ISOSPECTRAL GRAPH REDUCTIONS, ESTIMATES OF MATRICES' SPECTRA, AND EVENTUALLY NEGATIVE SCHWARZIAN SYSTEMS

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Benjamin Zachary Webb

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ISOSPECTRAL GRAPH REDUCTIONS, ESTIMATES OF MATRICES' SPECTRA, AND EVENTUALLY NEGATIVE SCHWARZIAN SYSTEMS

Approved by:

Dr. Leonid A. Bunimovich, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Yuri Bakhtin
School of Mathematics
Georgia Institute of Technology

Dr. Luca Dieci
School of Mathematics
Georgia Institute of Technology

Dr. Dana Randall
College of Computing
Georgia Institute of Technology

Dr. Howie Weiss
School of Mathematics
Georgia Institute of Technology

Date Approved: March 8, 2011

To my wife, Rebekah.

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SUMMARY

This dissertation can be essentially divided into two parts. The first, consisting of Chapters I, II, and III, studies the graph theoretic nature of complex systems. This includes the spectral properties of such systems and in particular their influence on the systems dynamics. In the second part of this dissertation, or Chapter IV, we consider a new class of one-dimensional dynamical systems or functions with an *eventual negative Schwarzian derivative* motivated by some maps arising in neuroscience.

To aid in understanding the interplay between the graph structure of a network and its dynamics we first introduce the concept of an *isospectral graph reduction* in Chapter I. Mathematically, an isospectral graph transformation is a graph operation (equivalently matrix operation) that modifies the structure of a graph while preserving the eigenvalues of the graphs weighted adjacency matrix. Because of their properties such reductions can be used to study graphs (networks) modulo any specific graph structure e.g. cycles of length n, cliques of size k, nodes of minimal/maximal degree, centrality, betweenness, etc.

The theory of isospectral graph reductions has also lead to improvements in the general theory of eigenvalue approximation. Specifically, such reductions can be used to improved the classical eigenvalue estimates of Gershgorin, Brauer, Brualdi, and Varga for a complex valued matrix. The details of these specific results are found in Chapter II. The theory of isospectral graph transformations is then used in Chapter III to study time-delayed dynamical systems and develop the notion of a *dynamical network expansion* and *reduction* which can be used to determine whether a network of interacting dynamical systems has a unique global attractor.

In Chapter IV we consider one-dimensional dynamical systems of an interval.

In the study of such systems it is often assumed that the functions involved have a negative Schwarzian derivative. Here we consider a generalization of this condition. Specifically, we consider the functions which have some iterate with a negative Schwarzian derivative and show that many known results generalize to this larger class of functions. This includes both systems with regular as well as chaotic dynamic properties.

CHAPTER I

ISOSPECTRAL GRAPH REDUCTIONS

1.1 Introduction

Real world networks, i.e. those found in nature and technology, typically have a complicated irregular structure and consist of a large number of highly interconnected dynamical units. Coupled biological and chemical systems, neural networks, social interacting species, and the Internet are only a few such examples [4, 23, 24, 42, 46, 50]. Because of their complex structure, the first approach to capturing the global properties of such systems has been to model them as graphs whose nodes represent elements of the network and the edges define a topology (graph of interactions) between these elements. This a principally a static approach to modeling networks in which only the structure of the network's interactions (and perhaps weights associated to them) are analyzed.

An additional line of research regarding such networks has been the investigation of their dynamical properties. This has been done by modeling networks via interacting dynamical systems [1, 6, 17]. Important processes studied within this framework include synchronization [16] or contact processes, such as opinion formation and epidemic spreading. These studies give strong evidence that the structure of a network can have a substantial impact on its dynamics [7].

Regarding this connection between the structure and dynamics of a network, the spectrum of the network's adjacency matrix has recently emerged as a key quantity in the study of a variety of dynamical networks. For example, systems of interacting dynamical units are known to synchronize depending only on the dynamics of the uncoupled dynamical systems and the spectral radius of the network adjacency matrix

[31, 30]. Moreover, the eigenvalues of a network are important for determining if the dynamics of a network is stable [6, 1].

To aid in understanding this interplay between the structure (graph of interactions) and dynamics of a network here we introduce the concept of an *isospectral graph transformation*. Such transformations allow one to modify a network at the level of a graph while maintaining properties related to the network's dynamics. Mathematically, an isospectral graph transformation is a graph operation that modifies the structure of a graph while preserving the eigenvalues of the graph's (weighted) adjacency matrix.

We note here that besides modifying interactions, such transforms can also reduce or increase the number of nodes in a graph (network). As not to violate the fundamental theorem of algebra, isospectral graph transformations preserve the spectrum of the graph (specifically number of eigenvalues) by allowing edges to be weighted by rational functions.

We note that by allowing such weights it may appear that our procedure is trading the complexity of the graph's structure for complex edge weights. However, this is not the case. In fact, it is often possible isospectrally reduce a graph in such a way that the reduced graph's edges weights belong to a particularly nice set e.g. positive integers, real numbers, or for unweighted graphs the weight set $\{1\}$ (see theorem 1.3.4).

For a graph G with vertex set V this is done by reducing or expanding G with respect to specific subsets of V known as structural sets (see section 1.3.1 for exact definitions). As a typical graph has many different structural sets it is possible to consider different isospectral transformations of the same graph as well as sequences of such transformations as a reduced graph generally will again have its own structural sets.

In this regard, the flexibility of this procedure is reflected in the fact that for a typical graph G it is possible to sequentially reduce G to a graph on any nonempty subset of its original vertex set. This follows from an existence and uniqueness result regarding sequences of isospectral reductions. That is a typical graph G can be uniquely reduced to a graph on any nonempty subset of its vertices (see theorem 1.3.8 and theorem 1.3.10). Because of this uniqueness, isospectral graph reductions can be used to induce new equivalence relations on the set of all graphs under some mild assumptions. Namely, two graphs (networks) are *spectrally equivalent* if they can by isospectrally reduced to one and the same graph. The implication then is that this procedure allows for the possibility of studying graphs (networks) modulo any graph feature that can be uniquely defined.

This itself is motivated by the need when dealing with complex systems to find ways of reducing complexity while maintaining some important network characteristic(s). Such reductions amount to course graining or finding the right scale at which to view the network. Isospectral graph transformations, specifically reductions, serve this need by allowing one to view a network as a smaller network with essentially the same spectrum. Finding the right scale here then amounts to finding the right structural set over which to reduce the network.

Chapter I is divided as follows. In section 1.2 we present notation and some general definitions. Section 1.3 contains the description of an isospectral graph transformation and results on sequences of such transformations. Graph transformations over fixed weight sets and spectral equivalence are treated in 1.4. The proof of results contained in sections 1.3 and 1.4 are then given in section 1.5. Section 1.7 contains some concluding remarks regarding the applications presented in chapters II and III.

1.2 Preliminaries

In this chapter we are primarily interested in the topology (graph structure) of dynamical networks. Here, we consider the most general class of networks (with fixed topology i.e. the graph of interactions does not change in time), namely those networks whose graph of interactions are finite, directed, with loops and with edge weights in the set $\mathbb{W}$. Such graphs form the class of graphs we denote by $\mathbb{G}$. A graph $G \in \mathbb{G}$ is an ordered triple $G = (V, E, \omega)$ where V and E are the *vertex set* and *edge set* of G respectively. Moreover, $\omega : E \to \mathbb{W}$ gives the *edge weights* of G where $\omega(e) = 0$ if and only if $e \notin \mathbb{W}$.

The set $\mathbb{W}$ is defined as follows. Let $\mathbb{C}[\lambda]$ be the set of polynomials in the complex variable λ with complex coefficients. The set $\mathbb{W}$ is the set of rational functions of the form $p(\lambda)/q(\lambda)$ where $p(\lambda), q(\lambda) \in \mathbb{C}[\lambda]$, $p(\lambda)$ and $q(\lambda)$ have no common factors (roots), and $q(\lambda) \neq 0$. Here, we note that the set $\mathbb{W}$ is a subfield of the field of rational functions over $\mathbb{C}$.

The vertex set of the graph $G = (V, E, \omega)$ is given the labelling $V = \{v_1, \ldots, v_n\}$ where we denote an edge from v_i to v_j by e_{ij}. The matrix $M(G) = M(G, \lambda)$ defined entrywise by

$$M(G)_{ij} = \omega(e_{ij})$$

is then the *weighted adjacency matrix* of G. The *spectrum* or *eigenvalues* of a matrix $A(\lambda) \in \mathbb{W}^{n\times n}$ are the solutions including multiplicities to the equation

$$\det(A(\lambda) - \lambda I) = 0. \tag{1}$$

For the graph G we let $\sigma(G)$ be the spectrum of $M(G)$.

The spectrum of a matrix with entries in $\mathbb{W}$ is therefore a generalization of the spectrum of a matrix with complex entries. Moreover, as $\det\big(M(G) - \lambda I\big)$ is the ratio of two polynomials $p(\lambda)/q(\lambda) \in \mathbb{W}$ we let $\sigma^{-1}(G)$ be the solutions including multiplicities of the equation $q(\lambda) = 0$.

Note both $\sigma(G)$ and $\sigma^{-1}(G)$ are *lists* of numbers. That is,

$$\sigma(G) = \Big\{ \ (\sigma_i, n_i) : 1 \leq i \leq p, \ \sigma_i \in \mathbb{C}, \ n_i \in \mathbb{N} \Big\}$$

where n_i is the multiplicity of the solutions σ_i to (1) i.e. eigenvalues, p the number of distinct solutions, and (σ_i, n_i) the elements in the list. The list $\sigma^{-1}(G)$ has a similar formulation.

1.3 Isospectral Graph Reductions and Expansions

In this section we formally describe the isospectral reduction process and the associated isospectral graph expansions. Once this is in place we then give some specific examples of this method and present some results regarding sequences of isospectral reductions and spectral equivalence of graphs.

1.3.1 Setup

As a typical graph $G \in \mathbb{G}$ cannot be reduced over any arbitrary subset of its vertices (by means of a single reduction) we define the set of *structural sets* of a graph or those vertex sets for which this is possible. If $S \subseteq V$ where V is the vertex set of a graph we write $\bar{S} = V - S$. Also, a *loop* is an edge that begins and ends at the same vertex.

Definition 1.3.1. *For $G = (V, E, \omega)$ let $\ell(G)$ be the graph G with all loops removed. The nonempty vertex set $S \subseteq V$ is a structural set of G if $\ell(G)|_{\bar{S}}$ contains no cycles and $\omega(e_{ii}) \neq \lambda$ for each $v_i \in \bar{S}$.*

We denote by $st(G)$ the set of all structural sets of G. To describe how a graph $G = (V, E, \omega)$ is reduced over the structural set $S \in st(G)$ we require the following standard terminology.

A *path* P in the graph $G = (V, E, \omega)$ is an ordered sequence of distinct vertices $v_1, \ldots, v_m \in V$ such that $e_{i,i+1} \in E$ for $1 \leq i \leq m-1$. In the case that the vetices $v_2, \ldots, v_{m-1}$ are distinct, but $v_1 = v_m$, then P is a *cycle.* Moreover, we call the vertices $v_2, \ldots, v_{m-1}$ of P the *interior* vertices of P.

Definition 1.3.2. *For $G = (V, E, \omega)$ with structural set $S = \{v_1, \dots, v_m\}$ let $\mathcal{B}_{ij}(G; S)$ be the set of paths or cycles with no interior vertices in S from v_i to v_j. Furthermore, let*

$$\mathcal{B}_S(G) = \bigcup_{1 \le i,j \le m} \mathcal{B}_{ij}(G; S).$$

We call the set $\mathcal{B}_S(G)$ the branches of G with respect to S.

If $\beta = v_1, \dots, v_m$ for $\beta \in \mathcal{B}_S(G)$ and $m > 2$ let

$$\mathcal{P}_\omega(\beta) = \omega(e_{12}) \prod_{i=2}^{m-1} \frac{\omega(e_{i,i+1})}{\lambda - \omega(e_{ii})}. \tag{2}$$

If $m = 2$ then $\mathcal{P}_\omega(\beta) = \omega(e_{12})$. We say $\mathcal{P}_\omega(\beta)$ is the branch product of β.

Definition 1.3.3. *Let $G = (V, E, \omega)$ with structural set $S = \{v_1 \dots, v_m\}$. The graph $\mathcal{R}_S(G) = (S, \mathcal{E}, \mu)$ where $e_{ij} \in \mathcal{E}$ if $\mathcal{B}_{ij}(G; S) \neq \emptyset$ and*

$$\mu(e_{ij}) = \sum_{\beta \in \mathcal{B}_{ij}(G;S)} \mathcal{P}_\omega(\beta), \quad 1 \le i, j \le m \tag{3}$$

is the isospectral reduction of G over S.

Theorem 1.3.4. *If $S \in st(G)$ then $\sigma\big(\mathcal{R}_S(G)\big) = \big(\sigma(G) \cup \sigma^{-1}(G|_{\bar{S}})\big) - \sigma(G|_{\bar{S}})$.*

To make sense of the formula in theorem 1.3.4 we note that

$$\det\big(M(G|_{\bar{S}}) - \lambda I\big) = \prod_{v_i \in \bar{S}} \big(\omega(e_{ii}) - \lambda\big) \tag{4}$$

since the graph $G|_{\bar{S}}$ contains no (nonloop) cycles. Hence, $\sigma(G|_{\bar{S}})$ are the values at which the loops weights of $\bar{S}$ equal λ and $\sigma^{-1}(G|_{\bar{S}})$ the values at which these weights are undefined. Hence, both $\sigma(G|_{\bar{S}})$ and $\sigma^{-1}(G|_{\bar{S}})$ can be read directly from G.

Figure 1 gives an example of an isospectral reduction of the graph G over the set $S = \{v_1, v_3\}$. Here, using our list notation, one can compute that the spectrum $\sigma(G) = \{(2, 1), (-1, 1), (1, 2), (0, 2)\}$. As $\bar{S}$ has loop weights $\omega(e_{22}) = 1$,$\omega(e_{44}) =$

0, $\omega(e_{55}) = 1$, and $\omega(e_{66}) = 0$ it also follows that $\sigma(G|_{\bar{S}}) = \{(1,2),(0,2)\}$ and $\sigma^{-1}(G|_{\bar{S}}) = \emptyset$. Theorem 1.3.4 then implies that $\sigma\big(\mathcal{R}_S(G)\big) = \{(2,1),(-1,1)\}$.

We note that it is also possible to reduce a graph without losing any eigenvalues. For instance, if $K_n = (V, E, \omega)$ where $M(K_n)_{ij} = 1$ for $1 \leq i, j \leq n$ then $\sigma(K_n) = \{(n,1),(0,n-1)\}$. Hence, for any $v \in V$, $\sigma(\mathcal{R}_{V-\{v\}}(K_n)) = \sigma(K_n)$ since $\sigma(K_n|_v) = \{(1,1)\}$ and $\sigma^{-1}(K_n|_v) = \emptyset$.

In terms of matrices an isospectral reduction of a graph can be described as follows. If $S \in st(G)$ then the vertices of G can be ordered such that $M(G) - \lambda I$ has the block form

$$M(G) - \lambda I = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5}$$

where A corresponds to the vertices in $\bar{S}$ and is a triangular matrix with nonzero diagonal. The matrix A can be made triangular by virtue of the fact that $G|_{\bar{S}}$ contains no (nonloop) cycles whereas the nonzero diagonal follows from the condition that $\omega(e_{ii}) \neq \lambda$ for $v_i \in \bar{S}$ (see *Frobenius normal form* in [12]). Hence, A is invertable.

Using the identity

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(A) \cdot \det(D - CA^{-1}B) \tag{6}$$

it then follows that

$$\det(D - CA^{-1}B) = \frac{\det(M(G) - \lambda I)}{\det(A)}.$$

Given $D - CA^{-1}B = R - \lambda I$, for some $R \in \mathbb{W}^{|S| \times |S|}$ the isospectral reduction, $\mathcal{R}_S(G)$ is the graph with adjacency matrix R (see proof of lemma 1.5.3 in section 5). Moreover, as A corresponds to the vertex set $\bar{S}$ then by equation (4)

$$\det(A) = \prod_{v_i \in \bar{S}} \big(\omega(e_{ii}) - \lambda\big).$$

Consequently, it is natural to define an *isospectral graph expansion* of a graph G as a graph H where $\mathcal{R}_T(H) = G$ for some $T \in st(H)$. Such expansions can be carried

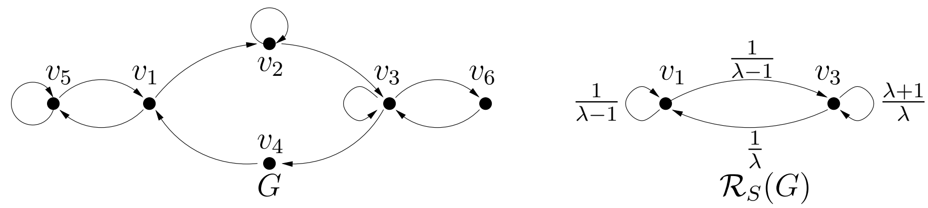


Figure 1: Reduction of G over $S = \{v_1, v_3\}$ where each edge in G has unit weight.

out by simply expanding edges into branches or multiple branches with the correct products and sums (e.g. (2) and (3)). However, in contrast to graph reductions, such expansions are nonunique. For unique expansions see section 4.1.

1.3.2 Sequential Reductions

As any reduction $\mathcal{R}_S(G)$ of a graph $G \in \mathbb{G}$ is again a graph in $\mathbb{G}$ it is natural to consider sequences of reductions of a graph as well as to what degree a graph can be reduced. To do so we extend our notation to an arbitrary sequence of reductions.

For $G = (V, E, \omega)$ suppose $S_m \subseteq S_{m-1} \subseteq \cdots \subseteq S_1 \subseteq V$ such that $S_1 \in st(G)$, $\mathcal{R}_1(G) = \mathcal{R}_{S_1}(G)$ and

$$S_{i+1} \in st(\mathcal{R}_i(G)) \text{ where } \mathcal{R}_{S_{i+1}}(\mathcal{R}_i(G)) = \mathcal{R}_{i+1}(G),\ 1 \leq i \leq m-1.$$

If this is the case we say $S_1, \ldots, S_m$ *induces a sequence of reductions* on G with *final vertex set* S_m and write $\mathcal{R}_m(G) = \mathcal{R}(G; S_1, \ldots, S_m)$.

Theorem 1.3.5. ***(Commutativity of Reductions)*** *For $G = (V, E, \omega)$ suppose $S_m \in st(G)$. If $S_m \subseteq S_{m-1} \subseteq \cdots \subseteq S_1 \subseteq V$ then $S_1, \ldots, S_{m-1}, S_m$ induces a sequence of reductions on G where $\mathcal{R}(G; S_1, \ldots, S_{m-1}, S_m) = \mathcal{R}_{S_m}(G)$.*

That is, the final vertex set in a sequence of reductions completely specifies the reduced graph irrespective of the specific sequence of reductions if this vertex set is a structural set. To address whether a graph $G = (V, E, \omega)$ may be reduced via some sequence of reductions to a graph on an arbitrary subset $\mathcal{V} \subseteq V$ we note the following.

If $\omega(e_{ii}) \neq \lambda$ for some $v_i \in V$ then $V - \{v_i\} \in st(G)$ since v_i induces no cycles in $\ell(G)$. Hence, any single vertex with this property can be removed from the graph. If it is somehow known a priori that no loop of G or any sequential reduction of G has weight λ then G can be sequentially reduced to a graph on any subset of its vertex set. To give such a condition we define the set $\mathbb{G}_\pi$.

Definition 1.3.6. *For $p(\lambda), q(\lambda) \in \mathbb{C}[\lambda]$ and $\omega = p(\lambda)/q(\lambda) \in \mathbb{W}$ define the function $\pi(\omega) = deg(p) - deg(q)$. Let $\mathbb{G}_\pi \subset \mathbb{G}$ be the set of graphs with the property that $G \in \mathbb{G}_\pi$ if $\pi(M(G)_{ij}) \leq 0$ for each entry of $M(G)$.*

Lemma 1.3.7. *If $G \in \mathbb{G}_\pi$ and $S \in st(G)$ then $\mathcal{R}_S(G) \in \mathbb{G}_\pi$. In particular, no loop of G and no loop of any reduction of G can have weight λ.*

By the reasoning above, if $G \in \mathbb{G}_\pi$ then G can be (sequentially) reduced to a graph on any subset of its vertex set. Moreover, the following holds.

Theorem 1.3.8. ***(Existence and Uniqueness of Reductions Over Any Set of Vertices)*** *Let $G = (V, E, \omega)$ be in $\mathbb{G}_\pi$. Then for any nonempty $\mathcal{V} \subseteq V$ any sequence of reductions on G with final vertex set $\mathcal{V}$ reduces G to the unique graph $\mathcal{R}_\mathcal{V}[G] = (\mathcal{V}, \mathcal{E}, \mu)$. Moreover, at least one such sequence always exists.*

The notation $\mathcal{R}_\mathcal{V}[G]$ in the previous theorem is intended to emphasize the fact that $\mathcal{V}$ need not be a structural set of G. Moreover, this notation is well defined given that there is a unique reduction of G over $\mathcal{V}$ in the sense of theorem 1.3.8.

Remark 1. *If $M(G) \in \mathbb{C}^{n\times n}$ then $G \in \mathbb{G}_\pi$. Therefore, any graph with complex weights can be uniquely reduced to a graph on any nonempty subset of its vertex set. This is of particular importance for the estimation of spectra of matrices with complex entries in [14] which is presented in chapter II.*

1.3.3 Unique Reductions

Here we consider specific types of vertex sets that are defined independent of any graph of $\mathbb{G}$. Our motivation is that every nonempty graph in $\mathbb{G}_\pi$ will have a unique reduction with respect to this type of vertex set. This will in turn allow us to partition the graphs in $\mathbb{G}_\pi - \emptyset$ according to their reductions.

Recall that two weighted digraphs $G_1 = (V_1, E_1, \omega_1)$, and $G_2 = (V_2, E_2, \omega_2)$ are *isomorphic* if there is a bijection $b : V_1 \to V_2$ such that there is an edge e_{ij} in G_1 from v_i to v_j if and only if there is an edge $\tilde{e}_{ij}$ between $b(v_i)$ and $b(v_j)$ in G_2 with $\omega_2(\tilde{e}_{ij}) = \omega_1(e_{ij})$. If the map b exists it is called an *isomorphism* and we write $G_1 \simeq G_2$. Moreover, we note that if two graphs are isomorphic then their spectra are identical.

Definition 1.3.9. *Let G and H be graphs such that $\mathcal{R}_S(G) = \mathcal{R}_T(H)$ for some $S \in st(G)$, $T \in st(H)$. Then we say G and H are spectrally equivalent.*

As an example, the graphs G in figure 1 and H in figure 4 are spectrally equivalent since $\mathcal{R}_S(G) = \mathcal{R}_S(H)$ for $S = \{v_1, v_3\}$.

Theorem 1.3.10. ***(Uniqueness and Equivalence Relations)*** *Suppose for any graph $G = (V, E, \omega)$ in $\mathbb{G}_\pi - \emptyset$ that τ is a rule that selects a unique nonempty subset $\tau(G) \subseteq V$. Then τ induces an equivalence relation $\sim$ on the set $\mathbb{G}_\pi - \emptyset$ where $G \sim H$ if $\mathcal{R}_{\tau(G)}[G] \simeq \mathcal{R}_{\tau(H)}[H]$.*

That is, the relation of being spectrally equivalent under some rule τ is an equivalence relation on the nonempty graphs in $\mathbb{G}_\pi$. As an example consider the following rule. For a graph $G = (V, E, \omega)$ where $G \in \mathbb{G}_\pi$ let $m(G) \subseteq V$ be the set of vertices of minimal out degree. If $m(G) \neq V$ then by theorem 1.3.8 $\mathcal{R}_{V-m(G)}[G]$ is uniquely defined and this process may be repeated until all vertices of the resulting graph have the same out degree. As the final vertex set of this sequence of reductions is uniquely defined then the relation of having an isomorphic reduction via this rule induces an equivalence relation on $\mathbb{G}_\pi$.

Note that the relation of simply having isomorphic reductions is not transitive. That is, if $\mathcal{R}_S[G] \simeq \mathcal{R}_T[H]$ and $\mathcal{R}_U[H] \simeq \mathcal{R}_V[K]$ it is not necessarily the case that

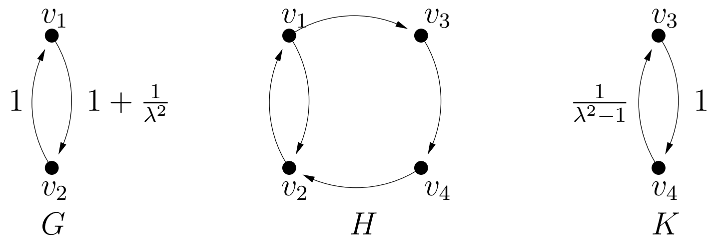


Figure 2: $G = \mathcal{R}_S[H]$ and $\mathcal{R}_T[H] = K$ for $S = \{v_1, v_2\}$ and $T = \{v_3, v_4\}$ but the graphs G and K do not have isomorphic reductions.

there are sets X and Y, subsets of the vertex sets of G and K respectively, such that $\mathcal{R}_X[G] \simeq \mathcal{R}_Y[K]$. For instance, in figure 2 both $\mathcal{R}_S[G] = \mathcal{R}_S[H]$ and $\mathcal{R}_T[H] = \mathcal{R}_T[K]$ where $S = \{v_1, v_2\}$ and $T = \{v_3, v_4\}$. However, one can quickly check that for no subsets $X \subseteq S$ and $Y \subseteq T$ is $\mathcal{R}_X[G] \simeq \mathcal{R}_Y[K]$.

Therefore, a general rule τ that selects a unique set of vertices from each graph is needed to overcome this intransitivity. However, once a general rule is given this allows us to study graphs (networks) modulo some particular graph structure. In the example above this structure is the vertices of strictly minimal out degree.

As an example of a reduction of a real world network, the (unweighted) graph on the left hand side of figure 3 represents part of the gene interaction network of a mouse lung [32]. The graph on the right hand side of figure 3 is the isospectral reduction this network over the its *hubs* (shown in red) or the vertices with large in or out degree relative to the other vertices. Alternately, this reduction could be viewed as reducing the network modulo its non-hub vertices.

1.4 Reductions and Expansions Over Fixed Weight Sets

The graph reductions and expansions of the previous section modify not only the graph structure but also the weight set associated with each graph. That is, if $G = (V, E, \omega)$ and $\mathcal{R}_S(G) = (S, \mathcal{E}, \mu)$ then typically $\omega(E) \neq \mu(\mathcal{E})$. Thus, one may think that our procedure simply shifts the complexity of the graph's structure to its set of edge weights. However, this is not the case.

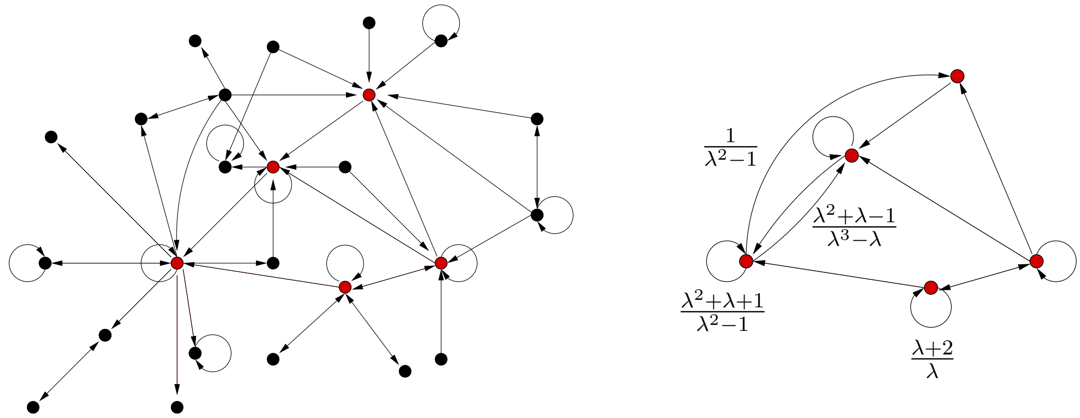


Figure 3: Gene interaction network of a mouse lung (left) reduced over its *hubs* (right).

In this section we introduce a procedure of reducing and expanding a graph while maintaining its set of edge weights where, as before, the procedure preserves the spectrum of the graph up to a known set (list). Such transformations are of particular importance in chapter III where *dynamical network expansions* are discussed.

1.4.1 Branch Expansions

Given a graph $G = (V, E, \omega)$ and $S \in st(G)$, two branches $\alpha, \beta \in \mathcal{B}_S(G)$ are said to be *independent* if they have no interior vertices in common. Moreover, if the branch $\beta = v_1, \ldots, v_m$ let $\Omega(\beta)$ be the ordered sequence

$$\Omega(\beta) = \omega(e_{12}), \omega(e_{22}), \ldots, \omega(e_{i-1,i}), \omega(e_{ii}), \omega(e_{i,i+1}), \ldots \omega(e_{m-1,m-1}), \omega(e_{m-1,m}).$$

Definition 1.4.1. *For $G = (V, E, \omega)$ with structural set $S = \{v_1, \ldots, v_m\}$ let $\mathcal{X}_S(G) = (X, \mathcal{E}, \mu)$ be the graph where $S \in st(\mathcal{X}_S(G))$ and the following holds:*

(i) There are bijections $b_{ij} : \mathcal{B}_{ij}(\mathcal{X}_S(G); S) \to \mathcal{B}_{ij}(G; S)$ for all $1 \leq i, j \leq m$ such that if $b_{ij}(\gamma) = \beta$ then $\Omega(\gamma) = \Omega(\beta)$.

(ii) The branches in $\mathcal{B}_S(\mathcal{X}_S(G))$ are pairwise independent.

(iii) Each vertex of $\mathcal{X}_S(G)$ is on a branch of $\mathcal{B}_S(\mathcal{X}_S(G))$.

As this uniquely defines $\mathcal{X}_S(G)$, up to a labeling of vertices, we call this graph the

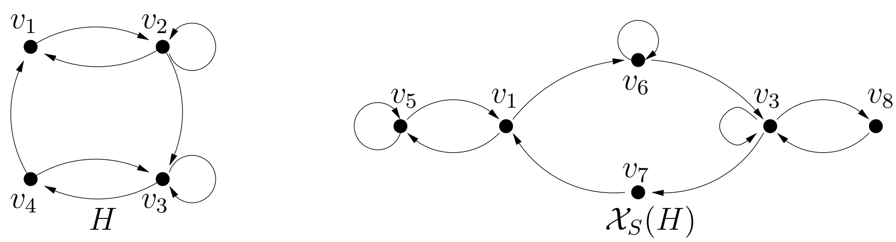


Figure 4: An expansion $\mathcal{X}_S(H)$ of H over $S = \{v_1, v_3\}$ where each edge in H and $\mathcal{X}_S(H)$ has unit weight.

branch expansion of G over S. Essentially, $\mathcal{X}_S(G)$ is the graph G in which every pair of branches with respest to S have been made independent.

Theorem 1.4.2. *Let $G = (V, E, \omega)$ and $S \in st(G)$. If each vertex of G is on a branch of $\mathcal{B}_S(G)$ then G and $\mathcal{X}_S(G)$ have the same weight set, $\mathcal{R}_S\big(\mathcal{X}_S(G)\big) = \mathcal{R}_S(G)$, and $|\mathcal{X}_S(G)| \geq |G|$. Moreover,*

$$\det\big(M(\mathcal{X}_S(G)) - \lambda I\big) = \det\big(M(G) - \lambda I\big) \prod_{v_i \in \bar{S}} \big(\omega(e_{ii}) - \lambda\big)^{n_i - 1}$$

where n_i is the number of branches in $\mathcal{B}_S(G)$ containing v_i.

As an example of a branch expansion consider the graph H in figure 4 where $S = \{v_1, v_3\}$. Given that $\sigma(H) = \{(2,1), (-1,1), (1,1), (0,1))\}$ one can compute $\sigma(\mathcal{X}_S(H)) = \sigma(H) \cup \{(1,1), (0,1)\}$ via theorem 1.4.2.

1.4.2 Transformations Over Fixed Weight Sets

Using techniques from both isospectral graph reductions and branch expansions it is often possible to reduce a graph over some fixed subset of weights $\mathbb{U} \subset \mathbb{W}$ while again preserving the graph's spectrum up to some known set (list). To make this precise we require the following.

Definition 1.4.3. *If G and H are spectrally equivalent, $|H| < |G|$, and both G and H have weight sets in some subset $\mathbb{U} \subseteq \mathbb{W}$ we say H is a reduction of G over the weight set $\mathbb{U}$. If $|H| \geq |G|$ then H is an expansion of G over the weight set $\mathbb{U}$.*

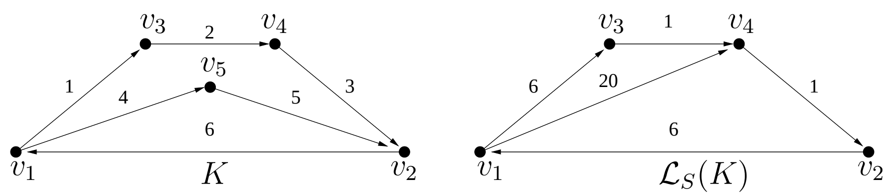


Figure 5: $\mathcal{L}_S(K)$ is a reduction of K over $S = \{v_1, v_2\}$ with weights in $\mathbb{Z}$.

For $G = (V, E, \omega)$ and any $\mathbb{U} \subseteq \mathbb{W}$ containing $\omega(E)$ an example of an expansion of G over the weight set $\mathbb{U}$ is any branch expansion of G. Finding reductions over some set $\mathbb{U}$ is more complicated. However, if $\mathbb{U}$ is a unital subring of $\mathbb{W}$ and we restrict our attention to specific types of structural sets it is often possible to simply construct a reduction of G over the set $\mathbb{U}$ as follows.

The $\mathcal{L}$-Construction

For $G = (V, E, \omega)$ let $st_0(G)$ be the collection of all structural sets of G such that $S \in st_0(G)$ if $\omega(e_{ii}) = 0$ for each $v_i \in \bar{S}$. Suppose $G = (V, E, \omega)$ and $S \in st_0(G)$ where each vertex of G is on a branch of $\mathcal{B}_S(G)$. If $S = \{v_1, \ldots, v_m\}$ then for

$$\mathcal{B}_j(\mathcal{X}_S(G)) = \bigcup_{1 \leq i \leq m} \mathcal{B}_{ij}(\mathcal{X}_S(G); S)$$

let $\beta^j \in \mathcal{B}_j(G)$ be a branch of maximal length for each $1 \leq j \leq m$.

As a first intermediate step let $\tilde{\mathcal{X}}$ be the graph $\mathcal{X}_S(G)$ with all but the branches β^j removed i.e.

$$\mathcal{B}_S(\tilde{\mathcal{X}}, S) = \{\beta^1, \ldots, \beta^m\}.$$

Suppose each β^j is denoted by the sequence of vertices $\beta^j_{\ell_j}, \ldots, \beta^j_0$ where $\ell^j = |\beta^j| - 1$ is the length of β^j or number of edges. Note each $\beta^j_{\ell_j} \in S$ and $\beta^j_0 = v_j$. We then reweight the edges of $\tilde{\mathcal{X}}$ in the following way.

Make the edge from $\beta^j_{\ell_j}$ to $\beta^j_{\ell_j - 1}$ have weight $\lambda^{|\beta^j|-2}\mathcal{P}_\omega(\beta^j)$ for each j and set all other edges of $\tilde{\mathcal{X}}$ to weight 1. Next, for each branch $\gamma \in \mathcal{B}_j(\mathcal{X}_S(G)) - \beta^j$ from v_i to v_j attach an edge from vertex v_i to vertex $\beta^j_{|\gamma|-2}$ in $\tilde{\mathcal{X}}$ with weight $\lambda^{|\gamma|-2}\mathcal{P}_\omega(\gamma)$. If two or more branches correspond to the same edge we reduce these to a single edge with weight equal to the sum of the corresponding branch products.

We call the resulting graph $\mathcal{L}_S(G)$ or the $\mathcal{L}-Construction$ of G over S. By construction, if G has weights in $\mathbb{U}$, some unital subring of $\mathbb{W}$, then so does $\mathcal{L}_S(G)$.

Theorem 1.4.4. *Let G have weights in the unital subring $\mathbb{U} \subseteq \mathbb{W}$ and $S \in st_0(G)$. Then G and $\mathcal{L}_S(G)$ are spectrally equivalent and have the same nonzero spectrum. Moreover, if $|G| > m + \sum_{j=1}^{m}(|\beta^j| - 2)$ then $\mathcal{L}_S(G)$ is a reduction of G over $\mathbb{U}$ otherwise it is an expansion.*

An example of an $\mathcal{L}-Construction$ is the graph $\mathcal{L}_S(K)$ in figure 5, which is constructed from the graph K over $S = \{v_1, v_2\} \in st_0(K)$ having weights in the unital subring $\mathbb{Z} \subset \mathbb{W}$.

1.5 Matrix Representation of Isospectral Graph Reductions and Proofs

In this section we give proofs of the theorems and lemma in sections 3 and 4 of this chapter. Our strategy will be to develop an analogous theory of matrix reductions from which such results will follow.

Definition 1.5.1. *Let $M \in \mathbb{W}^{n\times n}$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$ be nonempty. If the set $\mathcal{I} = \{i_1, \ldots, i_m\}$ where $i_j < i_{j+1}$ then the matrix $M|_{\mathcal{I}}$ given by $(M|_{\mathcal{I}})_{k\ell} = M_{i_k i_\ell}$ for all $1 \leq k, \ell \leq m \leq n$ is the principle submatrix of M indexed by $\mathcal{I}$.*

In what follows, we will use the convention that if $\mathcal{I} = \{i_1, \ldots, i_m\}$ is an indexing set of a matrix then $i_j < i_{j+1}$. Moreover, if $D = M|_{\mathcal{I}}$ is the principle submatrix of $M \in \mathbb{W}^{n\times n}$ indexed by $\mathcal{I}$ then there exists a unique permutation matrix P such that

$$PMP^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $A = M|_{\bar{\mathcal{I}}}$ is the principle submatrix of M indexed by $\bar{\mathcal{I}} = \{1, \ldots, n\} - \mathcal{I}$. Assuming A is invertable, then by (6)

$$\det(M) = \det(PMP^{-1}) = \det(A)\det(D - CA^{-1}B).$$

If this is the case we call the matrix $D - CA^{-1}B$ the reduction of M over $\mathcal{I}$ and write $r(M; \mathcal{I}) = D - CA^{-1}B$. In the case that $\mathcal{I} = \{1, \ldots, n\}$ we let the matrix $r(M; \mathcal{I}) = M$. Moreover, if $r(M; \mathcal{I})$ can be reduced over the index set $\mathcal{J}$ we write the reduction of $r(M; \mathcal{I})$ over $\mathcal{J}$ by $r(M; \mathcal{I}, \mathcal{J})$, and so on.

We formulate the following lemma related to the reduction of matrices.

Lemma 1.5.2. *Let $M \in \mathbb{W}^{n \times n}$ and $\mathcal{I} \subseteq \{1, \ldots, n\}$ be nonempty. If $M|_{\bar{\mathcal{I}}}$ is upper triangular with nonzero diagonal and the sets $\mathcal{I}_m \subseteq \mathcal{I}_{m-1} \subseteq \cdots \subseteq \mathcal{I}_1 \subseteq \{1, \ldots, n\}$ then $r(M; \mathcal{I}_1, \ldots, \mathcal{I}_{m-1}, \mathcal{I}_m) = r(M; \mathcal{I}_m)$.*

Proof. Without loss in generality suppose $\mathcal{I}_m \subseteq \{1, \ldots, n\}$ such that the matrix M has block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is an upper triangular principle submatrix of M indexed by $\bar{\mathcal{I}}_m$ with nonzero diagonal. Then for $\mathcal{I}_m \subseteq \mathcal{I}_1 \subseteq \{1, \ldots, n\}$ there exists a unique permutation matrix P such that

$$PMP^{-1} = \begin{pmatrix} A_0 & B_0 & B_2 \\ C_0 & A_1 & B_1 \\ C_2 & C_1 & D \end{pmatrix}$$

where A_0, A_1, and D are the principle submatrices indexed by $\{1, \ldots, n\} - \mathcal{I}_1, \mathcal{I}_1 - \mathcal{I}_m$, and $\mathcal{I}_m$ respectively. Moreover, both A_0 and A_1 are upper triangular matrices with nonzero diagonal as they are principle submatrices of A and are therefore invertable.

It then follows that

$$r(M; \mathcal{I}_1) = \begin{pmatrix} A_1 - C_0 A_0^{-1} B_0 & B_1 - C_0 A_0^{-1} B_2 \\ C_1 - C_2 A_0^{-1} B_0 & D - C_2 A_0^{-1} B_2 \end{pmatrix}. \tag{7}$$

The claim is then that the matrix $C_0A_0^{-1}B_0$ is an upper triangular matrix possessing a zero diagonal. To see this let the sets $\{1,\ldots,n\} - \mathcal{I}_1 = \{i_1,\ldots,i_s\}$, $\mathcal{I}_1 - \mathcal{I}_m = \{j_1,\ldots,j_t\}$, and $(A_0^{-1})_{ij} = \alpha_{ij}$. As the inverse of an upper triangular matrix is upper triangular $\alpha_{ij} = 0$ for $i > j$. Hence,

$$(C_0A_0^{-1}B_0)_{\ell k} = \sum_{p=1}^{s}\sum_{q=1}^{p}\left(A_{j_\ell i_q}\alpha_{qp}A_{i_p j_k}\right) \quad 1 \le \ell, k \le t.$$

As $1 \le q \le p$ then each $i_q \le i_p$. For each $k \le \ell$ note that similarly $j_k \le j_\ell$. Moreover, if $j_k < i_p$ then $A_{i_p j_k} = 0$ since A is upper triangular. If $j_k > i_p$ then $i_q < j_\ell$ implying $A_{j_\ell i_q} = 0$ for the same reason. Then, as $j_k \ne i_p$, it follows that

$$(C_0A_0^{-1}B_0)_{\ell k} = 0 \text{ for all } \ k \le \ell$$

verifying the claim.

$A_1 - C_0A_0^{-1}B_0$ is therefore an upper triangular matrix with nonzero diagonal. Hence, $A_1 - C_0A_0^{-1}B_0$ is invertible and $r(M;\mathcal{I}_1)$ can be reduced over $\mathcal{I}_m$. In particular,

$$r(M;\mathcal{I}_1,\mathcal{I}_m) = D - C_2A_0^{-1}B_2 - \big(C_1 - C_2A_0^{-1}B_0\big)\Gamma\big(B_1 - C_0A_0^{-1}B_2\big) \tag{8}$$

where $\Gamma = (A_1 - C_0A_0^{-1}B_0)^{-1}$.

To verify that $r(M;\mathcal{I}_1,\mathcal{I}_m) = r(M;\mathcal{I}_m)$ note that M has block form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ and } QAQ^{-1} = \begin{bmatrix} A_0 & B_0 \\ C_0 & A_1 \end{bmatrix}$$

where Q is the principle submatrix of P indexed by $\bar{\mathcal{I}}_m$. Therefore,

$$A^{-1} = Q^{-1}\begin{bmatrix} A_0 & B_0 \\ C_0 & A_1 \end{bmatrix}^{-1} Q = Q^{-1}\begin{bmatrix} (A_0 - B_0A_1^{-1}C_0)^{-1} & -A_0^{-1}B_0\Gamma^{-1} \\ -\Gamma^{-1}C_0A_0^{-1} & \Gamma^{-1} \end{bmatrix} Q.$$

Note the matrix $(A_0 - B_0A_1^{-1}C_0)^{-1} = A_0^{-1} - A_0^{-1}B_0\Gamma C_0A_0^{-1}$ by the Woodbury matrix identity [29] and is therefore well defined. From this

$$\begin{aligned} r(M;\mathcal{I}_m) =& D - CQ^{-1}\begin{bmatrix} A_0 & B_0 \\ C_0 & A_1 \end{bmatrix}^{-1} QB \\ =& D - \begin{bmatrix} C_2 & C_1 \end{bmatrix} \begin{bmatrix} (A_0 - B_0A_1^{-1}C_0)^{-1} & -A_0^{-1}B_0\Gamma^{-1} \\ -\Gamma^{-1}C_0A_0^{-1} & \Gamma^{-1} \end{bmatrix} \begin{bmatrix} B_2 \\ B_1 \end{bmatrix} \\ =& D - C_2(A_0^{-1} - A_0^{-1}B_0\Gamma C_0A_0^{-1})B_2 - C_1\Gamma C_0A_0^{-1}B_2 - \\ & C_2A_0^{-1}B_0\Gamma C_0A_0^{-1}B_1 + C_1\Gamma B_1. \end{aligned}$$

From (8) it follows that $r(M;\mathcal{I}_m) = r(M;\mathcal{I}_1,\mathcal{I}_m)$

For $\mathcal{I}_m \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_1$ the same argument can be repeated to show

$$r\big(r(M;\mathcal{I}_1);\mathcal{I}_m\big) = r\big(r(M;\mathcal{I}_1);\mathcal{I}_2;\mathcal{I}_m\big)$$

as $r(M;\mathcal{I}_1)$ has the same form as M, i.e. its upper left hand block is an upper triangular matrix with nonzero diagonal. Hence, $r(M;\mathcal{I}_m) = r(M;\mathcal{I}_1,\mathcal{I}_2;\mathcal{I}_m)$. The lemma follows by further extending $\mathcal{I}_m \subseteq \mathcal{I}_2 \subseteq \mathcal{I}_1 \subseteq \{1,\dots,n\}$ to the nested sequence $\mathcal{I}_m \subseteq \mathcal{I}_{m-1} \subseteq \cdots \subseteq \mathcal{I}_1 \subseteq \{1,\dots,n\}$ by repeated use of the above argument. □

To establish that graph reductions are analogous to matrix reductions we prove the following lemma.

Lemma 1.5.3. *If $S = \{v_{i_1},\dots,v_{i_p}\}$ is a structural set of $G \in \mathbb{G}$ then $\mathcal{R}_S(G)$ is the graph with adjacency matrix $r\big(M(G) - \lambda I; \{i_1,\dots,i_p\}\big) + \lambda I$.*

Proof. Without loss in generality let $\bar{S} = \{v_1,\dots,v_m\}$. From the discussion in section 1.3.1, if S is a structural set of $G = (V,E,\omega)$ then $M(G) - \lambda I$ has the block form

$$M(G) - \lambda I = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is the principle submatrix of $M(G) - \lambda I$ indexed by $\{1,\dots,m\}$ and is an upper triangular matrix with nonzero diagonal.

With this in mind, let $V_k = \{v_{k+1}, \ldots, v_n\}$ and $\mathcal{I}_k = \{k+1, \ldots, n\}$ for $1 \leq k \leq m$. If $M(G)_{ij} = \omega_{ij}$ for $1 \leq i, j \leq n$ then, proceeding by induction, for $k = 1$

$$r(M(G) - \lambda I; \mathcal{I}_1) = \big(D_1 - [\omega_{21}, \ldots, \omega_{n1}]^T \frac{1}{\omega_{11} - \lambda} [\omega_{12}, \ldots, \omega_{1n}]\big)$$

where D_1 is the principle submatrix of $M(G) - \lambda I$ indexed by $\mathcal{I}_1$. Hence,

$$r(M(G) - \lambda I; \mathcal{I}_1)_{ij} = \begin{cases} \omega_{ij} + \omega_{i1}\omega_{1j}/(\lambda - \omega_{11}) & i \neq j \\ \omega_{ij} + \omega_{i1}\omega_{1j}/(\lambda - \omega_{11}) - \lambda & i = j \end{cases} \tag{9}$$

for $2 \leq i, j \leq n$.

Conversely, note that the set $\mathcal{B}_{ij}(G; V_1)$ consists of at most the two branches $\beta^+ = v_i, v_1, v_j$ and $\beta^- = v_i, v_j$ i.e. the branches from v_i to v_j with and without interior vertex v_1. As $\mathcal{P}_\omega(\beta^-) = \omega_{ij}$ and $\mathcal{P}_\omega(\beta^+) = \omega_{i1}\omega_{1j}/(\lambda - \omega_{11})$ if the branches β^- and β^+ are respectively in $\mathcal{B}_{ij}(G; V_1)$ it then follows from (9) and (3) that $r\big(M(G) - \lambda I; \mathcal{I}_1\big) = M\big(\mathcal{R}_{V_1}(G)\big) - \lambda I$.

Suppose then for some $k \leq m$ that $r\big(M(G) - \lambda I; \mathcal{I}_{k-1}\big) = M\big(\mathcal{R}_{V_{k-1}}(G)\big) - \lambda I$. Since the principle submatrix of $M(G) - \lambda I$ indexed by $\mathcal{I}_k$ is upper triangular with nonzero diagonal and $\mathcal{I}_k \subseteq \mathcal{I}_{k-1} \subseteq \{1, \ldots, n\}$ an application of lemma 1.5.2 implies that $r(M(G) - \lambda I; \mathcal{I}_k) = r(M(G) - \lambda I; \mathcal{I}_{k-1}, \mathcal{I}_k)$. Hence,

$$r(M(G) - \lambda I; \mathcal{I}_k) = r\big(M(\mathcal{R}_{V_{k-1}}(G)) - \lambda I; \mathcal{I}_k\big).$$

Letting $\mathcal{M} = M(\mathcal{R}_{V_{k-1}}(G))$ then by the argument above

$$r(M(G) - \lambda I; \mathcal{I}_k)_{ij} = \begin{cases} \mathcal{M}_{ij} + \mathcal{M}_{ik}\mathcal{M}_{kj}/(\lambda - \mathcal{M}_{kk}) & i \neq j \\ \mathcal{M}_{ij} + \mathcal{M}_{ik}\mathcal{M}_{kj}/(\lambda - \mathcal{M}_{kk}) - \lambda & i = j \end{cases}$$

for $k + 1 \leq i, j \leq n$. As $\mathcal{M}_{ij} = M(\mathcal{R}_{V_{k-1}}(G))_{ij}$ then the entries of $\mathcal{M}$ are given by $\mathcal{M}_{ij} = \sum_{\beta \in \mathcal{B}_{ij}(G; V_{k-1})} \mathcal{P}_\omega(\beta)$.

Observe that

$$\sum_{\beta \in \mathcal{B}_{ij}(G; V_k)} \mathcal{P}_\omega(\beta) = \sum_{\beta \in \mathcal{B}^-_{ij}(G; V_k)} \mathcal{P}_\omega(\beta) + \sum_{\beta \in \mathcal{B}^+_{ij}(G; V_k)} \mathcal{P}_\omega(\beta)$$

where $\mathcal{B}^+_{ij}(G;V_k)$ and $\mathcal{B}^-_{ij}(G;V_k)$ are the branches in $\mathcal{B}_{ij}(G;V_k)$ that contain and do not contain the interior vertex v_k respectively. It then immediately follows that $\mathcal{B}^-_{ij}(G;V_k) = \mathcal{B}_{ij}(G;V_{k-1})$ implying $\sum_{\beta\in\mathcal{B}^-_{ij}(G;V_k)}\mathcal{P}_\omega(\beta) = \mathcal{M}_{ij}$.

On the other hand, any $\beta \in \mathcal{B}^+_{ij}(G;V_k)$ can be written as $\beta = v_i,\ldots,v_k,\ldots,v_j$ where $\beta_1 = v_i,\ldots,v_k \in \mathcal{B}_{ik}(G;V_{k-1})$ and $\beta_2 = v_k,\ldots,v_j \in \mathcal{B}_{kj}(G;V_{k-1})$. Hence, equation (2) implies

$$\sum_{\beta\in\mathcal{B}^+_{ij}(G;V_k)}\mathcal{P}_\omega(\beta) = \sum_{\beta\in\mathcal{B}^+_{ij}(G;V_k)}\frac{\mathcal{P}_\omega(\beta_1)\mathcal{P}_\omega(\beta_2)}{\lambda-\omega_{kk}}. \tag{10}$$

Conversely, if $\beta_1 = v_i,\ldots,v_k \in \mathcal{B}_{ik}(G;V_{k-1})$ and $\beta_2 = v_k,\ldots,v_j \in \mathcal{B}_{kj}(G;V_{k-1})$ then $\beta = v_i,\ldots,v_k,\ldots,v_j \in \mathcal{B}^+_{ij}(G;V_k)$. This follows from the fact that β_1 and β_2 share no interior vertices since otherwise $G|_{\bar{V}_k}$ would contain a cycle, which is not possible given $V_k \in st(G)$. Therefore,

$$\sum_{\beta\in\mathcal{B}^+_{ij}(G;V_k)}\mathcal{P}_\omega(\beta_1)\mathcal{P}_\omega(\beta_2) = \sum_{\beta_1\in\mathcal{B}_{ik}(G;V_{k-1})}\mathcal{P}_\omega(\beta_1)\sum_{\beta_2\in\mathcal{B}_{kj}(G;V_{k-1})}\mathcal{P}_\omega(\beta_2). \tag{11}$$

Moreover, $\omega_{kk} = \mathcal{M}_{kk}$ since $\mathcal{B}_{kk}(G;V_{k-1})$ contains at most the cycle v_k, v_k as $V_{k-1} \in st(G)$. From (10) and (11) it then follows that

$$\mathcal{M}_{ij} + \mathcal{M}_{ik}\mathcal{M}_{kj}/(\lambda - \mathcal{M}_{kk}) = \sum_{\beta\in\mathcal{B}_{ij}(G;V_k)}\mathcal{P}_\omega(\beta)$$

implying $r\big(M(G)-\lambda I;\mathcal{I}_k\big) = M\big(\mathcal{R}_{V_k}(G)\big) - \lambda I$.

By induction it follows that $r\big(M(G)-\lambda I;\mathcal{I}_m\big) = M\big(\mathcal{R}_S(G)\big) - \lambda I$ by setting $k = m$. $\square$

If $S = \{v_{i_1},\ldots,v_{i_m}\}$ is a structural set of G then let $\{i_1,\ldots,i_m\}$ be the index set associated with S. Hence, if $S \in st(G)$ is indexed by $\mathcal{I}$ then $\bar{S}$ is indexed by $\bar{\mathcal{I}}$ and there is a unique permutation matrix P such that

$$P\big(M(G)-\lambda I\big)P^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A is the principle submatrix of $M(G) - \lambda I$ indexed by $\bar{\mathcal{I}}$. Therefore, $A = M(G|_{\bar{S}}) - \lambda I$ and it follows from lemma 1.5.3 and (6) that

$$\det\big(M(G) - \lambda I\big) = \det\big(M(G|_{\bar{S}}) - \lambda I\big)\det\big(M(\mathcal{R}_S(G)) - \lambda I\big). \tag{12}$$

Letting $\det(M(G|_{\bar{S}}) - \lambda I) = p(\lambda)/q(\lambda) \in \mathbb{W}$ then theorem 1.3.4 follows by observing that $\sigma(G|_{\bar{S}})$ and $\sigma^{-1}(G|_{\bar{S}})$ are the solutions to $p(\lambda) = 0$ and $q(\lambda) = 0$ respectively.

We now give proofs of the theorems on isospectral transformations. For a proof of theorem 1.3.5 we have the following.

Proof. Suppose $S_m \subseteq S_{m-1} \subseteq \cdots \subseteq S_1 \subseteq V$ where S_m is a structural set of $G = (V, E, \omega)$ and let $\mathcal{I}_i$ be the index set associated with S_i for $1 \leq i \leq m$. By lemma 1.5.3 the graph $\mathcal{R}(G; S_1)$ has adjacency matrix $r(M(G) - \lambda I, \mathcal{I}_1) + \lambda I$. Hence, and another application of lemma 1.5.3, the graph $\mathcal{R}(G; S_1, S_2)$ has adjacency matrix

$$r\big(r(M(G) - \lambda I; \mathcal{I}_1) + (\lambda - \lambda)I; \mathcal{I}_2\big) + \lambda I = r(M(G) - \lambda I; \mathcal{I}_1, \mathcal{I}_2) + \lambda I.$$

Continuing in this manner, it follows that $\mathcal{R}(G; S_1, \ldots, S_m)$ has adjacency matrix given by $r(M(G) - \lambda I; \mathcal{I}_1, \ldots, \mathcal{I}_m) + \lambda I$ which by lemma 1.5.2 is equivalent to the matrix $r(M(G) - \lambda I; \mathcal{I}_m) + \lambda I$.

Given that $r(M(G) - \lambda I; \mathcal{I}_m) + \lambda I$ is the adjacency matrix of $\mathcal{R}_{S_m}(G)$ then again by use of lemma 1.5.3 it follows that $\mathcal{R}_{S_m}(G) = \mathcal{R}(G; S_1, \ldots, S_m)$. □

We now give a proof of lemma 1.3.7.

Proof. Let $w_1 = p_1/q_1$ and $w_2 = p_2/q_2$ be in $\mathbb{W}$ such that $\pi(w_1), \pi(w_2) \leq 0$. As

$$\begin{aligned}
\pi(w_1 + w_2) &= \pi\Big(\frac{p_1q_2 + p_2q_1}{q_1q_2}\Big) \leq \max\{\pi(w_1), \pi(w_2)\} \leq 0, \\
\pi(w_1w_2) &= \pi\Big(\frac{p_1p_2}{q_1q_2}\Big) = \pi(w_1) + \pi(w_2) \leq 0, \text{ and} \\
\pi\Big(\frac{1}{\lambda - w_1}\Big) &= \pi\Big(\frac{q_1}{(q_1\lambda - p_1)}\Big) < \pi(w_1) \leq 0
\end{aligned}$$

then the result follows from equation (2) and (3). □

In order to prove theorem 1.3.10 we require the following lemma that essentially implies that the order in which any two vertices are removed from a graph $G \in \mathbb{G}_\pi$ does not effect the final reduced graph.

Lemma 1.5.4. *Suppose $G \in \mathbb{G}_\pi$ with vertex set $V = \{v_1, \ldots, v_n\}$ for $n > 2$. If $T = \{v_1, v_2\}$ then $\mathcal{R}(G; V - \{v_1\}, T) = \mathcal{R}(G; V - \{v_2\}, T)$.*

Proof. Let $G \in \mathbb{G}_\pi$ and $M(G)_{ij} = \omega_{ij}$ for $1 \leq i, j \leq n$. Then $M(G) - \lambda I$ can be written in the block form

$$M(G) - \lambda I = \begin{pmatrix} \omega_{11} - \lambda & \omega_{12} & B_1 \\ \omega_{21} & \omega_{22} - \lambda & B_2 \\ C_1 & C_2 & D \end{pmatrix} \quad \text{where} \quad A = \begin{bmatrix} \omega_{11} - \lambda & \omega_{12} \\ \omega_{21} & \omega_{22} - \lambda \end{bmatrix}$$

is the principle submatrix of $M(G) - \lambda I$ indexed by $\bar{\mathcal{I}} = \{1, 2\}$.

Note the assumption $G \in \mathbb{G}_\pi$ implies that both $\omega_{11} - \lambda$, $\omega_{22} - \lambda \neq 0$. Moreover, if $\det(A) = 0$ then $\omega_{22} = \omega_{12}\omega_{21}(\omega_{11} - \lambda) + \lambda$. However, equation (9) implies in this case that

$$M\big(\mathcal{R}_{V-v_1}(G)\big)_{22} - \lambda = \omega_{22} + \frac{\omega_{21}\omega_{12}}{\lambda - \omega_{11}} - \lambda = 0.$$

Hence, $M\big(\mathcal{R}_{V-v_1}(G)\big)_{22} = \lambda$, which is not possible as it contradicts the conclusion of lemma 2.3.2. Therefore, $\det(A) \neq 0$ implying

$$R_{ij} = \left(D - \frac{1}{\omega_{jj} - \lambda} C_j B_j - \frac{\omega_{jj} - \lambda}{\det(A)} (C_i - \frac{\omega_{ji}}{\omega_{jj} - \lambda} C_j)(B_i - \frac{\omega_{ij}}{\omega_{jj} - \lambda} B_j)\right)$$

is a well defined quantity.

Letting $\mathcal{I}_i = \mathcal{I} \cup \{i\}$ for $i \neq j$ and $i, j \in \bar{\mathcal{I}}$ then repeated use of (6) implies $R_{ij} = r(M(G) - \lambda I; \mathcal{I}_i, \mathcal{I})$. Moreover, as $R_{12} = R_{21}$ then it follows that the graphs $\mathcal{R}(G; V - \{v_1\}, T) = \mathcal{R}(G; V - \{v_2\}, T)$ since they have the same adjacency matrices. $\square$

To simplify the proof of theorem 1.3.10 we note the following. If $S_1, \ldots, S_m$ induces a sequence of reductions on $G = (V, E, \omega)$ then $\mathcal{R}(G; S_1, \ldots, S_m)$ can alternately be written as $\mathcal{R}em(G; S_0 - S_1, \ldots, S_{m-1} - S_m)$ for $V = S_0$. This notation is meant to indicate that at the ith reduction we remove the vertices $S_{i-1} - S_i$ from the graph $\mathcal{R}em(G; S_0 - S_1, \ldots, S_{i-2} - S_{i-1})$ for each $1 \leq i \leq m$ where $S_{-1} - S_0 = \emptyset$.

With this notation in place we give a proof of theorem 1.3.10.

Proof. If $G = (V, E, \omega) \in \mathbb{G}_\pi$ and $\{v_1, v_m\} \subset V$ then by the discussion in section 1.3.2 the graph $\mathcal{R}em(G; \{v_1\}, \ldots, \{v_m\})$ is defined. Moreover, for any $1 \leq i < m$, let $G_i = \mathcal{R}em(G; \{v_1\}, \ldots, \{v_{i-1}\})$ where $G_1 = G$. Then by lemma 1.5.4 it follows that $\mathcal{R}em\big(G_i; \{v_i\}, \{v_{i+1}\}\big) = \mathcal{R}em\big(G_i; \{v_{i+1}\}, \{v_i\}\big)$ which in turn implies that

$$\mathcal{R}em\big(G; \{v_1\}, \ldots, \{v_i\}, \{v_{i+1}\}, \ldots \{v_m\}\big) = \\ \mathcal{R}em\big(G; \{v_1\}, \ldots, \{v_{i+1}\}, \{v_i\}, \ldots \{v_m\}\big).$$

By repeatedly switching the order of any two vertices as above it follows that for any bijection $b : \{v_1, \ldots, v_m\} \to \{v_1, \ldots, v_m\}$

$$\mathcal{R}em\big(G; \{v_1\}, \ldots \{v_m\}\big) = \mathcal{R}em\big(G; \{b(v_1)\}, \ldots \{b(v_m)\}\big). \tag{13}$$

Suppose $S_1, \ldots, S_m$ induces a sequence of reductions on G. If $S_0 = V$ and each $S_{i-1} - S_i = \{v_1^i, \ldots, v_{i_n}^i\}$ then theorem 2.3.5 implies

$$\mathcal{R}em(G; S_0 - S_1, \ldots, S_{m-1} - S_m) = \\ \mathcal{R}em(G; \{v_1^1\}, \ldots, \{v_{n_1}^1\}, \ldots, \{v_1^m\}, \ldots, \{v_{n_m}^m\}).$$

If $T_1, \ldots, T_p$ also induces sequence of reductions on G where $T_0 = V$ and each set $T_{i-1} - T_i = \{\tilde{v}_1^i, \ldots, \tilde{v}_{i_q}^i\}$ then similarly,

$$\mathcal{R}em(G; T_0 - T_1, \ldots, T_{p-1} - T_p) = \\ \mathcal{R}em(G; \{\tilde{v}_1^1\}, \ldots, \{\tilde{v}_{q_1}^1\}, \ldots, \{\tilde{v}_1^p\}, \ldots, \{\tilde{v}_{q_p}^p\}).$$

If $S_m = T_p$ then $\bar{S}_m = \bigcup_{i=1}^m \big(S_{i-1} - S_i\big) = \bigcup_{i=1}^p \big(T_{i-1} - T_i\big)$. There is then a bijection $\tilde{b} : \bar{S}_m \to \bar{S}_m$ such that $\mathcal{R}em(G; \{v_1^1\}, \ldots, \{v_{n_m}^m\}) = \mathcal{R}em(G; \{\tilde{b}(\tilde{v}_1^1)\}, \ldots, \{\tilde{b}(\tilde{v}_{q_p}^p)\})$ implying $\mathcal{R}(G; S_1, \ldots, S_m) = \mathcal{R}(G; T_1, \ldots, T_{p-1}, S_m)$ by use of equation (13) completing the proof. $\square$

The notation $\mathcal{R}_{S_m}[G] = \mathcal{R}(G; S_1, \ldots, S_m)$ is then well defined for any graph $G = (V, E, \omega)$ in $\mathbb{G}_\pi$ and nonempty $S_m \subseteq V$. Moreover, if for any $G \in \mathbb{G}_\pi - \emptyset$, $\tau(G) \subseteq V$ is nonempty then the relation $G \sim H$ if $\mathcal{R}_{\tau(G)}[G] \simeq \mathcal{R}_{\tau(H)}[H]$ is an equivalence relation on $\mathbb{G}_\pi$. To see this, note that as the rule τ specifies a unique nonempty graph $\mathcal{R}_{\tau(G)}[G]$ for any $G \in \mathbb{G}_\pi - \emptyset$, then $\sim$ is an equivalence relation given that $\simeq$ is reflexive, symmetric, and transitive. Hence, theorem 1.3.10 holds.

We now give a proof of theorem 1.4.2.

Proof. Let $G = (V, E, \omega)$ and $S \in st(G)$ such that each vertex of G is on a branch of $\mathcal{B}_S(G)$. It follows then that each $e \in E$ is part of the ordered sequence $\Omega(\beta)$ for some $\beta \in \mathcal{B}_{ij}(G; S)$. Given the bijection $b_{ij} : \mathcal{B}_{ij}(\mathcal{X}_S(G); S) \to \mathcal{B}_{ij}(G; S)$ in definition 1.4.1 if $\mathcal{X}_S(G) = (X, \mathcal{E}, \mu)$ then $\omega(e) \in \mu(\mathcal{E})$. Hence, $\omega(E) \subseteq \mu(\mathcal{E})$. Similarly, as each vertex of $\mathcal{X}_S(G)$ is on a branch of $\mathcal{B}_S(\mathcal{X}_S(G))$ then $\mu(\mathcal{E}) \subseteq \omega(E)$ implying the weight sets of G and $\mathcal{X}_S(G)$ are equal.

Moreover, the assumption that each vertex of G and $\mathcal{X}_S(G)$ lies on a branch of $\mathcal{B}_S(G)$ and $\mathcal{B}_S(\mathcal{X}_S(G))$ respectively implies together with (i) in definition 1.4.1 that $\mathcal{R}_S(G) = \mathcal{R}_S(\mathcal{X}_S(G))$. Similarly, $|\mathcal{X}_S(G)| \geq |G|$ follows from the fact that branches of equal length in $\mathcal{B}_S(G)$ and $\mathcal{B}_S(\mathcal{X}_S(G))$ are in bijective correspondence and that the branches in $\mathcal{B}_S(\mathcal{X}_S(G))$ are independent.

To compare the spectrum of G and its expansion $\mathcal{X}_S(G)$ note equation (12) and the fact that $\det\big(M(\mathcal{R}_S(G) - \lambda I)\big) = \det\big(M(\mathcal{R}_S(\mathcal{X}_S(G)) - \lambda I)\big)$ together imply

$$\frac{\det\big(M(G) - \lambda I\big)}{\det\big(M(G|_{\bar{S}}) - \lambda I\big)} = \frac{\det\big(M(\mathcal{X}_S(G)) - \lambda I\big)}{M(\mathcal{X}_S(G)|_{\bar{S}}) - \lambda I\big)}.$$

By (4) it then follows that $\det\big(M(\mathcal{X}_S(G)) - \lambda I\big) =$

$$\det\big(M(G) - \lambda I\big)\frac{\prod_{v_i \in X-S}\big(\mu(e_{ii}) - \lambda\big)}{\prod_{v_i \in V-S}\big(\omega(e_{ii}) - \lambda\big)} = \det\big(M(G) - \lambda I\big)\frac{\prod_{v_i \in V-S}\big(\omega(e_{ii}) - \lambda\big)^{n_i}}{\prod_{v_i \in V-S}\big(\omega(e_{ii}) - \lambda\big)}$$

where n_i is the number of distinct branches in $\mathcal{B}_S(G)$ containing v_i.

This completes the proof. $\square$

For theorem 1.4.4 we give the following proof.

Proof. Let $G = (V, E, \omega)$ and $S = \{v_1, \ldots, v_m\} \in st_0(G)$ where each vertex of G is on a branch of $\mathcal{B}_S(G)$. By construction, it follows that there exists bijections $\hat{b}_{ij} : \mathcal{B}_{ij}(\mathcal{L}_S(G); S) \to \mathcal{B}_{ij}(G; S)$ for all $1 \leq i, j \leq n$ such that if $\hat{b}_{ij}(\gamma) = \beta$ then $\Omega(\gamma) = \lambda^{|\beta|-2}\mathcal{P}_\omega(\beta), 0, 1, \ldots, 0, 1$ and $|\gamma| = |\beta|$. Hence, if $\mathcal{L}_S(G) = (X, \mathcal{E}, \mu)$ then $\mathcal{P}_\mu(\gamma) = \mathcal{P}_\omega(\beta)$ implying

$$\sum_{\gamma \in \mathcal{B}_{ij}(\mathcal{L}_S(G); S)} \mathcal{P}_\mu(\gamma) = \sum_{\beta \in \mathcal{B}_{ij}(G; S)} \mathcal{P}_\omega(\beta).$$

Therefore, $\mathcal{R}_S(G) = \mathcal{R}_S(\mathcal{L}_S(G))$ or G and $\mathcal{L}_S(G)$ are spectrally equivalent.

As $S \in st_0(G)$ and $S \in st_0(\mathcal{L}_S(G))$ then $\sigma(G|_{\bar{S}})$ and $\sigma(\mathcal{L}_S(G)|_{\bar{S}})$ are lists of zeros and both $\sigma^{-1}(G|_{\bar{S}})$ and $\sigma^{-1}(\mathcal{L}_S(G)|_{\bar{S}})$ are empty. By theorem 1.3.4 it then follows that $\sigma(G)$ and $\sigma(\mathcal{L}_S(G))$ differ by at most a list of zeros.

Moreover, since in the construction of $\mathcal{L}_S(G)$, $|\bar{\mathcal{X}}| = |\mathcal{L}_S(G)|$ then we have that $|\mathcal{L}_S(G)| = |S| + \sum_{j=1}^{m} \left(|\beta^j| - 2\right)$ as each β^j has $|\beta^j| - 2$ interior vertices and is independent of the branch β^i for each $i \neq j$ and $1 \leq j \leq m$. As $|S| = m$ then $\mathcal{L}_S(G)$ is a reduction of G over $\mathbb{U}$ if $|G| > m + \sum_{j=1}^{m} \left(|\beta^j| - 2\right)$ and an expansion otherwise for any unital subring $\mathbb{U} \subseteq \mathbb{W}$ containing $\omega(E)$. □

1.6 Concluding Remarks

The goal of this chapter is foremost to introduce the notion of *isospectral graph transformations*. Specifically, this chapter describes the general process of isospectral graph reductions in which a graph is collapsed around a specific set of vertices as well as the inverse process of isospectral expansion. We then considered sequences of isospectral reductions and showed that under mild assumptions a typical graph could be uniquely reduced to a graph on any subset of its vertex set.

As is shown in the following chapter, such reductions and sequences of reductions can be used to improve each of the eigenvalue estimates of Gershgorin et al. [10, 11, 27, 28, 48] for the general class of matrices with complex entries. That is, such eigenvalue estimates improve as a graph is reduced using the method of reduction introduced in this chapter. In this application of isospectral transformations, the flexibility and commutativity of such reductions is particularly useful.

Additionally, isospectral reductions are also relevant to the theory of networks as they provide a flexible means of course graining networks (i.e. viewing the network at some appropriate scale) by introducing new equivalence relations on the space of all networks. That is, viewing networks modulo some specified graph (or network) structure.

The second type of transformations we presented in this chapter were isospectral graph transformations over fixed weight sets. Such transformations allow one to transform an arbitrary (weighted) graph while preserving the graph's set of edge weights along with the its spectrum.

Motivated by this procedure, we will introduce dynamical network expansions in chapter III. Dynamical network expansions modify a dynamical network in a way that preserves network dynamics but alters its associated graph structure (graph of interactions). In this chapter III we will demonstrate that this procedure allows one to establish global stability of a more general class of dynamical networks than those previously considered.

Lastly, the results of the present chapter introduce various approaches to simplifying a graph's structure while maintaining its spectrum. Therefore, these techniques can be used for optimal design, in the sense of structure simplicity of dynamical networks with prescribed dynamical properties ranging from synchronizability to chaoticity [1, 6].

CHAPTER II

IMPROVED EIGENVALUE ESTIMATES

2.1 Introduction

A remarkable theorem due to Gershgorin [25] states that if the matrix $A \in \mathbb{C}^{n\times n}$ then the eigenvalues of A are contained in the union of the n discs

$$\bigcup_{i=1}^{n}\{\lambda \in \mathbb{C} : |\lambda - A_{ii}| \leq \sum_{j=1, j\neq i}^{n} |A_{ij}|\}.$$

This simple and geometrically intuitive result moreover implies a nonsingularity result for diagonally dominant matrices (see theorem 1.4 in [48]), which can be traced back to earlier work done by Lévy, Desplanques, Minkowski, and Hadamard [34, 22, 39, 27]. More recently, this result of Gershgorin has been improved upon by both Brauer and Varga [10, 48] whose results are similar in spirit to Gershgorin's in that each assigns to every matrix $A \in \mathbb{C}^{n\times n}$ a region of the complex plane containing the matrix's eigenvalues. Moreover, the same holds for a result of Brualdi [11] with the exception that the associated region is define for a proper subset of the matrices in $\mathbb{C}^{n\times n}$.

These improvements can be summarized as follows. If $A \in \mathbb{C}^{n\times n}$ let $\Gamma(A)$, $\mathcal{K}(A)$, and $B(A)$ denote the associated regions given respectively by Gershgorin, Brauer, and the improvement of Brualdi's theorem given by Varga. If $\sigma(A)$ denotes the eigenvalues of A then it is known that $\sigma(A) \subseteq B(A) \subseteq \mathcal{K}(A) \subseteq \Gamma(A)$ for any complex valued matrix A (see [48] for details). Furthermore, if the region $br(A)$ associated with Brualdi's original result is defined then $\sigma(A) \subseteq B(A) \subseteq br(A) \subseteq \mathcal{K}(A)$.

The main goal of this chapter is to improve upon each of the estimates of Gershgorin, Brauer, Brualdi, and Varga by considering reductions of the weighted digraphs associated to each matrix $A \in \mathbb{C}^{n\times n}$. To do so we first extend these classical results to a larger class of square matrices with entries in $\mathbb{W}$ consisting of complex rational functions. The motivation for considering this class of matrices arises from the isospectral graph reductions introduced in chapter I along with the observation that in the study of dynamical networks an important (dynamic) characteristic of a network is the spectrum of the network's adjacency matrix [1, 6, 40, 42] (see chapter III).

Recall that one of the main result of the previous chapter is that the eigenvalues of the adjacency matrix of a graph and the adjacency matrix of any one of its reductions differ at most by some set, which is known in advance (see theorem 1.3.4). What is novel about this process is that it equivalently allows for the reduction of an arbitrary matrix $A \in \mathbb{C}^{n\times n}$ to a smaller matrix $R \in \mathbb{W}^{m\times m}$ (where $m < n$) such that the eigenvalues of A and R again differ by at most some set, known in advance.

In the present chapter we show that by using such graph reductions (equivalently matrix reductions) one can improve each of the eigenvalue estimates associated with Gershgorin, Brauer, and Brualdi. Specifically, for $M(G) \in \mathbb{C}^{n\times n}$ the regions in the complex plane associated with both Gershgorin and Brauer shrink as the graph G is reduced (see theorems 2.5.1 and 2.5.3 for exact statements). For the estimates associated with Brualdi and Varga we give sufficient conditions under which such estimates also improve as the underlying graph is reduced (see theorems 2.5.4 and 2.5.5).

We also note that, for a given graph (equivalently matrix), many graph reductions are typically possible. Hence, this process is quite flexible. Moreover, as it is possible to sequentially reduce a graph G, graph reductions on G can be used to estimate the spectrum of $M(G)$ with increasing accuracy depending on the extent to which G is reduced.

With this in mind chapter II is organized as follows. Section 2.2 introduces the notation used in this chapter. Section 2.3 extends the results of Gershgorin, Brauer, Brualdi, and Varga to the class of matrices with entries in $\mathbb{W}$. Section 2.4 then summarizes and expands the theory of isospectral graph reductions of chapter I developed in [14] which will be used to improve the eigenvalue estimates of section 2.3. Section 2.5 contains the main results of the chapter demonstrating that isospectral graph reductions allow for improved eigenvalue estimates using the methods of Gershgorin, Brauer, Brualdi, and Varga. Section 2.6 gives some natural applications of the theorems of section 2.5. These include estimating the spectrum of a Laplacian matrix of graph, estimating the spectral radius of a matrix, and determining useful reductions to use for a given matrix (or equivalently, graph of a network).

2.2 Notation

In this chapter, as in the previous, we consider two equivalent mathematical objects. The first is the set of graphs consisting of all finite weighted digraphs with or without loops having no parallel edges and edge weights in the set $\mathbb{W}$ of complex rational functions (described below as well as in chapter I). We denote this class of graphs by $\mathbb{G}$ where $\mathbb{G}^n \subset \mathbb{G}$ is the set of graphs with n vertices. The second set of objects we consider are the weighted adjacency matrices associated with the graphs in $\mathbb{G}$. That is, the class of matrices $\mathbb{W}^{n\times n}$ for all $n \geq 1$.

As before we let the weighted digraph $G \in \mathbb{G}$ be the triple (V, E, ω) where for $V = \{v_1, \dots, v_n\}$ the edge from v_i to v_j is given by e_{ij}. Furthermore, the set of weights $\mathbb{W}$ are the set of rational functions of the form $p(\lambda)/q(\lambda)$ where $p(\lambda), q(\lambda) \in \mathbb{C}[\lambda]$ such that $p(\lambda)$ and $q(\lambda)$ have no common factors and the polynomial $q(\lambda)$ is nonzero. Additionally, the spectrum associated to a graph $G \in \mathbb{G}$ are the solutions to the equation $\det(M(G) - \lambda I) = 0$ including multiplicities.

As we are mainly concerned with the properties of the adjacency matrix of graphs in $\mathbb{G}$ we note, as we have previously suggested, that there is a one-to-one correspondence between the graphs in $\mathbb{G}^n$ and the matrices $\mathbb{W}^{n\times n}$. Therefore, we may talk of a graph $G \in \mathbb{G}^n$ associated with a matrix $M = M(G)$ in $\mathbb{W}^{n\times n}$ and vice-versa without ambiguity.

2.3 *Spectra Estimation of* $\mathbb{W}^{n\times n}$

Here we extend the classical results of Gershgorin, Brauer, Brualdi, and the more recent work of Varga (see for instance [48]) to matrices in $\mathbb{W}^{n\times n}$. To do so we will first define the notion of a *polynomial extension* of a graph $G \in \mathbb{G}$.

Definition 2.3.1. *If $G \in \mathbb{G}^n$ and $M(G)_{ij} = p_{ij}/q_{ij}$ where $p_{ij}, q_{ij} \in \mathbb{C}[\lambda]$ let $L_i(G)=\prod_{j=1}^n q_{ij}$ for $1 \le i \le n$. We call the graph $\bar{G}$ with adjacency matrix*

$$M(\bar{G})_{ij} = \begin{cases} L_i(G)M(G)_{ij} & i \neq j \\ L_i(G)\big(M(G)_{ij} - \lambda\big) + \lambda & i = j \end{cases}, \quad 1 \le i, j \le n$$

the polynomial extension of G.

To justify this name note that each $M(\bar{G})_{ij}$ is an element of $\mathbb{C}[\lambda]$ or $M(\bar{G})$ has complex polynomial entries. Moreover, we have the following result.

Lemma 2.3.2. *If $G \in \mathbb{G}$ then $\sigma(G) \subseteq \sigma(\bar{G})$.*

Proof. For $G \in \mathbb{G}^n$ note that the matrix $M(\bar{G}) - \lambda I$ is given by

$$(M(\bar{G}) - \lambda I)_{ij} = \begin{cases} L_i(G)M(G)_{ij} & i \neq j \\ L_i(G)\big(M(G)_{ij} - \lambda\big) & i = j \end{cases} \quad \text{for } 1 \le i \le n.$$

The matrix $M(\bar{G}) - \lambda I$ is then the matrix $M(G) - \lambda I$ whose ith row has been multiplied by $L_i(G)$. Therefore,

$$\det\big(M(\bar{G}) - \lambda I\big) = \Big(\prod_{i=1}^n L_i(G)\Big) \det\big(M(G) - \lambda I\big)$$

implying $\sigma(G) \subseteq \sigma(\bar{G})$. □

2.3.1 Gershgorin-Type Regions

As previously mentioned, a theorem of Gershgorin's, originating from [25], gives a simple method for estimating the eigenvalues of a square matrix with complex valued entries. This result is the following theorem which we formulate after introducing the following.

If $A \in \mathbb{C}^{n\times n}$ let

$$r_i(A) = \sum_{j=1,\ j\neq i}^{n} |A_{ij}|, \qquad 1 \leq i \leq n \tag{14}$$

be the *ith row sum* of A.

Theorem 2.3.3. ***(Gershgorin [25])*** *Let $A \in \mathbb{C}^{n\times n}$. Then all eigenvalues of A are contained in the set*

$$\Gamma(A) = \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C} : |\lambda - A_{ii}| \leq r_i(A)\}.$$

In order to extend theorem 2.3.3 to the class of matrices $\mathbb{W}^{n\times n}$ we use the following adaptation of the notation given in (14). For $G \in \mathbb{G}^n$ let

$$r_i(G) = \sum_{j=1, j\neq i}^{n} |M(G)_{ij}| \text{ for } 1 \leq i \leq n$$

be the *ith row sum* of $M(G)$.

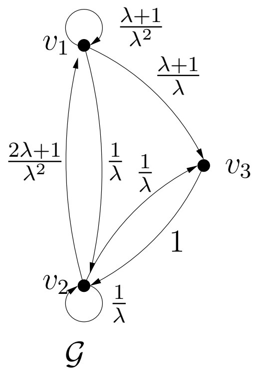
Note that as $M(\bar{G}) \in \mathbb{C}[\lambda]^{n\times n}$, for any $G \in \mathbb{G}$, we can view $M(\bar{G}) = M(\bar{G}, \lambda)$ as a function

$$M(\bar{G}, \cdot) : \mathbb{C} \to \mathbb{C}^{n\times n}$$

or entrywise $M(\bar{G}, \cdot)_{ij} : \mathbb{C} \to \mathbb{C}$. Likewise, we can consider $r_i(\bar{G}) = r_i(\bar{G}, \lambda)$ to be the function $r_i(\bar{G}, \cdot) : \mathbb{C} \to \mathbb{C}$. However, typically we will suppress the dependence of $M(\bar{G})$ and $r_i(\bar{G})$ on λ for ease of notation.

Theorem 2.3.4. *Let $G \in \mathbb{G}^n$. Then $\sigma(G)$ is contained in the set*

$$\mathcal{BW}_\Gamma(G) = \bigcup_{i=1}^{n} \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G})_{ii}| \leq r_i(\bar{G})\}.$$



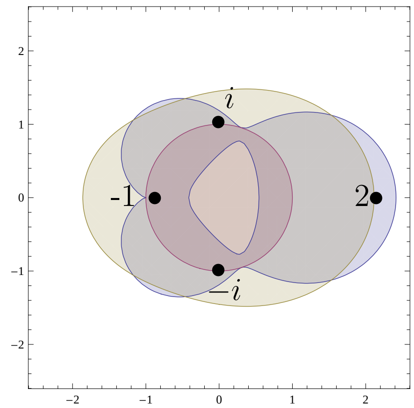


Figure 6: The graph $\mathcal{G}$ (left) and $\mathcal{BW}_\Gamma(\mathcal{G})$ (right) where $\sigma(\mathcal{G}) = \{-1,-1,2,-i,i\}$ is indicated.

Proof. First note that for $\alpha \in \sigma(G)$ the matrix $M(\bar{G},\alpha) \in \mathbb{C}^{n\times n}$. As Lemma 2.3.2 implies that α is an eigenvalue of the matrix $M(\bar{G},\alpha)$ then by an application of Gershgorin's theorem the inequality $|\alpha - M(\bar{G},\alpha)_{ii}| \leq r_i(\bar{G},\alpha)$ holds for some $1 \leq i \leq n$. Hence, $\alpha \in \mathcal{BW}_\Gamma(G)$. $\square$

Because it will be useful later in comparing different regions in the complex plane, for $G \in \mathbb{G}^n$ we denote

$$\mathcal{BW}_\Gamma(G)_i = \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G})_{ii}| \leq r_i(\bar{G})\} \text{ where } 1 \leq i \leq n$$

and call this the *ith Gershgorin-type region* of G. Similarly, we call the union $\mathcal{BW}_\Gamma(G)$ of these n sets the *Gershgorin-type region* of the graph $G \in \mathbb{G}$.

As an illustration of theorem 2.3.4 consider the following example. Let $\mathcal{G} \in \mathbb{G}$ be the graph with adjacency matrix

$$M(\mathcal{G}) = \begin{bmatrix} \frac{\lambda+1}{\lambda^2} & \frac{1}{\lambda} & \frac{\lambda+1}{\lambda} \\ \frac{2\lambda+1}{\lambda^2} & \frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & 1 & 0 \end{bmatrix}. \tag{15}$$

As $\det(M(\mathcal{G},\lambda) - \lambda I) = (-\lambda^5 + 2\lambda^3 + 2\lambda^2 + 3\lambda + 2)/(\lambda^2)$ it follows that $\sigma(\mathcal{G}) = \{-1,-1,i,-i,2\}$. The corresponding Gershgorin-type region $\mathcal{BW}_\Gamma(\mathcal{G})$ is shown in figure 6 where

$$M(\bar{\mathcal{G}}) = \begin{bmatrix} -\lambda^5+\lambda^3+\lambda^2+\lambda & \lambda^3 & \lambda^4+\lambda^3 \\ 2\lambda^3+\lambda^2 & -\lambda^5+\lambda^3+\lambda & \lambda^3 \\ 0 & 1 & 0 \end{bmatrix}.$$

We note here that $\mathcal{BW}_\Gamma(\mathcal{G})$ is the union of the three regions $\mathcal{BW}_\Gamma(\mathcal{G})_1$, $\mathcal{BW}_\Gamma(\mathcal{G})_2$, and $\mathcal{BW}_\Gamma(\mathcal{G})_3$ whose boundaries are shown in blue, red, and tan (if given in color). Additionally, the interior colors of these regions reflect their intersections and the eigenvalues of $M(\mathcal{G})$ are indicated as points. We will use the same technique to display similar regions in what follows.

2.3.2 Brauer-Type Regions

Following Gershgorin, Brauer was able to give the following eigenvalue inclusion result for matrices with complex valued entries.

Theorem 2.3.5. ***(Brauer [48])*** *Let $A \in \mathbb{C}^{n\times n}$ where $n \geq 2$. Then all eigenvalues of A are located in the set*

$$\mathcal{K}(A) = \bigcup_{\substack{1\leq i,j\leq n \\ i\neq j}} \{\lambda \in \mathbb{C} : |\lambda - A_{ii}||\lambda - A_{jj}| \leq r_i(A)r_j(A)\}. \tag{16}$$

The individual regions given by $\{\lambda \in \mathbb{C} : |\lambda - A_{ii}||\lambda - A_{jj}| \leq r_i(A)r_j(A)\}$ in equation (16) are known as Cassini ovals and may consists of one or two distinct components. Moreover, there are $\binom{n}{2}$ such regions for any $n \times n$ matrix with complex entries. As with Gershgorin's theorem we prove an extension to Brauer's theorem for matrices in $\mathbb{W}^{n\times n}$.

Theorem 2.3.6. *Let $G \in \mathbb{G}^n$ where $n \geq 2$. Then $\sigma(G)$ is contained in the set*

$$\mathcal{BW}_\mathcal{K}(G) = \bigcup_{\substack{1\leq i,j\leq n \\ i\neq j}} \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G})_{ii}||\lambda - M(\bar{G})_{jj}| \leq r_i(\bar{G})r_j(\bar{G})\}.$$

Also, $\mathcal{BW}_\mathcal{K}(G) \subseteq \mathcal{BW}_\Gamma(G)$.

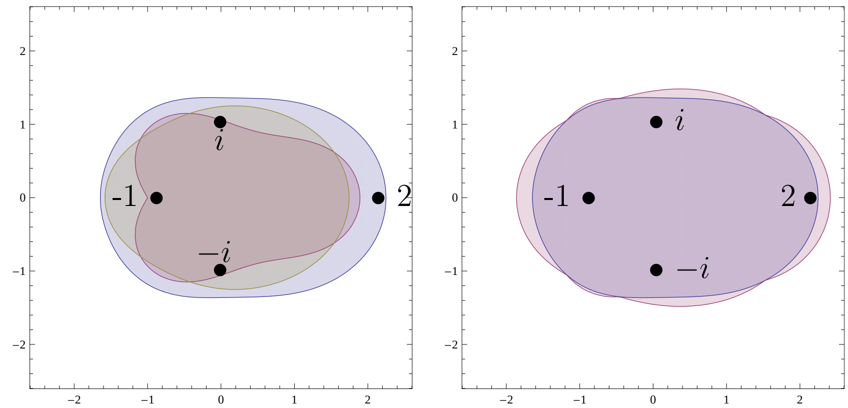


Figure 7: Left: The Brauer region $\mathcal{K}(\mathcal{G})$ for $\mathcal{G}$ in figure 6. Right: $\mathcal{K}(\mathcal{G}) \subseteq \Gamma(\mathcal{G})$.

Proof. As in the proof of theorem 2.3.4, if $\alpha \in \sigma(G)$ then $\alpha \in \sigma(\bar{G})$ and the matrix $M(\bar{G}, \alpha) \in \mathbb{C}^{n \times n}$. Brauer's theorem therefore implies that

$$|\alpha - M(\bar{G}, \alpha)_{ii}||\alpha - M(\bar{G}, \alpha)_{jj}| \leq r_i(\bar{G}, \alpha) r_j(\bar{G}, \alpha)$$

for some pair of distinct integers i and j. It then follows that, $\alpha \in \mathcal{BW}_{\mathcal{K}}(G)$ or $\sigma(G) \subseteq \mathcal{BW}_{\mathcal{K}}(G)$.

Following the proof in [48], to prove the assertion that $\mathcal{BW}_{\mathcal{K}}(G) \subseteq \mathcal{BW}_{\Gamma}(G)$ let

$$\mathcal{BW}_{\mathcal{K}}(G)_{ij} = \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}||\lambda - M(\bar{G}, \lambda)_{jj}| \leq r_i(\bar{G}, \lambda) r_j(\bar{G}, \lambda)\} \qquad (17)$$

for distinct i and j. The claim then is that $\mathcal{BW}_{\mathcal{K}}(G)_{ij} \subseteq \mathcal{BW}_{\Gamma}(G)_i \cup \mathcal{BW}_{\Gamma}(G)_j$. To see this, assume for a fixed λ that $\lambda \in \mathcal{BW}_{\mathcal{K}}(G)_{ij}$ or

$$|\lambda - M(\bar{G}, \lambda)_{ii}||\lambda - M(\bar{G}, \lambda)_{jj}| \leq r_i(\bar{G}, \lambda) r_j(\bar{G}, \lambda).$$

If $r_i(\bar{G}, \lambda) r_j(\bar{G}, \lambda) = 0$ then either $\lambda - M(\bar{G}, \lambda)_{ii} = 0$ or $\lambda - M(\bar{G}, \lambda)_{jj} = 0$. As $\lambda = M(\bar{G}, \lambda)_{ii}$ implies $\lambda \in \mathcal{BW}_{\Gamma}(G)_i$ and $\lambda = M(\bar{G}, \lambda)_{jj}$ implies $\lambda \in \mathcal{BW}_{\Gamma}(G)_j$ then $\lambda \in \mathcal{BW}_{\Gamma}(G)_i \cup \mathcal{BW}_{\Gamma}(G)_j$.

If $r_i(\bar{G}, \lambda) r_j(\bar{G}, \lambda) > 0$ then it follows that

$$\Big(\frac{|\lambda - M(\bar{G}, \lambda)_{ii}|}{r_i(\bar{G}, \lambda)}\Big)\Big(\frac{|\lambda - M(\bar{G}, \lambda)_{jj}|}{r_j(\bar{G}, \lambda)}\Big) \leq 1.$$

Since at least one of the two quotients on the left must be less than or equal to 1 then $\lambda \in \mathcal{BW}_{\Gamma}(G)_i \cup \mathcal{BW}_{\Gamma}(G)_j$ which verifies the claim and the result follows. $\square$

We call the region $\mathcal{BW}_{\mathcal{K}}(G)$ the *Brauer-type region* of the graph G and the region $\mathcal{BW}_{\mathcal{K}}(G)_{ij}$ given in (17) the *ijth Brauer-type region* of G. Using theorem 2.3.6 on the graph $\mathcal{G}$ given in figure 6 we have the Brauer-type region shown in the left hand side of figure 7. On the right is a comparison between $\mathcal{BW}_{\mathcal{K}}(\mathcal{G})$ and $\mathcal{BW}_{\Gamma}(\mathcal{G})$ where the inclusion $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}) \subseteq \mathcal{BW}_{\Gamma}(\mathcal{G})$ is demonstrated.

2.3.3 Brualdi-Type Regions

In this section we first extend a result of Varga [48], which is itself an extension of a result of Brualdi [11] relating the spectrum of a graph with complex weights to its cycle structure. We then show that the same can be done for the original result of Brualdi. To do so we require the following.

Recall a *path* P in the graph $G = (V, E, \omega)$ is a sequence of distinct vertices $v_1, \dots, v_m \in V$ such that $e_{i,i+1} \in E$ for $1 \leq i \leq m-1$. In the case that the vertices $v_1, \dots, v_m$ are distinct, with the exception that $v_1 = v_m$, then P is a *cycle*. If γ is a cycle of G we denote it by its ordered set of vertices. That is, if $e_{i,i+1} \in E$ for $1 \leq i \leq m-1$ and $e_{m1} \in E$ then we write this cycle as the ordered set of vertices $\{v_1, \dots, v_m\}$ up to cyclic permutation. Moreover, we call a cycle consisting of a single vertex a *loop*.

A *strong cycle* of G is a cycle $\{v_1, \dots, v_m\}$ such that $m \geq 2$. Furthermore, if $v_i \in V$ has no strong cycle passing through it then we define its associated *weak cycle* as $\{v_i\}$ irregardless of whether $e_{ii} \in E$. For $G \in \mathbb{G}$ we let $C_s(G)$ and $C_w(G)$ denote the set of strong and weak cycles of G respectively and let $C(G) = C_s(G) \cup C_w(G)$.

A directed graph is *strongly connected* if there is a path from each vertex of the graph to every other vertex. The *strongly connected components* of $G = (V, E)$ are its maximal strongly connected subgraphs. Moreover, the vertex set $V = \{v_1, \dots, v_n\}$ can always be labeled in such a way that $M(G)$ has the following triangular block structure

$$M(G) = \begin{bmatrix} M(\mathbb{S}_1(G)) & 0 & \dots & 0 \\ * & M(\mathbb{S}_2(G)) & & \vdots \\ \vdots & & \ddots & 0 \\ * & \dots & * & M(\mathbb{S}_m(G)) \end{bmatrix}$$

where $\mathbb{S}_i(G)$ is a strongly connected component of G and $*$ are block matrices with possibly nonzero entries (see [12], [28], or [48] or for more details).

As the strongly connected components of a graph are unique then for $G \in \mathbb{G}^n$ we define

$$\tilde{r}_i(G) = \sum_{j \in N_\ell, j \neq i} |M(\mathbb{S}_\ell(G))_{ij}| \text{ for } 1 \leq i \leq n$$

where $i \in N_\ell$ and N_ℓ is the set of indicies indexing the vertices in $\mathbb{S}_\ell(G)$. That is, $\tilde{r}_i(G)$ is $r_i(G)$ restricted to the strongly connected component containing v_i. Furthermore, we let $\tilde{r}_i(\bar{G}) = \tilde{r}_i(\bar{G}, \lambda)$ where we again consider $\tilde{r}_i(\bar{G}, \cdot) : \mathbb{C} \to \mathbb{C}$

If $A \in \mathbb{C}^{n \times n}$ then we write $\tilde{r}_i(G, \lambda) = \tilde{r}_i(A)$ where $A = M(G)$. This allows us to state the following theorem by Varga which, as previously mentioned, is an extension of Brualdi's original theorem [11].

Theorem 2.3.7. ***(Varga [48])*** *Let $A \in \mathbb{C}^{n \times n}$. Then the eigenvalues of A are contained in the set*

$$B(A) = \bigcup_{\gamma \in C(A)} \{\lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - A_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(A)\}.$$

As with the theorems of Gershgorin and Brauer this result can similarly be extended to matrices in $\mathbb{W}^{n \times n}$ as follows.

Theorem 2.3.8. *Let $G \in \mathbb{G}$. Then $\sigma(G)$ is contained in the set*

$$\mathcal{BW}_B(G) = \bigcup_{\gamma \in C(\bar{G})} \{\lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\bar{G})_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\bar{G})\}. \tag{18}$$

Also, $\mathcal{BW}_B(G) \subseteq \mathcal{BW}_{\mathcal{K}}(G)$.

For $G \in \mathbb{G}$ we call $\mathcal{BW}_B(G)$ the *Brualdi-type region* of the graph G and set

$$\mathcal{BW}_B(G)_\gamma = \{\lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\bar{G})_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\bar{G})\}$$

the Brualdi-type region associated with the cycle γ.

Proof. For $G \in \mathbb{G}^n$ let $\bar{G} = \bar{G}(\lambda)$ where for fixed $\alpha \in \mathbb{C}$, $\bar{G}(\alpha)$ is the graph with adjacency matrix $M(\bar{G}, \alpha) \in \mathbb{C}^{n \times n}$. Moreover, for any $\gamma = \{v_1, \ldots, v_m\}$ in $C(\bar{G})$ and fixed $\alpha \in \mathbb{C}$ let $\gamma(\alpha)$ be the set of vertices $\{v_1, \ldots, v_m\}$ in the graph $\bar{G}(\alpha)$.

Using this notation, if $\alpha \in \sigma(G)$ then by lemma 2.3.2 and theorem 2.3.7 there exists a $\gamma' \in C(\bar{G}(\alpha))$ such that

$$\prod_{v_i \in \gamma'} |\alpha - M(\bar{G}, \alpha)_{ii}| \leq \prod_{v_i \in \gamma'} \tilde{r}_i(\bar{G}, \alpha). \tag{19}$$

There are then two possibilities, either $\gamma' \in C(\bar{G})$ or it is not. If $\gamma' \in C(\bar{G})$ then the set of vertices $\gamma'(\alpha)$ is also a cycle in $\bar{G}$ in which case equation (18) and (19) imply $\alpha \in \mathcal{BW}_B(G)$. Suppose then that $\gamma' \notin C(\bar{G})$.

Note that if $\gamma' \in C_s(\bar{G}(\alpha))$ then as $M(\bar{G}, \alpha)_{ij} \neq 0$ implies $M(\bar{G}, \lambda)_{ij} \neq 0$ for $i \neq j$ then $\gamma' \in C_s(\bar{G})$, which is not possible. Hence, $\gamma' \in C_w(\bar{G}(\alpha))$ or γ' must be a loop of some vertex v_j where the graph induced by $\{v_j\}$ in $\bar{G}(\alpha)$ is a strongly connected component of $\bar{G}(\alpha)$. Therefore, equation (19) is equivalent to $|\alpha - M(\bar{G}, \alpha)_{jj}| \leq 0$ implying $\alpha = M(\bar{G}, \alpha)_{jj}$.

As some cycle $\gamma \in C(\bar{G})$ contains the vertex v_j then α is contained in the set

$$\{\lambda \in \mathbb{C} : \prod_{v_i \in \gamma} |\lambda - M(\bar{G}, \lambda)_{ii}| \leq \prod_{v_i \in \gamma} \tilde{r}_i(\bar{G}, \lambda)\}$$

implying that $\alpha \in \mathcal{BW}_B(G)$.

To show that $\mathcal{BW}_B(G) \subseteq \mathcal{BW}_\mathcal{K}(G)$ we again follow the proof in [48]. Let $\gamma \in C(\bar{G})$. Supposing that $\gamma \in C_w(\bar{G})$ then $\gamma = \{v_i\}$ for some vertex v_i of G and

$$\mathcal{BW}_B(G)_\gamma = \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G}, \lambda)_{ii}| = 0\}$$

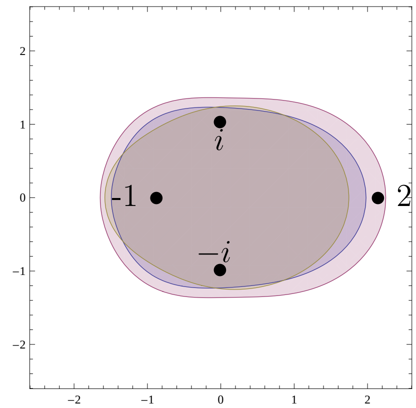


Figure 8: The Brualdi-type region $\mathcal{BW}_B(\mathcal{G})$ for $\mathcal{G}$ in figure 6.

as v_i is the vertex set of some strongly connected component of $\bar{G}$. It follows from (17) that $\mathcal{BW}_B(G)_\gamma \subseteq \mathcal{BW}_\mathcal{K}(G)_{ij}$ for any $1 \leq j \leq n$ where $i \neq j$. In particular, note that if $\tilde{r}_i(\bar{G}, \lambda) = 0$ then $\lambda \in \mathcal{BW}_\mathcal{K}(G)_{ij}$ for any $1 \leq j \leq n$ where $i \neq j$.

If on the other hand, $\gamma \in C_s(\bar{G})$ then for convenience let $\gamma = \{v_1, \dots, v_p\}$ where $p > 1$ and note that

$$\mathcal{BW}_B(G)_\gamma = \{\lambda \in \mathbb{C} : \prod_{i=1}^{p} |\lambda - M(\bar{G}, \lambda)_{ii}| \leq \prod_{i=1}^{p} \tilde{r}_i(\bar{G}, \lambda)\}. \tag{20}$$

Assuming $0 < \tilde{r}_i(\bar{G}, \lambda)$ for all $1 \leq i \leq p$ then for fixed $\lambda \in \mathcal{BW}_B(G)_\gamma$ it follows by raising both sides of the inequality in (20) to the $(p-1)st$ power that

$$\prod_{\substack{1 \leq i,j \leq p \\ i \neq j}} \Big(\frac{|\lambda - M(\bar{G}, \lambda)_{ii}||\lambda - M(\bar{G}, \lambda)_{jj}|}{\tilde{r}_i(\bar{G}, \lambda)\tilde{r}_j(\bar{G}, \lambda)} \Big) \leq 1 \tag{21}$$

As not all the terms of the product in (21) can exceed unity then for some pair of indices ℓ and k where $1 \leq \ell, k \leq p$ and $\ell \neq k$ it follows that

$$|\lambda - M(\bar{G}, \lambda)_{kk}||\lambda - M(\bar{G}, \lambda)_{\ell\ell}| \leq \tilde{r}_k(\bar{G}, \lambda)\tilde{r}_\ell(\bar{G}, \lambda). \tag{22}$$

Using the fact that $\tilde{r}_i(\bar{G}, \lambda) \leq r_i(\bar{G}, \lambda)$ for all $1 \leq i \leq n$ we conclude that $\lambda \in \mathcal{BW}_\mathcal{K}(G)_{k\ell}$ completing the proof. $\square$

The Brualdi-type region for the graph $\mathcal{G}$ with adjacency matrix (15) is shown in figure 8. We note that $\mathcal{BW}_B(\mathcal{G}) = \mathcal{BW}_\mathcal{K}(\mathcal{G})$ in this particular case.

We now consider Brualdi's original result which can be stated as follows.

Theorem 2.3.9. ***(Brualdi [11])*** *Let $A \in \mathbb{C}^{n\times n}$ where $C_w(A) = \emptyset$. Then the eigenvalues of A are contained in the set*

$$br(A) = \bigcup_{\gamma\in C(A)} \{\lambda \in \mathbb{C} : \prod_{v_i\in\gamma} |\lambda - A_{ii}| \leq \prod_{v_i\in\gamma} r_i(A)\}.$$

As with the theorems of Gershgorin, Brauer, and Varga this result generalizes to matrices with entries in $\mathbb{W}$ as follows.

Theorem 2.3.10. *Let $G \in \mathbb{G}$ where $C_w(G) = \emptyset$. Then $\sigma(G)$ is contained in the set*

$$\mathcal{BW}_{br}(G) = \bigcup_{\gamma\in C(\bar{G})} \{\lambda \in \mathbb{C} : \prod_{v_i\in\gamma} |\lambda - M(\bar{G})_{ii}| \leq \prod_{v_i\in\gamma} r_i(\bar{G})\}. \tag{23}$$

Also, $\mathcal{BW}_B(G) \subseteq \mathcal{BW}_{br}(G) \subseteq \mathcal{BW}_{\mathcal{K}}(G)$.

Proof. Note for any graph $G \in \mathbb{G}$ that $\tilde{r}_i(\bar{G}) \leq r_i(\bar{G})$ for all $\lambda \in \mathbb{C}$. Hence,

$$\mathcal{BW}_B(G) \subseteq \bigcup_{\gamma\in C(\bar{G})} \{\lambda \in \mathbb{C} : \prod_{v_i\in\gamma} |\lambda - M(\bar{G})_{ii}| \leq \prod_{v_i\in\gamma} r_i(\bar{G})\}.$$

Theorem 2.3.8 then implies that $\sigma(G)$ is contained in the set $\mathcal{BW}_{br}(G)$. Furthermore, if $\tilde{r}_i(G)$ is replaced by $r_i(G)$ in the proof of theorem 2.3.8 then in particular (22) implies that $\mathcal{BW}_{br}(G) \subseteq \mathcal{BW}_{\mathcal{K}}(G)$, completing the proof. □

We will refer to the region $\mathcal{BW}_{br}(G)$, given in (23), as the *original Brualdi-type* region of G.

2.4 Sequential Reductions and Principle Submatrices

In this section we extend the theory of isospectral reductions developed in chapter I. Specifically, to understand how sequential reductions effect the eigenvalues of a graph (or equivalently matrix) we prove the following.

Theorem 2.4.1. *If $G = (V, E, \omega) \in \mathbb{G}_\pi$ where $\mathcal{V} \subset V$ is nonempty then*

$$\det\big(M(\mathcal{R}_{\mathcal{V}}[G]) - \lambda I\big) = \frac{\det\big(M(G) - \lambda I\big)}{\det\big(M(G|_{\bar{\mathcal{V}}}) - \lambda I\big)}.$$

As a special case, if $M(G) \in \mathbb{C}^{n\times n}$ we have the following immediate corollary.

Corollary 1. *Let $G = (V, E, \omega)$. If $M(G) \in \mathbb{C}^{n\times n}$ then for any nonempty $\mathcal{V} \subset V$, $\sigma(\mathcal{R}_{\mathcal{V}}[G]) = \sigma(G) - \sigma(G|_{\bar{\mathcal{V}}})$.*

In terms of matrices theorem 2.4.1 can be stated as follows. If $M \in \mathbb{W}^{n\times n}$ then

$$\det\big(r[M;\mathcal{I}] - \lambda I\big) = \frac{\det(M - \lambda I)}{\det(M|_{\bar{\mathcal{I}}} - \lambda I)}$$

for any sequence of indexing sets $\mathcal{I} \subseteq \mathcal{I}_{m-1} \subseteq \cdots \subseteq \mathcal{I}_1 \subseteq \{1, \ldots, n\}$ for which the matrix $r[M, \mathcal{I}] = r(M; \mathcal{I}_1, \ldots, \mathcal{I}_{m-1}, \mathcal{I})$ is defined.

For a proof of theorem 2.4.1 we give the following.

Proof. For $G = (V, E, \omega) \in \mathbb{G}^n_\pi$ and $V = \{v_1, \ldots, v_n\}$ let $\mathcal{V}_m = \{v_1, \ldots, v_m\}$ for some fixed $1 \leq m < n$. For $1 \leq k \leq m$ denote the matrices $M_k = M(\mathcal{R}_{\bar{\mathcal{V}}_k}(G))$ and $M^k = M(G)|_{\{v_k\}}$. As $\bar{\mathcal{V}}_1 \in st(G)$ then equation (12) implies that

$$\det(M - \lambda I) = \det(M_1 - \lambda I)\det(M^1 - \lambda I).$$

As both $M_1 = M(\mathcal{R}_{\bar{\mathcal{V}}_1}(G))$ and $\bar{\mathcal{V}}_2 \in st(\mathcal{R}_{\bar{\mathcal{V}}_1}(G))$ equation (12) can again be used to infer that $\det\big(M_1 - \lambda I\big) = \det\big((M_1)_2 - \lambda I\big)\det\big((M_1)^2 - \lambda I\big)$. Given that $(M_1)_2 = M_2$ by theorem 1.3.10 then this implies

$$\det\big(M_1 - \lambda I\big) = \det\big(M_2 - \lambda I\big)\det\big((M_1)^2 - \lambda I\big).$$

As $\bar{\mathcal{V}}_i \in st(\mathcal{R}_{\mathcal{V}_{i-1}}[G])$ for $1 \leq i \leq m$ repeated use of both equation (12) and theorem 1.3.8 then imply

$$\det(M - \lambda I) = \det(M_m - \lambda I)\prod_{i=1}^{m}\det\big((M_{i-1})^i - \lambda I\big) \tag{24}$$

where $M_0 = M$.

Denoting the principle submatrix $M|_{\mathcal{V}_m} = \tilde{M}$ then, by the same argument, the characteristic equation of the $\tilde{M}$ is given by

$$\det(\tilde{M} - \lambda I) = \prod_{i=1}^{m}\det\big((\tilde{M}_{i-1})^i - \lambda I\big). \tag{25}$$

where $\tilde{M}_0 = \tilde{M}$. The claim then is that $(\tilde{M}_{i-1})^i = (M_{i-1})^i$ for all $1 \leq i \leq m$. To verify this we proceed by induction.

First, note that $(M_0)_{jk} = (\tilde{M}_0)_{jk}$ for all $1 \leq j, k \leq m$ as $\tilde{M}_0$ is the principle submatrix of M_0 consisting of its first m rows and columns. Therefore, assume that the entries $(M_i)_{jk} = (\tilde{M}_i)_{jk}$ for $1 \leq j, k \leq m - i$ and $i < \ell \leq m$. For the case $i = \ell$ it follows from this assumption that

$$(M_\ell)_{jk} = (M_{\ell-1})_{j+1,k+1} + \frac{(M_{\ell-1})_{j+1,1}(M_{\ell-1})_{1,k+1}}{\lambda - (M_{\ell-1})_{11}} = \tag{26}$$

$$(\tilde{M}_{\ell-1})_{j+1,k+1} + \frac{(\tilde{M}_{\ell-1})_{j+1,1}(\tilde{M}_{\ell-1})_{1,k+1}}{\lambda - (\tilde{M}_{\ell-1})_{11}} = (\tilde{M}_\ell)_{jk} \tag{27}$$

for all $1 \leq j, k \leq m - \ell$ where each quotient of (26) and (27) is defined on the basis that $G \in \mathbb{G}_\pi$. Hence, $(M_i)_{jk} = (\tilde{M}_i)_{jk}$ for $1 \leq j, k \leq m - i$ and $i \leq m$. This verifies the claim that $(\tilde{M}_{i-1})^i = (M_{i-1})^i$ for all $1 \leq i \leq m$.

Since $\det\big((\tilde{M}_{i-1})^i - \lambda I\big) = \det\big((M_{i-1})^i - \lambda I\big)$ for $1 \leq i \leq m$ equation (24) together with (25) imply

$$\det(M - \lambda I) = \det(M_m - \lambda I)\det(\tilde{M} - \lambda I).$$

As $M = M(G)$, $M_m = \mathcal{R}_{\bar{\mathcal{V}}_m}[G]$, and $\tilde{M} = M(G|_{\mathcal{V}_m})$ this verifies the proof once it is known that $\det(M(G|_{\mathcal{V}_m}) - \lambda I) \neq 0$.

To see this, note that each matrix $\tilde{M}_{i-1} = M(\mathcal{R}_{\mathcal{V}_{i-1}}[G|_{\mathcal{V}_m}])$. Given that the graph $\mathcal{R}_{\mathcal{V}_{i-1}}[G|_{\mathcal{V}_m}] \in \mathbb{G}_\pi$ then by lemma (1.3.7) the entry $(\tilde{M}_{i-1})^i \neq \lambda$. Hence, equation (25) implies that $\det(M(G|_{\mathcal{V}_m}) - \lambda I) \neq 0$ since the product of a number of nonzero rational functions (in this case $(\tilde{M}_{i-1})^i - \lambda$ written in the appropriate form) is nonzero. □

2.5 Main Results

In this section we give the main results of this paper. Specifically, we show that a reduced graph (equivalently reduced matrix) has a smaller Gershgorin and Brauer-type region respectively than the associated unreduced graph. Hence, the eigenvalue estimates given in section 2.3.1 and 2.3.2 can be improved via the process of isospectral graph reduction.

However, for both Brualdi and original Brualdi-type regions the situation is more complicated. For certain reductions the Brualdi-type (original Brualdi-type) region of a graph may decrease in size similar to Gershgorin and Brauer-type regions. In other cases the Brualdi-type (original Brualdi-type) region of a graph may do the opposite and increase in size when the graph is reduced. We give examples of both of these possibilities in section 2.5.3. Following this, we present sufficient conditions underwhich such estimates improve (see theorems 2.5.4 and 2.5.5) as the associated graph is reduced.

2.5.1 Improving Gershgorin-Type Estimates

We first consider the effect of reducing a graph on its associated Gershgorin region. Our main result in this direction is the following.

Theorem 2.5.1. ***(Improved Gershgorin Regions)*** *Let $G = (V, E, \omega)$ where $\mathcal{V}$ is any nonempty subset of V. If $G \in \mathbb{G}_\pi$ then $\mathcal{BW}_\Gamma(\mathcal{R}_\mathcal{V}[G]) \subseteq \mathcal{BW}_\Gamma(G)$.*

Gershgorin's original theorem can be thought of as estimating the spectrum of a graph by considering the paths of length 1 starting at each vertex. Heuristically, one can view graph reductions as allowing for better estimates by considering longer paths in the graph through the vertices which have been removed.

Theorem 2.5.1 together with theorem 2.4.1 have the following corollary.

Corollary 2. *If $G = (V, E, \omega) \in \mathbb{G}_\pi$ and $\mathcal{V}$ is a nonempty subset of V then $\sigma(G) \subseteq \mathcal{BW}_\Gamma(\mathcal{R}_\mathcal{V}[G]) \cup \sigma(G|_{\bar{\mathcal{V}}})$.*

In order to understand in which situations $\mathcal{BW}_\Gamma(\mathcal{R}_\mathcal{V}[G])$ is strictly contained in $\mathcal{BW}_\Gamma(G)$, i.e. under which conditions we have a strict improvement in approximating $\sigma(G)$, consider the following. For $G \in \mathbb{G}_\pi^n$ let

$$\partial\mathcal{BW}_\Gamma(G)_i = \{\lambda \in \mathbb{C} : |\lambda - M(\bar{G})_{ii}| = r_i(\bar{G})\} \quad \text{for} \quad 1 \leq i \leq n.$$

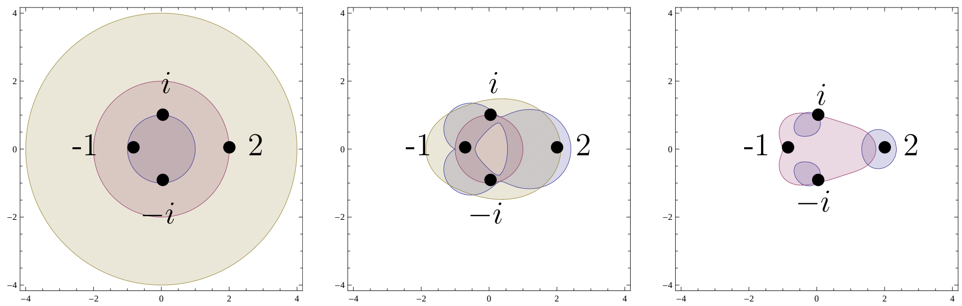


Figure 9: Left: $\mathcal{BW}_\Gamma(\mathcal{G}_0)$. Middle: $\mathcal{BW}_\Gamma(\mathcal{G}_1)$. Right: $\mathcal{BW}_\Gamma(\mathcal{G}_2)$, where in each the spectrum $\sigma(\mathcal{G}_0) = \{-1, -1, -i, i, 2\}$ is indicated.

We note here that the notation $\partial\mathcal{BW}_\Gamma(G)_i$ meant to signify the boundary of the region $\mathcal{BW}_\Gamma(G)_i$ is in fact an abuse of standard notation as the true topological boundary of this region may be a strict subset of $\partial\mathcal{BW}_\Gamma(G)_i$. However, the set $\partial\mathcal{BW}_\Gamma(G)_i$ always contains the boundary of the region $\mathcal{BW}_\Gamma(G)_i$.

Theorem 2.5.2. *Let $G = (V, E, \omega)$ be in $\mathbb{G}_\pi^n$ and suppose the subset of the boundary*

$$\partial\mathcal{BW}_\Gamma(G)_i \setminus \bigcup_{j=1, j\neq i}^{n} \mathcal{BW}_\Gamma(G)_j$$

contains an infinite set of points. Then $\mathcal{BW}_\Gamma(\mathcal{R}_\mathcal{V}[G]) \subset \mathcal{BW}_\Gamma(G)$ for any $\mathcal{V} \subset V$ with the property that $v_i \notin \mathcal{V}$.

Note that for a typical $G \in \mathbb{G}_\pi$ there is generically some region $\mathcal{BW}_\Gamma(G)_i$ whose boundary is not contained in the union of the other jth Gershgorin regions. In the nongeneric case this boundary can be a finite set of isolated points but otherwise, removing v_i from the graph G via an isospectral reduction strictly improves the estimates given by theorem 2.5.1.

As an example, consider the graph $\mathcal{G}_0 \in \mathbb{G}_\pi$ with adjacency matrix

$$M(\mathcal{G}_0) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

If $\mathcal{G}_1 = \mathcal{R}_{\{v_1,v_2,v_3\}}[\mathcal{G}_0]$ and $\mathcal{G}_2 = \mathcal{R}_{\{v_1,v_2\}}[\mathcal{G}_1]$ then one computes

$$M(\mathcal{G}_1) = \begin{bmatrix} \frac{\lambda+1}{\lambda^2} & \frac{1}{\lambda} & \frac{\lambda+1}{\lambda} \\ \frac{2\lambda+1}{\lambda^2} & \frac{1}{\lambda} & \frac{1}{\lambda} \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M(\mathcal{G}_2) = \begin{bmatrix} \frac{\lambda+1}{\lambda^2} & \frac{2\lambda+1}{\lambda^2} \\ \frac{2\lambda+1}{\lambda^2} & \frac{\lambda+1}{\lambda^2} \end{bmatrix}.$$

The Gershgorin regions of $\mathcal{G}_0, \mathcal{G}_1$, and $\mathcal{G}_2$ are shown in figure 9. Moreover, as

$$\partial\mathcal{BW}_\Gamma(\mathcal{G}_0)_5 \setminus \bigcup_{j=1}^{4} \mathcal{BW}_\Gamma(\mathcal{G}_0)_j \quad \text{and} \quad \partial\mathcal{BW}_\Gamma(\mathcal{G}_1)_3 \setminus \bigcup_{j=1}^{2} \mathcal{BW}_\Gamma(\mathcal{G}_1)_j$$

consist of curves in $\mathbb{C}$ (as can be seen in the the figure) this implies the strict inclusions

$$\mathcal{BW}_\Gamma(\mathcal{G}_2) \subset \mathcal{BW}_\Gamma(\mathcal{G}_1) \subset \mathcal{BW}_\Gamma(\mathcal{G}_0).$$

In addition, if $\mathcal{G}^1 = \mathcal{G}_0|_{\{v_4,v_5\}}$ and $\mathcal{G}^2 = \mathcal{G}_0|_{\{v_3,v_4,v_5\}}$ then

$$M(\mathcal{G}^1) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad M(\mathcal{G}^2) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

implying $\sigma(\mathcal{G}^1) = \sigma(\mathcal{G}^2) = \{0\}$ (not including multiplicities). As the point 0 is contained in both $\mathcal{BW}_\Gamma(\mathcal{G}_1)$ and $\mathcal{BW}_\Gamma(\mathcal{G}_2)$ then both $\mathcal{BW}_\Gamma(\mathcal{G}_1)$ and $\mathcal{BW}_\Gamma(\mathcal{G}_2)$ contain $\sigma(\mathcal{G}_0)$ by corollary 2. (Note here that $M(\mathcal{G}_1) = M(\mathcal{G})$ where $M(\mathcal{G})$ is given by (15).)

Also, an important implication of theorem 2.5.1 is that graph reductions on some $G \in \mathbb{G}_\pi$ can be used to obtain estimates of $\sigma(G)$ with increasing precision depending on how much one is willing to reduce the graph G.

With this in mind, suppose $v \in V$ is a vertex of G where $M(G) \in \mathbb{C}^{n\times n}$. Then the graph $\mathcal{R}_{\{v\}}[G] = (\{v\}, \mathcal{E}, \mu)$ consists of a single vertex v and possibly a loop. Moreover, $\mathcal{BW}_\Gamma(\mathcal{R}_{\{v\}}[G])$ is a finite set of points in the complex plane. As $\sigma(G|_{\{V\setminus v\}})$ consists of at most $n-1$ points this can be summarized as follows.

Remark 2. *If $G = (V, E, \omega)$ where $M(G) \in \mathbb{C}^{n\times n}$ and v is any vertex in V then $\sigma(G)$ is contained in the finite set of points $\sigma(\mathcal{R}_{\{v\}}[G]) \cup \sigma(G|_{\{V\setminus v\}})$. Furthermore,*

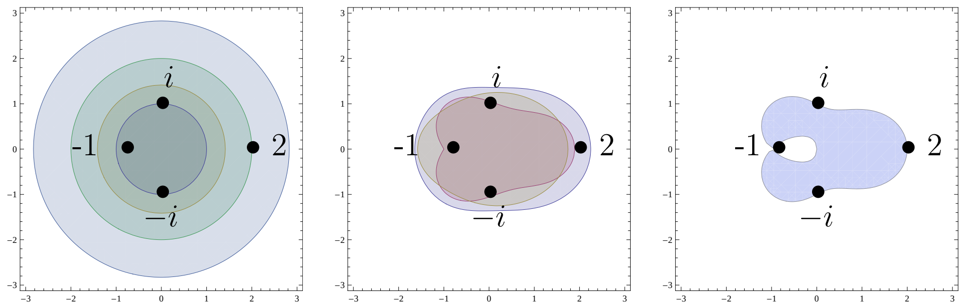


Figure 10: Left: $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}_0)$. Middle: $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}_1)$. Right: $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}_2)$, where in each the spectrum $\sigma(\mathcal{G}_0) = \{-1, -1, -i, i, 2\}$ is indicated.

$\sigma(G)$ and $\sigma(\mathcal{R}_{\{v\}}[G])$ differ at most by the set $\sigma(G|_{\{V\setminus v\}})$ which contains less than n points.

As an example, let $\mathcal{G}_3 = \mathcal{R}_{\{v_1\}}[\mathcal{G}_0]$ and $\mathcal{G}^3 = \mathcal{G}|_{\{}v_2, v_3, v_4, v_5\}$. It follows that $\sigma(\mathcal{G}_3) = \{-1, -1, -i, i, 2\}$ and $\sigma(\mathcal{G}^3) = \{0, 1.3247, -.6623 \pm 0.5622i\}$. Corollary 2 then implies $\sigma(\mathcal{G}_0) \subseteq \{-1, -i, i, 2, 0, 1.3247, -.6623 \pm 0.5622i\}$. We note that in this particular case $\sigma(\mathcal{G}_0) = \sigma(\mathcal{G}_3)$ or the spectrum of the reduced graph and the original are exactly the same.

2.5.2 Improving Brauer-Type Estimates

We now consider Brauer-type regions for which we give similar results.

Theorem 2.5.3. ***(Improved Brauer Regions)*** *Let $G = (V, E, \omega)$. If $G \in \mathbb{G}_\pi$ where $\mathcal{V} \subseteq V$ contains at least two vertices, then $\mathcal{BW}_{\mathcal{K}}(\mathcal{R}_{\mathcal{V}}[G]) \subseteq \mathcal{BW}_{\mathcal{K}}(G)$.*

Theorem 2.5.3 has the following corollary.

Corollary 3. *If $G = (V, E, \omega) \in \mathbb{G}_\pi$ and $\mathcal{V} \subseteq V$ contains at least two vertices then $\sigma(G) \subseteq \mathcal{BW}_{\mathcal{K}}(\mathcal{R}_{\mathcal{V}}[G]) \cup \sigma(G|_{\bar{\mathcal{V}}})$.*

Continuing our example, the Brauer-type regions of $\mathcal{G}_0, \mathcal{G}_1$, and $\mathcal{G}_2$ are shown in figure 10 where by theorem 2.5.3, $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}_2) \subseteq \mathcal{BW}_{\mathcal{K}}(\mathcal{G}_1) \subseteq \mathcal{BW}_{\mathcal{K}}(\mathcal{G}_0)$. Moreover, theorem 2.3.6 implies $\mathcal{BW}_{\mathcal{K}}(\mathcal{G}_i) \subseteq \mathcal{BW}_{\Gamma}(\mathcal{G}_i)$ for $i = 0, 1, 2$.

We also note that if a graph is reduced from n to m vertices then there are $\binom{n}{2} - \binom{m}{2}$ less ijth Brauer-type regions to calculate. Hence, the number of regions quickly decrease as a graph is reduced.

2.5.3 Brualdi-Type Estimates

Continuing on to Brualdi-type regions we note that in the example we have been considering it happens that we have the inclusions $\mathcal{BW}_B(\mathcal{G}_2) \subseteq \mathcal{BW}_B(\mathcal{G}_1) \subseteq \mathcal{BW}_B(\mathcal{G}_0)$ (see figure 11). However, it is not always the case that reducing a graph will improve its Brualdi-type region.

For example, consider the following graph $\mathcal{H} \in \mathbb{G}_\pi$ given in figure 12. If $\mathcal{H}$ is reduced over the sets $\mathcal{S} = \{v_2, v_3, v_4\}$ and $\mathcal{T} = \{v_1, v_2, v_3\}$ respectively then

$$M(\mathcal{R}_\mathcal{S}(\mathcal{H})) = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{10} & 0 \\ \frac{10}{\lambda} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad M(\mathcal{R}_\mathcal{T}(\mathcal{H})) = \begin{bmatrix} \frac{1}{\lambda} & \frac{1}{\lambda} & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this example we have the strict inclusions (see figure 12)

$$\mathcal{BW}_B(\mathcal{R}_\mathcal{T}(\mathcal{H})) \subset \mathcal{BW}_B(\mathcal{H}) \subset \mathcal{BW}_B(\mathcal{R}_\mathcal{S}(\mathcal{H})).$$

In particular, as $\mathcal{BW}_B(\mathcal{H}) \subset \mathcal{BW}_B(\mathcal{R}_\mathcal{S}(\mathcal{H}))$ then reducing the graph $\mathcal{H}$ over $\mathcal{S}$ increases the size of its Brualdi-type region. It follows that graph reductions do not always improve Brualdi-type estimates.

In order to give a sufficient condition under which a Brualdi-type region shrinks as the graph is reduced we require the following terminology. First, let $G = (V, E, \omega)$ where $V = \{v_1, \dots, v_n\}$ for some $n \geq 1$ and where G has strongly connected components $\mathbb{S}_1(G), \dots, \mathbb{S}_m(G)$. Define

$$E^{scc} = \{e \in E : e \in \mathbb{S}_i(G), 1 \leq i \leq m\}.$$

The cycle $\gamma \in C(G)$ is said to *adjacent* to $v_i \in V$ if $v_i \notin \gamma$ and there is some vertex $v_j \in \gamma$ such that $e_{ji} \in E^{scc}$.

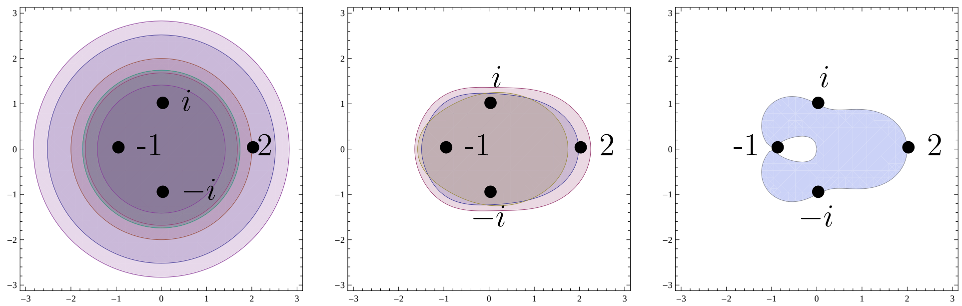


Figure 11: Left: $\mathcal{BW}_B(\mathcal{G}_0)$. Middle: $\mathcal{BW}_B(\mathcal{G}_1)$. Right: $\mathcal{BW}_B(\mathcal{G}_2)$, where in each the spectrum $\sigma(\mathcal{G}_0) = \{-1, -1, -i, i, 2\}$ is indicated.

Second, for any $v_i \in V$ denote

$$\mathcal{A}(v_i, G) = \{\gamma \in C(G) : \gamma \text{ is adjacent to } v_i\}.$$

Moreover, if $C(v_i, G) = \{\gamma \in C(G) : v_i \in \gamma\}$ then let $\mathcal{S}(v_i, G) \subseteq C(v_i, G)$ be the set containing the following cycles.

For $G \in \mathbb{G}_\pi^n$ and fixed $1 \leq i \leq n$, let $\gamma = \{v_{\alpha_1}, \ldots, v_{\alpha_m}\}$ be a cycle in $C(v_i, G)$ where $n \geq m \geq 1$ and $v_i = v_{\alpha_1}$. If $m = 1$, that is $\gamma = \{v_i\}$, then $\gamma \in \mathcal{S}(v_i, G)$. Otherwise, supposing $1 < m \leq n$ relabel the vertices of G such that v_{α_j} is v_j for $1 \leq j \leq m$ and denote this relabelled graph by $G_r = (V_r, E_r, \omega_r)$. Then $\gamma \in \mathcal{S}(v_i, G)$ if $e_{j1} \notin E_r$ for $1 < j < m$ and $e_{mk} \notin E_r^{scc}$ for $m < k \leq n$.

As it will be needed later, we furthermore define the set $\mathcal{S}_{br}(v_i, G)$ to be the set of cycles in $\mathcal{S}(v_i, G)$ where $\gamma \in \mathcal{S}_{br}(v_i, G)$ if $e_{j1} \notin E_r$ for $1 < j < m$ and $e_{mk} \notin E_r$ for $m < k \leq n$.

With this in place we state the following theorem.

Theorem 2.5.4. ***(Improved Brualdi Regions)*** *Let $G = (V, E, \omega)$ where $G \in \mathbb{G}_\pi$ and V contains at least two vertices. If $v \in V$ such that both $\mathcal{A}(v, G) = \emptyset$ and $C(v, G) = \mathcal{S}(v, G)$ then $\mathcal{BW}_B\big(\mathcal{R}_{V \setminus v}(G)\big) \subseteq \mathcal{BW}_B(G)$.*

That is, if the vertex v is adjacent to no cycle in $C(G)$ and each cycle passing through v is in $\mathcal{S}(v, G)$ then removing this vertex improves the Brualdi-type region of

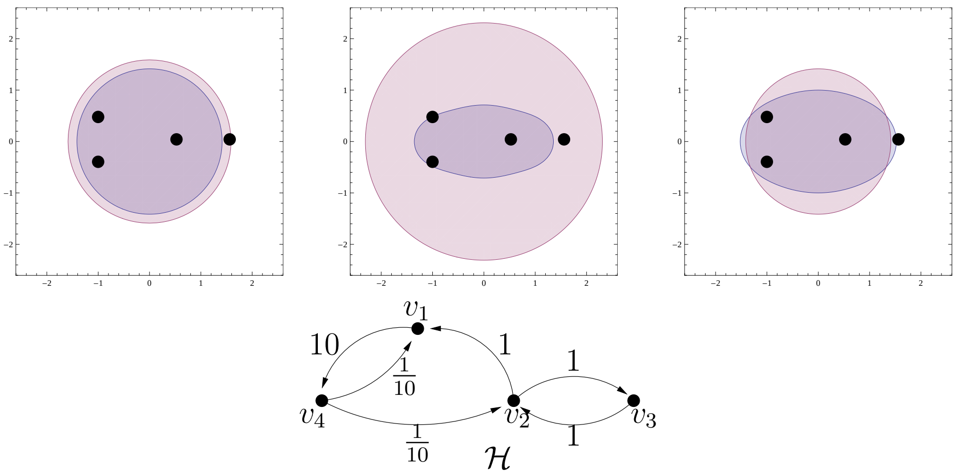


Figure 12: Top Left: $\mathcal{BW}_B(\mathcal{H})$. Top Middle: $\mathcal{BW}_B(\mathcal{R}_{\mathcal{S}}(\mathcal{H}))$. Top Right: $\mathcal{BW}_B(\mathcal{R}_{\mathcal{T}}(\mathcal{H}))$ where $\mathcal{S} = \{v_2, v_3, v_4\}$ and $\mathcal{T} = \{v_1, v_2, v_3\}$. $\sigma(\mathcal{H})$ is indicated.

G. We note that for the graph $\mathcal{H}$ in figure 12 the set $\mathcal{A}(v_1, \mathcal{H}) = \{v_2, v_3\} \neq \emptyset$. Hence, theorem 2.5.4 does not apply to the reduction of $\mathcal{H}$ over $\mathcal{S}$.

However, the vertex v_4 in $\mathcal{H}$ has the property that both $\mathcal{A}(v_4, \mathcal{H}) = \emptyset$ and $\mathcal{S}(v_4, \mathcal{H}) = C(v_4, \mathcal{H})$. Therefore, reducing $\mathcal{H}$ over the vertex set $\mathcal{T} = \{v_1, v_2, v_3\}$ improves the Brualdi-type region of this graph which can be seen on the upper right hand side of figure 12.

As an example for why the condition $C(v, G) = \mathcal{S}(v, G)$ is necessary in theorem 2.5.4 consider the following. Let $\mathcal{J}, \mathcal{R}_S(\mathcal{J}) \in \mathbb{G}$ be the matrices given by

$$M(\mathcal{J}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad \text{then} \quad M(\mathcal{R}_S(\mathcal{J})) = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\lambda} & 0 & 1 \\ \frac{1}{\lambda} & 0 & 0 \end{bmatrix}$$

where $S = \{v_2, v_3, v_4\}$. For this graph one can compute that $\mathcal{BW}_B(\mathcal{R}_S(\mathcal{J})) \nsubseteq \mathcal{BW}_B(\mathcal{J})$. Moreover, note that $\mathcal{A}(v_1, \mathcal{J}) = \emptyset$ but $\mathcal{S}(v_1, \mathcal{J})$ consists of the cycle $\{v_1, v_2, v_3\}$ whereas the cycle set $C(v_1, \mathcal{J}) = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_3, v_4\}\}$ i.e. $C(v_1, \mathcal{J}) \neq \mathcal{S}(v_1, \mathcal{J})$.

Graph reductions can furthermore increase, decrease or maintain the number of cycles a graph has in its cycle set. For instance the graph $\mathcal{G}_0$ in our previous example has 12 cycles in its cycle set whereas $\mathcal{G}_1$ has 3 and $\mathcal{G}_2$ has 1 (see figure 11). Conversely, let P, $\mathcal{R}_U(P) \in \mathbb{G}$ with adjacency matrices given by

$$M(P) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad M(\mathcal{R}_U(P)) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{\lambda} & 0 & \frac{1}{\lambda} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{\lambda} & 0 & \frac{1}{\lambda} & 0 \end{bmatrix}$$

where $U = \{v_1, v_2, v_3, v_4\}$. Here $C(P) = \{\{v_1, v_2, v_5\}, \{v_3, v_3, v_5\}\}$ whereas $C(\mathcal{R}_U(P)) = \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_1, v_2, v_3, v_4\}\}$. That is, reducing P over U increases the number of cycles needed to compute the associated Brualdi-type region from 2 to 3. Hence, reducing a graph may increase the number of regions needed to compute its Brualdi-type region. This is in contrast to Gershgorin and Brauer type regions which always decrease in number as the associated graph is reduced.

In adopting the result of theorem 2.5.4 to the case of Brualdi's original result (theorem 2.3.10) we must deal with the following complications. First, for a given graph $G \in \mathbb{G}_\pi$ where $C_w(G) = \emptyset$, it may not be the case that $C_w(\mathcal{R}_{V\setminus v}(G)) = \emptyset$. Furthermore, as the edges between strongly connected components play a role in the associated eigenvalue inclusion region (see (23)) this also complicates whether or not estimates given by the original Brualdi-type region improves as the graph is reduced. However, the following holds.

Theorem 2.5.5. ***(Improved Original Brualdi Regions)*** *Let $G = (V, E, \omega)$ be in $\mathbb{G}_\pi$ where $v \in V$. If $\mathcal{A}(v, G) = \emptyset$, $C(v, G) = \mathcal{S}_{br}(v, G)$ and both of the sets $C_w(G)$ and $C_w(\mathcal{R}_{V\setminus v}(G))$ are empty then $\mathcal{BW}_{br}(\mathcal{R}_{V\setminus v}(G)) \subseteq \mathcal{BW}_{br}(G)$.*

2.5.4 Proofs

In order to prove the theorems in section 2.5.1, 2.5.2, and 2.5.3 we will need to evaluate functions at some fixed $\lambda \in \mathbb{C}$. In each case we consider such functions first as elements in $\mathbb{W}$ with common factors removed then evaluated at the value λ. In fact most of these functions, once common factors are removed, will be polynomials in $\mathbb{C}[\lambda]$.

Moreover, to simplify notation we will use the following. For $G = (V, E, \omega)$ where $G \in \mathbb{G}_\pi^n$ and $n \geq 2$ first note that the vertex set $V \setminus \{v_1\} \in st(G)$. Therefore, let $\mathcal{R}_{V\setminus\{v_1\}}(G) = \mathcal{R}_1$, $L_k(G,\lambda) = L_k$, $L_k(\mathcal{R}_1, \lambda) = L_k^1$, $\lambda - \omega_{kk} = \lambda_{kk}$ and $M(G,\lambda)_{k\ell} = \omega_{k\ell}$. Also, let $\omega_{k\ell} = p_{k\ell}/q_{k\ell}$ for $p_{k\ell}, q_{k\ell} \in \mathbb{C}[\lambda]$ where we assume $q_{k\ell} = 1$ if $\omega_{k\ell} = 0$. Lastly, set $R_k(G) = \sum_{\ell=1,\ell\neq k} |\omega_{k\ell} L_k|$.

Before proceeding we state the following technical lemma.

Lemma 2.5.6. *If $G \in \mathbb{G}_\pi^n$ for $n \geq 2$ then $q_{11}q_{i1}L_i^1 = \big(q_{i1}(q_{11}\lambda - p_{11})\big)^{n-1} L_1 L_i$.*

Proof. First, note that

$$M(\mathcal{R}_1, \lambda)_{ij} = \frac{p_{i1}p_{1j}q_{ij}q_{11} + q_{i1}q_{1j}p_{ij}(q_{11}\lambda - p_{11})}{q_{i1}q_{1j}q_{ij}(q_{11}\lambda - p_{11})},\ 2 \leq i, j \leq n$$

from which $L_i^1 = \prod_{j=2}^{n} q_{i1}q_{1j}q_{ij}(q_{11}\lambda - p_{11})$. Therefore,

$$L_i^1 = \big(q_{i1}(q_{11}\lambda - p_{11})\big)^{n-1} \prod_{j=2}^{n} q_{1j} \prod_{j=2}^{n} q_{ij}. \tag{28}$$

As $L_k = \prod_{j=1}^{n} q_{kj}$ for $1 \leq k \leq n$ the result follows by multiplication of $q_{11}q_{i1}$. $\square$

A proof of theorem 2.5.1 is the following.

Proof. Suppose that $\lambda \in \mathcal{BW}_\Gamma(\mathcal{R}_1)_i$ for fixed $\lambda \in \mathbb{C}$ and $2 \leq i \leq n$. As each $M(\mathcal{R}_1)_{ij} = \omega_{ij} + \omega_{i1}\omega_{1j}/\lambda_{11}$ for $2 \leq j \leq n$ then

$$|(\lambda_{ii} - \frac{\omega_{i1}\omega_{1i}}{\lambda_{11}})L_i^1| \leq \sum_{j=2,j\neq i}^{n} |(\omega_{ij} + \frac{\omega_{i1}\omega_{1j}}{\lambda_{11}})L_i^1|.$$

Multiplying both sides of this inequality by $|\lambda_{11}q_{11}q_{i1}|$ implies, via lemma 2.5.6, that

$$Q_i(G)|\lambda_{11}L_1\lambda_{ii}L_i - \omega_{i1}\omega_{1i}L_1L_i| \le Q_i(G) \sum_{j=2,j\neq i}^{n} |(\omega_{ij}\lambda_{11} + \omega_{i1}\omega_{1j})L_1L_i|$$

where $Q_i(G) = |\big(q_{i1}(q_{11}\lambda - p_{11})\big)|^{n-1}$. If $Q_i(G) \neq 0$ then, by the triangle inequality,

$$|\lambda_{11}L_1\lambda_{ii}L_i| - |\omega_{i1}\omega_{1i}L_1L_i| \le \sum_{j=2,j\neq i}^{n} |\lambda_{11}L_1\omega_{ij}L_i| + \sum_{j=2,j\neq i}^{n} |\omega_{i1}L_i\omega_{1j}L_1|.$$

Therefore,

$$|\lambda_{11}L_1\lambda_{ii}L_i| - \sum_{j=1,j\neq i}^{n} |\lambda_{11}L_1\omega_{ij}L_i| \le \sum_{j=2}^{n} |\omega_{i1}\omega_{1j}L_1L_i| - |\omega_{i1}L_i\lambda_{11}L_1|.$$

By factoring

$$|\lambda_{11}L_1|\Big(|\lambda_{ii}L_i| - R_i(G)\Big) \le |\omega_{i1}L_i|\Big(R_1(G) - |\lambda_{11}L_1|\Big). \tag{29}$$

If we assume $\lambda \notin \mathcal{BW}_\Gamma(G)_i \cup \mathcal{BW}_\Gamma(G)_1$ then both

$$|\lambda_{ii}L_i| - R_i(G) > 0 \ \text{ and } \ R_1(G) - |\lambda_{11}L_1| < 0.$$

These inequalities together with (29) in particular imply that $\lambda_{11}L_1 = 0$. However, this in turn implies that $\lambda \in \mathcal{BW}_\Gamma(G)_1$, which is not possible.

Hence, $\lambda \in \mathcal{BW}_\Gamma(G)_i \cup \mathcal{BW}_\Gamma(G)_1$ unless $Q_i(G) = 0$. Supposing then that this is the case, i.e. $Q_i(G) = 0$ then note that if $L_{ij} = \prod_{\ell=1,\ell\neq j}^{n} q_{i\ell}$ for $1 \le i,j \le n$ then

$$\mathcal{BW}_\Gamma(G)_k = \{\lambda \in \mathbb{C} : |L_{kk}(q_{kk}\lambda - p_{kk})| \le \sum_{j=1,j\neq k}^{n} |p_{kj}L_{kj}|\} \ \text{ for } \ 1 \le k \le n. \tag{30}$$

Under the assumption $Q_i(G) = \big(q_{i1}(q_{11}\lambda - p_{11})\big)^{n-1} = 0$ note that if $q_{i1} = 0$ then $L_{ii} = 0$ implying $\lambda \in \mathcal{BW}_\Gamma(G)_i$. If $q_{11}\lambda - p_{11} = 0$ then $\lambda \in \mathcal{BW}_\Gamma(G)_1$ again by (30).

Therefore, $\mathcal{BW}_\Gamma(\mathcal{R}_1)_i \subseteq \mathcal{BW}_\Gamma(G)_1 \cup \mathcal{BW}_\Gamma(G)_i$ implying $\mathcal{BW}_\Gamma(\mathcal{R}_1) \subseteq \mathcal{BW}_\Gamma(G)$. The theorem follows by repeated use of theorem 1.3.8 as it is always possible to sequentially remove single vertices of a graph in order to remove an arbitrary set $\bar{\mathcal{V}}$ from V. $\square$

We now give a proof of theorem 2.5.2.

Proof. Let $\lambda \in \mathbb{C}$ be fixed such that

$$\lambda \in \partial\mathcal{BW}_{\Gamma}(G)_1 \setminus \bigcup_{j=2}^{n} \mathcal{BW}_{\Gamma}(G)_j. \tag{31}$$

Then both

$$|(\lambda_{11})L_1| = R_1(G), \tag{32}$$

$$|(\lambda_{ii})L_i| > R_i(G), \quad \text{for all} \;\; 1 < i \leq n. \tag{33}$$

Supposing $\lambda \in \mathcal{BW}_{\Gamma}(\mathcal{R}_1)_i$ for some fixed $1 < i \leq n$ and that $Q_i(G) \neq 0$ then (29) holds. Combining (29) with (32) it follows that

$$|\lambda_{11}L_1|\Big(|\lambda_{ii}L_i| - R_i(G)\Big) \leq 0.$$

Moreover, as $|\lambda_{ii}L_i| - R_i(G) > 0$ from equation (33) then this together with the previous inequality imply that $\lambda_{11}L_1$ must be zero. However, given that $\lambda_{11}L_1$ is a nonzero polynomial given that $G \in \mathbb{G}_{\pi}^{n}$ then this happens in at most finitely many values of $\lambda \in \mathbb{C}$. Similarly, the polynomial $Q_i(G) = 0$ on only a finite set of $\mathbb{C}$, hence the assumption that

$$\partial\mathcal{BW}_{\Gamma}(G)_1 \setminus \bigcup_{j=2}^{n} \mathcal{BW}_{\Gamma}(G)_j$$

is an infinite set in the complex plane yields a contradiction to assumption (31) for infinitely many points in this set. Hence, the result follows in the case that $\{v_1\} = \bar{\mathcal{V}}$. By sequentially removing single vertices of $\bar{\mathcal{V}}$ from the graph G repeated use of theorem 1.3.8 completes the proof. □

Next we give the proof of theorem 2.5.3.

Proof. Let $G = (V, E, \omega)$ where $G \in \mathbb{G}_{\pi}^{n}$ and $n \geq 3$. The claim then is that

$$\mathcal{BW}_{\mathcal{K}}(\mathcal{R}_1)_{ij} \subseteq \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij} \tag{34}$$

for any pair $2 \leq i, j \leq n$ where $i \neq j$.

To see this let $\lambda \in \mathcal{BW}_{\mathcal{K}}(\mathcal{R}_1)_{ij}$ for fixed i and j from which it follows that

$$
\begin{aligned}
|\big(\lambda_{ii} - \frac{\omega_{i1}\omega_{1i}}{\lambda_{11}}\big)L_i^1||\big(\lambda_{jj} - \frac{\omega_{j1}\omega_{1j}}{\lambda_{11}}\big)L_j^1| \leq \\
\Big(\sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\big(\omega_{i\ell} + \frac{\omega_{i1}\omega_{1\ell}}{\lambda_{11}}\big)L_i^1|\Big)\Big(\sum_{\substack{\ell=2\\ \ell\neq j}}^{n} |\big(\omega_{j\ell} + \frac{\omega_{j1}\omega_{1\ell}}{\lambda_{11}}\big)L_j^1|\Big)
\end{aligned}
\tag{35}
$$

Multiplying both sides of (35) by $|\lambda_{11}q_{11}q_{i1}|$ and $|\lambda_{11}q_{11}q_{j1}|$, lemma 2.5.6 implies

$$
\begin{aligned}
\prod_{k=i,j} Q_k(G)|\lambda_{kk}\lambda_{11}L_1L_k - \omega_{k1}\omega_{1k}L_1L_k| \leq \\
\prod_{k=i,j} Q_k(G)\Big(\sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\big(\omega_{k\ell}\lambda_{11} + \omega_{k1}\omega_{1\ell}\big)L_1L_k|\Big).
\end{aligned}
$$

Assuming for now that $Q_i(G)Q_j(G) \neq 0$ then from the triangle inequality

$$
\begin{aligned}
\prod_{k=i,j}\Big(|\lambda_{11}L_1\lambda_{kk}L_k| - |\omega_{1k}L_1\omega_{k1}L_k|\Big) \leq \\
\prod_{k=i,j}\Big(\sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\lambda_{11}L_1\omega_{k\ell}L_k| + \sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\omega_{1\ell}L_1\omega_{k1}L_k|\Big).
\end{aligned}
\tag{36}
$$

Suppose $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j}$. Then $|\lambda_{11}L_1||\lambda_{kk}L_k| > R_1(G)R_k(G)$ for $k = i, j$. Moreover, if $|\lambda_{11}L_1| \leq R_1(G)$ then from (36)

$$
\begin{aligned}
\prod_{k=i,j}\Big(R_1(G)R_k(G) - |\omega_{1k}L_1\omega_{k1}L_k|\Big) < \\
\prod_{k=i,j}\Big(R_1(G)\sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\omega_{k\ell}L_k| + \sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\omega_{k1}L_1\omega_{1\ell}L_k|\Big)
\end{aligned}
\tag{37}
$$

From the fact that

$$
\begin{aligned}
R_1(G)R_k(G) - |\omega_{k1}L_1\omega_{1k}L_k| = \\
R_1(G)\sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\omega_{k\ell}L_k| + \sum_{\substack{\ell=2\\ \ell\neq k}}^{n} |\omega_{k1}L_1\omega_{1\ell}L_k|
\end{aligned}
\tag{38}
$$

it follows that (37) cannot hold. Therefore, if $\lambda \in \mathcal{BW}_{\mathcal{K}}(\mathcal{R}_1)_{ij}$, $Q_i(G)Q_j(G) \neq 0$, and $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j}$ then $|\lambda_{11}L_1| > R_1(G)$.

Proceeding as above, we assume again that $\lambda \in \mathcal{BW}_{\mathcal{K}}(\mathcal{R}_1)_{ij}$, so in particular (35) holds. Note that if $\lambda_{11} = 0$ then $\lambda \in \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j}$ and claim (34) holds. In what follows we therefore assume that $\lambda_{11} \neq 0$. Moreover, if $Q_i(G)Q_j(G) \neq 0$ then multiplying both side of (35) by $|\lambda_{11}q_{11}q_{i1}|$ and $|\lambda_{ii}L_i q_{11}q_{j1}|$ gives

$$\begin{aligned}
&\Big(|\lambda_{11}L_1\lambda_{ii}L_i| - |\omega_{1i}L_1\omega_{i1}L_i|\Big)\Big(|\lambda_{ii}L_i\lambda_{jj}L_jL_1| - |\omega_{1j}L_1\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\Big) \leq \\
&\Big(\sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\lambda_{11}L_1\omega_{i\ell}L_i| + \sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\omega_{1\ell}L_1\omega_{i1}L_i|\Big)\cdot \\
&\qquad\qquad \Big(\sum_{\substack{\ell=2\\ \ell\neq j}}^{n} |\lambda_{ii}L_i\omega_{j\ell}L_jL_1| + \sum_{\substack{\ell=2\\ \ell\neq j}}^{n} |\omega_{1\ell}L_1\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\Big).
\end{aligned} \tag{39}$$

by use of the triangle inequality.

Furthermore, supposing that $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij}$ then both $R_1(G)R_i(G) < |\lambda_{11}L_1\lambda_{ii}L_i|$ and $R_i(G)R_j(G) < |\lambda_{ii}L_i\lambda_{jj}L_j|$. This together with (39) implies

$$\begin{aligned}
&\Big(R_1(G)R_i(G) - |\omega_{1i}L_1\omega_{i1}L_i|\Big)\Big(R_i(G)R_j(G)L_1 - |\omega_{1j}L_1\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\Big) < \\
&\Big(\sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\lambda_{11}L_1\omega_{i\ell}L_i| + \sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\omega_{1\ell}L_1\omega_{i1}L_i|\Big)\cdot \\
&\qquad \Big(|\lambda_{ii}L_iL_1|\big(R_j(G) - |\omega_{j1}L_j|\big) + |\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\big(R_1(G) - |\omega_{1j}L_1|\big)\Big).
\end{aligned}$$

If $|\lambda_{ii}L_i| \leq R_i(G)$ then

$$\begin{aligned}
&\Big(R_1(G)R_i(G) - |\omega_{1i}L_1\omega_{i1}L_i|\Big)\Big(R_i(G)R_j(G)L_1 - |\omega_{1j}L_1\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\Big) < \\
&\Big(\sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\lambda_{11}L_1\omega_{i\ell}L_i| + \sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\omega_{1\ell}L_1\omega_{i1}L_i|\Big)\cdot \\
&\qquad \Big(R_i(G)|L_1|\big(R_j(G) - |\omega_{j1}L_j|\big) + |\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\big(R_1(G) - |\omega_{1j}L_1|\big)\Big).
\end{aligned} \tag{40}$$

The claim then is that if $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j}$, which implies $|\lambda_{11}L_1| > R_1(G)$ by the above, then the second terms in each product of (40) have the relation

$$\begin{aligned}
&\qquad\qquad R_i(G)R_j(G) - |\omega_{1j}L_1\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}| \geq \\
&R_i(G)|L_1|\big(R_j(G) - |\omega_{j1}L_j|\big) + |\omega_{j1}L_j\frac{\lambda_{ii}L_i}{\lambda_{11}}|\big(R_1(G) - |\omega_{1j}L_1|\big).
\end{aligned} \tag{41}$$

To see this note that this is true if and only if

$$R_i(G)|\omega_{j1}L_jL_1| \geq |\omega_{j1}L_j\lambda_{ii}L_i|\frac{R_1(G)}{|\lambda_{11}|}.$$

As this is true if and only if $|\lambda_{11}L_1|R_i(G) \geq R_1(G)|\lambda_{ii}L_i|$ this verifies that (41) holds since both $R_i(G) \geq |\lambda_{ii}L_i|$ and $|\lambda_{11}L_1| > R_1(G)$. Then (40) and (41) together imply

$$R_1(G)R_i(G) - |\omega_{1i}L_1\omega_{i1}L_i| < \sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\lambda_{11}L_1\omega_{i\ell}L_i| + \sum_{\substack{\ell=2\\ \ell\neq i}}^{n} |\omega_{1\ell}L_1\omega_{i1}L_i|. \tag{42}$$

Rewriting the right-hand side of this inequality in terms of $R_k(G)$ (for $k = 1, i$) yields

$$R_1(G)R_i(G) < |\lambda_{11}L_1|R_i(G) - |\lambda_{11}L_1\omega_{i1}L_i| + |\omega_{i1}L_i|R_1(G).$$

This in turn implies that $R_i(G)\big(R_1(G) - |\lambda_{11}L_1|\big) < |\omega_{i1}L_i|\big(R_1(G) - |\lambda_{11}L_1|\big)$. However, then

$$R_i(G) = \sum_{\ell=1,\ell\neq i}^{n} |\omega_{i\ell}L_i| < |\omega_{i1}L_i|,$$

which is not possible.

Therefore, if both $Q_i(G)Q_j(G) \neq 0$ and $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij}$ then $|\lambda_{ii}L_i| > R_i(G)$. Moreover, as this argument is symmetric in the indices i and j then it can be modified to show that if both $Q_i(G)Q_j(G) \neq 0$ and it is the case that $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij}$ then $|\lambda_{jj}L_j| > R_j(G)$.

With this in mind, by multiplying (35) by $|q_{11}q_{i1}|$ and $|q_{11}q_{i1}|$ and assuming once again that $Q_i(G)Q_j(G) \neq 0$, then the triangle inequality implies

$$\begin{aligned}&\prod_{k=i,j}\Big(|\lambda_{kk}L_k||L_1| - |\frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_1L_k|\Big) \leq \\ &\prod_{k=i,j}\Big(\sum_{\substack{\ell=1\\ \ell\neq k}}^{n} |\omega_{k\ell}L_k||L_1| - |\omega_{k1}L_kL_1| + \sum_{\ell=2}^{n} |\frac{\omega_{k1}\omega_{1\ell}}{\lambda_{11}}L_kL_1| - |\frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_kL_1|\Big).\end{aligned} \tag{43}$$

Hence, if $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij}$ then from the previous calculations $R_k(G) < |\lambda_{kk}L_k|$ for $k = 1, i$, and j implying together with (43) that

$$\begin{aligned}&\prod_{k=i,j}\Big(R_k(G)|L_1| - |\frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_1L_k|\Big) < \\ &\qquad\prod_{k=i,j}\Big(R_k(G)|L_1| - |\omega_{k1}L_kL_1| + |\omega_{k1}L_k|\frac{R_1(G)}{|\lambda_{11}|} - |\frac{\omega_{k1}\omega_{1k}}{\lambda_{11}}L_kL_1|\Big).\end{aligned}$$

Hence, for either $k = i$ or $k = j$ it follows that

$$-|\omega_{k1} L_k L_1| + |\omega_{k1} L_k| \frac{R_1(G)}{|\lambda_{11}|} > 0.$$

Therefore, $R_1(G) > |\lambda_{11} L_1|$ which is not possible. Note that this implies that $\lambda \notin \mathcal{BW}_{\mathcal{K}}(G)_{1i} \cup \mathcal{BW}_{\mathcal{K}}(G)_{1j} \cup \mathcal{BW}_{\mathcal{K}}(G)_{ij}$, unless $Q_i(G)Q_j(G) = 0$. Therefore, suppose the this product $Q_i(G)Q_j(G)$ is in fact equal to zero.

Then note that by modifying equation (30)

$$\mathcal{BW}_{\mathcal{K}}(G)_{ij} = \{\lambda \in \mathbb{C} : \prod_{k=i,j} |L_{kk}(q_{kk}\lambda - p_{kk})| \leq \prod_{k=i,j} \Big(\sum_{j=1, j\neq k}^{n} |p_{kj} L_{kj}| \Big) \}$$

for $1 \leq k \leq n$. Hence, if $Q_k(G) = 0$ for either $k = i, j$ then by calculations analogous to those given in the proof of theorem 2.5.1 it follows that $\lambda \in \mathcal{BW}_{\mathcal{K}}(G)_{ik}$. This then verifies the claim given in (34).

As in the previous proofs, theorem 1.3.8 can then be invoked to generalize this result to the reduction over the set $\mathcal{V} \subseteq V$. □

In order to prove theorem 2.5.4 we first give the following lemma.

Lemma 2.5.7. *Let $G \in \mathbb{G}_{\pi}^n$ for $n \geq 2$ and suppose both $\mathcal{A}(v_1, G) = \emptyset$ and $C(v_1, G) = \mathcal{S}(v_1, G)$. Moreover, let $\gamma = \{v_1, \ldots, v_m\}$ and $\gamma' = \{v_2, \ldots, v_m\}$ for $m \geq 2$. If $\gamma \in C(G)$ and $\gamma' =\in C(\mathcal{R}_1(G))$ then $\mathcal{BW}_B(\mathcal{R}_1(G))_{\gamma'} \subseteq \mathcal{BW}_B(G)$.*

Proof. Suppose first that the hypotheses of the lemma hold. We then make the observation that the edges $e \in E^{scc}$ are not used to calculate $\mathcal{BW}_B(G)$. Furthermore, any cycle of G is contained in exactly one strongly connected component of this graph. This implies that the Brualdi-type region of the graph is the union of the Brualdi-type regions of its strongly connected components. Therefore, we may without loss in generality assume that G consists of a single strongly connected component.

Suppose that both $\gamma = \{v_1, \ldots, v_m\}$ and $\delta = \{v_1, v_m\}$ are cycles in $C(v_1, G)$ for some $1 < m \leq n$. Note the fact that $\gamma \in C(v_1, G)$ implies, in particular, that $\gamma' = \{v_2, \ldots, v_m\}$ is a cycle in $C(\mathcal{R}_1)$.

From the assumption that v_1 has no adjacent cycles it then follows that $\omega_{mi} = 0$ for $1 < i \leq m$ since otherwise $\{v_i, v_{i+1}, \ldots, v_m\} \in \mathcal{A}(v_1, G)$. Also, as $\gamma \in C(v_1, G) = \mathcal{S}(v_1, G)$ then $\omega_{i1} = 0$ for $1 < i < m$ as well as $\omega_{mi} = 0$ for $m < i \leq n$ as G is assumed to have one strongly connected component. Therefore,

$$\mathcal{BW}_B(G)_\gamma = \{\lambda \in \mathbb{C} : \prod_{i=1}^{m} |\lambda_{ii} L_i| \leq |\omega_{m1} L_m| \prod_{i=1}^{m-1} R_i(G)\}, \tag{44}$$

$$\mathcal{BW}_B(G)_\delta = \{\lambda \in \mathbb{C} : |\lambda_{11} L_1||\lambda_{mm} L_m| \leq |\omega_{m1} L_m| R_1(G)\}. \tag{45}$$

Suppose then that $\lambda \in \mathcal{BW}_B(\mathcal{R}_1)_{\gamma'}$. It then follows from this that

$$|(\lambda_{mm} - \frac{\omega_{m1}\omega_{1m}}{\lambda_{11}}) L_m^1| \prod_{i=2}^{m-1} |\lambda_{ii} L_i| \leq \sum_{i=2}^{m-1} |\frac{\omega_{m1}\omega_{1i}}{\lambda_{11}} L_m^1| \prod_{i=2}^{m-1} R_i(G). \tag{46}$$

Here, $L_i^1 = L_i$ for $1 < i < m$ since for each such i the edge $e_{i1} \notin E$.

Multiplying both sides of (46) by $|q_{11} q_{1m} \lambda_{11}|$ along with the triangle inequality implies

$$\begin{gathered} Q_m(G)\Big(|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{1m} L_1 \omega_{m1} L_m|\Big) \prod_{i=2}^{m-1} |\lambda_{ii} L_i| \leq \\ Q_m(G)\Big(|\omega_{m1} L_m| R_1(G) - |\omega_{1m} L_1 \omega_{m1} L_m|\Big) \prod_{i=2}^{m-1} R_i(G). \end{gathered} \tag{47}$$

Now by use of equation (30) we have

$$\mathcal{BW}_B(G)_\delta = \{\lambda \in \mathbb{C} : \prod_{k=1,m} |L_{kk}(q_{kk}\lambda - p_{kk})| \leq \prod_{k=1,m} \Big(\sum_{j=1, j\neq k}^{n} |p_{kj} L_{kj}|\Big)\}.$$

Hence, if $Q_m(G) = 0$ then by calculations analogous to those given in the proof of theorem 2.5.1 it follows that $\lambda \in \mathcal{BW}_B(G)_\delta$. Therefore, assume that $Q_m(G) \neq 0$.

Then if $\prod_{i=2}^{m-1} R_i(G) = 0$ it follows from (47) that either $\prod_{i=2}^{m-1} |\lambda_{ii} L_i| = 0$ or that $|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{1m} L_m| = 0$. If the first is the case then $\lambda \in \mathcal{BW}_B(G)_\gamma$. If the latter then $\lambda \in \mathcal{BW}_B(G)_\delta$ since $|\omega_{1m} L_1| \leq R_1(G)$.

If both $\prod_{i=2}^{m-1} R_i(G) \neq 0$ and $|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{1m} L_m| \neq 0$ then (47) implies

$$\frac{\prod_{i=2}^{m-1} |\lambda_{ii} L_i|}{\prod_{i=2}^{m-1} R_i(G)} \leq \frac{|\omega_{m1} L_m| R_1(G) - |\omega_{1m} L_1 \omega_{m1} L_m|}{|\lambda_{11} L_1 \lambda_{mm} L_m| - |\omega_{m1} L_1 \omega_{1m} L_m|} \tag{48}$$

Note that if

$$\frac{|\omega_{m1}L_m|R_1(G) - |\omega_{1m}L_1\omega_{m1}L_m|)}{|\lambda_{11}L_1\lambda_{mm}L_m| - |\omega_{m1}L_1\omega_{1m}L_m|} \leq \frac{|\omega_{m1}L_m|R_1(G)}{|\lambda_{11}L_1\lambda_{mm}L_m|}$$

then it follows from (48) together with (44) that $\lambda \in \mathcal{BW}_B(G)_\gamma$. On the other hand if this inequality does not hold then $|\lambda_{11}L_1||\lambda_{mm}L_m| < |\omega_{m1}L_m|R_1(G)$ implying $\lambda \in \mathcal{BW}_B(G)_\delta$. Therefore, $\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} \subseteq \mathcal{BW}_B(G)_\gamma \cup \mathcal{BW}_B(G)_\delta \subseteq \mathcal{BW}_B(G)$.

Conversely, if $\delta \notin C(G)$ then $\omega_{1m}L_1 = 0$. Equation (47) together with (44) then imply that $\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} \subseteq \mathcal{BW}_B(G)_\gamma$. Hence, $\mathcal{BW}_B(G)_{\gamma'} \subseteq \mathcal{BW}_B(G)$. □

We now give a proof of theorem 2.5.4.

Proof. First, as in the previous proof, suppose G consists of a single strongly connected component. Moreover, for the vertex $v_1 \in V$ suppose both $\mathcal{A}(v_1, G) = \emptyset$ and $C(v_1, G) = \mathcal{S}(v_1, G)$. Also let $\gamma' = \{v_2, \dots, v_m\}$ be a cycle in $C(\mathcal{R}_1)$ for some $1 < m \leq n$.

As $\mathcal{A}(v_1, G) = \emptyset$, if $\gamma' \in C(G)$ then $M(G,\lambda)_{ij} = M(\mathcal{R}_1,\lambda)_{ij}$ for $2 \leq i \leq m$ and $1 \leq j \leq n$ since γ' would otherwise be adjacent to v_1. From this it follows that $\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} = \mathcal{BW}_B(G)_{\gamma'} \subseteq \mathcal{BW}_B(G)$.

On the other hand, if $\gamma' \notin C(G)$ then at least one edge of the form $e_{i-1,i}$ for $3 \leq i \leq m$ or e_{m2} is not in E. If this is the case then without loss in generality assume for notational simplicity that $e_{m2} \notin E$. Furthermore, let

$$\mathcal{I} = \{i : e_{i-1,i} \notin E,\ 3 \leq i \leq m\} \cup \{2\}.$$

We give the set $\mathcal{I}$ the ordering $\mathcal{I} = \{i_1, \dots, i_\ell\}$ such that $i_j < i_k$ if and only if $j < k$. Then for each $1 \leq j \leq \ell$ the ordered sets

$$\gamma_j = \{v_1, v_{i_j}, v_{i_j+1}, \dots, v_{j_\alpha}\} \tag{49}$$

are cycles in $C(v_1, G)$ where $j_\alpha = i_{j+1} - 1$ and $\ell_\alpha = m$. Moreover, by removing the vertex v_1 from G it follows from (49) that each of the ordered sets

$$\gamma_j' = \{v_{i_j}, v_{i_j+1}, \dots, v_{j_\alpha}\}$$

are cycles in $C(\mathcal{R}_1)$. As both $\mathcal{A}(v_1, G) = \emptyset$ and $C(v_1, G) = \mathcal{S}(v_1, G)$, lemma 2.5.7 therefore implies that

$$\bigcup_{j=1}^{\ell} \mathcal{BW}_B(\mathcal{R}_1)_{\gamma'_j} \subseteq \mathcal{BW}_B(G).$$

The claim then is that the region

$$\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} \subseteq \bigcup_{j=1}^{\ell} \mathcal{BW}_B(\mathcal{R}_1)_{\gamma'_j}. \tag{50}$$

To see this, let $\lambda^1_{ii} = (\lambda - \omega_{ii} - \dfrac{\omega_{i1}\omega_{1i}}{\lambda_{11}})L^1_i$ and $R^1_i = \displaystyle\sum_{j=2, j\neq i}^{n} |M(\bar{\mathcal{R}}_1, \lambda)_{ij}|$. Then

$$\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} = \{\lambda \in \mathbb{C} : \prod_{i=2}^{m} |\lambda^1_{ii}| \leq \prod_{i=2}^{m} R^1_i\} \quad \text{and} \tag{51}$$

$$\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'_j} = \{\lambda \in \mathbb{C} : \prod_{i\in\gamma_j}^{m} |\lambda^1_{ii}| \leq \prod_{i\in\gamma_j}^{m} R^1_i\} \quad \text{for} \quad 1 \leq j \leq \ell. \tag{52}$$

As the vertex set γ' is the disjoint union of the vertex sets of the cycles γ'_j then the assumption that $\lambda \notin \mathcal{BW}_B(\mathcal{R}_1)_{\gamma'_j}$ for each $1 \leq j \leq \ell$ implies $\lambda \notin \mathcal{BW}_B(\mathcal{R}_1)_{\gamma'}$ by comparing the product of (52) over all $1 \leq j \leq \ell$ to (51). This verifies the claim given in (50), which implies that $\mathcal{BW}_B(\mathcal{R}_1)_{\gamma'} \subseteq \mathcal{BW}_B(G)$.

As γ' was an arbitrary cycle in $C(\mathcal{R}_1)$ then it follows that $\mathcal{BW}_B(\mathcal{R}_1) \subseteq \mathcal{BW}_B(G)$. This completes the proof. □

A proof of theorem 2.5.5 is the following.

Proof. If the conditions given in the theorem hold for $v = v_1$ then both $\mathcal{BW}_{br}(G)$ and $\mathcal{BW}_{br}(\mathcal{R}_1)$ exist since it is assumed that $C_w(G) = \emptyset$ and $C_w(\mathcal{R}_1) = \emptyset$. Moreover, if $\mathcal{S}(v_1, G)$ is replaced by $\mathcal{S}_{br}(v_1, G)$ and $\mathcal{BW}_B(\cdot)$ by $\mathcal{BW}_{br}(\cdot)$ then the conclusions of lemma 2.5.7 hold by the same proof with the exception that G is not assumed to have a single strongly connected component. As the same holds for the proof of theorem 2.5.4 the result follows. □

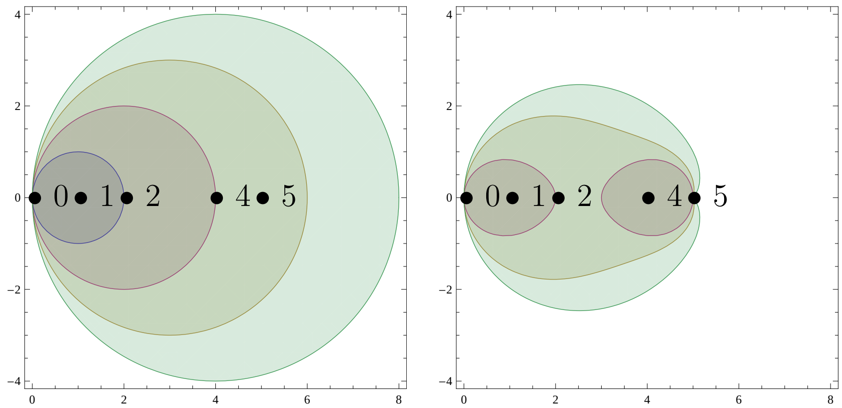


Figure 13: Left: $\mathcal{BW}_\Gamma\big(L(H)\big)$. Right: $\mathcal{BW}_\Gamma\big(\mathcal{R}_S(L(H))\big)$, where in each the spectrum $\sigma\big(L(H)\big) = \{0, 1, 2, 4, 5\}$ is indicated.

2.6 Some Applications

In this section we discuss some natural applications of using graph reductions to improve estimates of the spectra of certain graphs. Our first application deals with estimating the spectra of the Laplacian matrix of a given graph. Following this we give a method for estimating the spectral radius of a matrix using graph reductions. Last, we use the results of theorem 2.5.2 as well as some structural knowledge of a graph to identify particularly useful structural sets.

2.6.1 Laplacian Matrices

It is possible to reduce not only the graph G but also the graphs associated with both the combinatorial Laplacian matrix and the normalized Laplacian matrix of G. Such matrices are typically defined for undirected graphs without loops or weights but this definition can be extended to graphs in $\mathbb{G}$ (see remark 3 below). However, here we give the standard definitions as these are of interest in their own right (see [20, 21]).

Let $G = (V, E)$ be an unweighted undirected graph without loops, i.e. a *simple graph.* If G has vertex set $V = \{v_1, \ldots, v_n\}$ and $d(v_i)$ is the degree of vertex v_i then its *combinatorial Laplacian matrix* $M_L(G)$ of G is given by

$$M_L(G)_{ij} = \begin{cases} d(v_i) & if \quad i = j \\ -1 & if \quad i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

On the other hand the *normalized Laplacian matrix* $M_{\mathcal{L}}(G)$ of G is defined as

$$M_{\mathcal{L}}(G)_{ij} = \begin{cases} 1 & if \quad i = j \text{ and } d(v_j) \neq 0 \\ \frac{-1}{\sqrt{d(v_i)d(v_j)}} & if \quad v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

The interest in the eigenvalues of $M_L(G)$ is that $\sigma(M_L(G))$ gives structural information about G (see [20]). On the other hand knowing $\sigma(M_{\mathcal{L}}(G))$ is useful in determining the behavior of algorithms on the graph G among other things (see [21]).

Let $L(G)$ be the graph with adjacency matrix $M_L(G)$ and similarly let $\mathcal{L}(G)$ be the graph with adjacency matrix $M_{\mathcal{L}}(G)$. Since both $L(G), \mathcal{L}(G) \in \mathbb{G}_\pi$ either may be reduced over any subset of their respective vertex sets.

For example if $H \in \mathbb{G}_\pi$ is the simple graph with adjacency matrix

$$M(H) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

then the graph $L(H)$, has the structural set $S = \{v_1, v_2, v_3, v_4\}$. Reducing over this set yields $\mathcal{R}_S\big(L(H)\big)$ where

$$M\big(\mathcal{R}_S(L(H))\big) = \begin{bmatrix} \frac{\lambda-3}{\lambda-4} & \frac{1}{\lambda-4} & \frac{1}{\lambda-4} & \frac{1}{\lambda-4} \\ \frac{1}{\lambda-4} & \frac{2\lambda-7}{\lambda-4} & \frac{1}{\lambda-4} & \frac{-\lambda+5}{\lambda-4} \\ \frac{1}{\lambda-4} & \frac{1}{\lambda-4} & \frac{2\lambda-7}{\lambda-4} & \frac{-\lambda+5}{\lambda-4} \\ \frac{1}{\lambda-4} & \frac{-\lambda+5}{\lambda-4} & \frac{-\lambda+5}{\lambda-4} & \frac{3\lambda-11}{\lambda-4} \end{bmatrix}.$$

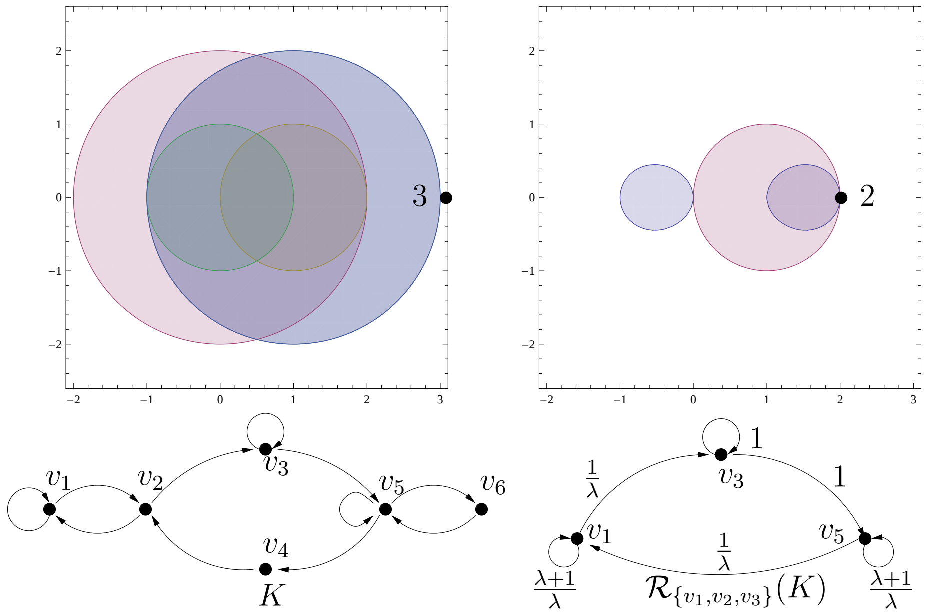


Figure 14: Top Left: $\mathcal{BW}_\Gamma(K)$ from which $\rho(K) \leq 3$. Top Right: $\mathcal{BW}_\Gamma(\mathcal{R}_{\{v_1,v_2,v_3\}}(K))$ from which $\rho(K) \leq 2$.

Figure 13 shows the Gershgorin regions for $L(H)$ as well as $\mathcal{R}_S(L(H))$.

Note that the adjacency matrix of H is symmetric so its eigenvalues must be real numbers. With this in mind we note that the Gershgorin-type region associated with simple graphs and their reductions can be reduced to intervals of the real number line.

Remark 3. *It is possible to generalize $M_L(G)$ to any $G \in \mathbb{G}$ if G has no loops and n vertices by setting $M_L(G)_{ij} = -M(G)_{ij}$ for $i \neq j$ and $M_L(G)_{ii} = \sum_{j=1,j\neq i}^{n} M(G)_{ij}$. This generalization is consistent with what is done for weighted digraphs in [52] for example.*

2.6.2 Estimating the Spectral Radius of a Matrix

For $G \in \mathbb{G}_\pi$ the spectral radius of G, denoted $\rho(G)$, is the maximum among the absolute values of the elements in $\sigma(G)$ i.e.

$$\rho(G) = \max_{\lambda \in \sigma(G)} |\lambda|.$$

For many graphs $G \in \mathbb{G}_\pi$ it is possible to find some structural set $S \in st(G)$ such

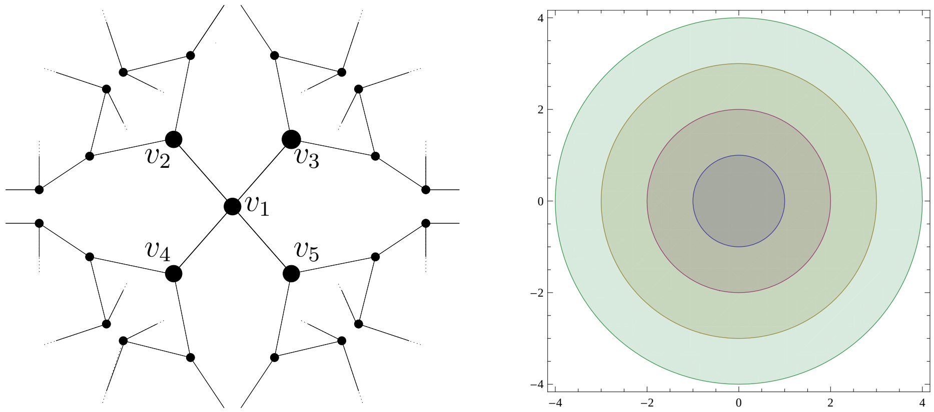


Figure 15: Left: The graph N. Right: $\mathcal{BW}_\Gamma(N)$.

that each vertex of $\bar{S}$ has no loop. Via corollary 2, if S is such a set then $\sigma(G)$ and $\sigma(\mathcal{R}_S(G))$ differ at most by some number of zeros implying that $\rho(G) = \rho(\mathcal{R}_S(G))$.

For example, in the graph K shown in figure 14 the vertices v_2, v_4, v_6 are the vertices of K without loops. As $\{v_1, v_3, v_5\} \in st(K)$ then $\rho(K) = \rho(\mathcal{R}_{\{v_1,v_3,v_5\}}(K))$. By employing the region $\mathcal{BW}_\Gamma(K)$ we can estimate $\rho(K) \leq 3$. However, using $\mathcal{BW}_\Gamma(\mathcal{R}_{\{v_1,v_3,v_5\}}(K))$ our estimate improves to $\rho(K) \leq 2$ (see the top left and right of figure 14).

It should be noted that for a given graph there is often no unique set of vertices without loops which is simultaneously a structural set. Therefore, there may be many ways to reduce a graph such that at each step only vertices without loops are removed ensuring, as above, that the spectral radius is maintained.

2.6.3 Targeting Specific Structural Sets

As a final application we consider reducing graphs over specific structural sets in order to improve eigenvalue estimates when some structural feature of the graph are known. To do so consider $G = (V, E, \omega)$.

If the sets $\mathcal{BW}_\Gamma(G)_i$ for $1 \leq i \leq n$ are known or can be estimated by some structural knowledge of G then it is possible to make decisions on which structural sets to reduce over. That is, it may be possible to identify structural sets $\mathcal{V} \subset V$ such that $v_i \notin \mathcal{V}$ and

$$\partial\mathcal{BW}_\Gamma(G)_i \nsubseteq \bigcup_{j\neq i}\mathcal{BW}_\Gamma(G)_j.$$

If this can be done, theorem 2.5.2 implies that a strictly better estimate of $\sigma(G)$ can be achieved by reducing over $\mathcal{V}$.

For example consider the graph $\mathcal{N}=(V,E,\omega)$ in the left hand side of figure 15 where $V=\{v_1,\dots,v_n\}$ for some $n>5$. If it is known for instance that $\mathcal{N}$ is a simple graph such that $d(v_1)=4$, $d(v_2)=d(v_3)=d(v_4)=d(v_5)=3$ and $d(v_i)\in\{0,1,2,3\}$ for all $6\leq i\leq n$ then the sets $\mathcal{BW}_\Gamma(G)_i$ are each discs of radius either 0,1,2,3 or 4 (see right hand side of figure 15). Moreover, as

$$\partial\mathcal{BW}_\Gamma(\mathcal{G})_1 \nsubseteq \bigcup_{i=2}^{n}\mathcal{BW}_\Gamma(\mathcal{G})_i=\{\lambda\in\mathbb{C}:|\lambda|=4\}$$

then theorem 2.5.2 implies that $\mathcal{R}_{V\setminus\{v_1\}}(\mathcal{N})$ has a strictly smaller Gershgorin-type region than does $\mathcal{N}$ which can be seen in figure 16. Considering the fact that n may be quite large this example is intended to illustrate that eigenvalues estimates can be improved with a minimal amount of effort if some simple structural feature(s) of the graph are known.

2.7 Concluding Remarks

The major goal of this chapter is to demonstrate that isospectral graph reductions can be used to improve each of the classical eigenvalue estimates of Gershgorin, Brauer, Brualdi, and the more recent extension of Brualdi's theorem by Varga. Of major importance is the fact that these graph reductions are general enough that this process can be applied to any graph with complex valued weights (or equivalently matrices with complex valued entries). Hence, the aforementioned eigenvalue estimates of all matrices in $\mathbb{C}^{n\times n}$ can be improved via the process of isospectral graph reduction. Additionally, this process can be repeated to improve such eigenvalue estimates to whatever degree is desired.

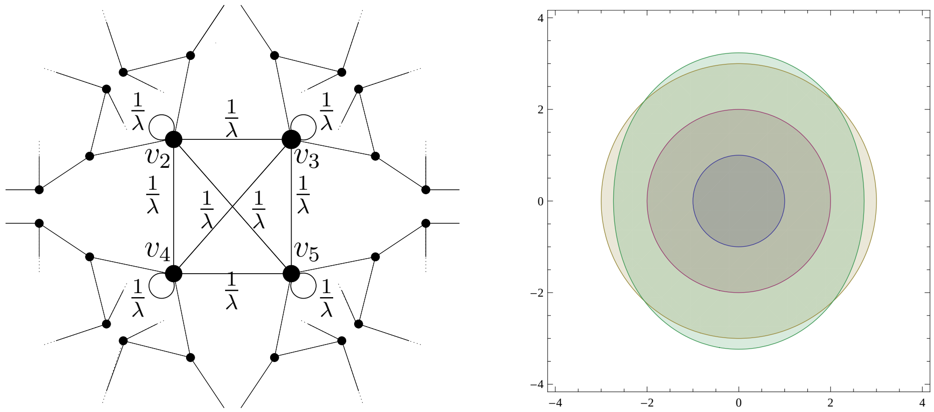


Figure 16: Left: $\mathcal{R}_{V\setminus\{v_1\}}(N)$. Right: $\mathcal{BW}_\Gamma(\mathcal{R}_{V\setminus\{v_1\}}(N))$.

Aside from this, graph reductions computationally do not seem to require much effort. In fact, the number of calculations required by such estimates may even be reduced by our procedure since nontrivial reductions typically produce fewer regions used to estimate the graph's spectrum.

Moreover, this chapter also raises new questions related to graph reductions and eigenvalue estimates. For instance, what algorithms related to choosing structural sets and sequences of structural sets can be developed to improve the speed or accuracy of such estimates, etc.

CHAPTER III

DYNAMICAL NETWORK EXPANSIONS

Of primary interest is the fact that the isospectral network transformations developed in chapter I suggest other useful transformations on networks of interacting dynamical systems i.e. dynamical networks. An important example introduced in this chapter is a *dynamical network expansion* in which a dynamical network is modified in a way that essentially preserves its dynamics but alters its associated graph structure.

Such transforms provide a new tool for the study of the interplay between the structure (topology) and dynamics of dynamical networks. Much as general dynamical systems are investigated via change of coordinates, isospectral network transforms introduce a mechanism for rearranging the specific (graph) structure of a system while preserving its dynamics in an essential way. By so doing the original network's dynamics can be investigated by studying the transformed dynamical network. This will allow us to generalize the results given in [1, 3] by demonstrating that the existence of a globally attracting fixed point of a network can be established by investigating one of its expansions when direct investigation of the network does not.

3.1 Dynamical Networks and Global Attractors

Dynamical networks or networks of interacting dynamical systems are composed of (i) local dynamical systems which have their own (local intrinsic) dynamics, (ii) interactions between these (elements of the network) local systems, and (iii) the graph of interactions (topology of the network).

Let $i, j \in \mathcal{I} = \{1, \ldots, n\}$ and $T_i : X_i \to X_i$ be maps on the compact metric space

(X_i, d) where $X_i = X_j$ and

$$L_i = \sup_{x_i \neq y_i \in X_i} \frac{d(T_i(x_i), T_i(y_i))}{d(x_i, y_i)} < \infty.$$

Let (T, X) denote the direct product of the local systems (T_i, X_i) over $\mathcal{I}$ on the compact metric space (X, d_{max}) where d_{max} is the ℓ^∞ metric.

Definition 3.1.1. *A map $F : X \to X$ is called an interaction if for every $j \in \mathcal{I}$ there exists a nonempty collection of indices $\mathcal{I}_j \subseteq \mathcal{I}$ and a continuous function*

$$F_j : \bigoplus_{i \in \mathcal{I}_j} X_i \to X_j,$$

that satisfies the following Lipschitz condition for constants $\Lambda_{ij} \geq 0$:

$$d\big(F_j(\{x_i\}), F_j(\{y_i\})\big) \leq \sum_{i \in \mathcal{I}_j} \Lambda_{ij} d(x_i, y_i) \tag{53}$$

for all $\{x_i\}, \{y_i\} \in \bigoplus_{i \in \mathcal{I}_j} X_i$ where $\{x_i\}$ is the restriction of $x \in X$ to $\bigoplus_{i \in \mathcal{I}_j} X_i$. Then the (interaction) map F is defined as follows:

$$F(x)_j = F_j(\{x_i\}), \quad j \in \mathcal{I}, \quad i \in \mathcal{I}_j.$$

The constants Λ_{ij} in definition 3.1.1 form the matrix $\Lambda \in \mathbb{R}^{n \times n}$ where the entry $\Lambda_{ij} = 0$ if $i \notin \mathcal{I}_j$. As in [1] we assume that interactions are Lipschitz. That is, each Λ_{ij} can be thought of as a Lipschitz constant of the function F measuring the maximal expansiveness of the function F_j in the ith coordinate. In particular, if the interaction F is continuously differentiable on X then the constants

$$\Lambda_{ij} = \max_{x \in X} |(DF)_{ji}(x)|$$

satisfy condition (53) where DF is the matrix of first partial derivatives of F.

Definition 3.1.2. *The superposition $N = F \circ T$ generates the dynamical system (N, X) which is a dynamical network.*

Let $M_N = \Lambda^T \cdot diag[L_1, \dots, L_n]$ i.e.

$$M_N = \begin{pmatrix} \Lambda_{11}L_1 & \dots & \Lambda_{n1}L_n \\ \vdots & \ddots & \vdots \\ \Lambda_{1n}L_1 & \dots & \Lambda_{nn}L_n \end{pmatrix}.$$

Moreover, for any $A \in \mathbb{W}^{n\times n}$ let $\rho(A)$ again denote the *spectral radius* of the matrix A.

Theorem 3.1.3. *If $\rho\big(M_N\big) < 1$ then the dynamical network (N, X) has a globally attracting fixed point.*

Proof. For $x, y \in X$ and $1 \leq j \leq n$

$$\begin{aligned} d\big(N(x)_j, N(y)_j\big) =& d\big(F_j(\{T(x)_i\}), F_j(\{T(y)_i\})\big) \\ \leq& \sum_{i\in\mathcal{I}} \Lambda_{ij} d\big(T_i(x_i), T_i(y_i)\big) \\ \leq& \sum_{i\in\mathcal{I}} \Lambda_{ij} L_i d(x_i, y_i). \end{aligned}$$

It then follows that enterywise

$$\begin{bmatrix} d\big(N(x)_1, N(y)_1\big) \\ \vdots \\ d\big(N(x)_n, N(y)_n\big) \end{bmatrix} \leq M_N \begin{bmatrix} d\big(x_1, y_1\big) \\ \vdots \\ d\big(x_n, y_n\big) \end{bmatrix}.$$

As $$\begin{bmatrix} d\big(N^2(x)_1, N^2(y)_1\big) \\ \vdots \\ d\big(N^2(x)_n, N^2(y)_n\big) \end{bmatrix} \leq M_N \begin{bmatrix} d\big(N(x)_1, N(y)_1\big) \\ \vdots \\ d\big(N(x)_n, N(y)_n\big) \end{bmatrix} \leq M_N^2 \begin{bmatrix} d\big(x_1, y_1\big) \\ \vdots \\ d\big(x_n, y_n\big) \end{bmatrix}$$

then by induction

$$d_{max}\big(N^k(x), N^k(y)\big) \leq ||M_N^k[d(x_1, y_1) \dots d(x_n, y_n)]^T||_\infty$$

for all $k > 0$.

Note that $\lim_{k\to\infty} M_N^k = 0$ if and only if $\rho(M_N) < 1$ (see [28] chapter 8). Then for any $x, y \in X$,

$$d_{max}(N^k(x), N^k(y)) \to 0 \ \text{ as } \ k \to \infty$$

under the assumption that $\rho(M_N) < 1$. Hence, $d_{max}(N^k(x), N^{k+1}(x)) \to 0$ as $k \to \infty$ implying the sequence $\{N^k(x)\}_{k\geq 1}$ is Cauchy and therefore convergent. If $N^k(x) \to \tilde{x}$ then it follows that $N^k(y) \to \tilde{x}$ for all $y \in X$ completing the proof. □

For an interaction $F : X \to X$ we say that F *stabilizes* the local systems (T, X) if the local systems are unstable i.e. $\max_{i\in\mathcal{I}}\{L_i\} \geq 1$ but the dynamical network $N = F \circ T$ has a globally attracting fixed point. If $\max_{i\in\mathcal{I}}\{L_i\} < 1$ and $N = F \circ T$ has a globally attracting fixed point we say F maintains the *stability* of (T, X). The following is then a corollary to theorem 3.1.3.

Corollary 4. *For the dynamical network $N = F \circ T$ let $\max_{i\in\mathcal{I}}\{L_i\} = L$. If the inequality $L\rho(\Lambda) < 1$ holds then the interaction F stabilizes (or maintains the stability of) the local systems (T, X).*

Before proving this we require the following. For $A, B \in \mathbb{R}^{n\times n}$ we write

$$A \leq B \ \text{ if } \ A_{ij} \leq B_{ij} \ \text{ for } \ 1 \leq i, j \leq n.$$

We note that if $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$ (see chapter 8 of [28]). We now give a proof of corollary 4.

Proof. As $0 \leq \Lambda^T \cdot diag[L_1, \ldots, L_n] \leq L\Lambda^T$ then $\rho(M_N) \leq \rho(L\Lambda^T)$. Moreover, given that $L\Lambda^T = \{L\lambda : \lambda \in \sigma(\Lambda^T)\}$ then $\rho(L\Lambda^T) = L\rho(\Lambda)$. This in turn implies $\rho(M_N) \leq L\rho(\Lambda)$. Hence, if $L\rho(\Lambda) < 1$ then by theorem 3.1.3 the interaction F stabilizes the local systems (T, X). □

The constant $\rho(\Lambda)$ related to the interaction F can then be thought of as an estimate of the interaction's *stability factor* or its ability to stabilize a set of local systems (T, X). In the following section we will develop techniques similar to branch expansions (see section 1.4.1) to obtain improved estimates on whether a dynamical network has a single global attractor.

3.2 Improved Stability Estimates via Dynamical Network Expansions

The goal of this section is to develop a method of transforming a general dynamical network to one with a different graph of interactions but dynamically equivalent to the original system in the sense described below. The transformed network can then be investigated in order to infer dynamical properties of the original untransformed network.

For $N = F \circ T$ the dynamical network N can itself be considered to be the interaction $F_{comp} = F \circ T$, i.e. the composition of the local systems and their interaction. For simplicity then, in this section we consider only the dynamical network F (or equivalently interaction F) with the understanding that all results extend to general dynamical networks of the form $N = F \circ T$.

To every dynamical network $F : X \to X$ and some choice of constants Λ_{ij} satisfying (53) there is a corresponding *graph of interactions* Γ_F which is the graph with adjacency matrix Λ. We note that as the entry $\Lambda_{ij} = 0$ if and only if $i \notin \mathcal{I}_j$ then at the level of an unweighted graph each Γ_F is equivalent for any choice of constants Λ_{ij} satisfying (53).

For $S \in st_0(\Gamma_F)$ we let $\mathcal{I}_S$ be the index set of S and let

$$C = \max\{|\beta| - 2 : \beta \in \mathcal{B}_S(\Gamma_F)\}.$$

Moreover, for $x \in X$ let $x^{k+1} = F(x^k)$ where $x^0 = x$.

Definition 3.2.1. *For $j \in \mathcal{I}_S$ let $(F|_S)_j$ be the function $F_j(\{x_{i_0}^C\})$ where each $x_{i_0}^C$ is replaced by $F_{i_0}(\{x_{i_1}^{C-1}\})$, each $x_{i_1}^{C-1}$ by $F_{i_1}(\{x_{i_2}^{C-2}\})$, and so on for all indices $i_\ell \notin \mathcal{I}_S$.*

We call the function

$$F|_S = \bigoplus_{j \in \mathcal{I}_S} (F|_S)_j$$

the restriction of the dynamical network F to S. As $\Gamma_F|_{\bar{S}}$ contains no cycles this recursion cannot continue indefinitely, hence $(F|_S)_j$ is well defined. Furthermore, the restriction of F to S is given by $F|_S : X_S \to X|_S$ where

$$X|_S = \bigoplus_{j \in \mathcal{I}_S} X_j, \ \ \textit{and} \ \ X_S = \Big\{(x^0|_S, \ldots, x^C|_S) : x \in X\Big\}.$$

As an example consider the dynamical network H given by

$$H(x) = \begin{bmatrix} H_1(x_2) \\ H_2(x_1, x_4) \\ H_3(x_1) \\ H_4(x_1, x_3) \end{bmatrix}. \tag{54}$$

Here, H has the graph of interactions $\Gamma_H = (V, E, \omega)$ given in figure 17 (left) where $e_{ij} \in E$ if H_i is a function of the variable x_j and each $e_{ij} \in E$ has weight Λ_{ij}. That is, $v_i \in V$ corresponds to the variable x_i where $e_{ij} \in E$ if F_j depends on x_i.

As $S = \{v_1, v_2\} \in st_0(\Gamma_H)$ with $C = 2$ then

$$H|_S(x^0|_S, x^1|_S, x^2|_S) = \begin{bmatrix} H_1(x_2^0) \\ H_2(x_1^2, H_4(x_1^1, H_3(x_1^0))) \end{bmatrix}.$$

Importantly, the function $H|_S : X_S \to X|_S$ is not strictly speaking a dynamical network as its range $X|_S$ is not a subset of its domain X_S. However, the functions H and $H|_S$ have the same dynamics restricted to S in the following sense.

Lemma 3.2.2. *For the dynamical network F suppose $S \in st_0(\Gamma_F)$. Then for any $x \in X$*

$$F|_S(x^k|_S, \ldots, x^{C+k}|_S) = F(x^{C+k})|_S$$

for $k \geq 0$.

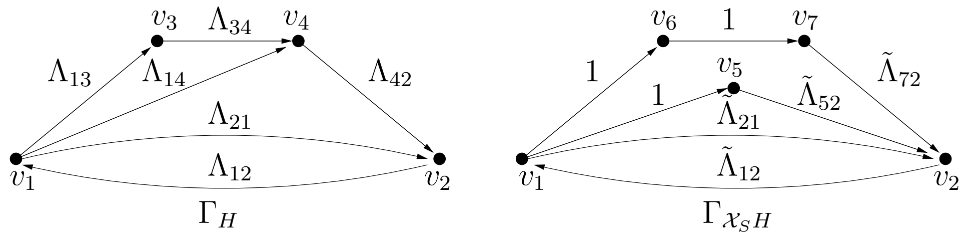


Figure 17: The interaction graphs of H (left) and its expansion $\mathcal{X}_S H$ over $S = \{v_1, v_2\}$ (right).

Proof. Let $x \in X$ and $j \in \mathcal{I}_S$. Note that by construction $x_{i_\ell}^{m+1} = F_{i_\ell}(\{x_i^m\})$. Hence, if each $F_{i_\ell}(\{x_i^m\})$ nested in $(F|_S)_j$ is recursively replaced by $x_{i_\ell}^{m+1}$ for all $i_\ell \notin \mathcal{I}_S$ then the result is the function $F_j(\{x_i^C\})$ where

$$F_j(\{x_i^C\}) = (F|_S)_j(x^0|_S, \ldots, x^C|_S).$$

As $F|_S = \bigoplus_{j \in \mathcal{I}_S} (F|_S)_j$ then $F|_S(x^0|_S, \ldots, x^C|_S) = F(x^C)|_S$. As $x \in X$ was arbitrary, replacing it by x^k for fixed $k \geq 0$ implies

$$F|_S(x^k|_S, \ldots, x^{C+k}|_S) = F(x^{C+k})|_S$$

completing the proof. $\square$

We now proceed to define the expansion of a dynamical network F with respect to S in terms of its restriction $F|_S$. To do so we first index the interior vertices of $\mathcal{B}_S(\Gamma_F)$ by

$$int(S) = \big\{(\beta, \ell) : \beta \in \mathcal{B}_S(\Gamma_F),\ 1 \leq \ell \leq |\beta| - 2\big\}$$

and let the map $\eta : int(S) \to \{n+1, \ldots, \mathcal{N}\}$ be a bijection. By abuse of notation we set $\eta(\beta, 0) = i$ if $\beta \in \mathcal{B}_{ij}(\Gamma_F; S)$.

For $j \in \mathcal{I}_S$ define $\mathcal{V}_j$ to be the variables that appear in the function $(F|_S)_j$ including multiplicities. Then each $x_{i_0}^m \in \mathcal{V}_j$ is nested in some sequence of functions $F_{i_1}, \ldots F_{i_m}, F_j$ inside of $(F|_S)_j$ corresponding to the branch

$$\beta = v_{i_0}, v_{i_1} \ldots, v_{i_m}, v_j \in \mathcal{B}_j(\Gamma_F; S).$$

Hence $x_{i_0}^m = x_{\eta(\beta,0)}^m$.

Modify the function $(F|_S)_j$ by changing this $x^m_{\eta(\beta,0)} \mapsto x_{\eta(\beta,L-m)}$. If this is done over all variables in $\mathcal{V}_j$ we call the resulting function $\tilde{F}_j$ and set

$$F_S = \bigoplus_{j\in\mathcal{I}_S} \tilde{F}_j \quad \text{and}$$

$$F_\beta = \bigoplus_{1\le\ell\le|\beta|-2} F_{\eta(\beta,\ell)}(x_{\eta(\beta,\ell-1)})$$

where each $F_{\eta(\beta,\ell)} = Id$ for $1 \le \ell \le |\beta| - 1$. Here $F_\beta = \emptyset$ if $|\beta| = 2$.

Definition 3.2.3. *For the dynamical network F suppose $S \in st_0(\Gamma_F)$. We call the function*

$$\mathcal{X}_S F = F_S \bigoplus_{\beta\in\mathcal{B}_S(\Gamma_F)} F_\beta$$

the expansion of F over S.

An expansion is then the dynamical network $\mathcal{X}_S F : X|_S \bigoplus B_S \to X|_S \bigoplus B_S$ where

$$B_S = \bigoplus_{n+1\le j\le\mathcal{N}} X_j.$$

Again, consider the dynamical network H with structural set $S = \{v_1, v_2\}$ in $st_0(\Gamma_H)$ (see figure 17). For this network $\beta_1 = v_1, v_4, v_2$; $\beta_2 = v_1, v_3, v_4, v_2$; $\beta_3 = v_2, v_1$; and $\beta_4 = v_1, v_2$ constitute the set $\mathcal{B}_S(\Gamma_F)$. Hence,

$$int(S) = \{(\beta_1, 1), (\beta_2, 1), (\beta_2, 2)\}.$$

Letting $\eta(\beta_i, \ell) = 3 + i + \ell$ then

$$H_S = \begin{bmatrix} H_1(x_2) \\ H_2(x_1, H_4(x_5, H_3(x_7))) \end{bmatrix},$$

where $H_{\beta_1} = H_5(x_1)$, $H_{\beta_2} = H_6(x_1) \bigoplus H_7(x_6)$, and $H_{\beta_3} = H_{\beta_4} = \emptyset$. The expansion of H over S is then given by

$$\mathcal{X}_S H(x) = \begin{bmatrix} H_1(x_2) \\ H_2(x_1, H_4(x_5, H_3(x_7))) \\ H_5(x_1) \\ H_6(x_1) \\ H_7(x_6) \end{bmatrix}.$$

If the constants $\tilde{\Lambda}_{ij}$ satisfy condition (53) for $\mathcal{X}_S H$ then $\Gamma_{\mathcal{X}_S H}$ is the graph in figure 17 (right) where $\tilde{\Lambda}_{16}$, $\tilde{\Lambda}_{67}$, $\tilde{\Lambda}_{15} = 1$, as each of $H_5, H_6, H_7 = Id$. Moreover, we note that as unweighted graphs $\Gamma_{\mathcal{X}_S H}$ is the branch expansion of Γ_H with respect to S.

Theorem 3.2.4. *For the dynamical network F suppose $S \in st_0(\Gamma_F)$. Then the following hold:*

(1) For $x \in X$ let $b \in B_S$ where $b_{\eta(\beta,\ell)} = x^{C-\ell}_{\eta(\beta,0)}$. If $\tilde{x}^C = x^C|_S \bigoplus b$ then the restriction $\mathcal{X}_S F(\tilde{x}^{C+k})|_S = F(x^{C+k})|_S$ for $k \geq 0$.

(2) If F satisfies (53) for the constants Λ_{ij} then there are constants $\tilde{\Lambda}_{ij}$ for $\mathcal{X}_S F$ satisfying (53) such that $\rho(\tilde{\Lambda}) \leq \rho(\Lambda)$.

We prove parts (1) and (2) of theorem 3.2.4 separately. For part (1) we give the following.

Proof. Let $j \in \mathcal{I}_S$ and $k \geq 0$. Then by changing each variable $x^m_{\eta(\beta,0)} \in \mathcal{V}_j$ to $x_{\eta(\beta,L-m)}$ in $(F|_S)_j$ we get the map $(F_S)_j$. However, note that $b_{\eta(\beta,\ell)} = x^{C-\ell}_{\eta(\beta,0)}$. Hence, evaluating $(F_S)_j$ at $\tilde{x}^k$ is equivalent to changing each $x_{\eta(\beta,\ell)}$ to $x^{C-\ell}_{\eta(\beta,0)}$ for each variable $x_{\eta(\beta,\ell)}$ of the function $(F_S)_j$.

As this sequence of operations changes each $x^m_{\eta(\beta,0)}$ to $x^m_{\eta(\beta,0)}$ then, for $x^k \in X$, it follows that

$$F|_S(x^k|_S, \ldots, x^{C+k}|_S) = \mathcal{X}_S F(\tilde{x}^{C+k})|_S.$$

Lemma 3.2.2 then implies that $\mathcal{X}_S(F(\tilde{x}^{C+k}))|_S = F(x^{C+k})|_S$ for $k \geq 0$. □

Before giving a proof of part (2) we introduce the following notation. For the branch $\beta = v_{i_0} \ldots, v_{i_m}, v_j \in \mathcal{B}_S(\Gamma_F)$ let

$$x_{i_0,\ldots,i_m,j} = x_{\eta(\beta,|\beta|-2)}.$$

Moreover, for $j \in \mathcal{I}_S$ define the sets $\mathcal{I}_j \cap \mathcal{I}_S = \mathcal{I}_j^+$, $\mathcal{I}_j - \mathcal{I}_S = \mathcal{I}_j^-$, and

$$\mathcal{B}_j^\ell = \{\beta \in \mathcal{B}_j(\Gamma(F); S) : |\beta| = \ell + 1\}.$$

Lastly, define $\tilde{F}_{i_1,j}$ to be the function in the i_1 argument of $\tilde{F}_j$. Similarly, define $\tilde{F}_{i_2,i_1,j}$ to be the function in the i_2 argument of $\tilde{F}_{i_1,j}$, and so on. We now give a proof of theorem 3.2.4 part (2).

Proof. For $x, y \in X_S \bigoplus B_S$ and $\beta \in \mathcal{B}_{ij}(\Gamma_F; S)$

$$d\big(F_{\eta(\beta,\ell)}(\{x_i\}), F_{\eta(\beta,\ell)}(\{y_i\})\big) = \begin{cases} d(x_i, y_i) & \ell = 1 \\ d(x_{\eta(\beta,\ell-1)}, y_{\eta(\beta,\ell-1)}) & 2 \leq \ell \leq |\beta| - 1 \end{cases}$$

as each $F_{\eta(\beta,\ell)}(x_{\eta(\beta,\ell-1)}) = Id$. As $F_{\eta(\beta,\ell)} = \mathcal{X}_S F_j$ for some choice of $(\beta, \ell) \in int(S)$ then the constants

$$\tilde{\Lambda}_{ij} = \begin{cases} 1 & \text{if } \ F_j = F_j(x_i) \\ 0 & \text{otherwise} \end{cases} \tag{55}$$

satisfy condition (53) for $\mathcal{X}_S F$ for each $j \in \{n+1, \ldots, \mathcal{N}\}$. Before considering the case where $j \in \mathcal{I}_S$ we first observe the following.

Let $\Gamma_F = (V, E, \omega)$. As $S \in st_0 \in st(\Gamma_F)$ then for $\beta = v_{i_1}, \ldots, v_{i_m}$

$$\mathcal{P}_\omega(\beta) = \frac{\prod_{\ell=1}^{|\beta|-1} \Lambda_{i_\ell, i_{\ell+1}}}{\lambda^{|\beta|-2}}.$$

For $j \in \mathcal{I}_S$ it follows that

$$
\begin{aligned}
d\big(\tilde{F}_j(\{x_i\}), \tilde{F}_j(\{y_i\})\big) &\leq \sum_{i_1 \in \mathcal{I}_j} \Lambda_{i_1,j} d(\tilde{F}_{i_1,j}(\{x_i\}), \tilde{F}_{i_1,j}(\{y_i\})) \\
= \sum_{i_1 \in \mathcal{I}_j^+} \Lambda_{i_1,j} d(x_{i_1,j}, y_{i_1,j}) &+ \sum_{i_1 \in \mathcal{I}_j^-} \Lambda_{i_1,j} d(\tilde{F}_{i_1,j}(\{x_i\}), \tilde{F}_{i_1,j}(\{y_i\})) \\
\leq \sum_{\beta \in \mathcal{B}_j^1} \mathcal{P}_\omega(\beta) d(x_{\eta(\beta,0)}, y_{\eta(\beta,0)}) + \\
&\sum_{i_1 \in \mathcal{I}_j^-} \Lambda_{i_1,j} \Big(\sum_{i_2 \in \mathcal{I}_{i_1}} \Lambda_{i_2,i_1} d(\tilde{F}_{i_2,i_1,j}(\{x_i\}), \tilde{F}_{i_2,i_1,j}(\{y_i\})) \Big).
\end{aligned}
$$

As the sum

$$
\begin{aligned}
&\sum_{i_1 \in \mathcal{I}_j^-} \sum_{i_2 \in \mathcal{I}_{i_1}} \Lambda_{i_2,i_1} \Lambda_{i_1,j} d(\tilde{F}_{i_2,i_1,j}(\{x_i\}), \tilde{F}_{i_2,i_1,j}(\{y_i\})) = \\
&\quad \sum_{i_1 \in \mathcal{I}_j^-} \sum_{i_2 \in \mathcal{I}_{i_1}^+} \Lambda_{i_2,i_1} \Lambda_{i_1,j} d(x_{i_2,i_1,j}, y_{i_2,i_1,j}) + \\
&\qquad \sum_{i_1 \in \mathcal{I}_j^-} \sum_{i_2 \in \mathcal{I}_{i_1}^-} \Lambda_{i_2,i_1} \Lambda_{i_1,j} d(\tilde{F}_{i_2,i_1,j}(\{x_i\}), \tilde{F}_{i_2,i_1,j}(\{y_i\})) = \\
&\quad \sum_{\beta \in \mathcal{B}_j^2} \mathcal{P}_\omega(\beta) \lambda \ d(x_{\eta(\beta,1)}, y_{\eta(\beta,1)}) + \\
&\qquad \sum_{i_1 \in \mathcal{I}_j^-} \sum_{i_2 \in \mathcal{I}_{i_1}^-} \Lambda_{i_2,i_1} \Lambda_{i_1,j} d(\tilde{F}_{i_2,i_1,j}(\{x_i\}), \tilde{F}_{i_2,i_1,j}(\{y_i\}))
\end{aligned}
$$

then continuing inductively

$$
d\big(\tilde{F}_j(\{x_{i_1}\}), \tilde{F}_j(\{y_{i_1}\})\big) \leq \sum_{\ell=1}^{B} \sum_{\beta \in \mathcal{C}_j^\ell} \mathcal{P}_\omega(\beta) \lambda^{|\beta|-2} \ d\big(x_{\eta(\beta,|\beta|-2)}, y_{\eta(\beta,|\beta|-2)}\big).
$$

As $\mathcal{X}_S F_j(\{x_i\}) = \tilde{F}_j(\{x_i\})$ then for $j \in \mathcal{I}_S$ the constants

$$
\tilde{\Lambda}_{ij} = \begin{cases} \mathcal{P}_\omega(\beta) \lambda^{|\beta|-2} & i = \eta(\beta, |\beta|-2) \\ 0 & \text{otherwise} \end{cases} \tag{56}
$$

satisfy condition (53) for $\mathcal{X}_S F$. Therefore, (55) and (56) give a complete set of the constants $\tilde{\Lambda}_{ij}$ for $\mathcal{X}_S F$.

Let $\Gamma_{\mathcal{X}_S F} = (\mathcal{V}, \mathcal{E}, \mu)$. By construction there are one-to-one correspondences

$$\tau_{ij} : \mathcal{B}_{ij}(\Gamma_F; S) \to \mathcal{B}_{ij}(\Gamma_{\mathcal{X}_S F}; S)$$

given as follows.

Suppose $\tau_{ij}(\beta) = \gamma$. If $|\beta| = 2$ then $\beta = v_i, v_j = \gamma$. Hence, $i, j \in \mathcal{I}_S$ implying $\tilde{\Lambda}_{ij} = \Lambda_{ij}$ by (56). Thus, $\mathcal{P}_\omega(\beta) = \mathcal{P}_\mu(\gamma)$. If $|\beta| > 2$ then γ is the branch corresponding to F_β. Equations (55) and (56) then imply

$$\Omega(\gamma) = 1, 0, 1, \ldots, 0, \mathcal{P}_\omega(\beta)\lambda^{|\beta|-2}.$$

As $|\beta| = |\gamma|$ then $\mathcal{P}_\omega(\beta) = \mathcal{P}_\mu(\gamma)$. Therefore, $\mathcal{R}_S(\Gamma_F) = \mathcal{R}_S(\Gamma_{\mathcal{X}_S F})$.

As $S \in st_0(\Gamma_F)$ and $S \in st_0(\Gamma_{\mathcal{X}_S F})$ by construction then theorem 1.3.4 implies that Γ_F and $\Gamma_{\mathcal{X}_S F}$ have the same nonzero spectrum. Hence, $\rho(\tilde{\Lambda}) = \rho(\Lambda)$ for this choice of constants. This completes the proof. □

Part (1) of theorem 3.2.4 states that the dynamical network F and its expansion $\mathcal{X}_S F$ have the same dynamics if we restrict each to the coordinates indexed by S. However, as there is no such correspondence between the coordinates indexed by $\bar{S}$ what is unclear is if the systems F and $\mathcal{X}_S F$ share any dynamic properties. With this in mind, part (2) of theorem 3.2.4 has the following important interpretation.

Remark 4. *Suppose Λ_{ij} are constants satisfying (53) for F. Then there always exist better constants $\tilde{\Lambda}_{ij}$ satisfying (53) for the expansion $\mathcal{X}_S F$ in the sense that $\rho(\tilde{\Lambda}) \leq \rho(\Lambda)$.*

Therefore, if one can show that F has a globally attracting fixed point via the condition $\rho(\Lambda) < 1$ then $\mathcal{X}_S F$ also has a globally attracting fixed point. Moreover, the converse of this statement holds.

Theorem 3.2.5. *If $\rho(\tilde{\Lambda}) < 1$ where $\tilde{\Lambda}_{ij}$ are constants satisfying (53) for $\mathcal{X}_S F$ then F has a globally attracting fixed point.*

Proof. Note that $\mathcal{X}_S F = \mathcal{X}_S F \circ T$ where $T = id$. As each

$$L_i = \sup_{x_i \neq y_i \in X_i} d(T_i(x_i), T_i(y_i))/d(x_i, y_i) = 1$$

then $L = \max_{i \in \mathcal{I}} L_i = 1$. Assuming then that $\rho(\tilde{\Lambda}) < 1$ corollary 4 implies the expansion $\mathcal{X}_S F$ has a globally attracting fixed point $\bar{x} \in X|_S \bigoplus B_S$. For $x \in X$ define $\tilde{x}^C = x^C|_S \bigoplus b$ where $b_{\eta(\beta,\ell)} = x^{C-\ell}_{\eta(\beta,0)}$ as in part (1) of theorem 3.2.4. Hence,

$$\mathcal{X}_S F(\tilde{x}^{C+k})|_S = F(x^{C+k})|_S \quad \text{for} \quad k \geq 0.$$

As $\mathcal{X}_S F(\tilde{x}^{C+k})|_S \to \bar{x}|_S$ then similarly $F(x^{C+k})|_S \to \bar{x}|_S$ as $k \to \infty$.

From definition 3.2.1 it follows for each $j \in \mathcal{I}$ that the function

$$(F|_S)_j = (F|_S)_j(x^0|_S, \ldots, x^{C_j}|_S)$$

for any $x^0 \in X$. Here $C_j + 2$ is the maximal length of any path or cycle from v_i to v_j with interior vertices in $\bar{S}$ where v_i is taken over all $i \in \mathcal{I}_S$.

By an argument similar to that in lemma 3.2.2 we have

$$x_j^{k+1} = (F|_S)_j(x^k|_S, \ldots, x^{C_j+k}|_S). \tag{57}$$

From the observation that $x^{\ell+k}|_S \to \bar{x}|_S$ as $k \to \infty$ for all $0 \leq \ell \leq C_j$ then this together with (57) implies

$$x_j^{k+1} \to (F|_S)_j(\bar{x}|_S, \ldots, \bar{x}|_S)$$

for all $j \in \mathcal{I}$ as $(F|_S)_j$ is continuous over its domain. Hence, F has a globally attracting fixed point. □

Note that we now have two ways for determining whether or not a given dynamical network F has a globally attracting fixed point. The first is by finding constants Λ_{ij} that satisfy (53) for F and computing $\rho(\Lambda)$. The second is by finding constants $\tilde{\Lambda}_{ij}$ that satisfy (53) for $\mathcal{X}_S F$ and computing $\rho(\tilde{\Lambda})$. If either $\rho(\Lambda) < 1$ or $\rho(\tilde{\Lambda}) < 1$

then theorem 3.1.3 or theorem 3.2.5 respectively imply that F has a unique global attractor.

However, given constants Λ_{ij} for F it is always possible to find constants $\tilde{\Lambda}_{ij}$ for the expanded network $\mathcal{X}_S F$ such that $\rho(\tilde{\Lambda}) \leq \rho(\Lambda_{ij})$ (see remark 4). This has the following immediate and important consequence that any dynamical network expansion $\mathcal{X}_S F$ can be used to obtain improved stability estimates of any dynamical network F.

Example 1. *Let $\mathcal{L}(x) = 4x(1-x)$ be the standard logistic map and let the map $\mathcal{Q}(x) = 1 - x^2$ where both are restricted to the interval $[0,1]$. Consider the network H given in (54) for*

$$H(x) = \begin{bmatrix} \frac{1}{4}\mathcal{L}(x_2) \\ \frac{1}{4}\mathcal{Q}(x_1) + \frac{1}{4}\mathcal{L}(x_4) \\ \frac{1}{4}\mathcal{Q}(x_1) \\ \frac{1}{4}\mathcal{Q}(x_1) + \frac{1}{4}\mathcal{Q}(x_3) \end{bmatrix}.$$

Note the constants $\Lambda_{ij} = \max_{x \in X} |(DH)_{ji}(x)|$ satisfy equation (53) for H and are given by

$$\Lambda = \begin{bmatrix} 0 & 1/2 & 1/2 & 1/2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

As $\rho(\Lambda) = 1.08$ theorem 3.1.3 does not imply that H has a globally attracting fixed point. However, the expansion over $S = \{v_1, v_2\}$ of H given by

$$\mathcal{X}_S H(x) = \begin{bmatrix} \frac{1}{4}\mathcal{L}(x_2) \\ \frac{1}{4}\mathcal{Q}(x_1) + \frac{1}{4}\mathcal{L}\big(\frac{1}{4}\mathcal{Q}(x_5) + \frac{1}{4}\mathcal{Q}(\frac{1}{4}\mathcal{Q}(x_7))\big) \\ H_5(x_1) \\ H_6(x_1) \\ H_7(x_6) \end{bmatrix}.$$

where $H_5(x_1), H_6(x_1), H_7(x_6) = Id$ *has constants* $\tilde{\Lambda}_{ij}$ *given by*

$$\tilde{\Lambda} = \begin{bmatrix} 0 & 1/2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0.265 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0.012 & 0 & 0 & 0 \end{bmatrix}, \quad \textit{satisfying equation (53).}$$

Here, as before, $\tilde{\Lambda}_{ij} = \max|(D\mathcal{X}_S H)_{ji}(x)|$*. As* $\rho(\tilde{\Lambda}) = 0.90$ *theorem 3.2.5 implies that* H *does in fact have a globally attracting fixed point.*

As a final remark, we note that it is possible to sequentially expand a dynamical network and thereby potentially improve ones estimate of whether the system has a globally attracting fixed point. Moreover, sequential expansions can also be used to obtain better estimates of an interaction's stability factor. For instance, if the network $\mathcal{X}_S H$ in example 1 (considered as an interaction) was again expanded over the vertex set $T = \{v_1, v_5, v_6, v_7\}$ then $\rho(\hat{\Lambda}) = 0.76$ for the constants $\hat{\Lambda} = \max(|D\mathcal{X}_T(\mathcal{X}_S H)_{ji}(x)|)$. That is, from this calculation the interaction H has a stability factor less than or equal to 0.76.

3.3 Time-Delayed Dynamical Systems

The dynamical network expansions of the previous section also have implications to time-delayed dynamical systems. This is because the restrictions used to define such expansions are themselves time-delayed systems. Using this connection as a starting point our goal is to study how time delays effect the dynamic stability of an arbitrary dynamical system.

One of the main results we obtain in this direction is that the introduction or removal of time delays does not in fact have an effect on a system's stability. Moreover, since any restriction of a dynamical network is itself a delayed dynamical system this will allow us to define *dynamical network reductions* by removing these delays.

As with dynamical network expansions such reductions will allow us to obtain improved estimates of the original network's dynamic stability i.e. whether it has a globally attracting fixed point. However, as the removal or addition of time-delays to a dynamical system does not correspond to any isospectral transformation the main obstacle in developing this theory is in determining how time delays effect the eigenvalues associated with such systems. Moreover, before we can do this we must first connect the theory developed in section 3.1 for dynamical networks to the more general class of time-delayed dynamical systems.

Definition 3.3.1. *A delayed dynamical system is a function of the form*

$$x^{k+1} = F(x^{k-T}, x^{k-T+1}, \ldots, x^k)$$

for a fixed integer $T \geq 0$ *where each* $x^j \in X = \bigoplus_{i=1}^n X_i$ *for* X_i *a compact metric space. Moreover, the orbit of the initial condition* $(x^{-T}, x^{-T+1}, \ldots, x^0) \in \bigoplus_{i=0}^T X$ *is the sequence* $\{x^i\}_{i \geq -T}$*.*

For each delayed dynamical system $F = F(x^{k-T}, x^{k-T+1}, \ldots, x^k)$ there is a corresponding undelayed dynamical network which can be constructed as follows. Let $\mathcal{V}_j(F)$ be the variables of the form x_i^{k-m} appearing in $F_j(x^{k-T}, x^{k-T+1}, \ldots, x^k)$ for all $0 < m \leq T$ and $1 \leq i, j \leq n$. Furthermore, let

$$\mathcal{I}(F) = \{(i, j, l, m) : \ 1 \leq j \leq n, \ x_i^{k-m} \in \mathcal{V}_j(F), \ 1 \leq \ell \leq m\}.$$

In F_j replace each $x_i^{k-m} \in \mathcal{V}_j(F)$ by x_{ij}^{mm} and each x_i^k by x_i. If this is done over all $1 \leq j \leq n$ we call the resulting function F^*. We then define the function

$$\mathcal{N}F = F^* \bigoplus_{(i,j,l,m) \in \mathcal{I}(F)} F_{ij}^{m\ell}$$

where each $F_{ij}^{m\ell}(x_{ij}^{m,\ell-1}) = x_{ij}^{m,\ell-1}$ and $x_{ij}^{m0} = x_i$. Moreover, let

$$X_F = \left\{ x \in X \bigoplus_{(i,j,l,m) \in \mathcal{I}(F)} X_{ij}^{m\ell} : x_{ij_1}^{m_1\ell} = x_{ij_2}^{m_2\ell}, \ (i, j_1, \ell, m_1), (i, j_2, \ell, m_2) \in \mathcal{I}(F) \right\}.$$

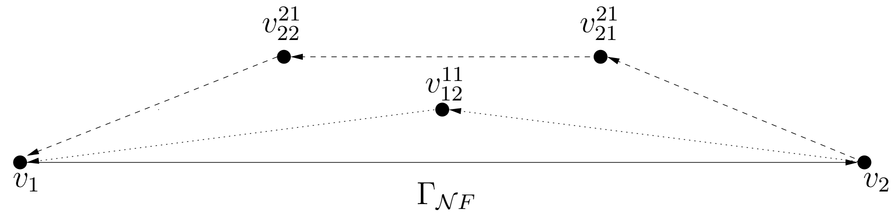


Figure 18: The graph of the dynamical network $\mathcal{N}F$. Here dashed paths correspond to the delays in the original system $F = F(x^{k-2}, x^{k-1}, x^k)$

We note that $\mathcal{N}F : X_F \to X_F$ by observing that each $F_{ij}^{m\ell}(x_{ij}^{m,\ell-1}) = x_{ij}^{m,\ell-1}$. Hence, both the sequences of functions $F_{ij_1}^{m_1 1}, F_{ij_1}^{m_1 2}, \ldots, F_{ij_1}^{m_1 m_1}$ and $F_{ij_2}^{m_2 1}, F_{ij_2}^{m_2 2}, \ldots, F_{ij_2}^{m_2 m_2}$ sequentially pass the value of x_i through the coordinates $x_{ij_1}^{m_1 1}, x_{ij_1}^{m_1 2}, \ldots, x_{ij_1}^{m_1 m_1}$ and $x_{ij_2}^{m_2 1}, x_{ij_2}^{m_2 2}, \ldots, x_{ij_2}^{m_2 m_2}$ in m_1 and m_2 steps respectively. Hence, if $\tilde{y} \in X_F$ then $F_{ij_1}^{m_1 \ell}(\tilde{y}_{ij_1}^{m_1,\ell-1}) = F_{ij_2}^{m_2 \ell}(\tilde{y}_{ij_2}^{m_2,\ell-1})$ for $(i, j_1, \ell, m_1), (i, j_2, \ell, m_2) \in \mathcal{I}(F)$ implying that $F(\tilde{y}) \in X_F$.

We call the function $\mathcal{N}F : X_F \to X_F$ the *undelayed dynamical network* associated with F. As an example consider

$$F(x^{k-2}, x^{k-1}, x^k) = \begin{bmatrix} F_1(x_2^{-2}, x_2^{-1}) \\ F_2(x_1) \end{bmatrix}.$$

The undelayed dynamical network associated with F is then given by

$$\mathcal{N}F(x) = \begin{bmatrix} F_1(x_{21}^{11}, x_{21}^{22}) \\ F_2(x_1) \\ F_{21}^{11}(x_2) \\ F_{21}^{21}(x_2) \\ F_{21}^{22}(x_{21}^{21}) \end{bmatrix}$$

where $F_{21}^{11}(x_2) = x_2$, $F_{21}^{21}(x_2) = x_2$, and $F_{21}^{22}(x_{21}^{21}) = x_{21}^{21}$. The graph $\Lambda_{\mathcal{N}F}$ corresponding to $\mathcal{N}F$ is given in figure 18.

The motivation behind this construction is the following. Recall that the functions $F_{ij}^{m1}, F_{ij}^{m2}, \ldots, F_{ij}^{mm}$ sequentially pass the value of x_i through the coordinates $x_{ij}^{m1}, x_{ij}^{m2}, \ldots, x_{ij}^{mm}$ in m steps which is the same function performed by the variable

$x_i^{k-m} \in \mathcal{V}_j(F)$ in the time-delayed system $F = F(x^{-T}, \ldots, x^k)$. Hence, the substitution of x_{ij}^{mm} for x_i^{k-m} in F_j with the addition of the functions $F_{ij}^{m1}, F_{ij}^{m2}, \ldots, F_{ij}^{mm}$ removes the delay associated with x_i^{k-m} from F. Moreover, we note that there is a one-to-one correspondence between the domain of the delayed system F and the dynamical network $\mathcal{N}F$. This can be seen as follows.

For $F = F(x^{k-T}, \ldots, x^k)$ let $\delta : X_F \to \bigoplus_{i=1}^{T} X$ be the function given by $\delta(\tilde{y})_i^0 = \tilde{y}_i$ and $\delta(\tilde{y})_i^{-\ell} = \tilde{y}_{ij}^{m\ell}$ for all $(i, j, \ell, m) \in \mathcal{I}(F)$. As the value of any $\tilde{y}_{ij}^{m\ell}$ depends only on the indicies i and ℓ then the function δ is a bijection. This together with the discussion above implies that for any $\tilde{y} \in X_F$ where $(y^{-T}, y^{-T+1}, \ldots, y^0) = \delta(\tilde{y})$, then the component $F_j(y^{-T}, y^{-T+1}, \ldots, y^0) = \mathcal{N}F_j(\tilde{y})$ for $1 \leq j \leq n$. This can be phrased as follows.

Lemma 3.3.2. *Let $F = F(x^{k-T}, \ldots, x^k)$. Then $F^k(y^{-T}, y^{-T+1}, \ldots, y^0) = \mathcal{N}F^k(\tilde{y})|_V$ for $V = \{v_1, \ldots, v_n\}$ and any $(y^{-T}, y^{-T+1}, \ldots, y^0) = \delta(\tilde{y})$.*

That is, the dynamical network $\mathcal{N}F$ restricted to its first n coordinates has the same dynamics as the delayed dynamical system F. Recall that the original motivation behind the construction of $\mathcal{N}F$ was to study delayed systems using the theory established in sections 3.1 and 3.2.

With this in mind, a *fixed point* of a delayed dynamical system $F = F(x^{k-T}, \ldots, x^k)$ is a point $\tilde{x} \in X$ such that $\tilde{x} = F(\tilde{x}, \tilde{x}, \ldots, \tilde{x})$. Moreover, the point $\tilde{x} \in X$ is a global attractor of the delayed system if for any initial condition $(y^{-T}, y^{-T+1}, \ldots, y^0)$ we have

$$\lim_{k \to \infty} ||y^k - \tilde{x}||_\infty = 0.$$

Theorem 3.3.3. *The delayed dynamical system $F = F(x^{k-T}, \ldots, x^k)$ has a globally attracting fixed point if and only if the dynamical network $\mathcal{N}F$ has a globally attracting fixed point.*

Proof. Suppose $\tilde{x} \in X_F$ is a global attractor of $\mathcal{N}F$. Then for any $\tilde{y} \in X_F$ this implies in particular that

$$\lim_{k\to\infty} d\big(\mathcal{N}F_i^k(\tilde{y}),\tilde{x}_i\big) = 0 \text{ for } 1 \le i \le n.$$

By use of lemma 3.3.2 it follows that $\tilde{x}|_V$ is a global attractor for the delayed system F.

Conversely, suppose $\bar{x} \in X$ is a global attractor of F. Then for any $\tilde{y} \in X_F$ where $\delta(\tilde{y}) = (y^{-T},\dots,y^0)$ we have

$$\lim_{k\to\infty} d\big(F_i^k(y^{-T},\dots,y^0),\bar{x}_i\big) = 0 \text{ for } 1 \le i \le n$$

implying by lemma 3.3.2 that $\lim_{k\to\infty} \mathcal{N}F_i^k(\tilde{y}) = \bar{x}_i$.

As $\big(\mathcal{N}F^k(\tilde{y})\big)_{ij}^{\ell m} = \mathcal{N}F_i^{k-\ell}(\tilde{y})$ for each $(i,j,\ell,m) \in \mathcal{I}(F)$ then similarly it follows that $\lim_{k\to\infty} \big(\mathcal{N}F^k(\tilde{y})\big)_{ij}^{\ell m} = \bar{x}_i$. Hence, the point $\delta^{-1}(\bar{x},\dots,\bar{x}) \in F_X$ is the unique global attractor of $\mathcal{N}F$. $\square$

Because of theorem 3.3.3 it is possible to investigate the dynamic stability of any delayed dynamical system F via its associated dynamical network $\mathcal{N}F$. However, before doing so we note the following.

Let $F = F(x^{k-T},\dots,x^k)$. If we say Λ_{ij} are constants satisfying equation (53) for $\mathcal{N}F$ then by definition 3.1.1 we mean that for $x, y \in X_F$

$$d\big(\mathcal{N}F_j(x),\mathcal{N}F_j(y)\big) \le \sum_{i=1}^{n} \Lambda_{ij} d(x_i,y_i) + \sum_{x_i^{k-m} \in \mathcal{V}_j(F)} \big(\Lambda_{ij}^{mm}\big)_j d\big(x_{ij}^{mm},y_{ij}^{mm}\big), \quad 1 \le j \le n$$

$$\text{and} \qquad d\big(\mathcal{N}F_{ij}^{m\ell}(x),\mathcal{N}F_{ij}^{m\ell}(y)\big) \le \big(\Lambda_{ij}^{m,\ell-1}\big)_{ij}^{m\ell} d(x_{ij}^{m,\ell-1},y_{ij}^{m,\ell-1}), \quad (i,j,\ell,m) \in \mathcal{I}(F).$$

In particular, the constants of the form $\big(\Lambda_{ij}^{m,\ell}\big)_j$, $\big(\Lambda_i\big)_{ij}^{m\ell}$, and $\big(\Lambda_{i_1 j_1}^{m_1,\ell_1}\big)_{i_2 j_2}^{m_2 \ell_2}$ that do not appear in these inequalities are zero.

Theorem 3.3.3 then has the following corollary.

Corollary 5. *For the delayed dynamical system $F = F(x^{k-T},\dots,x^k)$ suppose Λ_{ij} are constants satisfying (53) for $\mathcal{N}F$. If $\rho(\Lambda) < 1$ then F has a globally attracting fixed point.*

The main goal of this section is to extend as well as improve upon this condition by investigating how modifying the delays of a dynamical system effects the systems dynamic stability.

Definition 3.3.4. *For the delayed dynamical system* $F = F(x^{k-T}, x^{k-T+1}, \dots, x^k)$ *let* $U_F : X \to X$ *be the undelayed dynamical network given by* $U_F(x) = F(x, x, \dots, x)$.

Simply put, the system U_F is the delayed dynamical system F with its delays removed. The following result relates the dynamic stability of the undelayed system U_F to the delayed dynamical system F.

Theorem 3.3.5. *Let* $F = F(x^{k-T}, x^{k-T+1}, \dots, x^k)$ *be a delayed dynamical system. If* Λ_{ij} *are constants satisfying (53) for* $\mathcal{N}F$ *and* $\rho(\Lambda) < 1$ *then* U_F *has a globally attracting fixed point. Coversely, if* $\tilde{\Lambda}_{ij}$ *are constants satisfying (53) for* U_F *and* $\rho(\tilde{\Lambda}) < 1$ *then* $\mathcal{N}F$ *has a globally attracting fixed point.*

That is, if either of the dynamical networks $\mathcal{N}F$ or U_F are known to have a single global attractor (via theorem 3.1.3) then the same holds for the other as well as for the original delayed system F. Moreover, the converse of this statement holds in the following sense.

Definition 3.3.6. *The function* $F = F(x^{k-T}, x^{k-T+1}, \dots, x^k)$ *is a delayed version of the dynamical network* $H : X \to X$ *if* $H(x) = F(x, x, \dots, x)$.

An immediate corollary of theorem 3.3.5 is the following.

Corollary 6. *If* Λ_{ij} *are constants satisfying (53) for the dynamical network* H *and* $\rho(\Lambda) < 1$ *then any delayed version of* H *has a globally attracting fixed point.*

To summarize, if a system is known to have a unique global attractor via theorem 3.1.3 then the removal or addition of delays does not change this property. In order to prove this we will use the well known theorem of Perron and Frobenius and the following standard terminology.

If $A \in \mathbb{R}^{n\times n}$ then A is said to be *irreducible* if the graph associated with A is strongly connected. Moreover, A is *nonnegative* if $A \geq 0$.

Theorem 3.3.7. ***(Perron-Frobenius)*** *Let $A \in \mathbb{R}^{n\times n}$ and suppose that A is irreducible and nonnegative. Then*

(a) $\rho(A) > 0$;

(b) $\rho(A)$ is an eigenvalue of A;

(c) $\rho(A)$ is an algebraically simple eigenvalue of A; and

(d) the left and right eigenvectors x and y associated with $\rho(A)$ have strictly positive entries.

Recall that if a graph is not strongly connected then it has strongly connected components given by $\mathbb{S}(G)_1 \dots, \mathbb{S}(G)_N$ where

$$\sigma(G) = \bigcup_{\ell=1}^{N} \sigma\big(\mathbb{S}_\ell(G)\big). \tag{58}$$

We call a strongly connected component $\mathbb{S}_k(G)$ *trivial* if it consists of a single vertex without loop in which case $\sigma\big(\mathbb{S}_k(G)\big) = \{0\}$.

For $G = (V, E, \omega)$ with $V = \{v_1, \dots, v_n\}$ and $e_{ij} \in E$ define

$$G_{ij}^m(t, \theta) = (V \cup \{v_{n+1}, \dots, v_{n+m}\}, \mathcal{E}, \mu) \ \text{ for } \ t \in [0, 1]$$

as the graph G in which the weight $\omega(e_{ij})$ is replaced by $(1-t)\omega(e_{ij})$ and a path $v_i, v_{n+1}, \dots, v_{n+m}, v_j$ is added with weights $\mu(e_{i,n+1}) = \theta$, $\mu(e_{n+\ell-1,n+\ell}) = \theta$ for each $1 \leq \ell \leq m$ and $\mu(e_{n+m,j}) = t\omega(e_{ij})$ (see figure 19). In the following if we write $G_{ij}^m(t, \theta)$ we will implicitly assume $e_{ij} \in E$. With this in place we state the following lemma.

Lemma 3.3.8. *If $M(G) \geq 0$ then $\rho(G) = \rho\big(G_{ij}^m(t, \rho(G))\big)$ for all $t \in [0, 1]$.*

Proof. For simplicity let $G_{ij}^m(t, \rho(G)) = G_{ij}^m(t)$. For $m = 1$ the claim is that $\rho(G) \in \sigma(G_{ij}^1(t))$ for all $t \in [0, 1]$. To see this suppose that the graph G is strongly connected

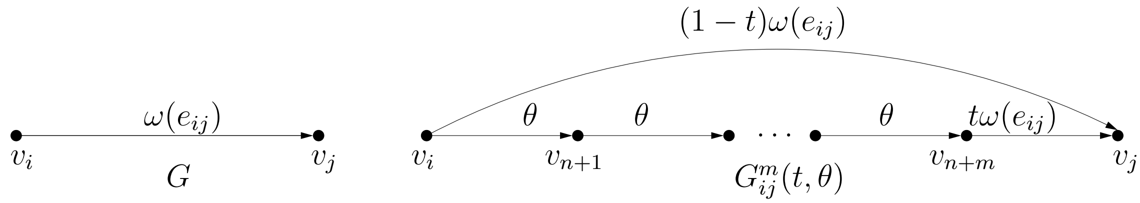


Figure 19: The edge e_{ij} of G (left) and its replacement in $G_{ij}^m(t,\theta)$ (right).

i.e. $M(G)$ is irreducible. Since the Perron-Frobenius theorem implies that the spectral radius of any nonnegative irreducible matrix is a strictly positive eigenvalue of the matrix then both $\rho(G) > 0$ and $\det(M(G) - \rho(G)I) = 0$.

Note that as reducing $G_{ij}^1(t)$ over $V = \{v_1, \ldots, v_n\}$ implies

$$M\big(\mathcal{R}_V(G_{ij}^1(t))\big)_{pq} = \begin{cases} \big(1 - t(1 - \frac{\rho(G)}{\lambda})\big)M(G)_{pg} & \text{for} \quad p = i, \ q = j \\ M(G)_{pq} \quad \text{otherwise} \end{cases} \tag{59}$$

then for $\lambda = \rho(G)$ it follows that

$$\det(M\big(\mathcal{R}_V(G_{ij}^1(t))\big) - \lambda I) = \det(M(G) - \lambda I) = 0$$

or $\rho(G) \in \sigma(\mathcal{R}_V(G_{ij}^1(t))$. Hence, $\rho(G) \in \sigma(G_{ij}^1(t))$ for all $t \in [0,1]$ since $\sigma(\mathcal{R}_V(G_{ij}^1(t)) = \sigma(G_{ij}^1(t)) - \{0\}$ by theorem 1.3.4. By use of equation (59) this furthermore implies that the spectrum $\sigma(G_{ij}^1(0)) = \sigma(G) \cup \{0\}$ and in particular that $\rho(G_{ij}^1(0)) = \rho(G)$.

Next, the assumption that G is strongly connected implies that $G_{ij}^1(t)$ is also strongly connected for all $t \in (0,1]$. Therefore, as $\rho(G) > 0$ then $M(G_{ij}^1(t))$ is a nonnegative irreducible matrix and the Perron-Frobenius theorem implies

(i) $\rho(G_{ij}^1(t)) > 0$;

(ii) $\rho(G_{ij}^1(t))$ is an eigenvector of $M(G_{ij}^1(t))$; and

(iii) $\rho(G_{ij}^1(t))$ is a algebraically simple eigenvector of $G_{ij}^1(t)$

for all $t \in (0,1]$. Additionally, (i), (ii), and (iii) hold for $t = 0$ since $M(G)$ satisfies the conditions of the Perron-Frobenius theorem and $\sigma(G_{ij}^1(0)) = \sigma(G) \cup \{0\}$.

Let $\lambda_1(t), \ldots, \lambda_{n+1}(t)$ denote the eigenvalues of $G_{ij}^1(t)$ for $t \in [0,1]$. Then (i) through (iii) imply for fixed $t \in [0,1]$ that $\rho(G_{ij}^1(t)) = \lambda_p(t)$ where $\lambda_p(t)$ is the unique eigenvalue such that $Re\lambda_p(t) > Re\lambda_q(t)$ for all $q \neq p$. Hence, as each $Re\lambda_q(t)$ is a continuous function of t, then assuming the ordering

$$\rho(G) = Re\lambda_1(0) > Re\lambda_2(0) \geq \cdots \geq Re\lambda_{n+1}(0)$$

there can be no first $t_1 \in [0,1]$ such that $Re\lambda_q(t_1) = Re\lambda_1(t_1)$ for all $q > 1$. Therefore, $Re\lambda_1(t) > Re\lambda_q(t)$ for $q > 1$ and $t \in [0,1]$ implying that $\lambda_1(t) = \rho(G^1_{ij}(t))$.

Moreover, as $\rho(G) \in \sigma(G^1_{ij}(t))$ for all $t \in [0,1]$ and $\rho(G) = \lambda_1(0)$ then the continuity of each $Re\lambda_q(t)$ coupled with the fact that $\lambda_1(t) > Re\lambda_q(t)$ for $q > 1$ implies that $\lambda_1(t) = \rho(G)$ for all $t \in [0,1]$. Hence, $\rho(G^1_{ij}(t)) = \rho(G)$ for $t \in [0,1]$ under the assumption that G is strongly connected.

If G is not strongly connected then it has strongly connected components given by $\mathbb{S}_1 \ldots, \mathbb{S}_N$. If the edge e_{ij} belongs to the strongly connected component $\mathbb{S}_k$ where $\rho(\mathbb{S}_k) = \rho(G)$ then the previous argument implies that $(\mathbb{S}_k)^1_{ij}(t, \rho(G)) = \rho(G)$. Given that

$$\sigma\big(G^1_{ij}(t)\big) = \sigma\big((\mathbb{S}_k)^1_{ij}(t, \rho(G))\big) \bigcup_{\ell=1, \ell\neq k}^{N} \sigma(\mathbb{S}_\ell) \tag{60}$$

then in this case $\rho(G^1_{ij}(t)) = \rho(G)$. However, if the spectral radius $\rho(\mathbb{S}_k) < \rho(G)$ we note the following.

For a square matrix $M \leq 0$ and $\alpha \geq 0$, $\rho(\alpha M) = \alpha\rho(M)$. Hence, if $\mathbb{T}$ is the graph $\mathbb{S}_k$ where each edge weight is scaled by the constant $\rho(G)/\rho\big(\mathbb{S}_k\big)$ then $\rho(\mathbb{T}) = \rho(G)$. As $\mathbb{T}$ is strongly connected on the basis that $\mathbb{S}_k$ is strongly connected then $\rho\big(\mathbb{T}^1_{ij}(t, \rho(G))\big) = \rho(G)$. However, given that $\rho(G)/\rho(\mathbb{S}_k) > 1$ then $M\big(\mathbb{T}^1_{ij}(t, \rho(G))\big) \geq M\big((\mathbb{S}_k)^1_{ij}(t, \rho(G))\big) \geq 0$ from which it follows that the spectral radius $\rho\big(\mathbb{T}^1_{ij}(t, \rho(G))\big) \geq \rho\big((\mathbb{S}_k)^1_{ij}(t, \rho(G))\big)$ (see chapter 8 [28]). Therefore, $\rho(G) \geq \rho\big((\mathbb{S}_k)^1_{ij}(t, \rho(G))\big)$ and equation (60) again implies that $\rho(G^1_{ij}(t)) = \rho(G)$ for $t \in [0,1]$.

Lastly, if e_{ij} does not belong to a strongly connected component then it follows that $\rho(G^1_{ij}(t)) = \rho(G)$ since G and $G^1_{ij}(t)$ have the same nontrivial strongly connected components. This completes the proof for the case $m = 1$.

If $m = 2$ consider the fact that for $H = G^1_{ij}(t)$ the graph

$$G^2_{ij}(t, \rho(G)) = H^1_{i,n+1}(1, \rho(G)).$$

Therefore, $\rho\big(G^2_{ij}(t, \rho(G))\big) = \rho(G)$ by the previous argument. If we continue in this manner the result follows for any finite m. $\square$

Hence, $G^m_{ij}(t, \theta)$ is a graph transformation that preserves the spectral radius of the graph G if $\theta = \rho(G)$. However, to relate such transformations to adding and removing time delays to a given dynamical system we will need the following lemma.

Lemma 3.3.9. *Suppose $A, B \in \mathbb{R}^{n\times n}$ where B is nonnegative and irreducible. If $0 \leq A \leq B$ and $A_{rs} < B_{rs}$ for some $1 \leq r, s \leq n$ then $\rho(A) < \rho(B)$.*

To prove this lemma we use the following theorem from [28] (see theorem 6.3.12) which we restate here for completeness. Here, the notation y^* denotes the conjugate transpose of the vector y.

Theorem 3.3.10. *Let $A(\tau) \in \mathbb{R}^{n\times n}$ be differentiable at $\tau = 1$. Assume that λ is an algebraically simple eigenvalue of $A(1)$ and that $\lambda(\tau)$ is an eigenvalue of $A(\tau)$, for small τ, such that $\lambda(1) = \lambda$. Let x be a right eigenvector of A and let y be a left eigenvector of A. Then*

$$\lambda'(1) = \frac{y^*A'(1)x}{y^*x}.$$

By way of notation, if $A \in \mathbb{R}^{n\times n}$ let $A_{rs}(\tau)$ be the matrix A in which the rs-entry A_{rs} is multiplied by $\tau \in \mathbb{R}$. With in place we give a proof of lemma 3.3.11.

Proof. Suppose that $M \in \mathbb{R}^{n\times n}$ such that the entry $M_{rs} > 0$. Under the assumption that $M \in \mathbb{R}^{n\times n}$ is nonnegative and irreducible then the same holds for $M_{rs}(\tau)$ for any $\tau > 0$. Then the Perron-Frobenius theorem implies that $\rho(M_{rs}(\tau))$ is an algebraically simple eigenvalue of $M_{rs}(\tau)$ with strictly positive left and right eigenvectors $y(\tau)$ and $x(\tau)$ respectively. Via theorem 3.3.10 it follows that

$$\rho'\big(M_{rs}(\tau)\big) = \frac{y(\tau)^*M'_{rs}(\tau)x(\tau)}{y(\tau)^*x(t)} \quad \text{for} \quad \tau > 0.$$

As both $y(\tau)^* M'_{rs}(\tau)x(\tau) = y_r(\tau)M_{rs}x_s(\tau) > 0$ and $y(\tau)^*x(\tau) > 0$ it follows that $\rho'\big(M_{rs}(\tau)\big) > 0$. The lemma follows by noting that if $0 < \tau_1 < \tau_2$ then this implies that $\rho\big(M_{rs}(\tau_1)\big) < \rho\big(M_{rs}(\tau_2)\big)$. $\square$

Hence, the spectral radius of a nonnegative irreducible matrix depends monotonically on each of its nonnegative entries. With this in place we prove the following lemma.

Lemma 3.3.11. *Suppose $M(G) \geq 0$ and $\theta > 0$. Then $\rho(G) < \theta$ if and only if $\rho\big(G^m_{ij}(t,\theta)\big) < \theta$.*

Proof. For $G = (V, E, \omega)$ such that $M(G) \geq 0$ denote $\hat{G} = G^m_{ij}\big(t, \rho(G)\big)$. Lemma 3.3.8 then implies that $\rho(\hat{G}) = \rho(G)$. For $\theta > 0$ and assuming $0 < \rho(G)$ let $\hat{G}(c)$ be the graph $\hat{G}$ in which every edge weight has been multiplied by $c = \theta/\rho(G) > 0$. As $cM(\hat{G}) = M\big(\hat{G}(c)\big) \geq 0$ and $c > 0$ then it follows that $\rho(\hat{G}(c)) = c\rho(G) = \theta$.

Next, observe that the matrix $M(\hat{G}(c))$ is given by

$$M(\hat{G}(c))_{pq} = \begin{cases} cM\big(G^m_{ij}(t,\theta)\big)_{pq} & 1 \leq p, q \leq n \text{ and } p = n+m,\ q = j \\ M\big(G^m_{ij}(t,\theta)\big)_{pq} & \text{otherwise} \end{cases} \tag{61}$$

from which it follows that $\hat{G}(c)$ is strongly connected if and only if $G^m_{ij}(t,\theta)$ is strongly connected. If both $\hat{G}(c)$ and $G^m_{ij}(t,\theta)$ are strongly connected then there exist $1 \leq r, s \leq n$ such that $M(\hat{G}(c))_{rs} = cM\big(G^m_{ij}(t,\theta)\big)_{rs} > 0$ as $\hat{G}(c)$ and $G^m_{ij}(t,\theta)$ would otherwise contain no cycles. In this case lemma 3.3.9 together with (61) imply that

(I) if $c > 1$ then $\rho\big(\hat{G}(c)\big) > \rho\big(G^m_{ij}(t,\theta)\big)$; and

(II) if $c \leq 1$ then $\rho\big(\hat{G}(c)\big) \leq \rho\big(G^m_{ij}(t,\theta)\big)$.

As $c = \theta/\rho(G)$ and $\rho\big(\hat{G}(c)\big) = \theta$ then it follows that $\rho(G) < \theta$ if and only if $\rho\big(G^m_{ij}(t,\theta)\big) < \theta$.

If $\hat{G}(c)$ is not strongly connected it has strongly connected components given by $\mathbb{S}_1 \ldots, \mathbb{S}_N$ where (58) holds. Hence, if e_{ij} belongs to the strongly connected component of $\mathbb{S}_k$ it follows by substituting $\mathbb{S}_k$ for G in (I) and (II) that

(III) if $\rho(\mathbb{S}_k) < \theta$ then $\rho\big((\mathbb{S}_k)_{ij}^m(t,\theta)\big) < \theta$; and

(IV) if $\rho(\mathbb{S}_k) \geq \theta$ then $\rho\big((\mathbb{S}_k)_{ij}^m(t,\theta)\big) \geq \theta$.

Suppose then that $\rho(G) < \theta$. As $\rho(\mathbb{S}_k) \leq \rho(G)$ then (III) implies $\rho\big((\mathbb{S}_k)_{ij}^m(t,\theta)\big) < \theta$. Given that

$$\sigma\big(G_{ij}^m(t,\theta)\big) = \sigma\big((\mathbb{S}_k)_{ij}^m(t,\theta)\big) \bigcup_{\ell=1,\ell\neq k}^{N} \sigma(\mathbb{S}_\ell)$$

then $\rho\big(G_{ij}^m(t,\theta)\big) < \theta$ since $\max_{1\leq \ell\leq N}\rho(\mathbb{S}_\ell) = \rho(G)$. Similarly, if $\rho(G) \geq \theta$ then from (IV) it follows that $\rho\big(G_{ij}^m(t,\theta)\big) \geq \theta$. Hence, $\rho(G) < \theta$ if and only if $\rho\big(G_{ij}^m(t,\theta)\big) < \theta$ in $\hat{G}(c)$ is not strongly connected.

If e_{ij} does not belong to any strongly connected component then $\rho(G) = \rho\big(G_{ij}^m(t,\theta)\big)$ as both graphs have the same nontrivial strongly connected components. In this case the conclusion of the lemma follows immediately.

For the case when $\rho(G) = 0$, note that by the Perron-Frobenius theorem a matrix $M(\mathcal{G}) \geq 0$ has spectral radius 0 only if $\mathcal{G}$ has no cycles since $\mathcal{G}$ would otherwise have a nontrivial strongly connected component with positive spectral radius. Given that G has no cycles if and only if $G_{ij}^m(t,\theta)$ has no cycles it follows that $\rho(G) = 0$ if and only if $\rho\big(G_{ij}^m(t,\theta)\big) = 0$. Again, this implies $\rho(G) < \theta$ if and only if $\rho\big(G_{ij}^m(t,\theta)\big) < \theta$ completing the proof. $\square$

The major idea behind lemma 3.3.11 is the following. Adding delays to a system corresponds to modifying the network's graph structure as in figure 19 from G to $G_{ij}^m(t,1)$. Conversely, removing delays from a system has the opposite effect (see the following proof). Lemma 3.3.11 is then the statement that the removal or addition of time delays does not change the dynamic stability of the system.

Before we give a proof of theorem 3.3.5 we introduce the following. For ease of notation if $\mathcal{N}F : X_F \to X_F$ let $v_i = v_{ij}^{m0}$, $v_j = v_{ij}^{m,m+1}$ for all $x_i^{k-m} \in \mathcal{V}_j(F)$. A proof of theorem 3.3.5 is the following.

Proof. Suppose $F = F(x^{k-T}, \ldots, x^k)$ and Λ_{ij} are constants satisfying equation (53) for $\mathcal{N}F$. Let $\mathcal{F}$ be the function F in which the variable $x_i^{k-m} \in \mathcal{V}_j(F)$ is replaced by x_i and let the constants $\overline{\Delta}$ be the matrix given by

$$\overline{\Delta}_{pg} = \begin{cases} \Lambda_{pg} & 1 \leq p, q \leq n, \quad p \neq i, \quad q \neq j \\ \\ \Lambda_{pg} + (\Lambda_{pg}^{mm})_{pq}^{m,m+1} & p = i, \quad q = j \end{cases} \qquad \text{and}$$

$\big(\overline{\Delta}_{pq}^{m,\ell-1}\big)_{pq}^{m\ell} = \big(\Lambda_{pq}^{m,\ell-1}\big)_{pq}^{m\ell}$ for all $x_p^{k-m} \in \mathcal{V}_q(\mathcal{F})$ and zero otherwise. The claim then is that $\overline{\Delta}_{pg}$ are constants satisfying (53) for $\mathcal{N}\mathcal{F}$.

To see this note that as F and $\mathcal{F}$ are identical with the exception that the variable $x_i^{k-m} \in \mathcal{V}_j(F)$ has been replaced by x_i in $\mathcal{F}$ then this claim follows immediately apart from showing that $\overline{\Delta}_{ij} = \Lambda_{ij} + (\Lambda_{ij}^{mm})_{ij}^{m,m+1}$ satisfies equation (53). To see that this holds as well we note that under the assumption that the constants Λ_{pq} satisfy (53) for $\mathcal{N}F$ then for $x, y \in X_F$

$$d\big(\mathcal{N}F_j(x), \mathcal{N}F_j(y)\big) \leq \sum_{p=1}^{n} \Lambda_{pj} d(x_p, y_p) + \sum_{(p,j,M,M) \in \mathcal{I}(F)} (\Lambda_{pj}^{MM})_{pj}^{M,M+1} d(x_{pj}^{MM}, y_{pj}^{MM}).$$

As substituting x_i for $x_i^{k-m} \in \mathcal{V}_j(F)$ in F_j substitutes x_i for x_{ij}^{mm} and y_i for y_{ij}^{mm} in the previous inequality then for $\tilde{x}, \tilde{y} \in X_{\mathcal{F}}$

$$\begin{aligned} d\big(\mathcal{N}\mathcal{F}_j(\tilde{x}), \mathcal{N}\mathcal{F}_j(\tilde{y})\big) \leq & \big(\Lambda_{ij} + (\Lambda_{ij}^{mm})_{ij}^{m,m+1}\big) d(\tilde{x}_i, \tilde{y}_i) + \\ & \sum_{p=1, p\neq i}^{n} \Lambda_{pj} d(\tilde{x}_p, \tilde{y}_p) + \sum_{(p,j,M,M) \in \mathcal{I}(\mathcal{F})} (\Lambda_{pj}^{MM})_{pj}^{M,M+1} d(\tilde{x}_{pj}^{MM}, \tilde{y}_{pj}^{MM}). \end{aligned}$$

verifying the claim.

Moreover, note that the graphs $\Gamma_{\mathcal{N}F}$ and $\Gamma_{\mathcal{N}\mathcal{F}}$ with adjacency matrices Λ and $\overline{\Delta}$ respectively satisfy the relation

$$\Gamma_{\mathcal{N}F} = (\Gamma_{\mathcal{N}\mathcal{F}})_{ij}^{m}(t, 1) \;\; \text{for} \;\; t = \frac{(\Lambda_{ij}^{mm})_{ij}^{m,m+1}}{\Lambda_{ij} + (\Lambda_{ij}^{mm})_{ij}^{m,m+1}}$$

Lemma 3.3.11 then implies that if $\rho(\Lambda) < 1$ then $\rho(\Delta) < 1$.

Note that if each variable of the form $x_i^{k-m} \in \mathcal{V}_j(F)$ is replaced by x_i in F then by repeated use of lemma 3.3.11 the resulting dynamical network $U_F(x) = F(x,\ldots,x)$ has the following property. The constants

$$\tilde{\Lambda}_{pq} = \Lambda_{pq} + \sum_{x_p^{k-M} \in \mathcal{V}_q(F)} (\Lambda_{pq}^{MM})_{ij}^{M,M+1} \quad \text{for} \quad 1 \leq p,q \leq n$$

satisfy equation (53) for U_F and moreover $\rho(\tilde{\Lambda}) < 1$. Hence, under the assumption that $\rho(\Lambda) < 1$ it follows that the undelayed network U_F has a globally attracting fixed point.

Conversely, suppose $\tilde{\Lambda}_{pq}$ are constants satisfying (53) for U_F such that $\rho(\tilde{\Lambda}) < 1$. Let $\mathcal{U}$ be the function U_F in which the x_i in mth slot of $(U_F)_j$ is replaced by x_i^{k-m}. Since each $\mathcal{N}\mathcal{U}_{ij}^{m\ell}(x_{ij}^{m,\ell-1}) = x_{ij}^{m,\ell-1}$ for all $1 \leq \ell \leq m$ then

$$d\big(\mathcal{N}\mathcal{U}_{ij}^{m\ell}(x_{ij}^{m,\ell-1}), \mathcal{N}\mathcal{U}_{ij}^{m\ell}(y_{ij}^{m,\ell-1})\big) = d\big(x_{ij}^{m,\ell-1}, y_{ij}^{m,\ell-1}\big) \quad \text{for} \quad x,y \in X_F.$$

Hence, the constants $(\Lambda_{ij}^{m,\ell-1})_{ij}^{m\ell} = 1$ satisfy (53) for $\mathcal{N}\mathcal{U}$ for all $1 \leq \ell \leq m$.

Then, by the previous calculations, the matrix of constants $\underline{\Delta}$ given by

$$\underline{\Delta}_{pg} = \begin{cases} \tilde{\Lambda}_{pg} & 1 \leq p,q \leq n, \quad p \neq i, \quad q \neq j \\ (1-t)\tilde{\Lambda}_{pg} & p = i, \quad q = j \end{cases},$$

$$\big(\underline{\Delta}_{ij}^{m,\ell-1}\big)_{ij}^{m\ell} = \begin{cases} 1 & 1 \leq \ell \leq m \\ t\tilde{\Lambda}_{ij} & \ell = m+1 \end{cases}, \quad \text{and zero otherwise}$$

satisfy (53) for $\mathcal{N}\mathcal{U}$. Again, the graphs $\Gamma_{\mathcal{N}\mathcal{U}}$ and Γ_{U_F} with adjacency matrices $\underline{\Delta}$ and $\tilde{\Lambda}$ respectively have the relation $\Gamma_{\mathcal{N}\mathcal{U}} = (\Gamma_{U_F})_{ij}^m(t,1)$. Therefore, under the assumption that $\rho(\tilde{\Lambda}) < 1$ it follows that $\rho(\underline{\Delta}) < 1$ via lemma 3.3.11.

Therefore, by sequentially replacing each x_i in $(U_F)_j$ with the appropriate variable $x_i^{k-m} \in \mathcal{V}_j(F)$ it follows that $\mathcal{N}F$ has a globally attracting fixed point if $\rho(\tilde{\Lambda}) < 1$. This completes the proof. $\square$

Combining the ideas present in this and the previous section it is possible to define a *dynamical network reduction* of a dynamical network $F : X \to X$. This is similar to defining a restriction of a dynamical network and is done as follows.

Definition 3.3.12. *Let $S \in st_0(\Gamma_F)$ for the dynamical network $F : X \to X$. For $j \in \mathcal{I}_S$ let $(\mathcal{R}_S F)_j$ be the function $F_j(\{x_{i_0}\})$ where each x_{i_0} is replaced by $F_{i_0}(\{x_{i_1}\})$, each x_{i_1} by $F_{i_1}(\{x_{i_2}\})$, and so on for all indices $i_\ell \notin \mathcal{I}_S$. We call the function*

$$\mathcal{R}_S F : X|_S \to X|_S$$

the reduction of the dynamical network F over S.

As an example, for the dynamical network H given in equation (54) the reduction of the dynamical network H over $S = \{v_1, v_2\}$ is given by

$$\mathcal{R}_S H(x_1, x_2) = \begin{bmatrix} H_1(x_2) \\ H_2\big(x_1, H_4(x_1, H_3(x_1))\big) \end{bmatrix}.$$

Theorem 3.3.13. *For $S \in st_0(\Gamma_F)$ suppose $\tilde{\Lambda}_{ij}$ are constants satisfying (53) for $\mathcal{R}_S F$. If $\rho(\tilde{\Lambda}) < 1$ then the dynamical network F has a globally attracting fixed point.*

Proof. Let $S \in st_0(\Gamma_F)$ for the dynamical network F. Lemma 3.2.2 then implies that

$$\mathcal{X}_S F|_S(x^{k-T}|_S, \ldots, x^k|_S) = \big(\mathcal{X}_S F(x^k)\big)|_S \tag{62}$$

for some $T \geq 0$. Hence, if the expansion $\mathcal{X}_S F$ has a globally attracting fixed point then the same holds for the restriction $\mathcal{X}_S F|_S$.

Conversely, suppose that $\mathcal{X}_S F|_S$ has a globally attracting fixed point. If this point is given by $(\tilde{x}|_S, \ldots \tilde{x}|_S) \in \bigoplus_{i=0}^{T} X|_S$ then by (62)

$$\lim_{k\to\infty} \big(\mathcal{X}_S F^k(y)\big)|_S = \tilde{x}|_S \text{ for any } y \in X.$$

Note that for any $\ell \in \mathcal{I}_{\bar{S}}$ that v_ℓ lies on a $\beta \in \mathcal{B}_{ij}(\Gamma_F, S)$. By construction then $\mathcal{X}_S F_\ell(x^{k+\tilde{T}}) = x_i^k$ for all $k \geq 0$ and some $\tilde{T} \leq T$. Hence, $\lim_{k\to\infty} \mathcal{X}_S F_\ell(y) = \tilde{x}_i$

implying $\mathcal{X}_S F$ has a globally attracting fixed point. Therefore, $\mathcal{X}_S F|_S$ has a globally attracting fixed point if and only if $\mathcal{X}_S F$ has a globally attracting fixed point.

As theorem 3.2.4 implies $\mathcal{X}_S F|_S = F|_S$ then by theorem 3.3.3 the network $\mathcal{N}(F|_S)$ has a globally attracting fixed point if and only if the expansion $\mathcal{X}_S F$ has a globally attracting fixed point. Suppose then that the constants $\tilde{\Lambda}_{ij}$ satisfy equation (53) for $\mathcal{R}_S F$ such that $\rho(\tilde{\Lambda}) < 1$. As $U_{F|_S} = \mathcal{R}_S F$, which can be seen by comparing definitions 3.2.1 and 3.3.12, then the second half of theorem 3.3.5 implies $\mathcal{N}(F|_S)$ has a globally attracting fixed point. This in turn implies that $\mathcal{X}_S F$ has a globally attracting fixed point.

Lastly, the argument in the proof of theorem 3.2.5 shows that if $\mathcal{X}_S F$ has a globally attracting fixed point the same holds for F. This completes the proof. □

Theorem 3.3.14. *Let $S \in st_0(\Gamma_F)$ and suppose Λ_{ij} are constants satisfying (53) for the expansion $\mathcal{X}_S F$. If $\rho(\Lambda) < 1$ then there exist constants $\tilde{\Lambda}_{ij}$ satisfying (53) for $\mathcal{R}_S F$ such that $\rho(\tilde{\Lambda}) < 1$.*

Proof. Let Λ_{ij} be constants satisfying (53) for $\mathcal{X}_S F$. By modifying the argument in the proof of theorem 3.3.5 it follows that the constants

$$\tilde{\Lambda}_{ij} = \Lambda_{ij} + \sum_{\beta \in \mathcal{B}_{ij}(\Gamma_{\mathcal{X}_S F}, S)} \Lambda_\beta$$

satisfy (53) for $U_{F|_S}$ where Λ_β denote the weights of the final edge on the path β. Moreover, from the same argument it follows that if $\rho(\Lambda) < 1$ then $\rho(\tilde{\Lambda}) < 1$.

As $U_{F|_S} = \mathcal{R}_S F$ the result follows. □

In the previous section, dynamical network expansion were introduced to give improved estimates of a dynamical network's stability. Theorem 3.3.13 together with theorem 3.3.14 imply that dynamical network reductions can be used for the same purpose. However, the main advantage to using a reduction rather than an expansion is that $\mathcal{R}_S F$ is much easier to construct then $\mathcal{X}_S F$ making it easier to find $\rho(\tilde{\Lambda})$ verses $\rho(\Lambda)$.

As an illustration let $H(x)$ be the dynamical network given in example 1 of section 3.2 where again $S = \{v_1, v_2\}$. Here, it can be computed that

$$\mathcal{R}_S H(x_1, x_2) = \begin{bmatrix} \frac{1}{4}\mathcal{L}(x_2) \\ \frac{1}{4}\mathcal{Q}(x_1) + \frac{1}{4}\mathcal{L}\big(\frac{1}{4}\mathcal{Q}(x_1) + \frac{1}{4}\mathcal{Q}(\frac{1}{4}\mathcal{Q}(x_1))\big) \end{bmatrix}$$

having the associated matrix of constants $\tilde{\Lambda} = \begin{bmatrix} 0 & 3/4 \\ 1 & 0 \end{bmatrix}$ satisfying equation (53).

The constants $\tilde{\Lambda}_{ij} = \max |(D\mathcal{R}_S F)_{ji}(x)|$ from which it follows that $\rho(\tilde{\Lambda}) = \frac{\sqrt{3}}{2} < 1$. Theorem 3.3.14 then implies that H does in fact have a globally attracting fixed point.

Note that as with dynamical network expansions it is possible to sequentially reduce a network to gain better stability estimates. However, if Λ_{ij} are constants satisfying (53) for the expansion $\mathcal{X}_S F$ and $\rho(\Lambda) < 1$ then using $\mathcal{R}_S F$ to estimate the stability of F may give worse estimates than simply using F itself.

For example let $K(x_1, x_2) = \begin{bmatrix} \mathcal{Q}(x_2) \\ \mathcal{L}(x_1) \end{bmatrix}$ in which case the matrix of constants $\Delta = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}$ satisfy (53) for K. Hence, $\rho(\Delta) = \sqrt{8} \approx 2.828$.

For $S = \{v_1\}$ the expansion $\mathcal{X}_S K(x_1, x_2) = \begin{bmatrix} \mathcal{Q}(\mathcal{L}(x_2)) \\ x_1 \end{bmatrix}$ where the matrix of constants $\Lambda = \begin{bmatrix} 0 & 3.079 \\ 1 & 0 \end{bmatrix}$ satisfy (53) for $\mathcal{X}_S K$. Note that here $\rho(\Lambda) = 1.754$.

However, the reduced dynamical network $\mathcal{R}_S K(x_1) = \mathcal{Q}(\mathcal{L}(x_1))$ in which case the best we can do is $\rho(\tilde{\Lambda}) = 3.079$.

3.4 Concluding Remarks

The major goal of this chapter has been to use the theory of isospectral graph transformations developed in chapter I to investigate dynamical networks. This has additionally lead to investigations of time-delayed dynamical systems specifically the effect of such delays on the system's dynamics.

Of particular interest in this section is the fact that to both the graph reductions and graph expansions of chapter I there are analogous dynamical network reductions and expansions. Moreover, both network transformations are useful in obtaining improved estimates of the dynamical stability of any given network. In this way both types of transformations can be seen as a new tool for investigating interplay between the dynamics of a network and the network's structure.

Additionally, as time delays naturally arise in such dynamical network transformations, this provides a framework for investigating the strictly larger class of systems with time delays. As it is shown in the second half of this chapter time delays do not effect the dynamic stability of a system. That is, the addition or removal of delays to a system does not effect whether the system has a globally attracting fixed point.

CHAPTER IV

EVENTUALLY NEGATIVE SCHWARZIAN SYSTEMS

4.1 Introduction

Singer was the first to observe that if a function has a negative Schwarzian derivative then this property is preserved under iteration and moreover that this property puts restrictions on the type and number of periodic orbits the function can have [45]. These properties, as well as those results derived from them, essentially rely on the global structure functions with a negative Schwarzian derivative have. For a full list of such properties see [38].

Later it was found that such functions also possess local properties useful in establishing certain distortion bounds. For the most part these properties are concerned with the way in which functions with a negative Schwarzian derivative increase cross ratios. Because of these special properties some results are known only in the case where a function has a negative Schwarzian derivative. An exception, however, to this is the result by Kozlovski [33] where it was shown that the assumption of a negative Schwarzian derivative is superfluous in the case of any C^3 unimodal map with nonflat critical point. For such functions there is always some interval around their critical value on which the first return map has a negative Schwarzian derivative. That is locally all such C^3 maps behave as maps with a negative Schwarzian derivative. More recently using the same technique van Strien and Vargas have generalized this result to multimodal functions [47]. Also Graczyk, Sands and Swiatek have shown that any C^3 unimodal map with only repelling periodic points is analytically conjugate to a map with a negative Schwarzian derivative [26]. The main purpose of these results is to relate functions without a negative Schwarzian derivative to functions with this property.

In this chapter we consider those C^2 functions on a finite interval of the real line having some iterate with a negative Schwarzian derivative. This class, which we call functions with an *eventual negative Schwarzian derivative*, was originally introduced by L. Bunimovich (2007) in an attempt to describe some one-dimensional maps which appear in neuroscience [36, 37]. It is noteworthy that this class of functions is broader than those previously considered in the study of unimodal and multimodal maps related to functions with a negative Schwarzian derivative (see for example [33, 47, 26]). To demonstrate this we will present examples of such functions and further examples can also be found in [35]. Moreover, as having an eventual negative Schwarzian derivative is not an asymptotic condition, verifying whether a function has this property can often be done by direct computation. Hence, the concept of an eventual negative Schwarzian derivative has potential to be very useful in applications.

Since this class of functions contains functions with a negative Schwarzian derivative as a subset we do not attempt to prove stronger results then have already been proved for this smaller class of functions. With this in mind the large majority of the results we will present in this chapter will simply be restatements of those already known with the modification that only some iterate of our function need have a negative Schwarzian derivative. We note that this by no means is meant to be an exhaustive list of such results as the purpose of this chapter is to give further evidence that the useful properties possessed by functions with negative Schwarzian derivative are not limited to this family of functions.

This chapter is organized as follows. In the next section we formally introduce the class of functions we are considering and present our main results. Section 3.3 presents the proof of those results that are topological in nature. Specifically, we prove an analogue of Singer's theorem in [45] and mention some important concepts and corollaries that will be useful in what follows. Section 3.4 is comprised of those proofs which are more measure theoretic. Specifically, we generalize the main results in [13, 41, 49] to this larger class of functions. In section 3.5 we give a partial characterization of functions that have an eventual negative Schwarzian derivative as well as some examples. The next section is devoted to an application of these results to a one-parameter family of maps that model the electrical activity in a neuronal cell near the transition to bursting [36, 37]. Section 3.7 contains some concluding remarks.

4.2 Iterates and the Schwarzian Derivative

Here we consider only C^2 functions $f : I \to I$ on some nontrivial compact interval I of real numbers having a nonempty but finite set of critical points $\mathcal{C}(f)$ and use the notation f' to denote the derivative with respect to the spacial variable.

Definition 4.2.1. *A C^2 function $f : I \to I$ is said to have a negative Schwarzian derivative if on any open interval $J \subset I$, not containing critical points of f, $|f'|^{-1/2}$ is strictly convex on J.*

If $f : I \to I$ is C^3 then its *Schwarzian derivative* is defined off of $\mathcal{C}(f)$ by

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\Big(\frac{f''(x)}{f'(x)}\Big)^2 \tag{63}$$

If $S(f)(x)$ is strictly negative on open intervals not containing critical points of f then $|f'(x)|^{-1/2}$ is strictly convex on these sets or f has a negative Schwarzian derivative. However, the converse does not always hold.

Definition 4.2.2. *We say a C^2 function $f : I \to I$ has an eventual negative Schwarzian derivative if there exists $k \in \mathbb{N}$ such that f^k has a negative Schwarzian derivative. The smallest such number k is said to be the order of the derivative.*

Definition 4.2.3. *A C^2 map $f : I \to I$ is called S-multimodal if:*

(i) $\mathcal{C}(f)$ is nonempty and finite.

(ii) For every $c \in \mathcal{C}(f)$, c is nonflat. That is, there exists some $\ell \in (1, \infty)$ and $L \in (0, \infty)$ such that

$$\lim_{x \to c} \frac{|f'(x)|}{|x-c|^{\ell-1}} = L, \tag{64}$$

where ℓ is the order of the critical point.

(iii) f has a negative Schwarzian derivative on I.

If f has a single critical point c we call the map S-unimodal.

Definition 4.2.4. *A function $f : I \to I$ is S^k-multimodal if there exists a smallest $k \in \mathbb{N}$ such that $f^k(x)$ is S-multimodal. If f is unimodal and has this property we say it is S^k-unimodal*

Note that by the terminology above S-unimodal functions are special S-multimodal functions and S^k-unimodal functions are special S^k-multimodal functions.

In what follows we will make use of the following standard terminology. The *basin* of a periodic point x is the set of points that converge to the orbit of x, and x is said to be *attracting* if its basin contains an open set. The *immediate basin* of x is the union of connected components of its basin that contain a point of the orbit of x. Furthermore, we say that the periodic point x of order p is a *hyperbolic attractor* if $|(f^p)'(x)| < 1$, a *hyperbolic repeller* if $|(f^p)'(x)| > 1$, and *neutral* if $|(f^p)'(x)| = 1$. Note that it is possible for neutral periodic points to be attracting from one or both sides. We now state the main results of this chapter.

Theorem 4.2.5. *If $f : I \to I$ is S^k-multimodal, then the immediate basin of attraction of any attracting periodic point contains either a critical point of f or a boundary point of I. Furthermore, any neutral periodic point of f, except possibly on the boundary of I, is attracting and there exist no interval of periodic points.*

This type of theorem, often called Singer's theorem as it resembles the result in [45], is proved in [38] under the assumption that f is C^3 with $S(f)(x) < 0$. In [18] theorem 4.2.5 is proved in the case that f is S^1-multimodal.

We note here that theorem 4.2.5 indicates a few of the properties that sets this collection of functions apart from those considered elsewhere. In fact the property of having an eventual negative Schwarzian derivative cannot be generalized by looking at first return maps in the sense that it is global in nature and as first return maps generically introduce discontinuities, i.e. more boundary points, this global structure is not preserved.

The next result is not a generalization of a known result rather it is a corollary to the main result in [13] using the restriction on the periodic orbits obtained in the previous theorem to simplify the hypothesis.

Denote $D_n(c) = |(f^n)'(f(c))|$. Also, for some measure μ let φ and ψ be bounded Hölder continuous functions on I and denote the n^{th} *correlation function* by

$$C_n = C_n(\varphi, \psi) = \left| \int (\varphi \circ f^n)\psi d\mu - \int \varphi d\mu \int \psi d\mu \right|.$$

Furthermore, if for a multimodal $f : I \to I$ there is a closed proper subinterval J of I and an $n \geq 2$ such that

(i) the interiors of $J, \ldots, f^{n-1}(J)$ are disjoint,

(ii) $f^n(J) \subset J$, $f^n(\partial J) \subset \partial J$,

(iii) at least one of the intervals $J, \ldots, f^{n-1}(J)$ contains a point of $\mathcal{C}(f)$,

then f is called *renormalizable* on any set $\tilde{J}$ where $J \subseteq \tilde{J} \subseteq I$.

Theorem 4.2.6. *Let $f : I \to I$ be C^3 with an eventual negative Schwarzian derivative, and a finite nonempty critical set $\mathcal{C}(f)$. If every point of $\mathcal{C}(f)$ has order $\ell \in (1, \infty)$ and f satisfies*

$$\sum_n D_n^{-1/(2\ell-1)}(c) < \infty \text{ for all } c \in \mathcal{C}(f) \tag{65}$$

then there exists an f-invariant probability measure μ with support supp(μ) absolutely continuous with respect to Lebesgue measure (an "acip"). Furthermore, if f is not renormalizable on supp(μ) then (supp(μ), μ, f) is mixing and C_n decays at the following rates:

Polynomial: If there is $C > 0$, $\alpha > 2\ell - 1$ such that $D_n(c) \geq Cn^{\alpha}$ for all $c \in \mathcal{C}(f)$ and $n \geq 1$, then for any $\tilde{\alpha} < \frac{\alpha-1}{\ell-1} - 1$ we have $C_n = \mathcal{O}(n^{-\tilde{\alpha}})$.

Exponential: If there is $C > 0$, $\beta > 0$ such that $D_n(c) \geq Ce^{\beta n}$ for all $c \in \mathcal{C}(f)$ and $n \geq 1$, then there is a $\tilde{\beta} > 0$ such that $C_n = \mathcal{O}(e^{-\tilde{\beta}})$.

In the original result given in [13] the function f was assumed to have no attracting or neutral periodic points. In effect, the previous theorem says that instead of requiring the function to have no such periodic orbits we may assume that it has an eventual negative Schwarzian derivative. The advantage here is that having an eventual negative Schwarzian derivative is a nonasymptotic condition so it is potentially easier to verify a function has this property than to show it has no attracting or neutral periodic points by some other means.

However, if all that is needed is the existence of an acip, we have different orders of critical points, or the function is C^2 but not C^3 then we may use the following result which generalizes the main result in [41] to functions with an eventual negative Schwarzian derivative.

Theorem 4.2.7. *If f is an S^k-multimodal function and satisfies the condition*

$$\sum_n D_n^{-1/(\ell_{max})}(c) < \infty \text{ for all } c \in \mathcal{C}(f) \qquad (66)$$

where ℓ_{max} is the largest order of the critical points in $\mathcal{C}(f)$ then f admits an absolutely continuous invariant probability measure.

The next theorem deals with one-parameter families of maps. In the theory presented in [49] there is a special class of functions denoted by $\mathcal{M}$ having strongly expansive properties. This set of functions is helpful in proving under what conditions one-parameter families of functions have absolutely continuous invariant measures for positive Lebesgue measure sets of parameters. Specifically, a technical but generically satisfied transversality condition in the parameter which will be denoted by (PT) is required for this to be the case. We refer the reader to the article for details.

Theorem 4.2.8. *Let $f_a : I \to I$ be a one-parameter family of C^3 functions where a belongs to some interval A of the real line. If, for some parameter value $\beta \in A$, f_β has a finite nonempty critical set $\mathcal{C}(f_\beta)$ and*

(i) f_β has an eventual negative Schwarzian derivative of order k

(ii) $f''_\beta(c) \neq 0$, $c \in \mathcal{C}(f_\beta)$

(iii) if $f^m_\beta(x) = x$, then $|(f^m_\beta)'(x)| > 1$

(iv) $\inf_{i>0} d(f^i_\beta(c), \mathcal{C}(f_\beta)) > 0$, $c \in \mathcal{C}(f_\beta)$

then f_β has an absolutely continuous invariant measure. In particular $f^k_\beta \in \mathcal{M}$ and if f^k_β satisfies the condition (PT) then on a positive Lebesgue measure set of parameters the family of functions f_a has an absolutely continuous invariant probability measure.

Theorem 4.2.8 is an extension of the results in [49] with the modification that we require only an eventual negative Schwarzian derivative of order $k \geq 1$ instead of a negative Schwarzian derivative.

4.3 Topological Properties

In this section we prove theorem 4.2.5 along with some corollaries that will be needed in the following sections but are of interest in their own right. We now give a proof of theorem 4.2.5.

Proof. Let $f : I \to I$ be an S^k-multimodal function of order $k \geq 1$. From the proof of Singer's theorem in [18] the results of theorem 4.2.5 holds for f^k. Hence, if $\tilde{x} \in I$ is an attracting periodic point of f^k then in its immediate basin of attraction $B(\tilde{x})$ there is either an endpoint of I or a critical point $\tilde{c}$ of f^k. Since any point that is attracted to the orbit of $\tilde{x}$ under f^k is also attracted to the orbit of $\tilde{x}$ under f the immediate basin of attraction of $\tilde{x}$ under f contains $B(\tilde{x})$ hence either an endpoint of I or $\tilde{c}$. As the critical set of f^k is given by

$$\mathcal{C}(f^k) = \{x \in I : \exists \ 0 \leq i \leq k-1 \text{ where } f^i(x) \in \mathcal{C}(f)\}$$

then $\tilde{c}$ is the preimage of some critical point of f and the first and second statement of theorem 4.2.5 follows from the fact that f has an attracting periodic orbit containing $\tilde{x}$ if and only if the same is true of f^k. Since this is true for general periodic points the result follows. □

Definition 4.3.1. *Let $f : I \to I$ be a C^1 map with nonempty critical set $\mathcal{C}(f)$. Let*

$$a = \inf_{n \geq 0}\{f^n(c) : \ c \in \mathcal{C}(f)\}, b = \sup_{n \geq 0}\{f^n(c) : c \in \mathcal{C}(f)\}.$$

We call the interval $[a, b]$ the critical interval of the function f. That is, the smallest closed interval that contains the forward orbit of all critical points of f.

By a simple argument it follows that if f has a critical interval $\tilde{I}$ then $f(\tilde{I}) \subseteq \tilde{I}$. A more complicated result is the following.

Lemma 4.3.2. *Let $f : I \to I$ be C^1 with a nonempty critical set. Then the endpoints of the critical interval of f are either attracting periodic points of period 1 or 2 that attract some critical point of f or else lie on the orbit of a critical point of f.*

Proof. For simplicity let $I = [0, 1]$ which can always be achieved by some affine change of coordinates. Let $I = L \cup C \cup R$ where $C = [c_l, c_r]$ is the smallest interval containing $\mathcal{C}(f)$, $L = [0, c_l]$, and $R = [c_r, 1]$. Also c_{max} and c_{min} are the critical points of f with largest and smallest function values respectively. For the critical interval $[a, b]$ suppose in the following cases that b is not on the orbit of any critical point.

Case 1: Let f be increasing on L and R. If $f^2(c_{max}) \leq f(c_{max})$ then it follows that $f([0, f(c_{max})]) \subseteq [0, f(c_{max})]$ so either $b = f(c_{max})$ or $b = c_r$, which violates the supposition. As it follows then that $f^2(c_{max}) > f(c_{max})$ then $c_r < f(c_{max})$ implying that f is strictly increasing on $[f^2(c_{max}), 1]$. Hence, by monotonicity any orbit containing a point in $[f^2(c_{max}), 1]$ is attracted to a fixed point of this interval. The least of these fixed points p must be b since c_{max} is attracted to it and as $f([0,p]) \subseteq [0,p]$.

Case 2: Let f be decreasing on L and R. Consider f^2 where we define L^2, C^2, R^2, c^2_{max} analogous to L, C, R and c_{max} for f^2. Note that $f^2(c^2_{max}) = f(c_{max}) < b$. Also if f^2 is decreasing on L^2 this implies that $f(L) \subseteq [c_l, c_r]$ and f^2 decreasing on R^2 implies that $f(R) \subseteq [c_l, c_r]$. If either of these is the case then either $f(c_{max})$ or $f^2(c_{min})$ is equal to b. As neither of these is possible then f^2 is increasing on both L^2 and R^2 and the analysis reduces to that of Case 1. Therefore, c^2_{max} is attracted to b which implies the critical point c_{max} of f is attracted to a two cycle of f containing b.

Case 3: Let f be decreasing on L and increasing on R. If $f^2(c_{max}) \leq f(c_{max})$ and $f^2(c_{min}) \leq f(c_{max})$ then $f([f(c_{min}), f(c_{max})]) \subseteq [f(c_{min}), f(c_{max})]$ implying b lies on the orbit of c_{max} or c_r, which violates the supposition. If $f^2(c_{max}) > f(c_{max})$ then as in Case 1 either c_{min} or c_{max} is attracted to a fixed point in the interval $[f^2(c_{max}, 1)]$ which must be b. If $f^2(c_{min}) > f(c_{max})$ but $f^2(c_{max}) \leq f(c_{max})$ then $f^2(c_{min}) = b$.

Case 4: Let f be increasing on L and decreasing on R. This however implies that $b = f(c_{max})$.

Repeating this argument with appropriate modifications implies the same is true for the endpoint a. □

Corollary 7. *An S^k-multimodal map f can have at most $|\mathcal{C}(f)|+2$ attracting periodic orbits. If no critical point of f is attracted to a periodic orbit then all periodic points in the critical interval are hyperbolic repelling.*

Proof. The first statement is immediate from theorem 4.2.5. To prove the second note that lemma 4.3.2 implies f^k restricted to its critical interval $\tilde{I}$ has the property that each of its attracting periodic orbits attracts a critical point. Hence, if no critical point of f^k is attracted to a periodic orbit, f^k can have no attracting periodic points in its critical interval, in particular no hyperbolic attractors as well as no neutral periodic points in the interior of $\tilde{I}$. If an endpoint of $\tilde{I}$ is a neutral periodic point not on the orbit of a critical point it attracts an open set of points in the critical interval and the corollary follows for f^k. Hence, all periodic orbits of f in $\tilde{I}$ are hyperbolic repelling and as the critical interval of f is contained in $\tilde{I}$ the proof follows. $\square$

4.4 *Measure Theoretic Properties*

In this section we prove theorems 4.2.6, 4.2.7, and 4.2.8, that is those results that are more measure theoretic in nature. We also give some related corollaries that will be used in section 3.6 in our discussion of neuronal models. We now give the proof of theorem 4.2.6.

Proof. Assuming condition (65) on f implies the critical interval of f is not a single point as a critical point of f cannot be a fixed point of the function. Let $\tilde{f}$ be the restriction of f to its critical interval and suppose $\tilde{x}$ is either a hyperbolic attracting or neutral fixed point of $\tilde{f}$. Then theorem 4.2.5 together with lemma 4.3.2 imply that some critical point $\tilde{c} \in \mathcal{C}(\tilde{f})$ is attracted to $\tilde{x}$. In both the hyperbolic attracting and neutral case there is an N and a closed interval $J \ni \tilde{x}$ on which $|(\tilde{f})'(x)| \leq 1$ containing $\tilde{f}^n(\tilde{c})$ for $n \geq N$. Hence, there is a finite $C \geq 0$ such that for $n \geq N$, $D_n(\tilde{c}) \leq C$ implying

$$\sum_n D_n^{-1/(2\ell-1)}(\tilde{c}) \geq \sum_{n \geq N} C^{-1} = \infty.$$

As this violates (65) it follows that every fixed point of $\tilde{f}$ is hyperbolic repelling and a slight modification of this argument implies the same for all periodic orbits of $\tilde{f}$ as well. The main result of [13] then implies the result of the theorem for $\tilde{f}$. This result can then be trivially extended to f. $\square$

For the proof of theorem 4.2.7 we require the following lemma.

Lemma 4.4.1. *Let f be C^1 with a finite critical set. If for some $k \in \mathbb{N}$ there is an f^k-invariant probability measure μ absolutely continuous with respect to Lebesgue measure (acip) then f admits an invariant measure also absolutely continuous with respect to Lebesgue measure.*

Proof. Suppose f, k, and μ are as above. Consider the measure υ given by

$$\upsilon(A) = \frac{1}{k} \sum_{i=0}^{k-1} \mu(f^{-i}(A)).$$

To see that υ is f-invariant note

$$\upsilon(f^{-1}(A)) = \frac{1}{k} \sum_{i=1}^{k-1} \mu(f^{-i}(A)) + \frac{\mu(f^{-k}(A))}{k} = \frac{1}{k} \sum_{i=1}^{k-1} \mu(f^{-i}(A)) + \frac{\mu(A)}{k} = \upsilon(A).$$

For absolute continuity of the measure note that a C^1 function with a finite critical set is non-singular. Therefore, absolute continuity of the measure υ follows from that of μ. □

We now give the proof of theorem 4.2.7

Proof. For any $\tilde{c} \in \mathcal{C}(f^k)$ only one of $\tilde{c}, f(\tilde{c}), f^2(\tilde{c}), \ldots, f^{k-1}(\tilde{c})$ can be a critical point of f since if f eventually maps any point of $\mathcal{C}(f)$ back to this set then condition (66) does not hold. Hence, for any $\tilde{c} \in \mathcal{C}(f^k)$ there exists a unique $m < k$ such that $f^m(\tilde{c}) = c$ where $c \in \mathcal{C}(f)$. Consider

$$\lim_{x \to \tilde{c}} \frac{|(f^k)'(x)|}{|x - \tilde{c}|^{\ell-1}} = \prod_{i=0, i \neq m}^{k-1} |f'(f^i(\tilde{c}))| \lim_{x \to \tilde{c}} \frac{|f'(f^m(x))|}{|x - \tilde{c}|^{\ell-1}}. \tag{67}$$

Let $A = \prod_{i=0, i \neq m}^{k-1} |f'(f^i(\tilde{c}))|$ which is strictly positive and note that

$$\lim_{x \to \tilde{c}} \frac{|f'(f^m(x))|}{|f^m(x) - f^m(\tilde{c})|^{\ell-1}} = \lim_{x \to f^m(\tilde{c})} \frac{|f'(x)|}{|x - f^m(\tilde{c})|^{\ell-1}} > 0$$

where the inequality follows from definition 2.3(ii). Setting this limit to B, the right side of equation (67) is

$$A \lim_{x \to \tilde{c}} \frac{|f'(f^m(x))|}{|f^m(x) - f^m(\tilde{c})|^{\ell-1}} \frac{|f^m(x) - f^m(\tilde{c})|^{\ell-1}}{|x - \tilde{c}|^{\ell-1}} = AB \lim_{x \to \tilde{c}} \Big(\frac{|f^m(x) - f^m(\tilde{c})|}{|x - \tilde{c}|} \Big)^{\ell-1}.$$

An application of L'Hospital's rule implies that this limit is strictly positive. That is, the collection of orders of $\mathcal{C}(f)$ is the same as those for $\mathcal{C}(f^k)$ implying their maximum ℓ_{max} is equal. In what follows let

$$\begin{aligned} D_{n,k}(\tilde{c}) &= |((f^k)^n)'(f^k(\tilde{c}))|, \\ C_{m,k}(\tilde{c}) &= \prod_{i=1}^{k-m-1} |f'(f^i(\tilde{c}))| \end{aligned}$$

and note by repeated use of the chain rule that

$$C_{m,k}(\tilde{c})D_{n,k}(\tilde{c}) = \prod_{i=1}^{nk-m-1} |f'(f^i(c))| = D_{nk-m-1}(c), \ n \geq 1.$$

The important observation here is $C_{m,k}(\tilde{c}) \neq 0$ and does not depend on n. Since all the quantities involved are positive it follows that

$$C_{m,k}^{-\ell_{max}}(\tilde{c}) \sum_n D_{n,k}^{-\ell_{max}}(\tilde{c}) \leq \sum_n D_n^{-\ell_{max}}(c) < \infty$$

or f^k satisfies condition (66). This implies f^k has an acip via the main result in [41] and lemma 4.4.1 implies the same for f. $\square$

We now proceed to the proof of theorem 4.2.8.

Proof. As each point in $\mathcal{C}(f_\beta^k)$ is the preimage of some critical point c of f_β then $\inf_{i>0} d((f_\beta^k)^i(c), \mathcal{C}(f_\beta^k)) > 0$ for all $c \in \mathcal{C}(f_\beta^k)$ as this would otherwise violate condition (iv) on f. Second, for $c \in \mathcal{C}(f_\beta)$ note that $(f_\beta)''(c) \neq 0$ implies c is nonflat with order $\ell = 2$. As $\inf_{i>0} d((f_\beta^k)^i(c), \mathcal{C}(f_\beta^k)) > 0$ for all $c \in \mathcal{C}(f_\beta^k)$ then the same argument used in the proof of theorem 4.2.7 implies that the critical points of f_β^k are also nonflat of order $\ell = 2$. Therefore, $(f_\beta^k)''(c) \neq 0$ for all $c \in \mathcal{C}(f_\beta^k)$. Also note that property (iii) of theorem 4.2.8 means $f_\beta(x)$ has no attracting or neutral periodic orbit. But this is true if and only if $f_\beta^k(x)$ has none itself.

From the assumption that f_β^k has a negative Schwarzian derivative it follows from [49] that $f_\beta^k \in \mathcal{M}$. In particular, f^k admits an invariant absolutely continuous measure

μ. And an application then of lemma 4.4.1 implies that f_β also has an acip. Moreover, if f_β^k satisfies (PT) then on a positive Lebesgue measure set of parameters the family of functions f_a^k has an acip and lemma 4.4.1 can again be used to show the same for f_a. □

As we will be concerned specifically with unimodal maps in section 3.6 we give the following corollary.

Corollary 8. *Let $f : I \to I$ be C^3 and S^k-unimodal with critical point c of order $\ell = 2$ to the left of which f is increasing and to the right of which is decreasing. If the orbit of c contains a repelling periodic orbit and f has no fixed points on the boundary of its critical interval or outside of its critical interval then $f^k \in \mathcal{M}$.*

Proof. Assuming these conditions then $f(c) > c$ since c is otherwise attracted to the rightmost hyperbolic attracting or neutral fixed point of f. Also if $f^2(c) \geq c$ the forward orbit of c is contained in the interval $[c, f(c)]$ so c is attracted to a fixed point or a periodic cycle neither of which can be repelling. Similarly, if $f^3(c) < f^2(c)$ then c is also attracted to a nonrepelling fixed point, this implies that the critical interval $\tilde{I}$ of f is $[f^2(c), f(c)]$.

Note that if f has no fixed points on the boundary or outside $\tilde{I}$ then there is some $n \in \mathbb{N}$ such that for all $x \in I \setminus \tilde{I}$, $f_\beta^n(x) \in \tilde{I}$ since $\tilde{I}$ is forward invariant and all orbits fall on this set after some bounded number of iterations. Therefore, f has no periodic points outside $\tilde{I}$ and the proof of theorem 4.2.6 implies that on I, f satisfies condition (iii) of theorem 4.2.8 as c is mapped to an unstable periodic orbit. Condition (iv) follows for the same reason, (ii) follows from the proof of theorem 4.2.8 and condition (i) is assumed to hold. □

4.5 Characterizing S^k-multimodal Functions

As not every C^2 function has the characteristics given in theorem 4.2.5 it is not the case that every function will be either a function with a negative Schwarzian derivative or have an eventual negative Schwarzian derivative. That is, it is possible for a function to have a Schwarzian derivative that is mixed for all of its iterates. If however a function does have an eventual negative Schwarzian derivative it would be useful to have a way of identifying this. Specifically, we would like to have sufficient conditions under which a function has this property.

The simplest case to consider is the one in which $|f'|^{-1/2}$ is convex but not strictly convex. To do so we mention the following.

Definition 4.5.1. *Let $g : J \to K$ be continuous and monotone where $U \subset V \subset J$ are open bounded intervals in K. If $V \setminus U$ consists of the intervals L and R the cross ratio of the intervals U and V is given by*

$$CR(U, V) = \frac{|U||V|}{|L||R|}.$$

The function g is said to expand cross ratios on J if for any intervals $U \subset V$ in J, $CR(U, V) < CR(g(U), g(V))$.

Suppose f is C^2 on the open, bounded interval J containing no critical points of f. It is known that f increases cross ratios on J if and only if it has a negative Schwarzian derivative on this set (see [18]). Furthermore, if a function is a Möbius transform, that is a function of the form $g(x) = (ax + b)/(cx + d)$ where $ad - bc \neq 0$, then this function preserves cross ratios i.e. $CR(U, V) = CR(g(U), g(V))$ (see [38]).

Proposition 1. *Let $f : I \to I$ be C^2. Suppose $M \subset I$ is a finite union of closed intervals on which f is a Möbius transform and off of which f has a negative Schwarzian derivative. Furthermore, assume there is a $k \geq 2$ such that for every $x \in M$, $\{f^i(x) : 1 \leq i < k\} \cap I \setminus M \neq \emptyset$. Then f has an eventual negative Schwarzian derivative of order less than or equal to k.*

Proof. If B are the boundary points of M let $B^k = \{x \in I : f^i(x) \in B \text{ for some } 0 \leq i < k\}$. Let $J \subset I$ be an open interval containing no points in $\mathcal{C}(f^k)$ or B^k. Then there is a first $0 \leq \ell < k$ such that $f^{\ell}(J) \subset I \setminus M$. As the composition of a Möbius transform with itself is again such a function then $f^{\ell}(x)$ restricted to J does not change cross ratios. However, $f^{\ell}(J) \cap \mathcal{C}(f) = \emptyset$ and $f^{\ell}(J) \subseteq I \setminus M$ on which f has a negative Schwarzian derivative implying $f^{\ell+1}$ increases cross ratios on J. Since cross ratios on J are either maintained or increased by further iteration of f it follows that f^k also increases cross ratios on J or $|(f^k)'|^{-1/2}$ is strictly convex on J. Note that as $|(f^k)'|^{-1/2}$ is C^1 on J it then has a strictly increasing derivative on this interval. Since B^k is a finite set it follows that on any open interval of I, not containing critical points of f^k, $|(f^k)'|^{-1/2}$ is also strictly convex or f^k has a negative Schwarzian derivative on I. □

It follows directly from this proposition that the C^2 family of unimodal functions

$$f_a(x) = \begin{cases} (x - 1/2) + a, & x \in [0, 1/2] \\ -4(2a+1)(x-1/2)^3 + (x - 1/2) + a, & x \in (1/2, 1] \end{cases} \tag{68}$$

has an eventual negative Schwarzian derivative for $a \in (1/2, 7/8]$ (see Fig. 1). A C^3 example of a similar family is given by $g_a(x) = (ax - 1/4) + (ax - 1/4)^4 + 4a/11$ for $x \in [0, 1]$ and $a \in [1, 11/8]$. The reason for (68) is that it serves as a simple example of a family of functions for which many results, which refer only to C^3 functions, are not applicable. For example, as $f_a(x)$ is only C^2 it is not directly possible to use the main results in [26] to show this family is conjugate to a function with a negative Schwarzian derivative.

It is also worthwhile to recall, as mentioned in the introduction, the motivation as well as the inspiration for the study of this new class of functions with eventual negative Schwarzian derivatives comes from the analysis of one-dimensional maps which appear in some models in neuroscience. These maps, given in [36, 37] in particular, have a part that is linear (or almost linear). Therefore, our example above also includes this feature although, as implied by proposition 1, this is not a necessary

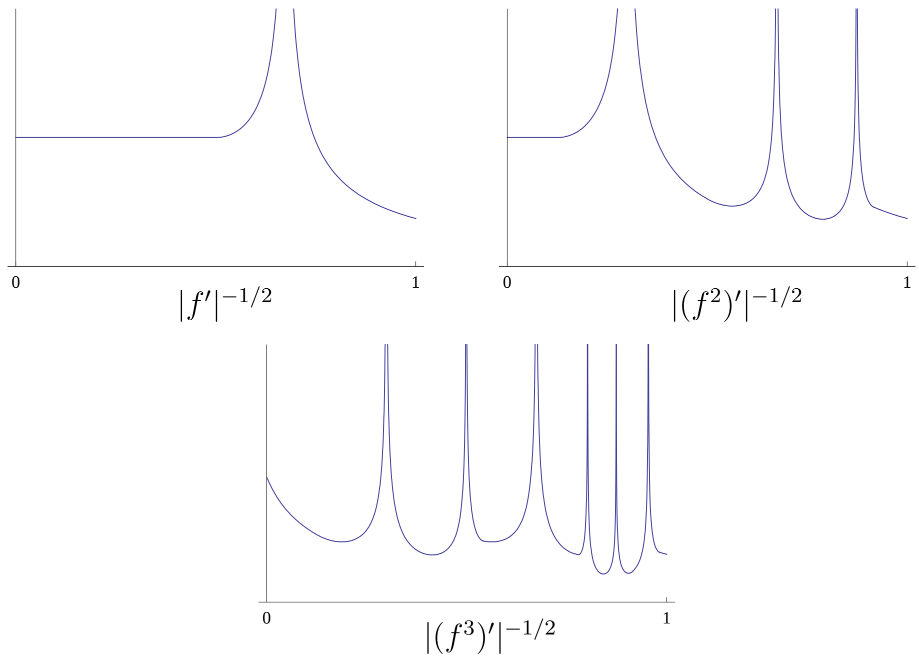


Figure 20: $f = f_{7/8}$ in equation (68)

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condition to have an eventual negative Schwarzian derivative.

From (63) it follows that if f is C^3 then

$$S(f^k)(x) = \sum_{i=0}^{k-1} \Big((f^i)'(x) \Big)^2 \Big(S(f)(f^i(x)) \Big). \tag{69}$$

Note that if in the previous proposition the function was assumed C^3 with $S(f)(x) < 0$ on $I \setminus M$, this equation would have immediately implied the result. However, this equation suggests a method for identifying C^3 functions which have an eventual negative Schwarzian derivative.

If $I = \bigcup_{i=1}^n I_i$, where the I_i are nonintersecting intervals, let $A = (a_{ij})$ be the *transition matrix* of this *partition* with respect to f. That is, $a_{ij} = 1$ if there is an $x \in I_i$ such that $f(x) \in I_j$ and $a_{ij} = 0$ otherwise. Let a sequence $\bar{x} = (x_0 x_1 x_2 \ldots x_{k-1})$ of length k be *admissible* with respect to this partition if each $x_i \in \{1, 2, \ldots, n\}$ and x_j can follow x_i in this sequence if and only if $a_{x_i x_j} = 1$. Let $T(i) = \sup_{x \in I_i} \{S(f)(x)\}$, $m(i) = \inf_{x \in I_i} \{|f'(x)|^2\}$, $M(i) = \sup_{x \in I_i} \{|f'(x)|^2\}$ and for some admissible $\bar{x}$ of length

k and $0 < j < k$ define

$$R_j(\bar{x}) = \begin{cases} \left[\prod_{i=0}^{j-1} m(x_i)\right] T(x_j), & T(x_j) \leq 0 \\ \left[\prod_{i=0}^{j-1} M(x_i)\right] T(x_j), & \text{otherwise} \end{cases}$$

letting $R_0(\bar{x}) = T(x_0)$. From equation (69) we have the following proposition.

Proposition 2. *Let $f : I \to I$ be C^3 and $I = \bigcup_{i=1}^{n} I_i$ a partition of I. If there is a $k \geq 1$ such that for every admissible sequence $\bar{x}$ of length k,*

$$\sum_{i=0}^{k-1} R_i(\bar{x}) < 0$$

then f has an eventual negative Schwarzian derivative of order less than or equal to k.

Using this proposition it can be shown that the one parameter family of functions

$$g_a(x) = 1 - a \tan\left(\frac{\pi}{4}x^2\right), \ x \in [-1, 1], \tag{70}$$

which mimics the logistic function, has an eventual negative Schwarzian derivative of order $k = 2$ in a parameter neighborhood of $a = 1.7$. This can be done by using the partition with endpoints $\{-1, -0.95, -0.7, -0.47, -0.18, 0.18, 0.47, 0.7, 0.95, 1\}$ for instance.

4.6 Application to a Neuronal Model

The motivation for considering functions having some iterate with a negative Schwarzian derivative comes from a model for the electrical activity in neural cells specifically in behavior described as bursting. This model given first in [36] and later in [37] is a reduction of a system of three nonlinear differential equations to a 1-d map.

The model is initially given by the following fast-slow system of three differential equations which describe the dynamics of the membrane potential υ and two gating variables η and ω of a neuronal cell:

$$\epsilon\dot{\upsilon} = f(\upsilon, \eta, \omega; \delta) \tag{71}$$

$$\dot{\eta} = g(\upsilon, \eta) \tag{72}$$

$$\dot{\omega} = \beta h(\eta, \omega) \tag{73}$$

Here the parameter δ can be viewed as a control parameter of the full system where $\delta \in [\delta_{min}, \delta_{max}]$. Also the time constant β represents the slowest time scale in the dynamics of (10)-(12) so in the limit $\beta \to 0^+$ the system uncouples into a fast subsystem (10), (11) and a slow subsystem (12). As is explained in [37] the trajectory of the full system is drawn towards a surface foliated by periodic orbits of the fast subsystem where the evolution along this surface is determined by the dynamics of the slow subsystem.

As the state of the fast subsystem depends on the value of the slow variable ω, it is sufficient to know how ω changes after each oscillation of the fast subsystem. Knowing these changes is precisely the reduction of the system to a 1-d map denoted by F_δ. To achieve this a Poincaré section P_δ is placed transversal to the surface of periodic orbits and the map is defined by $F_\delta(\omega_n) = \omega_{n+1}$ where ω_{n+1} is the ω-coordinate of the next point on the flow to pass through P_δ.

In [36] F_δ is shown to have the following properties (see Fig. 2):

I For fixed δ, $F_\delta(x)$ is a piecewise C^0 map with two intervals of continuity, $I_1 = I^- \cup I^0$ and $I_2 = I^+$, between which there is a single discontinuity $d(\delta)$.

II F_δ is unimodal on $0 \leq x < d(\delta) = I_1$, with critical point $c(\delta)$ of order $\ell = 2$.

III F_δ is nearly linear on its left outer region I^- with slope slightly less than 1 and in the limit $\beta \to 0^+$ is linear with slope 1.

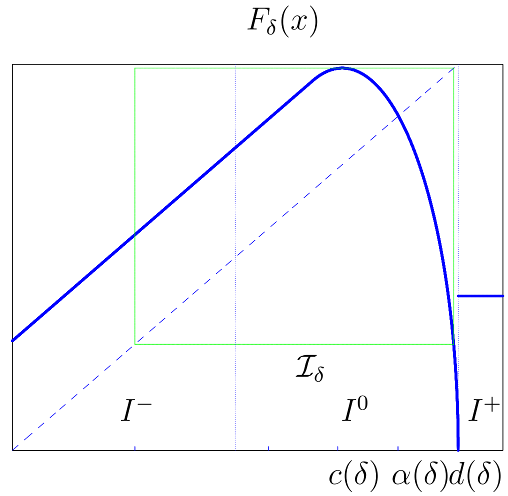

Figure 21: First Return Map Near Bursting $\delta \approx \delta_b$

IV On the inner region I^0, F_δ is unimodal with slope tending toward $-\infty$ as $d(\delta)$ is approached from the left.

V In the second region I^+ the function is nearly constant and between 0 and $F_\delta(c(\delta))$.

VI F_δ has a unique fixed point $\alpha(\delta)$ continuous in δ on the interval I^0.

VII There is a $\delta_0 \in (\delta_{min}, \delta_{max})$ where $\alpha(\delta_0) = c(\delta_0)$ and for $\delta \in (\delta_0, \delta_{max}]$, $c(\delta) < \alpha(\delta)$.

VIII There is a $\delta_b \in (\delta_0, \delta_{max})$ such that $F_{\delta_b}(c(\delta_b)) = d(\delta_b)$ and for $\delta \in [\delta_0, \delta_b)$, $F_\delta(c(\delta)) < d(\delta)$.

IX There is a $\delta_n > \delta_0$ such that for $\delta \in (\delta_n, \delta_b)$, $F'_\delta(\alpha(\delta)) < -1$.

Remark 5. *As the map F_δ depends on the placement of P_δ then locally F_δ also corresponds to this placement. This variability in the placement of P_δ is one of the main obstacles in verifying whether F_δ has an eventual negative Schwarzian derivative. However, at a global level the nearly linear part of F_δ on I^- and the fact that orbits* *leave this interval in a finite number of iterations suggests that if off I^- the function has a negative Schwarzian derivative then for β small enough F_δ will have an eventual negative Schwarzian derivative (see (68) for the linear case).*

Our aim in this section is to give sufficient conditions under which this one parameter family of functions has an acip for a positive Lebesgue measure set of its parameters. Also under what conditions these functions exhibit a mixing property with respect to these measures. These conditions will ultimately include the assumption that F_δ has an eventual negative Schwarzian derivative in order to illustrate the usefulness of this concept.

To simplify notation denote $F_\delta^n(c(\delta)) = c^n(\delta)$, F_δ restricted to its critical interval by $\tilde{F}_\delta$, and let $\mathcal{O} = \{(\delta, x) : \delta \in (\delta_0, \delta_b),\ x \in (0, c^1(\delta))\}$. In what follows we will often consider $F_\delta(x) = F(\delta, x)$ to be a function of two variables on $\mathcal{O}$. It will be the set $\mathcal{O}$ on which we will focus our attention. The reason being is that for parameter values $\delta \leq \delta_0$ the map F_δ has a global attracting fixed point and for parameters larger than δ_b the map F_δ is not continuous so the previous theory does not apply. Specifically, we make the observation that so long as $\delta \in (\delta_0, \delta_b)$ then $c^1(\delta) < d(\delta)$ implying the critical interval $\mathcal{I}_\delta$ of F_δ is either

(i) $[c(\delta), c^1(\delta)]$ if $c(\delta) \leq c^2(\delta)$ in which case $\tilde{F}_\delta$ is a diffeomorphism or

(ii) $[c^2(\delta), c^1(\delta)]$ if $c(\delta) > c^2(\delta)$ and $\tilde{F}_\delta$ is unimodal.

The following lemma is meant to establish basic continuity properties of the family of functions F_δ under variation of parameters.

Lemma 4.6.1. *Suppose $F_\delta(x) = F(\delta, x)$ is C^2 on $\mathcal{O}$ and there exist a closed interval J with endpoints $\delta_1, \delta_2 \in (\delta_0, \delta_b)$ such that for all $\delta \in J$, $c^3(\delta) < \alpha(\delta)$. If there is also an $m \in \mathbb{N}$ where $c^m(\delta_1) = c^1(\delta_1)$ and $c^m(\delta_2) = \alpha(\delta_2)$ then on an infinite set of parameters $\Delta \subset J$ the orbit of $c(\delta)$ contains $\alpha(\delta)$.*

Proof. If $\delta \in J$ then $c(\delta) > c^2(\delta)$ implying $\mathcal{I}_\delta = [c^2(\delta), c^1(\delta)]$ on which F_δ is unimodal. Also every point in $(\alpha(\delta), c^1(\delta))$ has exactly two preimages; one preimage in $l(\delta) =$

$(c^2(\delta), c(\delta))$ and the other in $r(\delta) = (c(\delta), \alpha(\delta))$. As every point in $l(\delta)$ and $r(\delta)$ has a unique preimage in $s(\delta) = (\alpha(\delta), c^1(\delta))$ it is possible to specify a preimage of $\alpha(\delta)$ by some finite sequence made up of l, r, and s which stand for whether the fixed point is reached by tracing its forward orbit through these sets in this manner. We say a finite sequence of l, r, and s is admissible if and only if an s separates every l or r, s does not follow itself, and the sequence ends with an l. This corresponds to the structure above and uniquely defines a preimage of $\alpha(\delta)$. Note we need not assign a symbol to $c(\delta)$, $c^1(\delta)$, $c^2(\delta) \in \mathcal{I}_\delta$ since no preimage of $\alpha(\delta)$ with an admissible sequence has an orbit containing these points for $\delta \in J$.

Under the assumption that $F_\delta(x) = F(\delta, x)$ is C^2 the Implicit Function Theorem applied on the open set $\{(\delta, x) : \delta \in (\delta_0, \delta_b),\ x \in (0, \alpha(\delta))\}$ guarantees that $c(\delta)$ is a C^1 function of $\delta \in (\delta_0, \delta_b)$ as the critical point of F_δ is to leading order quadratic. As this implies $\mathcal{O}$ is open, similar calculations imply the same is true for $\alpha(\delta)$, and $c^m(\delta)$ for all $m \geq 1$. Also important is that this is true for any preimage of $\alpha(\delta)$ having an admissible sequence. This follows as well from the Implicit Function Theorem using the fact that the orbit of these particular preimages cannot contain the critical point.

As there are infinitely many sequences which can end in either l, r, s then infinitely many admissible preimages of the fixed point $\alpha(\delta)$ are in each of $l(\delta)$, $r(\delta)$, and $s(\delta)$ and each vary continuously in δ. As the first letter in the sequence of a preimage determines whether it is in $l(\delta)$, $r(\delta)$, or $s(\delta)$ it must stay in that interval for all $\delta \in J$.

Continuity, specifically of the admissible preimages in $s(\delta)$, and the assumption that $c^m(\delta_1) = c^1(\delta_1)$ and $c^m(\delta_2) = \alpha(\delta_2)$ then imply the existence of the infinite set $\Delta \subseteq [\delta_1, \delta_2]$ on which the orbit of the critical point $c(\delta)$ contains the fixed point $\alpha(\delta)$. $\square$

Theorem 4.6.2. *Suppose $F_\delta(x) = F(\delta, x)$ is C^0 on $\mathcal{O}$ and assume that there are $\delta_1, \delta_2 \in (\delta_0, \delta_b)$ and $m \geq 3$ such that* *(i)* $c(\delta_1) \leq c^m(\delta_1)$*,*

(ii) $c^m(\delta_2) \leq c(\delta_2)$ *and* $c^3(\delta_2)$*,* $c^{m+1}(\delta_2) < \alpha(\delta_2)$*.*

Then there is an infinite set $\Delta \subset [\delta_1, \delta_2]$ *on which the orbit of* $c(\delta)$ *contains* $\alpha(\delta)$*. Furthermore, if for some* $\delta \in \Delta$

(iii) $\tilde{F}_\delta$ *has an eventual negative Schwarzian derivative of order* k *and*

(iv) $\delta > \delta_n$ *then the following is true:*

(A) $\tilde{F}_\delta$ *has an acip.*

(B) If $\tilde{F}_\delta$ *is* C^3 *in* x *then some iterate of* $\tilde{F}_\delta$ *is mixing with exponential decay of correlations.*

Proof. As in the proof of lemma 4.6.1 the assumption that F_δ is C^2 implies $\alpha(\delta)$, $c(\delta)$, and $c^i(\delta)$ for $i \geq 1$ all vary continuously in the parameter. Without loss in generality if $\delta_1 < \delta_2$ since both $c(\delta_1) \leq c^m(\delta_1)$ and $c^m(\delta_2) \leq c(\delta_2)$ then there is a largest $\delta_* \in [\delta_1, \delta_2]$ such that $c(\delta_*) = c^m(\delta_*)$. As this implies $c^1(\delta_*) = c^{m+1}(\delta_*)$ then $\delta_* \neq \delta_2$. Note that if at some $\beta \in [\delta_*, \delta_2]$, $c^3(\beta) \geq \alpha(\beta)$ the assumption $c^3(\delta_2) < \alpha(\delta_2)$ implies there is a $\gamma \in [\beta, \delta_2]$ at which $c^3(\gamma) = \alpha(\gamma)$ in turn implying $c^m(\gamma) = \alpha(\gamma) > c(\gamma)$ contradicting the maximality of δ_*. Lemma 4.6.1 therefore guarantees the existence of the set $\Delta \subset [\delta_*, \delta_2]$ as $c^{m+1}(\delta_*) = c^1(\delta_*)$ and $c^{m+1}(\delta_2) < \alpha(\delta_2)$.

If $\delta \in \Delta$ and condition *(iv)* holds then there is $C > 0$ such that $D_n(c(\delta)) = C|\tilde{F}_\delta'(\alpha(\delta))|^n$ for n large enough where $|\tilde{F}_\delta'(\alpha(\delta))| > 1$. Hence, inequality (66) of theorem 4.2.7 holds. This together with condition *(iii)* implies *(A)* via theorem 4.2.7.

For *(B)* if $\tilde{F}_\delta$ is C^3 in x then from the calculation of $D_n(c(\delta))$ above theorem 4.2.6 implies some iterate $\tilde{F}_\delta$ is mixing with exponential decay of correlations. □

Corollary 9. *For* $\delta \in (\delta_0, \delta_b)$ *let* $h_\delta(x) = (b_\delta - a_\delta)x + a_\delta$ *where* $[a_\delta, b_\delta]$ *is the critical interval of* F_δ*. Also denote by* $H_\delta : [0,1] \to [0,1]$ *the family of functions* $\tilde{F}_\delta$ *conjugated by* h_δ*. If* $\tilde{F}_\delta$ *satisfies conditions (i)-(iv) of theorem 4.6.2, is* C^3*, and for some* $\delta \in \Delta$*,* H_δ^k *has property (PT), then on a positive set of parameters* F_δ *has an acip.*

Proof. Note that the one parameter family of functions $H_\delta = h_\delta^{-1} \circ F_\delta \circ h_\delta$ when restricted to the interval $[0,1]$ is a linearly scaled version of the family F_δ restricted to their critical intervals. It follows by a simple calculation that $S(H_\delta^k)(x) = S(F_\delta^k)(h_\delta(x))$ or the property of having an eventual negative Schwarzian derivative is preserved under this change of coordinates. Furthermore, H_δ^k has a finite number of nonflat critical points and as the orbit of $c(\delta)$ contains $\alpha(\delta)$ corollary 8 implies that $H_\delta^k \in \mathcal{M}$. It follows directly from [49] that if H_δ^k satisfies property *(PT)* then on a positive set of parameters H_δ has an acip which implies the same for F_δ. $\square$

Numerically, as $\delta \to \delta_b$ the number of iterates the orbit of a point can stay on $\mathcal{I}_\delta \bigcap I^-$ increases from 1 near δ_0 to around 5 near δ_b (see Fig.3 in [37]). From this point of view conditions *(i)* and *(ii)* of the previous theorem are a natural way to ensure this family of maps have a parameter set Δ mentioned above. More than this, however, it allows for some latitude in selecting how close Δ is to δ_b which is useful since for parameters near δ_b, the map F_δ is more likely to have a repelling fixed point. Again, because of the variability in the placement of P_δ it would take some work to verify whether or not for a particular placement condition *(iii)* holds.

It should be noted that functions with the property that their critical points are mapped to repelling periodic orbits, often called *Misiurewicz maps*, are very rare both in a topological and metric sense [44]. From this the previous theorem seems to imply that mixing is a rare event for F_δ. However, the conditions of theorem 4.2.6 are generically satisfied on a larger set of parameters than just Δ. The problem which is generally encountered is in showing for a particular parameter value or set of values that the derivatives along the orbits of the critical points have the requisite growth. However, one reason to expect that (B) holds for a much larger set of parameters than just Δ is the very large negative slope of F_δ near $d(\delta)$, for $\delta \approx \delta_b$.

4.7 Concluding Remarks

The goal of this chapter was first and foremost to study a general class of functions that behave in significant ways like those with a negative Schwarzian derivative. It is surprising that this class of functions with an eventual negative Schwarzian derivative has not been previously considered as it naturally combines the nondynamic condition of having a negative Schwarzian derivative with the dynamics of the function. One result is that this enlarges the class of known functions with local and global properties similar to those of a function with a negative Schwarzian derivative. Moreover, as having an eventual negative Schwarzian derivative is not an asymptotic property, verification of this can often be done by direct computation. This last property makes functions with an eventual negative Schwarzian derivative a potentially useful tool in dealing with applications. Recall, that the introduction of this class was motivated by the maps arising in some models in neuroscience.

CHAPTER V

FUTURE WORK

In the preceding chapters we have considered principally two types of results; those related to *isospectral graph reductions* and those related to functions with an *eventual negative Schwarzian derivative*. For the latter it is possible that many other results known to hold for functions with a negative Schwarzian derivative could be extended to this larger class of functions. Following the pattern established in Chapter IV, the most natural way of establishing such results would likely involve proving that a certain property of f^k can in some way be pulled back to the original function f if $f \in S^k$-multimodal. We note that this strategy would almost certainly lead to further resultsin this direction.

However, the research that seems the most promising involves exploring applications and extentions of our theory developed in chapters I, II, and III. Natural extensions of isospectral transformations include transformations that preserve specific subsets of a graph's adjacency matrix as opposed to its entire spectrum. An important example currently being investigated are so called *isoradial graph transformations* that modify a graph while preserving its spectral radius (a subset of which are discussed in section 3.3). Such transformations have potential to lead to more general mechanisms than those considered in chapter III for investigating the stability of dynamical networks. More general graph transforms that preserve or modify subsets of a graphs spectrum would moreover have implications for instance in the study of finite Markov chains as the spectrum of such chains influences important aspects of such processes.

In terms of applications of *isospectral graph reductions* one of the most natural is to systematically investigate the statistics of real as well as theoretic graphs modulo some structure. For instance, isospectral graph reductions could be used to investigate the statistical properties of *scale-free networks* reduced over their *hubs* as is demonstrated in figure 3.

Other current research involves extending the work done in chapter III to continuous time dynamical networks i.e. systems of differential equation. This would again involves determining under what conditions such continuous time systems have a unique global attractor which includes investigating systems of *delayed differential equations*. However, we note that the major difference in investigating such systems is that time delays in this setting corrrespond to path of infinite length. Hence, a our theory would need to be extended to a framework involving the spectrum of infinite graphs.

Moreover, as *network synchronization* can be interpreted as a globally attracting fixed point of an associated dynamical network it is possible to investigate this type of dynamic behavior via the theory developed in this dissertation. However, the graph theoretic nature of such investigations is different than in the networks studied here requiring some new theory.

Additionally, one of the main objectives in future work is to use isospectral graph transformations and the transforms they induce on dynamical networks to investigate how the structural evolution of a network influences its dynamics. This is natural for two reasons. The first is that isospectral transformations and those they induce evolve the structure of a dynamical network. Second, such transformations preserve characteristics associated with the dynamical behavior of the network much as real networks to large degree preserve their functionality as their structure evolves. In this way, isospectral transformations may provide a useful tool in understanding the structural evolution of networks.

Beyond this, it is possible to investigate open dynamical systems or systems in which orbits can escape through holes using techniques developed for network analysis. This is possible since any open system with a Markovian hole can be considered a dynamical network in which orbits escape through particular nodes [2]. Specifically, future work here would involve using techniques analogous to the dynamical network expansions of chapter III to estimate the survival properties in such open dynamical systems.

Lastly, many computational questions related to isospectral graph reductions have yet to be resolved. For example, how to optimally select structural sets when estimating the spectra of a matrix by Gershgorin, Brauer, and Brualdi-type methods or minimizing the difference between the spectrum of a graph and its possible reductions.

REFERENCES

[1] V. Afriamovich and L. Bunimovich 2007 Dynamical networks: interplay of topology, interactions, and local dynamics, *Nonlinearity* **20** 1761-1771

[2] V. S. Afraimovich, L. A. Bunimovich 2010 Which Hole is Leaking the Most: a Topological Approach to Study Open Systems Nonlinearity **23** 643656

[3] V. Afriamovich, L. Bunimovich, and Moreno S 2010 Dynamical networks: Continuous time and general discrete time models *Regular and Chaotic Dynamics* **15** 129-147

[4] R. Albert and A-L Barabási 2002 Statistical mechanics of complex networks *Rev. Mod. Phys.* **74** 47-97

[5] V. M. Alekseyev, Symbolic Dynamics, Translations of the AMS(Ser.2) vol. 116, Providence, RI: AMS, 1-113,1981.

[6] M. Blank and L. Bunimovich 2006 Long range action in networks of chaotic elements, *Nonlinearity* **19** 329-344

[7] S. Boccaletti, V. Latora, Y. Moreno, M. Chavez and D. Hwang 2006 Complex Networks: Structure and Dynamics *Physics Reports* **424** 175-308.

[8] B. Bollobas, C. Borgs, J. Chayes and O Riordan 2010 Percolation on Dense Graph Sequences, *Ann. Probab.* **38** Number 1, 150-183

[9] M. Boyle and D. Lind, Small polynomial matrix presentations of nonnegative matrices, Linear Algebra Applications 355 (2002), 49-70.

[10] A. Brauer 1947 Limits for the characteristic roots of a matrix II, *Duke Math J.* **14** 21-26

[11] R. Brualdi 1982 Matrices, Eigenvalues, and Directed Graphs *Lin. Multilin. Alg.* **11** 143-165

[12] R. Brualdi and H. Ryser 1991 Combinatorial Matrix Theory *Cambridge University Press Melbourne*

[13] H. Bruin, S. Luzzatto, and Sebastian van Strien, *Decay of correlations in one-dimensional dynamics*, Ann. Sci. Ec. Norm. Sup., **36** (2003), 621-646.

[14] L. Bunimovich and B. Webb 2009 Dynamical Networks, Isospectral Reductions, and Generalization of Gershgorin's Theorem *preprint*

[15] L. A. Bunimovich, B. Z. Webb, *Dynamical Networks, Isospectral Graph Reductions, and Improved Estimates of Matrices Spectra*, arXiv, arXiv:0911.2453v1, submitted.

[16] M. Chavez, D. Hwang and S. Boccaletti 2007 Synchronization Processes in Complex Networks *Eur. Phys. J. Special Topics* **146** 129-144

[17] J-R. Chazottes, B. Fernandes (ed.), *Dynamics of Coupled Map Lattices and Related Spatially Extended Systems*, Lect. Notes in Physics, v. 671, Berlin: Springer, 2005.

[18] S. Cedervall, "Invariant Measure and Correlation Decay for S-multimodal Interval Maps", PhD thesis, University of London, 2006.

[19] L. Chau (ed), *Synchronization in Coupled Chaotic Circuts and Systems*, vol. 41, World Scientific Series on Nonlinear Sciences, Series A, World Scientific, Singapore, 2002.

[20] F. R. K. Chung, *Spectral Graph Theory*, American Mathematical Society, (1997).

[21] F. Chung and L. Lu, *Complex Graphs and Networks*, Providence, RI: American Mathematical Society, 2006.

[22] J. Desplanques, Théorème d'algébre, J. Math. Spéc. (3) 1(1887), 12-13.

[23] S. N. Dorogovtsev, J.F.F. Mendes, *Evolution of Networks: From Biological Networks to the Internet and WWW*, Oxford: Oxford Univ. Press, 2003.

[24] M. Faloutsos, P. Faloutsos, C. Faloutsos, *On power-law relationship of the internet topology*, ACMSIGCOMM,'99, Comput. Commun. Rev. **29**, 251-263 (1999).

[25] S. Gershgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk SSSR Ser. Mat. 1, 749-754 (1931).

[26] J. Graczyk, D. Sands, and G. Swiatek, *La dérivée Schwarzienne en dynamique unimodale*, Comptes Rendus de l'Académie des Sciences Série I, **332** (2001), 329-332.

[27] J. Hadamard, Leçons sur la propagation des ondes, Hermann et fils, Paris, (1949) reprinted by Chelsea, New York.

[28] R. A. Horn, C. R. Johnson, *Matrix Analysis*, Cambridge University Press, Cambridge 1990.

[29] W. Hager 1989 Updating the inverse of a matrix, *SIAM Review* **31** 221-239

[30] B. Hunt, E. Ott and J. Restrepo 2006 Characterizing the Dynamical Importance of Network Nodes and Links *Phys. Rev. Lett.* **97** 094102

[31] B. Hunt, E. Ott and J. Restrepo 2005 Onset of Synchronization in Large Networks of Coupled Oscillators *Phys. Rev. E* **71** Issue 3

[32] John C. Kash, Terrence M. Tumpey, Sean C. Proll, Victoria Carter, Olivia Perwitasari, Matthew J. Thomas, Christopher F. Basler, Peter Palese, Jeffery K. Taubenberger, Adolfo Garca-Sastre, David E. Swayne and Michael G. Katze, *Genomic analysis of increased host immune and cell death responses induced by 1918 influenza virus*, Nature **443** (2006), 578-581.

[33] O. S. Kozlovski, *Getting rid of the negative Schwarzian derivative condition*, Ann. of Mathematics, 2nd Ser., **152** (2000), 743-762.

[34] L. Levy, Sur la possibilité de l'équilibre électrique, Comptes Rendus C. R. Acad. Sci. Paris 93 (1881) 706-708.

[35] Mathematica Demonstrations Project, cited 2008: The Schwarzian Derivative of Iterated Functions [Available online at http://demonstrations.wolfram.com/TheSchwarzianDerivativeOfIteratedFunctions.]

[36] G. S. Medvedev, *Reduction of a model of an excitable cell to a one-dimensional map*, Physica D, **202** (2005), 37-59.

[37] G. S. Medvedev, *Transition to bursting via deterministic chaos*, Physical Review Letters, **97** (2006), 048102.

[38] W. de Melo and Sebastian van Strien, *"One Dimensional Dynamics"*, Springer-Verlag, Berlin, 1993.

[39] H. Minkowski, Zur Theorie der Einheiten in den algebraischen Zahlkörpern, Göttinger Nachr. (1900), 90-93. (Ges. Abh. 1, 316-317). Diophantische Approximationen (Leipzig, 1907), 143-144.

[40] A. E. Motter, Bounding network spectra for network design, New J. of Physics **9**, 2-17 (2007).

[41] T. Nowicki and S. van Strien, *Existence of acips for multimodal maps, in "Global Analysis of Dynamical Systems"* (eds. Henk Broer, Bernd Krauskopf and Gert Vegter), CRC Press, (2001), 433-449.

[42] M. Newman, A-L. Barabási, D. J. Watts (ed), *The Structure of Dynamic Networks*, Princeton: Princeton Univ. Press, 2006.

[43] M. Porter, J. Onnela and P. Mucha Communities in Networks *AMS Notices* **56** 1082-1097, 1164-1166

[44] D. Sands, *Misiurewicz maps are rare*, Comm. Math. Phys., **197** (1998), 109-129.

[45] D. Singer, *Stable orbits and bifurcation of maps of the interval*, SIAM J. Appl. Math., **35** (1978), 260-267.

[46] S. Strogatz, *Sync: The Emerging Science of Spontaneous Order*, New York: Hyperion, 2003.

[47] S. van Strien and E. Vargas, *Real bounds, ergodicity and negative Schwarzian derivative for multimodal maps*, J. Amer. Math. Soc., **17** (2004), 749-782.

[48] R. S. Varga, *Gershgorin and His Circles*, Germany: Springer-Verlag Berlin Heidelberg, 2004.

[49] Q. Wang and L. Young *Nonuniformly expanding 1D maps*, Comm. Math. Phys., **264** (2006) 255-282.

[50] D. J. Watts, *Small Worlds: The Dynamics of Networks Between Order and Randomness*, Princeton: Princeton Univ. Press, 1999.

[51] B. Z. Webb, *Dynamics of Functions with an Eventual Negative Schwarzian Derivative*, *Discrete and Continuous Dynamical System Ser. A*, 24, 4 August 2009, 1393-1408.

[52] C. W. Wu, Synchronization in networks of nonlinear dynamical systems coupled via a direct graph, *Nonlinearity* **18**, 1057-1064 (2005).