THE OBSERVABILITY OF SYSTEMS WITH LINEAR DYNAMICS AND QUADRATIC OUTPUT

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THE OBSERVABILITY OF SYSTEMS WITH LINEAR

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NOMENCLATURE

$\underline{\underline{A}}$	A matrix having the dimension n x n, unless otherwise noted.
$\underline{a}$	A vector having n components.
a_{ij}	A scalar, the j^{th} component of the i^{th} row of the matrix $\underline{\underline{A}}$.
a_i	A scalar, the i^{th} component of the vector $\underline{a}$.
$\forall$	To be read as "for all".
$\simeq$	To be read as "has the same rank as".
ε	To be read as "on the".

SUMMARY

The objective of this thesis is to present a statement of conditions for the observability of quadratic systems. A quadratic system is a system having dynamics represented by linear, stationary, and deterministic differential equations and an output composed of a summation of the quadratic functions of the components of the state vector. In order to facilitate the investigation of this nonlinear system, an extended definition of observability is presented along with a general theorem for the investigation of system observability. The extended definition and the general observability theorem are used to develop conditions of observability for quadratic systems. Finally, a brief discussion is given on the relationship between system observability and system state reconstruction. A method of state reconstruction for the quadratic system is given.

CHAPTER I

INTRODUCTION

Definition of the Problem

Modern control theory has, in recent years, led system analysts to formulate their problems in terms of the state-space notation. Some advantages of this notational convention are: (1) a simplification of the mathematical analysis of the system operation, and (2) an improved understanding of the interaction of the individual system components in the creation of the overall system's response. The development of the theories of observability and controllability is an example of this increased understanding of system operation.

The present work is concerned with the application of the notion of "observability" to the class of nonlinear systems represented by:

$$\dot{\underline{x}} = A\,\underline{x} + \underline{b}\,u(t), \text{ and} \tag{1.1}$$

$$y(t) = \underline{x}^T\,Q\,\underline{x}. \tag{1.2}$$

This system has a dynamical representation as a linear, stationary, deterministic, and ordinary differential equation. The output relationship is a summation of the quadratic functions of components of the state vector. This system will be referred to as a "quadratic" system.

Historically, a system is termed "observable" if the state vector of the system can be reconstructed from a knowledge of the system's inputs, outputs, and mathematical structure. For example, Kriendler and Sarachik[1] have defined "observability" in the following manner:

> Definition 1: An unforced system is said to be completely observable on $[t_o, t_f]$ if given t_o and t_f every state $\underline{x}(t_o)$ in X can be determined from the knowledge of y(t) on $[t_o, t_f]$.

The usual approach in determining conditions of observability is to solve for the system's response in terms of an undetermined initial condition, $\underline{x}(t_o) = \underline{x}_o$, as:

$$\underline{y}(t) = \underline{g}(\underline{x}_o, t), \tag{1.3}$$

and then attempt a straightforward solution of Equation 1.3 for the unknown initial condition, $\underline{x}_o$. Conditions which allow such a solution are the conditions of observability.

If an unspecified input, $\underline{u}(t)$, is included in the response relationship, the problem may be viewed as one of determining under what conditions Equation 1.4 may be solved for the unknown value $\underline{x}_o$:

$$\underline{y}(t) = \underline{g}(\underline{x}_o, \underline{u}(t), t). \tag{1.4}$$

For completely linear systems, the addition of the unspecified input, $\underline{u}(t)$, does not complicate the procedure just described. However, in the case of a nonlinear system, the difficulty of finding an analytic solution for the initial state, $\underline{x}_o$, of an unforced system, Equation 1.3, may become exceedingly difficult by the inclusion of an unspecified input, $\underline{u}(t)$, as in Equation 1.4.

Since observability is a system property, conditions of observability must be determinable without specifying a particular method of solving the problem represented by Equation 1.4. A direct approach for solving this problem for the quadratic system (Equations 1.1 and 1.2) has not produced an analytic solution for $\underline{x}_o$. Therefore, an additional aspect of the problem for determining the observability of the quadratic system will be to develop a new means of investigating system observability independent of a straightforward solution for the unknown initial condition.

Purpose of the Research

The specific purpose of this research was to determine under what conditions the state of the quadratic system (Equations 1.1 and 1.2) may be reconstructed. Consider the system of Figure 1, in which the output or observed response of the system, the quantity y(t), is proportional to the power dissipated by the viscous damper. Such a system is a quadratic system of the form of Equations 1.1 and 1.2. If such a system were observable, it would then be possible to reconstruct both the displacement and velocity histories of the two masses, M_1 and M_2. Under certain conditions such a system is observable.

An additional purpose of this work was to provide a general framework for the investigation of nonlinear systems. In order to accomplish this, it has been necessary to investigate the basic notion involved in defining observability. This problem has resulted in the proposal of an Extended Definition of Observability. Furthermore, A General Observability

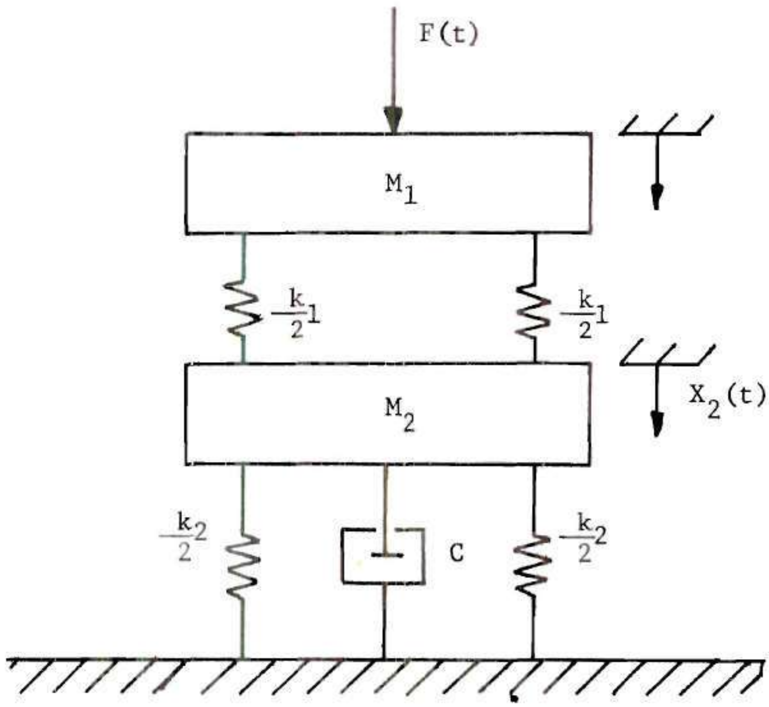

Figure 1. Example of a System Possessing Linear Dynamics and Non-linear Output y(t). (y(t)=Power Dissipated by Viscous Damper C)

Theorem based upon the Extended Definition will be stated and proved. These investigating tools will first be applied to the completely linear system to demonstrate their usefulness. Secondly, the quadratic system will be investigated for its conditions of observability.

Literature Review

The notion of observability, originally introduced by Kalman[2], was presented as a mathematical dual to the notion of controllability developed in the same reference. Kalman[3] later expanded the understanding of observability by developing an explanation independent of the dualization process. This concept was further discussed by Kriendler and Sarachik[1] and others. Wonham and Johnson [4] are among those who have shown the relationship of controllability and observability to the problem of optimizing quadratic cost functions for linear dynamical systems. Kalman, Falb and Arbib[5] present a concise review of the conditions of observability for linear systems.

Kalman, Falb, and Arbib[5] also discuss a method of reconstructing the state of a system by means of an asymptotic state estimator. The output of this state reconstructor approaches that of the system being observed asymptotically with respect to the independent variable. Luenberger[6-8] gives a more detailed discussion of the same topic. Another method of state reconstruction is presented by Gilchrist[9] in which information on the inputs and outputs of the system is collected at discrete values of the independent variable. Once sufficient information is collected, the present state of the system is calculated by direct substitution of the collected data into the system equations of motion. Al'brekt and

Krasoviskii[10] have considered the problem of observing a nonlinear system when the motion of the system is in the neighborhood of a given motion. Roitenberg[11-12] has developed an indirect method of calculating the state of both linear and nonlinear systems based upon information of the system's inputs and outputs. The quadratic system being discussed in the present work has not been previously analyzed with respect to its quality of observability. However, McClamrock and Aggarwal[13] have discussed the invariance of the quadratic system to changes in the state description.

Contribution

In the present work, the definition of observability has been restated in order to facilitate its usefulness in analyzing non-linear systems. The extended definition and its associated observability theorem allow nonlinear systems to be investigated without linearizing the non-linear system or otherwise approximating its reponse. Unlike most previous non-linear observability work, this thesis treats observability in-the-large rather than for small deviations from a specified state trajectory. As an example of their usefulness, the extended definition and general theorem are used to develop necessary and sufficient conditions for the observability of quadratic systems. Conditions for the observability of quadratic systems have not been previously defined.

Organization of the Work

The organization of this work is reduced to three main topics. The first topic is the subject of general system observability, covered in Chapter II. Here, motivation will be given for an extended definition of observability. Also, a general observability theorem will be stated and proven. This theorem will facilitate the investigation of observability of nonlinear systems. In Chapter III, using the proposed general observability theorem,conditions for the observability of quadratic systems will be stated and proven. In addition, it will be shown that under certain conditions the state of the quadratic system may be uniquely observed. Also, a straightforward calculatory method of determining observability will be presented. In Chapter IV, a discussion will be presented showing the relationship between system observability and state reconstruction. A numerical example of the system shown in Figure 1 will be presented, demonstrating that the state of a quadratic system can be uniquely reconstructed.

The Bibliography is divided into two parts. The first contains references which have been cited in the text of this work; the second contains those references which have not been specifically noted in the text but which have been useful in developing a background and basis for this work.

CHAPTER II

SYSTEM OBSERVABILITY

Objective

The objective of this chapter is to present a precise definition of observability, consistent with the present definition but extending the applicability of the observability notion to nonlinear systems while allowing for a clear distinction between observability and state reconstruction. In addition, this new definition will allow all systems to be compared as to their degree or quality of observability,regardless of their degree of linearity.

A general observability theorem will be stated and proved.based upon the Extended Definition of Observability. Because of the vast numbers of different systems which may be encountered, the General Observability Theorem will not specify a specific test to be applied to every system, but, rather, will indicate a general approach to determining whether or not a system is observable.

The conditions necessary for a linear system to be observable are well established in the literature. In an effort to demonstrate the usefulness of the Extended Definition of Observability and the General Observability Theorem, these ideas will be applied to a completely linear system. Such an analysis does not reveal new results but does demonstrate the distinct difference between a system which is observable and a system which is state reconstructable.

q-Point Observability

Figure 2 presents a block diagram of a dynamical system for which it is necessary to design a state reconstruction device. The output of the state reconstruction device is a state vector, $\underline{x}^*(t)$, which, if the process of reconstruction is "perfect", will satisfy:

$$\underline{x}^*(t) = \underline{x}(t) \tag{2.1}$$

for all time on the interval $[t_o, t_f]$. If the system of Figure 2 were observable, then the system state could be reconstructed by some technique. Conditions of observability do not necessarily indicate a method of state reconstruction, but, rather, only assure that the state may be formally reconstructed; that is, the state may be reconstructed only in mathematical form.

Consider the system of Figure 2 having a dynamic plant represented by the differential equation (Equation 2.2) and an output relationship given by Equation 2.3:

$$\dot{\underline{x}}(t) = \underline{f}[\underline{x}(t), \underline{u}(t), t], \text{ and} \tag{2.2}$$

$$\underline{y}(t) = \underline{g}[\underline{x}(t), \underline{u}(t), t]. \tag{2.3}$$

where:

$\underline{x}(t)$ represents an n-dimensional vector, termed the "state of the system";

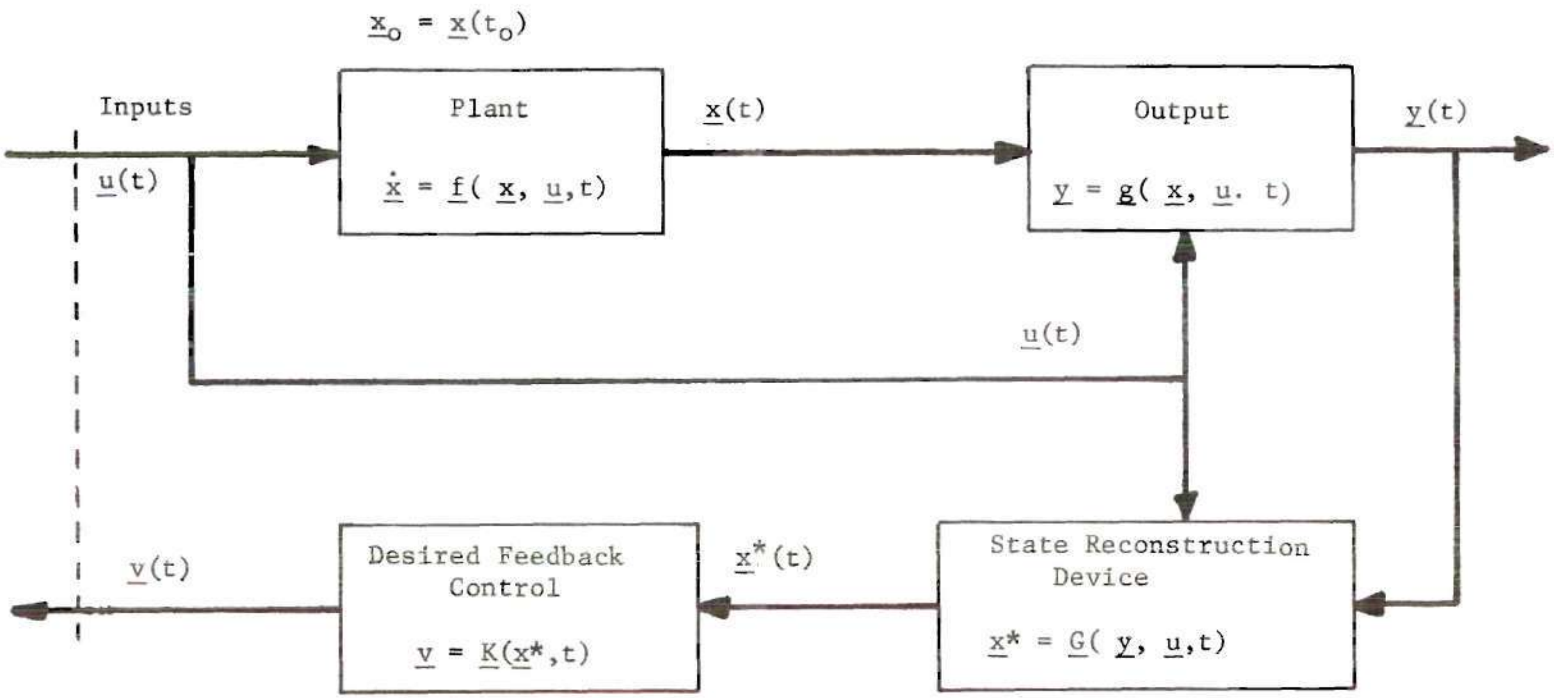

Figure 2. Dynamic System with State-Reconstruction Device.

$\underline{u}(t)$ an r-dimensional vector, representing the input to the system;

$\underline{y}(t)$ an m-dimensional vector ($m \leq n$), representing the output of the system.

Assume that the unique solution of Equation 2.2 may be written as:

$$\underline{x}(t) = \underline{X} \left[\underline{x}_o, \underline{u}(t), t \right] . \tag{2.4}$$

By direct substitution of Equation 2.4 into Equation 2.3, the output of the system may be found as:

$$\underline{y}(t) = \underline{F} \left[\underline{x}_o, \underline{u}(t), t \right] . \tag{2.5}$$

Traditionally, the system composed of Equations 2.2-2.3 would be termed "observable", if given the structure of the system (i.e., Equations 2.2-2.3), the input, $\underline{u}(t)$, and output, $\underline{y}(t)$, as explicit functions of t on the interval $[t_o, t_f]$, it is possible to determine the value of the initial state $\underline{x}(t = t_o) = \underline{x}_o$, which is consistent with the input-output relationship, Equation 2.5.

The procedure is very straightforward. Given the structure of the system, find the functional form of Equation 2.5. Then, knowing the responses, $\underline{y}(t)$, as a result of some known input, $\underline{u}(t)$, and an unknown initial condition, $\underline{x}_o$, find the initial condition, $\underline{x}_o$, which satisfies Equation 2.5. With this information in hand, the state of the system may be reconstructed by direct substitution of $\underline{u}(t)$ and $\underline{x}_o$ into Equation 2.4 to yield:

$$\underline{x}(t) = \underline{X}(t). \tag{2.6}$$

The solution of Equation 2.5 may or may not be simple or even straightforward. However, an important comment on Equation 2.5 is that several distinct results may occur in attempting to solve for $\underline{x}_o$. First, it may be possible to find a single value for $\underline{x}_o$ which satisfies the equality. Secondly, there may be several values for $\underline{x}_o$ which satisfy the equality; and, finally, it may turn out that there are an infinite number of initial conditions for $\underline{x}_o$ which satisfy the equality.

If there is only a single condition for $\underline{x}_o$ that satisfies Equation 2.5, then the system may be declared observable; if there are an infinite number of values for $\underline{x}_o$ that solve Equation 2.5, then the system may be declared unobservable. However, if there are several values for $\underline{x}_o$ which satisfy Equation 2.5 it is certainly true the system is not observable in the traditional sense, although it is <u>more</u> observable than if there were an infinite number of such values for $\underline{x}_o$. It is this quality of the nonlinear system which the following definition is designed to take into account:

Definition 2: Extended Definition of Observability (q-Point Observability)

A system having an n-dimensional state vector, $\underline{x}(t)$ and input u(t), will be said to be q-point observable on the interval $[t_o, t_f]$ if there are q and only q distinct values of the initial conditions, $\underline{x}_o$, consistent with the system's input-output relationship.

The only difference between this definition and that given earlier (Definition 1, page 2) lies in the possibility of defining as observable those systems having a finite number of multiplicities of initial conditions consistent with the input-output relationship. The reasons for wishing to define such systems as observable rest on the knowledge that in almost all cases the control engineer knows more about the system and its operation than that which is expressed in Equation 2.2 and 2.3. This additional knowledge may help in eliminating some of the candidate initial conditions from consideration. Failing to reduce the possible number of candidate initial conditions to one, a trial-and-error procedure might be used. At the very least the uncertainty as to the correct initial conditions is bounded to a finite set.

General Observability Theorem

The definition of q-point observability given in the previous section suggests a very simple theorem for determining the degree of observability of a system. The basic premise of this theorem is that if the structure of a system is known, then, for any specific input-output pair, $\underline{u}(t)$ and $\underline{y}(t)$, there must exist at least one value of $\underline{x}_o$ that satisfies Equation 2.5. An additional question is: How many solutions for $\underline{x}_o$ are there to Equation 2.5?

A straightforward attempt to solve the input-output relation (Equation 2.5) for $\underline{x}_o$ in terms of an unspecified input, $\underline{u}(t)$, may lead to conditions of observability. For problems such as the completely linear system for which Equation 2.5 has the form:

$$\underline{y}(t) = \underline{F}_1 (\underline{x}_o, t) + \underline{F}_2 (\underline{u}(t)), \tag{2.7}$$

this straightforward procedure may yield results. More complicated problems, however, make the straightforward solution difficult if not impossible to achieve.

Another approach is implied by the Extended Definition. Two distinct initial conditions, $\underline{x}_o$ and $\underline{x}_o'$ can be related as: $\underline{x}_o' = \underline{x}_o + \delta\underline{x}_o$, where the vector, $\delta\underline{x}_o$, is given as: $\delta\underline{x}_o = \underline{x}_o' - \underline{x}_o$. The responses of a system to each of these initial conditions and the same unspecified input, $\underline{u}(t)$, may be written as:

$$\underline{y}_1(t) = \underline{F}\left[\underline{x}_o, \underline{u}(t), t\right] \quad , \underline{x}(t_o) = \underline{x}_o, \text{ and} \tag{2.8}$$

$$\underline{y}_2(t) = \underline{F}\left[\underline{x}_o + \delta\underline{x}_o, \underline{u}(t), t\right] \quad , \underline{x}(t_o) = \underline{x}_o + \delta\underline{x}_o. \tag{2.9}$$

In general, the response (Equation 2.9) is different from that of Equation 2.8. For convenience, this difference shall be defined as a vector, $\Delta\underline{y}(t)$:

$$\Delta\underline{y}(t) = \underline{F}\left[\underline{x}_o + \delta\underline{x}_o, \underline{u}(t), t\right] - \underline{F}\left[\underline{x}_o, \underline{u}(t), t\right] \tag{2.10}$$

Therefore, $\Delta\underline{y}(t)$ is a measure of the difference in $\underline{y}_1(t)$ and $\underline{y}_2(t)$ resulting from a variation of the initial conditions. Using the definitions just stated, the following theorem is proposed:

Theorem I: A General Theorem of System Observability

A system having state $\underline{x}(t)$ ε n-dimensional state space X^n and output $\underline{y}(t)$ ε m-dimensional output spaoe Y^m $(m \leq n)$, will be q-point observable on the finite interval $[t_o, t_f]$ if there are (q-1) and only (q-1) nonzero and distinct vectors, $\delta\underline{x}_o$, for each $\underline{x}_o$, such that:

$$\Delta\underline{y}(t) = \underline{F}\,[\underline{x}_o + \delta\underline{x}_o, \underline{u}(t), t] - \underline{F}[\underline{x}_o, \underline{u}(t)] = 0. \qquad (2.11)$$

everywhere on $[t_o, t_f]$.

Proof: (Sufficiency) Assume there are (q-1) distinct nonzero values, $\delta\underline{x}_o$, such that $\Delta\underline{y}(t) = 0 \;\forall\; t \in [t_o, t_f]$. Then, there are q and only q values, $\underline{x}(t_o)$, which may satisfy the output relationship, namely: $\underline{x}(t_o) = \underline{x}_o$, $\underline{x}_o + \underline{x}_o^1$, $\underline{x}_o + \underline{x}_o^2$, . . ., $\underline{x}_o + \underline{x}_o(q-1)$.

(Necessity) Assume the system to be q-point observable and let there be more {fewer} than (q-1) distinct nonzero values, $\delta\underline{x}_o$, which satisfy $\Delta\underline{y}(t) = 0 \;\forall\; t \in [t_o, t_f]$. If this were so then there would be more {fewer} than q values of $\underline{x}(t_o)$ which could create the same output response $\underline{y}(t) = \underline{F}(\underline{x}_o, \underline{u}(t), t)$, hence the system would not be q-point observable.

Q.E.D.

The theorem just stated and proved is not a direct method of approaching the problem of observability, That is, when the value of has been determined there is no explicit guarantee that the unknown initial condition(s) will have been found .The theorem only states that property of the system response the investigator is striving to uncover and leaves the development of an algorithm to accomplish the reconstruction as a later problem.

Observability of the Linear Systems

In the previous sections a new definition of observability was proposed and a general theorem for determining the degree of observability was stated and proven. Before discussing the solution to the problem of quadratic system observability, it will be useful to demonstrate the application of the new definition and general theorem to the problem of linear system observability. This treatment will not produce new results, but will demonstrate the technique of investigation to be used on the quadratic system.

The following linear system having an n-dimensional state, $\underline{x}(t)$, an m-dimensional output, $\underline{y}(t)$ $(m \leq n)$, and an input, $\underline{u}(t)$, will be considered:

$$\dot{\underline{x}}(t) = \underline{\underline{A}}(t)\ \underline{x}(t) + \underline{\underline{B}}(t)\ \underline{u}(t), \text{ and} \tag{2.12}$$

$$\underline{y}(t) = \underline{\underline{C}}(t)\ \underline{x}(t) + \underline{\underline{D}}(t)\ \underline{u}(t). \tag{2.13}$$

Following the procedure outlined in Theorem I, three steps will be necessary in order to investigate the quality of observability of this system. First, it will be necessary to write the solution to Equations 2.12-2.13 in terms of the input, $\underline{u}(t)$, and initial condition, $\underline{x}(t_o)$. Secondly, Equation 2.11 will be formed in terms of the system parameters and the two initial conditions: $\underline{x}(t_o) = \underline{x}_o$ and $\underline{x}(t_o) = \underline{x}_o + \delta\underline{x}_o$. Finally, the quality of observability will be determined from an investigation of the equation: $\Delta\underline{y}(t) = 0$.

The state, $\underline{x}(t)$, may be written as:

$$\underline{x}(t) = \underline{\underline{\Phi}}(t,t_o)\ \left[\, \underline{x}(t_o) \,\right] + \underline{f}(t,t_o). \tag{2.14}$$

where $\underline{\underline{\Phi}}(t,t_o)$ represents the state-transition matrix satisfying the relationship:

$$\dot{\underline{\underline{\Phi}}}(t,t_o) = \underline{\underline{A}}(t) \quad \underline{\underline{\Phi}}(t,t_o), \tag{2.15}$$

$$\underline{\underline{\Phi}}(t_o,t_o) = \underline{\underline{I}}, \tag{2.16}$$

and the vector function, $\underline{f}(t,t_o)$, is given by:

$$\underline{f}(t,t_o) = \int_{t_o}^{t} \underline{\underline{\Phi}}\,(t-\tau,\ t_o)\underline{\underline{B}}(\tau)\underline{u}(\tau)d\tau,$$

$$\forall\ t \in \left[\ t_o,t_f\ \right]. \tag{2.17}$$

Substituting Equation 2.14 into Equation 2.13 gives:

$$\underline{y}(t) = \underline{\underline{C}}(t) \quad \underline{\underline{\Phi}}(t,t_o)\ \underline{x}(t_o) + \underline{\underline{C}}(t)\underline{f}(t,t_o) + \underline{\underline{D}}(t)\underline{u}(t). \tag{2.18}$$

The result, Equation 2.18, can now be used to find $\Delta\,\underline{y}(t) = 0$ as:

$$\Delta\underline{y}(t) = \underline{\underline{C}}(t) \quad \underline{\underline{\Phi}}\,(t,t_o) \quad \delta\underline{x}_o = 0,$$

$$\forall\ t\ \in\ \left[t_o,t_f\right]\ . \tag{2.19}$$

The problem is now reduced to one of finding what if any values of $\delta\underline{x}_o$ may satisfy Equation 2.19 for all t on the interval $\left[\ t_o,t_f\ \right]$. Two possibilities present themselves: The columns of the matrix $\underline{\underline{C}}(t)\ \ \underline{\underline{\Phi}}(t,t_o)$ may or may not be linearly independent of each other. If this matrix does not have linearly independent columns then there exists an infinite number of constant vectors, $\delta\underline{x}_o$, such that Equation

2.19 is satisfied. On the other hand, if the columns of $\underline{\underline{C}}(t)\ \underline{\underline{\Phi}}\ (t,t_o)$ are linearly independent, there is one and only one value of $\delta\underline{x}_o$ which satisfies Equation 2.19, namely: $\delta\underline{x}_o = 0$. If $\delta\underline{x}_o = 0$ is the only value to satisfy Equation 2.19, then the degree of observability is $q = 1$.

In accordance with the foregoing discussion, it is possible to postulate the following theorem for linear systems of the form Equations 2.12-2.13.

> Theorem II: Observability of the Linear Systems
>
> The linear system, Equations 2.12-2.13, having state transition matrix $\underline{\underline{\Phi}}(t,t_o)$ defined by Equations 2.15-2.16 is one point observable if and only if the columns of the matrix $\underline{\underline{C}}(t)\underline{\underline{\Phi}}(t,t_o)$ are linearly independent on $[t_o,t_f]$.

A direct result of Theorem II is Corollary II-1.

> Corollary II-1: If a linear system is observable, it is one-point observable, That is, an observable linear system possesses the minimal degree of observability.

These results are not new and are well documented in the literature: Kriendler and Sarachik,[1], Chen and Desoer,[14] Athens and Falb,[15] DeRusso, Roy and Close.[16] It is well known that the observability property of linear systems is independent of the system input, $\underline{u}(t)$. However, this result does not extend to all systems. That is, there are systems wherein the conditions of observability may be dependent upon the system's inputs. However, even if the conditions of observability are input independent, as in the case of the linear systems, it is never possible to reconstruct the state of any system without explicit knowledge of the system's input. This calculation will be dealt with in Chapter IV.

Finally, the result of Theorem II may be restated for the special case in which the system matrices $\underline{A}$ and $\underline{C}$ of Equations 2.12-2.13 are time invariant.

Corollary II-2: Observability of Linear Time Invariant Systems

The linear system, Equations 2.12-2.13 having constant coefficient matrices $\underline{A}$ and $\underline{C}$, is one-point observable if and only if the composite matrix, $\underline{M}$, has rank, n, equal to the dimension of the system state.

$$\underline{\underline{M}} = [\ \underline{\underline{C}}^T \ \vdots\ \underline{\underline{A}}^T \underline{\underline{C}}^T \ \vdots\ (\underline{\underline{A}}^T)^2\ \underline{\underline{C}}^T \ \vdots\ ---\ \vdots\ (\underline{\underline{A}}^T)^{n-1}\ \underline{\underline{C}}^T\ ]$$

This extension of the results of Theorem II is also well documented in the literature.(1, 14, 15, 16)

Summary

In summary, this chapter has been devoted to the statement of an Extended Definition of Observability, the proposal and proof of a general theorem based upon the Extended Definition, and finally, a demonstration of the use of the general observability theorem by a brief investigation of linear system observability. The Extended Definition is based upon the observability criterion proposed by Kalman but has been modified to aid the investigation of observability of nonlinear systems. Although the investigation of linear system observability did not produce new results, it did demonstrate the procedure which will be used in the investigation of the nonlinear quadratic system.

CHAPTER III

QUADRATIC SYSTEM OBSERVABILITY

Objective

This chapter will be devoted to a detailed investigation of the quality of observability of a general quadratic system. The discussion will also demonstrate how both the Extended Definition of Observability and the General Observability Theorem of the previous chapter may be applied to a nonlinear system.

Although the quadratic system defined here possesses a convenient mathematical form, the resulting algebraic problem does not admit to a completely tractable solution as in the case of the totally linear system. For this reason, the results of this investigation will be presented in three parts. The first will discuss a necessary condition for q-point observability applicable to all quadratic systems; the second result is a necessary and sufficient condition for q-point observability applicable to a restricted class of quadratic systems; finally, a method will be demonstrated whereby the quality of observability of any quadratic system may be determined by means of a straightforward solution to Equation 2.11, Page 20 .

Problem Definition

The quadratic system is defined as a system having a linear, stationary, and deterministic dynamical plant and having a nonlinear output relationship:

$$\dot{\underline{x}}(t) = \underline{\underline{A}}\,\underline{x} + \underline{b}\,u(t), \text{ and} \tag{3.1}$$

$$y(t) = \underline{x}^T \underline{\underline{Q}}\,\underline{x}. \tag{3.2}$$

The state of the system, $\underline{x}(t)$, is an n-dimensional vector; $u(t)$, the system input is a scalar function of the independent variable, t; and $y(t)$ is the system output, also a scalar. The matrix $\underline{\underline{A}}$ is of dimension n x n, the column vector $\underline{b}$ is of dimension n, and the matrix, $\underline{\underline{Q}}$, is symmetric and positive semidefinite, n x n in dimension, and may always be written as $\underline{\underline{Q}} = \underline{\underline{H}}^T\underline{\underline{H}}$, where $\underline{\underline{H}}$ is also an n x n matrix. The quadratic system of Equations 3.1-3.2 is defined for all t's on the finite interval $[t_o, t_f]$. Furthermore, the initial state of the system is $\underline{x}(t=t_o)$ and the final state of the system is $\underline{x}(t=t_f)$.

Using Theorem I, Page 14, the problem of determining under what conditions the quadratic system (Equations 3.1-3.2) is observable and to what degree of multiplicity the system may be observed is reduced to finding the solution or solutions to Equation 2.11, Page 15. Restating the procedure: For every input-output pair, $u(t)$ and $y(t)$, related by Equations 3.1-3.2, there must be an initial state, $\underline{x}(t=t_o)$, which satisfies the output relationship. By convention, the as-yet-undetermined initial state, $\underline{x}(t=t_o)$ will be assigned the designation $\underline{x}_o$. The proposition put forth by Theorem I is: Knowing $\underline{x}_o$ to exist, are there other values for the initial state which satisfy the input-output relation. The remainder of this section will be devoted to laying the groundwork for answering this question.

In terms of an arbitrary initial condition, $\underline{x}(t_o)$, and an unspecified but known input, u(t), the solution of Equation 3.1 for the state, $\underline{x}(t)$, may be written as:

$$\underline{x}(t) = \underline{\underline{\Phi}}(t,t_o) \left[\underline{x}(t_o) + \underline{f}(t,t_o) \right] \;\forall\; t \;\varepsilon (t_o,t_f), \tag{3.3}$$

where $\underline{\underline{\Phi}}(t,t_o)$ represents ths state transition matrix satisfying:

$$\dot{\underline{\underline{\Phi}}}(t,t_o) = \underline{\underline{A}}\; \underline{\underline{\Phi}}(t,t_o), \tag{3.4}$$

$$\underline{\underline{\Phi}}(t_o,t_o) = \underline{\underline{I}}, \tag{3.5}$$

and where $\underline{f}(t,\ t_o)$, the forcing function is defined by:

$$\underline{f}(t,t_o) = \int_{t_o}^{t} \underline{\underline{\Phi}}^{-1} (\tau)\underline{b}u(\tau)d\tau,$$

$$\forall\; t \;\varepsilon \left[t_o,t_f \right] . \tag{3.6}$$

Equations 3.6 tacitly assumes that u(t) is a continuous function of t on $[t_o,t_f]$. Using the result of Equation 3.3, the response or output of the system, y(t), is given as:

$$y(t) = \left[\underline{x}(t_o) + \underline{f}(t,t_o) \right]^T \underline{\underline{\Phi}}^T (t,t_o)\; \underline{\underline{Q}}\; \underline{\underline{\Phi}}(t,t_o) \left[\underline{x}(t_o) + \underline{f}(t,t_o) \right] ,$$

$$\forall\; t \;\varepsilon\; (t_o,t_f). \tag{3.7}$$

By the use of Equation 2.11, Page 15, the function, $\Delta y(t)$, may be found in terms of $\underline{x}_o$ and $\underline{x}_o + \delta\underline{x}_o$, as:

$$\Delta y(t) = [\delta\underline{x}_o + 2\underline{x}_o + 2\underline{f}(t,t_o)]^T \underline{\underline{\Phi}}^T(t,t_o)\underline{\underline{Q}}\,\underline{\underline{\Phi}}(t,t_o)\delta\underline{x}_o \tag{3.8}$$

The essence of the remainder of this chapter is to find all possible values of $\delta\underline{x}_o$ which make $\Delta y(t) = 0 \;\forall\; t \in [t_o,t_f]$.

By inspection, it can be seen that $\delta\underline{x}_o = 0$ will satisfy the condition $\Delta y(t) = 0$ everywhere on the interval. However, this solution is discussed as being trivial since, by hypothesis, it has been assumed there is an initial state for the system, namely, $\underline{x}_o$. Solving Equation 3.8 for nonzero values of $\delta\underline{x}_o$ which make $\Delta y(t) = 0$ everywhere on the interval, will tell whether or not the system is observable and if observable to what degree of multiplicity this observability may be determined.

As previously stated, the determination of the quality of observability for the quadratic system has been reduced to a straightforward algebraic problem. For any particular numerical example, this algebraic problem may be quite simple. However, it should be remembered that, although the problem may be straightforward this does not guarantee that the solution is simple.

Conditions for Quadratic System Observability

A Necessary Condition of q-Point Observablility

Making use of the relationship $\underline{\underline{Q}} = \underline{\underline{H}}^T\underline{\underline{H}}$, Equation 3.8 may be equated to zero and written as:

$$[\delta \underline{x}_o + 2\underline{x}_o + 2\underline{f}(t,t_o)]^T \underline{\underline{\phi}}^T(t,t_o)\ \underline{\underline{H}}^T\underline{\underline{H}}\ \underline{\underline{\phi}}(t,t_o)\ \delta\underline{x}_o = 0. \quad (3.9)$$

This expression can be written as the inner product of two vectors:

$$[\underline{\underline{H}}\ \underline{\underline{\phi}}(t,t_o)\ (\delta\underline{x}_o + 2\underline{x}_o + 2\underline{f}(t,t_o))]^T\ [\underline{\underline{H}}\ \underline{\underline{\phi}}(t,t_o)\ \delta\underline{x}_o] = 0. \quad (3.10)$$

Recognizing that the system, Equations 3.1-3.2, Page 21, will be unobservable if Equation 3.10 possesses an infinite number of nontrivial solutions for $\delta\underline{x}_o$ (Via viz Theorem I, Page 14), the following theorem may be proven:

Theorem III: Necessary Condition for q-Point Observability of a Quadratic System

The quadratic system, Equations 3.1-3.2, is q-point observable only if the composite matrix, $\underline{\underline{M}}$, has rank, n, equal to the dimension of the system state, or:

$$\underline{\underline{M}} = [\ \underline{\underline{H}}^T \mid \underline{\underline{A}}^T\underline{\underline{H}}^T \mid (\underline{\underline{A}}^T)^2\ \underline{\underline{H}}^T \mid - - - \mid (\underline{\underline{A}}^T)^{n-1}\ \underline{\underline{H}}^T\ ] \quad (3.11)$$

Proof: Assume the system, Equations 3.1-3.2, to be q-point observable with matrix, $\underline{\underline{M}}$, having rank, $m < n$. Then, by Theorem II, Page 18, the matrix, $\underline{\underline{H}}\ \underline{\underline{\Phi}}\ (t,t_o)$, does not have n-linearly independent columns; hence, there are an infinite number of nontrivial vectors, $\delta\underline{x}_o$, such that:

$$\underline{\underline{H}}\ \underline{\underline{\Phi}}(t,t_o)\ \delta\underline{x}_o = 0, \quad (3.12)$$

for all t's on the interval. Therefore, there are an infinite number of nontrivial values, $\delta\underline{x}_o$, which satisfy Equation 3.10 everywhere on the interval, implying that the system, Equations

3.1-3.2, is not q-point observable, contrary to the hypothesis.

Q. E. D.

The test stated by Equation 3.11 is very similiar to the corresponding test for linear systems. It can be noted that if $\underline{\underline{Q}}$ is positive definite then the matrix $\underline{\underline{M}}$ (Equation 3.11) will have rank n regardless of the form of $\underline{\underline{A}}$. Therefore positive definiteness of $\underline{\underline{Q}}$ is sufficient to satisfy the necessary condition of Theorem III. It is, however, necessary to evaluate the rank of $\underline{\underline{M}}$ (Equation 3.11) for cases in which the quadratic form $\underline{x}^T\underline{\underline{Q}}\underline{x}$ is zero for some value of $\underline{x} \neq \underline{0}$.

Furthermore, this entire discussion, beginning with page 23, is equally valid for quadratic systems in which the quadratic form $\underline{x}^T\underline{\underline{Q}}\underline{x}$ is negative semi-definite. Therefore, Theorem III applies to all quadratic systems in which the output equation is semi-definite or definite.

Observability of Unforced Quadratic Systems

By applying the General Observability Theorem of Chapter II to the quadratic system, Equations 3.1-3.2, page 21, it has been possible to develop a necessary condition for the quadratic system to be q-point observable (Theorem III). An observability criterion will now be developed for a quadratic system having zero input everywhere on the interval.

The unforced quadratic system presents a situation in which it is not possible to observe the system uniquely. That is, there are always a multiplicity of possible initial conditions which may have created any given response. This situation can be easily seen by inspecting Equation 3.9 which, for the unforced case, becomes:

$$[\,\delta\underline{x}_o + 2\underline{x}_o]^T\,\underline{\underline{\Phi}}^T(t,t_o)\,\underline{\underline{H}}^T\underline{\underline{H}}\,\underline{\underline{\Phi}}(t,t_o)\,\delta\underline{x}_o = 0. \qquad (3.13)$$

By inspection there are always at least two values of the vector, $\delta\underline{x}_o$, which satisfy Equation 3.13, namely: the trivial solution $\delta\underline{x}_o=0$, and $\delta\underline{x}_o = -2\underline{x}_o$. Therefore, if the initial state, $\underline{x}(t_o) = \underline{x}_o$, is a candidate initial condition then the initial state, $\underline{x}(t_o) = -\underline{x}_o$, will equally well satisfy the system output relationship. Within the definition of multipoint observability, the unforced quadratic system can at best be two-point observable.

For a restricted class of unforced quadratic systems, namely those for which the system matrix, $\underline{\underline{A}}$, has distinct eigen-values, the following theorem and corollaries may be proved:

Theorem IV: Observability of Unforced Quadratic Systems

The quadratic system, Equations 3.1-3.2, having input $u(t)=0$ $\forall\ t \in [t_o,t_f]$ and having matrix $\underline{\underline{A}}$ with distinct eigen-values, will be observable to q-possible initial states if and only if the pair $(\underline{\underline{A}},\underline{\underline{H}})$ is observable in the sense of Theorem II.

Proof: (Necessity) The proof of necessity has already been given in Theorem III.

(Sufficiency) By hypothesis, the matrix $\underline{\underline{A}}$ of the system Equation 3.1-3.2, has distinct eigen-values; therefore, there exists a unique transformation $\underline{x} = \underline{\underline{T}}\ \underline{z}$, such that Equation 3.1-3.2 may be written:

$$\dot{\underline{z}}(t) = \underline{\underline{T}}^{-1}\ \underline{\underline{A}}\ \underline{\underline{T}}\ \underline{z}(t) = \underline{\underline{\Lambda}}_A\ \underline{z}(t), \text{ and} \tag{3.14}$$

$$y(t) = \underline{z}^T\ (t)\ \underline{\underline{K}}^T\underline{\underline{K}}\ \underline{z}(t), \tag{3.15}$$

where $\underline{\underline{K}} = \underline{\underline{H}}\ \underline{\underline{T}}$ and the matrix, $\underline{\underline{\Lambda}}_A = \underline{\underline{T}}^{-1}\ \underline{\underline{A}}\ \underline{\underline{T}}$, is a diagonal matrix. Therefore, Equation 3.9 may be written as:

$$(\delta \underline{x}_o + 2\underline{x}_o)^T \underline{\underline{\Phi}}^T(t,t_o)\ \underline{\underline{K}}^T \underline{\underline{K}}\ \underline{\underline{\Phi}}(t,t_o)\ \delta \underline{x}_o = 0. \tag{3.16}$$

Defining the columns of the matrix, $\underline{\underline{K}}$, as vectors, $\underline{k}_i$:

$$\underline{\underline{K}} = [\underline{k}_1 \quad \underline{k}_2 \quad --- \quad \underline{k}_n],$$

and the diagonal elements of $\underline{\underline{\Lambda}}_A$ as:

$$\underline{\underline{\Lambda}}_A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

The expansion of Equation 3.16 can then be written as:

$$\sum_{i=1}^{n} \delta x_{io}\,(\delta x_{io} + 2x_{io})\,(\underline{k}_i{}^T \underline{k}_i)\, e^{2\lambda_i (t-t_o)} \quad +$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \left[\delta x_{io}\,(\delta x_{jo} + 2x_{jo}) + \right.$$

$$\left. \delta x_{jo}\,(\delta x_{io} + 2x_{io})\right] \left[\underline{k}_i{}^T \underline{k}_j\right] e^{(\lambda_i+\lambda_j)(t-t_o)} = 0. \tag{3.17}$$

Equation 3.17 contains all the information as to the number of vectors, $\delta \underline{x}_o$, which satisfy Equation 3.17. The more terms which appear in Equation 3.17, the more restricted will be the number of $\delta \underline{x}_o$'s satisfying the equality. Likewise, the maximum number of $\delta \underline{x}_o$'s satisfying the equality will occur when the equality, Equation 3.17, has a minimal number of terms. Assuming the system satisfies the conditions of the theorem, a minimal number of terms will appear in Equation 3.17 if all columns of $\underline{\underline{K}}$ are mutually orthogonal. Under these conditions, Equation 3.17 becomes:

$$\sum_{i=1}^{n} \delta x_{io} \; (\delta x_{io} + 2x_{io}) \; (\underline{k}_i^T \underline{k}_i) \; e^{2\lambda_i (t-t_o)} = 0. \qquad (3.18)$$

Since this equation is composed of n linearly independent functions of t, it will only be satisfied by equating each of the coefficients to zero. That is:

$$\begin{aligned} \delta x_{10}(\delta x_{10}+2x_{10}) &= 0, \\ \delta x_{20}(\delta x_{20}+2x_{20}) &= 0, \\ &\vdots \\ \delta x_{n0}(\delta x_{n0}+2x_{n0}) &= 0. \end{aligned} \qquad (3.19)$$

Since the vector, $\delta \underline{x}_o$, is composed of n-elements (δx_{10}, δx_{20}, . . ., δx_{n0}), there are a total of $q = 2^n$ distinct choices of $\delta \underline{x}_o$ satisfying Equation 3.19. Therefore, the system is q-point observable.

Q.E.D.

Equation 3.17 has been written as two summations, the first involving products of δx_{i0} and ($\delta x_{i0} + \delta 2x_{i0}$), the second involving what will be called the "cross-product" terms. If the vector components of the matrix $\underline{\underline{K}}$ are not mutually orthogonal, that is $\underline{k}_i^T \; \underline{k}_j \neq 0$, $i \neq j$, then the cross-product terms of Equation 3.17 must be satisfied along with those already noted in Equation 3.18. Since the set (Equation 3.19) allows a total of $q = 2^n$ possible solutions for $\delta \underline{x}_o$, the further restriction of satisfying any cross-product requirements can only reduce the number of values of $\delta \underline{x}_o$ that satisfy Equation 3.17. This relationship can easily be shown by the following example.

Example:

Given a system defined by Equations 3.14-3.15, where n = 3:

$$\underline{\underline{\Lambda}}_A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \text{ and}$$

$$\begin{aligned} \underline{k}_1^T \underline{k}_1 &= a & \qquad \underline{k}_1^T \underline{k}_2 &= 0 \\ \underline{k}_2^T \underline{k}_2 &= b & \qquad \underline{k}_1^T \underline{k}_3 &= d \\ \underline{k}_3^T \underline{k}_3 &= c & \qquad \underline{k}_2^T \underline{k}_3 &= 0 \end{aligned}$$

and the constants a, b, c, d $\neq 0$. Equation 3.17 may be expanded as:

$$\begin{aligned} & a(\delta x_{10})(\delta x_{10}+2x_{10})e^{2\lambda_1(t-t_o)} + \\ & b(\delta x_{20})(\delta x_{20}+2x_{20})e^{2\lambda_2(t-t_o)} + \\ & c(\delta x_{30})(\delta x_{30}+2x_{30})e^{2\lambda_3(t-t_o)} + \\ & d\left[\delta x_{10}(\delta x_{30}+2x_{30})+\delta x_{30}(\delta x_{10}+2x_{10})\right]e^{(\lambda_1+\lambda_3)(t-t_o)} = 0. \end{aligned} \tag{3.20}$$

Since the values λ_1, λ_2, and λ_3 are distinct, the first three terms of Equation 3.20 are linearly independent functions of t everywhere on $[t_o, t_f]$.

The function $e^{(\lambda_1+\lambda_3)(t-t_o)}$ may or may not be linearly independent of the other functions in Equation 3.20; however, this possibility does not change the nature of the results previously mentioned. To show this, two cases will be discussed.

Case 1: Assume $\lambda_1 + \lambda_3 \neq 2\lambda_2$.

In this instance the four linearly independent functions of t must sum to zero for the entire interval $[t_o,t_f]$. This result is only true if the coefficients of the functions $e^{2\lambda_1(t-t_o)}$, $e^{2\lambda_2(t-t_o)}$, $e^{2\lambda_3(t-t_o)}$, and $e^{(\lambda_1+\lambda_3)(t-t_o)}$ are identically zero. That is:

$$\delta x_{10}(\delta x_{10} + 2x_{10}) = 0, \tag{3.21-1}$$

$$\delta x_{20}(\delta x_{20} + 2x_{20}) = 0, \tag{3.21-2}$$

$$\delta x_{30}(\delta x_{30} + 2x_{30}) = 0, \text{ and} \tag{3.21-3}$$

$$\delta x_{10}(\delta x_{30} + 2x_{30}) + \delta x_{30}(\delta x_{10} + 2x_{10}) = 0. \tag{3.21-4}$$

By inspection of Equations 3.21-1, 2, 3, there are eight choices of $\underline{x}_o$ which simultaneously satisfy these three equations:

$$\delta \underline{x}_o = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}^*, \begin{bmatrix} 0 \\ 0 \\ -2x_{30} \end{bmatrix}, \begin{bmatrix} 0 \\ -2x_{20} \\ 0 \end{bmatrix}^*, \begin{bmatrix} 0 \\ -2x_{20} \\ -2x_{30} \end{bmatrix},$$

$$\begin{bmatrix} -2x_{10} \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} -2x_{10} \\ 0 \\ -2x_{30} \end{bmatrix}^*, \quad \begin{bmatrix} -2x_{10} \\ -2x_{20} \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -2x_{10} \\ -2x_{20} \\ -2x_{30} \end{bmatrix}^* \tag{3.22}$$

The cross-product relationship (Equation 3.21-4) will only be satisfied by four of these eight choices of the vector, $\delta\underline{x}_o$, namely: those having an asterisk above them.

Case 2: Assume $\lambda_1 + \lambda_3 = 2\lambda_2$.

Equations 3.21-1, 2, 3, then becomes:

$$\delta x_{10}(\delta x_{10}+2x_{10}) = 0, \tag{3.23-1}$$

$$b\,\delta x_{20}(\delta x_{20}+2x_{20}) + d\,\delta x_{10}(\delta x_{30}+2x_{30}) + d\,\delta x_{30}(\delta x_{10}+2x_{10}) = 0, \tag{3.23-2}$$

$$\delta x_{30}(\delta x_{30}+2x_{30}) = 0. \tag{3.23-3}$$

By inspection, it can be noted that four values of $\delta\underline{x}_o$ marked by an asterisk in Equation 3.22 also satisfy Equations 3.23. There are, however, at most four additional values of $\delta\underline{x}_o$ which also satisfy Equations 3.23. These are:

$$\begin{bmatrix} 0 \\ -x_{20} \pm (x_{20}^2 + 4\frac{d}{b}\, x_{10}x_{30})^{1/2} \\ -2x_{30} \end{bmatrix}, \quad \begin{bmatrix} -2x_{10} \\ -x_{20} \pm (x_{20}^2 + 4\frac{d}{b}\, x_{10}x_{30})^{1/2} \\ 0 \end{bmatrix}$$

As a consequence of Theorem IV, Page 26, the following corollary can be stated:

Corollary IV-1: If the unforced quadratic system of Equations 3.14-3.15, having state vector $\underline{x}(t)$ of dimension n, satisfies the conditions of Theorem IV, then the system will be q-point observable where $2^n \geq q \geq 2$.

Justification for this corollary can be made by noting that there are always at least two values of $\delta\underline{x}_o$ which satisfy Equation 3.16. Likewise, upon expanding Equation 3.16, it can be noted that there are at most 2^n values of $\delta\underline{x}_o$ which satisfy the set of equations designated as Equation 3.19, Page 28.

From the example it can be noted that if one independent cross-product term exists, the basic set of possible solutions is reduced by a factor of 2. If the matrix, $\underline{\underline{Q}}$, is defined as:

$$\underline{\underline{Q}} = \underline{\underline{K}}^T\underline{\underline{K}} \tag{3.24}$$

the number of independent cross-product terms which are present in the expansion of Equation 3.17 can be determined as R. The number R is fully defined in Appendix I.

Corollary IV-2: Given the quadratic system:

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}, \text{ and} \tag{3.25}$$

$$y = \underline{x}^T\underline{\underline{Q}}\,\underline{x}, \tag{3.26}$$

having matrix $\underline{\underline{A}}$ with n-distinct eigen-values, a nonsingular transformation, $\underline{x} = \underline{\underline{T}}\ z$, exists such that Equation 3.25-3.26 become :

$$\underline{\dot{z}} = \underline{\underline{\Lambda}}_A\ \underline{z}, \text{ and} \tag{3.27}$$

$$y = \underline{z}^T \underline{\underline{\tilde{Q}}}\ \underline{z}, \tag{3.28}$$

where:

$$\underline{\underline{\Lambda}}_A = \underline{\underline{T}}^{-1}\ \underline{\underline{A}}\ \underline{\underline{T}}, \text{ and } \ \underline{\underline{\tilde{Q}}} = \underline{\underline{T}}^T \underline{\underline{Q}}\ \underline{\underline{T}}.$$

The system, Equation 3.25-3.26, will be q-point observable if and only if the diagonal terms of $\underline{\underline{\tilde{Q}}}$ are all nonzero. Furthermore, the value of q is given by:

$$q \leq \begin{cases} 2^{n-R} & n-R > 1 \\ 2 & n-R \leq 1 \end{cases} \tag{3.29}$$

where:

n   represents the dimension of the system state vector, $\underline{x}(t)$, and

R   the total number of independent cross-product terms as defined in Appendix I.

The inequality in Equation 3.29 holds in those instances when there are repeated solutions for the candidate values, $\underline{x}_o$. In order to show the usefulness of Theorem IV and its corollaries, an example will be worked to demonstrate these results.

Example 3-B:

Given the system:

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -5 & 2 & 2 & 2 \\ -4 & 1 & 0 & 4 \end{bmatrix} \underline{x}, \text{ and} \tag{3.30}$$

$$y = \underline{x}^T \begin{bmatrix} 4 & 0 & -2 & -2 \\ 0 & 1 & -1 & 0 \\ -2 & -1 & 2 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \underline{x}, \tag{3.31}$$

determine if the system is q-point observable by means of Theorem IV.

From Theorem IV, the system, Equations 3.30-3.31, will be q-point observable if and only if it satisfies the necessary condition of Theorem III. That is, the matrix, $\underline{\underline{M}}$, must have rank n = 4.

$$\underline{\underline{M}} = \left[\underline{\underline{H}}^T \,\vdots\, \underline{\underline{A}}^T\underline{\underline{H}}^T \,\vdots\, (\underline{\underline{A}}^T)^2\underline{\underline{H}}^T \,\vdots\, (\underline{\underline{A}}^T)^3\underline{\underline{H}}^T \right], \tag{3.32}$$

where $\underline{\underline{A}}$ is the system matrix of Equation 3.30 and H is the matrix such that $\underline{\underline{Q}} = \underline{\underline{H}}^T\underline{\underline{H}}$. From Equation 3.31, the matrix, $\underline{\underline{Q}}$, may be written as:

$$\underline{\underline{Q}} = \begin{bmatrix} 4 & 0 & -2 & -2 \\ 0 & 1 & -1 & 0 \\ -2 & -1 & 2 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & 0 & 1 & 1 \end{bmatrix} \qquad (3.33)$$

Performing the indicated operation in Equation 3.32:

$$\underline{\underline{M}} = \left[\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & -2 & -4 & -3 & -4 & -11 & -22 & -19 & -22 & -5 & -100 & -93 & -100 & -221 \\ 0 & -1 & 0 & 0 & 1 & -1 & 1 & 3 & 7 & 3 & 7 & 19 & 37 & 29 & 37 & 93 \\ 0 & 1 & 0 & 1 & 0 & 2 & 0 & 2 & 0 & 4 & 0 & 4 & 0 & 8 & 0 & 8 \\ 1 & 0 & 1 & 1 & 4 & 2 & 4 & 6 & 16 & 12 & 16 & 28 & 64 & 56 & 64 & 120 \end{array}\right]$$

which has the same rank as:

$$M \simeq \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \text{rank } n = 4.$$

Therefore, it can be stated that the system is q-point observable where $16 \geq q \geq 2$. To determine the value of q, the system must be put in the canonical form indicated by Equation 3.27-3.28 of Corollary IV-2. Using the transformation:

$$\underline{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \underline{z}. \tag{3.34}$$

the system of Equations 3.30-3.31 can be written as:

$$\dot{\underline{z}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \underline{z}, \text{ and} \tag{3.35}$$

$$y = \underline{z}^T \begin{bmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 2 & 2 & 4 & 5 \\ 2 & 3 & 5 & 7 \end{bmatrix} \underline{z}. \tag{3.36}$$

As discussed in Appendix I, for this example R = 2. The degree of observability can now be calculated as:

$$q = (2)^{n-R} = 2^{4-2} = 4. \tag{3.37}$$

Therefore, the system is four-point observable. The validity of this result can be shown by a direct expansion of Equation 3.16 and solving the resulting set of simultaneously nonlinear equations for all possible values of $\delta \underline{x}_o$ which satisfy the equality. In the canonical form (Equations

3.35-3.36), Equation 3.16 becomes:

$$
\begin{aligned}
&2(\delta z_{10})(\delta z_{10}+2z_{10})e^{2t} + 2(\delta z_{20})(\delta z_{20}+2z_{20})e^{4t} + \\
&4(\delta z_{30})(\delta z_{30}+2z_{30})e^{6t} + 7(\delta z_{40})(\delta z_{40}+2z_{40})e^{8t} + \\
&2\left[\delta z_{10}(\delta z_{30}+2z_{30}) + \delta z_{30}(\delta z_{10}+2z_{10})\right]e^{4t} + \\
&2\left[\delta z_{10}(\delta z_{40}+2z_{40}) + \delta z_{40}(\delta z_{10}+2z_{10})\right]e^{5t} + \\
&2\left[\delta z_{20}(\delta z_{30}+2z_{30}) + \delta z_{30}(\delta z_{20}+2z_{20})\right]e^{5t} + \\
&3\left[\delta z_{20}(\delta z_{40}+2z_{40}) + \delta z_{40}(\delta z_{20}+2z_{20})\right]e^{6t} + \\
&5\left[\delta z_{30}(\delta z_{40}+2z_{40}) + \delta z_{40}(\delta z_{30}+2z_{30})\right]e^{7t} = 0.
\end{aligned} \tag{3.38}
$$

Collecting like terms:

$$
\begin{aligned}
&2\left[(\delta z_{10})(\delta z_{10}+2z_{10})\right] e^{2t} + \\
&2\left[(\delta z_{20})(\delta z_{20}+2z_{20})+\delta z_{10}(\delta z_{30}+2z_{30})+\delta z_{30}(\delta z_{10}+2z_{10})\right] e^{4t} + \\
&2\left[(\delta z_{10})(\delta z_{40}+2z_{40})+\delta z_{40}(\delta z_{10}+2z_{10})+\delta z_{20}(\delta z_{30}+2z_{30})+\delta z_{30}(\delta z_{20}+2z_{20})\right]e^{5t}+ \\
&4\left[4(\delta z_{30})(\delta z_{30}+2z_{30})+3\delta z_{20}(\delta z_{40}+2z_{40})+3\delta z_{40}(\delta z_{20}+2z_{20})\right] e^{6t} + \\
&5\left[(\delta z_{30})(\delta z_{40}+2z_{40})+\delta z_{40}(\delta z_{30}+2z_{30})\right] e^{7t} + \\
&7\left[(\delta z_{40})(\delta z_{40}+2z_{40})\right] e^{8t} = 0.
\end{aligned} \tag{3.39}
$$

Since Equation 3.39 is the summation of six linearly independent functions of t, the equality can only be maintained if the set of Equations 3.40 are all simultaneously satisfied:

$$\delta z_{10}(\delta z_{10}+2z_{10}) = 0,$$

$$\delta z_{20}(\delta z_{20}+2z_{20})+\delta z_{10}(\delta z_{30}+2z_{30})+\delta z_{30}(\delta z_{10}+2z_{10}) = 0,$$

$$\delta z_{10}(\delta z_{40}+2z_{40})+\delta z_{40}(\delta z_{10}+2z_{10})+\delta z_{20}(\delta z_{30}+2z_{30})+\delta z_{30}(\delta z_{20}+2z_{20}) = 0,$$

$$4\delta z_{30}(\delta z_{30}+2z_{30})+3\delta z_{20}(\delta z_{40}+2z_{40})+3\delta z_{40}(\delta z_{20}+2z_{20}) = 0,$$

$$\delta z_{30}(\delta z_{40}+2z_{40})+\delta z_{40}(\delta z_{30}+2z_{30}) = 0, \text{ and}$$

$$\delta z_{40}(\delta z_{40}+2z_{40}) = 0. \tag{3.40}$$

There are only four distinct values of the vector, $\delta \underline{z}_o$, which will satisfy these six equations:

$$\delta \underline{z}_o = \begin{bmatrix} \delta z_{10} \\ \delta z_{20} \\ \delta z_{30} \\ \delta z_{40} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2z_{10} \\ -2z_{20} \\ -2z_{30} \\ -2z_{40} \end{bmatrix}, \begin{bmatrix} -2x_{10} \\ 0 \\ 0 \\ 0 \end{bmatrix}_{\substack{x_{30}=0 \\ x_{40}=0}}, \begin{bmatrix} 0 \\ -2x_{20} \\ -2x_{30} \\ 0 \end{bmatrix}_{x_{40}=0} \tag{3.41}$$

In this example, it has been shown that by the use of Theorem IV and its corollaries, it is not only possible to determine whether or not a system is q-point observable, but it is also possible to determine the degree of observability. Finally, it has been shown that a straightforward if not difficult solution of the basic equation (Equation 3.9) resulting from the General Observability Theorem, will also yield the desired results.

q-Point Observability - A Direct Method

All conditions for the observability of quadratic systems thus far have made use of mathematical insight into what conditions are necessary and/or sufficient to solve Equation 3.9. There is, of course, a more direct method of determining quadratic system observability, namely, solve Equation 3.9 for $\delta \underline{x}_o$.

The procedure is to expand the equation into a summation of scalar terms, each term containing a function of the independent variable, t, which is linearly independent of all other functions in the summation. The coefficient of each of the functions of t will itself be a function of the unknown δx_{i0}'s. Since the summation must be identically zero for the entire interval, $[t_o, t_f]$, each coefficient may be set equal to zero, thereby forming a set of nonlinear algebraic equations in the unknown δx_{i0}'s. Solving this set of equations will determine not only the degree of observability but will also give the relationship between all the candidate initial conditions.

To demonstrate the technique just described, a detailed example will be worked:

<u>Example 3-C:</u>

$$\dot{\underline{x}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 3 & 0 & 0 \\ -5 & 2 & 2 & 2 \\ -4 & 1 & 0 & 4 \end{bmatrix} \underline{x} + \begin{bmatrix} c_1 \\ c_1 - c_3 \\ c_1 + c_2 + c_4 \\ c_1 + c_3 + c_4 \end{bmatrix} u(t), \text{ and} \tag{3.42}$$

$$y(t) = \underline{x}^T \begin{bmatrix} 4 & 0 & -2 & -2 \\ 0 & 1 & -1 & 0 \\ -2 & -1 & 2 & 1 \\ -2 & 0 & 1 & 3 \end{bmatrix} \underline{x}. \tag{3.43}$$

This system has the same structure as the system of the previous example except that the system now has a scalar input, u(t), and a column vector composed of combinations of constants c_1, c_2, c_3 and c_4. It is possible to calculate the state transition matrix, $\underline{\underline{\Phi}}(t,t_o)$, and the forcing functions, $\underline{f}(t,t_o)$ associated with Equations 3.42-3.43, by substituting these values directly into Equation 3.9, and proceeding with the solution of the problem. However, the complexity of the resulting algebraic problem is more burdensome than need be. If, instead, the system is transformed to that canonical representation in which the system matrix is diagonal, the complexity in solving the problem is greatly reduced. Let:

$$\underline{x} = \underline{\underline{T}}\,\underline{z} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \underline{z}. \tag{3.44}$$

Then, by direct substitution of Equation 3.44 into Equations 3.42-3.43, the system is presented in terms of the state vector $\underline{z}$, as:

$$\dot{\underline{z}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \underline{z} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} u(t), \text{ and} \tag{3.45}$$

$$y = \underline{z}^T \begin{bmatrix} 2 & 0 & 2 & 2 \\ 0 & 2 & 2 & 3 \\ 2 & 2 & 4 & 5 \\ 2 & 3 & 5 & 7 \end{bmatrix} \underline{z}. \tag{3.46}$$

The forcing function, $\underline{f}(t)$, is found from Equation 3.5 as:

$$\underline{f}(t,t_o) = \begin{bmatrix} f_1(t,t_o) \\ f_2(t,t_o) \\ f_3(t,t_o) \\ f_4(t,t_o) \end{bmatrix} = \begin{bmatrix} \int_{t_o}^{t} c_1 e^{-\tau} u(\tau) d\tau \\ \int_{t_o}^{t} c_2 e^{-2\tau} u(\tau) d\tau \\ \int_{t_o}^{t} c_3 e^{-3\tau} u(\tau) d\tau \\ \int_{t_o}^{t} c_4 e^{-4\tau} u(\tau) d\tau \end{bmatrix}. \tag{3.47}$$

It was determined in the previous example that the system satisfied the necessary condition for q-point observability. It is the purpose of this example to show that the degree of system observability may be determined by a straightforward, if not difficult, solution to Equation 3.9, page 24. That is:

$$\left[\delta \underline{x}_o + 2\underline{x}_o + 2\underline{f}(t,t_o)\right]^T \underline{\underline{\Phi}}^T(t,t_o)\ \underline{\underline{Q}}\ \underline{\underline{\Phi}}(t,t_o)\ \delta\underline{x}_o = 0. \tag{3.48}$$

By direct substitution the state transition matrix, $\underline{\underline{\Phi}}(t,t_o)$, of the system represented by Equation 3.45, the forcing function given by Equation 3.47 and the matrix, $\underline{\underline{Q}}$, from Equation 3.46, Equation 3.48 becomes:

$$\begin{bmatrix} \delta x_{10}+2x_{10}+2f_1(t,t_o) \\ \delta x_{20}+2x_{20}+2f_2(t,t_o) \\ \delta x_{30}+2x_{30}+2f_3(t,t_o) \\ \delta x_{40}+2x_{40}+2f_4(t,t_o) \end{bmatrix}^T \begin{bmatrix} 2e^{2(t-t_o)} & 0 & 2e^{4(t-t_o)} & 2e^{5(t-t_o)} \\ 0 & 2e^{4(t-t_o)} & 2e^{5(t-t_o)} & 3e^{6(t-t_o)} \\ 2e^{4(t-t_o)} & 2e^{5(t-t_o)} & 4e^{6(t-t_o)} & 5e^{7(t-t_o)} \\ 2e^{5(t-t_o)} & 3e^{6(t-t_o)} & 5e^{7(t-t_o)} & 7e^{8(t-t_o)} \end{bmatrix} \begin{bmatrix} \delta x_{10} \\ \delta x_{20} \\ \delta x_{30} \\ \delta x_{40} \end{bmatrix} = 0. \tag{3.49}$$

In order to simplify the problem, assume that $c_2=c_3=c_4=0$ and $c_1\neq 0$. Then expanding Equation 3.49 gives:

$$\begin{aligned}
&\delta x_{10}(\delta x_{10}+2x_{10})\left[2e^{2(t-t_o)}\right]+\delta x_{20}(\delta x_{20}+2x_{20})\left[2e^{4(t-t_o)}\right] + \\
&\left[\delta x_{30}(\delta x_{30}+2x_{30})+\delta x_{40}(\delta x_{40}+2x_{40})\right]\left[7e^{4(t-t_o)}\right] + \\
&\left[\delta\, x_{10}(\delta x_{30}+2x_{30})+\delta x_{30}(\delta x_{10}+2x_{10})\right] \left[2e^{4(t-t_o)}\right] + \\
&\left[\delta\, x_{10}(\delta x_{40}+2x_{40})+\delta x_{40}(\delta x_{10}+2x_{10})+\delta x_{20}(\delta x_{30}+2x_{30})+\right. \\
&\quad \left.\delta x_{30}(\delta x_{20}+2x_{20})\right] \left[2e^{5(t-t_o)}\right] + \\
&\left[\delta x_{20}(\delta x_{40}+2x_{40})+\delta x_{40}(\delta x_{20}+2x_{20})\right] \left[3e^{6(t-t_o)}\right] + \\
&\left[\delta x_{30}(\delta x_{40}+2x_{40})+\delta x_{40}(\delta x_{30}+2x_{30})\right] \left[5e^{7(t-t_o)}\right]+ \\
&4\delta x_{10}f\ (t,t_o)e^{2(t-t_o)}+ 4\delta x_{30}f\ (t,t_o)e^{4(t-t_o)}+ \\
&4\delta x_{40}f\ (t,t_o)e^{5(t-t_o)} = 0.
\end{aligned} \tag{3.50}$$

Collecting terms, Equation 3.50 may be written as:

$$\begin{aligned}
&\left[\delta x_{10}(\delta x_{10}+2x_{10})\right] \left[2e^{2(t-t_o)}\right] + \left[\delta x_{20}(\delta x_{20}+2x_{20}) + \right. \\
&\quad \left.\delta x_{10}(\delta x_{30}+2x_{30}) + \delta x_{30}(\delta x_{10}+2x_{10})\right] \left[2e^{4(t-t_o)}\right] + \\
&\left[\delta x_{10}(\delta x_{40}+2x_{40}) + \delta x_{40}(\delta x_{10}+2x_{10}) + \delta x_{20}(\delta x_{30}+2x_{30}) + \right. \\
&\quad \left.\delta x_{30}(\delta x_{20}+2x_{20})\right] \left[2e^{5(t-t_o)}\right] + \left[4\delta x_{30}(\delta x_{30}+2x_{30}) + \right. \\
&\quad \left.3\delta x_{20}(\delta x_{40}+2x_{40}) + 3\delta x_{40}(\delta x_{20}+2x_{20})\right] \left[3e^{6(t-t_o)}\right] +
\end{aligned}$$

$$\left[\delta x_{30}(\delta x_{40}+2x_{40}) + \delta x_{40}(\delta x_{30}+2x_{30})\right] \left[5e^{7(t-t_o)}\right] +$$
$$\left[\delta x_{40}(\delta x_{40}+2x_{40})\right] \left[7e^{8(t-t_o)}\right] + 4\delta x_{10}e^{2(t-t_o)}f_1(t,t_o) +$$
$$4\delta x_{30}e^{4(t-t_o)}f_1(t,t_o) + 4\delta x_{40}e^{5(t-t_o)}\ f_1(t,t_o) = 0. \qquad (3.51)$$

It can be immediately recognized that Equation 3.51 is made up of a summation of nine functions of t, linearly independent of each other everywhere on the interval $[t_o, t_f]$. If the summation is to equal zero everywhere on the interval, each coefficient of the linearly independent functions must be identically zero. That is:

$$\delta x_{10}(\delta x_{10}+2x_{10}) = 0, \qquad (3.52\text{-}1)$$
$$\delta x_{20}(\delta x_{20}+2x_{20})+\delta x_{10}(\delta x_{30}+2x_{30})+\delta x_{30}(\delta x_{10}+2x_{10}) = 0, \qquad (3.52\text{-}2)$$
$$\delta x_{10}(\delta x_{40}+2x_{40})+\delta x_{40}(\delta x_{10}+2x_{10})+\delta x_{20}(\delta x_{30}+2x_{30}) +$$
$$\delta x_{30}(\delta x_{20}+2x_{20}) = 0, \qquad (3.52\text{-}3)$$
$$4\delta x_{30}(\delta x_{30}+2x_{30})+3\delta x_{20}(\delta x_{40}+2x_{40})+3\delta x_{40}(\delta x_{20}+2x_{20}) = 0, \qquad (3.52\text{-}4)$$
$$\delta x_{30}(\delta x_{40}+2x_{40})+\delta x_{40}(\delta x_{30}+2x_{30}) = 0, \qquad (3.52\text{-}5)$$
$$\delta x_{40}(\delta x_{40}+2x_{40}) = 0, \qquad (3.52\text{-}6)$$
$$4\delta x_{10} = 0, \qquad (3.52\text{-}7)$$
$$4\delta x_{20} = 0, \text{ and} \qquad (3.52\text{-}8)$$
$$4\delta x_{40} = 0. \qquad (3.52\text{-}9)$$

This set of equations must all be simultaneously satisfied in order to satisfy the conditicns implied by assuming that the same output trajectory may be created by more than one initial condition. Relationships 3.52-7, 8, 9, restrict δx_{10}, δx_{30}, and δx_{40} to be each identically zero, leaving only the determination of δx_{20} to specify the degree of observability of the system. By direct substitution of $\delta x_{10} = \delta x_{30} = \delta x_{40} = 0$, the 3.52 Equations become:

$$\delta x_{20}(\delta x_{20} + 2x_{20}) = 0, \tag{3.53-1}$$

$$\delta x_{20}(\ 2x_{30}) = 0, \text{ and} \tag{3.53-2}$$

$$\delta x_{20}(\ 2x_{40}) = 0. \tag{3.53-3}$$

The only value of δx_{20} which simultaneously satisfies all these relationships is $\delta x_{20} = 0$. Therefore, it can be concluded that the only vector of $\delta \underline{x}_o$ which satisfies the 3.52 Equations is $\delta \underline{x}_o = 0$, thereby showing the system to be uniquely observable.

It should be noted that this example system does satisfy the conditions stated in Appendix II. That is, the pair $(\underline{A},\underline{H})$ is observable in the sense of Theorem II. Likewise, it should be noted that the linear portion of the system is not controllable. As pointed out in Appendix II, the condition that $u(t) \neq 0$ for all t on the interval of observation implies that the system have at least one controllable state. If the system had no controllable states it would be tantamount to having no input. The example system has one controllable state.

Summary

A number of important results pertaining to quadratic systems have been discussed in this chapter. The primary result is that quadratic systems may be uniquely observable. In addition, very simple tests have been stated for determining the degree of observability for quadratic systems having a zero forcing function. Also, a necessary and sufficient condition for unique observability has been proven. Finally, a direct method for determining observability has been given in an example by directly solving the equation $\Delta y(t) = 0$.

CHAPTER IV

STATE RECONSTRUCTION

Objective

In this chapter, a discussion will be presented which shows the relationship between the observability theory and reconstruction of the system state. A detailed example of the unique reconstruction of the state of a quadratic system will be used to illustrate the discussion. A distinction will be drawn between formal reconstruction of a system state and practical or on-line methods discussed in the literature.

Observability and State Reconstruction

Throughout this work, a distinction has been maintained between observability and state reconstruction. Observability is a system property; state reconstruction is a technique or mathematical algorithm by which the state of an observable system may be reconstructed. If a system is determined to be uniquely observable, this does not imply that just any reconstruction algorithm may be used to reconstruct the state. Usually, implementation of a particular reconstruction technique places requirements on the system in addition to those of observability stated by Theorem I, page 14.

This situation has been discussed by Luenberger [6, 7, 8], and Gilchrist.[9, 17] In the case of the former, the system state may be reconstructed on-line,with precision, only if the system structure, system input, system output, and the initial state of the system are known. If the initial state of the system is unknown, the reconstructed state of the system approaches the real value of the system state in an asymptotic fashion. Likewise, with the method of state reconstruction suggested by Gilchrist, the investigator must be very sensitive to the manner in which data are collected and manipulated. As a result of this restriction on data collection, it is necessary to define observability in order to satisfy the conditions necessary for implementing the reconstruction technique.

The Extended Definition of Observability and the General Observability Theorem stated in Chapter II suggest a reconstruction technique which may be implemented to produce the exact state of the system. Unfortunately, the mathematical precision or knowledge of the system structure, its inputs and its outputs necessary to implement the technique, may only be achieved formally. Such mathematical insight into the system and its behavior nearly precludes the use of this method of reconstruction in the implementation of a control strategy. The value of formal state reconstruction lies in its use as a tool in determining the observability of a system without confusing the issue with restrictions associated with on-line or practical reconstruction techniques.

Formal Reconstruction of the State of a Quadratic System

Formal state reconstruction will be demonstrated by an example in which the dynamics of the system are linear but the output of the system is of quadratic form. Such a system is shown in Figure 3. The basic question being asked in this problem is: Is the system state observable if the

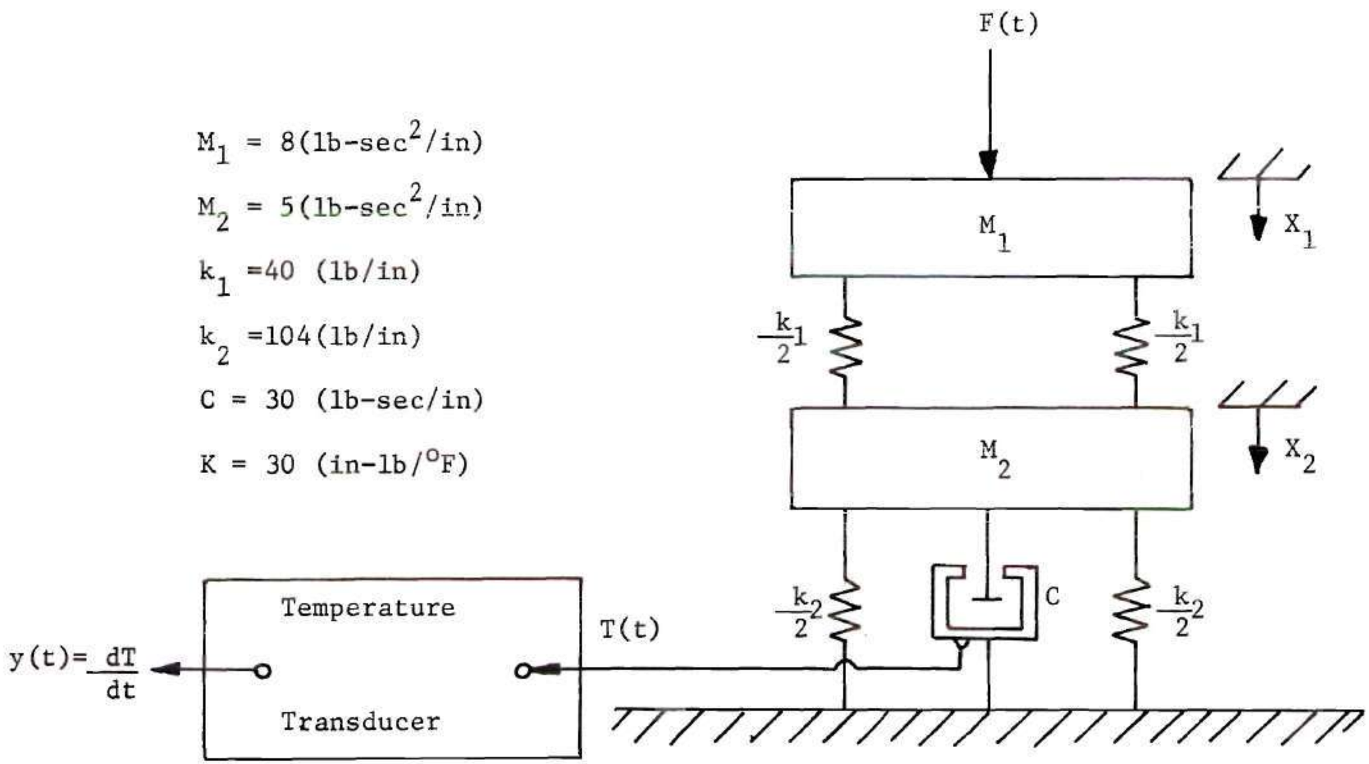

Figure 3. A Quadratic System

power dissipated by the viscous damper is monitored? For the purpose of illustration, numerical values will be assigned to the physical parameters of the system.

If it is assumed that the operation of the viscous damper is adiabatic, the work done on the damper simply causes an increase in its internal energy. That is:

$$\Delta w = \Delta E \ . \tag{4.1}$$

But, the change in internal energy of the damper results in a change in its temperature, T. That is:

$$\Delta E = K \Delta T. \tag{4.2}$$

The time rate of change of work done on the damper is a measure of the power being dissipated by it. That is:

$$\frac{\Delta w}{\Delta t} = \frac{\Delta E}{\Delta t} = K\frac{\Delta T}{\Delta t} \ . \tag{4.3}$$

In the limit (i.e., as $\Delta t \rightarrow 0$), Equation 4.3 becomes:

$$\frac{dw}{dt} = K\frac{dT}{dt}\text{, or} \tag{4.4}$$

$$\frac{dT}{dt} = \frac{1}{K}\frac{dw}{dt} \tag{4.5}$$

Also:

$$\frac{dw}{dt} = \frac{d}{dt} \int_{o}^{x_2(t)} C\dot{x}_2(t)\,dx_2(t) = C\dot{x}_2^2 \quad (t). \tag{4.6}$$

Referring to Figure 3, the transducer output, y(t), may be written in terms of the velocity of the mass, M_2, as:

$$y(t) = \frac{C}{K} \; \dot{x}_2^2 \; (t). \tag{4.7}$$

Using the state-space notation, the equations of motion for the system are:

$$\dot{\underline{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m_1} & 0 & \frac{k_1}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_1}{m_1} & 0 & \frac{k_2}{m_2} & \frac{-C}{m_2} \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ \frac{1}{m_1} \\ 0 \\ 0 \end{bmatrix} F(t), \text{ and} \tag{4.8}$$

$$y(t) = \underline{z}^T(t) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{C}{K} \end{bmatrix} \underline{z}(t), \tag{4.9}$$

where:

$$\underline{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ \dot{x}_1(t) \\ x_2(t) \\ \dot{x}_2(t) \end{bmatrix} \tag{4.10}$$

Evaluating, numerically, the algebraic coefficients of Equations 4.8-4.9 gives:

$$\dot{\underline{z}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 13 & -6 \end{bmatrix} \underline{z} + \begin{bmatrix} 0 \\ \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} F(t), \text{ and} \tag{4.11}$$

$$y(t) = \underline{z}^T(t) \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{z}(t) \tag{4.12}$$

Before proceeding with the reconstruction of the state vector, it will be useful to determine the degree of observability of the system. The System 4.11-4.12 is of the form given by Equations 3.1-3.2, Page 21 where:

$$\underline{\underline{A}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 8 & 0 & 13 & -6 \end{bmatrix}, \quad \underline{\underline{H}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} \tag{4.13}$$

The requirements of a uniquely observable quadratic system are that the pair $(\underline{\underline{A}},\underline{\underline{H}})$ be observable in the sense of Theorem II.

The pair $(\underline{\underline{A}},\underline{\underline{H}})$ is observable if and only if the partitioned matrix, $\underline{\underline{M}}$, is of rank n = 4.

$$\underline{\underline{M}} = \left[\underline{\underline{H}}^T \;\middle|\; \underline{\underline{A}}^T\underline{\underline{H}}^T \;\middle|\; (\underline{\underline{A}}^T)^2\underline{\underline{H}}^T \;\middle|\; (\underline{\underline{A}}^T)^3\underline{\underline{H}}^T \right]. \tag{4.14}$$

By direct substitution from Equation 4.13, $\underline{\underline{M}}$ is:

$$\underline{\underline{M}} = \left[\begin{array}{cccc|cccc|cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -224 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 0 & 0 & 0 & -78 & 0 & 0 & 0 & -259 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -23 & 0 & 0 & 0 & 60 \end{array}\right], \text{ or} \tag{4.15}$$

$$\underline{\underline{M}} \simeq \begin{bmatrix} 0 & 8 & -48 & -224 \\ 0 & 0 & 8 & -48 \\ 0 & 13 & -78 & -259 \\ 1 & 6 & -23 & 60 \end{bmatrix}. \tag{4.16}$$

The matrix of Equation 4.16 has rank $n = 4$; therefore, the pair $(\underline{\underline{A}},\underline{\underline{H}})$ is observable.

The process of formally reconstructing the state of the quadratic system requires three pieces of information: First, the input of the system as an explicit function of time must be known. This information is not a requirement peculiar to the quadratic system, but is required whenever the state of any system is reconstructed.

Secondly, it is necessary to know the exact mathematical structure of the system. In this instance, Equations 4.11-4.12 supply this information. However, there is an assumption implicit with the knowledge of the system input and its structure, namely that these two pieces of information allow the investigator to write an explicit function of the form:

$$y(t) = g(t,\underline{x}_o), \tag{4.17}$$

where $y(t)$ is the system output (in this case a scalar function), t is the independent variable, and $\underline{x}_o$ is the initial value of the state of the system, i.e. $\underline{x}(t=t_o) = \underline{x}_o$. The value of the initial state is not known as a numerical quantity but rather is included in $g(t,\underline{x}_o)$ as an algebraic quantity.

The third piece of necessary information is the output of the system. The system output must also be known as an explicit function of the independent variable, t. Such information must be consistent with the unknown initial condition of the system and the system input, $F(t)$. These data may be thought of as having originated by means of an experiment performed on the mathematical system; however, such an experiment should not be confused with a laboratory test which may require approximations in modeling, or assumptions about instrumentation, or exactness in the fitting of a mathematical form to the data collected.

For example, assume that a mathematical experiment has been performed on the system represented by Equations 4.11-4.12 and the response, y(t), was found to be:

$$\begin{aligned} y(t) = \frac{1}{25} \Big[& 576e^{-4t} \cos^2 t - 3936e^{-4t} \cos t \sin t + \\ & 2064e^{-3t} \cos t \sin 2t - 1152e^{-3t} \cos t \cos 2t + \\ & 6724e^{-4t} \sin^2 t - 7052e^{-3t} \sin t \sin 2t + \\ & 3936e^{-3t} \sin t \cos 2t + 1849e^{-2t} \sin^2 2t - \\ & 2064e^{-2t} \cos 2t \sin 2t + 576e^{-2t} \cos^2 2t \Big] \mu(t). \end{aligned} \tag{4.18}$$

Likewise, knowing that the system input is a step function, as shown in Figure 4, the function g(·) of Equation 4.17 may be calculated as:

$$\begin{aligned} g(t,\underline{x}_o) = \Big[& (\tfrac{4}{5} - 4x_{10} + 4x_{30} + 2x_{40})^2 e^{-4t} \cos^2 t + \\ & (\tfrac{8}{5} - 8x_{10} + 8x_{30} + 4x_{40})(-\tfrac{2}{5} + 8x_{10} - 18x_{30} - 4x_{40}) e^{-4t} \cos t \sin t + \\ & (\tfrac{8}{5} - 8x_{10} + 8x_{30} + 4x_{40})(-\tfrac{2}{5} - 2x_{10} + 2x_{20} + 9x_{30} - x_{40}) e^{-3t} \cos t \sin 2t + \\ & (\tfrac{8}{5} - 8x_{10} + 8x_{30} + 4x_{40})(-\tfrac{4}{5} + 4x_{10} - 4x_{30} - x_{40}) e^{-3t} \cos t \cos 2t + \\ & (\tfrac{8}{5} + 8x_{10} - 4x_{20} - 18x_{30} - 4x_{40})^2 e^{-4t} \sin^2 t + \end{aligned}$$

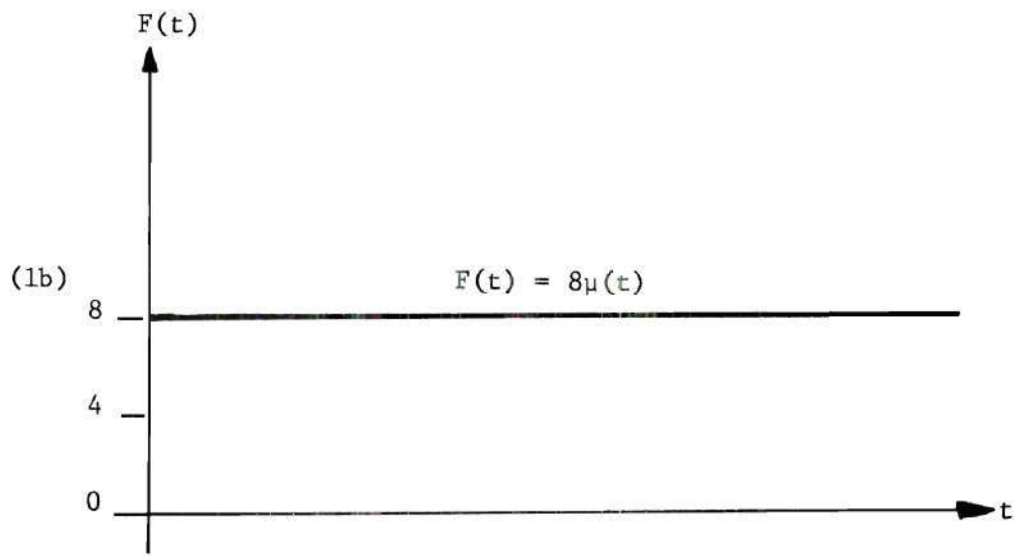

Figure 4. System Input F(t).

$$(\frac{16}{5}+16x_{10}-8x_{20}-36_{30}-8x_{40})(-\frac{2}{5}-2x_{10}+2x_{20}+9x_{30}-x_{40})e^{-2t}\sin t \sin 2t+$$

$$(\frac{16}{5}+16x_{10}-8x_{20}-36_{30}-8x_{40})(-\frac{4}{5}+4x_{10}-4x_{30}-x_{40})e^{-2t}\cos 2t \sin 2t+$$

$$(-\frac{2}{5}-2x_{10}+2x_{20}+9x_{30}-x_{40})^2e^{-2t}\sin^2 2t +$$

$$(-\frac{4}{5}-4x_{10}+4x_{20}+18x_{30}-2x_{40})(-\frac{4}{5}+4x_{10}-4x_{30}-x_{40})e^{-2t}\cos 2t \sin 2t+$$

$$(-\frac{4}{5}+4x_{10}-4x_{30}-x_{40})^2e^{-2t}\cos^2 2t\,]\,\mu(t). \tag{4.19}$$

The "experimental" data of Equation 4.18 may be equated to the analytical expression for $g(t,\underline{x}_o)$ given by Equation 4.19. Since both sides of that equality are composed of like, linearly independent functions of t, the equality may be satisfied by equating the coefficients of like terms to yield:

$$\frac{576}{25} = \frac{4}{5}-4x_{10}+4x_{30}+2x_{40})^2\ , \tag{4.20-1}$$

$$-\frac{3936}{25} = (\frac{8}{5}-8x_{10}+8x_{30}+4x_{40})(\frac{8}{5}+8x_{10}-4x_{20}-18x_{30}-4x_{40}) \tag{4.20-2}$$

$$\frac{2064}{25} = (\frac{8}{5}-8x_{10}+8x_{30}+4x_{40})(-\frac{2}{5}-2x_{10}+2x_{20}+9x_{30}-x_{40})\ , \tag{4.20-3}$$

$$-\frac{1152}{25} = (\frac{8}{5}-8x_{10}+8x_{30}+4x_{40})(-\frac{4}{5}+4x_{10}-4x_{30}-x_{40})\ , \tag{4.20-4}$$

$$\frac{6724}{25} = \left(\frac{8}{5} + 8x_{10} - 4x_{20} - 18x_{30} - 4x_{40}\right)^2, \qquad (4.20\text{-}5)$$

$$\frac{7052}{25} = \left(\frac{16}{5} + 16x_{10} - 8x_{20} - 36x_{30} - 8x_{40}\right)\left(-\frac{2}{5} - 2x_{10} + 2x_{20} + 9x_{30} - x_{40}\right) \qquad (4.20\text{-}6)$$

$$\frac{3936}{25} = \left(\frac{16}{5} + 16x_{10} - 8x_{20} - 36x_{30} - 8x_{40}\right)\left(-\frac{4}{5} + 4x_{10} - 4x_{30} - x_{40}\right) \qquad (4.20\text{-}7)$$

$$\frac{1849}{25} = \left(-\frac{2}{5} - 2x_{10} + 2x_{20} + 9x_{30} - x_{40}\right)^2, \qquad (4.20\text{-}8)$$

$$-\frac{2064}{25} = \left(-\frac{4}{5} - 4x_{10} + 4x_{20} + 18x_{30} - 2x_{40}\right)\left(-\frac{4}{5} + 4x_{10} - 4x_{30} - x_{40}\right) \qquad (4.20\text{-}9)$$

$$\frac{576}{25} = \left(-\frac{4}{5} + 4x_{10} - 4x_{30} - x_{40}\right)^2, \qquad (4.20\text{-}10)$$

These ten equations relate four unknown quantities, namely: the four unknown components of the initial state of the system. An acceptable solution to the set is one which satisfies all ten of the equalities. By trial and error it can be seen that the solution:

$$\underline{x}_o = \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (4.21)$$

does satisfy Equation 4.20. Further study reveals that this solution is the only solution to the equality.

Having determined the initial condition of the system and having been assured that the solution is a unique one, it is now a simple task to reconstruct the state of the system by direct substitution of $\underline{x}_o$ into Equation 4.11, page 51 . Therefore:

$$\underline{z}(t) = \underline{\underline{\Phi}}(t) \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \int_0^t \underline{\underline{\Phi}}(t-\tau) \begin{bmatrix} 0 \\ \frac{1}{8} \\ 0 \\ 0 \end{bmatrix} 8\mu(\tau)\,d\tau \tag{4.22}$$

where the state transition matrix is given by:

$$\underline{\underline{\Phi}}(t) = e^{\underline{\underline{A}}t} \tag{4.23}$$

and the matrix $\underline{\underline{A}}$ is given by Equation 4.13, page 52 .

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The basic conclusion of this work is that quadratic systems may be observable. As shown by example in Chapter III conditions do exist which, if satisfied, allow the state of a quadratic system to be uniquely observed. Unlike the linear system, however, the quadratic system may be observed in a multipoint sense. That is, by making use of the Extended Definition of Observability it is possible to state conditions for which there are a finite number of distinct trajectories which may produce the same output or response.

The Extended Definition of Observability and the General Observability Theorem have proved to be useful in the investigation of the quadratic systemand should prove equally useful in the investigation of other nonlinear systems.

Finally, it may be concluded that the quality of observability is indeed a system property. If a system is observable, then its state may be reconstructed by means of a "formal" state reconstruction. Actual state reconstruction is not guaranteed by conditions of observability.

Recommendations

Having determined that forced quadratic systems frequently are uniquely observable, there is now a basis for proposing further investigations. The first of these concerns the task of practical state reconstruction. Although practical state reconstruction has not been discussed in this work, it is evident that once a system is determined to be observable, the usefulness of the system output for control purposes must depend upon the quality of the state reconstructor. Two different techniques have already been cited in the literature: asymptotic state reconstruction and discrete point reconstruction. Both methods may be applicable to nonlinear systems. The former method, discussed by Luenberger,[6, 7, 8] may not be practical for all nonlinear systems. The latter method, discussed by Gilchrist,[9, 17] should be applicable to most problems. However, the algebraic problem of solving large numbers of nonlinear equations for n-unknowns most certainly will present problems.

If a state reconstruction device can be developed for the quadratic system, then quadratic output instruments may be put to more purposeful uses. For example, radar systems are most accurate in determining the radial distance to the objective and are less accurate in determining the relative direction cosines of the objective. With the development of a state reconstructor for quadratic systems, it may well be possible to improve the accuracy of such radar systems.

A second example is an omnidirectional strain gauge. This instrument produces a voltage output without respect to the direction of the strain imparted to the object to which the gauge is attached. The usefulness of the device is in its high-frequency response which, in conjunction with a state reconstructor, may allow the directional strain components to be resolved.

A third means of instrumentation is the measurement of power, as in the example in the text. Although the text example may seem to be contrived, it does demonstrate that a single power measurement of one element in a dynamic network may be sufficient for control purposes. Since power is oftentimes the cost function used in optimization, an additional savings may be experienced by using the same instrumentation for both cost-evaluation and state-variable feedback.

Finally, it is recommended that additional work may be done on the observability of systems having nonlinear differential equations and quadratic outputs. One such system has already been discussed in the literature by Kostyukovskii.[19,20]

Summary

In this chapter it has been shown that for systems which are observable the state may be reconstructed. However, the method of reconstruction is at best a difficult procedure to implement and in many cases may not be applicable as a practical means of reconstructing a system state.

APPENDIX I

COUNTING OF CROSS-PRODUCT TERMS

The number R necessary to predict the degree of observability by means of Corollary IV-2, page 32, is found by counting the number of independent cross-product terms which will be found in the expansion of Equation 3.17, page 27.

Using the matrices of Example 3-B, page 34,

$$R = r - k$$

where:

r the number of non-zero terms above the diagonal of the matrix $\tilde{Q}$

k the number of eigen-value combinations (λ_i, λ_j) corresponding to the non-zero terms $\tilde{q}_{ij}$ $(i \neq j)$ such that:

$$\lambda_i + \lambda_j = \lambda_k + \lambda_l$$

except for $\{i,j\} = \{k,l\}$.

With reference to Equation 3.36, page 36, it can be noted that $r = 5$. From Equation 3.35, define $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$, and $\lambda_4 = 4$. Since the terms $\tilde{q}_{12} = 0$, the cross-product terms involving the summation of

λ_1 and λ_2 will not appear in the expansion of Equation 3.17; therefore $\lambda_1 + \lambda_2$ is not considered in determining the value k.

Therefore,

$$\lambda_1 + \lambda_3 = \lambda_2 + \lambda_2$$

$$\lambda_2 + \lambda_4 = \lambda_3 + \lambda_3$$

$$\lambda_2 + \lambda_3 = \lambda_1 + \lambda_4$$

The value of k is the three (3) since all three sets of eigenvalues are present in the system's response. The total number of independent cross-products is:

$$\begin{aligned} R &= r - k \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

APPENDIX II

A PROPOSED OBSERVABILITY THEOREM

The purpose of this Appendix is to present several of the author's views on the observability of forced quadratic systems. It has been shown that quadratic systems having a zero input on the interval of observation may never be uniquely observable, At best, such a system may be two-point observable. Furthermore, it has been shown that some quadratic systems, having non-zero inputs on the interval of observation, may be uniquely observable. Although it has not been possible to prove conditions both necessary and sufficient for unique observability, it is the author's opinion that such conditions do exist.

L. B. Rall[18] has presented necessary and sufficient conditions for the unique solution of quadratic equations. Unfortunately, the technique does not produce usable results for systems formulated in terms of an orthogonal state space such as that used in this work.

Observability of quadratic systems with non-zero inputs is dependent upon the input of the system. This can be seen from the form of Equation 3.9, Page 24,

$$\delta\underline{x}_o^T \underline{\underline{\Phi}}^T(t,t_o)\ \underline{\underline{H}}^T\underline{\underline{H}}\ \underline{\underline{\Phi}}(t,t_o)[\delta\underline{x}_o + 2\underline{f}(t,t_o)] = 0. \qquad \text{(A-1)}$$

Since $\underline{f}(t,t_o)$ is a function of the system input, the system's input must play a role in creating situations in which non-zero values of $\delta\underline{x}_o$ will or will not satisfy Equation (A-1) everywhere on the interval of observation. Furthermore, conditions of controllability may well play an important part in the problem.

By working example problems it is possible to show that quadratic systems which satisfy the conditions of Theorem III, have non-zero inputs on the interval of observation and which have at least one controllable state are almost always uniquely observable. Heuristically the reason can be seen by re-writing Equation (A-1) as:

$$\delta \underline{x}_o^T \underline{\underline{\Phi}}^T \underline{\underline{H}}^T \underline{\underline{H}}\, \underline{\underline{\Phi}} \delta \underline{x}_o + 2\, \delta \underline{x}_o^T\, \underline{\underline{\Phi}}^T\, \underline{\underline{H}}^T \underline{\underline{H}} \underline{\underline{\Phi}} (\underline{x}_o + \underline{f}(t,t_o)) = 0. \qquad \text{(A-2)}$$

If the system is at least one state controllable then the second term in Equation (A-2) will contribute a linear term δx_{i0} for each i = 1, 2, . . ., n. Under such conditions, it appears unlikely that a single non-zero set of values δx_{i0}, i = 1, 2, . . ., n could be found which satisfy the equality throughout the interval $[t_o, t_f]$. This can be seen in Example 3-C, Page 14 and the example problem discussed in Chapter IV.

This heuristic argument may be re-stated as a Theorem:

Proposed Theorem for Unique Observability

> A quadratic system of the form of Equations 3.1-3.2, page 21, satisfying the conditions of Theorem III, page 25, will be uniquely observable on the interval $[t_o,t_f]$ if and only if the forcing function $\underline{f}(t,t_o) \neq 0$ for all t on $[t_o,t_f]$.

For systems having a scalar input the requirement that $\underline{f}(t,t_o) \neq 0$ $\forall$ t on $[t_o,t_f[$ is equivalent to requiring that the system have at least one controllable state. Although this work has not been concerned with systems having multi-dimensional inputs, the Proposed Theorem should be equally applicable to the multi-dimensional input system.

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