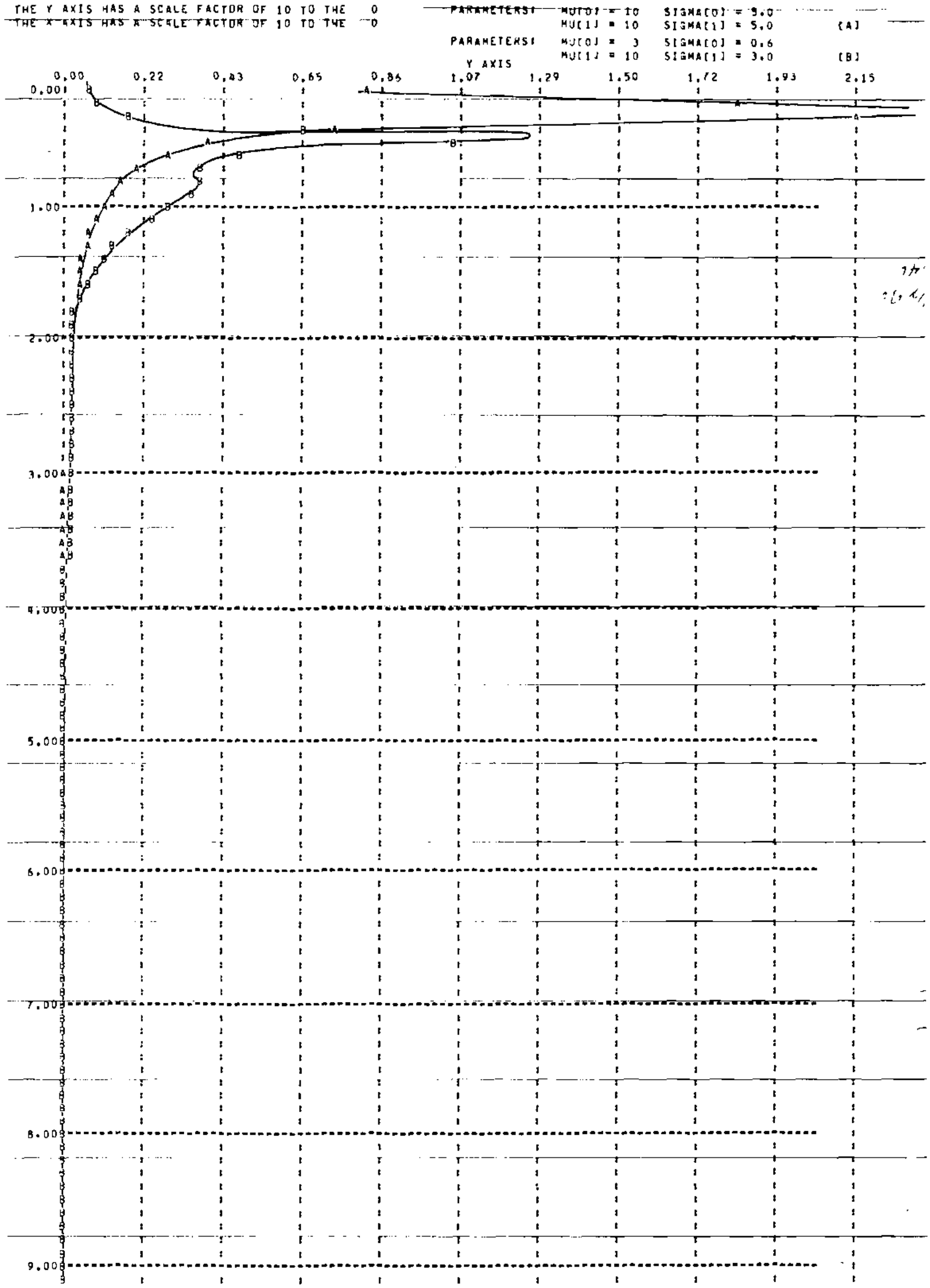

Figure 97. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \approx N(\mu_0,\sigma_0^2)$ and $V \approx N(\mu_1,\sigma_1^2)$.

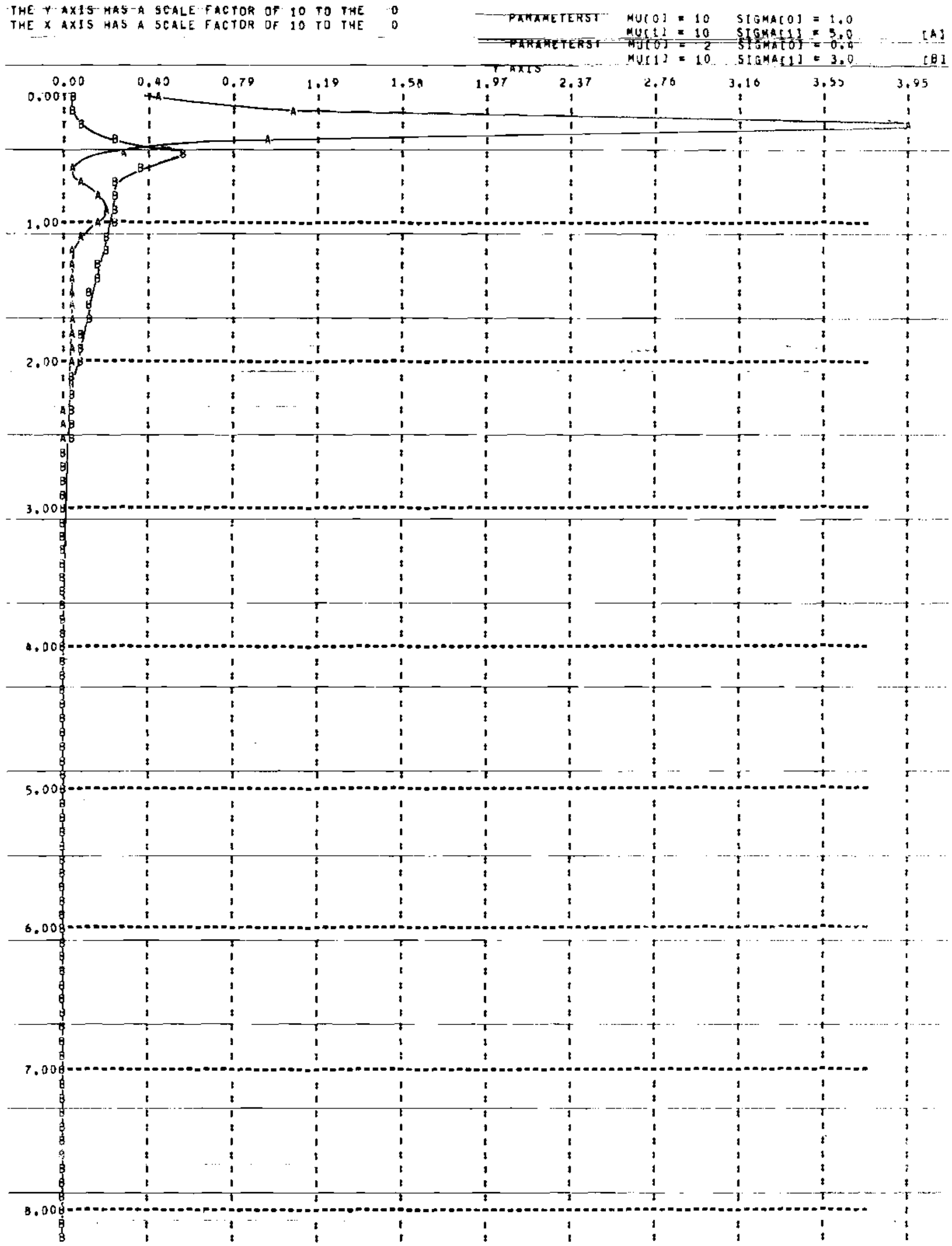

Figure 98. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq N(\mu_0,\sigma_0^2)$ and $V \simeq N(\mu_1,\sigma_1^2)$.

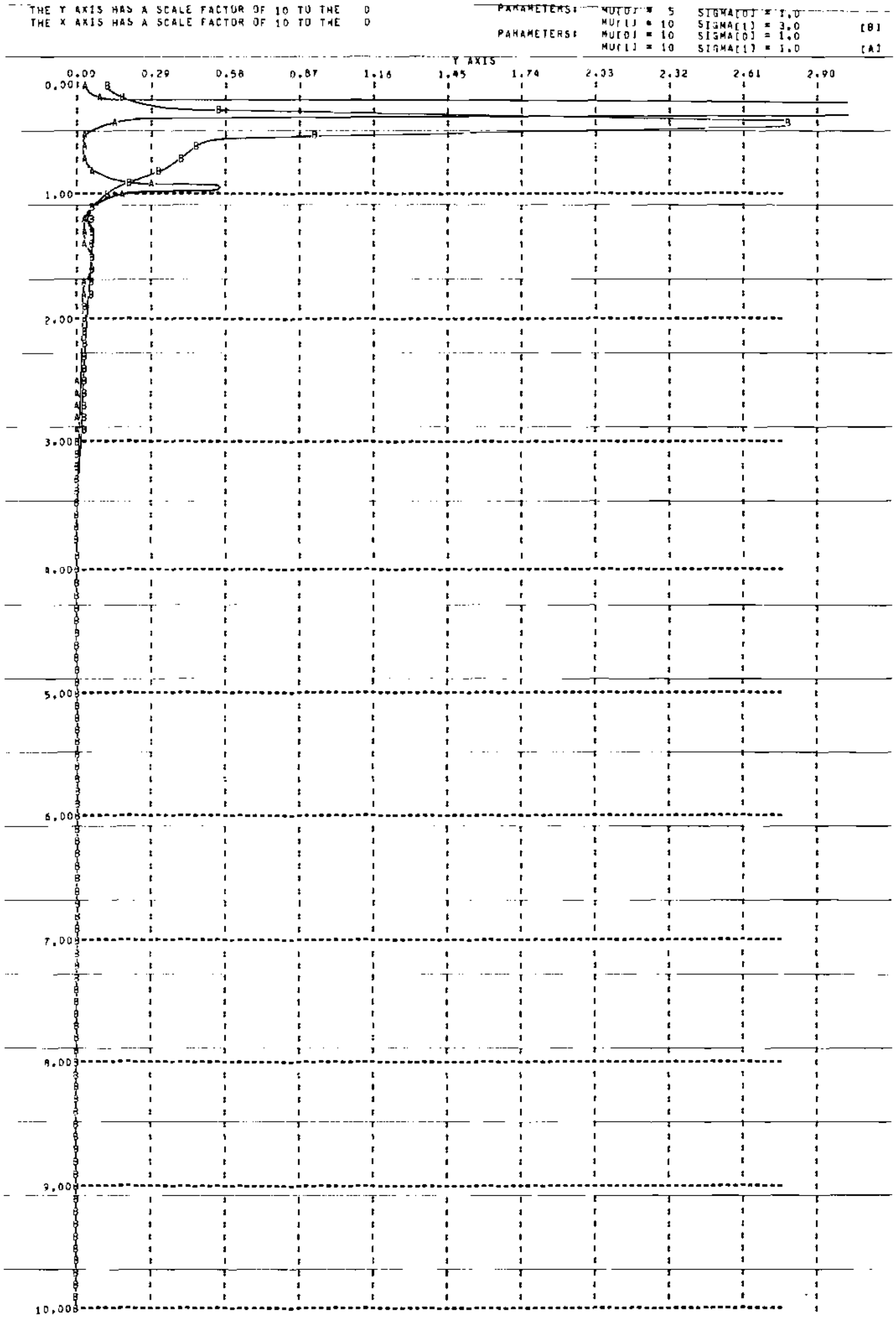

Figure 99. Spectral Density Function $S(\omega)$ [Y-Axis] over ω[X-Axis] for $U \simeq N(\mu_0,\sigma_0^2)$ and $V \simeq N(\mu_1,\sigma_1^2)$.

APPENDIX C

This Appendix accumulates the computer output of the numerical values of autocovariances and autocorrelation functions. The graphs in Figures 5 through 64 were plotted on the basis of these results.

It is to be observed that the error magnitude of $\rho(\tau)$ was fixed to 0.004 or less and that numerical values can be affected by this error to a differing degree. This is of particular importance in those cases where the absolute value of the autocorrelation function falls below $|\rho(\tau)| \leq 0.004$. No conclusions can be drawn about periods and amplitudes at that stage, it is not even to be derived from the output whether the autocorrelation function is positive or negative at any particular instance.

Table 16. Algol Program: Calculation of Autocovariance and Autocorrelation Function for the Case of Normal Distributions of Span Lengths.

```
BEGIN
INTEGER ULO,MU0,MU1 ;
REAL SIG0,S,W,P,S101,S201,S301,S001,SN01,R01,S102,S202,S302,SN02,R02,R0,
     TAU,S11,S21,S31,SN1,RTAU1,UL,H,P1,S1,S2,S3,SN,RTAU2,RTAU,COR,SIG1 ;
  FILE IN IMPIN (2,10) ;
  FILE OUT IMPOUT 16(2,15) ;
FORMAT HEAD (X3,"AUTOCOVARIANCE R(TAU) AND AUTOCORRELATION FUNCTION RHO(
TAU)",/,X3,"FOR U=N(MU0,SIG0) AND V=N(MU1,SIG1)",X5,"[ERROR S 0.004]"),
       PARAM (/,X3,"PARAMETERS:",X3,"MU[0] = ",I2,X3,"SIGMA[0] = ",
         F3.1,/,X17,"MU[1] = ",I2,X3,"SIGMA[1] = ",F3.1) ,
     RESH (//,X11,"TAU",X5,"R(TAU)",X9,"RHO(TAU)") ,
     RESULTS1 (/,X10," 0.0",X4,E9.3,X7,"1.000@+00"),
     RESULTS2 (X10,F5.1,X4,E9.3,X7,E9.3) ;

PROCEDURE  TIMING (MCGOUT) ; % GIVES VARIOUS TIMES (SEE FORMAT).
FILE       MCGOUT ;  % MCGOUT IS THE OUTPUT FILE ID.
BEGIN      % WRITTEN BY MASON C. GIBSON.
REAL       COST ;
ARRAY      T[0:2] ;
FORMAT     TMP1("PROCESS = ",F10.5," SECS.",/,"I/O = ",F10.3," SECS.",/,
              "RUN TIME = ",F12.5," SECS.",/,"CHARGES = $",F6.2) ;
           T[0] ← TIME (2) / 60 ;
           T[1] ← TIME (3) / 60 ;
           T[2] ← T[0] + T[1] ;
           COST ← ((140/60)×(TIME(2)/3600))+(140/60)×((TIME(3)/3600)/3) ;
           WRITE(MCGOUT,TMP1,T[0],T[1],T[2],COST) ;
END OF TIMING ;

    MU0←20; MU1← 4; SIG0←2.0; SIG1←1.0;
    WRITE (IMPOUT,HEAD) ;
    WRITE (IMPOUT,PARAM,MU0,SIG0,MU1,SIG1);
    S ← 0 ;
    W ← 0.01 ;
    WHILE W<2 DO
    BEGIN
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S← (S+P) ;
      W← (W+0.02) ;
    END ;
    S101 ← (4×S) ;
    S ← 0 ;
    W ← 0.02 ;
    WHILE W<2 DO
    BEGIN
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S←(S+P) ;
      W←(W+0.02) ;
    END ;
    S201←(2×S) ;
    W ← 2 ;
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
    S301←P ;
    S001←((SIG0*2×MU1*2+SIG1*2×MU0*2)/(3.1416×((MU0+MU1)*3))) ;
    SN01←(S001+S101+S201+S301) ;
    R01←(0.01×SN01/3)  ;
    ULO ← (ENTIER(500/(3.1416×(MU0+MU1)))) ;
    S ← 0 ;
    W ← 2.1 ;
    WHILE W<ULO  DO
    BEGIN
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S←(S+P) ;
      W←(W+0.2) ;
    END ;
    S102 ← (4×S) ;
    S ← 0 ;
    W ← 2.2 ;
    WHILE W<ULO  DO
    BEGIN
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S←(S+P);
      W←(W+0.2) ;
    END ;
    S202 ← (2×S) ;
    W ← ULO  ;
      P←((2×(1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×      %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
    S302 ← P ;
    SN02 ← (S302+S202+S102+S301) ;
    R02 ← (0.1×SN02/3) ;
    R0 ← (R01+R02) ;
    WRITE (IMPOUT,RESH) ;
    WRITE (IMPOUT,RESULTS1,R0) ;
    FOR TAU←(0.1×(MU0+MU1)) STEP (0.1×(MU0+MU1)) UNTIL (5.5×(MU0+MU1))
    DO
    BEGIN
      S ← 0 ;
      W ← 0.01 ;
      WHILE W < 2 DO
      BEGIN
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
        S ← (S+P1) ;
        W ← (W+0.02) ;
      END ;
      S11 ← (4×S) ;
      S ← 0 ;
      W ← 0.02 ;
      WHILE W < 2 DO
      BEGIN
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
        S ← (S+P1) ;
        W ← (W+0.02) ;
      END ;
      S21 ← (2×S) ;
      W ← 2 ;
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S31 ← P ;
      SN1 ← (S001+S11+S21+S31) ;
      RTAU1← (0.01×SN1/3) ;
      UL ← (500/(3.1416×(MU0+MU1))) ;
      IF TAU < 10 THEN H←(3.1416/(12×TAU)) ELSE H←(3.1416/(5×TAU)) ;
      S ← 0 ;
      W ←(2+H) ;
      WHILE W<(UL-H) DO
      BEGIN
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
        S ← (S+P1) ;
        W ← (W+(2×H)) ;
      END ;
      S1 ← (4×S) ;
      S ← 0 ;
      W ← (2+(2×H)) ;
      WHILE W<UL DO
      BEGIN
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
        S ← (S+P1) ;
        W ← (W+(2×H)) ;
      END ;
      S2 ← (2×S) ;
      W ← 2  ;
      WHILE W<UL DO W ← (W+(2×H)) ;
      P1←((2×COS(W×TAU)×
          (1-EXP(-(W*2)×(SIG0*2+SIG1*2))-(EXP(-(W*2/2)×SIG0*2)×        %1
        COS(MU0×W)×(1-EXP(-(W*2×SIG1*2))))-(EXP(-(W*2/2)×SIG1*2)×      %2
        COS(MU1×W)×(1-EXP(-(W*2×SIG0*2))))))/(3.1416×(MU0+MU1)×(W*2)×  %3
        (1-(2×EXP(-((W*2/2)×(SIG0*2+SIG1*2)))×COS(W×(MU0+MU1)))+       %4
        EXP(-(W*2×(SIG0*2+SIG1*2)))))) ;
      S3 ← P1 ;
      SN ← (S3+S2+S1+S31) ;
      RTAU2 ← (H×SN/3) ;
      RTAU ←(RTAU1+RTAU2) ;
      COR ← (RTAU/R0) ;
      WRITE (IMPOUT,RESULTS2,TAU,RTAU,COR) ;
    END ;
  TIMING (IMPOUT) ;
END.
```

Table 17. Autocovariance and Autocorrelation Function for U = Constant and V ≈ EXP(λ) and for Indicated Parameters.

PARAMETERS: U = 1 LAM = 1.000

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.1	2.066E-01	8.332E-01
0.2	1.620E-01	6.531E-01
0.3	1.203E-01	4.852E-01
0.4	8.512E-02	3.432E-01
0.5	5.325E-02	2.147E-01
0.6	2.442E-02	9.847E-02
0.7	-1.715E-03	-6.915E-03
0.8	-2.533E-02	-1.022E-01
0.9	-4.672E-02	-1.884E-01
1.0	-6.505E-02	-2.623E-01
1.1	-3.832E-02	-1.545E-01
1.2	-1.753E-02	-7.070E-02
1.3	-2.600E-03	-1.048E-02
1.4	7.353E-03	2.965E-02
1.5	1.321E-02	5.325E-02
1.6	1.558E-02	6.282E-02
1.7	1.516E-02	6.113E-02
1.8	1.237E-02	4.987E-02
1.9	7.753E-03	3.127E-02
2.0	1.597E-03	6.438E-03
2.1	-3.420E-03	-1.379E-02
2.2	-5.704E-03	-2.300E-02
2.3	-6.049E-03	-2.439E-02
2.4	-5.217E-03	-2.104E-02
2.5	-3.696E-03	-1.490E-02
2.6	-1.956E-03	-7.886E-03
2.7	-2.823E-04	-1.138E-03
2.8	1.068E-03	4.307E-03
2.9	1.931E-03	7.786E-03
3.0	2.200E-03	8.871E-03
3.1	1.868E-03	7.534E-03
3.2	1.244E-03	5.018E-03
3.3	5.449E-04	2.197E-03
3.4	-3.871E-05	-1.561E-04
3.5	-4.715E-04	-1.901E-03
3.6	-6.845E-04	-2.760E-03
3.7	-7.331E-04	-2.956E-03
3.8	-6.128E-04	-2.471E-03
3.9	-4.146E-04	-1.672E-03

PARAMETERS: U = 2 LAM = 0.500

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.2	2.066E-01	8.332E-01
0.4	1.620E-01	6.532E-01
0.6	1.203E-01	4.853E-01
0.8	8.512E-02	3.432E-01
1.0	5.325E-02	2.148E-01
1.2	2.442E-02	9.848E-02
1.4	-1.715E-03	-6.916E-03
1.6	-2.533E-02	-1.022E-01
1.8	-4.672E-02	-1.884E-01
2.0	-6.505E-02	-2.623E-01
2.2	-3.832E-02	-1.545E-01
2.4	-1.753E-02	-7.071E-02
2.6	-2.600E-03	-1.048E-02
2.8	7.353E-03	2.965E-02
3.0	1.321E-02	5.326E-02
3.2	1.558E-02	6.282E-02
3.4	1.516E-02	6.113E-02
3.6	1.237E-02	4.987E-02
3.8	7.753E-03	3.127E-02
4.0	1.597E-03	6.435E-03
4.2	-3.420E-03	-1.379E-02
4.4	-5.704E-03	-2.300E-02
4.6	-6.049E-03	-2.439E-02
4.8	-5.217E-03	-2.104E-02
5.0	-3.696E-03	-1.490E-02
5.2	-1.956E-03	-7.886E-03
5.4	-2.823E-04	-1.138E-03
5.6	1.068E-03	4.307E-03
5.8	1.931E-03	7.786E-03
6.0	2.200E-03	8.872E-03
6.2	1.868E-03	7.535E-03
6.4	1.244E-03	5.018E-03
6.6	5.449E-04	2.197E-03
6.8	-3.871E-05	-1.561E-04
7.0	-4.715E-04	-1.902E-03
7.2	-6.845E-04	-2.760E-03
7.4	-7.331E-04	-2.957E-03
7.6	-6.128E-04	-2.471E-03
7.8	-4.146E-04	-1.672E-03
8.0	-1.659E-04	-6.691E-04

PARAMETERS: U = 3 LAM = 0.333

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.3	2.066E-01	8.331E-01
0.6	1.620E-01	6.531E-01
0.9	1.203E-01	4.852E-01
1.2	8.512E-02	3.432E-01
1.5	5.325E-02	2.147E-01
1.8	2.442E-02	9.847E-02
2.1	-1.715E-03	-6.915E-03
2.4	-2.533E-02	-1.021E-01
2.7	-4.672E-02	-1.884E-01
3.0	-6.505E-02	-2.623E-01
3.3	-3.832E-02	-1.545E-01
3.6	-1.753E-02	-7.070E-02
3.9	-2.600E-03	-1.048E-02
4.2	7.353E-03	2.965E-02
4.5	1.321E-02	5.325E-02
4.8	1.558E-02	6.282E-02
5.1	1.516E-02	6.112E-02
5.4	1.237E-02	4.987E-02
5.7	7.753E-03	3.126E-02
6.0	1.597E-03	6.438E-03
6.3	-3.420E-03	-1.379E-02
6.6	-5.704E-03	-2.300E-02
6.9	-6.049E-03	-2.439E-02
7.2	-5.217E-03	-2.104E-02
7.5	-3.696E-03	-1.490E-02
7.8	-1.956E-03	-7.886E-03
8.1	-2.823E-04	-1.138E-03
8.4	1.068E-03	4.307E-03
8.7	1.931E-03	7.786E-03
9.0	2.200E-03	8.871E-03
9.3	1.868E-03	7.534E-03
9.6	1.244E-03	5.018E-03
9.9	5.449E-04	2.197E-03
10.2	-3.821E-05	-1.541E-04
10.5	-4.720E-04	-1.903E-03
10.8	-6.848E-04	-2.761E-03
11.1	-7.329E-04	-2.955E-03
11.4	-6.143E-04	-2.477E-03
11.7	-4.150E-04	-1.674E-03

PARAMETERS: U = 4 LAM = 0.250

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.4	2.066E-01	8.332E-01
0.8	1.620E-01	6.532E-01
1.2	1.203E-01	4.853E-01
1.6	8.512E-02	3.432E-01
2.0	5.325E-02	2.148E-01
2.4	2.442E-02	9.848E-02
2.8	-1.715E-03	-6.916E-03
3.2	-2.533E-02	-1.022E-01
3.6	-4.672E-02	-1.884E-01
4.0	-6.505E-02	-2.623E-01
4.4	-3.832E-02	-1.545E-01
4.8	-1.753E-02	-7.071E-02
5.2	-2.600E-03	-1.048E-02
5.6	7.353E-03	2.965E-02
6.0	1.321E-02	5.326E-02
6.4	1.558E-02	6.282E-02
6.8	1.516E-02	6.113E-02
7.2	1.237E-02	4.987E-02
7.6	7.753E-03	3.127E-02
8.0	1.597E-03	6.438E-03
8.4	-3.420E-03	-1.379E-02
8.8	-5.704E-03	-2.300E-02
9.2	-6.049E-03	-2.439E-02
9.6	-5.217E-03	-2.104E-02
10.0	-3.696E-03	-1.490E-02
10.4	-1.952E-03	-7.870E-03
10.8	-2.816E-04	-1.136E-03
11.2	1.068E-03	4.307E-03
11.6	1.927E-03	7.770E-03
12.0	2.200E-03	8.872E-03
12.4	1.868E-03	7.533E-03
12.8	1.247E-03	5.029E-03
13.2	5.465E-04	2.204E-03
13.6	-3.821E-05	-1.541E-04
14.0	-4.720E-04	-1.903E-03
14.4	-6.848E-04	-2.761E-03
14.8	-7.329E-04	-2.955E-03
15.2	-6.143E-04	-2.477E-03
15.6	-4.150E-04	-1.674E-03

Table 18. Autocovariance and Autocorrelation Function for U = Constant and V ≈ EXP(λ) and for Indicated Parameters.

PARAMETERS: U = 1 LAM = 0.500

TAU	R(TAU)	RHO(TAU)
0.0	2.202E-01	1.000E+00
0.1	2.006E-01	9.109E-01
0.2	1.591E-01	7.226E-01
0.3	1.293E-01	5.873E-01
0.4	1.013E-01	4.601E-01
0.5	7.476E-02	3.394E-01
0.6	4.948E-02	2.247E-01
0.7	2.532E-02	1.150E-01
0.8	2.395E-03	1.087E-02
0.9	-1.926E-02	-8.747E-02
1.0	-3.910E-02	-1.775E-01
1.1	-2.803E-02	-1.273E-01
1.2	-1.827E-02	-8.296E-02
1.3	-1.035E-02	-4.702E-02
1.4	-4.210E-03	-1.912E-02
1.5	2.671E-04	1.213E-03
1.6	3.265E-03	1.482E-02
1.7	4.929E-03	2.238E-02
1.8	5.358E-03	2.433E-02
1.9	4.668E-03	2.120E-02
2.0	2.985E-03	1.355E-02
2.1	1.191E-03	5.409E-03
2.2	8.794E-06	3.993E-05
2.3	-6.795E-04	-3.085E-03
2.4	-9.902E-04	-4.496E-03
2.5	-1.033E-03	-4.689E-03
2.6	-8.920E-04	-4.050E-03
2.7	-6.403E-04	-2.907E-03
2.8	-3.532E-04	-1.604E-03
2.9	-8.991E-05	-4.082E-04
3.0	1.062E-04	4.822E-04
3.1	2.008E-04	9.118E-04
3.2	2.159E-04	9.802E-04
3.3	1.871E-04	8.498E-04
3.4	1.381E-04	6.271E-04
3.5	7.960E-05	3.616E-04
3.6	2.728E-05	1.239E-04
3.7	-9.675E-06	-4.393E-05
3.8	-3.385E-05	-1.537E-04
3.9	-4.418E-05	-2.006E-04
4.0	-3.994E-05	-1.813E-04
4.1	-2.960E-05	-1.344E-04
4.2	-1.918E-05	-8.710E-05
4.3	-7.882E-06	-3.579E-05
4.4	1.633E-06	7.414E-06
4.5	5.273E-06	2.394E-05
4.6	7.624E-06	3.462E-05
4.7	8.965E-06	4.071E-05
4.8	6.500E-06	2.952E-05
4.9	3.546E-06	1.610E-05
5.0	2.515E-06	1.142E-05
5.1	5.438E-07	2.469E-06
5.2	-1.835E-06	-8.331E-06
5.3	-1.112E-06	-5.047E-06
5.4	-1.060E-06	-4.814E-06
5.5	-2.063E-06	-9.367E-06
5.6	-9.695E-07	-4.402E-06
5.7	1.252E-07	5.683E-07
5.8	-4.794E-07	-2.177E-06
5.9	-2.196E-07	-9.973E-07

PARAMETERS: U = 2 LAM = 0.250

TAU	R(TAU)	RHO(TAU)
0.0	2.202E-01	1.000E+00
0.2	2.006E-01	9.109E-01
0.4	1.591E-01	7.226E-01
0.6	1.293E-01	5.873E-01
0.8	1.013E-01	4.601E-01
1.0	7.476E-02	3.394E-01
1.2	4.948E-02	2.247E-01
1.4	2.532E-02	1.150E-01
1.6	2.395E-03	1.087E-02
1.8	-1.926E-02	-8.747E-02
2.0	-3.910E-02	-1.775E-01
2.2	-2.803E-02	-1.273E-01
2.4	-1.827E-02	-8.296E-02
2.6	-1.035E-02	-4.702E-02
2.8	-4.210E-03	-1.912E-02
3.0	2.671E-04	1.213E-03
3.2	3.265E-03	1.482E-02
3.4	4.929E-03	2.238E-02
3.6	5.358E-03	2.433E-02
3.8	4.668E-03	2.120E-02
4.0	2.985E-03	1.355E-02
4.2	1.191E-03	5.409E-03
4.4	8.794E-06	3.993E-05
4.6	-6.795E-04	-3.085E-03
4.8	-9.902E-04	-4.496E-03
5.0	-1.033E-03	-4.689E-03
5.2	-8.920E-04	-4.050E-03
5.4	-6.403E-04	-2.907E-03
5.6	-3.532E-04	-1.604E-03
5.8	-8.991E-05	-4.082E-04
6.0	1.062E-04	4.822E-04
6.2	2.008E-04	9.118E-04
6.4	2.159E-04	9.802E-04
6.6	1.871E-04	8.498E-04
6.8	1.381E-04	6.271E-04
7.0	7.960E-05	3.616E-04
7.2	2.728E-05	1.239E-04
7.4	-9.675E-06	-4.393E-05
7.6	-3.385E-05	-1.537E-04
7.8	-4.418E-05	-2.006E-04
8.0	-3.994E-05	-1.813E-04
8.2	-2.960E-05	-1.344E-04
8.4	-1.918E-05	-8.710E-05
8.6	-7.882E-06	-3.579E-05
8.8	1.633E-06	7.414E-06
9.0	5.273E-06	2.394E-05
9.2	7.624E-06	3.462E-05
9.4	8.965E-06	4.071E-05
9.6	6.500E-06	2.952E-05
9.8	3.546E-06	1.610E-05
10.0	2.515E-06	1.142E-05
10.2	6.063E-07	2.753E-06
10.4	-1.326E-06	-6.020E-06
10.6	-1.547E-06	-7.022E-06
10.8	-1.097E-06	-4.982E-06
11.0	-1.744E-06	-7.918E-06
11.2	-1.037E-06	-4.709E-06
11.4	-2.920E-08	-1.326E-07
11.6	-4.784E-07	-2.172E-06
11.8	-2.820E-07	-1.281E-06
12.0	8.605E-07	3.907E-06

PARAMETERS: U = 3 LAM = 0.167

TAU	R(TAU)	RHO(TAU)
0.0	2.202E-01	1.000E+00
0.5	1.669E-01	7.578E-01
1.0	1.198E-01	5.441E-01
1.5	7.476E-02	3.395E-01
2.0	3.329E-02	1.511E-01
2.5	-4.886E-03	-2.219E-02
3.0	-3.910E-02	-1.775E-01
3.5	-2.127E-02	-9.657E-02
4.0	-8.098E-03	-3.677E-02
4.5	2.671E-04	1.213E-03
5.0	4.511E-03	2.048E-02
5.5	5.240E-03	2.379E-02
6.0	2.985E-03	1.356E-02
6.5	3.448E-04	1.566E-03
7.0	-8.167E-04	-3.709E-03
7.5	-1.033E-03	-4.698E-03
8.0	-7.320E-04	-3.327E-03
8.5	-2.619E-04	-1.189E-03
9.0	1.062E-04	4.823E-04
9.5	2.187E-04	9.930E-04
10.0	1.734E-04	7.875E-04
10.5	7.972E-05	3.620E-04
11.0	-7.029E-08	-3.192E-07
11.5	-3.961E-05	-1.799E-04
12.0	-4.002E-05	-1.817E-04

PARAMETERS: U = 4 LAM = 0.125

TAU	R(TAU)	RHO(TAU)
0.0	2.202E-01	1.000E+00
0.5	1.826E-01	8.292E-01
1.0	1.441E-01	6.542E-01
1.5	1.083E-01	4.918E-01
2.0	7.476E-02	3.395E-01
2.5	4.326E-02	1.965E-01
3.0	1.369E-02	6.218E-02
3.5	-1.407E-02	-6.388E-02
4.0	-3.910E-02	-1.776E-01
4.5	-2.548E-02	-1.157E-01
5.0	-1.409E-02	-6.398E-02
5.5	-5.606E-03	-2.546E-02
6.0	2.671E-04	1.213E-03
6.5	3.812E-03	1.731E-02
7.0	5.291E-03	2.403E-02
7.5	4.943E-03	2.245E-02
8.0	2.985E-03	1.356E-02
8.5	8.390E-04	3.810E-03
9.0	-3.905E-04	-1.773E-03
9.5	-9.443E-04	-4.288E-03
10.0	-1.033E-03	-4.690E-03
10.5	-8.336E-04	-3.785E-03
11.0	-4.974E-04	-2.259E-03
11.5	-1.490E-04	-6.768E-04
12.0	1.061E-04	4.817E-04
12.5	2.087E-04	9.476E-04
13.0	2.055E-04	9.331E-04
13.5	1.494E-04	6.784E-04
14.0	7.972E-05	3.620E-04
14.5	1.772E-05	8.049E-05
15.0	-2.348E-05	-1.066E-04
15.5	-4.188E-05	-1.902E-04
16.0	-4.002E-05	-1.817E-04

Table 19. Autocovariance and Autocorrelation Function for U = Constant and V ≃ EXP(λ) and for Indicated Parameters.

PARAMETERS: U = 1 LAM = 0.333

TAU	R(TAU)	RHO(TAU)
0.0	1.855E-01	1.000E+00
0.2	1.390E-01	7.496E-01
0.4	9.393E-02	5.064E-01
0.6	5.149E-02	2.776E-01
0.8	1.199E-02	6.462E-02
1.0	-2.413E-02	-1.301E-01
1.2	-1.295E-02	-6.983E-02
1.4	-4.698E-03	-2.533E-02
1.6	3.208E-04	1.730E-03
1.8	2.271E-03	1.224E-02
2.0	1.717E-03	9.259E-03
2.2	3.702E-04	1.996E-03
2.4	-1.752E-04	-9.443E-04
2.6	-3.020E-04	-1.628E-03
2.8	-1.760E-04	-9.489E-04
3.0	-2.924E-05	-1.576E-04
3.2	3.360E-05	1.811E-04
3.4	4.141E-05	2.233E-04
3.6	1.007E-05	5.429E-05
3.8	8.255E-06	4.450E-05
4.0	-1.341E-05	-7.229E-05

PARAMETERS: U = 2 LAM = 0.167

TAU	R(TAU)	RHO(TAU)
0.0	1.855E-01	1.000E+00
0.4	1.390E-01	7.497E-01
0.8	9.393E-02	5.065E-01
1.2	5.149E-02	2.777E-01
1.6	1.199E-02	6.463E-02
2.0	-2.413E-02	-1.301E-01
2.4	-1.295E-02	-6.985E-02
2.8	-4.698E-03	-2.533E-02
3.2	3.208E-04	1.730E-03
3.6	2.271E-03	1.224E-02
4.0	1.717E-03	9.261E-03
4.4	3.702E-04	1.996E-03
4.8	-1.752E-04	-9.445E-04
5.2	-3.020E-04	-1.629E-03
5.6	-1.760E-04	-9.491E-04
6.0	-2.924E-05	-1.576E-04
6.4	3.360E-05	1.812E-04
6.8	4.141E-05	2.233E-04
7.2	1.007E-05	5.431E-05
7.6	8.255E-06	4.451E-05
8.0	-1.341E-05	-7.231E-05

PARAMETERS: U = 3 LAM = 0.111

TAU	R(TAU)	RHO(TAU)
0.0	1.855E-01	1.000E+00
0.6	1.390E-01	7.497E-01
1.2	9.393E-02	5.065E-01
1.8	5.149E-02	2.777E-01
2.4	1.199E-02	6.463E-02
3.0	-2.413E-02	-1.301E-01
3.6	-1.295E-02	-6.985E-02
4.2	-4.698E-03	-2.533E-02
4.8	3.208E-04	1.730E-03
5.4	2.271E-03	1.224E-02
6.0	1.717E-03	9.261E-03
6.6	3.702E-04	1.996E-03
7.2	-1.752E-04	-9.445E-04
7.8	-3.020E-04	-1.628E-03
8.4	-1.760E-04	-9.491E-04
9.0	-2.924E-05	-1.576E-04
9.6	3.360E-05	1.812E-04
10.2	4.608E-05	2.484E-04
10.8	1.290E-05	6.954E-05
11.4	9.071E-06	4.891E-05
12.0	-1.424E-05	-7.676E-05

PARAMETERS: U = 4 LAM = 0.083

TAU	R(TAU)	RHO(TAU)
0.0	1.854E-01	1.000E+00
0.8	1.390E-01	7.500E-01
1.6	9.393E-02	5.067E-01
2.4	5.149E-02	2.777E-01
3.2	1.199E-02	6.465E-02
4.0	-2.413E-02	-1.301E-01
4.8	-1.295E-02	-6.987E-02
5.6	-4.698E-03	-2.534E-02
6.4	3.208E-04	1.730E-03
7.2	2.271E-03	1.225E-02
8.0	1.717E-03	9.264E-03
8.8	3.702E-04	1.997E-03
9.6	-1.752E-04	-9.448E-04
10.4	-3.020E-04	-1.629E-03
11.2	-1.831E-04	-9.875E-04
12.0	-2.923E-05	-1.577E-04
12.8	3.262E-05	1.759E-04
13.6	4.608E-05	2.485E-04
14.4	1.290E-05	6.957E-05
15.2	9.071E-06	4.893E-05
16.0	-1.424E-05	-7.679E-05

Table 20. Autocovariance and Autocorrelation Function for U = Constant and V ≈ EXP(λ) and for Indicated Parameters.

PARAMETERS: U = 1 LAM = 0.250

TAU	R(TAU)	RHO(TAU)
0.0	1.580E-01	1.000E+00
0.25	1.115E-01	7.061E-01
0.50	6.600E-02	4.178E-01
0.75	2.321E-02	1.469E-01
1.00	-1.594E-02	-1.009E-01
1.25	-7.746E-03	-4.903E-02
1.50	-1.915E-03	-1.213E-02
1.75	8.716E-04	5.517E-03
2.00	9.849E-04	6.235E-03
2.25	1.977E-04	1.251E-03
2.50	-8.830E-05	-5.590E-04
2.75	-9.816E-05	-6.214E-04
3.00	-2.556E-05	-1.618E-04
3.25	1.208E-05	7.645E-05
3.50	8.026E-06	5.081E-05
3.75	4.734E-06	2.997E-05
4.00	-3.737E-06	-2.366E-05

PARAMETERS: U = 2 LAM = 0.125

TAU	R(TAU)	RHO(TAU)
0.0	1.580E-01	1.000E+00
0.50	1.115E-01	7.061E-01
1.00	6.600E-02	4.178E-01
1.50	2.321E-02	1.469E-01
2.00	-1.594E-02	-1.009E-01
2.50	-7.746E-03	-4.903E-02
3.00	-1.915E-03	-1.213E-02
3.50	8.716E-04	5.517E-03
4.00	9.849E-04	6.235E-03
4.50	1.977E-04	1.251E-03
5.00	-8.830E-05	-5.590E-04
5.50	-9.816E-05	-6.214E-04
6.00	-2.556E-05	-1.618E-04
6.50	1.208E-05	7.645E-05
7.00	8.026E-06	5.081E-05
7.50	4.734E-06	2.997E-05
8.00	-3.737E-06	-2.366E-05

PARAMETERS: U = 3 LAM = 0.083

TAU	R(TAU)	RHO(TAU)
0.0	1.580E-01	1.000E+00
0.75	1.114E-01	7.061E-01
1.50	6.600E-02	4.178E-01
2.25	2.321E-02	1.469E-01
3.00	-1.594E-02	-1.009E-01
3.75	-7.748E-03	-4.903E-02
4.50	-1.915E-03	-1.213E-02
5.25	8.716E-04	5.517E-03
6.00	9.849E-04	6.235E-03
6.75	1.977E-04	1.251E-03
7.50	-8.830E-05	-5.590E-04
8.25	-9.816E-05	-6.214E-04
9.00	-2.556E-05	-1.618E-04
9.75	1.208E-05	7.645E-05
10.50	5.696E-06	3.606E-05
11.25	4.729E-06	2.994E-05
12.00	-4.004E-06	-2.538E-05

PARAMETERS: U = 4 LAM = 0.063

TAU	R(TAU)	RHO(TAU)
0.0	1.579E-01	1.000E+00
1.00	1.115E-01	7.064E-01
2.00	6.600E-02	4.180E-01
3.00	2.321E-02	1.470E-01
4.00	-1.594E-02	-1.010E-01
5.00	-7.746E-03	-4.905E-02
6.00	-1.915E-03	-1.213E-02
7.00	8.716E-04	5.520E-03
8.00	9.849E-04	6.237E-03
9.00	1.977E-04	1.252E-03
10.00	-8.830E-05	-5.592E-04
11.00	-9.534E-05	-6.038E-04
12.00	-2.524E-05	-1.599E-04
13.00	1.207E-05	7.644E-05
14.00	5.696E-06	3.607E-05
15.00	4.729E-06	2.995E-05
16.00	-4.009E-06	-2.539E-05

Table 21. Autocovariance and Autocorrelation Function for U = Constant and V ≈ U(t) and for Indicated Parameters.

PARAMETERS: U = 1 T = 2

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.2	1.483E-01	5.980E-01
0.4	6.676E-02	2.692E-01
0.6	-5.005E-03	-2.018E-02
0.8	-6.979E-02	-2.814E-01
1.0	-1.239E-01	-4.997E-01
1.2	-7.981E-02	-3.218E-01
1.4	-4.367E-02	-1.761E-01
1.6	-1.551E-02	-6.255E-02
1.8	5.653E-03	2.279E-02
2.0	2.082E-02	8.397E-02
2.2	3.066E-02	1.236E-01
2.4	3.456E-02	1.394E-01
2.6	3.167E-02	1.277E-01
2.8	2.123E-02	8.563E-02
3.0	2.605E-03	1.051E-02
3.2	-1.494E-02	-6.023E-02
3.4	-2.279E-02	-9.189E-02
3.6	-2.258E-02	-9.106E-02
3.8	-1.604E-02	-6.468E-02
4.0	-4.944E-03	-1.994E-02
4.2	4.512E-03	1.819E-02
4.4	8.653E-03	3.489E-02
4.6	9.218E-03	3.717E-02
4.8	7.677E-03	3.096E-02
5.0	5.227E-03	2.108E-02
5.5	-1.248E-03	-5.033E-03
6.0	-4.903E-03	-1.977E-02
6.5	-5.774E-04	-2.328E-03
7.0	2.737E-03	1.104E-02
7.5	7.727E-04	3.116E-03
8.0	-8.115E-04	-3.272E-03
8.5	-8.168E-04	-3.294E-03
9.0	4.582E-05	1.848E-04
9.5	6.025E-04	2.429E-03
10.0	1.080E-04	4.357E-04
10.5	-2.746E-04	-1.107E-03
11.0	-1.292E-04	-5.211E-04
11.5	6.512E-05	2.626E-04
12.0	1.163E-04	4.689E-04
12.5	3.827E-06	1.543E-05
13.0	-6.936E-05	-2.797E-04
13.5	-1.834E-05	-7.394E-05
14.0	2.507E-05	1.011E-0[illegible]
14.5	2.082E-05	8.396E-0[illegible]

PARAMETERS: U = 2 T = 4

TAU	R(TAU)	RHO(TAU)
0.0	2.480E-01	1.000E+00
0.4	1.483E-01	5.980E-01
0.8	6.676E-02	2.692E-01
1.2	-5.005E-03	-2.018E-02
1.6	-6.979E-02	-2.814E-01
2.0	-1.239E-01	-4.997E-01
2.4	-7.981E-02	-3.218E-01
2.8	-4.367E-02	-1.761E-01
3.2	-1.551E-02	-6.255E-02
3.6	5.653E-03	2.280E-02
4.0	2.082E-02	8.397E-02
4.4	3.066E-02	1.236E-01
4.8	3.456E-02	1.394E-01
5.2	3.167E-02	1.277E-01
5.6	2.123E-02	8.563E-02
6.0	2.605E-03	1.051E-02
6.4	-1.494E-02	-6.024E-02
6.8	-2.279E-02	-9.189E-02
7.2	-2.258E-02	-9.106E-02
7.6	-1.604E-02	-6.468E-02
8.0	-4.944E-03	-1.994E-02
8.4	4.512E-03	1.819E-02
8.8	8.653E-03	3.489E-02
9.2	9.218E-03	3.717E-02
9.6	7.677E-03	3.096E-02
10.0	5.227E-03	2.109E-02

PROCESS = 266.93333 SECS.
I/O = 8.71667 SECS.
RUN TIME = 275.65000 SECS.

PARAMETERS: U = 3 T = 6

TAU	R(TAU)	RHO(TAU)
0.0	2.492E-01	1.000E+00
0.5	1.480E-01	5.962E-01
1.0	1.007E-01	4.057E-01
1.5	3.192E-02	1.286E-01
2.0	-2.815E-02	-1.134E-01
2.5	-7.993E-02	-3.221E-01
3.0	-1.239E-01	-4.993E-01
3.5	-8.673E-02	-3.495E-01
4.0	-5.480E-02	-2.208E-01
4.5	-2.864E-02	-1.154E-01
5.0	-7.701E-03	-3.103E-02
5.5	8.580E-03	3.457E-02
6.0	2.082E-02	8.390E-02
6.5	2.942E-02	1.185E-01
7.0	3.399E-02	1.370E-01
7.5	3.402E-02	1.371E-01
8.0	2.906E-02	1.171E-01
8.5	1.871E-02	7.538E-02
9.0	2.605E-03	1.050E-02
9.5	-1.273E-02	-5.128E-02
10.0	-2.115E-02	-8.524E-02
10.5	-2.361E-02	-9.512E-02
11.0	-2.103E-02	-8.473E-02
11.5	-1.445E-02	-5.823E-02
12.0	-4.945E-03	-1.993E-02
12.5	3.348E-03	1.349E-02
13.0	7.747E-03	3.122E-02

PARAMETERS: U = 1 T = 4

TAU	R(TAU)	RHO(TAU)
0.0	2.207E-01	1.000E+00
0.4	8.876E-02	4.030E-01
0.8	-1.806E-02	-8.190E-02
1.2	-5.452E-02	-2.474E-01
1.6	-3.370E-02	-1.530E-01
2.0	-2.430E-02	-1.104E-01
2.4	-1.869E-02	-8.495E-02
2.8	-1.099E-02	-4.992E-02
3.2	-2.329E-03	-1.058E-02
3.6	6.921E-03	3.143E-02
4.0	1.694E-02	7.691E-02
4.4	2.115E-02	9.605E-02
4.8	1.296E-02	5.884E-02
5.2	-4.290E-03	-1.948E-02
5.6	-1.184E-02	-5.374E-02
6.0	-8.869E-03	-4.027E-02
6.4	-3.749E-03	-1.702E-02
6.8	-1.473E-03	-6.689E-03
7.2	9.205E-05	4.180E-04
7.6	1.566E-03	7.112E-03
8.0	2.512E-03	1.141E-02
8.4	2.751E-03	1.249E-02
8.8	2.210E-03	1.003E-02
9.2	7.888E-04	3.582E-03
9.6	-1.003E-03	-4.553E-03
10.0	-2.031E-03	-9.221E-03

PROCESS = 361.95000 SECS.
I/O = 6.26667 SECS.
RUN TIME = 368.21667 SECS.

PARAMETERS: U = 2 T = 8

TAU	R(TAU)	RHO(TAU)
0.0	2.207E-01	1.000E+00
0.8	8.876E-02	4.031E-01
1.6	-1.806E-02	-8.201E-02
2.4	-5.452E-02	-2.476E-01
3.2	-3.370E-02	-1.531E-01
4.0	-2.430E-02	-1.100E-01
4.8	-1.869E-02	-8.487E-02
5.6	-1.099E-02	-4.993E-02
6.4	-2.329E-03	-1.058E-02
7.2	6.921E-03	3.144E-02
8.0	1.694E-02	7.693E-02
8.8	2.115E-02	9.608E-02
9.6	1.296E-02	5.886E-02
10.4	-4.294E-03	-1.950E-02
11.2	-1.183E-02	-5.375E-02
12.0	-8.870E-03	-4.029E-02
12.8	-3.749E-03	-1.703E-02
13.6	-1.472E-03	-6.688E-03
14.4	9.197E-05	4.177E-04
15.2	1.566E-03	7.115E-03
16.0	2.512E-03	1.141E-02
16.8	2.751E-03	1.249E-02
17.6	2.210E-03	1.004E-02
18.4	7.889E-04	3.583E-03
19.2	-1.003E-03	-4.555E-03

PROCESS = 190.90000 SECS.
I/O = 6.73333 SECS.
RUN TIME = 197.63333 SECS.

PARAMETERS: U = 3 T = 12

TAU	R(TAU)	RHO(TAU)
0.0	2.212E-01	1.000E+00
1.0	1.102E-01	4.981E-01
2.0	1.761E-02	7.962E-02
3.0	-6.851E-02	-3.097E-01
4.0	-5.607E-02	-2.533E-01
5.0	-3.137E-02	-1.418E-01
6.0	-2.430E-02	-1.099E-01
7.0	-1.980E-02	-8.952E-02
8.0	-1.372E-02	-6.202E-02
9.0	-6.727E-03	-3.042E-02
10.0	6.816E-04	3.082E-03
11.0	8.623E-03	3.899E-02
12.0	1.690E-02	7.640E-02
13.0	2.131E-02	9.633E-02
14.0	1.708E-02	7.722E-02
15.0	4.225E-03	1.910E-02
16.0	-8.133E-03	-3.677E-02
17.0	-1.200E-02	-5.426E-02
18.0	-8.870E-03	-4.010E-02
19.0	-4.341E-03	-1.963E-02
20.0	-2.058E-03	-9.305E-03
21.0	-7.021E-04	-3.174E-03
22.0	6.188E-04	2.798E-03
23.0	1.768E-03	7.995E-03
24.0	2.512E-03	1.136E-02
25.0	2.762E-03	1.249E-02
26.0	2.488E-03	1.125E-02
27.0	1.608E-03	7.271E-03
28.0	1.778E-04	8.021E-04
29.0	-1.256E-03	-5.679E-03
30.0	-2.039E-03	-9.180E-03

PROCESS = 214.38333 SECS.
I/O = 7.06667 SECS.
RUN TIME = 221.45000 SECS.

Table 22. Autocovariance and Autocorrelation Function for U = Constant and V ≈ U(t) and for Indicated Parameters.

PARAMETERS: U = 1 T = 6

TAU	R(TAU)	RHO(TAU)
0.0	1.855E-01	1.000E+00
0.5	6.361E-02	3.429E-01
1.0	-4.120E-02	-2.221E-01
1.5	-2.595E-02	-1.399E-01
2.0	-1.966E-02	-1.060E-01
2.5	-1.722E-02	-9.285E-02
3.0	-1.384E-02	-7.463E-02
3.5	-1.017E-02	-5.481E-02
4.0	-6.275E-03	-3.383E-02
4.5	-2.063E-03	-1.112E-02
5.0	2.467E-03	1.330E-02
5.5	7.313E-03	3.942E-02
6.0	1.254E-02	6.761E-02
6.5	1.295E-02	6.982E-02
7.0	3.350E-03	1.806E-02
7.5	-6.076E-03	-3.275E-02
8.0	-6.335E-03	-3.415E-02
8.5	-3.780E-03	-2.038E-02
9.0	-2.516E-03	-1.357E-02
9.5	-1.373E-03	-7.402E-03
10.0	-3.409E-04	-1.838E-03
10.5	5.020E-04	2.706E-03
11.0	1.122E-03	6.050E-03
11.5	1.474E-03	7.946E-03
12.0	1.532E-03	8.256E-03
12.5	1.239E-03	6.682E-03
13.0	5.371E-04	2.895E-03
13.5	-4.742E-04	-2.556E-03
14.0	-1.143E-03	-6.163E-03
14.5	-9.617E-04	-5.184E-03
15.0	-4.572E-04	-2.465E-03

PROCESS = 505.95000 SECS.
I/O = 5.05000 SECS.
RUN TIME = 511.00000 SECS.

PARAMETERS: U = 2 T = 12

TAU	R(TAU)	RHO(TAU)
0.0	1.855E-01	1.000E+00
0.5	1.650E-01	8.897E-01
1.0	6.361E-02	3.430E-01
1.5	1.391E-02	7.497E-02
2.0	-4.120E-02	-2.221E-01
2.5	-3.233E-02	-1.743E-01
3.0	-2.595E-02	-1.399E-01
3.5	-2.164E-02	-1.167E-01
4.0	-1.966E-02	-1.060E-01
4.5	-1.861E-02	-1.003E-01
5.0	-1.722E-02	-9.286E-02
5.5	-1.561E-02	-8.414E-02
6.0	-1.384E-02	-7.463E-02
6.5	-1.203E-02	-6.484E-02
7.0	-1.017E-02	-5.482E-02
7.5	-8.257E-03	-4.452E-02
8.0	-6.275E-03	-3.383E-02
8.5	-4.211E-03	-2.270E-02
9.0	-2.063E-03	-1.112E-02
9.5	1.648E-04	8.883E-04
10.0	2.467E-03	1.330E-02
10.5	4.846E-03	2.613E-02
11.0	7.313E-03	3.943E-02
11.5	9.877E-03	5.325E-02
12.0	1.254E-02	6.761E-02
12.5	1.400E-02	7.548E-02
13.0	1.295E-02	6.983E-02
13.5	9.399E-03	5.067E-02
14.0	3.350E-03	1.806E-02
14.5	-2.617E-03	-1.411E-02
15.0	-6.076E-03	-3.276E-02
15.5	-7.243E-03	-3.905E-02
16.0	-6.335E-03	-3.416E-02
16.5	-4.796E-03	-2.586E-02
17.0	-3.780E-03	-2.038E-02
17.5	-3.083E-03	-1.662E-02
18.0	-2.516E-03	-1.357E-02
18.5	-1.944E-03	-1.048E-02
19.0	-1.373E-03	-7.404E-03
19.5	-8.345E-04	-4.499E-03
20.0	-3.409E-04	-1.838E-03
20.5	1.052E-04	5.670E-04
21.0	5.020E-04	2.707E-03
21.5	8.436E-04	4.548E-03
22.0	1.122E-03	6.051E-03
22.5	1.333E-03	7.188E-03
23.0	1.474E-03	7.947E-03
23.5	1.542E-03	8.314E-03
24.0	1.532E-03	8.257E-03
24.5	1.434E-03	7.730E-03
25.0	1.239E-03	6.683E-03
25.5	9.421E-04	5.079E-03
26.0	5.371E-04	2.896E-03
26.5	3.748E-05	2.021E-04
27.0	-4.742E-04	-2.556E-03
27.5	-9.002E-04	-4.853E-03
28.0	-1.143E-03	-6.163E-03
28.5	-1.140E-03	-6.144E-03
29.0	-9.617E-04	-5.185E-03
29.5	-7.058E-04	-3.805E-03

PARAMETERS: U = 3 T = 18

TAU	R(TAU)	RHO(TAU)
0.0	1.849E-01	1.000E+00
1.5	6.361E-02	3.440E-01
3.0	-4.120E-02	-2.228E-01
4.5	-2.595E-02	-1.403E-01
6.0	-1.966E-02	-1.063E-01
7.5	-1.722E-02	-9.313E-02
9.0	-1.384E-02	-7.485E-02
10.5	-9.761E-03	-5.278E-02
12.0	-6.305E-03	-3.409E-02
13.5	-2.129E-03	-1.151E-02
15.0	2.514E-03	1.360E-02
16.5	7.303E-03	3.949E-02
18.0	1.253E-02	6.776E-02
19.5	1.296E-02	7.007E-02
21.0	3.347E-03	1.810E-02
22.5	-6.077E-03	-3.286E-02
24.0	-6.335E-03	-3.426E-02
25.5	-3.781E-03	-2.044E-02
27.0	-2.516E-03	-1.361E-02
28.5	-1.374E-03	-7.430E-03
30.0	-3.413E-04	-1.845E-03
31.5	5.020E-04	2.715E-03
33.0	1.121E-03	6.064E-03
34.5	1.474E-03	7.969E-03
36.0	1.532E-03	8.281E-03
37.5	1.239E-03	6.698E-03
39.0	5.369E-04	2.903E-03
40.5	-4.742E-04	-2.564E-03
42.0	-1.144E-03	-6.185E-03
43.5	-9.619E-04	-5.201E-03
45.0	-4.572E-04	-2.472E-03

PROCESS = 214.05000 SECS.
I/O = 9.71667 SECS.
RUN TIME = 223.76667 SECS.

Table 23. Autocovariance and Autocorrelation Function for U = Constant and V ≃ U(t) and for Indicated Parameters.

PARAMETERS: U = 1 T = 8

TAU	R(TAU)	RHO(TAU)
0.0	1.580E-01	1.000E+00
0.5	6.152E-02	3.894E-01
1.0	-2.549E-02	-1.614E-01
1.5	-1.818E-02	-1.151E-01
2.0	-1.450E-02	-9.179E-02
2.5	-1.336E-02	-8.457E-02
3.0	-1.186E-02	-7.506E-02
3.5	-1.024E-02	-6.480E-02
4.0	-8.525E-03	-5.397E-02
4.5	-6.715E-03	-4.251E-02
5.0	-4.801E-03	-3.039E-02
5.5	-2.777E-03	-1.758E-02
6.0	-6.371E-04	-4.033E-03
6.5	1.626E-03	1.030E-02
7.0	4.020E-03	2.545E-02
7.5	6.550E-03	4.147E-02
8.0	9.227E-03	5.841E-02
8.5	8.931E-03	5.654E-02
9.0	2.549E-03	1.613E-02
9.5	-3.792E-03	-2.401E-02
10.0	-4.483E-03	-2.838E-02
10.5	-3.161E-03	-2.001E-02
11.0	-2.462E-03	-1.559E-02
11.5	-1.822E-03	-1.154E-02
12.0	-1.206E-03	-7.638E-03
12.5	-6.499E-04	-4.114E-03
13.0	-1.578E-04	-9.988E-04
13.5	2.613E-04	1.654E-03
14.0	5.976E-04	3.783E-03
14.5	8.386E-04	5.308E-03
15.0	9.731E-04	6.160E-03
15.5	9.888E-04	6.260E-03
16.0	8.702E-04	5.509E-03
16.5	6.025E-04	3.814E-03
17.0	1.688E-04	1.069E-03
17.5	-3.835E-04	-2.428E-03
18.0	-7.489E-04	-4.741E-03
18.5	-6.860E-04	-4.343E-03
19.0	-4.372E-04	-2.768E-03
19.5	-2.501E-04	-1.583E-03
20.0	-1.111E-04	-7.032E-04

PROCESS = 390.53333 SECS.
I/O = 6.90000 SECS.
RUN TIME = 397.43333 SECS.

PARAMETERS: U = 2 T = 16

TAU	R(TAU)	RHO(TAU)
0.0	1.574E-01	1.000E+00
1.0	6.152E-02	3.909E-01
2.0	-2.549E-02	-1.620E-01
3.0	-1.818E-02	-1.155E-01
4.0	-1.450E-02	-9.214E-02
5.0	-1.336E-02	-8.490E-02
6.0	-1.186E-02	-7.535E-02
7.0	-1.024E-02	-6.505E-02
8.0	-8.525E-03	-5.417E-02
9.0	-6.715E-03	-4.267E-02
10.0	-4.801E-03	-3.051E-02
11.0	-2.829E-03	-1.798E-02
12.0	-6.621E-04	-4.208E-03
13.0	1.661E-03	1.056E-02
14.0	4.020E-03	2.555E-02
15.0	6.539E-03	4.156E-02
16.0	9.227E-03	5.864E-02
17.0	8.936E-03	5.679E-02
18.0	2.548E-03	1.619E-02
19.0	-3.794E-03	-2.411E-02
20.0	-4.483E-03	-2.849E-02
21.0	-3.161E-03	-2.009E-02
22.0	-2.462E-03	-1.565E-02
23.0	-1.822E-03	-1.158E-02
24.0	-1.206E-03	-7.667E-03
25.0	-6.499E-04	-4.130E-03
26.0	-1.578E-04	-1.003E-03
27.0	2.613E-04	1.661E-03
28.0	5.976E-04	3.798E-03
29.0	8.386E-04	5.329E-03
30.0	9.731E-04	6.184E-03
31.0	9.888E-04	6.284E-03
32.0	8.702E-04	5.530E-03
33.0	6.025E-04	3.829E-03
34.0	1.688E-04	1.073E-03
35.0	-3.835E-04	-2.437E-03
36.0	-7.489E-04	-4.759E-03
37.0	-6.860E-04	-4.359E-03
38.0	-4.372E-04	-2.778E-03
39.0	-2.501E-04	-1.589E-03
40.0	-1.111E-04	-7.059E-04

PROCESS = 310.21667 SECS.
I/O = 8.45000 SECS.
RUN TIME = 318.66667 SECS.

PARAMETERS: U = 3 T = 24

TAU	R(TAU)	RHO(TAU)
0.0	1.603E-01	1.000E+00
1.5	6.152E-02	3.839E-01
3.0	-2.549E-02	-1.591E-01
4.5	-1.818E-02	-1.134E-01
6.0	-1.450E-02	-9.048E-02
7.5	-1.336E-02	-8.336E-02
9.0	-1.186E-02	-7.398E-02
10.5	-1.045E-02	-6.521E-02
12.0	-8.835E-03	-5.513E-02
13.5	-6.444E-03	-4.021E-02
15.0	-4.774E-03	-2.979E-02
16.5	-2.829E-03	-1.766E-02
18.0	-6.621E-04	-4.131E-03
19.5	1.661E-03	1.037E-02
21.0	4.020E-03	2.509E-02
22.5	6.539E-03	4.080E-02
24.0	9.227E-03	5.757E-02
25.5	8.936E-03	5.576E-02
27.0	2.548E-03	1.590E-02
28.5	-3.794E-03	-2.368E-02
30.0	-4.483E-03	-2.797E-02
31.5	-3.161E-03	-1.972E-02
33.0	-2.462E-03	-1.536E-02
34.5	-1.822E-03	-1.137E-02
36.0	-1.206E-03	-7.528E-03
37.5	-6.499E-04	-4.056E-03
39.0	-1.578E-04	-9.845E-04
40.5	2.613E-04	1.630E-03
42.0	5.976E-04	3.729E-03
43.5	8.386E-04	5.232E-03
45.0	9.731E-04	6.072E-03
46.5	9.888E-04	6.170E-03
48.0	8.702E-04	5.430E-03
49.5	6.025E-04	3.760E-03
51.0	1.688E-04	1.053E-03
52.5	-3.835E-04	-2.393E-03
54.0	-7.489E-04	-4.673E-03
55.5	-6.860E-04	-4.280E-03
57.0	-4.372E-04	-2.728E-03
58.5	-2.501E-04	-1.560E-03
60.0	-1.111E-04	-6.931E-04

PROCESS = 293.71667 SECS.
I/O = 8.56667 SECS.
RUN TIME = 302.28333 SECS.

Table 24. Autocovariance and Autocorrelation Function for U = Constant and V ≃ U(t) and for Indicated Parameters

PARAMETERS: U = 1 T = 10

TAU	R(TAU)	RHO(TAU)
0.0	1.369E-01	1.000E+00
1.0	-1.879E-02	-1.373E-01
2.0	-1.070E-02	-7.814E-02
3.0	-9.400E-03	-6.865E-02
4.0	-7.697E-03	-5.622E-02
5.0	-5.745E-03	-4.196E-02
6.0	-3.666E-03	-2.678E-02
7.0	-1.341E-03	-9.792E-03
8.0	1.171E-03	8.553E-03
9.0	3.943E-03	2.880E-02
10.0	6.974E-03	5.094E-02
11.0	1.962E-03	1.433E-02
12.0	-3.284E-03	-2.399E-02
13.0	-2.105E-03	-1.537E-02
14.0	-1.334E-03	-9.743E-03
15.0	-6.559E-04	-4.791E-03
16.0	-7.214E-05	-5.269E-04
17.0	3.580E-04	2.614E-03
18.0	6.281E-04	4.587E-03
19.0	6.873E-04	5.020E-03
20.0	5.036E-04	3.678E-03
21.0	2.392E-05	1.747E-04
22.0	-5.355E-04	-3.911E-03
23.0	-3.646E-04	-2.663E-03
24.0	-1.615E-04	-1.179E-03
25.0	-9.732E-06	-7.108E-05

PROCESS = 231.88333 SECS.
I/O = 6.98333 SECS.
RUN TIME = 238.86667 SECS.

PARAMETERS: U = 1 T = 20

TAU	R(TAU)	RHO(TAU)
0.0	8.050E-02	1.000E+00
2.0	-3.117E-03	-3.872E-02
4.0	-3.229E-03	-4.011E-02
6.0	-2.773E-03	-3.445E-02
8.0	-2.215E-03	-2.751E-02
10.0	-1.594E-03	-1.980E-02
12.0	-9.279E-04	-1.153E-02
14.0	-2.108E-04	-2.619E-03
16.0	6.028E-04	7.488E-03
18.0	1.493E-03	1.854E-02
20.0	2.478E-03	3.078E-02
22.0	-1.074E-03	-1.334E-02
24.0	-7.513E-04	-9.332E-03
26.0	-5.024E-04	-6.241E-03
28.0	-2.723E-04	-3.382E-03
30.0	-7.537E-05	-9.362E-04
32.0	7.469E-05	9.278E-04
34.0	1.704E-04	2.116E-03
36.0	2.059E-04	2.557E-03
38.0	1.717E-04	2.133E-03
40.0	3.718E-05	4.619E-04
42.0	-1.728E-04	-2.147E-03
44.0	-1.136E-04	-1.411E-03
46.0	-4.718E-05	-5.861E-04
48.0	3.508E-07	4.357E-06
50.0	2.646E-05	3.286E-04

PROCESS = 222.13333 SECS.
I/O = 9.10000 SECS.
RUN TIME = 231.23333 SECS.

PARAMETERS: U = 2 T = 20

TAU	R(TAU)	RHO(TAU)
0.0	1.368E-01	1.000E+00
2.0	-1.879E-02	-1.373E-01
4.0	-1.070E-02	-7.818E-02
6.0	-9.400E-03	-6.869E-02
8.0	-7.697E-03	-5.624E-02
10.0	-5.745E-03	-4.198E-02
12.0	-3.621E-03	-2.646E-02
14.0	-1.374E-03	-1.004E-02
16.0	1.195E-03	8.730E-03
18.0	3.935E-03	2.876E-02
20.0	6.972E-03	5.095E-02
22.0	1.962E-03	1.434E-02
24.0	-3.284E-03	-2.400E-02
26.0	-2.105E-03	-1.538E-02
28.0	-1.334E-03	-9.748E-03
30.0	-6.559E-04	-4.793E-03
32.0	-7.214E-05	-5.272E-04
34.0	3.580E-04	2.616E-03
36.0	6.281E-04	4.590E-03
38.0	6.873E-04	5.023E-03
40.0	5.036E-04	3.680E-03
42.0	2.392E-05	1.748E-04
44.0	-5.355E-04	-3.913E-03
46.0	-3.646E-04	-2.664E-03
48.0	-1.615E-04	-1.180E-03
50.0	-9.732E-06	-7.111E-05

PROCESS = 198.43333 SECS.
I/O = 5.13333 SECS.
RUN TIME = 203.56667 SECS.

PARAMETERS: U = 2 T = 40

TAU	R(TAU)	RHO(TAU)
0.0	8.134E-02	1.000E+00
4.0	-3.188E-03	-3.919E-02
8.0	-3.234E-03	-3.975E-02
12.0	-2.755E-03	-3.386E-02
16.0	-2.208E-03	-2.715E-02
20.0	-1.599E-03	-1.965E-02
24.0	-9.339E-04	-1.148E-02
28.0	-1.998E-04	-2.456E-03
32.0	6.073E-04	7.466E-03
36.0	1.494E-03	1.836E-02
40.0	2.471E-03	3.038E-02
44.0	-1.077E-03	-1.324E-02
48.0	-7.482E-04	-9.198E-03
52.0	-4.979E-04	-6.121E-03
56.0	-2.725E-04	-3.350E-03
60.0	-7.888E-05	-9.697E-04
64.0	7.205E-05	8.857E-04
68.0	1.729E-04	2.125E-03
72.0	2.088E-04	2.567E-03
76.0	1.689E-04	2.076E-03
80.0	3.536E-05	4.347E-04
84.0	-1.713E-04	-2.106E-03
88.0	-1.107E-04	-1.360E-03
92.0	-4.745E-05	-5.834E-04
96.0	-1.832E-06	-2.252E-05

PARAMETERS: U = 3 T = 30

TAU	R(TAU)	RHO(TAU)
0.0	1.325E-01	1.000E+00
2.5	3.737E-03	2.821E-02
5.0	-1.201E-02	-9.066E-02
7.5	-1.023E-02	-7.722E-02
10.0	-8.853E-03	-6.682E-02
12.5	-7.406E-03	-5.590E-02
15.0	-5.918E-03	-4.467E-02
17.5	-3.922E-03	-2.960E-02
20.0	-2.170E-03	-1.638E-02
22.5	-1.164E-04	-8.783E-04
25.0	2.088E-03	1.576E-02
27.5	4.426E-03	3.341E-02
30.0	6.972E-03	5.262E-02
32.5	3.931E-03	2.967E-02
35.0	-3.226E-03	-2.435E-02
37.5	-2.525E-03	-1.906E-02
40.0	-1.847E-03	-1.394E-02
42.5	-1.221E-03	-9.215E-03
45.0	-6.559E-04	-4.951E-03
47.5	-1.679E-04	-1.268E-03
50.0	2.279E-04	1.720E-03
52.5	5.137E-04	3.877E-03
55.0	6.696E-04	5.054E-03
57.5	6.771E-04	5.110E-03
60.0	5.036E-04	3.801E-03
62.5	1.250E-04	9.434E-04
65.0	-4.078E-04	-3.078E-03
67.5	-5.074E-04	-3.830E-03
70.0	-2.834E-04	-2.139E-03
72.5	-1.275E-04	-9.625E-04
75.0	-9.732E-06	-7.345E-05

PROCESS = 229.41667 SECS.
I/O = 7.41667 SECS.
RUN TIME = 236.83333 SECS.

PARAMETERS: U = 3 T = 60

TAU	R(TAU)	RHO(TAU)
0.0	8.287E-02	1.000E+00
5.0	-4.256E-03	-5.136E-02
10.0	-3.416E-03	-4.122E-02
15.0	-3.842E-03	-4.636E-02
20.0	-2.483E-03	-2.996E-02
25.0	-2.156E-03	-2.601E-02
30.0	-1.594E-03	-1.923E-02
35.0	-1.051E-03	-1.268E-02
40.0	-4.676E-04	-5.642E-03
45.0	2.070E-04	2.498E-03
50.0	8.853E-04	1.068E-02
55.0	1.646E-03	1.986E-02
60.0	2.478E-03	2.990E-02
65.0	-9.628E-04	-1.162E-02
70.0	-8.394E-04	-1.013E-02
75.0	-6.158E-04	-7.430E-03
80.0	-4.196E-04	-5.064E-03
85.0	-2.401E-04	-2.897E-03
90.0	-7.537E-05	-9.095E-04
95.0	5.037E-05	6.078E-04

Table 25. Autocovariance and Autocorrelation Function for U = Constant and V ≃ $N(\mu,\sigma^2)$ and for Indicated Parameters.

Table 25

PARAMETERS: U = 1 MU = 0 SIGMA = 1.00

TAU	R(TAU)	RHO(TAU)
0.0	9.302E-01	1.000E+00
0.1	8.289E-01	8.911E-01
0.2	7.338E-01	7.889E-01
0.3	6.417E-01	6.899E-01
0.4	5.495E-01	5.908E-01
0.5	4.573E-01	4.917E-01
0.6	3.654E-01	3.928E-01
0.7	2.734E-01	2.939E-01
0.8	1.814E-01	1.951E-01
0.9	8.931E-02	9.601E-02
1.0	-9.745E-04	-1.048E-03
1.1	4.339E-03	4.665E-03
1.2	1.120E-02	1.204E-02
1.3	1.743E-02	1.874E-02
1.4	2.276E-02	2.447E-02
1.5	2.716E-02	2.920E-02
1.6	3.029E-02	3.256E-02
1.7	3.228E-02	3.471E-02
1.8	3.292E-02	3.539E-02
1.9	3.248E-02	3.492E-02
2.0	3.094E-02	3.326E-02
2.1	2.869E-02	3.084E-02
2.2	2.582E-02	2.776E-02
2.3	2.274E-02	2.445E-02
2.4	1.955E-02	2.101E-02
2.5	1.724E-02	1.853E-02
2.6	1.436E-02	1.544E-02
2.7	1.168E-02	1.256E-02
2.8	9.238E-03	9.932E-03
2.9	7.075E-03	7.606E-03
3.0	5.232E-03	5.624E-03
3.1	3.684E-03	3.960E-03
3.2	2.466E-03	2.651E-03
3.3	1.491E-03	1.603E-03
3.4	8.071E-04	8.677E-04
3.5	2.916E-04	3.135E-04
3.6	-4.430E-06	-4.762E-06
3.7	-2.053E-04	-2.207E-04
3.8	-2.604E-04	-2.800E-04
3.9	-2.777E-04	-2.986E-04
4.0	-2.149E-04	-2.311E-04
4.1	-1.531E-04	-1.646E-04
4.2	-6.270E-05	-6.741E-05
4.3	1.569E-05	1.687E-05
4.4	8.399E-05	9.030E-05
4.5	1.421E-04	1.527E-04
4.6	1.749E-04	1.881E-04
4.7	2.079E-04	2.235E-04
4.8	2.031E-04	2.183E-04
4.9	2.136E-04	2.297E-04
5.0	1.850E-04	1.989E-04
5.2	1.392E-04	1.497E-04
5.4	8.864E-05	9.530E-05
5.6	4.617E-05	4.964E-05
5.8	1.390E-05	1.495E-05
6.0	-1.744E-06	-1.875E-06
6.2	-6.707E-06	-7.211E-06
6.4	-5.517E-06	-5.931E-06
6.6	-2.188E-06	-2.352E-06
6.8	1.357E-06	1.459E-06
7.0	3.647E-06	3.921E-06
7.2	3.620E-06	3.892E-06
7.4	1.873E-06	2.014E-06
7.6	-5.713E-07	-6.142E-07
7.8	-9.026E-07	-9.704E-07
8.0	-3.101E-06	-3.334E-06
8.2	-4.077E-06	-4.383E-06
8.4	-3.662E-06	-3.937E-06
8.6	-2.099E-06	-2.257E-06
8.8	-2.090E-06	-2.247E-06
9.0	7.572E-08	8.141E-08
9.2	2.152E-06	2.313E-06
9.4	3.575E-06	3.843E-06
9.6	3.988E-06	4.288E-06
9.8	3.848E-06	4.137E-06
10.0	3.190E-06	3.430E-06

PARAMETERS: U = 1 MU = 2 SIGMA = 0.30

TAU	R(TAU)	RHO(TAU)
0.0	2.182E-01	1.000E+00
0.3	1.223E-01	5.603E-01
0.6	2.281E-02	1.045E-01
0.9	-7.744E-02	-3.549E-01
1.2	-1.110E-01	-5.087E-01
1.5	-1.093E-01	-5.010E-01
1.8	-9.573E-02	-4.387E-01
2.1	-5.243E-02	-2.403E-01
2.4	2.472E-02	1.133E-01
2.7	1.058E-01	4.848E-01
3.0	1.426E-01	6.534E-01
3.3	1.059E-01	4.855E-01
3.6	2.474E-02	1.134E-01
3.9	-5.230E-02	-2.397E-01
4.2	-9.433E-02	-4.323E-01
4.5	-1.008E-01	-4.619E-01
4.8	-8.199E-02	-3.758E-01
5.1	-3.823E-02	-1.752E-01
5.4	2.530E-02	1.159E-01
5.7	8.535E-02	3.911E-01
6.0	1.102E-01	5.052E-01
6.3	8.525E-02	3.907E-01
6.6	2.546E-02	1.167E-01
6.9	-3.705E-02	-1.698E-01
7.2	-7.746E-02	-3.550E-01
7.5	-8.757E-02	-4.013E-01
7.8	-6.993E-02	-3.205E-01
8.1	-2.966E-02	-1.359E-01
8.4	2.295E-02	1.052E-01
8.7	6.905E-02	3.164E-01
9.0	8.762E-02	4.016E-01
9.3	6.925E-02	3.174E-01
9.6	2.362E-02	1.082E-01
9.9	-2.736E-02	-1.254E-01
10.2	-6.324E-02	-2.898E-01
10.5	-7.391E-02	-3.387E-01
10.8	-5.896E-02	-2.702E-01
11.1	-2.421E-02	-1.110E-01
11.4	1.950E-02	8.935E-02
11.7	5.625E-02	2.578E-01
12.0	7.075E-02	3.242E-01
12.3	5.657E-02	2.593E-01
12.6	2.078E-02	9.522E-02
12.9	-2.060E-02	-9.439E-02
13.2	-5.135E-02	-2.353E-01
13.5	-6.164E-02	-2.825E-01
13.8	-4.958E-02	-2.272E-01
14.1	-2.012E-02	-9.219E-02
14.4	1.615E-02	7.404E-02
14.7	4.586E-02	2.102E-01
15.0	5.740E-02	2.631E-01
15.3	4.647E-02	2.130E-01
15.6	1.807E-02	8.283E-02
15.9	-1.575E-02	-7.219E-02
16.2	-4.184E-02	-1.917E-01

PROCESS = 508.01667 SECS.
I/O = 6.66667 SECS.
RUN TIME = 514.68333 SECS.

Table 26. Autocovariance and Autocorrelation Function for U = Constant and V ≃ $N(\mu,\sigma^2)$ and for Indicated Parameters.

PARAMETERS: U = 1 MU = 1 SIGMA = 1.00

TAU	R(TAU)	RHO(TAU)
0.0	3.203E-01	1.000E+00
0.2	2.254E-01	7.039E-01
0.4	1.379E-01	4.305E-01
0.6	5.318E-02	1.660E-01
0.8	-2.778E-02	-8.674E-02
1.0	-1.041E-01	-3.249E-01
1.2	-8.207E-02	-2.562E-01
1.4	-5.782E-02	-1.805E-01
1.6	-3.483E-02	-1.087E-01
1.8	-1.550E-02	-4.838E-02
2.0	-6.528E-04	-2.038E-03
2.2	9.017E-03	2.815E-02
2.4	1.424E-02	4.445E-02
2.6	1.565E-02	4.885E-02
2.8	1.432E-02	4.470E-02
3.0	1.106E-02	3.452E-02
3.2	6.793E-03	2.121E-02
3.4	2.479E-03	7.738E-03
3.6	-1.090E-03	-3.404E-03
3.8	-3.283E-03	-1.025E-02
4.0	-4.128E-03	-1.289E-02
4.2	-3.829E-03	-1.195E-02
4.4	-3.025E-03	-9.446E-03
4.6	-1.999E-03	-6.241E-03
4.8	-1.079E-03	-3.368E-03
5.0	-1.913E-04	-5.971E-04
5.2	5.271E-04	1.646E-03
5.4	1.057E-03	3.301E-03
5.6	1.219E-03	3.806E-03
5.8	1.032E-03	3.221E-03
6.0	6.161E-04	1.924E-03
6.2	1.912E-04	5.968E-04
6.4	-4.775E-05	-1.491E-04
6.6	-1.279E-04	-3.993E-04
6.8	-1.003E-04	-3.131E-04
7.0	-1.501E-04	-4.685E-04
7.2	-2.179E-04	-6.802E-04
7.4	-3.009E-04	-9.393E-04
7.6	-2.295E-04	-7.164E-04
7.8	-8.317E-05	-2.597E-04
8.0	1.124E-04	3.509E-04

PARAMETERS: U = 5 MU = 5 SIGMA = 1.00

TAU	R(TAU)	RHO(TAU)
0.0	2.460E-01	1.000E+00
1.0	1.500E-01	6.100E-01
2.0	5.019E-02	2.041E-01
3.0	-4.932E-02	-2.005E-01
4.0	-1.417E-01	-5.763E-01
5.0	-2.083E-01	-8.471E-01
6.0	-1.417E-01	-5.763E-01
7.0	-4.932E-02	-2.005E-01
8.0	4.848E-02	1.971E-01
9.0	1.335E-01	5.426E-01
10.0	1.704E-01	6.930E-01
11.0	1.335E-01	5.429E-01
12.0	4.921E-02	2.001E-01
13.0	-4.433E-02	-1.802E-01
14.0	-1.219E-01	-4.956E-01
15.0	-1.539E-01	-6.256E-01
16.0	-1.220E-01	-4.961E-01
17.0	-4.582E-02	-1.863E-01
18.0	4.096E-02	1.666E-01
19.0	1.104E-01	4.489E-01
20.0	1.374E-01	5.588E-01
21.0	1.109E-01	4.509E-01
22.0	4.370E-02	1.777E-01
23.0	-3.618E-02	-1.471E-01
24.0	-1.002E-01	-4.073E-01
25.0	-1.249E-01	-5.078E-01
26.0	-1.011E-01	-4.109E-01
27.0	-4.012E-02	-1.631E-01
28.0	3.268E-02	1.329E-01
29.0	9.044E-02	3.677E-01
30.0	1.126E-01	4.579E-01
31.0	9.180E-02	3.732E-01
32.0	3.741E-02	1.521E-01
33.0	-2.878E-02	-1.170E-01
34.0	-8.178E-02	-3.325E-01
35.0	-1.023E-01	-4.161E-01
36.0	-8.363E-02	-3.400E-01
37.0	-3.431E-02	-1.395E-01
38.0	2.577E-02	1.048E-01
39.0	7.381E-02	3.001E-01
40.0	9.255E-02	3.763E-01
41.0	7.603E-02	3.091E-01
42.0	3.173E-02	1.290E-01
43.0	-2.275E-02	-9.248E-02
44.0	-6.669E-02	-2.711E-01
45.0	-8.403E-02	-3.417E-01
46.0	-6.926E-02	-2.816E-01
47.0	-2.911E-02	-1.184E-01
48.0	2.030E-02	8.254E-02
49.0	6.021E-02	2.448E-01
50.0	7.609E-02	3.094E-01
51.0	6.299E-02	2.561E-01
52.0	2.683E-02	1.091E-01
53.0	-1.797E-02	-7.305E-02
54.0	-5.437E-02	-2.211E-01
55.0	-6.906E-02	-2.808E-01

PROCESS = 356.50000 SECS.
I/O = 6.90000 SECS.
RUN TIME = 363.40000 SECS.

Table 27. Autocovariance and Autocorrelation Function for U = Constant and V ≃ $N(\mu,\sigma^2)$ and for Indicated Parameters.

PARAMETERS: U = 5 MU = 10 SIGMA = 1.50

TAU	R(TAU)	RHO(TAU)
0.0	2.166E-01	1.000E+00
1.5	1.210E-01	5.583E-01
3.0	2.189E-02	1.010E-01
4.5	-7.708E-02	-3.558E-01
6.0	-1.096E-01	-5.060E-01
7.5	-1.079E-01	-4.980E-01
9.0	-9.484E-02	-4.378E-01
10.5	-5.218E-02	-2.409E-01
12.0	2.428E-02	1.121E-01
13.5	1.047E-01	4.832E-01
15.0	1.411E-01	6.514E-01
16.5	1.048E-01	4.839E-01
18.0	2.451E-02	1.131E-01
19.5	-5.158E-02	-2.381E-01
21.0	-9.312E-02	-4.298E-01
22.5	-9.941E-02	-4.589E-01
24.0	-8.096E-02	-3.737E-01
25.5	-3.796E-02	-1.752E-01
27.0	2.457E-02	1.134E-01
28.5	8.394E-02	3.874E-01
30.0	1.088E-01	5.021E-01
31.5	8.429E-02	3.890E-01
33.0	2.531E-02	1.168E-01
34.5	-3.633E-02	-1.677E-01
36.0	-7.603E-02	-3.509E-01
37.5	-8.599E-02	-3.969E-01
39.0	-6.887E-02	-3.179E-01
40.5	-2.960E-02	-1.366E-01
42.0	2.205E-02	1.018E-01
43.5	6.752E-02	3.117E-01
45.0	8.602E-02	3.971E-01
46.5	6.816E-02	3.146E-01
48.0	2.354E-02	1.087E-01
49.5	-2.633E-02	-1.215E-01
51.0	-6.151E-02	-2.839E-01
52.5	-7.225E-02	-3.335E-01
54.0	-5.795E-02	-2.675E-01
55.5	-2.418E-02	-1.116E-01
57.0	1.845E-02	8.514E-02
58.5	5.441E-02	2.511E-01
60.0	6.885E-02	3.178E-01
61.5	5.546E-02	2.560E-01
63.0	2.088E-02	9.637E-02
64.5	-1.938E-02	-8.945E-02
66.0	-4.944E-02	-2.282E-01
67.5	-5.968E-02	-2.755E-01
69.0	-4.834E-02	-2.231E-01
70.5	-2.020E-02	-9.324E-02
72.0	1.472E-02	6.795E-02
73.5	4.368E-02	2.016E-01
75.0	5.533E-02	2.554E-01
76.5	4.524E-02	2.088E-01
78.0	1.816E-02	8.381E-02
79.5	-1.427E-02	-6.586E-02
81.0	-3.944E-02	-1.821E-01
82.5	-4.874E-02	-2.250E-01

PROCESS = 307.45000 SECS.
I/O = 4.96667 SECS.
RUN TIME = 312.41667 SECS.

PARAMETERS: U = 5 MU = 15 SIGMA = 1.00

TAU	R(TAU)	RHO(TAU)
0.0	1.561E-01	1.000E+00
2.0	6.512E-02	4.172E-01
4.0	-2.207E-02	-1.414E-01
6.0	-5.446E-02	-3.489E-01
8.0	-3.850E-02	-2.466E-01
10.0	-3.296E-02	-2.112E-01
12.0	-3.993E-02	-2.558E-01
14.0	-5.213E-02	-3.340E-01
16.0	-1.952E-02	-1.251E-01
18.0	6.329E-02	4.054E-01
20.0	1.206E-01	7.727E-01
22.0	6.526E-02	4.181E-01
24.0	-1.599E-02	-1.024E-01
26.0	-4.797E-02	-3.073E-01
28.0	-3.717E-02	-2.381E-01
30.0	-3.274E-02	-2.097E-01
32.0	-4.093E-02	-2.622E-01
34.0	-4.833E-02	-3.096E-01
36.0	-1.576E-02	-1.010E-01
38.0	5.851E-02	3.748E-01
40.0	1.041E-01	6.666E-01
42.0	6.244E-02	4.000E-01
44.0	-8.593E-03	-5.505E-02
46.0	-4.015E-02	-2.572E-01
48.0	-3.523E-02	-2.257E-01
50.0	-3.244E-02	-2.078E-01
52.0	-4.113E-02	-2.635E-01
54.0	-4.532E-02	-2.904E-01
56.0	-1.306E-02	-8.365E-02
58.0	5.281E-02	3.383E-01
60.0	9.145E-02	5.858E-01
62.0	5.870E-02	3.760E-01
64.0	-2.291E-03	-1.467E-02
66.0	-3.299E-02	-2.114E-01
68.0	-3.252E-02	-2.083E-01
70.0	-3.187E-02	-2.042E-01
72.0	-4.085E-02	-2.617E-01
74.0	-4.294E-02	-2.751E-01
76.0	-1.153E-02	-7.386E-02
78.0	4.717E-02	3.022E-01
80.0	8.099E-02	5.189E-01
82.0	5.498E-02	3.522E-01
84.0	2.803E-03	1.796E-02
86.0	-2.640E-02	-1.691E-01
88.0	-2.929E-02	-1.876E-01
90.0	-3.088E-02	-1.978E-01
92.0	-4.031E-02	-2.582E-01
94.0	-4.114E-02	-2.635E-01
96.0	-1.102E-02	-7.062E-02
98.0	4.179E-02	2.677E-01
100.0	7.209E-02	4.618E-01
102.0	5.146E-02	3.297E-01
104.0	6.897E-03	4.418E-02
106.0	-2.034E-02	-1.303E-01
108.0	-2.563E-02	-1.642E-01
110.0	-2.946E-02	-1.887E-01

PROCESS = 293.36667 SECS.
I/O = 6.68333 SECS.
RUN TIME = 300.05000 SECS.

Table 28. Autocovariance and Autocorrelation Function for U = Constant and V ≈ N(μ,σ^2) and for Indicated Parameters.

PARAMETERS: U = 1 MU = 1 SIGMA = 0.10

TAU	R(TAU)	RHO(TAU)
0.0	2.459E-01	1.000E+00
0.2	1.534E-01	6.238E-01
0.4	4.860E-02	1.976E-01
0.6	-5.113E-02	-2.079E-01
0.8	-1.493E-01	-6.070E-01
1.0	-2.293E-01	-9.322E-01
1.2	-1.496E-01	-6.081E-01
1.4	-5.011E-02	-2.038E-01
1.6	4.985E-02	2.027E-01
1.8	1.492E-01	6.065E-01
2.0	2.101E-01	8.542E-01
2.2	1.491E-01	6.062E-01
2.4	5.000E-02	2.033E-01
2.6	-4.998E-02	-2.032E-01
2.8	-1.470E-01	-5.978E-01
3.0	-2.018E-01	-8.205E-01
3.2	-1.470E-01	-5.978E-01
3.4	-4.991E-02	-2.030E-01
3.6	4.987E-02	2.028E-01
3.8	1.449E-01	5.894E-01
4.0	1.935E-01	7.869E-01
4.2	1.450E-01	5.894E-01
4.4	4.989E-02	2.029E-01
4.6	-4.960E-02	-2.017E-01
4.8	-1.421E-01	-5.780E-01
5.0	-1.872E-01	-7.612E-01
5.2	-1.421E-01	-5.780E-01
5.4	-4.962E-02	-2.018E-01
5.6	4.938E-02	2.008E-01
5.8	1.393E-01	5.665E-01
6.0	1.808E-01	7.354E-01
6.2	1.393E-01	5.664E-01
6.4	4.941E-02	2.009E-01
6.6	-4.885E-02	-1.986E-01
6.8	-1.363E-01	-5.542E-01
7.0	-1.755E-01	-7.136E-01
7.2	-1.363E-01	-5.542E-01
7.4	-4.891E-02	-1.989E-01
7.6	4.834E-02	1.965E-01
7.8	1.333E-01	5.420E-01
8.0	1.702E-01	6.920E-01
8.2	1.333E-01	5.420E-01
8.4	4.845E-02	1.970E-01
8.6	-4.758E-02	-1.935E-01
8.8	-1.303E-01	-5.298E-01
9.0	-1.655E-01	-6.728E-01
9.2	-1.303E-01	-5.298E-01
9.4	-4.773E-02	-1.941E-01
9.6	4.687E-02	1.906E-01
9.8	1.273E-01	5.176E-01
10.0	1.607E-01	6.536E-01
10.2	1.273E-01	5.176E-01
10.4	4.711E-02	1.915E-01
10.6	-4.595E-02	-1.868E-01
10.8	-1.243E-01	-5.055E-01
11.0	-1.565E-01	-6.363E-01

PROCESS = 732.56667 SECS.
I/O = 7.38333 SECS.
RUN TIME = 739.95000 SECS.

PARAMETERS: U = 5 MU = 20 SIGMA = 1.25

TAU	R(TAU)	RHO(TAU)
0.0	2.006E-01	1.000E+00
2.5	9.718E-02	4.845E-01
5.0	-2.161E-02	-1.077E-01
7.5	-5.194E-02	-2.590E-01
10.0	-7.727E-02	-3.852E-01
12.5	-8.828E-02	-4.401E-01
15.0	-7.936E-02	-3.957E-01
17.5	-5.458E-02	-2.721E-01
20.0	-6.476E-03	-3.228E-02
22.5	9.515E-02	4.744E-01
25.0	1.648E-01	8.215E-01
27.5	9.840E-02	4.905E-01
30.0	-8.403E-04	-4.189E-03
32.5	-4.840E-02	-2.413E-01
35.0	-7.531E-02	-3.754E-01
37.5	-8.846E-02	-4.410E-01
40.0	-8.152E-02	-4.064E-01
42.5	-5.556E-02	-2.770E-01
45.0	-1.008E-03	-5.024E-03
47.5	9.153E-02	4.563E-01
50.0	1.484E-01	7.400E-01
52.5	9.803E-02	4.887E-01
55.0	1.027E-02	5.119E-02
57.5	-4.316E-02	-2.152E-01
60.0	-7.335E-02	-3.657E-01
62.5	-8.883E-02	-4.428E-01
65.0	-8.360E-02	-4.168E-01
67.5	-5.580E-02	-2.782E-01
70.0	2.210E-03	1.102E-02
72.5	8.726E-02	4.350E-01
75.0	1.364E-01	6.799E-01
77.5	9.700E-02	4.836E-01
80.0	1.912E-02	9.531E-02
82.5	-3.716E-02	-1.852E-01
85.0	-7.125E-02	-3.552E-01
87.5	-8.935E-02	-4.454E-01
90.0	-8.556E-02	-4.265E-01
92.5	-5.591E-02	-2.787E-01
95.0	3.965E-03	1.977E-02
97.5	8.289E-02	4.132E-01
100.0	1.269E-01	6.327E-01
102.5	9.587E-02	4.779E-01
105.0	2.650E-02	1.321E-01
107.5	-3.094E-02	-1.542E-01
110.0	-6.891E-02	-3.435E-01
112.5	-8.999E-02	-4.486E-01
115.0	-8.746E-02	-4.360E-01
117.5	-5.610E-02	-2.797E-01
120.0	4.688E-03	2.337E-02
122.5	7.863E-02	3.920E-01
125.0	1.193E-01	5.946E-01
127.5	9.485E-02	4.728E-01
130.0	3.282E-02	1.636E-01
132.5	-2.473E-02	-1.233E-01
135.0	-6.633E-02	-3.307E-01
137.5	-9.073E-02	-4.523E-01

PROCESS = 242.56667 SECS.
I/O = 5.93333 SECS.
RUN TIME = 248.50000 SECS.

Table 29. Autocovariance and Autocorrelation Function for U ≈ EXP (λ) and V ≈ N(μ,σ^2) and for Indicated Parameters.

PARAMETERS: MU = 3 SIGMA = 0.1 LAMBDA = 0.1250

TAU	R(TAU)	RHO(TAU)
0.0	9.487E-02	1.000E+00
0.2	6.791E-02	7.173E-01
0.4	5.235E-02	5.530E-01
0.6	3.306E-02	3.492E-01
0.8	1.460E-02	1.543E-01
1.0	-5.855E-04	-6.184E-03
1.2	-2.059E-03	-2.175E-02
1.4	-6.652E-04	-6.914E-03
1.6	6.775E-04	7.156E-03
1.8	1.056E-03	1.115E-02
2.0	7.839E-04	8.281E-03
2.2	4.824E-04	5.096E-03
2.4	-1.155E-04	-1.220E-03
2.6	-2.997E-04	-3.166E-03
2.8	-6.423E-04	-6.785E-03
3.0	-6.615E-04	-6.985E-03
3.2	-6.672E-04	-7.047E-03
3.4	-5.756E-04	-6.091E-03
3.6	-3.732E-04	-3.934E-03
3.8	-1.902E-04	-2.009E-03
4.0	1.386E-04	1.464E-03
4.2	2.457E-04	2.542E-03
4.4	4.526E-04	4.781E-03
4.6	4.754E-04	5.021E-03
4.8	4.709E-04	4.975E-03
5.0	4.126E-04	4.358E-03
5.2	2.244E-04	2.371E-03
5.4	1.260E-04	1.331E-03
5.6	-1.164E-04	-1.230E-03
5.8	-1.967E-04	-2.077E-03
6.0	-3.498E-04	-3.695E-03
6.2	-3.692E-04	-3.900E-03
6.4	-3.559E-04	-3.759E-03
6.6	-3.090E-04	-3.264E-03
6.8	-1.541E-04	-1.625E-03
7.0	-7.290E-05	-7.690E-04
7.2	1.229E-04	1.277E-03
7.4	1.878E-04	1.984E-03
7.6	3.051E-04	3.223E-03
7.8	3.202E-04	3.382E-03
8.0	2.999E-04	3.167E-03
8.2	2.577E-04	2.722E-03
8.4	1.226E-04	1.295E-03
8.6	5.092E-05	5.379E-04
8.8	-1.117E-04	-1.180E-03
9.0	-1.697E-04	-1.793E-03
9.2	-2.632E-04	-2.780E-03
9.4	-2.745E-04	-2.899E-03
9.6	-2.489E-04	-2.629E-03
9.8	-2.095E-04	-2.211E-03
10.0	-8.752E-05	-9.245E-04

PROCESS = 197.25000 SECS.
I/O = 10.65000 SECS.
RUN TIME = 207.90000 SECS.

PARAMETERS: MU = 3 SIGMA = 0.1 LAMBDA = 1.0000

TAU	R(TAU)	RHO(TAU)
0.0	2.440E-01	1.000E+00
0.2	1.420E-01	5.805E-01
0.4	7.860E-02	3.196E-01
0.6	2.203E-02	8.955E-02
0.8	-2.474E-02	-1.006E-01
1.0	-4.684E-02	-1.904E-01
1.2	-1.830E-02	-7.442E-02
1.4	7.315E-03	2.974E-02
1.6	1.619E-02	6.584E-02
1.8	1.314E-02	5.341E-02
2.0	4.034E-03	1.640E-02
2.2	-3.264E-03	-1.327E-02
2.4	-4.711E-03	-1.915E-02
2.6	-2.907E-03	-1.182E-02
2.8	-7.453E-04	-3.030E-03
3.0	2.094E-04	8.471E-04
3.2	-1.513E-04	-6.152E-04
3.4	-4.544E-04	-1.873E-03
3.6	-1.139E-03	-4.632E-03
3.8	-7.578E-04	-3.081E-03
4.0	-7.876E-05	-3.202E-04
4.2	6.088E-04	2.474E-03
4.4	9.706E-04	3.946E-03
4.6	1.076E-03	4.373E-03
4.8	9.314E-04	3.795E-03
5.0	7.212E-04	2.932E-03
5.2	4.332E-04	1.761E-03
5.4	1.440E-04	5.856E-04
5.6	-1.618E-04	-6.577E-04
5.8	-4.343E-04	-1.766E-03
6.0	-6.403E-04	-2.603E-03
6.2	-7.418E-04	-3.094E-03
6.4	-7.140E-04	-2.903E-03
6.6	-5.874E-04	-2.388E-03
6.8	-3.538E-04	-1.439E-03
7.0	-9.933E-05	-4.038E-04
7.2	1.833E-04	7.451E-04
7.4	3.988E-04	1.621E-03
7.6	5.675E-04	2.307E-03
7.8	6.109E-04	2.520E-03
8.0	5.955E-04	2.421E-03
8.2	4.621E-04	1.879E-03
8.4	2.775E-04	1.129E-03
8.6	4.652E-05	1.891E-04
8.8	-1.702E-04	-7.285E-04
9.0	-3.710E-04	-1.508E-03
9.2	-5.027E-04	-2.044E-03
9.4	-5.430E-04	-2.208E-03
9.6	-5.091E-04	-2.070E-03
9.8	-3.838E-04	-1.560E-03
10.0	-2.173E-04	-8.835E-04
10.2	-1.874E-05	-7.620E-05
10.4	3.720E-04	1.512E-03
10.6	7.043E-04	2.868E-03
10.8	9.042E-04	3.686E-03
11.0	9.706E-04	3.965E-03

PROCESS = 751.80000 SECS.
I/O = 16.01667 SECS.
RUN TIME = 767.81667 SECS.

PARAMETERS: MU = 4 SIGMA = 0.1 LAMBDA = 0.0625

TAU	R(TAU)	RHO(TAU)
0.0	1.555E-01	1.000E+00
0.5	1.343E-01	8.640E-01
1.0	1.129E-01	7.230E-01
1.5	8.841E-02	5.712E-01
2.0	6.630E-02	4.264E-01
2.5	4.446E-02	2.859E-01
3.0	2.270E-02	1.460E-01
3.5	2.814E-03	1.810E-02
4.0	-1.428E-02	-9.182E-02
4.5	-1.172E-02	-7.535E-02
5.0	-7.686E-03	-4.943E-02
5.5	-4.365E-03	-2.808E-02
6.0	-2.086E-03	-1.342E-02
6.5	-3.015E-04	-1.939E-03
7.0	8.635E-04	5.554E-03
7.5	1.312E-03	8.440E-03
8.0	1.132E-03	7.282E-03
8.5	5.597E-04	3.600E-03
9.0	1.389E-04	8.927E-04
9.5	-6.082E-05	-3.912E-04
10.0	-1.583E-04	-1.018E-03
10.5	2.746E-05	1.766E-04
11.0	1.368E-04	8.799E-04
11.5	4.431E-05	2.850E-04
12.0	-6.439E-05	-4.142E-04
12.5	-1.447E-04	-9.308E-04
13.0	-8.471E-05	-5.449E-04
13.5	9.333E-05	6.003E-04
14.0	1.736E-04	1.116E-03
14.5	1.431E-04	9.206E-04
15.0	1.456E-05	9.364E-05
15.5	-1.234E-04	-7.939E-04
16.0	-8.007E-05	-5.451E-04
16.5	2.514E-05	1.617E-04
17.0	1.214E-04	7.811E-04
17.5	1.436E-04	9.233E-04
18.0	3.138E-05	2.019E-04
18.5	-8.043E-05	-5.173E-04
19.0	-9.047E-05	-5.818E-04
19.5	-2.421E-05	-1.557E-04
20.0	9.906E-05	6.371E-04

PROCESS = 106.98333 SECS.
I/O = 9.45000 SECS.
RUN TIME = 116.43333 SECS.

PARAMETERS: MU = 3 SIGMA = 0.1 LAMBDA = 0.1111

TAU	R(TAU)	RHO(TAU)
0.0	1.834E-01	1.000E+00
0.2	1.814E-01	9.891E-01
0.4	1.547E-01	8.433E-01
0.6	1.397E-01	7.616E-01
0.8	1.234E-01	6.731E-01
1.0	1.089E-01	5.937E-01
1.2	9.394E-02	5.122E-01
1.4	7.952E-02	4.336E-01
1.6	6.552E-02	3.573E-01
1.8	5.144E-02	2.805E-01
2.0	3.521E-02	2.043E-01
2.2	2.480E-02	1.353E-01
2.4	1.193E-02	6.505E-02
2.6	-5.319E-04	-2.901E-03
2.8	-1.317E-02	-7.180E-02
3.0	-2.154E-02	-1.175E-01
3.2	-2.092E-02	-1.141E-01
3.4	-1.646E-02	-8.974E-02
3.6	-1.323E-02	-7.212E-02
3.8	-9.821E-03	-5.355E-02
4.0	-7.147E-03	-3.897E-02
4.2	-4.764E-03	-2.598E-02
4.4	-2.624E-03	-1.431E-02
4.6	-1.155E-03	-6.296E-03
4.8	3.410E-04	1.859E-03
5.0	1.166E-03	6.360E-03
5.2	1.892E-03	1.032E-02
5.4	2.247E-03	1.225E-02
5.6	2.261E-03	1.233E-02
5.8	2.173E-03	1.185E-02
6.0	1.609E-03	8.772E-03
6.2	1.234E-03	6.727E-03
6.4	7.229E-04	3.942E-03
6.6	3.874E-04	2.112E-03
6.8	1.709E-04	9.317E-04
7.0	-8.872E-05	-4.838E-04
7.2	-1.117E-04	-6.090E-04
7.4	-2.899E-04	-1.581E-03
7.6	-2.423E-04	-1.321E-03
7.8	-2.745E-04	-1.497E-03
8.0	-2.611E-04	-1.424E-03
8.2	-1.831E-04	-9.986E-04
8.4	-2.154E-04	-1.174E-03
8.6	-8.324E-05	-4.539E-04
8.8	-1.094E-04	-5.963E-04
9.0	-2.330E-05	-1.270E-04
9.2	9.363E-06	5.105E-05

PARAMETERS: MU = 2 SIGMA = 0.1 LAMBDA = 0.2500

TAU	R(TAU)	RHO(TAU)
0.0	2.181E-01	1.000E+00
0.4	1.513E-01	6.991E-01
0.8	1.017E-01	4.640E-01
1.2	5.235E-02	2.400E-01
1.6	5.262E-03	2.412E-02
2.0	-3.207E-02	-1.470E-01
2.4	-1.872E-02	-8.579E-02
2.8	-5.959E-03	-2.708E-02
3.2	1.481E-03	6.788E-03
3.6	4.333E-03	1.986E-02
4.0	3.165E-03	1.451E-02
4.4	1.075E-03	4.928E-03
4.8	1.868E-04	8.562E-04
5.2	-2.959E-04	-1.356E-03
5.6	-5.850E-04	-2.682E-03
6.0	-6.739E-04	-3.089E-03
6.4	-6.225E-04	-2.854E-03
6.8	-3.002E-04	-1.376E-03
7.2	2.451E-04	1.124E-03
7.6	6.365E-04	2.918E-03
8.0	6.324E-04	2.899E-03
8.4	3.001E-04	1.376E-03
8.8	-1.922E-04	-8.812E-04
9.2	-5.858E-04	-2.685E-03
9.6	-5.842E-04	-2.678E-03
10.0	-4.869E-04	-2.232E-03
10.4	4.292E-04	1.968E-03
10.8	1.048E-03	4.805E-03
11.2	1.033E-03	4.736E-03
11.6	3.957E-04	1.814E-03
12.0	-4.489E-04	-2.058E-03
12.4	-9.617E-04	-4.409E-03
12.8	-9.747E-04	-4.328E-03
13.2	-3.005E-04	-1.377E-03
13.6	4.473E-04	2.050E-03
14.0	8.998E-04	4.125E-03
14.4	7.820E-04	3.585E-03
14.8	2.188E-04	1.003E-03

PROCESS = 200.20000 SECS.
I/O = 7.38333 SECS.
RUN TIME = 207.58333 SECS.

Table 30. Autocovariance and Autocorrelation Function for $U \approx EXP(\lambda)$ and $V \approx N(\mu,\sigma^2)$ and for Indicated Parameters.

PARAMETERS: MU = 2 SIGMA = 1.0 LAMBDA = 1.0000

TAU	R(TAU)	RHO(TAU)
0.0	2.214E-01	1.000E+00
0.3	1.284E-01	5.798E-01
0.6	7.588E-02	3.427E-01
0.9	3.664E-02	1.655E-01
1.2	1.024E-02	4.623E-02
1.5	-6.105E-03	-2.757E-02
1.8	-1.012E-02	-4.575E-02
2.1	-1.619E-02	-7.312E-02
2.4	-1.402E-02	-6.332E-02
2.7	-9.756E-03	-4.406E-02
3.0	-4.845E-03	-2.188E-02
3.3	-6.478E-04	-2.926E-03
3.6	2.293E-03	1.035E-02
3.9	3.738E-03	1.688E-02
4.2	3.897E-03	1.760E-02
4.5	3.161E-03	1.427E-02
4.8	1.936E-03	8.742E-03
5.1	6.593E-04	2.978E-03
5.4	-4.684E-04	-2.115E-03
5.7	-1.181E-03	-5.332E-03
6.0	-1.489E-03	-6.726E-03
6.3	-1.325E-03	-5.985E-03
6.6	-8.911E-04	-4.024E-03
6.9	-2.556E-04	-1.155E-03
7.2	3.151E-04	1.423E-03
7.5	7.378E-04	3.332E-03
7.8	8.613E-04	3.890E-03
8.1	7.240E-04	3.270E-03
8.4	3.647E-04	1.646E-03
8.7	-6.912E-05	-3.122E-04
9.0	-4.373E-04	-1.975E-03
9.3	-6.319E-04	-2.857E-03
9.6	-5.943E-04	-2.684E-03
9.9	-3.747E-04	-1.690E-03
10.2	-3.349E-05	-1.513E-04
10.5	6.040E-04	2.728E-03
10.8	1.027E-03	4.638E-03
11.1	1.053E-03	4.757E-03
11.4	7.397E-04	3.340E-03
11.7	1.742E-04	7.868E-04
12.0	-4.144E-04	-1.872E-03
12.3	-8.461E-04	-3.821E-03
12.6	-9.582E-04	-4.327E-03
12.9	-7.413E-04	-3.348E-03
13.2	-2.842E-04	-1.283E-03
13.5	2.660E-04	1.201E-03
13.8	6.940E-04	3.134E-03
14.1	8.797E-04	3.973E-03
14.4	7.448E-04	3.364E-03
14.7	3.530E-04	1.594E-03
15.0	-1.292E-04	-5.837E-04
15.3	-5.548E-04	-2.505E-03
15.6	-7.848E-04	-3.543E-03
15.9	-7.361E-04	-3.325E-03
16.2	-4.371E-04	-1.974E-03

PROCESS = 413.86667 SECS.
I/O = 8.71667 SECS.
RUN TIME = 422.58333 SECS.

PARAMETERS: MU = 2 SIGMA = 1.0 LAMBDA = 2.0000

TAU	R(TAU)	RHO(TAU)
0.0	1.584E-01	1.000E+00
0.3	2.813E-02	1.817E-01
0.5	3.420E-02	2.199E-01
0.8	1.024E-02	6.568E-02
1.0	-2.675E-03	-1.449E-02
1.3	-8.754E-03	-5.615E-02
1.5	-1.041E-02	-6.697E-02
1.8	-9.800E-03	-6.284E-02
2.0	-7.629E-03	-4.817E-02
2.3	-4.869E-03	-3.125E-02
2.5	-2.257E-03	-1.425E-02
2.8	-1.411E-04	-9.058E-04
3.0	1.318E-03	8.460E-03
3.3	2.017E-03	1.273E-02
3.5	2.156E-03	1.382E-02
3.8	1.813E-03	1.157E-02
4.0	1.394E-03	8.159E-03
4.3	5.161E-04	3.313E-03
4.5	3.445E-04	2.175E-03
4.8	1.899E-05	1.199E-04
5.0	-2.455E-04	-1.550E-03
5.3	-3.892E-04	-2.419E-03
5.5	-5.044E-04	-3.107E-03
5.8	-5.314E-04	-3.388E-03
6.0	-5.301E-04	-3.404E-03
6.3	-4.473E-04	-2.824E-03
6.5	-3.159E-04	-1.995E-03
6.8	-1.107E-04	-6.987E-04
7.0	6.851E-05	4.220E-04
7.3	2.980E-04	1.886E-03
7.5	4.391E-04	2.766E-03
7.8	4.802E-04	3.032E-03
8.0	4.312E-04	2.735E-03
8.3	2.745E-04	1.719E-03
8.5	8.283E-05	5.210E-04
8.8	-1.481E-04	-9.350E-04
9.0	-3.109E-04	-1.962E-03
9.3	-4.281E-04	-2.733E-03
9.5	-4.157E-04	-2.625E-03
9.8	-3.301E-04	-2.084E-03
10.0	-1.247E-04	-7.940E-04
10.3	7.310E-05	4.621E-04
10.5	4.506E-04	2.845E-03
10.8	7.057E-04	4.356E-03
11.0	7.805E-04	4.928E-03
11.3	6.726E-04	4.247E-03
11.5	3.948E-04	2.493E-03
11.8	4.944E-05	3.137E-04
12.0	-3.101E-04	-1.958E-03
12.3	-5.680E-04	-3.556E-03
12.5	-6.974E-04	-4.453E-03
12.8	-6.367E-04	-4.020E-03
13.0	-4.426E-04	-2.794E-03
13.3	-1.324E-04	-8.359E-04
13.5	1.889E-04	1.193E-03
13.8	4.657E-04	2.940E-03

PROCESS = 466.76667 SECS.
I/O = 8.35000 SECS.
RUN TIME = 475.11667 SECS.

PARAMETERS: MU = 4 SIGMA = 1.0 LAMBDA = 1.0000

TAU	R(TAU)	RHO(TAU)
0.0	1.549E-01	1.000E+00
0.5	7.896E-02	5.044E-01
1.0	3.455E-02	2.216E-01
1.5	4.219E-03	2.949E-02
2.0	-1.134E-02	-7.307E-02
2.5	-2.092E-02	-1.342E-01
3.0	-2.311E-02	-1.442E-01
3.5	-1.855E-02	-1.190E-01
4.0	-8.716E-03	-5.590E-02
4.5	2.334E-03	1.497E-02
5.0	1.043E-02	6.692E-02
5.5	1.262E-02	8.222E-02
6.0	1.004E-02	6.441E-02
6.5	4.527E-03	2.901E-02
7.0	-9.495E-04	-6.390E-03
7.5	-4.668E-03	-2.940E-02
8.0	-5.904E-03	-3.845E-02
8.5	-5.144E-03	-3.393E-02
9.0	-2.940E-03	-1.898E-02
9.5	-1.693E-04	-1.066E-03
10.0	2.022E-03	1.297E-02
10.5	3.466E-03	2.223E-02
11.0	3.296E-03	2.114E-02
11.5	1.836E-03	1.178E-02
12.0	-1.142E-04	-7.450E-04
12.5	-1.305E-03	-8.652E-03
13.0	-1.848E-03	-1.198E-02
13.5	-1.314E-03	-8.426E-03
14.0	-4.448E-04	-2.853E-03
14.5	1.818E-04	1.166E-03

PARAMETERS: MU = 4 SIGMA = 1.0 LAMBDA = 2.0000

TAU	R(TAU)	RHO(TAU)
0.0	9.474E-02	1.000E+00
0.4	1.049E-02	1.264E-01
0.9	6.448E-03	6.848E-02
1.3	-3.842E-03	-4.098E-02
1.8	-8.146E-03	-8.599E-02
2.2	-9.673E-03	-1.047E-01
2.7	-9.335E-03	-9.854E-02
3.1	-6.874E-03	-7.256E-02
3.6	-2.538E-03	-2.679E-02
4.0	2.208E-03	2.330E-02
4.5	5.808E-03	6.130E-02
4.9	6.753E-03	7.128E-02
5.4	5.071E-03	5.356E-02
5.8	1.845E-03	1.948E-02
6.3	-1.302E-03	-1.374E-02
6.7	-3.234E-03	-3.414E-02
7.2	-3.631E-03	-3.833E-02
7.6	-2.768E-03	-2.922E-02
8.1	-1.309E-03	-1.382E-02
8.5	2.798E-05	2.953E-04
9.0	1.326E-03	1.400E-02
9.4	1.871E-03	1.975E-02
9.9	1.747E-03	1.844E-02
10.3	1.225E-03	1.293E-02
10.8	4.593E-04	4.848E-03
11.2	-3.851E-04	-4.065E-03
11.7	-1.086E-03	-1.146E-02
12.1	-1.319E-03	-1.414E-02
12.6	-1.064E-03	-1.123E-02
13.0	-3.861E-04	-4.076E-03
13.5	3.844E-04	4.058E-03
13.9	8.870E-04	9.363E-03
14.4	9.150E-04	9.658E-03
14.8	5.324E-04	5.620E-03
15.3	3.110E-05	3.282E-04
15.7	-3.636E-04	-3.838E-03
16.2	-4.472E-04	-4.721E-03
16.6	-3.112E-04	-3.284E-03
17.1	-8.660E-05	-9.141E-04

PARAMETERS: MU = 8 SIGMA = 1.0 LAMBDA = 1.0000

TAU	R(TAU)	RHO(TAU)
0.0	8.578E-02	1.000E+00
1.0	2.726E-02	3.178E-01
2.0	4.035E-03	4.704E-02
3.0	-5.475E-03	-6.383E-02
4.0	-8.057E-03	-9.392E-02
5.0	-9.136E-03	-1.065E-01
6.0	-9.876E-03	-1.152E-01
7.0	-9.500E-03	-1.108E-01
8.0	-5.446E-03	-6.035E-02
9.0	3.619E-03	4.219E-02
10.0	1.409E-02	1.643E-01
11.0	1.446E-02	1.686E-01
12.0	4.389E-03	5.448E-02
13.0	-1.106E-03	-1.289E-02
14.0	-4.321E-03	-6.203E-02
15.0	-7.840E-03	-9.140E-02
16.0	-8.519E-03	-9.931E-02
17.0	-6.455E-03	-7.525E-02
18.0	-2.102E-03	-2.451E-02
19.0	3.898E-03	4.544E-02
20.0	8.513E-03	9.924E-02
21.0	8.504E-03	9.914E-02
22.0	4.629E-03	5.396E-02
23.0	1.847E-04	2.154E-03
24.0	-3.665E-03	-4.272E-02
25.0	-6.102E-03	-7.114E-02
26.0	-6.176E-03	-7.199E-02
27.0	-4.103E-03	-4.783E-02
28.0	-8.869E-04	-1.034E-02
29.0	3.046E-03	3.551E-02
30.0	4.643E-03	6.378E-02
31.0	5.380E-03	6.272E-02
32.0	3.264E-03	3.805E-02
33.0	3.095E-04	3.608E-03
34.0	-2.690E-03	-3.136E-02
35.0	-4.386E-03	-5.113E-02
36.0	-4.224E-03	-4.924E-02
37.0	-2.843E-03	-3.314E-02
38.0	-3.797E-04	-4.426E-03
39.0	2.327E-03	2.713E-02
40.0	3.749E-03	4.370E-02
41.0	3.566E-03	4.140E-02
42.0	2.302E-03	2.683E-02
43.0	1.425E-04	1.661E-03
44.0	-1.993E-03	-2.323E-02
45.0	-2.976E-03	-3.469E-02
46.0	-2.926E-03	-3.411E-02
47.0	-1.944E-03	-2.266E-02
48.0	-7.268E-05	-8.473E-04
49.0	1.685E-03	1.964E-02
50.0	2.472E-03	2.882E-02
51.0	2.442E-03	2.847E-02
52.0	1.575E-03	1.836E-02
53.0	-4.161E-05	-4.851E-04
54.0	-1.343E-03	-1.566E-02
55.0	-2.015E-03	-2.349E-02

PROCESS = 359.88333 SECS.
I/O = 14.28333 SECS.
RUN TIME = 374.16667 SECS.

Table 31. Autocovariance and Autocorrelation Function for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$ and for Indicated Parameters.

Table 31

PARAMETERS: MU[0] = 4 SIGMA[0] = 3.0
MU[1] = 1 SIGMA[1] = 0.5

TAU	R(TAU)	RHO(TAU)
0.0	1.724E-01	1.000E+00
0.5	7.967E-02	4.621E-01
1.0	1.782E-02	1.034E-01
1.5	-1.107E-02	-6.418E-02
2.0	-1.720E-02	-9.977E-02
2.5	-1.689E-02	-9.798E-02
3.0	-1.453E-02	-8.428E-02
3.5	-1.104E-02	-6.401E-02
4.0	-6.924E-03	-4.016E-02
4.5	-3.540E-03	-2.054E-02
5.0	-1.293E-03	-7.499E-03
5.5	-1.987E-04	-1.153E-03
6.0	5.115E-04	2.967E-03
6.5	1.257E-03	7.293E-03
7.0	2.087E-03	1.211E-02
7.5	2.570E-03	1.491E-02
8.0	2.337E-03	1.355E-02
8.5	1.500E-03	8.704E-03
9.0	4.898E-04	2.841E-03
9.5	-1.029E-04	-5.970E-04
10.0	-4.066E-04	-2.359E-03
10.5	2.577E-04	1.495E-03
11.0	4.816E-04	2.794E-03
11.5	2.319E-05	1.345E-04
12.0	-7.735E-04	-4.487E-03
12.5	-1.119E-03	-6.493E-03
13.0	-7.353E-04	-4.265E-03
13.5	9.467E-05	5.491E-04
14.0	6.724E-04	3.900E-03
14.5	5.611E-04	3.255E-03
15.0	-2.045E-05	-1.186E-04
15.5	-5.213E-04	-3.024E-03
16.0	-4.502E-04	-2.611E-03
16.5	7.969E-05	4.622E-04
17.0	5.950E-04	3.451E-03
17.5	5.823E-04	3.378E-03
18.0	1.014E-04	5.883E-04
18.5	-4.495E-04	-2.607E-03
19.0	-5.551E-04	-3.220E-03
19.5	-1.730E-04	-1.004E-03
20.0	3.535E-04	2.050E-03
20.5	5.156E-04	2.991E-03
21.0	2.026E-04	1.175E-03
21.5	-2.940E-04	-1.705E-03
22.0	-5.191E-04	-3.011E-03
22.5	-2.610E-04	-1.514E-03
23.0	2.154E-04	1.249E-03
23.5	4.917E-04	2.852E-03
24.0	3.103E-04	1.800E-03
24.5	-1.309E-04	-7.593E-04
25.0	-4.506E-04	-2.614E-03
25.5	-3.343E-04	-1.939E-03
26.0	7.220E-05	4.188E-04
26.5	4.149E-04	2.406E-03
27.0	3.644E-04	2.114E-03
27.5	-3.793E-06	-2.200E-05

PROCESS = 493.38333 SECS.
I/O = 9.91667 SECS.
RUN TIME = 503.30000 SECS.

PARAMETERS: MU[0] = 10 SIGMA[0] = 1.0
MU[1] = 10 SIGMA[1] = 5.0

TAU	R(TAU)	RHO(TAU)
0.0	2.497E-01	1.000E+00
2.0	1.567E-01	6.274E-01
4.0	6.422E-02	2.572E-01
6.0	-2.050E-02	-8.209E-02
8.0	-9.328E-02	-3.735E-01
10.0	-1.350E-01	-5.404E-01
12.0	-1.040E-01	-4.166E-01
14.0	-4.823E-02	-1.931E-01
16.0	4.546E-03	1.820E-02
18.0	4.451E-02	1.782E-01
20.0	6.401E-02	2.563E-01
22.0	6.084E-02	2.436E-01
24.0	4.009E-02	1.605E-01
26.0	1.053E-02	4.216E-02
28.0	-1.699E-02	-6.805E-02
30.0	-3.386E-02	-1.356E-01
32.0	-3.709E-02	-1.485E-01
34.0	-2.790E-02	-1.117E-01
36.0	-1.168E-02	-4.676E-02
38.0	4.873E-03	1.951E-02
40.0	1.673E-02	6.700E-02
42.0	2.107E-02	8.437E-02
44.0	1.786E-02	7.153E-02
46.0	9.941E-03	3.981E-02
48.0	3.679E-04	1.473E-03
50.0	-7.478E-03	-2.994E-02
52.0	-1.135E-02	-4.546E-02
54.0	-1.109E-02	-4.439E-02
56.0	-7.367E-03	-2.950E-02
58.0	-2.009E-03	-8.043E-03
60.0	2.801E-03	1.122E-02
62.0	5.906E-03	2.365E-02
64.0	6.577E-03	2.634E-02
66.0	4.959E-03	1.986E-02
68.0	2.266E-03	9.073E-03
70.0	-7.018E-04	-2.810E-03
72.0	-2.881E-03	-1.154E-02
74.0	-3.649E-03	-1.461E-02
76.0	-3.226E-03	-1.292E-02
78.0	-1.860E-03	-7.450E-03
80.0	-1.049E-04	-4.201E-04
82.0	1.215E-03	4.864E-03
84.0	1.997E-03	7.998E-03
86.0	2.004E-03	8.023E-03
88.0	1.297E-03	5.194E-03
90.0	4.461E-04	1.786E-03
92.0	-4.501E-04	-1.802E-03
94.0	-1.049E-03	-4.200E-03
96.0	-1.117E-03	-4.474E-03
98.0	-9.097E-04	-3.643E-03
100.0	-4.428E-04	-1.773E-03
102.0	1.463E-04	5.857E-04
104.0	4.555E-04	1.824E-03
106.0	6.571E-04	2.631E-03
108.0	6.083E-04	2.436E-03
110.0	3.068E-04	1.228E-03

PROCESS = 376.70000 SECS.
I/O = 10.16667 SECS.
RUN TIME = 386.86667 SECS.

Table 32. Autocovariance and Autocorrelation Function for $U \simeq N(\mu_0, \sigma_0^2)$ and $V \simeq N(\mu_1, \sigma_1^2)$ and for Indicated Parameters.

PARAMETERS: MU[0] = 10 SIGMA[0] = 5.0
MU[1] = 10 SIGMA[1] = 5.0

TAU	R(TAU)	RHO(TAU)
0.0	2.531E-01	1.000E+00
2.0	1.619E-01	6.398E-01
4.0	7.590E-02	2.999E-01
6.0	4.316E-03	1.705E-02
8.0	-4.625E-02	-1.827E-01
10.0	-7.302E-02	-2.885E-01
12.0	-7.511E-02	-2.968E-01
14.0	-5.807E-02	-2.294E-01
16.0	-3.112E-02	-1.230E-01
18.0	-3.177E-03	-1.255E-02
20.0	1.813E-02	7.164E-02
22.0	2.887E-02	1.140E-01
24.0	2.969E-02	1.173E-01
26.0	2.267E-02	8.956E-02
28.0	1.179E-02	4.657E-02
30.0	1.102E-03	4.355E-03
32.0	-7.207E-03	-2.847E-02
34.0	-1.136E-02	-4.488E-02
36.0	-1.148E-02	-4.534E-02
38.0	-8.820E-03	-3.485E-02
40.0	-4.554E-03	-1.799E-02
42.0	-2.633E-04	-1.040E-03
44.0	2.781E-03	1.099E-02
46.0	4.500E-03	1.778E-02
48.0	4.510E-03	1.782E-02
50.0	3.358E-03	1.327E-02
52.0	1.806E-03	7.134E-03
54.0	7.626E-05	3.013E-04
56.0	-1.152E-03	-4.550E-03
58.0	-1.686E-03	-6.660E-03
60.0	-1.790E-03	-7.073E-03
62.0	-1.313E-03	-5.187E-03
64.0	-6.232E-04	-2.462E-03
66.0	-7.973E-05	-3.150E-04
68.0	4.922E-04	1.944E-03
70.0	6.923E-04	2.735E-03
72.0	6.515E-04	2.574E-03
74.0	5.554E-04	2.194E-03
76.0	2.481E-04	9.804E-04
78.0	-1.797E-05	-7.098E-05
80.0	-1.268E-04	-5.011E-04
82.0	-3.058E-04	-1.208E-03
84.0	-2.612E-04	-1.032E-03
86.0	-1.631E-04	-6.446E-04
88.0	-1.439E-04	-5.686E-04
90.0	3.115E-05	1.231E-04
92.0	7.928E-05	3.132E-04
94.0	7.348E-05	2.903E-04
96.0	1.433E-04	5.660E-04
98.0	7.078E-05	2.796E-04
100.0	1.554E-05	6.141E-05
102.0	5.017E-05	1.982E-04
104.0	-6.074E-05	-2.400E-04
106.0	-3.194E-05	-1.262E-04
108.0	-7.263E-06	-2.870E-05
110.0	-6.550E-05	-2.588E-04

PROCESS = 323.08333 SECS.
I/O = 19.20000 SECS.
RUN TIME = 342.28333 SECS.

PARAMETERS: MU[0] = 2 SIGMA[0] = 0.4
MU[1] = 10 SIGMA[1] = 3.0

TAU	R(TAU)	RHO(TAU)
0.0	1.349E-01	1.000E+00
1.2	4.035E-02	2.992E-01
2.4	-2.464E-02	-1.827E-01
3.6	-2.708E-02	-2.008E-01
4.8	-2.445E-02	-1.813E-01
6.0	-2.165E-02	-1.605E-01
7.2	-1.499E-02	-1.112E-01
8.4	-6.299E-03	-4.670E-02
9.6	3.226E-03	2.392E-02
10.8	1.197E-02	8.877E-02
12.0	1.512E-02	1.121E-01
13.2	1.322E-02	9.802E-02
14.4	7.319E-03	5.427E-02
15.6	-9.408E-04	-6.975E-03
16.8	-6.360E-03	-4.715E-02
18.0	-8.973E-03	-6.653E-02
19.2	-8.303E-03	-6.156E-02
20.4	-4.542E-03	-3.367E-02
21.6	-9.666E-04	-7.167E-03
22.8	2.707E-03	2.007E-02
24.0	4.795E-03	3.555E-02
25.2	4.617E-03	3.423E-02
26.4	3.654E-03	2.709E-02
27.6	1.263E-03	9.361E-03
28.8	-9.874E-04	-7.321E-03
30.0	-2.177E-03	-1.614E-02
31.2	-3.022E-03	-2.240E-02
32.4	-2.246E-03	-1.665E-02
33.6	-1.101E-03	-8.162E-03
34.8	-3.567E-05	-2.645E-04
36.0	1.307E-03	9.691E-03
37.2	1.489E-03	1.104E-02
38.4	1.444E-03	1.070E-02
39.6	1.037E-03	7.686E-03
40.8	1.670E-05	1.238E-04
42.0	-3.680E-04	-2.728E-03
43.2	-8.407E-04	-6.233E-03
44.4	-9.959E-04	-7.384E-03
45.6	-4.762E-04	-3.531E-03
46.8	-3.630E-04	-2.692E-03
48.0	1.880E-04	1.394E-03
49.2	5.066E-04	3.756E-03
50.4	3.785E-04	2.806E-03
51.6	5.551E-04	4.116E-03
52.8	1.656E-04	1.228E-03
54.0	-6.942E-05	-5.147E-04
55.2	-7.630E-05	-5.657E-04
56.4	-4.128E-04	-3.060E-03
57.6	-2.007E-04	-1.488E-03
58.8	-1.316E-04	-9.755E-04
60.0	-1.448E-04	-1.074E-03
61.2	2.226E-04	1.651E-03
62.4	6.690E-05	4.960E-04
63.6	1.605E-04	1.190E-03
64.8	2.049E-04	1.519E-03
66.0	-7.773E-05	-5.763E-04

PROCESS = 427.83333 SECS.
I/O = 15.23333 SECS.
RUN TIME = 443.06667 SECS.

Table 33. Autocovariance and Autocorrelation Function for $U \approx N(\mu_0,\sigma_0^2)$ and $V \approx N(\mu_1,\sigma_1^2)$ and for Indicated Parameters.

PARAMETERS: MU[0] = 3 SIGMA[0] = 0.6
MU[1] = 10 SIGMA[1] = 3.0

TAU	R(TAU)	RHO(TAU)
0.0	1.735E-01	1.000E+00
1.3	7.800E-02	4.495E-01
2.6	-1.509E-02	-8.695E-02
3.9	-5.037E-02	-2.903E-01
5.2	-4.855E-02	-2.798E-01
6.5	-4.230E-02	-2.438E-01
7.8	-3.024E-02	-1.743E-01
9.1	-1.322E-02	-7.620E-02
10.4	6.915E-03	3.985E-02
11.7	2.369E-02	1.365E-01
13.0	3.123E-02	1.800E-01
14.3	2.710E-02	1.562E-01
15.6	1.344E-02	7.748E-02
16.9	-2.421E-03	-1.395E-02
18.2	-1.469E-02	-8.465E-02
19.5	-1.944E-02	-1.121E-01
20.8	-1.683E-02	-9.701E-02
22.1	-9.574E-03	-5.518E-02
23.4	-5.131E-04	-2.957E-03
24.7	6.875E-03	3.963E-02
26.0	1.094E-02	6.307E-02
27.3	1.087E-02	6.266E-02
28.6	7.212E-03	4.157E-02
29.9	2.036E-03	1.174E-02
31.2	-3.156E-03	-1.819E-02
32.5	-6.225E-03	-3.588E-02
33.8	-6.861E-03	-3.954E-02
35.1	-5.147E-03	-2.967E-02
36.4	-1.977E-03	-1.140E-02
37.7	1.105E-03	6.369E-03
39.0	3.474E-03	2.002E-02
40.3	4.152E-03	2.393E-02
41.6	3.471E-03	2.000E-02
42.9	1.784E-03	1.028E-02
44.2	-2.549E-04	-1.469E-03
45.5	-1.730E-03	-9.972E-03
46.8	-2.525E-03	-1.455E-02
48.1	-2.263E-03	-1.304E-02
49.4	-1.353E-03	-7.799E-03
50.7	-2.073E-04	-1.195E-03
52.0	9.188E-04	5.296E-03
53.3	1.401E-03	8.076E-03
54.6	1.493E-03	8.604E-03
55.9	9.895E-04	5.703E-03
57.2	2.851E-04	1.643E-03
58.5	-3.408E-04	-1.964E-03
59.8	-8.575E-04	-4.942E-03
61.1	-8.606E-04	-4.960E-03
62.4	-7.281E-04	-4.197E-03
63.7	-2.830E-04	-1.631E-03
65.0	1.419E-04	8.179E-04
66.3	4.121E-04	2.375E-03
67.6	5.926E-04	3.415E-03
68.9	4.382E-04	2.525E-03
70.2	2.730E-04	1.574E-03

PROCESS = 411.06667 SECS.
I/O = 7.61667 SECS.
RUN TIME = 418.68333 SECS.

PARAMETERS: MU[0] = 9 SIGMA[0] = 0.9
MU[1] = 1 SIGMA[1] = 0.1

TAU	R(TAU)	RHO(TAU)
0.0	8.577E-02	1.000E+00
1.0	-4.664E-03	-5.438E-02
2.0	-9.138E-03	-1.065E-01
3.0	-1.084E-02	-1.264E-01
4.0	-9.879E-03	-1.152E-01
5.0	-9.682E-03	-1.129E-01
6.0	-1.034E-02	-1.206E-01
7.0	-9.711E-03	-1.132E-01
8.0	-4.408E-03	-5.140E-02
9.0	1.385E-02	1.615E-01
10.0	2.992E-02	3.488E-01
11.0	1.502E-02	1.751E-01
12.0	-4.759E-03	-5.549E-02
13.0	-9.912E-03	-1.156E-01
14.0	-9.674E-03	-1.127E-01
15.0	-1.007E-02	-1.175E-01
16.0	-1.005E-02	-1.172E-01
17.0	-7.242E-03	-8.443E-02
18.0	-2.128E-04	-2.481E-03
19.0	1.214E-02	1.416E-01
20.0	1.994E-02	2.325E-01
21.0	1.281E-02	1.493E-01
22.0	-3.951E-04	-4.606E-03
23.0	-7.384E-03	-8.610E-02
24.0	-9.495E-03	-1.107E-01
25.0	-1.009E-02	-1.177E-01
26.0	-8.847E-03	-1.032E-01
27.0	-5.372E-03	-6.263E-02
28.0	1.247E-03	1.453E-02
29.0	1.031E-02	1.202E-01
30.0	1.493E-02	1.740E-01
31.0	1.029E-02	1.199E-01
32.0	1.497E-03	1.745E-02
33.0	-5.310E-03	-6.191E-02
34.0	-8.860E-03	-1.033E-01
35.0	-9.443E-03	-1.101E-01
36.0	-7.699E-03	-8.976E-02
37.0	-4.255E-03	-4.961E-02
38.0	1.852E-03	2.159E-02
39.0	8.793E-03	1.025E-01
40.0	1.159E-02	1.352E-01
41.0	8.484E-03	9.892E-02
42.0	2.231E-03	2.602E-02
43.0	-4.041E-03	-4.711E-02
44.0	-7.870E-03	-9.176E-02
45.0	-8.383E-03	-9.774E-02
46.0	-6.868E-03	-8.007E-02
47.0	-3.459E-03	-4.033E-02
48.0	2.133E-03	2.487E-02
49.0	7.393E-03	8.620E-02
50.0	9.232E-03	1.076E-01
51.0	7.250E-03	8.453E-02
52.0	2.319E-03	2.704E-02
53.0	-3.272E-03	-3.815E-02
54.0	-6.672E-03	-7.779E-02
55.0	-7.366E-03	-8.589E-02

PROCESS = 536.48333 SECS.
I/O = 22.03333 SECS.
RUN TIME = 558.51667 SECS.

Table 34. Autocovariance and Autocorrelation Function for U ≃ $N(\mu_0,\sigma_0^2)$ and V ≃ $N(\mu_1,\sigma_1^2)$ and for Indicated Parameters.

PARAMETERS: MU[0] = 5 SIGMA[0] = 1.0
MU[1] = 10 SIGMA[1] = 3.0

TAU	R(TAU)	RHO(TAU)
0.0	2.181E-01	1.000E+00
1.5	1.230E-01	5.643E-01
3.0	2.317E-02	1.063E-01
4.5	-6.181E-02	-2.835E-01
6.0	-9.747E-02	-4.470E-01
7.5	-8.947E-02	-4.103E-01
9.0	-6.498E-02	-2.980E-01
10.5	-2.795E-02	-1.282E-01
12.0	1.436E-02	6.587E-02
13.5	4.995E-02	2.291E-01
15.0	6.536E-02	2.998E-01
16.5	5.570E-02	2.554E-01
18.0	2.690E-02	1.234E-01
19.5	-7.444E-03	-3.414E-02
21.0	-3.340E-02	-1.532E-01
22.5	-4.363E-02	-2.001E-01
24.0	-3.727E-02	-1.709E-01
25.5	-2.002E-02	-9.183E-02
27.0	1.313E-03	6.023E-03
28.5	1.870E-02	8.575E-02
30.0	2.779E-02	1.274E-01
31.5	2.601E-02	1.193E-01
33.0	1.611E-02	7.387E-02
34.5	1.816E-03	8.328E-03
36.0	-1.067E-02	-4.893E-02
37.5	-1.802E-02	-8.264E-02
39.0	-1.788E-02	-8.201E-02
40.5	-1.204E-02	-5.521E-02
42.0	-2.785E-03	-1.277E-02
43.5	5.784E-03	2.653E-02
45.0	1.134E-02	5.198E-02
46.5	1.211E-02	5.555E-02
48.0	8.837E-03	4.053E-02
49.5	3.015E-03	1.383E-02
51.0	-2.926E-03	-1.342E-02
52.5	-6.999E-03	-3.210E-02
54.0	-8.167E-03	-3.745E-02
55.5	-6.332E-03	-2.904E-02
57.0	-2.768E-03	-1.270E-02
58.5	1.350E-03	6.189E-03
60.0	4.218E-03	1.934E-02
61.5	5.461E-03	2.505E-02
63.0	4.456E-03	2.044E-02
64.5	2.351E-03	1.078E-02
66.0	-4.830E-04	-2.215E-03
67.5	-2.470E-03	-1.133E-02
69.0	-3.597E-03	-1.650E-02
70.5	-3.093E-03	-1.419E-02
72.0	-1.873E-03	-8.590E-03
73.5	2.670E-05	1.224E-04
75.0	1.427E-03	6.543E-03
76.5	2.325E-03	1.066E-02
78.0	2.156E-03	9.887E-03
79.5	1.416E-03	6.492E-03
81.0	2.164E-04	9.923E-04
82.5	-8.169E-04	-3.746E-03

PROCESS = 412.01667 SECS.
I/O = 7.05000 SECS.
RUN TIME = 419.06667 SECS.

PARAMETERS: MU[0] = 10 SIGMA[0] = 1.0
MU[1] = 10 SIGMA[1] = 1.0

TAU	R(TAU)	RHO(TAU)
0.0	2.388E-01	1.000E+00
2.0	1.449E-01	6.071E-01
4.0	4.871E-02	2.040E-01
6.0	-4.706E-02	-1.971E-01
8.0	-1.427E-01	-5.976E-01
10.0	-2.035E-01	-8.522E-01
12.0	-1.450E-01	-6.071E-01
14.0	-5.019E-02	-2.102E-01
16.0	4.503E-02	1.886E-01
18.0	1.376E-01	5.764E-01
20.0	1.869E-01	7.828E-01
22.0	1.417E-01	5.936E-01
24.0	5.198E-02	2.177E-01
26.0	-4.281E-02	-1.793E-01
28.0	-1.309E-01	-5.484E-01
30.0	-1.740E-01	-7.287E-01
32.0	-1.372E-01	-5.747E-01
34.0	-5.316E-02	-2.227E-01
36.0	4.009E-02	1.679E-01
38.0	1.237E-01	5.181E-01
40.0	1.633E-01	6.840E-01
42.0	1.322E-01	5.539E-01
44.0	5.389E-02	2.257E-01
46.0	-3.670E-02	-1.537E-01
48.0	-1.166E-01	-4.882E-01
50.0	-1.538E-01	-6.443E-01
52.0	-1.271E-01	-5.325E-01
54.0	-5.443E-02	-2.280E-01
56.0	3.313E-02	1.388E-01
58.0	1.095E-01	4.588E-01
60.0	1.451E-01	6.078E-01
62.0	1.223E-01	5.124E-01
64.0	5.470E-02	2.291E-01
66.0	-2.941E-02	-1.232E-01
68.0	-1.025E-01	-4.293E-01
70.0	-1.372E-01	-5.746E-01
72.0	-1.177E-01	-4.929E-01
74.0	-5.475E-02	-2.293E-01
76.0	2.549E-02	1.068E-01
78.0	9.575E-02	4.010E-01
80.0	1.298E-01	5.436E-01
82.0	1.132E-01	4.741E-01
84.0	5.488E-02	2.299E-01
86.0	-2.157E-02	-9.032E-02
88.0	-8.921E-02	-3.736E-01
90.0	-1.227E-01	-5.140E-01
92.0	-1.090E-01	-4.567E-01
94.0	-5.501E-02	-2.304E-01
96.0	1.773E-02	7.426E-02
98.0	8.276E-02	3.466E-01
100.0	1.161E-01	4.863E-01
102.0	1.051E-01	4.403E-01
104.0	5.512E-02	2.308E-01
106.0	-1.380E-02	-5.781E-02
108.0	-7.650E-02	-3.204E-01
110.0	-1.099E-01	-4.602E-01

PROCESS = 509.91667 SECS.
I/O = 10.36667 SECS.
RUN TIME = 520.28333 SECS.

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