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# MINIMIZING FLOWS FOR THE MONGE–KANTOROVICH PROBLEM*

SIGURD ANGENENT†, STEVEN HAKER‡, AND ALLEN TANNENBAUM§

**Abstract.** In this work, we formulate a new minimizing flow for the optimal mass transport (Monge–Kantorovich) problem. We study certain properties of the flow, including weak solutions as well as short- and long-term existence. Optimal transport has found a number of applications, including econometrics, fluid dynamics, cosmology, image processing, automatic control, transportation, statistical physics, shape optimization, expert systems, and meteorology.

**Key words.** optimal transport, gradient flows, weak solutions, image registration, medical imaging

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**1. Introduction.** In this paper, we derive a novel gradient descent flow for the computation of the optimal transport map (when it exists) in the Monge–Kantorovich framework. Besides being quite useful for the efficient computation of the transport map, we believe that the flow presented here is quite interesting from a theoretical point of view as well. In the present work, we undertake a study of some of its key properties.

The *mass transport problem* was first formulated by Monge in 1781 and concerned finding the optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one site to another. This problem was given a modern formulation in the work of Kantorovich [13] and so is now known as the *Monge–Kantorovich problem.* We recall the formulation of the Monge–Kantorovich problem for smooth densities and domains in Euclidean space. For more general measures, see [1]. Let $\Omega_0$ and $\Omega_1$ be two diffeomorphic connected subdomains of $\mathbb{R}^d$, with smooth boundaries, and let $\mu_0$, $\mu_1$ be Borel measures on $\Omega_0$ and $\Omega_1$, each with a positive density function $\mu_0$ and $\mu_1$, respectively. We assume

$$\mu_0(\Omega_0) = \mu_1(\Omega_1),$$

i.e.,

$$\int_{\Omega_0} \mu_0(x)dx = \int_{\Omega_0} \mu_1(x)dx,$$

so that the same total mass is associated with $\Omega_0$ and $\Omega_1$, and we consider diffeomorphisms $u : \Omega_0 \to \Omega_1$ which map one density to the other in the sense that

$$\mu_0 = \det(\nabla u)\, \mu_1 \circ u, \tag{1}$$

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†Department of Mathematics, University of Wisconsin, Madison, WI 53706 (angenent@math.wisc).

‡Department of Radiology, Surgical Planning Laboratory, Brigham and Women's Hospital, Boston, MA 02115 (haker@bwh.harvard.edu).

§Departments of Electrical and Computer and Biomedical Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250 (tannenba@ece.gatech.edu).

where $\det \nabla u$ denotes the determinant of the Jacobian map $\nabla u$. This is the well-known *Jacobian equation*, which constrains the mapping $u$ to be mass preserving with respect to $\mu_0$ and $\mu_1$.

There may be many such mappings, and we want to pick out an optimal one in some sense. Accordingly, we define a generalized Monge–Kantorovich functional as

$$M(u) = \int \Phi(x, u(x))\, \mathrm{d}\mu_0(x),$$

where $\Phi : \bar{\Omega}_0 \times \bar{\Omega}_1 \to \mathbb{R}$ is a positive $C^1$ cost function. A $\Phi$-*optimal mass preserving map*, when it exists, is a diffeomorphism which satisfies (1) and minimizes this integral.

In particular, the $L^2$ Monge–Kantorovich problem, corresponding to the cost function $\Phi(x, \xi) = \frac{1}{2}|x-\xi|^2$, has been studied in statistics, functional analysis, atmospheric sciences, automatic control, computer vision, statistical physics, and expert systems. See [3, 5, 8, 15, 16] and the references therein. This functional is seen to place a quadratic penalty on the distance the map $u$ moves each bit of material, weighted by the material's mass. A fundamental theoretical result for the $L^2$ case [14, 4, 9] is that there is a unique optimal mass preserving $u$, and that this $u$ is characterized as the gradient of a convex function $p$, i.e., $u = \nabla p$.

**1.1. Reallocation measures.** It turns out to be very convenient to use Kantorovich's generalization of the notion of a measure preserving map $u : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$. Instead of considering a map $u$ we introduce its graph

$$\{(x, u(x)) \mid x \in \Omega_0\} \subset \Omega_0 \times \Omega_1$$

and, more importantly, the measure

$$\gamma_u = (\mathrm{id} \times u)_{\#}\mu_0 \tag{2}$$

on $\Omega_0 \times \Omega_1$ supported on this graph.

The measures $\gamma$ that arise in this way all satisfy[1]

$$\big(p_0\big)_{\#}\gamma = \mu_0 \quad \text{and} \quad \big(p_1\big)_{\#}\gamma = \mu_1. \tag{3}$$

We define $\mathfrak{X}$ to be the space of nonnegative Borel measures on $\bar{\Omega}_0 \times \bar{\Omega}_1$ which satisfy (3).

The Monge–Kantorovich cost functional extends in a natural way to the space of measures $\mathfrak{X}$ by

$$M(\gamma) = \int_{\Omega_0 \times \Omega_1} \Phi(x, y)\, \mathrm{d}\gamma(x, y).$$

We may think of a measure $\gamma \in \mathfrak{X}$ as a "multivalued map," which, rather than sending a point $x \in \Omega_0$ to *one* other point $u(x)$, assigns a probability measure $P_x$ on the range space $\Omega_1$ and "smears the point $x$ out over $\Omega_1$ according to the probability measure $P_x$." The measure $\gamma$ is reconstructed from the family of probability measures $\{P_x\}$ by specifying

$$\int_{\Omega_0 \times \Omega_1} \phi(x, y)\, \mathrm{d}\gamma(x, y) = \int_{\Omega_0} \left\{ \int_{\Omega_1} \phi(x, y)\, \mathrm{d}P_x(y) \right\} \mathrm{d}\mu_0(x). \tag{4}$$

[1] If $X$ and $Y$ are sets with $\sigma$-algebras $\mathcal{M}$ and $\mathcal{N}$, and if $f : X \to Y$ is a measurable map, then we write $f_{\#}\mu$ for the pushforward of any measure $\mu$ on $(X, \mathcal{M})$, i.e., for any measurable $E \subset Y$ we define $f_{\#}\mu(E) = \mu\big(f^{-1}(E)\big)$.

See [1] for a rigorous measure-theoretic account of this way of decomposing $\gamma$. In this paper we will write for any bounded Borel measurable function $\phi : \Omega_0 \times \Omega_1 \to \mathbb{R}$

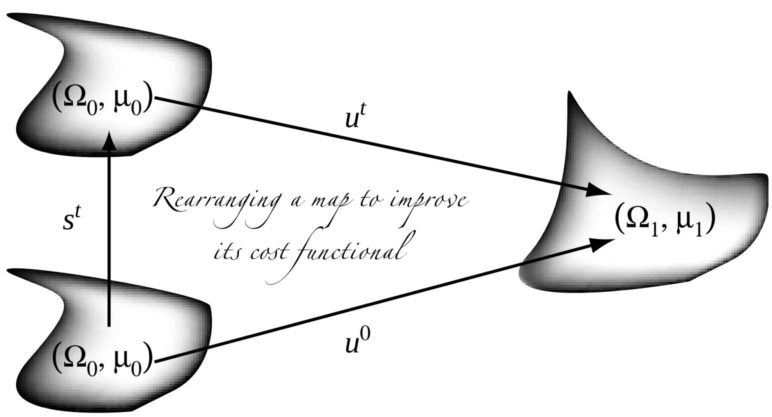
$$\mathbb{E}_\gamma(\phi(x, y) \mid x) = \int_{\Omega_1} \phi(x, y)\, \mathrm{d}P_x(y)$$

for the expectation of $\phi(x, \cdot)$ with respect to the probability measure $P_x$. See Lemma 3.1, where we define this expectation directly without using the probability measures $P_x$. The principal role of the expectation $\mathbb{E}_\gamma(\phi(x, y) \mid x)$ is as generalization of the expression $\phi(x, u(x))$. Indeed, when $\gamma = (\mathrm{id} \times u)_{\#}\mu_0$, then both expressions coincide.

**1.2. The gradient flow.** To reduce the Monge–Kantorovich cost $M(u)$ of a map $u^0 : \Omega_0 \to \Omega_1$ we "rearrange the points in the domain of the map"; i.e., we replace the map $u^0$ by a family of maps $u^t$ for which one has $u^t \circ s^t = u^0$ for some family of diffeomorphisms $s^t : \Omega_0 \to \Omega_0$ (see figure below). If the initial map $u^0$ sends the measure $\mu_0$ to $\mu_1$ (if it satisfies (1)), and if the diffeomorphisms $s^t$ preserve the measure $\mu_0$, then the maps $u^t = u^0 \circ \left(s^t\right)^{-1}$ will also send $\mu_0$ to $\mu_1$. Thus the group $\mathrm{Diff}^1_{\mu_0}(\Omega_0)$ of $C^1$, $\mu_0$ preserving diffeomorphisms acts on the space of measure preserving maps $u : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$. The group action of $\mathrm{Diff}^1_{\mu_0}(\Omega_0)$ can be extended to an action on $\mathfrak{X}$ by

$$s \cdot \gamma = \left(s \times \mathrm{id}_{\Omega_1}\right)_{\#}\gamma.$$

Any sufficiently smooth family of diffeomorphism $s^t : \Omega_0 \to \Omega_0$ is determined by its velocity field, defined by $\partial_t s^t = v^t \circ s^t$.




In section 3 we compute the change in $M(\gamma^t)$ for measures $\gamma^t = s^t \cdot \gamma^0 \in \mathfrak{X}$ obtained by letting a family of diffeomorphisms $s^t \in \mathrm{Diff}^1_{\mu_0}(\Omega_0)$ act on an initial measure $\gamma^0 \in \mathfrak{X}$. We find that steepest descent is achieved by a family $s^t \in \mathrm{Diff}^1_{\mu_0}(\Omega_0)$, whose velocity is given by

$$v^t = -\frac{1}{\mu_0(x)} \mathcal{P}\left(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\right). \tag{5}$$

Here $\mathcal{P}$ is the Helmholtz projection, which extracts the divergence-free part of vector fields on $\Omega_0$ (see section 7).

In the special case where the measures $\gamma^t$ are given by graph measures $\gamma^t = \gamma_{u^t}$ as in (2) we get the following equations for the evolution of the map $u^t$. From $u^0 = u^t \circ s^t$ we get the transport equation

$$\frac{\partial u^t}{\partial t} + v^t \cdot \nabla u^t = 0. \tag{6}$$

The velocity field is still given by (5) above, but this now simplifies to

$$v^t = \frac{-1}{\mu_0(x)} \mathcal{P}\big\{\Phi_x(x, u^t(x))\big\}.$$

This equation, together with the transport equation (6), determines an initial value problem for the map $u^t$.

We will show the following.

THEOREM 1.1. *Let $0 < \alpha < 1$. For any $C^{1,\alpha}$, measure preserving initial map $u^0$ a smooth ($C^{1,\alpha}$) family of maps $\{u^t \mid 0 \le t < T\}$ exists such that the maps $s^t$ generated by the vector field $v^t$ given by* (5) *satisfy $u^0 = u^t \circ s^t$.*

*The existence time $T$ of the smooth solution depends on the $C^{1,\alpha}$ norm of the initial map $u^0$.*

See Lemma 11.1 for more detail.

It is not clear if these smooth solutions exist for *all* $t > 0$ (we make no geometric assumptions on $\Omega_0$ or the cost function $\Phi$ at all). To construct global solutions we modify the equation by introducing a smoothing operator $\mathcal{A}$. This operator acts on the space $\mathfrak{H}$ of all $L^2$ vector fields on $\Omega_0$. We choose $\mathcal{A}$ to be an operator which approximates the identity and for which $\mathcal{A}w$ will always be smooth for all $w \in \mathfrak{H}$. The operators $\mathcal{A}$ we use are versions of a parabolic operator $\mathcal{A} = e^{\varepsilon\Delta}$. See section 8 for more detail.

Instead of considering the gradient flow generated by the velocity field (5), we smooth out $v^t$ and consider

$$v^t = -\frac{1}{\mu_0(x)} \mathcal{P}\mathcal{A}^2\mathcal{P}(W^t) = -\frac{1}{\mu_0(x)} \mathcal{P}\mathcal{A}^2\mathcal{P}\left(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\right). \tag{7}$$

We refer to the corresponding initial value problem as *the regularized problem.* Since the velocity field here is smooth for any possible $\gamma^t \in \mathfrak{X}$, no singularities can occur, and we can prove the following.

THEOREM 1.2. *Under appropriate assumptions on the smoothing operator $A$ solutions to the initial value problem exist for all time $t \ge 0$ and for any initial measure $\gamma^0 \in \mathfrak{X}$.*

See Theorem 9.1 for a more precise statement.

In fact, assuming the smoothing operator is injective, it follows that the initial value problem corresponding to the velocity field (7) generates a continuous semiflow on $\mathfrak{X}$ and that the Monge–Kantorovich functional $M(\gamma)$ acts as a Lyapunov function for this flow. Thus all orbits exist for all $t > 0$, and all orbits have $\omega$-limit sets consisting of critical points only. Here a critical point of the flow is measure $\gamma \in \mathfrak{X}$ whose velocity field defined in (7) vanishes. Injectivity of $\mathcal{A}$ implies that critical points can be characterized independently of the smoothing operator $\mathcal{A}$: *A critical point is a measure $\gamma \in \mathfrak{X}$ for which*

$$\mathbb{E}_\gamma\big(\Phi_x \mid x\big) = \nabla p$$

*for some Lipschitz continuous function* $p : \Omega_0 \to \mathbb{R}$. These are precisely the measures $\gamma \in \mathfrak{X}$ whose Monge–Kantorovich cost $M(\gamma)$ cannot be reduced by the action of some $s \in \mathrm{Diff}^1_{\mu_0}(\Omega_0)$ infinitesimally close to the identity.

If the measure $\gamma$ is given by $\gamma = (\mathrm{id} \times u)_{\#}\mu_0$ for some measure preserving $u : \Omega_0 \to \Omega_1$, then $\gamma = \gamma_u$ is a critical point exactly when the map $u$ satisfies

$$\Phi_x(x, u(x)) = \nabla p(x) \quad \text{a.e. on } \Omega_0$$

for some Lipschitz function $p : \Omega_0 \to \mathbb{R}$. This is very important since it motivates our approach for finding a flow which in a certain sense kills the curl of a vector field (see our discussion in section 3.2). In particular, if the cost function is quadratic, $\Phi(x, y) = \frac{1}{2}|x - y|^2$, then a measure preserving map $u : \Omega_0 \to \Omega_1$ whose reallocation measure $\gamma_u \in \mathfrak{X}$ is a critical point also satisfies

$$u(x) = x - \nabla p(x)$$

for some Lipschitz function $p$.

Our gradient flows (both regularized and unregularized) move measures $\gamma \in \mathfrak{X}$ around on orbits of the group action $\mathrm{Diff}^1_{\mu_0}(\Omega_0) \times \mathfrak{X} \to \mathfrak{X}$.

A pertinent example is the group orbit of a $C^1$ diffeomorphism $\hat{u} : \bar{\Omega}_0 \to \bar{\Omega}_1$, or, rather, the measure $\gamma_u$ associated to such a map. This orbit consists of all measures of the form $s \cdot \gamma_u = \gamma_{u \circ s^{-1}}$. Since any other diffeomorphism $\tilde{u} : \bar{\Omega}_0 \to \bar{\Omega}_1$ is of the form $\tilde{u} = \hat{u} \circ s^{-1}$ for some $s \in \mathrm{Diff}^1_{\mu_0}(\Omega_0)$ we see that the set

$$\big\{\gamma_u \mid u : \bar{\Omega}_0 \to \bar{\Omega}_1 \text{ is a } C^1 \text{ measure preserving diffeomorphism}\big\}$$

is exactly one orbit of the group action. So, if we have an initial measure $\gamma = \gamma_u$ which is generated by some map $u$ and solve the initial value problem, then the solution we get will again consist of measures of the form $\gamma^t = \gamma_{u^t}$.

Unfortunately, such group orbits are not always closed, so if $\{\gamma^t = s^t \cdot \gamma^0 \mid t \geq 0\}$ is a trajectory of one of our gradient flows, then its $\omega$-limit set might not be contained in the same orbit of the group action; i.e., if $\hat{\gamma}$ belongs to the $\omega$-limit set, then it is possible that $\hat{\gamma}$ is not of the form $s \cdot \gamma^0$ for any $s \in \mathrm{Diff}^1_{\mu_0}(\Omega_0)$. In particular, if we start with $\gamma^0 = \gamma_{u^0}$, then the corresponding solution $\gamma^t$ to the regularized flow will be of the form $\gamma^t = \gamma_{u^t}$ for a family of maps $u^t = u^0 \circ (s^t)^{-1}$, but, as $t \nearrow \infty$, the $\gamma^t$ might converge to a measure $\tilde{\gamma} \in \mathfrak{X}$, which does not correspond to any map. (For example, if $\Omega_0 = \Omega_1$ is the unit disc, $\mu_0 = \mu_1$ is Lebesgue measure, $u^0 = \mathrm{id}_{\Omega_0}$, and $s^t$ is defined by $s^t(z) = e^{it|z|}z$ in complex notation, then the measures $\gamma_{u^t}$ converge weakly to $\bar{\gamma} \in \mathfrak{X}$. The limiting measure $\bar{\gamma}$ is described as in (4) with $P_x$ the uniform distribution on the circle of radius $|x|$. In other words, instead of corresponding to a map, the limiting measure $\bar{\gamma}$ takes each point $x \in \Omega_0$ and spreads it out uniformly over the circle through $x$, centered at the origin. See Figure 1.)

We must therefore study the gradient flow(s) on all of $\mathfrak{X}$.

It turns out that there is always one stationary measure, namely,

$$\gamma_\times = \mu_0 \times \mu_1. \tag{8}$$

This measure takes each point in $\Omega_0$ and spreads it out evenly (with a probability measure proportional to $\mu_1$) over $\Omega_1$. This measure must be a critical measure, for it is a fixed point for the group action; i.e., for all $s \in \mathrm{Diff}^1_{\mu_0}$ one has

$$s \cdot \gamma_\times = \big(s \times \mathrm{id}_{\Omega_1}\big)_{\#}\gamma_\times = (s_{\#}\mu_0) \times \mu_1 = \mu_0 \times \mu_1 = \gamma_\infty.$$

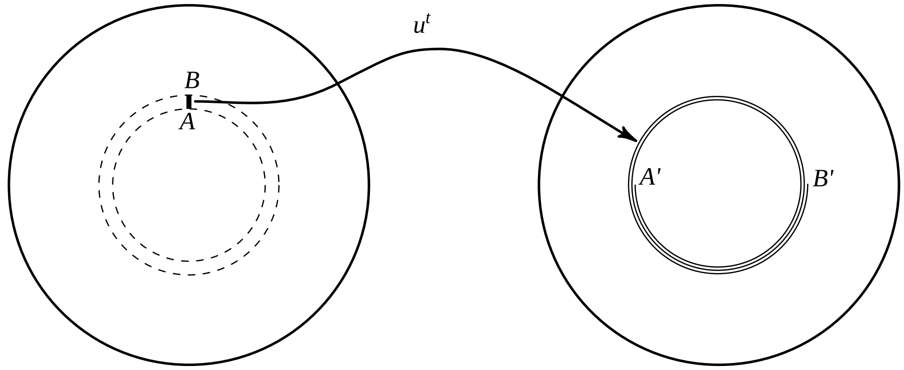



FIG. 1. *The map $u^t$ spreads the short line segment $AB$ out over the spiral $A'B'$.*

Therefore any of the gradient flows $\gamma \mapsto s^t \cdot \gamma$ we construct here act trivially on $\gamma_\times$.

Although we have no global existence result for the unregularized flow, we can choose a family of smoothing operators $\mathcal{A}_\varepsilon = e^{\varepsilon\Delta}$ which approximate the identity operator as $\varepsilon \searrow 0$ and consider the solutions $\{\gamma_\varepsilon^t \mid t \geq 0\}$ of the regularized flows whose existence we have already proved. We then show in section 10.2 that the $\gamma_\varepsilon^t$ converge weakly to a family of measures $\tilde{\gamma}^t$ whose Monge–Kantorovich cost is decreasing and whose $\omega$-limit set consists of critical measures (Proposition 10.1.)

**1.3. Computations.** Our interest in Monge–Kantorovich arose because of certain problems in computer vision and image processing, including image registration and image warping [2, 11, 12]. Image registration is the process of establishing a common geometric reference frame between two or more data sets possibly taken at different times. In [11, 12], we present a method for computing elastic registration maps based on the Monge–Kantorovich problem of optimal mass transport.

For image registration, it is natural to take $\Phi(x,y) = \frac{1}{2}|x-y|^2$ and $\Omega_0 = \Omega_1$ to be a rectangle. Extensive numerical computations show that the solution to the unregularized flow converges to a limiting map for a large choice of measures and initial maps. Indeed, in this case, we can write the minimizing flow in the following "nonlocal" form:

$$\frac{\partial u^t}{\partial t} = -\frac{1}{\mu_0}\left(u^t - \nabla\Delta^{-1}\operatorname{div}(u^t)\right)\cdot\nabla u^t. \tag{9}$$

In section 12, we give some details on our numerical methods as well as some illustrative examples.

**2. Reallocation measures.** The search for minimizers of $M(u)$ simplifies greatly if one suitably generalizes the notion of "mapping from $\Omega_0$ to $\Omega_1$." The standard way to do this in the present context is to identify the measure preserving map $u : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$ with its graph, or, rather, with the Borel measure $\gamma_u$ on $\Omega_0 \times \Omega_1$ defined by

$$\gamma_u(E) = \mu_0\big(\{x \in \Omega_0 : (x, u(x)) \in E\}\big).$$

This measure is supported on the graph of the map $u$; it is the pushforward of $\mu_0$ under the map $\mathrm{id} \times u$, so $\gamma_u = (\mathrm{id} \times u)_\#(\mu_0)$.

The map $u$ is measure preserving if and only if the measure $\gamma_u$ satisfies $(p_0)_\#(\gamma_u) = \mu_0$ and $(p_1)_\#(\gamma_u) = \mu_1$, where $p_j : \Omega_0 \times \Omega_1 \to \Omega_j$ is the canonical projection. This prompts us to consider the space

$$\mathfrak{X} = \{\text{Borel measures } \gamma \geq 0 \text{ on } \Omega_0 \times \Omega_1 \;\mid (p_j)_\#\gamma = \mu_j \text{ for } j = 0, 1\}.$$

If the measure $\gamma$ has a density, so that $d\gamma(x, y) = \mu(x, y)\mathrm{d}x\mathrm{d}y$, then $\gamma \in \mathfrak{X}$ exactly when

$$\int_{\Omega_0} \mu(x, y)\mathrm{d}x = \mu_1(y) \quad \text{for } \mu_1 \text{ almost all } y \in \Omega_1 \tag{10}$$

and

$$\int_{\Omega_1} \mu(x, y)\mathrm{d}y = \mu_0(x) \quad \text{for } \mu_0 \text{ almost all } x \in \Omega_0. \tag{11}$$

All measures $\gamma \in \mathfrak{X}$ have total mass

$$\gamma(\Omega_0 \times \Omega_1) = \mu_0(\Omega_0) = \mu_1(\Omega_1). \tag{12}$$

The space $\mathfrak{X}$ with the weak* topology is a compact metrizable space. (It is a closed and convex subset of the dual of $C^0(\Omega_0 \times \Omega_1)$.) The Monge–Kantorovich cost functional is linear on $\mathfrak{X}$. It is simply given by

$$M(\gamma) = \langle \gamma, \Phi \rangle = \int_{\Omega_0 \times \Omega_1} \Phi(x, y) d\gamma(x, y).$$

As such, there is always a minimizer for the cost functional (although in general it is only known to be a measure $\gamma \in \mathfrak{X}$, and it does not follow from general principles that $\gamma$ is of the form $\gamma_u$ for some measure preserving map).

**3. Steepest descent.** The group $\mathcal{G}$ of $\mu_0$ measure preserving transformations on $s : \Omega_0 \to \Omega_0$ acts on $\mathfrak{X}$ by $s \cdot \gamma \mapsto (s \times \mathrm{id}_{\Omega_1})\gamma$. We propose to study a cost reducing flow on $\mathfrak{X}$ which is defined by the group action $\mathcal{G} \times \mathfrak{X} \to \mathfrak{X}$. Rather than applying arbitrary measurable maps $s \in \mathcal{G}$, we restrict ourselves to smooth ($C^1$) orientation preserving diffeomorphisms $s : \Omega_0 \to \Omega_0$.

**3.1. The first variation.** If we have a one-parameter family of $\mu_0$ preserving $C^1$ diffeomorphisms $s^t : \Omega_0 \to \Omega_0$ with velocity field $v^t$, and we write $\gamma^t = s^t \cdot \gamma$ for some $\gamma \in \mathfrak{X}$, then the first variation of the Monge–Kantorovich cost functional is

$$\begin{aligned} \frac{dM(\gamma^t)}{dt} &= \frac{d}{dt} \int_{\Omega_0 \times \Omega_1} \Phi(x, y)\mathrm{d}(s^t \times \mathrm{id})_\#\gamma(x, y) \\ &= \frac{d}{dt} \int_{\Omega_0 \times \Omega_1} \Phi(s^t(x), y)\mathrm{d}\gamma(x, y) \\ &= \int_{\Omega_0 \times \Omega_1} v^t(s^t(x)) \cdot \Phi_x(s^t(x), y)\mathrm{d}\gamma(x, y) \\ &= \int_{\Omega_0 \times \Omega_1} v^t(x) \cdot \Phi_x(x, y)\mathrm{d}\gamma^t(x, y). \end{aligned} \tag{13}$$

LEMMA 3.1. *For any bounded measurable function $F : \Omega_0 \times \Omega_1 \to \mathbb{R}$ there exists a bounded measurable function $\tilde{F} : \Omega_0 \to \mathbb{R}$ for which*

$$\int_{\Omega_0 \times \Omega_1} \phi(x)F(x, y)\mathrm{d}\gamma(x, y) = \int_{\Omega_0} \phi(x)\tilde{F}(x)\mathrm{d}\mu_0(x)$$

*holds for all* $\phi \in L^1(\Omega_0; \mathrm{d}\mu_0)$.

*Proof.* The left-hand side defines a bounded linear functional on $L^1(\Omega_0, \mathrm{d}\mu_0)$, so the existence and uniqueness of $\tilde{F}$ is guaranteed. □

We will denote the function $\tilde{F}$ by

$$\tilde{F}(x) = \mathbb{E}_\gamma(F \mid x) \quad \text{or} \quad \tilde{F}(x) = \mathbb{E}_\gamma(F(x,y) \mid x). \tag{14}$$

If the measure $\gamma$ has a density $\mu(x,y)$, then $\tilde{F}(x)$ is given by

$$\mathbb{E}_\gamma(F \mid x) = \int_{\Omega_1} F(x,y)\frac{\mu(x,y)}{\mu_0(x)}\mathrm{d}y. \tag{15}$$

Fubini's theorem implies that this integral exists for $\mu_0$ almost all $x \in \Omega_0$. The condition $\mu(x,y)\mathrm{d}x\mathrm{d}y \in \mathfrak{X}$ implies that $\frac{\mu(x,y)}{\mu_0(x)}\mathrm{d}y$ is a probability measure on $\Omega_1$ for every $x \in \Omega_0$, and $\tilde{F}(x)$ is just the expectation of $F(x,y)$ for this probability measure. This justifies the notation in (14).

If the measure is of the form $\gamma = \gamma_u$ for some measure preserving map $u : \Omega_0 \to \Omega_1$, then $\tilde{F}$ is given by

$$\mathbb{E}_\gamma(F \mid x) = F(x, u(x)). \tag{16}$$

One may think of (16) as a special case of (15) in which the "density" $\mu(x,y)$ is given by $\mu(x,y) = \mu_0(x)\delta(y - u(x))$ ($\delta$ being the Dirac delta-function). Here the probability measure $\frac{\mu(x,y)}{\mu_0(x)}\mathrm{d}y$ puts probability one at $y = u(x)$, and thus the expectation of $F(x,y)$ for this measure is just $F(x, u(x))$.

With this notation we now complete our computation (13) of the first variation:

$$\frac{dM(\gamma^t)}{dt} = \int_{\Omega_0} v^t(x) \cdot W^t(x)\mathrm{d}\mu_0(x), \tag{17}$$

where

$$W^t(x) = \mathbb{E}_{\gamma^t}(\Phi_x(x,y) \mid x). \tag{18}$$

When the measure $\gamma \in \mathfrak{X}$ is of the form $\gamma = \gamma_u$ for some map $u : \Omega_0 \to \Omega_1$, one has $\gamma^t = \gamma_{u^t}$ with $u^t \circ s^t = u$, and thus (18) reduces to

$$W^t(x) = \Phi_x(x, u^t(x)).$$

In the case of a quadratic cost function $\Phi(x,y) = \frac{1}{2}|x-y|^2$ but general measure $\gamma^t \in \mathfrak{X}$, one has

$$W^t(x) = \mathbb{E}_{\gamma^t}(x - y \mid x) = x - Y^t(x),$$

where

$$Y^t(x) = \mathbb{E}_{\gamma^t}(y|x)$$

is the expected $y$ value to which the measure $\gamma^t$ reallocates the point $x$.

If the cost function is quadratic, *and* if the measure $\gamma^t$ is of the form $\gamma_{u^t}$, then we get $Y^t(x) = u^t(x)$, and hence

$$W^t(x) = x - u^t(x).$$

**3.2. Steepest descent.** To reduce the cost functional we choose the velocity field $v^t$ so as to minimize $\int v^t \cdot W^t \mathrm{d}\mu_0$ subject to a constraint on $\|v^t\|_{L^2}$ (or some similar quadratic norm) and subject to the constraint that $v^t$ preserve the measure $\mu_0$, i.e., $\operatorname{div} \mu_0 v^t = 0$.

To this end we use the Helmholtz projection to split $W^t$ into a gradient and its divergence-free part,

$$W^t = \nabla p^t + \mathcal{P}(W^t),$$

where

$$\operatorname{div} \mathcal{P}(W^t) = 0,$$

and where $\mathcal{P}(W^t)\,|_{\partial\Omega_0}$ is tangential to the boundary of $\Omega_0$. Such a decomposition is always possible, and $\mathcal{P}$ can be interpreted as orthogonal projection in $L^2(\Omega_0) \otimes \mathbb{R}^d$. See section 7, where we discuss $\mathcal{P}$ in more detail.

If the velocity field satisfies $\operatorname{div} \mu_0 v^t = 0$, then we get

$$\begin{aligned} \frac{dM(\gamma^t)}{dt} &= \int_{\Omega_0} \mu_0(x) v^t(x) \cdot W^t \mathrm{d}x \\ &= \int_{\Omega_0} \mu_0(x) v^t(x) \cdot \{\nabla p^t + \mathcal{P}(W^t)\} \mathrm{d}x \\ &= \int_{\Omega_0} \{-p^t \nabla \cdot (\mu_0 v^t) + \mu_0 v^t \cdot \mathcal{P}(W^t)\} \mathrm{d}x \\ &= \int_{\Omega_0} \mu_0(x) v^t(x) \cdot \mathcal{P}(W^t) \mathrm{d}x. \end{aligned} \tag{19}$$

We choose the following velocity field:

$$v^t = -\frac{1}{\mu_0(x)} \mathcal{P}\mathcal{A}^2\mathcal{P}(W^t) = -\frac{1}{\mu_0(x)} \mathcal{P}\mathcal{A}^2\mathcal{P}\left(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\right). \tag{20}$$

Here $\mathcal{A}$ is an operator on the Hilbert space

$$\mathfrak{H} \stackrel{\text{def}}{=} L^2(\Omega_0) \otimes \mathbb{R}^d.$$

Throughout this paper we will assume that $\mathcal{A}$ satisfies

(21) $\mathcal{A}$ is a *bounded, symmetric, and injective* operator on $\mathfrak{H}$.

Thus $\mathcal{A}^2$ is positive definite, and $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is positive definite on divergence-free vector fields on $\Omega_0$.

The most natural choice for $\mathcal{A}$ would be $\mathcal{A} = I_{\mathfrak{H}}$, the identity operator on $\mathfrak{H}$. In that case $\mathcal{P}\mathcal{A}^2\mathcal{P} = \mathcal{P}$, so that

$$v^t = -\frac{1}{\mu_0(x)} \mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x \mid x).$$

In what follows we are also interested in the case where the operator $\mathcal{A}$ is an approximate identity, e.g., $\mathcal{A}$ could be defined by running a heat equation for a short time, $\mathcal{A}f = e^{\varepsilon\Delta} f$. In section 8 we specify a class of operators $\mathcal{A}$ to which the theory in this paper is applicable.

**3.3. Evolution equation for the measure $\gamma^t$.** Let $\gamma^t = (s^t \times \mathrm{id})_{\#}\gamma^0$ for some initial measure $\gamma^0 \in \mathfrak{X}$. Here we compute the distributional time derivative of the $\gamma^t$ assuming the diffeomorphisms $s^t$ have velocity field $v^t$ given by (20).

Let $\varphi \in C^1(\bar\Omega_0 \times \bar\Omega_1)$ be a test function. Then $\langle \gamma^t, \varphi\rangle = \big\langle \gamma^0, \varphi \circ (s^t \times \mathrm{id})\big\rangle$, so that one has

$$\begin{aligned}\frac{d}{dt}\langle \gamma^t, \varphi\rangle &= \big\langle \gamma^0, (v^t \cdot \nabla_x \varphi) \circ (s^t \times \mathrm{id})\big\rangle \\ &= \big\langle \gamma^t, v^t \cdot \nabla_x \varphi\big\rangle \\ &= \big\langle -\nabla \cdot (v^t \gamma^t), \varphi\big\rangle,\end{aligned} \tag{22}$$

where $\nabla_x f(x, y)$ represents the gradient in the $x \in \Omega_0$ variable for any function $f : \Omega_0 \times \Omega_1 \to \mathbb{R}$.

Thus we have found that the family of measures $\gamma^t$ satisfies

$$\frac{\partial \gamma^t}{\partial t} + \nabla_x \cdot \big(v^t(x)\gamma^t\big) = 0 \tag{23}$$

in the sense of distributions. This equation, combined with (20), which prescribes $v^t$ in terms of $\gamma^t$, gives an initial value problem for $\gamma^t$,

$$\boxed{\frac{\partial \gamma^t}{\partial t} = \nabla_x \cdot \left(\frac{1}{\mu_0(x)}\mathcal{P}\mathcal{A}^2\mathcal{P}\big(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\big)\gamma^t\right).} \tag{24}$$

**3.4. A PDE for the map $u^t$.** If the measure $\gamma^t$ is given by $\gamma^t = \gamma_{u^t}$ for some family of measure preserving maps $u^t : (\Omega_0, \mu_0) \to (\Omega_1, \mu_1)$, then we have $u^0 = u^t \circ s^t$, so that the $u^t$ satisfy the transport equation

$$\frac{\partial u^t}{\partial t} + v^t \cdot \nabla u^t = 0. \tag{25}$$

Since for $\gamma^t = \gamma_{u^t}$ one has

$$\mathbb{E}_{\gamma^t}(\Phi_x \mid x) = \Phi_x(x, u^t(x)),$$

the velocity field is given by

$$v^t = \frac{-1}{\mu_0(x)}\mathcal{P}\mathcal{A}^2\mathcal{P}\big\{\Phi_x(x, u^t(x))\big\}. \tag{26}$$

Together, (25) and (26) determine an evolution equation for the map $u^t$.

**3.5. Evolution of the rearrangement $s^t$.** We return to the case where $\gamma^t$ is a general measure in $\mathfrak{X}$. Let us assume that the operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ can be represented as an integral operator with kernel $K(x, \xi)$, so that for any vector field $W \in L^2(\Omega_0; \mathbb{R}^n)$ one has

$$(\mathcal{P}\mathcal{A}^2\mathcal{P}W)(x) = \int_{\Omega_0} K(x, \xi) \cdot W(\xi)\mathrm{d}\xi. \tag{27}$$

Here d$y$ is the Lebesgue measure, $K(x, y)$ is an $n \times n$ matrix-valued function, and $K(x, y) \cdot W(y)$ is pointwise matrix multiplication.

Self-adjointness of the operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ implies

$$K(x,\xi) = K(\xi,x)^T. \tag{28}$$

When $\mathcal{A}$ is the identity operator on $\mathfrak{H}$, the kernel $K(x,\xi)$ is a singular integral kernel. When $\mathcal{A}$ is given by solving a heat equation, $\mathcal{A}f = e^{\varepsilon\Delta}f$, then the kernel $K(x,\xi)$ is a $C^{1,\alpha}$ function on $\bar{\Omega}_0 \times \bar{\Omega}_1$. (See section 8 for more details.)

The velocity field is given by

$$\begin{aligned} v^t(x) &= \frac{-1}{\mu_0(x)} \int_{\Omega_0} K(x,\xi)\mathbb{E}_{\gamma^t}(\Phi_x(\xi,\eta) \mid \xi)\,\mathrm{d}\xi \\ &= \frac{-1}{\mu_0(x)} \int_{\Omega_0\times\Omega_1} K(x,\xi)\cdot\Phi_x(\xi,\eta)\,\frac{\mathrm{d}\gamma^t(\xi,\eta)}{\mu_0(\xi)} \\ &= -\int_{\Omega_0\times\Omega_1} \frac{K(x,\xi)}{\mu_0(x)\mu_0(\xi)}\cdot\Phi_x(\xi,\eta)\,\mathrm{d}\gamma^t(\xi,\eta). \end{aligned} \tag{29}$$

Since the rearrangement maps $s^t : \Omega_0 \to \Omega_0$ are related to the velocity field $v^t$ by $\partial_t s^t = v^t \circ s^t$, we find the following integral-differential equation for $s^t$:

$$\begin{aligned} \frac{\partial s^t}{\partial t} &= -\int_{\Omega_0\times\Omega_1} \frac{K(s^t(x),\xi)}{\mu_0(s^t(x))\mu_0(\xi)}\cdot\Phi_x(\xi,\eta)\,\mathrm{d}\gamma^t(\xi,\eta) \\ &= -\int_{\Omega_0\times\Omega_1} \frac{K(s^t(x),s^t(\xi))}{\mu_0(s^t(x))\mu_0(s^t(\xi))}\cdot\Phi_x(s^t(\xi),\eta)\,\mathrm{d}\gamma^0(\xi,\eta), \end{aligned} \tag{30}$$

where we have used $\gamma^t = (s^t \times \mathrm{id})_\# \gamma^0$, with $\gamma^0$ the initial measure.

**3.6. An alternative steepest descent flow.** We can also derive a related flow in the following manner. Instead of using the Helmholtz projection to get a divergence-free vector field out of $W^t$, as we did in section 3.2, we set

$$\mu_0 v^t = \nabla \mathrm{div} W^t - \Delta W^t. \tag{31}$$

It is straightforward to check that in this case $\mathrm{div}(\mu_0 v^t) = 0$ and

$$\begin{aligned} M_t &= -\int_{\Omega_0} W^t\cdot\mu_0 v^t\,\mathrm{d}x \\ &= -\int_{\Omega_0} W^t\cdot\left(\nabla(\nabla\cdot W^t) - \Delta W^t\right)\,\mathrm{d}x \\ &= -\int_{\Omega_0} (W^t)^k((W^t)^l_{lk} - (W^t)^k_{ll})\,\mathrm{d}x, \end{aligned}$$

where we've used superscripts to denote vector components and subscripts for spatial derivatives, with the standard convention of summation over repeated indices. Integrating by parts, and ignoring the boundary for the sake of exposition, gives

$$\begin{aligned} M_t &= -\int_{\Omega_0} -(W^t)^k_l((W^t)^l_k - (W^t)^k_l)\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\Omega_0} ((W^t)^l_k - (W^t)^k_l)^2\,\mathrm{d}x \\ &= -\frac{1}{2}\int_{\Omega_0} |\,\mathrm{curl}\,W^t|^2\,\mathrm{d}x \\ &\le 0. \end{aligned} \tag{32}$$

If the measures $\gamma^t$ are of the form $\gamma^t = (\mathrm{id}\times u^t)_\#(\mu_0)$, then we have $W^t = \Phi_x(x, u^t(x))$, resulting in the evolution equation

$$\frac{\partial u^t}{\partial t} = -\frac{1}{\mu_0(x)}\big(\nabla \mathrm{div}\Phi_x(x,u^t) - \Delta\Phi_x(x,u^t)\big)\cdot\nabla u^t \tag{33}$$

for $u^t$ corresponding to (31), and (32) shows that at optimality we must again have curl $W^t = 0$, so $W^t = \nabla p$ for some function $p$.

For the quadratic cost function $\Phi(x,\xi) = \frac{1}{2}|x-\xi|^2$ we have $\Phi_x(x,\xi) = x - \xi$, so we get the following PDE:

$$\frac{\partial u^t}{\partial t} = \frac{1}{\mu_0(x)}\big(\nabla(\nabla\cdot u^t) - \Delta u^t\big)\cdot\nabla u^t.$$

We plan to study this equation in future work.

**4. Weak solutions.** Let $\gamma^t = (s^t\times\mathrm{id})_\#\gamma^0$ for some smooth family of diffeomorphisms $s^t : \bar\Omega_0 \to \bar\Omega_0$, whose velocity field satisfies (20).

In section 3.3 we observed that for any test function $\varphi \in C^1(\bar\Omega_0\times\bar\Omega_1)$ one has

$$\frac{d}{dt}\left\langle\gamma^t,\varphi\right\rangle = \left\langle\gamma^t, v^t\cdot\nabla_x\varphi\right\rangle.$$

Using (20) we get

$$\frac{d}{dt}\left\langle\gamma^t,\varphi\right\rangle = \left\langle\gamma^t, \frac{-1}{\mu_0(x)}\varphi_x\cdot\mathcal{P}\mathcal{A}^2\mathcal{P}\big(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\big)\right\rangle,$$

which implies

$$\begin{aligned}\frac{d}{dt}\left\langle\gamma^t,\varphi\right\rangle &= \big(\mathbb{E}_{\gamma^t}(\varphi_x\mid x), \mathcal{P}\mathcal{A}^2\mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x\mid x)\big)_{\mathfrak{H}} \\ &= \big(\mathcal{A}\mathcal{P}\mathbb{E}_{\gamma^t}(\varphi_x\mid x), \mathcal{A}\mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x\mid x)\big)_{\mathfrak{H}}.\end{aligned} \tag{34}$$

Integrate this in time, and you get

$$\int_{t_0}^{t_1}\big(\mathcal{A}\mathcal{P}\mathbb{E}_{\gamma^t}(\varphi_x\mid x), \mathcal{A}\mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x\mid x)\big)_{\mathfrak{H}}\,\mathrm{d}t = \left\langle\gamma^{t_0},\varphi\right\rangle - \left\langle\gamma^{t_1},\varphi\right\rangle. \tag{35}$$

For any measure $\gamma\in\mathfrak{X}$ and any $\varphi\in C^1(\bar\Omega_0\times\bar\Omega_1)$ the functions $\mathbb{E}_\gamma(\varphi_x\mid x)$ are bounded and measurable, and hence $\mathbb{E}_{\gamma^t}(\varphi_x\mid x)\in\mathfrak{H} = L^2(\Omega_0;\mathbb{R}^d)$ will always hold. Since $\mathcal{P}$ and $\mathcal{A}$ are bounded operators on $\mathfrak{H}$, both sides of the equation in (35) are defined for any weak* continuous family of measures $\gamma^t\in\mathfrak{X}$.

Definition 4.1 (weak solution). *A* weak solution *to the initial value problem* (24) *is a map* $t\in[0,T)\mapsto\gamma^t\in\mathfrak{X}$ *which is weak** *continuous, and which satisfies* (35) *for all test functions* $\varphi\in C^1(\bar\Omega_0\times\bar\Omega_1)$ *and for all* $0\le t_0<t_1<T$.

If $\{\gamma^t, 0\le t<T\}$ is a weak solution, then (35) implies that $\langle\gamma^t,\varphi\rangle$ is an absolutely continuous function of $t$ and that (34) holds for almost all $t$.

We could also introduce the notion of *classical solution* by requiring a classical solution to be a family of measures $\{\gamma^t, t\in[0,T)\}$ which is of the form $\gamma^t = (s^t\times\mathrm{id})_\#\gamma^0$ for some family of $C^1$ diffeomorphisms $s^t:\Omega_0\to\Omega_0$ whose velocity field $v^t = (\partial_t s^t)\circ(s^t)^{-1}$ satisfies $\mu_0 v^t = -\mathcal{P}\mathcal{A}^2\mathcal{P}\left\{\mathbb{E}_{\gamma^t}(\Phi_x\mid x)\right\}$.

LEMMA 4.2. *If the kernel $K(x,\xi)$ of the operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is $C^1$, and if $\{\gamma^t, 0 \le t < T\}$ is a weak solution, then there is a $C^1$ family of diffeomorphisms $s^t : \bar{\Omega}_0 \to \bar{\Omega}_0$ such that $\gamma^t = (s^t \times \mathrm{id})_\# \gamma^0$; i.e., $\{\gamma^t\}$ is a classical solution.*

*Proof.* The velocity field $v^t$ defined by the first line in (30), i.e.,

$$v^t(s) = -\int_{\Omega_0 \times \Omega_1} \frac{K(s,\xi)}{\mu_0(s)\mu_0(\xi)} \cdot \Phi_x(\xi,\eta)\,\mathrm{d}\gamma^t(\xi,\eta),$$

is $C^1$ in $s \in \bar{\Omega}_0$. Therefore the ODE $\dot{s} = v^t(s)$ defines a unique family of diffeomorphisms $s^t$, $0 \le t < T$, with $s^0(x) \equiv x$. We now verify that $\gamma^t = (s^t \times \mathrm{id})_\# \gamma^0$.

Consider the measure $\lambda^t = \big((s^t)^{-1} \times \mathrm{id}\big)_\# \gamma^0$. We have $\gamma^t = (s^t \times \mathrm{id})_\# \lambda^t$, and $\lambda^0 = \gamma^0$. We will show that $\lambda^t$ is constant.

For any test function $\varphi \in C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)$ we differentiate

$$\frac{d}{dt}\left\langle \lambda^t, \varphi \circ (s^t \times \mathrm{id})\right\rangle = \frac{d}{dt}\left\langle \gamma^t, \varphi\right\rangle$$

(using the fact that $\gamma^t$ is a weak solution)

$$\begin{aligned}
&= \left\langle \gamma^t, \varphi_x(x,y) \cdot v^t(x)\right\rangle \\
&= \left\langle \gamma^t, \frac{\partial \varphi \circ (s^t \times \mathrm{id})}{\partial t} \circ ((s^t)^{-1} \times \mathrm{id})\right\rangle \\
&= \left\langle \lambda^t, \frac{\partial}{\partial t}\big(\varphi \circ (s^t \times \mathrm{id})\big)\right\rangle.
\end{aligned}$$

On the other hand we also have

$$\frac{d}{dt}\left\langle \lambda^t, \varphi \circ (s^t \times \mathrm{id})\right\rangle = \left\langle \frac{\partial \lambda^t}{\partial t}, \varphi \circ (s^t \times \mathrm{id})\right\rangle + \left\langle \lambda^t, \frac{\partial}{\partial t}\big(\varphi \circ (s^t \times \mathrm{id})\big)\right\rangle.$$

We see that $\langle \partial_t \lambda^t, \varphi \circ (s^t \times \mathrm{id})\rangle$ vanishes for arbitrary test functions $\varphi$. Since $s^t$ is $C^1$, this implies that $\tilde{\varphi} = \varphi \circ (s^t \times \mathrm{id})$ can also be any $C^1$ test function, and we conclude that $\partial_t \lambda^t = 0$.

Since $\lambda^0 = \gamma^0$, we get $\lambda^t = \gamma^0$ for all $t$, and finally, $\gamma^t = (s^t \times \mathrm{id})_\# \lambda^t = (s^t \times \mathrm{id})_\# \gamma^0$, as claimed. ◻

**5. General energy bounds.** By setting $\varphi = \Phi$ in (35) we get the following.

LEMMA 5.1 (energy identity). *For any weak solution $\{\gamma^t, t \in [0,T)\}$ and any $0 \le t_0 < t_1 < T$ one has*

$$M(\gamma^{t_1}) + \int_{t_0}^{t_1} \|\mathcal{A}\mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\|_{\mathfrak{H}}^2 \,\mathrm{d}t = M(\gamma^{t_0}).$$

This immediately leads to the following lemma.

LEMMA 5.2. *For any weak solution $\{\gamma^t, t \in [0,T)\}$ the Monge–Kantorovich cost functional is nonincreasing. It remains constant if and only if $\mathcal{P}\mathbb{E}_{\gamma^t}(\Phi_x \mid x) = 0$ for almost all $t \in [0,T)$, i.e., if and only if*

$$\mathbb{E}_{\gamma^t}(\Phi_x \mid x) = \nabla p^t$$

*for some function $p^t : \Omega_0 \to \mathbb{R}$ and almost all $t \in [0,T)$.*

*Proof.* It is clear that $M(\gamma^t)$ cannot increase. If $M(\gamma^{t_1}) = M(\gamma^{t_0})$ for certain $t_0 < t_1$, then (35) implies that $\mathcal{A}\mathcal{P}\left(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\right) = 0$ for almost all $t_0 < t < t_1$. Since we assume the smoothing operator $\mathcal{A}$ is injective, this forces $\mathcal{P}\left(\mathbb{E}_{\gamma^t}(\Phi_x \mid x)\right) = 0$. ◻

LEMMA 5.3 (uniform Lipschitz bound). *If $\{\gamma^t, 0 \le t < T\}$ is a weak solution to* (24)*, then for any test function $\varphi \in C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)$ the function $t \mapsto \langle \gamma^t, \varphi\rangle$ is Lipschitz continuous, with*

$$\left|\frac{d\left\langle \mu_n^t, \varphi\right\rangle}{dt}\right| \le \|\mathcal{A}\|^2_{L(\mathfrak{H})} \left\|\frac{\partial \varphi}{\partial x}\right\|_{L^\infty} \left\|\frac{\partial \Phi}{\partial x}\right\|_{L^\infty}. \tag{36}$$

One could formulate this lemma as follows: weak solutions $\gamma^t$ are uniformly Lipschitz continuous functions of $t$ with values in $\left(C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)\right)^*$ (the dual of $C^1$ functions on $\bar{\Omega}_0 \times \bar{\Omega}_1$), with Lipschitz constant depending only on the smoothing operator $\|\mathcal{A}\|$ and the cost function $\Phi$.

*Proof.* This follows directly from (34) and the fact that for almost all $x \in \Omega_0$

$$|\mathbb{E}_\gamma(f(x,y) \mid x)| \le \operatorname*{ess\,sup}_{y \in \Omega_1} |f(x,y)|$$

for any $f \in L^\infty(\Omega_0 \times \Omega_1)$. ◻

LEMMA 5.4 (equicontinuity). *For any $\varphi \in C^0(\bar{\Omega}_0 \times \bar{\Omega}_1)$ there is a modulus of continuity $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ which depends only on $\varphi$, $\|\mathcal{A}\|_{L(\mathfrak{H})}$, $\|\Phi_x\|_{L^\infty}$, and the total mass $\mu_0(\Omega_0)$ such that*

$$\left|\left\langle \gamma^{t_1}, \varphi\right\rangle - \left\langle \gamma^{t_0}, \varphi\right\rangle\right| \le \sigma(|t_1 - t_0|).$$

*Proof.* For test functions $\tilde{\varphi} \in C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)$ the previous lemma gives us a uniform Lipschitz bound. We now approximate our given $\varphi \in C^0$ by a $\tilde{\varphi} \in C^1$ and compute

$$\begin{aligned}
\left|\left\langle \gamma^{t_1}, \varphi\right\rangle - \left\langle \gamma^{t_0}, \varphi\right\rangle\right| &= \left|\left\langle \gamma^{t_1} - \gamma^{t_0}, \varphi\right\rangle\right| \\
&\le \left|\left\langle \gamma^{t_1} - \gamma^{t_0}, \tilde{\varphi}\right\rangle\right| + \left|\left\langle \gamma^{t_1} - \gamma^{t_0}, \tilde{\varphi} - \varphi\right\rangle\right| \\
&\le C\|\tilde{\varphi}_x\|_\infty \delta + 2\mu_0(\Omega_0)\|\varphi - \tilde{\varphi}\|_\infty,
\end{aligned}$$

where $C = \|\mathcal{A}\|^2\|\Phi_x\|_\infty$, and where we have used the fact that all measures $\gamma^t$ have the same total mass $\gamma^t(\Omega_0 \times \Omega_1) = \mu_0(\Omega_0)$ (see (12)) to estimate the second term.

Thus we see that the modulus of continuity $\sigma$ is given by

$$\sigma(\delta) = \inf_{\tilde{\varphi} \in C^1} \left\{\|\mathcal{A}\|^2\|\Phi_x\|_\infty\|\tilde{\varphi}_x\|_\infty\delta + 2\mu_0(\Omega_0)\|\varphi - \tilde{\varphi}\|_\infty\right\}.$$

Clearly $\sigma(\delta)$ is monotone in $\delta > 0$, and $\lim_{\delta\to 0}\sigma(\delta) = 0$, which makes $\sigma$ a modulus of continuity. ◻

We note that the modulus of continuity $\sigma(\delta)$ is actually bounded by

$$\sigma(\delta) \le C \sup_{|x-x'|+|y-y'|<\delta} |\varphi(x,y) - \varphi(x',y')|,$$

where $C$ depends only on $\|A\|_{L(\mathfrak{H})}$ and $\|\Phi_x\|_\infty$.

**6. Weak compactness.** In this section, we study limits of sequences of weak solutions.

LEMMA 6.1. *Let $\mathcal{A}_n$ be a sequence of operators satisfying* (21) *and let $\Phi_n \in C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)$ be a sequence of cost functions. Assume the $\mathcal{A}_n$ are uniformly bounded in $L(\mathfrak{H})$ and the $\Phi_n$ are uniformly bounded in $C^1(\bar{\Omega}_0 \times \bar{\Omega}_1)$.*

*Given a family of weak solutions* $\{\gamma_n^t, t \in [0,T)\}$, *to* (24) *with* $\mathcal{A} = \mathcal{A}_n$, $\Phi = \Phi_n$, *there is a subsequence such that* $\gamma_{n_k}^t \rightharpoonup \gamma_\infty^t$ *for some weak* continuous family of measures* $\{\gamma_\infty^t, t \in [0,T)\}$.

*Proof.* Given any $t \in [0,T)$ weak* compactness of $\mathfrak{X}$ enables us to find a subsequence of $\{\gamma_n^t\}$ which weak* converges. Given a finite subset $\{t_1, \dots, t_m\} \subset [0,T)$ we can repeat this argument $m$ times and obtain a subsequence $n_k \in \mathbb{N}$ such that $\mu_{n_k}^t$ weak* converges for $t = t_1, t_2, \dots, t_m$. A diagonalization trick then gives us a further subsequence $n_k \in \mathbb{N}$ such that $\mu_{n_k}^t$ weak* converges for all rational $t \in [0,T)$.

We now argue that this subsequence $\gamma_{n_k}^t$ actually weak* converges for *all* $t \in [0,T)$ rather than just for all rational $t$.

Let $\varphi \in C^0(\bar{\Omega}_0 \times \bar{\Omega}_1)$ and some $t \in [0,T)$ be given. By Lemma 5.4 the functions $t \mapsto \langle \gamma_{n_k}^t, \varphi \rangle$ are equicontinuous. By Ascoli–Arzelà they form a precompact subset of $C^0([0,T))$. Since they converge pointwise on a dense subset of the interval $[0,T)$ they must converge uniformly for $0 \le t < T$.

It follows that $\langle \gamma_{n_k}^t, \varphi \rangle$ converges for all $t \in [0,T)$, as claimed.

To complete the proof we check weak* continuity in time of the limit measures $\gamma^t$. But this is immediate since the $\mu_n^t$ all share the same modulus of continuity from Lemma 5.4. Passing to the limit we find that $\gamma^t$ also has this modulus of continuity. ∎

PROPOSITION 6.2. *Let* $\gamma_n^t$, $\mathcal{A}_n$, *and* $\Phi_n$ *be as in the previous lemma.*

*Assume that the operators* $\mathcal{A}_n$ *converge strongly to some operator* $\mathcal{A}_\infty$. *Assume also that the cost functions* $\Phi_n$ *converge in* $C^1$ *to some* $\Phi_\infty \in C^1$.

*Then* $W_n^t \stackrel{\text{def}}{=} \mathbb{E}_{\gamma_n^t}(\Phi_{n,x} \mid x)$ *weak* converges in* $L^\infty(\Omega_0; \mathbb{R}^d) = L^1(\Omega_0; \mathbb{R}^d)^*$ *to* $W_\infty^t \stackrel{\text{def}}{=} \mathbb{E}_{\gamma_\infty^t}(\Phi_{\infty,x} \mid x)$.

*The limiting family* $\gamma_\infty^t$ *satisfies the energy inequality*

$$M(\gamma^{t_1}) + \int_{t_0}^{t_1} \left\| \mathcal{A}_\infty \mathcal{P} W_\infty^t \right\|_{L^2}^2 \,\mathrm{d}t \le M(\gamma^{t_0}) \tag{37}$$

*for all* $0 \le t_0 < t_1 < T$.

*Proof.* For any $\zeta \in L^1(\Omega_0)$ we have

$$\begin{aligned}
\int W_n^t(x) \cdot \zeta(x) \mathrm{d}\mu_0(x) &= \iint_{\Omega_0 \times \Omega_1} \zeta(x) \cdot \frac{\partial \Phi_n(x,y)}{\partial x} \mathrm{d}\mu_n^t(x,y) \\
&\to \iint_{\Omega_0 \times \Omega_1} \zeta(x) \cdot \frac{\partial \Phi_\infty(x,y)}{\partial x} \mathrm{d}\mu_\infty^t(x,y) \qquad \text{as } n \to \infty \\
&= \int W_\infty^t(x) \cdot \zeta(x) \mathrm{d}\mu_0(x),
\end{aligned}$$

which establishes the weak* convergence of the $W_n^t$.

Since $M(\gamma_n^t) = \langle \gamma_n^t, \Phi_n \rangle$ weak* convergence of the measures $\gamma_n^t$ directly implies convergence of the corresponding costs,

$$\lim_{n \to \infty} M(\gamma_n^t) = M(\gamma_\infty^t),$$

for all $t \in [0,T)$.

To prove the energy inequality (37) we need the following.

LEMMA 6.3. $\mathcal{A}_n \mathcal{P} W_n^t$ *converges weakly to* $\mathcal{A}_\infty \mathcal{P} W_\infty^t$ *in* $\mathfrak{H}$, *and hence*

$$\|\mathcal{A}_\infty \mathcal{P} W_\infty^t\| \le \liminf_{n \to \infty} \|\mathcal{A}_n \mathcal{P} W_n^t\|.$$

Given this lemma, we use Fatou's lemma to get

$$\int_{t_0}^{t_1} \left\|\mathcal{A}\mathcal{P}W_\infty^t\right\|_{L^2}^2 \,\mathrm{d}t \le \liminf_{k\to\infty} \int_{t_0}^{t_1} \left\|\mathcal{A}\mathcal{P}W_k^t\right\|_{L^2}^2 \,\mathrm{d}t.$$

The energy identity in Lemma 5.1 then directly leads to the energy inequality (37).

It therefore remains only to verify Lemma 6.3. To this end we recall that the $W_n^t$ converge in the weak* topology on $L^\infty(\Omega_0)\otimes\mathbb{R}^d$ and hence converge weakly in $L^2(\Omega_0)\otimes\mathbb{R}^d = \mathfrak{H}$.

The Helmholtz projection $\mathcal{P}$ is bounded on $\mathfrak{H}$, so $\mathcal{P}W_n^t$ converges weakly to $\mathcal{P}W_\infty^t$.

The operators $\mathcal{A}_n$ converge strongly to $\mathcal{A}_\infty$, so for an arbitrary $f \in \mathfrak{H}$ we have $\|\mathcal{A}_n f - \mathcal{A}_\infty f\|_{\mathfrak{H}} \to 0$.

Altogether this gives us

$$(f, \mathcal{A}_n\mathcal{P}W_n^t)_{\mathfrak{H}} = (\mathcal{A}_n f, \mathcal{P}W_n^t)_{\mathfrak{H}} \to (\mathcal{A}_\infty f, \mathcal{P}W_\infty^t)_{\mathfrak{H}} = (f, \mathcal{A}_\infty\mathcal{P}W_\infty^t)_{\mathfrak{H}}$$

as $n\to\infty$ and for arbitrary $f\in\mathfrak{H}$; i.e., we find that $\mathcal{A}_n\mathcal{P}W_n^t$ converges weakly to $\mathcal{A}_\infty\mathcal{P}W_\infty^t$, and we are done. ∎

At this point it is not clear whether the limiting family $\{\gamma_\infty^t, t\in[0,T)\}$ is a weak solution.

Lemma 6.4. *Assume that the integral kernel $K(x,\xi)$ of the operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is a continuous function on $\bar\Omega_0\times\bar\Omega_1$.*

*Then any weak* limit $\{\gamma_\infty^t\}$ of weak solutions $\{\gamma_k^t\}$ is again a weak solution.*

*Proof.* Since the $\gamma_k^t$ weak* converge to $\gamma_\infty^t$, the product measures $\gamma_k^t\times\gamma_k^t$ also weak* converge[2] to $\gamma_\infty^t\times\gamma_\infty^t$.

The measures $\gamma_n^t$ are uniformly bounded (their total mass is fixed by (12)), so, using the dominated convergence theorem, one easily shows that the Borel measures $\mathrm{d}\gamma_n^t\times\mathrm{d}\gamma_n^t\times\mathrm{d}t$ on $\bar\Omega_0\times\bar\Omega_0\times[0,T)$ converge weakly to $\mathrm{d}\gamma_\infty^t\times\mathrm{d}\gamma_\infty^t\times\mathrm{d}t$.

To prove that $\gamma_\infty^t$ is a weak solution to (24) we must show that $\gamma_\infty^t$ satisfies (35) for all test functions $\varphi$. Using the integral kernel $K(x,\xi)$ we rewrite (35) as

$$\int_{t_0}^{t_1}\int_{\Omega_0} \mathbb{E}_{\gamma^t}(\varphi_x \mid x)\cdot\mathcal{P}\mathcal{A}^2\mathcal{P}\{\mathbb{E}_{\gamma^t}(\Phi_x\mid x)\}\,\mathrm{d}x\,\mathrm{d}t = \left\langle\gamma^{t_0},\varphi\right\rangle - \left\langle\gamma^{t_1},\varphi\right\rangle,$$

i.e.,

$$\int_{t_0}^{t_1}\iint_{\Omega_0\times\Omega_0} \mathbb{E}_{\gamma^t}(\varphi_x\mid x)\cdot K(x,\xi)\cdot\mathbb{E}_{\gamma^t}(\Phi_x\mid\xi)\,\mathrm{d}x\mathrm{d}\xi\,\mathrm{d}t = \left\langle\gamma^{t_0},\varphi\right\rangle - \left\langle\gamma^{t_1},\varphi\right\rangle.$$

Using the definition of $\mathbb{E}_{\gamma^t}(\cdot\mid\cdot)$ we see that (35) is equivalent to

$$\int_{t_0}^{t_1}\iint_{(\Omega_0\times\Omega_1)^2} \frac{\varphi_x(x,y)\cdot K(x,\xi)\cdot\Phi_x(\xi,\eta)}{\mu_0(x)\mu_0(\xi)}\,\mathrm{d}\gamma^t(x,y)\mathrm{d}\gamma^t(\xi,\eta)\,\mathrm{d}t = \left\langle\gamma^{t_0},\varphi\right\rangle - \left\langle\gamma^{t_1},\varphi\right\rangle. \tag{38}$$

[2]If Borel measures $\mu_n$ and $\nu_n$ on compact Hausdorff spaces $X$ and $Y$, respectively, converge weakly to measures $\gamma$ and $\nu$, then the product measures $\mu_n\times\nu_n$ converge weakly to $\gamma\times\nu$. Indeed, the $\mu_n\times\nu_n$ are uniformly bounded so one only has to check $\langle\mu_n\times\nu_n,f\rangle\to\langle\gamma\times\nu,f\rangle$ for a dense set of $f\in C^0(X\times Y)$. By the Stone–Weierstraß theorem we may therefore assume that $f(x,y) = g_1(x)h_1(y)+\cdots+g_k(x)h_k(y)$ for continuous functions $g_i$ and $h_i$. By linearity we may assume that $k=1$. But if $f(x,y)=g_1(x)h_1(y)$, then $\langle\mu_n\times\nu_n,f\rangle = \langle\mu_n,g_1\rangle\langle\nu_n,h_1\rangle$, which converges to $\langle\gamma,g_1\rangle\langle\nu,h_1\rangle = \langle\gamma\times\nu,f\rangle$.

All measures $\gamma_n^t$ satisfy (38) since they are weak solutions. Our hypotheses are such that the integrand in the triple integral in (38) is a continuous function for any choice of the test function $\varphi$. Weak* convergence of the measures $\mathrm{d}\gamma_n^t \times \mathrm{d}\gamma_n^t \times \mathrm{d}t$ then allows us to complete the proof by passing to the limit in (35). ∎

**7. The Helmholtz decomposition.** Any vector field $w : \Omega_0 \to \mathbb{R}^d$ can be decomposed into a divergence-free part and a gradient; i.e., one can find a vector field $\mathcal{P}w$ and a function $p = p_w$ such that

$$w = \mathcal{P}w + \nabla p, \qquad \operatorname{div} \mathcal{P}w = 0 \tag{39}$$

holds. We call $\mathcal{P}w$ the *Helmholtz projection* of $w$, and by analogy with fluid dynamics we will call $p_w$ the corresponding pressure. The pressure $p_w$ is determined by (39) up to an additive constant, at best. We can remove this freedom by imposing some normalization on $p_w$, such as

$$\int_{\Omega_0} p_w(x)\,\mathrm{d}x = 0.$$

To uniquely specify $\mathcal{P}w$ and $p_w$ we must impose boundary conditions: we will always require $\mathcal{P}w$ to be tangential to the boundary. Thus if $\nu$ denotes the outward unit normal to $\partial\Omega_0$, then we require

$$\nu \cdot \nabla(\mathcal{P}w) = 0, \quad \text{or, equivalently,} \quad \nu \cdot \nabla p_w = \nu \cdot w \quad \text{on } \partial\Omega_0. \tag{40}$$

A brief construction of $\mathcal{P}w$ uses Hilbert space theory. Indeed, let $\mathfrak{H}_{\mathrm{div}}$ be the closed subspace of $\mathfrak{H} = L^2(\Omega; \mathbb{R}^d)$ determined by

$$\mathfrak{H}_{\mathrm{div}} = \{w \in \mathfrak{H} \mid (w, \nabla\varphi)_{\mathfrak{H}} = 0 \text{ for all } \varphi \in C^1(\bar{\Omega})\}.$$

Then the Helmholtz projection $\mathcal{P}$ is simply the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_{\mathrm{div}}$. This implies the following.

LEMMA 7.1. *The operator $\mathcal{P}$ is bounded on $\mathfrak{H}$, with $\|\mathcal{P}\|_{L(\mathfrak{H})} = 1$.*

This construction does not show how $\mathcal{P}$ preserves smoothness of the vector field $w$. Therefore we now recall a different description of the Helmholtz decomposition.

**7.1. Smooth domains.** The defining equations (39) and (40) imply

$$\begin{cases} \Delta p_w = \operatorname{div} w \ \text{ on } \Omega_0, \\ \nu \cdot \nabla p_w = \nu \cdot w \text{ on } \partial\Omega_0. \end{cases} \tag{41}$$

Standard elliptic theory tells us that under minimal smoothness assumptions on $\partial\Omega_0$ and $w$ a solution in the weak sense exists for this boundary value problem. Moreover, the vector field $\mathcal{P}w$ defined by

$$\mathcal{P}w := w - \nabla p_w$$

is divergence free and tangential to the boundary.

LEMMA 7.2. *Assume the boundary $\partial\Omega_0$ is $C^{1,\alpha}$ smooth. Then the Helmholtz projection is a bounded operator on $C^{1,\alpha}(\Omega_0; \mathbb{R}^d)$; i.e., for any vector field $w \in C^{1,\alpha}(\Omega_0; \mathbb{R}^d)$ one has $\mathcal{P}w \in C^{1,\alpha}(\Omega_0; \mathbb{R}^d)$ and $\|\mathcal{P}w\|_{C^{1,\alpha}} \le C\|w\|_{C^{1,\alpha}}$.*

*Proof.* If $w \in C^{1,\alpha}$, then $w \in C^{0,\alpha}$, while $\nu \cdot w \in C^{1,\alpha}$. Furthermore the data satisfy

$$\int_{\partial\Omega_0} \nu \cdot w = \int_{\Omega_0} \operatorname{div} w,$$

so the boundary value problem (41) has a unique solution $p_w \in C^{2,\alpha}(\Omega_0)$ with $\int_{\Omega_0} p_w \, dx = 0$ (see [10]). One has

$$\begin{aligned}\|p_w\|_{C^{2,\alpha}} &\le C_{\alpha,\Omega} \left\{\|\nu \cdot \nabla p_w\|_{C^{1,\alpha}} + \|\Delta p_w\|_{C^\alpha}\right\} \\ &= C_{\alpha,\Omega} \left\{\|\nu \cdot w\|_{C^{1,\alpha}(\partial\Omega_0)} + \|\operatorname{div} w\|_{C^\alpha(\Omega_0)}\right\}\end{aligned}$$

for some constant $C_{\alpha,\Omega_0}$. The representation $\mathcal{P}w = w - \nabla p_w$ then implies the lemma. ◻

**7.2. Helmholtz decomposition on rectangles.** In the case that $\Omega_0$ is a rectangle, i.e., $\Omega_0 = [0, L_1] \times \cdots \times [0, L_d]$, we can give a more explicit representation of the Helmholtz projection by using Fourier series.

Assume for simplicity of notation that all sides of $\Omega_0$ have length $\pi$, i.e., $L_j = \pi$. Then one can write any $w \in \mathfrak{H}$ as a series,

$$w(x) = \sum_{j=1}^{d} \sum_{\ell_1,\dots,\ell_d \ge 0} \hat{w}_{j,\ell_1,\dots,\ell_d} \cos(\ell_1 x_1) \cdots \sin(\ell_j x_j) \cdots \cos(\ell_d x_d) \mathbf{e}_j. \tag{42}$$

Observe that due to the presence of the factor $\sin \ell_j x_j$ the term with $\ell = 0$, i.e., with $\ell_1 = \cdots = \ell_d = 0$, is absent from the sum. We will not try to incorporate this fact into our notation, but it will allow us to divide by $|\ell|$ in what follows.

The $L^2(\Omega_0; \mathbb{R}^d) = \mathfrak{H}$-norm of such a vector field is

$$\|w\|_{\mathfrak{H}}^2 = \left(\frac{\pi}{2}\right)^d \sum_{j=1}^{d} \sum_{\ell_1,\dots,\ell_d \ge 0} 2^{\mathsf{C}_\ell} \left|\hat{w}_{j,\ell_1,\dots,\ell_d}\right|^2, \tag{43}$$

where $\mathsf{C}_\ell$ denotes the number of components of $\ell = (\ell_1, \dots, \ell_d)$ which vanish.

Any $L^2$ vector field given by (42) extends to a vector field on all of $\mathbb{R}^d$ which is $2\pi$ periodic in each of the variables $x_1, \dots, x_d$. Moreover, any $w$ given by (42) has the symmetry

$$w(R_j x) = R_j\big(w(x)\big) \quad \text{for } j = 1, \dots, d, \tag{44}$$

in which $R_j$ is the reflection $R_j(x_1, \dots, x_d) = (x_1, \dots, x_{j-1}, -x_j, x_{j+1}, \dots, x_d)$. See Figure 2. Conversely, any vector field $w \in L^2\big([-\pi, \pi]^d; \mathbb{R}^d\big)$ which has the symmetries (44) can be written as a Fourier series of the form (42).

The Helmholtz projection of $w$ is then given by

$$\mathcal{P}w(x) = \sum_{j=1}^{d} \sum_{\ell_1,\dots,\ell_d \ge 0} \widehat{(\mathcal{P}w)}_{j,\ell_1,\dots,\ell_d} \cos(\ell_1 x_1) \cdots \sin(\ell_j x_j) \cdots \cos(\ell_d x_d) \mathbf{e}_j, \tag{45}$$

with

$$\widehat{(\mathcal{P}w)}_{j,\ell} = \sum_{k=1}^{d} \left(\delta_{jk} - \frac{\ell_j \ell_k}{|\ell|^2}\right) \hat{w}_{j,\ell}$$

and where $|\ell|^2 = \ell_1^2 + \cdots + \ell_d^2$. The corresponding "pressure" is given by

$$p_w = \sum_{|\ell| > 0} \widehat{(p_w)}_\ell \cos(\ell_1 x_1) \cdots \cos(\ell_d x_d),$$

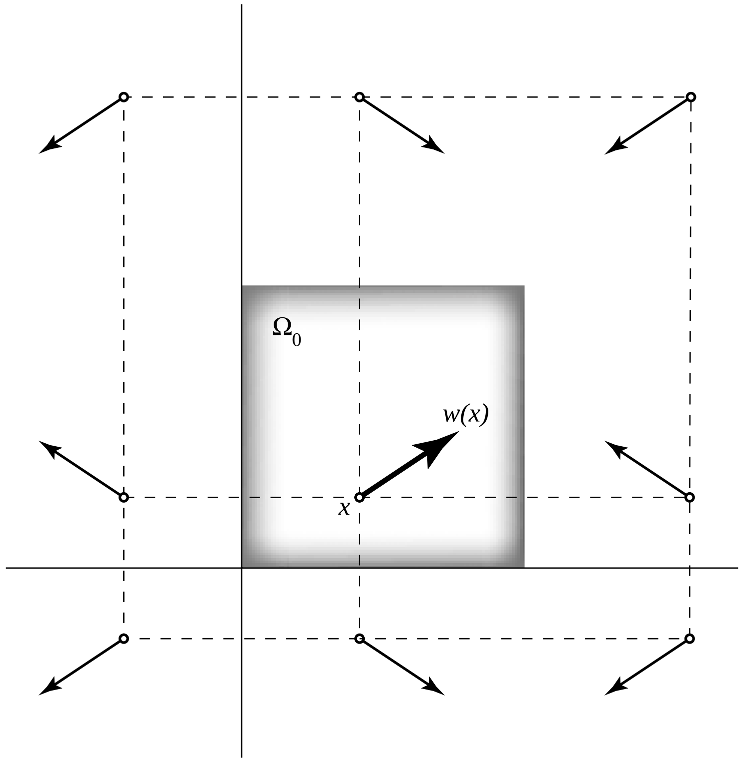



FIG. 2. *The symmetry* (44).

with

$$\widehat{(p_w)}_\ell = \frac{\ell_1 \hat{w}_{1,\ell} + \cdots + \ell_d \hat{w}_{d,\ell}}{|\ell|^2}.$$

Let $C^{1,\alpha}(\mathbb{T}^d;\mathbb{R}^d)$ be the space of all $C^{1,\alpha}$ vector fields on $\mathbb{R}^d$ which are $2\pi$ periodic in all variables $x_1, \dots, x_d$, and let $\mathfrak{C}^{1,\alpha}$ be the closed subspace of $C^{1,\alpha}(\mathbb{T}^d;\mathbb{R}^d)$ consisting of all vector fields which have the symmetry (44).

LEMMA 7.3. *The Helmholtz projection is a bounded operator on $C^{1,\alpha}(\mathbb{T}^d;\mathbb{R}^d)$ which leaves the subspace $\mathfrak{C}^{1,\alpha}$ invariant.*

*Proof.* We can write $\mathcal{P}$ on $C^{1,\alpha}(\mathbb{T}^d;\mathbb{R}^d)$ as $\mathcal{P} = \mathrm{I} - \mathrm{grad} \circ (\Delta)^{-1} \circ \mathrm{div}$, where for any $f$ with $\int_{[-\pi,\pi]^d} f \mathrm{d}x = 0$ we define $u = (\Delta)^{-1} f$ to be the unique solution of $\Delta u = f$ with $\int_{[-\pi,\pi]^d} u \mathrm{d}x = 0$. Classical Schauder estimates imply that $(\Delta)^{-1}$ is bounded from $C^{0,\alpha}$ to $C^{2,\alpha}$. This implies that $\mathcal{P} = \mathrm{I} - \mathrm{grad} \circ (\Delta)^{-1} \circ \mathrm{div}$ is bounded on $C^{1,\alpha}$ vector fields.

The Helmholtz decomposition is easily seen to commute with the symmetries (44), so that $\mathfrak{C}^{1,\alpha}$ is an invariant subspace. ∎

**8. The smoothing operator $\mathcal{A}$.** In this section, we exhibit smoothing operators $\mathcal{A}$ which satisfy all assumptions made so far.

**8.1. Smoothing vector fields on $C^{1,\alpha}$ domains.** Many different smoothing operators can be constructed. The following is one choice.

LEMMA 8.1. *Let $\Omega_0$ be a domain with $C^{1,\alpha}$ boundary, and let $\mathcal{A}_\varepsilon$ be the operator*

$$\mathcal{A}_\varepsilon = e^{\varepsilon\Delta_N}, \text{ i.e., } \mathcal{A}_\varepsilon w = \big(e^{\varepsilon\Delta_N} w_1, \dots, e^{\varepsilon\Delta_N} w_d\big),$$

*where $\Delta_N$ is the Neumann–Laplacian on $\Omega_0$.*

*Then $\mathcal{A}_\varepsilon$ is a bounded, injective, self-adjoint operator on $\mathfrak{H}$, with $\|A_\varepsilon\|_{L(\mathfrak{H})} \le 1$ for any $\varepsilon > 0$.*

*The operator $\mathcal{A}_\varepsilon$ is also bounded from $C^{1,\alpha}$ to $C^{1,\alpha}$, with $\|\mathcal{A}_\varepsilon\|_{L(C^{1,\alpha})} \le C$ for some $C$ that does not depend on $\varepsilon > 0$.*

*Proof.* The operator $\mathcal{A}_\varepsilon$ acts on each individual component $w_i$ of a vector field $w$ in the same way. The fact that $e^{\varepsilon\Delta_N}$ is a contraction of $L^2$ and uniformly bounded on $C^{1,\alpha}$ respectively follows from linear parabolic theory.

The Neumann Laplacian is well known to be a self-adjoint operator on $L^2(\Omega_0)$, so that $\mathcal{A}_\varepsilon$ is self-adjoint. Self-adjointness of $\Delta_N$ implies via the spectral theorem for self-adjoint operators that $e^{\varepsilon\Delta_N}$ is injective for all $\varepsilon > 0$. □

LEMMA 8.2. *Let $\mathcal{A}$ be as above.*

*The operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is bounded from $\mathfrak{H}$ to $C^{1,\alpha}(\Omega; \mathbb{R}^d)$ for any $0 < \alpha < 1$.*

*The operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ has an integral kernel $K \in C^{1,\alpha}(\bar\Omega_0 \times \bar\Omega_0)$.*

*Proof.* Boundedness of $\mathcal{A} : \mathfrak{H} \to C^{1,\alpha}$ follows from the smoothing property of the heat equation. (But the operator norm $\|\mathcal{A}\|_{L(\mathfrak{H}, C^{1,\alpha})}$ blows up as $\varepsilon \searrow 0$.) Since $\mathcal{P}$ is bounded on both $\mathfrak{H}$ and $C^{1,\alpha}$ it follows that $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is bounded from $\mathfrak{H}$ to $C^{1,\alpha}$.

To study the kernel of the operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ we write $\mathcal{P}\mathcal{A}^2\mathcal{P}$ as $T \circ T^*$, where $T = \mathcal{P}\mathcal{A}$, and show that $T$ has an integral kernel.

Let $\Gamma^y_\varepsilon(x)$ be the heat kernel for $\Delta_N$; i.e., for any function $\phi \in L^1(\Omega_0)$ one has

$$e^{\varepsilon\Delta_N} f(x) = \int_{\Omega_0} \phi(y)\Gamma^y_\varepsilon(x)\,\mathrm{d}y. \tag{46}$$

Then $(x, y) \mapsto \Gamma^y(x)$ is a $C^{1,\alpha}$ function.

We expand the vector-valued function $f$ into its components, $f(x) = f_1(x)\mathbf{e}_1 + \cdots + f_d(x)\mathbf{e}_d$, $f_1, \dots, f_d$ being scalar $L^2$ functions. From (46) we then get the following representation for $\mathcal{P}\mathcal{A}f$:

$$\mathcal{P}\mathcal{A}f = \sum_{j=1}^{d} \int_{\Omega_0} f_j(y)\mathcal{P}(\Gamma^y_\varepsilon \otimes \mathbf{e}_j)\,\mathrm{d}y.$$

Let $N^y_j(x)$ be the function $N^y_j = \mathcal{P}(\Gamma^y_\varepsilon \otimes \mathbf{e}_j)$. We get

$$\mathcal{P}\mathcal{A}f(x) = \int_{\Omega_0} \sum_{j=1}^{d} N^y_j(x) f_j(y)\,\mathrm{d}y,$$

which means that $\mathcal{P}\mathcal{A}$ is an integral operator with matrix-valued kernel

$$\mathsf{N}(x, y) = \big[N^y_1(x), \dots, N^y_d(x)\big];$$

i.e., the $j$th column of the matrix $\mathsf{N}(x, y)$ is $N^y_j(x)$.

For each $y \in \Omega_0$ we have $\Gamma^y_\varepsilon \in C^{1,\alpha}(\bar\Omega_0)$, so by Lemma 7.2 we get $\mathcal{P}\Gamma^y_\varepsilon \in C^{1,\alpha}(\bar\Omega_0; \mathbb{R}^d)$. Moreover, $\Gamma^y_\varepsilon \in C^{1,\alpha}(\bar\Omega_0; \mathbb{R}^d)$ depends $C^{1,\alpha}$ smoothly on $y$, so we see that the kernel $\mathsf{N}$ is a $C^{1,\alpha}$ function on $\bar\Omega_0 \times \bar\Omega_0$.

The operator $\mathcal{P}\mathcal{A}^2\mathcal{P} = (\mathcal{P}\mathcal{A})(\mathcal{P}\mathcal{A})^*$ must now also be an integral operator, and its kernel must be

$$K(x,\xi) = \int_{\Omega_0} \mathsf{N}(x,y)\mathsf{N}(\xi,y)^T \, \mathrm{d}y.$$

This is clearly again a $C^{1,\alpha}$ function on $\bar{\Omega}_0 \times \bar{\Omega}_0$. ◻

The operators $\mathcal{A}_\varepsilon$ do not preserve the boundary condition $n \cdot w = 0$, so they will not commute with the Helmholtz projection $\mathcal{P}$.

**8.2. Smoothing operator on a rectangle.** In the case that $\Omega_0$ is a rectangle, i.e., $\Omega_0 = [0,\pi]^d$, we can construct a different smoothing operator by using the Fourier series (42).

We define the smoothing operator $\mathcal{A}_\varepsilon$ by

$$\mathcal{A}_\varepsilon w = \left(e^{\varepsilon\Delta_1} w_1, \ldots, e^{\varepsilon\Delta_d} w_d\right),$$

in which $\Delta_j$ is the Laplacian with Neumann boundary conditions on the sides $x_j = 0$ and $x_j = \pi$, and Dirichlet boundary conditions on all other sides of the rectangle $\Omega_0$.

An equivalent description of $\mathcal{A}_\varepsilon$ goes like this: to compute $\mathcal{A}_\varepsilon w$ for some vector field $w$ on $\Omega_0 = [0,\pi]^d$ extend $w$ to a vector field $\tilde{w}$ on all of $\mathbb{R}^d$ by imposing the symmetries (44) and by requiring the extension to be $2\pi$ periodic in all variables. We then set $\mathcal{A}_\varepsilon = e^{\varepsilon\Delta}\tilde{w}$, in which $e^{\varepsilon\Delta}$ is the standard heat semigroup on $\mathbb{R}^d$; i.e., we have

$$\mathcal{A}_\varepsilon w(x) = \left(4\pi\varepsilon\right)^{-d/2} \int_{\mathbb{R}^d} e^{-|x-\xi|^2/4\varepsilon} \tilde{w}(\xi) \, \mathrm{d}\xi. \tag{47}$$

If $w$ is given by the Fourier series (42), then $\mathcal{A}_\varepsilon w$ is given by

$$\mathcal{A}_\varepsilon w(x) = \sum_{j=1}^{d} \sum_{\ell_1,\ldots,\ell_d \geq 0} \widehat{(\mathcal{A}_\varepsilon w)}_{j,\ell_1,\ldots,\ell_d} \cos(\ell_1 x_1) \cdots \sin(\ell_j x_j) \cdots \cos(\ell_d x_d) \mathbf{e}_j,$$

with

$$\widehat{(\mathcal{A}_\varepsilon w)}_{j,\ell} = e^{-\varepsilon|\ell|^2} \hat{w}_{j,\ell}. \tag{48}$$

Lemma 8.3. *The smoothing operators $\mathcal{A}_\varepsilon$ are uniformly bounded on $\mathfrak{H}$ and $\mathfrak{C}^{1,\alpha}$. They are self-adjoint and injective, and they commute with the Helmholtz projection.*

*Proof.* The statements concerning the behavior of the operators on the Hilbert space $\mathfrak{H}$ follow directly from the series expansion (42), the Fourier multiplier descriptions (45) and (47) of $\mathcal{P}$ and $\mathcal{A}_\varepsilon$, respectively, and the Plancherel identity (43).

The $C^{1,\alpha}$ bounds follow from the representation (47). ◻

Lemma 8.4. *Let $\mathcal{A} = \mathcal{A}_\varepsilon$ be as above.*

*The operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ is bounded from $\mathfrak{H}$ to $\mathfrak{C}^{1,\alpha}(\Omega;\mathbb{R}^d)$ for any* $0 < \alpha < 1$.

*The operator $\mathcal{P}\mathcal{A}^2\mathcal{P}$ has an integral kernel $K \in C^{1,\alpha}(\bar{\Omega}_0 \times \bar{\Omega}_0)$.*

*The kernel $K(x,\xi)$ satisfies*

$$K(R_j x, \xi) = R_j K(x,\xi), \qquad j = 1, \ldots, d. \tag{49}$$

*Proof.* This lemma is analogous to Lemma 8.2, and its proof proceeds along the same lines.

Boundedness from $\mathfrak{H}$ to $\mathfrak{C}^{1,\alpha}$ again follows from the smoothing property of the heat equation, i.e., of $e^{\varepsilon\Delta}$. The integral kernel is constructed in the same way, starting from the explicit representation (47) of $e^{\varepsilon\Delta}w$.

For any vector field $f \in \mathfrak{H}$ the smoothed-out projection $w = \mathcal{P}\mathcal{A}^2\mathcal{P}f$ belongs to $\mathfrak{C}^{1,\alpha}$. We therefore may conclude from $w(R_jx) \equiv R_jw(x)$ that

$$\int_{\Omega_0} K(R_jx,\xi)f(\xi)\,\mathrm{d}\xi = R_j\int_{\Omega_0} K(x,\xi)f(\xi)\,\mathrm{d}\xi$$

for all $x \in \mathbb{R}^d$, $j = 1, \ldots ,d$, and all $f \in \mathfrak{H}$. This implies (49). $\square$

**9. Existence and well-posedness for the regularized flow.** In this section, we construct classical solutions $\gamma^t$ of the initial value problem (24) by writing them as $\gamma^t = (s^t \times \mathrm{id})_\#(\gamma^0)$ and solving the initial value problem (30)

$$\left.\begin{aligned} &\frac{\partial s^t}{\partial t} = -\int_{\Omega_0\times\Omega_1} \frac{K(s^t(x), s^t(\xi))}{\mu_0(s^t(x))\mu_0(s^t(\xi))}\cdot\Phi_x(s^t(\xi),\eta)\,\mathrm{d}\gamma^0(\xi,\eta), && (50)\\ &s^0(x) = x \quad (x\in\Omega) && (51)\end{aligned}\right\}$$

for $s^t$.

THEOREM 9.1. *Let the cost function $\Phi$ be $C^1$. Assume also that the smoothing operator $\mathcal{A}$ is such that the kernel $K$ is $C^{1,\alpha}$. Then for any initial measure $\gamma^0 \in \mathfrak{X}$ the initial value problem* (50) *has a solution $\{s^t \in C^{1,\alpha}(\bar\Omega_0;\bar\Omega_0) : 0 \le t < \infty\}$.*

*If the cost function $\Phi$ is $C^2$, then the solution $\{s^t : t \geq 0\}$ is unique.*

We begin the proof by observing that (50) is of the form

$$\frac{\partial s^t}{\partial t} = \int_{\Omega_0\times\Omega_1} F(s^t(x), s^t(\xi);\xi,\eta)\,\mathrm{d}\gamma^0(\xi,\eta), \tag{52}$$

where the map $F : \Omega_0 \times \Omega_0 \times \Omega_0 \times \Omega_1 \to \mathbb{R}^d$ is given by

$$F(s,\sigma;\xi,\eta) = \frac{K(s,\sigma)\cdot\Phi_x(\sigma,\eta)}{\mu_0(s)\mu_0(\sigma)}. \tag{53}$$

If $\Omega_0 = [0,\pi]^d$ is a rectangle, then $K(s,\sigma)$ is defined for all $s \in \mathbb{R}^d$. We agree to extend $\mu_0(x)$ to be $2\pi$ periodic and even in each variable so that $F(s,\sigma;\xi,\eta)$ is also defined for $s \in \mathbb{R}^d$.

LEMMA 9.2.

(A) *The map $(s,\sigma;\xi,\eta) \mapsto F(s,\sigma;\xi,\eta)$ is continuous.*

(B) *$F(s,\sigma;\xi,\eta)$ is also $C^1$ in $s \in \Omega_0$, and the partial derivative $\frac{\partial F}{\partial s}$ is uniformly bounded:*

$$\left|\frac{\partial F}{\partial s}(s,\sigma;\xi,\eta)\right| \le C, \tag{54}$$

*with $C < \infty$ independent of $s$, $\sigma$, $\xi$, and $\eta$.*

(C1) *If $\partial\Omega_0$ is $C^{1,\alpha}$ smooth, and if $s \in \partial\Omega_0$, then $s\cdot F(s,\sigma,\xi,\eta) = 0$ for all $\sigma,\xi \in \Omega_0$ and $\eta \in \Omega_1$.*

(C2) *If $\Omega_0 = [0,\pi]^d$, then $F(s,\sigma;\xi,\eta)$ is $2\pi$ periodic in each component of $s = (s_1,\ldots,s_d)$, and $s \mapsto F(s,\sigma;\xi,\eta)$ satisfies the symmetries* (44), *i.e.,*

$$F(R_js,\sigma;\xi,\eta) = R_jF(s,\sigma;\xi,\eta).$$

*Proof.* (A) and (B) are immediate from the representation (53), the known continuity and smoothness properties of the kernel $K$, and the cost function $\Phi$.

(C1). The kernel $K(x,\xi)$ satisfies $x \cdot K(x,\xi) = 0$, since for any vector field $w \in L^2(\Omega;\mathbb{R}^d)$ the vector field $\mathcal{P}\mathcal{A}^2\mathcal{P}w(x) = \int_{\Omega_0} K(x,\xi)w(\xi)d\xi$ is everywhere tangent to $\partial\Omega$. This implies

$$x \cdot \mathcal{P}\mathcal{A}^2\mathcal{P}w(x) = \int_{\Omega_0} x \cdot K(x,\xi) \cdot w(\xi)d\xi = 0$$

for arbitrary $w$, which can happen only if $x \cdot K(x,\xi) \equiv 0$. Equation (53) then implies (C1).

The kernel $K(s,\sigma)$ and the density $\mu_0(s)$ are periodic and have the appropriate symmetries, so (C2) follows immediately from (53). ◻

**9.1. Construction of a solution to (50).** We regard the initial value problem (50) as a fixed point problem for the map $\mathcal{F} : \sigma \mapsto s$, where $s = \mathcal{F}(\sigma)$ is the solution of the ODE

$$\frac{\partial s^t(x)}{\partial t} = \int_{\Omega_0} F(s^t(x), \sigma^t(\xi), \xi, \eta)\, \mathrm{d}\gamma^0(\xi,\eta),$$

with initial data $s^0 = \mathrm{id}$, i.e., $s^0(x) \equiv x$.

To set up a fixed point argument (and, in particular, to use the Brouwer–Leray–Schauder fixed point theorem) we must overcome a technical difficulty, namely, the space of maps $\{s^t : \Omega_0 \to \Omega_0, 0 \le t \le T\}$ is not a linear space, since the target $\Omega_0$ is not a vector space. To deal with this we extend the domain of the definition of the nonlinear map $F(s,\sigma;\xi,\eta)$ to include all $(s,\sigma,\xi,\eta) \in \mathbb{R}^d \times \mathbb{R}^d \times \Omega_0 \times \Omega_1$; in other words, we lift the restriction $s,\sigma \in \Omega_0$. Then we can regard maps $s^t : \Omega_0 \to \Omega_0$ as maps $s^t : \Omega_0 \to \mathbb{R}^d$, and the space of such maps (defined below as $\mathcal{C}_T$) is a vector space.

We only have to go through this extension process in the case where $\Omega_0$ is a smoothly bounded domain. When $\Omega_0$ is a rectangle the function $F(s,\sigma;\xi,\eta)$ is already defined for all $s,\sigma \in \mathbb{R}^d$.

**9.1.1. Extending *F*.** We choose a defining function $\varrho \in C^{1,\alpha}(\mathbb{R}^d)$ for $\Omega_0$. This means that $\Omega_0 = \{x \in \mathbb{R}^d : \varrho(x) > 0\}$ and $\nabla\varrho \ne 0$ on $\partial\Omega_0$. We can choose $\varrho$ so that $\nabla\varrho(x) \ne 0$ if $-1 \le \varrho(x) \le 1$, while $\varrho(x) = -2$ outside some compact set $K \supset \bar{\Omega}_0$.

Let $U = \{x \in \mathbb{R}^d : \varrho(x) \ge -1\}$, and choose a retraction $\pi : U \to \bar{\Omega}_0$. One possible choice is $\pi(x_0) = x_0$ for $x_0 \in \bar{\Omega}_0$, and

$$\pi(x_0) = \left\{ \begin{matrix} \text{the first point in } \bar{\Omega}_0 \text{ on the orbit } x(t) \text{ of} \\ \text{the gradient flow } \dot{x} = \nabla\varrho(x) \text{ which starts} \\ \text{at } x(0) = x_0 \in U \setminus \bar{\Omega}_0 \end{matrix} \right\}$$

for all $x_0 \in U \setminus \bar{\Omega}_0$. The retraction $\pi$ is Lipschitz continuous on $U$ and even $C^1$ on $U \setminus \partial\Omega_0$.

The extension $F_*$ of $F$ will now be defined in several stages. First we introduce a map $F_1 : U \times U \times \Omega_0 \times \Omega_1 \to \mathbb{R}^d$ given by

$$F_1(s,\sigma,\xi,\eta) = F(\pi(s),\pi(\sigma),\xi,\eta).$$

Next, let $\chi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz cutoff function, e.g., $\chi(t) = 1$ for $t \ge 0$, $1 + t$ for $-1 \le t \le 0$, and $0$ for $t \ge -1$. Put

$$F_2(s,\sigma,\xi,\eta) = \chi(\varrho(s))\chi(\varrho(\sigma))F_1(s,\sigma,\xi,\eta)$$

when $(s, \sigma) \in U \times U$, and $F_2(s, \sigma, \xi, \eta) = 0$ otherwise. Then $F_2 : \mathbb{R}^d \times \mathbb{R}^d \times \Omega_0 \times \Omega_1 \to \mathbb{R}^d$ is an extension of $F$ which is uniformly Lipschitz continuous in $(s, \sigma) \in \mathbb{R}^d \times \mathbb{R}^d$ and continuous in $\xi \in \Omega_0$, $\eta \in \Omega_1$.

Finally, we define $F_* : \mathbb{R}^d \times \mathbb{R}^d \times \Omega_0 \times \Omega_1 \to \mathbb{R}^d$ by setting

$$F_*(s, \sigma, \xi, \eta) = \begin{cases} F_2 & \text{for } s \in \bar{\Omega}_0, \\ F_2 - \dfrac{\nabla \varrho(s) \cdot F_2}{|\nabla \varrho(s)|^2} \nabla \varrho(s) & \text{for } s \in U \setminus \bar{\Omega}_0, \\ 0 & \text{for } s \in \mathbb{R}^d \setminus U, \end{cases} \tag{55}$$

where $F_2 = F_2(s, \sigma, \xi, \eta)$.

Lemma 9.3. *The extension $F_* : \mathbb{R}^d \times \mathbb{R}^d \times \Omega_0 \times \Omega_1 \to \mathbb{R}^d$ of $F$ is continuous in $(s, \sigma)$ and uniformly Lipschitz in $s \in \mathbb{R}^d$; i.e., for all $\sigma \in \mathbb{R}^d$ and $\xi \in \Omega_0$, $\eta \in \Omega_1$ one has*

$$|F_*(s, \sigma, \xi, \eta) - F_*(s', \sigma, \xi, \eta)| \le M|s - s'| \tag{56}$$

*for some $M < \infty$. Furthermore, $F_*$ is uniformly bounded,*

$$|F_*(s, \sigma, \xi, \eta)| \le M' \tag{57}$$

*for some constant $M' < \infty$ and for all $s, \sigma \in \mathbb{R}^d$ and $\xi \in \Omega_0$, $\eta \in \Omega_1$.*

*If the cost function $\Phi$ is $C^2$, then $F_*(s, \sigma, \xi, \eta)$ is uniformly Lipschitz in $(s, \sigma) \in \mathbb{R}^d$; i.e., for some finite $M''$ one has*

$$|F_*(s, \sigma, \xi, \eta) - F_*(s', \sigma', \xi, \eta)| \le M''\{|s - s'| + |\sigma - \sigma'|\} \tag{58}$$

*for all $s, s', \sigma, \sigma' \in \mathbb{R}^d$ and $\xi \in \Omega_0$, $\eta \in \Omega_1$.*

*Finally, $F_*$ satisfies*

$$\nabla \varrho(s) \cdot F_*(s, \sigma, \xi, \eta) = 0 \quad \textit{when} \quad \varrho(s) \le 0. \tag{59}$$

**9.1.2. The fixed point argument.** We prove existence and uniqueness of solutions for the case where $\partial\Omega_0$ is $C^{1,\alpha}$ smooth. The same arguments with minor modifications apply to the case $\Omega_0 = [0, \pi]^d$.

With the extended $F$ in hand we can set up the fixed point problem. Let $\mathcal{C}_T$ be the Banach space

$$\mathcal{C}_T = C^0([0, T] \times \Omega_0; \mathbb{R}^d).$$

Lemma 9.4 (definition of $\mathcal{F}$). *Let $\sigma \in \mathcal{C}_T$ be given. Define $s = \mathcal{F}(\sigma)$ to be the solution of*

$$\frac{\partial s^t}{\partial t} = \int_{\Omega_0} F_*(s^t(x), \sigma^t(\xi); \xi, \eta)\, \mathrm{d}\gamma^0(\xi, \eta), \qquad s^0(x) = x. \tag{60}$$

*Then*

$$\left| \frac{\partial s^t}{\partial t} \right| \le M'|\Omega_0| \tag{61}$$

*and*

$$|s^t(x) - s^t(x')| \le e^{M|\Omega_0|t}|x - x'|. \tag{62}$$

*Proof.* Equation (60) is an ODE for $s^t(x)$ of the form $\partial_t s^t = v^t(s^t)$, where

$$v^t(s) = \int_{\Omega_0} F_*(s, \sigma^t(\xi); \xi, \eta)\, \mathrm{d}\gamma^0(\xi, \eta).$$

The estimates (56) and (57) for $F_*$ imply that $|v^t(s)| \le M'|\Omega_0|$ and $|v^t(s) - v^t(s')| \le M|\Omega_0||s - s'|$. Standard theorems for ODEs then imply (61), (62). □

LEMMA 9.5. *Let $s^t$, $0 \le t \le T$, be the solution of* (60) *for some $\sigma \in \mathcal{C}_T$. Then the $s^t$ are $C^1$ diffeomorphisms of $\bar\Omega_0$.*

*Proof.* The $s^t$ are the flow of a vector field $v^t$, so we only have to show that $s^t(\bar\Omega_0) = \bar\Omega_0$. But our construction of $F_*$ is such that $v^t(s) \cdot \nabla\varrho(s) = 0$ whenever $\varrho(s) \le 0$. Indeed, one has

$$v^t(s) \cdot \nabla\varrho(s) = \int_{\Omega_0} \nabla\varrho(s) \cdot F_*(s, \sigma^t(\xi); \xi, \eta)\, \mathrm{d}\gamma^0(\xi, \eta) = 0$$

by (59). Therefore $\varrho$ is a conserved quantity outside of $\Omega_0$, and in particular $\partial\Omega_0$ is invariant under the flow of $v^t$. So $s^t(\bar\Omega_0) = \bar\Omega_0$.

Since the vector field $v^t$ is $C^1$ on $\bar\Omega$ the flow $s^t$ is also $C^1$. □

*Existence*. The estimates (61) and (62) imply that $\mathcal{F}$ maps all of $\mathcal{C}_T$ into a compact subset of $\mathcal{C}_T$. Hence the Brouwer–Leray–Schauder fixed point theorem applies, and we can conclude the existence of a fixed point $s_T \in \mathcal{C}_T$ for $\mathcal{F}$. The initial value problem for the rearrangement map $s^t$ therefore has a solution on any finite time interval $0 \le t \le T$. Since we have not established uniqueness of the solution, the solutions $s_T$ might actually depend on $T$. However, they all satisfy the a priori estimates (61), (62) so, as $T \nearrow \infty$, one can extract a subsequence which converges uniformly on any finite time interval. The limit of this subsequence is then a global solution $\{s^t\}_{t \ge 0}$.

*Uniqueness*. If the cost function $\Phi$ is $C^2$, then there is only one solution. To see this let $s, \bar s \in \mathcal{C}_T$ be any two solutions and consider their difference $w^t(x) = s^t(x) - \bar s^t(x)$.

Both $s$ and $\bar s$ are solutions to (52), so subtracting the two equations we get

$$|\partial_t w^t(x)| \le M''|\Omega_0| \sup_{\xi \in \Omega_0} |w^t(\xi)|,$$

where we have used that $F_*(s, \sigma, \xi, \eta)$ is uniformly Lipschitz in $(s, \sigma) \in \mathbb{R}^d \times \mathbb{R}^d$ (by (58)).

This implies that $\sup|w^t| \le e^{M''|\Omega_0|t} \sup|w^0|$. Since $w^0 = s^0 - \bar s^0 = 0$ we find that $w^t \equiv 0$.

**9.2. The regularized flow on $\mathfrak{X}$.** If the cost function $\Phi$ is $C^2$, then there is another way of proving existence and uniqueness of solutions to (50). Namely, we observe that (50) is an ODE on a Banach space. One can write (50) as

$$\frac{\partial s^t}{\partial t} = \mathrm{V}(s),$$

where V is given by

$$\mathrm{V}(s)(x) = \iint_{\Omega_0 \times \Omega_1} F_*(s^t(x), s^t(\xi); \xi, \eta)\, \mathrm{d}\gamma^0(\xi, \eta).$$

Here $F_*$ is the extension of $F$ constructed in section 9.1.1.

The properties of $F_*$ derived in Lemma 9.3 imply that V is a globally Lipschitz vector field on the Banach space $\mathfrak{Z} = C^0(\bar{\Omega}_0; \mathbb{R}^d)$. It follows immediately that $\partial_t s^t = \mathrm{V}(s^t)$ generates a global flow on $\mathfrak{Z}$, since $t \mapsto s^t(x)$ is a solution of the ODE

$$\frac{ds}{dt} = v^t(s),$$

where

$$v^t(\sigma) = \iint_{\Omega_0 \times \Omega_1} F_*(\sigma, s^t(\xi); \xi, \eta)\, \mathrm{d}\gamma^0(\xi, \eta).$$

Thus $s^t = S^t \circ s^0$, where $S^t$ is the flow of the vector field $v^t$.

**9.3. $\omega$-limit sets of the regularized flow.** Let $\gamma^0 \in \mathfrak{X}$ be any initial measure, and let $\{\gamma^t, t \geq 0\}$ be a solution of (24) starting at $\gamma^0$. In dynamical systems one defines the $\omega$-limit set of the solution $\{\gamma^t\}$ to be

$$\omega(\{\gamma^t\}) = \left\{\lambda \in \mathfrak{X} \mid \exists t_k \nearrow \infty : \gamma^{t_k} \rightharpoonup \lambda\right\} = \bigcap_{s \geq 0} \overline{\{\gamma^t \mid t \geq s\}}.$$

The second description shows that $\omega(\{\gamma^t\})$ is a closed (hence compact) and connected subset of $\mathfrak{X}$.

PROPOSITION 9.6. *$\omega(\{\gamma^t\})$ consists of critical points for* (24).

*Proof.* For given $\lambda \in \omega(\{\gamma^t\})$ we choose a sequence $t_k \nearrow \infty$ with $\gamma^{t_k} \rightharpoonup \lambda$ and consider the weak solutions $\lambda_k^t = \gamma^{t_k + t}$. By Lemma 6.4 we can find a subsequence $t_{k_j}$ for which the $\lambda_k^t$ weak* converge to a new weak solution $\lambda_\dagger^t$. The $\lambda_\dagger^t$, being weak solutions, satisfy the energy identity from Lemma 5.1. Furthermore,

$$M(\lambda_\dagger^t) = \lim_{j \to \infty} M(\lambda_{k_j}^t) = \lim_{j \to \infty} M(\gamma^{t_{k_j} + t}) = \lim_{t \to \infty} M(\gamma^t),$$

where the latter limit must exist since $M(\gamma^t)$ is a nonincreasing and bounded quantity.

The energy identity for $\lambda_\dagger^t$ together with constancy of $M(\lambda_\dagger^t)$ imply that the $\lambda_\dagger^t$ are critical points. In particular, $\lambda_\dagger^0 = \lim_{j \to \infty} \gamma^{t_{k_j}} = \lambda$ must be a critical point, as claimed. ∎

**10. The unregularized flow.** In the unregularized case, where one takes $\mathcal{A} = I_{\mathfrak{H}}$, one can try to construct weak solutions of (35) by solving the equation for a sequence of smoothing operators $\mathcal{A}$ which approximate the identity, and extract a weak limit of the solutions of the regularized equations. In this section, we study the limits of weak solutions which arise in this way. Although we do not show they are weak solutions, these limits still have many of the properties of weak solutions.

**10.1. Choice of $\mathcal{A}_\varepsilon$.** We let $\mathcal{A}_\varepsilon$ be given by the heat equation with Neumann boundary conditions, $\mathcal{A}_\varepsilon = e^{\varepsilon \Delta_N}$. It is classical that the heat equation defines a strongly continuous semigroup on $L^2(\Omega_0; \mathbb{R}^d)$, so that the $\mathcal{A}_\varepsilon$ converge strongly to the identity operator on $\mathfrak{H}$ as $\varepsilon \searrow 0$.

If $\Omega_0$ is a rectangle, then we choose $\mathcal{A}_\varepsilon$ as in (47), (48).

**10.2. Construction of a generalized solution.** Let $\gamma^0 \in \mathfrak{X}$ be a given initial measure, and denote by $\{\gamma_\varepsilon^t, t \geq 0\}$ the global solutions to (35) with $\mathcal{A} = \mathcal{A}_\varepsilon$ which exist by Theorem 9.1.

Lemma 6.1 provides us with a convergent sequence $\gamma_{\varepsilon_k}^t$: write $\gamma_\dagger^t$ for the weak limit. We declare this family of measures to be a generalized solution of (24).

By Proposition 6.2 any generalized solution we construct in this way satisfies the energy inequality (37). Thus a generalized solution decreases the cost functional at least as fast as a smooth solution would, i.e.,

$$\frac{d}{dt}M(\gamma_\dagger^t) \leq -\left\|\mathcal{P}\mathbb{E}_{\gamma_\dagger^t}(\Phi_x \mid x)\right\|_{\mathfrak{H}}^2.$$

**10.3. $\omega$-limit sets of generalized solutions.** We define

$$\omega(\{\gamma_\dagger^t, t \geq 0\}) = \left\{\lambda \in \mathfrak{X} \mid \exists t_j \nearrow \infty : \gamma_\dagger^{t_j} \rightharpoonup \lambda\right\} = \bigcap_{s\geq 0} \overline{\{\gamma_\dagger^t \mid t \geq s\}}.$$

PROPOSITION 10.1. *The $\omega$-limit set of a generalized solution is a closed and connected subset of $\mathfrak{X}$ which consists of critical points for the Monge–Kantorovich functional.*

*Proof.* Connectedness and closedness follow through entirely conventional arguments from the second description of $\omega(\{\gamma_\dagger^t\})$ given above.

Let $\nu \in \omega(\{\gamma_\dagger^t\})$ be given. Choose a sequence of times $t_k \nearrow \infty$ from which $\gamma_\dagger^{t_k} \rightharpoonup \nu$, and consider the families of measures $\nu_k^t = \gamma_\dagger^{t_k+t}$, $t \in \mathbb{R}$. The arguments in the proof of Proposition 6.2 imply that we can select a subsequence $\nu_{k_j}^t$ which weak* converges for all $t$. The limit $\nu_\dagger^t$ of this subsequence again satisfies the energy inequality. Moreover, the cost functional is constant on $\nu_\dagger^t$, since

$$M(\nu_\dagger^t) = \lim_{k\to\infty} M(\gamma_\dagger^{t_k+t}) = \lim_{t\to\infty} M(\gamma_\dagger^t).$$

The last limit must exist because $M(\gamma_\dagger^t)$ is a nonincreasing bounded quantity.

The energy inequality for $\nu_\dagger^t$ states that

$$\int_{t_0}^{t_1} \|\mathcal{P}\mathbb{E}_{\nu_\dagger^t}(\Phi_x \mid x)\|_{\mathfrak{H}}^2 \,\mathrm{d}t \leq M(\nu_\dagger^{t_0}) - M(\nu_\dagger^{t_1}) = 0$$

so that $\mathcal{P}\mathbb{E}_{\nu_\dagger^t}(\Phi_x \mid x) = 0$ for almost all $t$. Weak* continuity of $\nu^t$ with respect to $t$ strengthens this to $\mathcal{P}\mathbb{E}_{\nu_\dagger^t}(\Phi_x \mid x) = 0$ for all $t$.

Recalling that $\nu = \nu_\dagger^0$, we conclude that $\mathcal{P}\mathbb{E}_\nu(\Phi_x \mid x) = 0$; i.e., $\nu \in \omega(\{\gamma_\dagger^t\})$ is a critical point. $\square$

**11. The unregularized flow—smooth solutions.** If we omit the smoothing operator, i.e., if we set $\mathcal{A} = I_{\mathfrak{H}}$, then (30) for the rearrangement map $s^t$,

$$\frac{\partial s^t(x)}{\partial t} = -\int_{\Omega_0\times\Omega_1} \frac{K(s^t(x), s^t(\xi))}{\mu_0(s^t(x))\mu_0(s^t(\xi))} \cdot \Phi_x(s^t(\xi), \eta)\,\mathrm{d}\gamma^0(\xi,\eta),$$

is highly singular, since the kernel $K$ now is the kernel of the Helmholtz projection. The fixed point arguments of section 9 no longer work. Nonetheless, it turns out that a short time existence theorem for solutions of this equation *does* hold if one assumes the initial data are sufficiently regular. In this section we prove such a theorem.

We will assume in this section that the measures $\gamma^t$ are all defined by measure preserving maps $u^t : \Omega_0 \to \Omega_1$, i.e., $\gamma^t = (\mathrm{id} \times u^t)_\#(\mu_0)$.

Our strategy will be to consider the regularized equation in which $\mathcal{A} = \mathcal{A}_\varepsilon$ is given by a heat operator, as in section 10.1. For each positive $\varepsilon$ we have already shown that a global solution exists. The heart of this section is an estimate for how fast the $C^{1,\alpha}$

norm of the map $u^t_\varepsilon$ grows with time. The estimate is independent of the mollifying parameter $\varepsilon$ if the initial data is smooth. Letting $\varepsilon \searrow 0$ then gives an estimate and short time existence result in $C^{1,\alpha}$ for the unregularized equation.

LEMMA 11.1. *Let $s^t : \Omega_0 \to \Omega_0$ be a solution of the regularized equation* (30) *with $s^0 = \mathrm{id}$. If the initial map $u^0 : \Omega_0 \to \Omega_1$ is $C^{1,\alpha}$, then $s^t$ remains $C^{1,\alpha}$ for a short time $T_* > 0$, and one has $\|\mathrm{d}s^t\|_{0,\alpha} \le C_*$ for $0 \le t \le T_*$, where $C_*$ and $T_*$ depend on the initial data but not on $\varepsilon > 0$.*

**11.1. Notation for Hölder norms.** For any map $f : \Omega_0 \to \mathbb{R}^N$, we write

$$[f]_\alpha = \sup_{x,x'\in\Omega_0} \frac{|f(x)-f(x')|}{|x-x'|^\alpha},$$
$$\|f\|_{0,\alpha} = \|f\|_\infty + [f]_\alpha,$$
$$\|f\|_{1,\alpha} = \|f\|_\infty + \|\mathrm{d}f\|_\infty + [f]_\alpha.$$

The Hölder seminorm $[\cdot]_\alpha$ satisfies the "product-rule estimate,"

$$[f\cdot g]_\alpha \le \|f\|_\infty [g]_\alpha + \|g\|_\infty [f]_\alpha,$$

which one easily derives from $f(x)g(x) - f(y)g(y) = f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)$. One then also finds

$$\|f\cdot g\|_{0,\alpha} \le \|f\|_{0,\alpha}\, \|g\|_{0,\alpha}.$$

**11.2. Estimates of inverses and compositions.** The following proposition shows that we will never have to bother with the case of small $\|s\|_\infty$.

PROPOSITION 11.2. *If $s : \bar\Omega_0 \to \bar\Omega_0$ is a $C^1$ diffeomorphism, then $\|\mathrm{d}s\|_\infty \ge 1$.*

*Consequently we also have $\|s\|_{1,\alpha} \ge \|\mathrm{d}s\|_\infty \ge 1$.*

*Proof.* Since $s(\partial\Omega_0) = \partial\Omega_0$, $s$ cannot be a contraction on all of $\partial\Omega_0$, so somewhere on $\partial\Omega_0$ one has $|\mathrm{d}s| \ge 1$. ∎

Let $s : \Omega_0 \to \Omega_0$ be a $C^1$ diffeomorphism which preserves $\mu_0$, i.e., for which

$$\mu_0(s(x)) \det \mathrm{d}s(x) = \mu_0(x) \tag{63}$$

holds.

LEMMA 11.3. *Assume the diffeomorphism $s : \Omega_0 \to \Omega_0$ satisfies* (63). *If*

$$K = \max_{x,x'\in\Omega_0} \frac{\mu_0(x)}{\mu_0(x')},$$

*then*

$$\sup_{\Omega_0} |(\mathrm{d}s(x))^{-1}| \le C_d \left( K \sup_{\Omega_0} |\mathrm{d}s(x)| \right)^{d-1}$$

*for some constant $C_d$ which only depends on the dimension $d$. In particular, if $s$ is Lipschitz continuous with Lipschitz constant $L$, then $s^{-1}$ is Lipschitz continuous with constant at most $C_d(KL)^{d-1}$.*

*Proof.* We represent $\mathrm{d}s(x)$ as a $d \times d$ matrix. Then

$$(\mathrm{d}s(x))^{-1} = \frac{1}{\det \mathrm{d}s(x)} (\mathrm{d}s(x))^{\#} = \frac{\mu_0(x)}{\mu_1(s(x))} (\mathrm{d}s(x))^{\#}, \tag{64}$$

where $\mathrm{d}s^t(x)^\#$ is the cofactor matrix. This matrix is a polynomial of degree $d-1$ in the entries of the matrix $\mathrm{d}s^t$, hence the lemma. ◻

LEMMA 11.4. *Assume the diffeomorphism $s : \Omega_0 \to \Omega_0$ satisfies (63). If $s \in C^{1,\alpha}$, then $s^{-1} \in C^{1,\alpha}$, and*

$$\|s^{-1}\|_{1,\alpha} \le C\|s\|_{1,\alpha}^{(d-1)(2+\alpha)+1}, \tag{65}$$

*where the constant $C$ only depends on $\mu_0$ and the dimension $d$.*

*Proof.* We will henceforth write $\mathrm{d}s(x) = \mathrm{d}s_x$ if it seems to improve the notation.

We get an estimate for the supremum norm of $\mathrm{d}(s^{-1})$ from the inverse function theorem, which says $\mathrm{d}(s^{-1}) = (\mathrm{d}s)^{-1} \circ s^{-1}$, so that $\|\mathrm{d}(s^{-1})\|_\infty = \|(\mathrm{d}s)^{-1}\|_\infty \le C\|\mathrm{d}s\|_\infty^{d-1}$.

To estimate the Hölder seminorm $[\mathrm{d}s^{-1}]_\alpha$ we compute for $x, y \in \Omega_0$

$$\begin{aligned}
|\mathrm{d}(s^{-1})(x) - \mathrm{d}(s^{-1})(y)| &= \left|(\mathrm{d}s)^{-1}_{s^{-1}(x)} - (\mathrm{d}s)^{-1}_{s^{-1}(y)}\right| \\
&= \left|(\mathrm{d}s)^{-1}_{s^{-1}(y)} \left\{\mathrm{d}s_{s^{-1}(x)} - \mathrm{d}s_{s^{-1}(y)}\right\} \mathrm{d}(s^{-1})_{s^{-1}(x)}\right| \\
&\le \|(\mathrm{d}s)^{-1}\|_\infty^2 \left\|\mathrm{d}s_{s^{-1}(x)} - \mathrm{d}s_{s^{-1}(y)}\right\| \\
&\le \|(\mathrm{d}s)^{-1}\|_\infty^2 [\mathrm{d}s]_\alpha \left|s^{-1}(x) - s^{-1}(y)\right|^\alpha \\
&\le \|(\mathrm{d}s)^{-1}\|_\infty^{2+\alpha} [\mathrm{d}s]_\alpha |x-y|^\alpha \\
&\le C\|\mathrm{d}s\|_\infty^{(d-1)(2+\alpha)} [\mathrm{d}s]_\alpha |x-y|^\alpha.
\end{aligned}$$

Hence we get

$$[d(s^{-1})]_\alpha \le C\|\mathrm{d}s\|_\infty^{(d-1)(2+\alpha)}[\mathrm{d}s]_\alpha \le C\|s\|_{1,\alpha}^{(d-1)(2+\alpha)+1}.$$

To estimate the full $C^{1,\alpha}$ norm of $s^{-1}$ we add the lower order terms,

$$\begin{aligned}
\|s^{-1}\|_{1,\alpha} &= \|s^{-1}\|_\infty + \|\mathrm{d}s^{-1}\|_\infty + [d(s^{-1})]_\alpha \\
&\le C + C\|\mathrm{d}s\|_\infty^{d-1} + C\|s\|_{1,\alpha}^{(d-1)(2+\alpha)+1} \\
&\le C + C\|s\|_{1,\alpha}^{(d-1)(2+\alpha)+1}.
\end{aligned}$$

Finally we use $\|s\|_{1,\alpha} \ge 1$ to get (65). ◻

We will occasionally use the following crude estimate for the $C^{1,\alpha}$ norm of the composition of two maps.

LEMMA 11.5. *For two $C^{1,\alpha}$ maps $f, g$ one has*

$$\begin{aligned}
[f \circ g]_\alpha &\le \|\mathrm{d}g\|_\infty \cdot [f]_\alpha, \\
\|f \circ g\|_{0,\alpha} &\le \left(1 + \|\mathrm{d}g\|_\infty^\alpha\right) \|f\|_{0,\alpha}, \\
\|f \circ g\|_{1,\alpha} &\le 3\big(1 + \|g\|_{1,\alpha}^{1+\alpha}\big)\|f\|_{1,\alpha}.
\end{aligned}$$

*Proof.* The first inequality follows directly from

$$|f(g(x)) - f(g(y))| \le [f]_\alpha |g(x) - g(y)|^\alpha \le [f]_\alpha \|\mathrm{d}g\|_\infty^\alpha\, |x-y|^\alpha.$$

The second inequality follows from

$$\begin{aligned}
\|f \circ g\|_{0,\alpha} &= \|f \circ g\|_\infty + [f \circ g]_\alpha \\
&\le \|f\|_\infty + \|\mathrm{d}g\|_\infty^\alpha \cdot [f]_\alpha \\
&\le \|f\|_{0,\alpha} + \|\mathrm{d}g\|_\infty^\alpha \|f\|_{0,\alpha}.
\end{aligned}$$

To prove the third inequality, we compute

$$\begin{aligned}\|f\circ g\|_{1,\alpha} &= \|f\circ g\|_\infty + \|(\mathrm{d}f\circ g)\cdot \mathrm{d}g\|_{0,\alpha}\\ &\le \|f\|_\infty + \|\mathrm{d}f\circ g\|_{0,\alpha}\|\mathrm{d}g\|_{0,\alpha}\\ &\le \|f\|_\infty + \big(1+\|\mathrm{d}g\|_\infty^\alpha\big)\|\mathrm{d}f\|_{0,\alpha}\|\mathrm{d}g\|_{0,\alpha}\\ &\le \big(1+\|\mathrm{d}g\|_{0,\alpha}+\|\mathrm{d}g\|_{0,\alpha}^{1+\alpha}\big)\|f\|_{1,\alpha}\\ &\le 3\big(1+\|\mathrm{d}g\|_{0,\alpha}^{1+\alpha}\big)\|f\|_{1,\alpha}\end{aligned}$$

since $1+x+x^{1+\alpha}\le 3(1+x^{1+\alpha})$ for all $x\ge 0$. ∎

**11.3. Proof of Lemma 11.1.** We summarize the relations that define the maps $u^t$.

First, $u^t$ and the initial map $u^0$ are related by

$$u^t = u^0\circ (s^t)^{-1}. \tag{66}$$

The rearrangement maps $s^t$ move with velocity field $v^t$. This gives two equations, one for $s^t$ and one for its space derivative:

$$\frac{\partial s^t}{\partial t} = v^t\circ s^t, \qquad \frac{\partial \mathrm{d}s^t}{\partial t} = (\mathrm{d}v^t\circ s^t)\cdot \mathrm{d}s^t. \tag{67}$$

The velocity field $v^t$ is determined by the map $u^t$ via

$$v^t(x) = \frac{-1}{\mu_0(x)}\mathcal{P}\mathcal{A}_\varepsilon^2\mathcal{P}W^t, \tag{68}$$

while $W^t$ is given by $W^t = \mathbb{E}_{\gamma^t}(\Phi_x \mid x)$, i.e., by

$$W^t(x) = \Phi_x(x, u^t(x)). \tag{69}$$

We begin our estimate of $\|\partial_t s^t\|_{1,\alpha}$ as follows:

$$\begin{aligned}\left\|\frac{\partial}{\partial t}s^t\right\|_{1,\alpha} &= \|\partial_t s^t\|_\infty + \left\|\frac{\partial}{\partial t}\mathrm{d}s^t\right\|_{0,\alpha}\\ &= \|v^t\|_\infty + \|(\mathrm{d}v^t\circ s^t)\cdot \mathrm{d}s^t\|_{0,\alpha}\\ &\le \|v^t\|_\infty + \|\mathrm{d}v^t\circ s^t\|_{0,\alpha}\cdot\|\mathrm{d}s^t\|_{0,\alpha}\\ &\le \|v^t\|_\infty + \|\mathrm{d}v^t\|_{0,\alpha}\big(1+\|\mathrm{d}s^t\|_\infty^\alpha\big)\|\mathrm{d}s^t\|_{0,\alpha}\\ &\le \|v^t\|_\infty + 2\|\mathrm{d}v^t\|_{0,\alpha}\|\mathrm{d}s^t\|_{0,\alpha}^{1+\alpha}\\ &\le \|v^t\|_\infty + 2\|v^t\|_{1,\alpha}\|s^t\|_{1,\alpha}^{1+\alpha}\\ &\le 3\|v^t\|_{1,\alpha}\|s^t\|_{1,\alpha}^{1+\alpha},\end{aligned} \tag{70}$$

where we have used $\|\mathrm{d}s^t\|_\infty\ge 1$ again.

Next, we estimate the $C^{1,\alpha}$ norm of the velocity field:

$$\begin{aligned}\|v^t\|_{1,\alpha} &= \left\|\frac{-1}{\mu_0(x)}\mathcal{P}\mathcal{A}^2\mathcal{P}\big(\Phi_x(x,u^t(x))\big)\right\|_{1,\alpha}\\ &\le C\|\Phi_x(x,u^t(x))\|_{1,\alpha}\\ &\le 3C\|\Phi\|_{2,\alpha}\big(1+\|u^t\|_{1,\alpha}^{1+\alpha}\big)\\ &\le C\big(1+\|u^t\|_{1,\alpha}^{1+\alpha}\big).\end{aligned} \tag{71}$$

Here we have used the facts that $\mu_0 \in C^{1,\alpha}$, that the smoothing operators $\mathcal{A}$ are uniformly bounded from $C^{1,\alpha}$ to $C^{1,\alpha}$, and that the Helmholtz decomposition is also bounded on $C^{1,\alpha}$.

Finally we estimate $\|u^t\|_{1,\alpha}$ in terms of $\|s^t\|_{1,\alpha}$. We have

$$
\begin{aligned}
\|u^t\|_{1,\alpha} &= \|u^0 \circ \left(s^t\right)^{-1}\|_{1,\alpha} \\
&\le 3\left(1 + \|(s^t)^{-1}\|_{1,\alpha}^{1+\alpha}\right) \|u^0\|_{1,\alpha} \\
&\le C \, \|(s^t)^{-1}\|_{1,\alpha}^{1+\alpha} \, \|u^0\|_{1,\alpha} \\
&\le C \, \|s^t\|_{1,\alpha}^{(d-1)(2+\alpha)(1+\alpha)+1+\alpha}.
\end{aligned} \tag{72}
$$

Combining (70), (71), and (72), we arrive at

$$
\frac{d}{dt}\|s^t\|_{1,\alpha} \le \|\partial_t s^t\|_{1,\alpha} \le C \left(\|s^t\|_{1,\alpha}\right)^{\kappa}, \tag{73}
$$

where

$$
\begin{aligned}
\kappa &= \{(d-1)(2+\alpha)(1+\alpha) + 1 + \alpha\}(1+\alpha) + 1 + \alpha \\
&= \big((d-1)\alpha^2 + 3(d-1)\alpha + 2d\big)(1+\alpha).
\end{aligned}
$$

Integrate this ODE, using the initial data $\|s^0\|_{1,\alpha} = 1$ which derives from $s^0 = \mathrm{id}$, and you get

$$
\|s^t\|_{1,\alpha} \le \left(1 - Ct\right)^{-\frac{1}{\kappa-1}}.
$$

The constant $C$ depends on the initial map $u^0$ but not on the smoothing parameter $\varepsilon > 0$, as claimed in Lemma 11.1.

**12. Numerical methods and examples.** In this section, we describe some of the techniques we use to numerically solve (9), as well as how we compute the initial mapping. Briefly, we have employed an upwinding scheme when computing $\nabla u^t$ and the FFT when inverting the Laplacian on a rectangular grid. Standard centered differences were used for the other spatial derivatives. In practice, we iterate until the mean absolute curl is sufficiently small. More details of the numerical implementation for solving (9) are given below. See also [11, 12].

**12.1. Finding an initial mapping.** In this section, we describe our procedure for finding the initial mass preserving mapping $u$ for (9). We work here on the unit square. An initial mapping for general domains can also be obtained using a method of Moser [6].

So we work in $\mathbb{R}^2$ and assume $\Omega_0 = \Omega_1 = [0,1]^2$, the generalization to higher dimensions being straightforward. The idea of this construction is that we solve a family of one-dimensional mass transport problems. In one dimension, the optimal transport map can be found by simple quadrature. We first transport mass along lines parallel to the $x$-axis, and then afterward transport mass along lines parallel to the $y$-axis. Accordingly, we define a function $a = a(x)$ by the equation

$$
\int_0^{a(x)} \int_0^1 \mu_1(\eta, y) \, dy \, d\eta = \int_0^x \int_0^1 \mu_0(\eta, y) \, dy \, d\eta, \tag{74}
$$

which gives by differentiation with respect to $x$

$$a'(x) \int_0^1 \mu_1(a(x), y)\ dy = \int_0^1 \mu_0(x, y)\ dy. \tag{75}$$

We may now define a function $b = b(x, y)$ by the equation

$$a'(x) \int_0^{b(x,y)} \mu_1(a(x), \rho)\ d\rho = \int_0^y \mu_0(x, \rho)\ d\rho \tag{76}$$

and set $u(x, y) = (a(x), b(x, y))$. Since $a_y = 0$, $|Du| = a_x b_y$, and differentiating (76) with respect to $y$ we find

$$a'(x)\ b_y(x, y)\ \mu_1(a(x), b(x, y)) = \mu_0(x, y),$$
$$|Du|\ \mu_1 \circ u = \mu_0,$$

which is the mass preserving property we need. In practice, $a$ and $b$ can be found with simple numerical integration techniques.

**12.2. Defining the warping map.** Typically in elastic registration, one wants to see an explicit warping which smoothly deforms one image into the other [11]. This can easily be done using the solution of the Monge–Kantorovich problem. Thus, we assume now that we have applied our gradient descent process as described above and that it has converged to the optimal $L^2$ Monge–Kantorovich mapping $u_{MK}$.

Following the work of Benamou and Brenier [3] (see also [9]), we consider the related problem

$$\inf \int \int_0^1 \mu(t, x)|v(t, x)|^2\, dt\, dx \tag{77}$$

over all time varying densities $\mu$ and velocity fields $v$ satisfying

$$\frac{\partial \mu}{\partial t} + \operatorname{div}(\mu v) = 0, \tag{78}$$

$$\mu(0, \cdot) = \mu_0, \quad \mu(1, \cdot) = \mu_1. \tag{79}$$

It is shown in [3] that this infimum is attained for some $\mu_{min}$ and $v_{min}$, and that it is equal to the $L^2$ Kantorovich–Wasserstein distance between $\mu_0$ and $\mu_1$. Recall that this distance is defined by

$$d_2(\mu_0, \mu_1)^2 := \inf_u \int_{\Omega_0} |u(x) - x|^2\ dx,$$

the infimum taken over all diffeomorphisms which satisfy the Jacobian condition (1). Further, the flow $X = X(x, t)$ corresponding to the minimizing velocity field $v_{min}$ via

$$X(x, 0) = x, \quad X_t = v_{min} \circ X \tag{80}$$

is given simply as

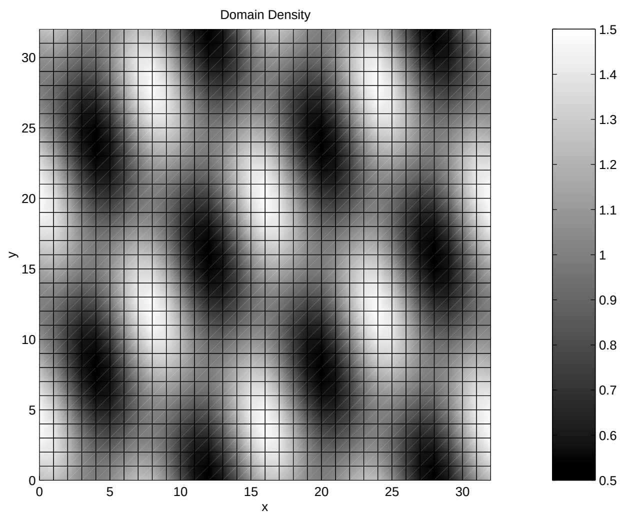



FIG. 3. *Density* $\mu_1$ *on* $\Omega_0$.

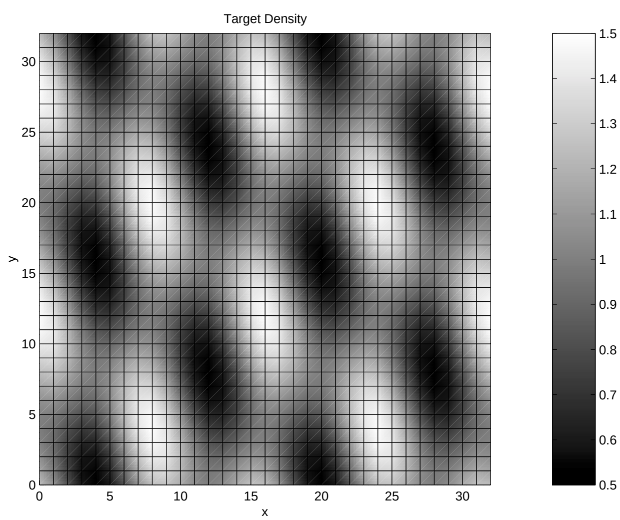



FIG. 4. *Density* $\mu_1$ *on* $\Omega_1$.

$$X(x,t) = x + t\,(u_{MK}(x) - x). \tag{81}$$

Note that when $t = 0$, $X$ is the identity map, and when $t = 1$, it is the solution $u_{MK}$ to the Monge–Kantorovich problem. This analysis provides appropriate justification for using (81) to *define* our continuous warping map $X$ between the densities $\mu_0$ and $\mu_1$.

**13. Implementation and examples.** We illustrate our methods with the following examples. The first is the mapping of one synthetic density onto another. Figure 3 shows a mass distribution $\mu_0$ on $\Omega_0$, with dark regions representing little mass, lighter regions representing more. Similarly, Figure 4 indicates the density $\mu_1$ on $\Omega_1$. Figure 5 represents the initial mapping $u$, which was obtained by the method described above. The shading in this figure represents the Jacobian of $u$. Figure 6

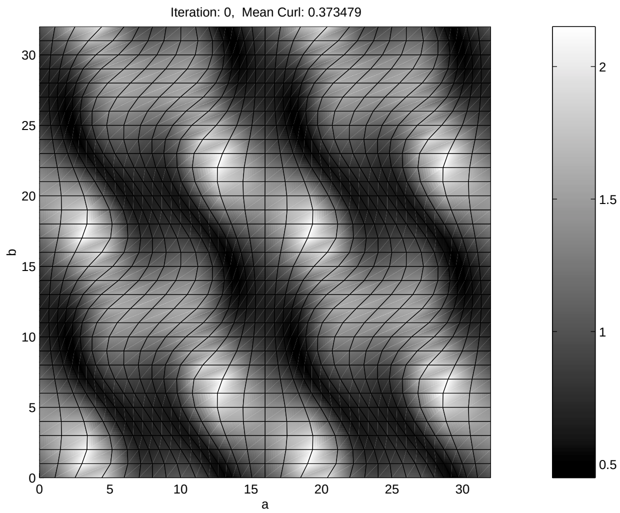



FIG. 5. *Initial mapping from* $\Omega_0$ *to* $\Omega_1$.

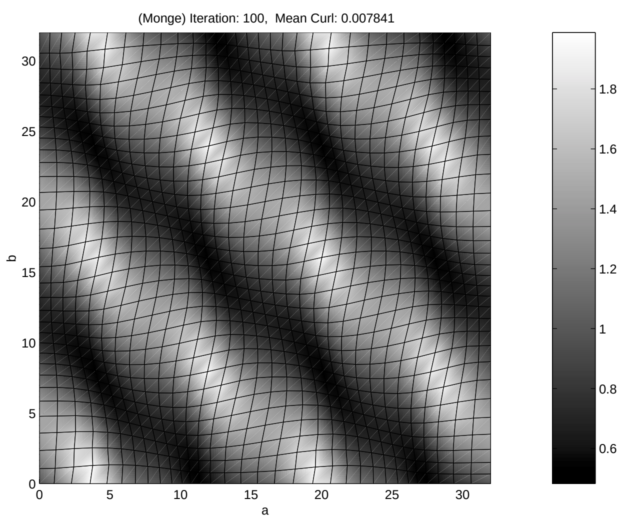



FIG. 6. *Final Monge–Kantorovich mapping from* $\Omega_0$ *to* $\Omega_1$.

shows the nearly optimal Monge–Kantorovich mapping obtained using the nonlocal first order equation (9). One can see that the effect of removing the curl is to straighten out the grid lines somewhat. On a Sun Ultra10, this process took just a few seconds.

In Figures 7 through 10 we show a brain deformation sequence obtained with MRI. The first and last images were given, and the intermediate two were found using our process. This type of elastic brain deformation occurs during surgery, after the skull is opened. These two-dimensional slices were extracted from an original three-dimensional data set ($256 \times 256 \times 124$) to which the registration algorithm was applied. We should note that in contrast to other elastic approaches based on fluid and continuum mechanics ideas (see [17], especially Chapters 1 and 18 for a general discussion) in which the computations may take hours, in our case the three-dimensional set was processed in about half an hour with very reasonable results.

FIG. 7. *Brain warping:* $t = 0.00$.

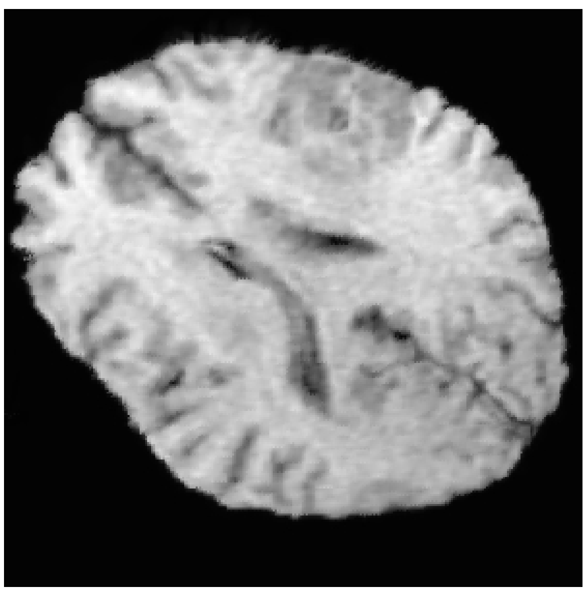


FIG. 8. *Brain warping:* $t = 0.33$.

In general, the target domain $\Omega_1$ need not be rectangular when using the nonlocal method. However, we note that if the periodic boundary conditions are used on the displacement, as in section 7.1, then the Laplacian in (9) can be inverted using the FFT alone, without the need to solve a subsequent matrix system. For the brain warp, this reduced the processing time by about 1/3.

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FIG. 9. *Brain warping:* $t = 0.66$.

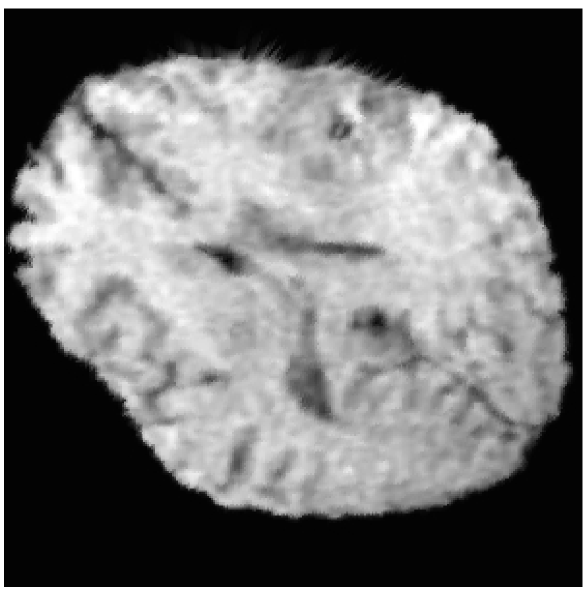

FIG. 10. *Brain warping:* $t = 1.00$.

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