Braces and Pfaffian Orientations

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May 10, 2012

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4. Given a digraph, does it have no directed cycle of even length?
5. Given a digraph does it have a subdivision with no even cycle?

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2. When is a real square matrix sign non-singular?
3. When does a bipartite graph have a Pfaffian Orientation?
4. Given a digraph, does it have no directed cycle of even length?
5. Given a digraph does it have a subdivision with no even cycle?

All of these are equivalent!

The Theorem

Theorem (McCuaig, Robertson, Seymour, Thomas, 1996)

A brace has a Pfaffian orientation if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the 4*-sum operation.*

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A brace has a Pfaffian orientation if and only if either it is isomorphic to the Heawood graph, or it can be obtained from planar braces by repeated application of the 4-sum operation.

The hard part of this theorem boils down to:

Theorem (Robertson, Seymour, Thomas, 1996)

Let G be a nonplanar brace. Then G contains an even subdivision of $K_{3,3}$, the Heawood graph, or Rotunda with a perfect matching in the complement.

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Definition

A bipartite graph G is a *brace* if every matching of size 2 can be extended to a perfect matching. Equivalently, $G = (A, B)$ is a brace if for every set $X \subseteq A$, $|N(X)| \geq |X| + 2$ and similarly for B.

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Definition

An *even subdivision* of a graph is a subdivision in which each edge is evenly subdivided (or equivalently in which each edge is replaced by an odd path).

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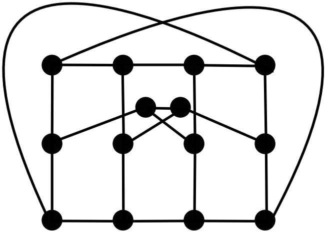
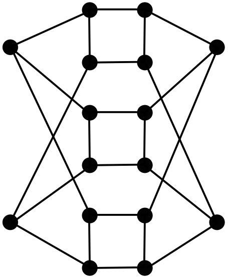


The Heawood Graph



Rotunda

Our Theorem

Theorem (Robertson, Seymour, Thomas, 1996)

Let G be a nonplanar brace. Then G contains an even subdivision of $K_{3,3}$, the Heawood graph, or Rotunda with a perfect matching in the complement.

We prove the same theorem, but with a completely different set of methods.

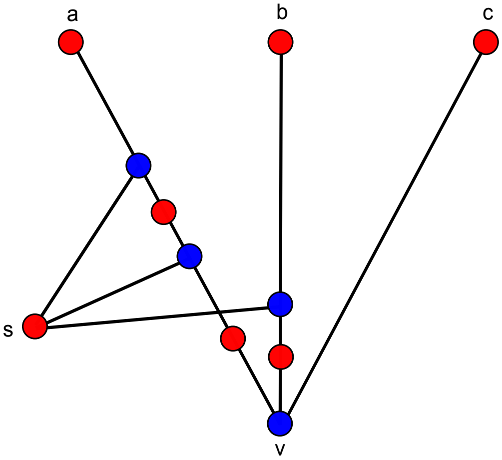
The Theorem

Theorem (Robertson, Seymour, Thomas, 1996)

Let G be a nonplanar brace. Then G contains an even subdivision of $K_{3,3}$, the Heawood graph, or Rotunda with a perfect matching in the complement.

1. Structural lemma
2. Find an even $K_{3,3}$
3. Find a good vertex
4. Look at outcomes from a good vertex (Heawood and Rotunda)
5. Prove perfect matching in complement

Structural Lemma and Bicontraction



Find a Good Unmatched Vertex

Take a maximum matching, M, in the complement of our $K_{3,3}$.

Some A vertex is unmatched. Chase M-alternating paths until we hit the $K_{3,3}$ to get three paths (might have to reroute M, but won't shrink it), then apply the lemma.

Pick everything so that the unmatched vertex hits paths incident with as many different B vertices of the $K_{3,3}$ as possible.

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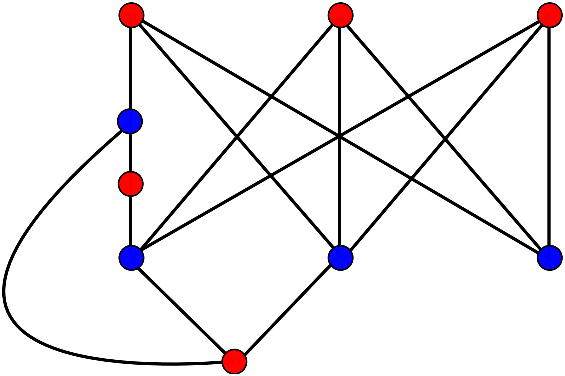

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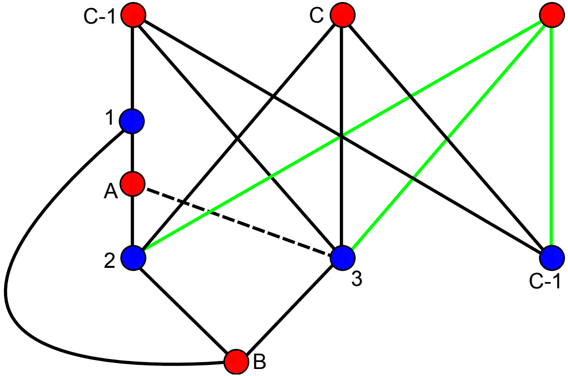
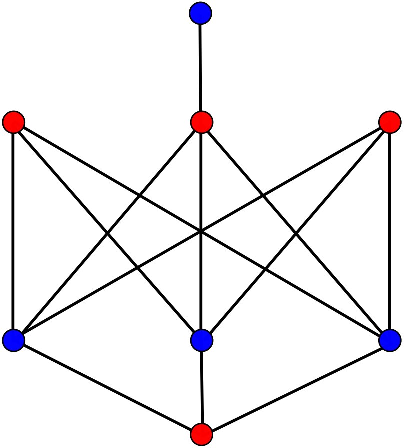
Pick everything so that the unmatched vertex hits paths incident with as many different *B* vertices of the $K_{3,3}$ as possible.



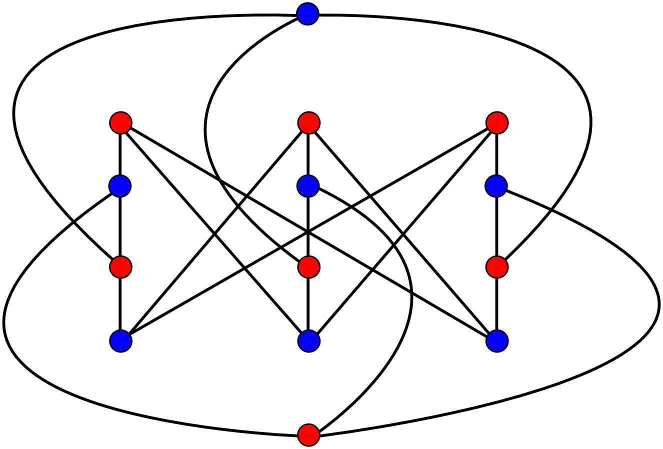
Find Rotunda or Heawood

Now the first unmatched vertex hits all three B vertices. Look at the corresponding B unmatched vertex. Suppose one of its ends contracts to an A of the $K_{3,3}$.



Find Rotunda or Heawood

So they fight everywhere. A couple of cases give a better matching. The other two cases are:



The Heawood Graph

Find Rotunda or Heawood

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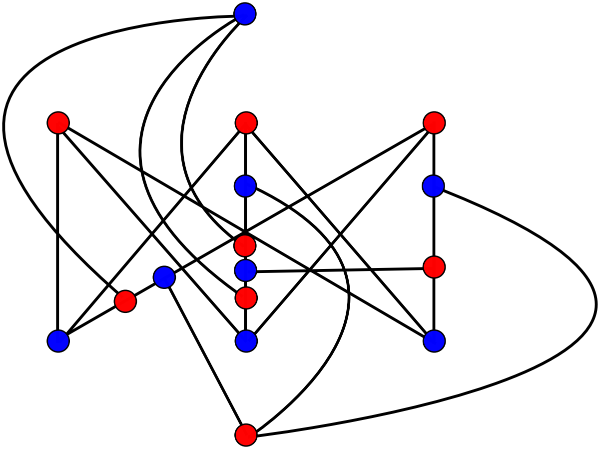

Rotunda

Finish Up

Now given Rotunda or the Heawood Graph, look at another unmatched vertex. Contracts to two different vertices, so like adding an edge between two vertices of the same parity. Fairly quick to see that this gives a better $K_{3,3}$, Rotunda, or Heawood Graph.

Find an even $K_{3,3}$

For finding the even $K_{3,3}$, braces may not even be necessary:

Conjecture

Let G be an internally four-connected non-planar bipartite graph. Then G contains an even subdivision of $K_{3,3}$.

Thank you

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