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OFFICE OF CONTRACT ADMINISTRATION
SPONSORED PROJECT INITIATION

Date: 4/10/78

Project Title: Nonlinear Stability Analysis of Unbraced Frames Subjected to Static and Dynamic Loads

Project No: E-23-635

Project Director: Dr. George J. Simitses

Sponsor: National Science Foundation

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(Grant Period--12-month budget period plus 6-month flexibility period

Type Agreement: Grant No. ENG77-22443

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Sponsor Contact Person (s):

Technical Matters
Dr. Charles A. Babendreier, Director
Structural, Materials and Geotechnical Engineering Program
Engineering Mechanics Section
Division of Engineering
National Science Foundation
Washington, D.C. 20550
(202) 632-5787

Contractual Matters
(thru OCA)
Ms. Mary Frances O'Connell
Grants Manager
MPE/BBS/SE Branch
Division of Grants and Contracts
National Science Foundation
Washington, D.C. 20550
(202) 632-2858

Defense Priority Rating: n/a

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SPONSORED PROJECT TERMINATION

Date: November 28, 1979

Project Title: Nonlinear Stability Analysis of Unbraced Frames Subjected to Static and Dynamic Loads

Project No: E-23-635

Project Director: Dr. G. J. Simitses

Sponsor: National Science Foundation

Effective Termination Date: 9/30/79

Clearance of Accounting Charges: ----

Grant/Contract Closeout Actions Remaining:

TERMINATED

_ Final Invoice and Closing Documents
X Final Fiscal ~~XXXXX~~ Accounting (FCTR)
X Final Report of Inventions
_ Govt. Property Inventory & Related Certificate
_ Classified Material Certificate
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CA–4 (1/79)

E-23-635

FINAL TECHNICAL REPORT

Georgia Institute of Technology
Atlanta, Georgia 30332

by

George J. Simitses, Professor
of Engineering Science and Mechanics

on

NSF Grant ENG 77-22443
(4/1/78 - 9/30/79)

entitled

NONLINEAR STABILITY ANALYSIS OF UNBRACED FRAMES SUBJECTED TO STATIC AND DYNAMIC LOADS

Abstract

The interest in the analysis of plane frames is understandable, because of the many uses of such configurations in the design of buildings, bridges and offshore structures. Many linear buckling analyses of rigid-jointed, unbraced plane frames have been reported in the open literature. For a historical sketch and review on the subject, the reader is referred to Bleich's text (Ref. 1) and Ref. 2. On the other hand, the nonlinear

analyses reported in the open literature are very few and of limited applicability, because of simplifying assumptions and load characteristics, such as no load eccentricity, use of extreme boundary conditions (either simply supported or clamped) and others (Refs. 3 and 4).

A nonlinear solution methodology has been developed for the complete analysis of plane frames (prebuckling and postbuckling). The method is based on nonlinear kinematic relations and linear constitutive equations. This methodology is fully described and demonstrated in the attached publications.

Moreover, from the studies it is concluded that

(1) Two-bar frames are, in general, subject to limit point instability under static application of the applied load. Also, there is a critical condition under sudden application of the load (dynamic buckling, see the second paper in the list of publications).

(2) Portal frames exhibit postbuckling strength (stable postbuckling branch) and thus they are insensitive to either initial geometric imperfections or initial load eccentricities. Moreover, these configurations do not buckle under sudden application of the load.

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Scientific Collaborators

1. Professor A. N. Kounadis, Visiting Associate Professor, School of ESM, Georgia Institute of Technology. Professor Kounadis is Professor of Civil Engineering at the National Technical University of Athens, Athens, Greece.

2. Dr. J. Giri, Research Engineer, School of ESM, Georgia Institute of Technology.

List of Publications

1. "Buckling of Imperfect Rigid-Jointed Frames", Journal of the Engineering Mechanics Division, ASCE, Vol. 104, EM3, 1978, pp. 569-586, (with A. N. Kounadis).

2. "Dynamic Buckling of Simple Frames Under a Step-Load", Journal of the Engineering Mechanics Division, ASCE, to appear in the October, 1979 issue (with A. N. Kounadis and J. Giri).

3. "Nonlinear Analysis of Elastically Restrained and Eccentrically Loaded Portal Frames", Journal of the Engineering Mechanics Division, ASCE, submitted for publication (with A. N. Kounadis and J. Giri).

4. "Nonlinear Analysis of Portal Frames", Journal of the Structural Division, ASCE, submitted for publication (with J. Giri and A. N. Kounadis).

5. "Nonlinear Analysis of Unbraced Portal Frames of Variable Geometry", Int'l Journal of Nonlinear Mechanics, submitted for publication (with J. Giri).

Presentations

1. "Nonlinear Analysis of Portal Frames", presented at the SSRC (Structural Stability Research Council) Annual Meeting, April 23-25, 1979, Pittsburgh, Pennsylvania.

2. "Nonlinear Analysis of Unbraced Portal Frames of Variable Geometry", presented at the 16th Midwestern Mechanics Conference, Sept. 19-21, 1979, Manhattan, Kansas.

DYNAMIC BUCKLING OF SIMPLE FRAMES

UNDER A STEP-LOAD

George J. Simitses*, Anthony N. Kounadis**, and Jagannath Giri***

INTRODUCTION

Since most loads on structural systems induce dynamic effects, an effort has been exerted, in the past twenty-five years, to answer some of the problems associated with stability under dynamic conditions. These efforts have been on specific problems; and no unifying concept has been developed to the point that, criteria for stability, estimates of critical conditions, and the response phenomena under dynamic load themselves are clearly understood by the practicing engineer.

One particular class of problems that has received wide attention is the stability of shallow arches and shallow spherical caps under impulsive loads and suddenly applied constant loads of infinite duration. The former studies started with the early work of Hoff and Bruce [7] and the latter with Budiansky and Roth [2]. In the case of shallow arches, the initial work of Hoff and Bruce [7] relates dynamic critical conditions with characteristic of the total potential surface. This idea was extended independently by Hsu and his collaborators [8-12] and by Simitses [15,16,18]. Most of the investigations that followed, on the shallow arch, are listed in [5]. In the case of spherical caps, Budiansky and Roth [2] defined the load to

*Prof., School of Eng. Sci. and Mechanics, Georgia Institute of Technology Atlanta, Georgia.
**Assoc. Prof. School of Civil Engineering, Nat'l Technical U. of Athens Athens, Greece.
***Res. Engineer, School of Eng. Sci. and Mechanics, Georgia Institute of Technology, Atlanta, Georgia.

be critical, when the transient response increases suddenly with very little increase in the magnitude of the load. This concept was adopted by numerous investigators (for a review see [16] and [1]) in the subsequent years, because it is tractable to computer solutions. This same concept was employed by Budiansky and Hutchinson [3] in estimating the critical load (suddenly applied) for systems that are imperfection sensitive. Through this criterion they related the dynamic critical load to the static one (in an approximate sense). The concept was improved and generalized in a subsequent paper by Budiansky [4] in attempting to predict critical conditions for imperfection sensitive structures under time-dependent loads. Independently, Thompson [19] outlined an enery based procedure for estimating a critical suddenly applied load on imperfection sensitive structures. Finally, Lo and Masur [14] present a finite element discretization solution to the dynamic buckling of shallow arches by employing a criterion similar to that of Budiansky and Roth.

The present note presents critical conditions for three simple two-bar frames, loaded eccentrically and suddenly by a constant load of infinite duration. The criterion used is similar to that of [7,15] and the critical load corresponds to a lower bound. The complete static stability analysis for all three models is available in [13,17] and experimental evidence has been reported [19] for one of them (model A). The three models are shown in Fig. 1. The symbols used are the same as in [13,17].

NUMERICAL RESULTS AND DISCUSSION

On the basis of the criterion established, critical loads are computed for all three frames and for a large practical range of load eccentricities ($-0.01 \le \bar{e} \le 0.01$) and of slenderness ratios ($\lambda = 40, 80, \infty$). The results are presented graphically in Figs. 2-4, and discussed separately for each frame (Model).

Model A: The results for this model are presented graphically on Fig. 2. It is observed that, as in the static case, there is a small positive eccentricity, $\bar{e}_{cr}$, such that for $\bar{e} \le \bar{e}_{cr}$ there is dynamic instability, while for $\bar{e} > \bar{e}_{cr}$ there is not. This $\bar{e}_{cr}$ is λ-dependent and identical to the corresponding static case. For all λ-values considered, except $\lambda \to \infty$, the difference between $\beta^2_{cr_D}$ and $\beta^2_{cr_{st}}$ is the largest at $\bar{e} = \bar{e}_{cr}$ and it diminishes as $\bar{e}$ increases negatively. On the contrary, for $\lambda \to \infty$ this effect is reversed and more specifically, the difference is close to zero at $\bar{e} = \bar{e}_{cr}$ and it increases as $\bar{e}$ increases negatively. In addition, eccentricity has a destabilizing effect regardless of the value of the slenderness ratio. This effect is less pronounced for the static case.

Finally, dynamic instability takes place with a trajectory corresponding to a positive joint rotation φ. Because of this, of course, the compressive force in the vertical bar, k_1, is higher than the applied load, β^2, at the instant of dynamic snap-through.

Note that the experimental results of Thompson ($\lambda = 1275$) agree very well with the $\lambda \to \infty$ theoretical prediction. The largest discrepancy between theory and experiment is approximately 1.5%.

Model B: This is the only model, which exhibits bifurcational buckling (through an unstable branch) under static application of the load. The results are presented graphically in Fig. 3.

It is seen from Fig. 3 that the effect of slenderness on the dynamic critical load is appreciable while its effect on the static critical load [17] (limit point load) is negligible. In addition, for all λ, except $\lambda \to \infty$, the difference between the static and dynamic critical loads is the largest at $\bar{e} = 0$ and decreases as $|\bar{e}|$ increases. Furthermore, at $\bar{e} = 0$ and for a given λ, except $\lambda \to \infty$, there are two dynamic critical loads, one corresponding to a negative rotation φ trajectory (the lower) and one corresponding to a positive φ trajectory (the upper). Definitely the system for $\bar{e} = 0$, buckles in the mode associated with the lower load and it should be designed for this lower dynamic critical load. But the results indicate that a small positive eccentricity, in this case, has a stabilizing effect, because it forces the system to dynamically buckle through a positive rotation φ trajectory and therefore it can carry a higher load. In general, though, eccentricity has a destabilizing effect. This means that as $|\bar{e}|$ increases the dynamic critical load decreases.

Model C: The results for this model are presented graphically in Fig. 4. The observations for this model are very similar to those corresponding to model A.

CONCLUSIONS

Among the most important conclusions of this investigation, one may list the following.

1. In general, for frames which under static conditions exhibit limit point instability, there is a positive critical eccentricity, $\bar{e}_{cr}$, such that a system with $\bar{e} < \bar{e}_{cr}$ buckles dynamically, while with $\bar{e} > \bar{e}_{cr}$ there is no instability. This observation is also true for static loading.

2. For all three frames, increase in $|\bar{e}|$ resulted into a decrease in the dynamic critical load.

3. The effect of slenderness ratio upon the dynamic critical load is appreciable, even in the case (Model B) in which this effect was negligible for the corresponding static loading.

4. The correlation between theory and experimental results (limited in availability) is excellent. The discrepancy is smaller than 1.5%.

ACKNOWLEDGEMENT

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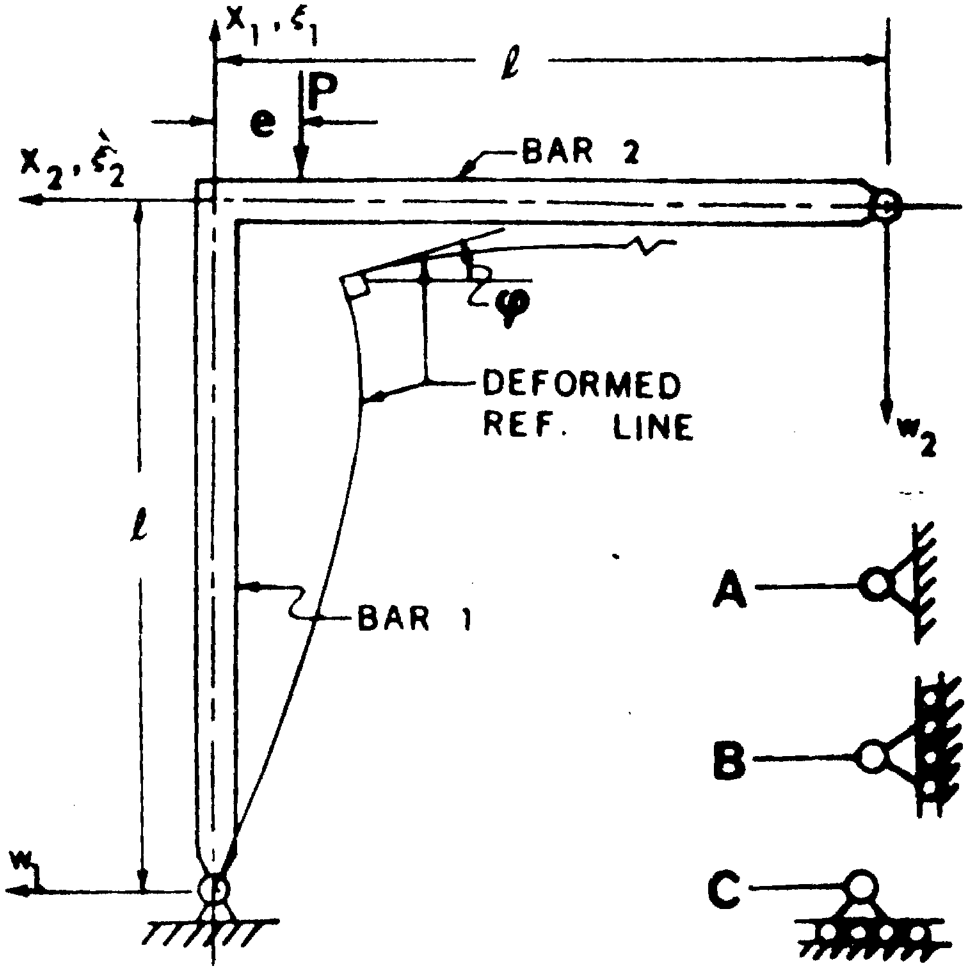

Figure 1. Geometry and Sign Convention

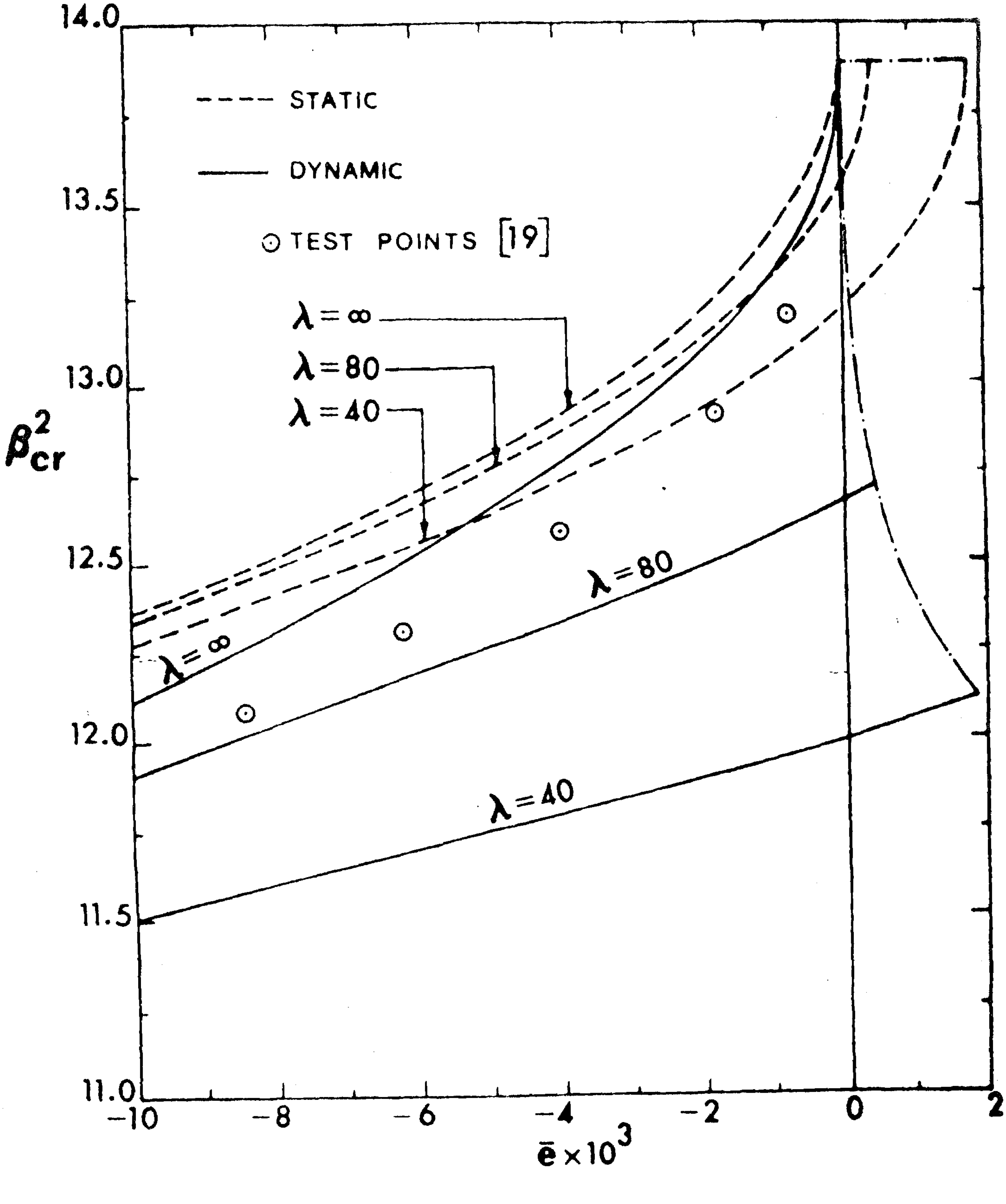

Figure 2. Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model A)

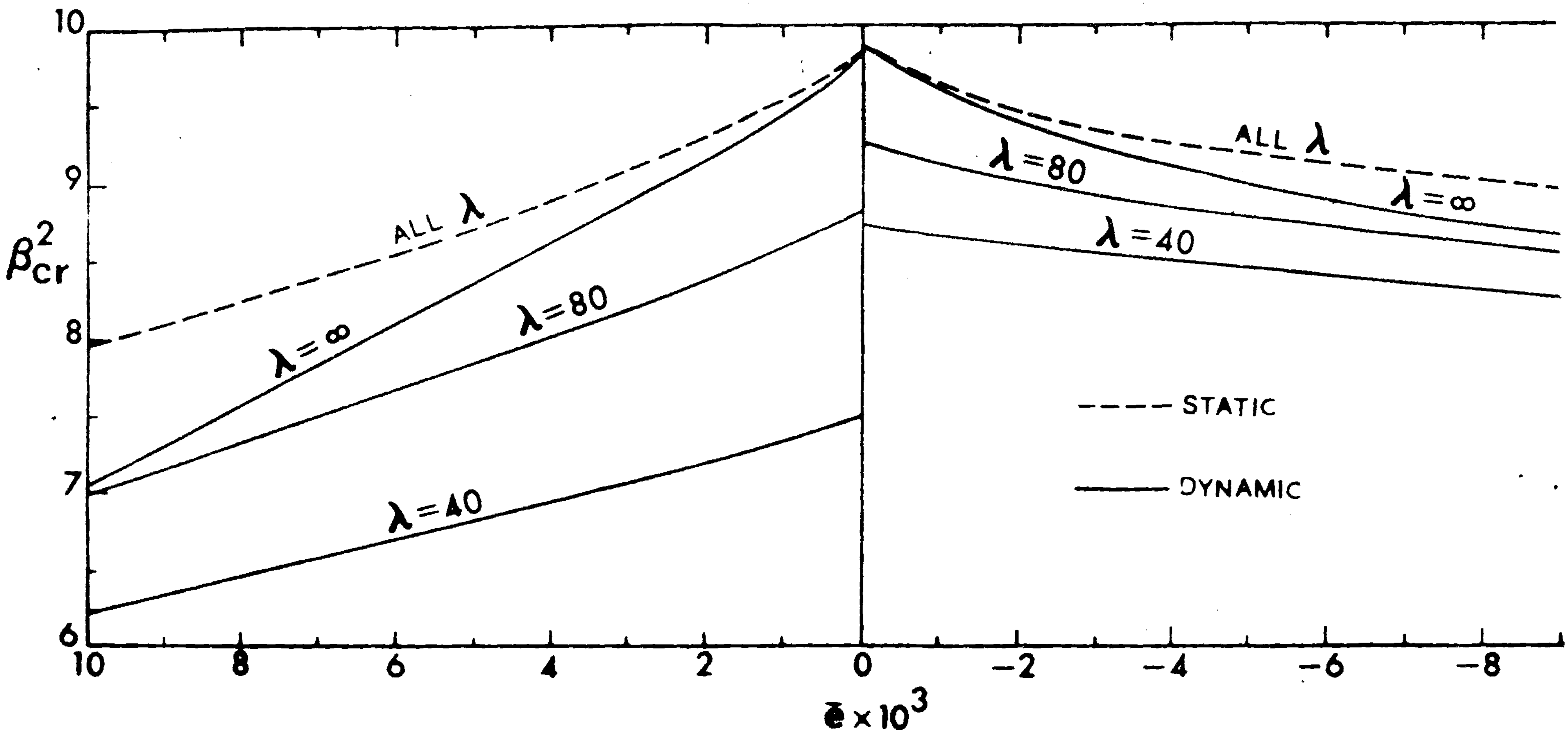

Figure 3. Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model B)

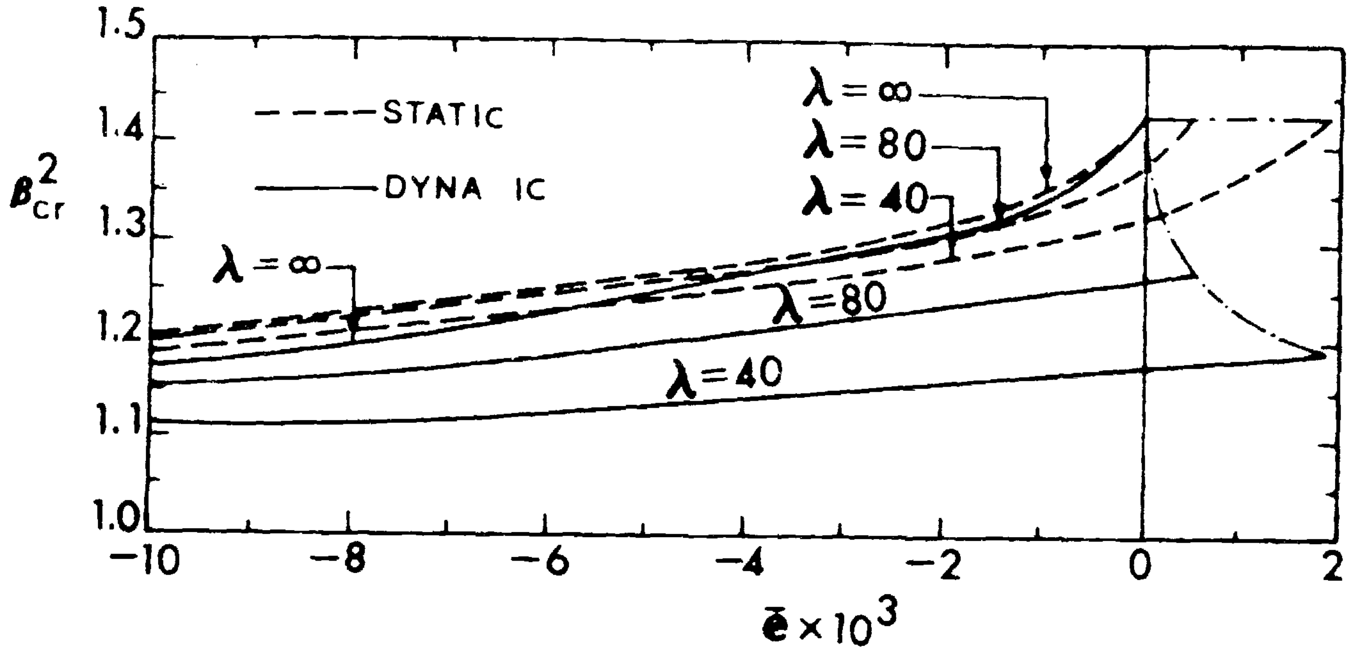

Figure 4. Effect of Eccentricity, $\bar{e}$, and Slenderness ratio, λ, on the Static and Dynamic Critical Loads, β^2. (Model C)

NONLINEAR ANALYSIS OF ELASTICALLY RESTRAINED AND ECCENTRICALLY LOADED PORTAL FRAMES

By George J. Simitses[1], Jagannath Giri[2], and Anthony N. Kounadis[3]

INTRODUCTION

Buckling of a portal frame, which is loaded eccentrically and elastically restrained at the base against rotation, is considered herein. A kinematically nonlinear analysis is performed, with the primary goal being the assessment of the effect of load eccentricity and amount of rotational restraint on the response characteristics of the frame (including the possibility of buckling).

The interest in plane frame analysis is understandable, because of the many uses of this configuration in the design of buildings, bridges, and offshore structures. Many linear buckling analyses of rigid-jointed plane frameworks have been reported in the open literature. For a historical sketch and review on the subject, the reader is referred to Bleich's (1) text and to Ref. 5. On the other hand, the nonlinear analyses available in the open literature are very few and of limited applicability because of simplifying assumptions and load characteristics, such as no load eccentricity, extreme boundary conditions (either simply supported or clamped) and others. (2,3)

There are two important considerations in the present investigation: (a) to demonstrate the applicability of the developed kinematically nonlinear analysis to both the postbuckling range for the perfectly

[1] Professor, School of Engineering Science and Mechanics; Georgia Institute of Technology

[2] Research Engineer, School of Engineering Science and Mechanics; Georgia Institute of Technology, Atlanta, Georgia

[3] Associate Professor, School of Civil Engineering; National Technical University of Athens, Greece.

loaded configuration (no load eccentricity) as well as to the entire response range of the eccentrically loaded configurations. In the latter case, the possibility of the existence of limit point instability is considered; (b) to establish whether or not eccentrically loaded portal frames are sensitive or insensitive to imperfections (load eccentricities). In addition, the effects of rotational restraint ($0 \leq \beta < \infty$) and bar slenderness ratio are assessed.

The analysis is based on nonlinear kinematic relations (moderate rotations) and linearly elastic material behavior. Finally the effect of transverse shear on deformations is neglected.

MATHEMATICAL FORMULATION

Consider the portal frame shown on Fig. 1. Each bar is of length ℓ_k, constant cross-sectional area A_k, constant cross-sectional second moment of area I_k, and has displacement components u_k (in-plane) and w_k (transverse) $k = 1,2,3$. The sign convention used is given on the figure. The loads Q_i are eccentrically applied (eccentricity e_i is shown in the positive sense) and the supports of the portal frame are rotationally restrained, β. Note that $\ell_1 = \ell_2$ and the eccentricity, e_i, is small ($e_i/\ell_1 \ll 1$).

The equilibrium equations are given by

$$P_{k,x} = (EA)_k \left(u_{k,x} + \frac{1}{2} w^2_{k,x}\right)_{,x} = 0$$

$$(EI)_k w_{k,xxxx} - P_k w_{k,xx} = 0 \tag{1}$$

$k = 1,2,3$; and P_k is positive in tension

The associated boundary and joint conditions are:

Boundaries 1 and 2

$$
\begin{aligned}
&u_k(0) = 0 \qquad k = 1,2 \\
&w_k(0) = 0 \\
&(EI)_k w_{k,xx}(0) = \beta w_{k,x}(0)
\end{aligned}
\tag{2}
$$

Joint 3

$$
\begin{aligned}
&u_1(\ell_1) = -w_3(\ell_3); \; u_3(\ell_3) = w_1(\ell_1); \; w_{1,x}(\ell_1) = w_{3,x}(\ell_3); \\
&e_1 Q_1 - (EI)_1 w_{1,xx}(\ell_1) - (EI)_3 w_{3,xx}(\ell_3) = 0 \\
&-P_1 - Q_1 + P_3 w_{3,x}(\ell_3) - (EI)_3 w_{3,xxx}(\ell_3) = 0 \\
&-P_3 - P_1 w_{1,x}(\ell_1) + (EI)_1 w_{1,xxx}(\ell_1) = 0
\end{aligned}
\tag{3}
$$

Joint 4

$$
\begin{aligned}
&u_3(0) = w_2(\ell_2); \; u_2(\ell_2) = -w_3(0); \; w_{2,x}(\ell_2) = w_{3,x}(0); \\
&e_2 Q_2 + (EI)_3 w_{3,xx}(0) - (EI)_2 w_{2,xx}(\ell_2) = 0 \\
&-P_2 - Q_2 - P_3 w_{3,x}(0) + (EI)_3 w_{3,xxx}(0) = 0 \\
&-P_3 + P_2 w_{2,x}(\ell_2) - (EI)_2 w_{2,xxx}(\ell_2) = 0
\end{aligned}
\tag{4}
$$

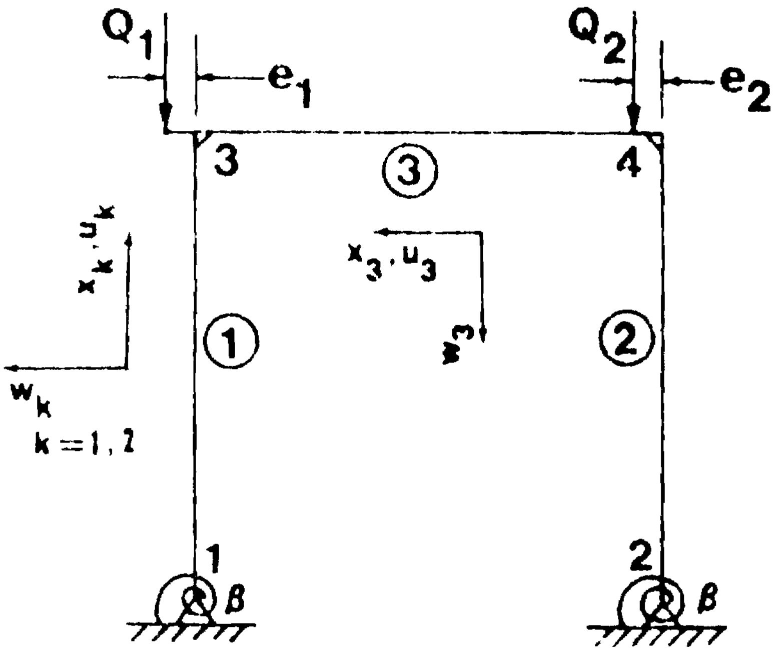

Fig. 1 Geometry and Sign Convention

Before proceeding with the solution, the following nondimensionalization is introduced.

$$r_k = \frac{(EI)_k}{(EI)_1}\ ; \qquad r_k = \frac{\ell_k}{\ell_1}\ ; \qquad X = \frac{x}{\ell_k}\ ; \qquad U_k = \frac{u_k}{\ell_k}\ ;$$

$$W_k = \frac{w_k}{\ell_k}\ ; \qquad \rho_k^2 = \frac{I_k}{A_k}\ ; \qquad \lambda_k = \frac{\ell_k}{\rho_k}\ ; \qquad k_k^2 = \pm\frac{P_k \ell_k^2}{(EI)_k}$$

(+ for tension, P_k positive; - for compression, P_k negative) (5)

$$k = 1,2,3.;\ \bar{Q}_i = \frac{Q_i \ell_1^2}{(EI)_1}\ ; \qquad e_i = \frac{e_i}{\ell_i}\ ; \qquad i = 1,2.;$$

$$\bar{\beta} = \beta\ell_i/(EI)_1 .$$

Moreover, it is observed that the horizontal bar, bar 3, can be either in tension or in compression, while the vertical bars, bars 1 and 2, are always in compression. Thus, the governing equations and some of the joint conditions are different, depending on the sense of the axial force in bar 3.

The governing equations (equilibrium equations), Eqs (1) and the associated boundary and joint conditions are given below in nondimensionalized form.

In-plane equilibrium

$$U_{k,X} + \frac{1}{2}W_{k,X}^2 = -k_k^2/\lambda_k^2\ , \qquad k = 1,2.$$

$$U_{3,X} + \frac{1}{2}W_{3,X}^2 = -k_3^2/\lambda_3^2 \qquad \text{(compression)} \tag{6}$$

$$U_{3,X} + \frac{1}{2}W_{3,X}^2 = k_3^2/\lambda_3^2 \qquad \text{(tension)}$$

Transverse equilibrium

$$W_{k,XXXX} + k_k^2 W_{k,XX} = 0 \quad , \quad k = 1,2$$

$$W_{3,XXXX} + k_3^2 W_{3,XX} = 0 \quad \text{(compression)}$$

$$W_{3,XXXX} - k_3^2 W_{3,XX} = 0 \quad \text{(tension)} \tag{7}$$

Boundaries 1 and 2

$$U_k(0) = 0 \ ; \quad W_k(0) = 0 \ ; \quad W_{k,XX}(0) = \bar{\beta} W_{k,X}(0) \tag{8}$$

Joint 3

$$U_1(1) = -W_3(1) \ ; \quad U_3(1) = W_1(1) \ ; \quad W_{1,X}(1) = W_{3,X}(1) \tag{9}$$

$$\bar{e}_1 \bar{Q}_1 - W_{1,XX}(1) - \left(\frac{r_3}{\mu_3}\right) W_{3,XXX}(1) = 0$$

$$k_1^2 - \bar{Q}_1 - \left(\frac{r_3}{\mu_3^2}\right) \left[k_3^2 W_{3,X}(1) + W_{3,XXX}(1)\right] = 0$$

$$\left(\frac{r_3}{\mu_3^2}\right) k_3^2 + \left[k_1^2 W_{1,X}(1) + W_{1,XXX}(1)\right] = 0 \qquad \text{(compression)} \tag{10}$$

$$\bar{e}_1 \bar{Q}_1 - W_{1,XX}(1) - \left(\frac{r_3}{\mu_3}\right) W_{3,XXX}(1) = 0$$

$$k_1^2 - \bar{Q}_1 - \left(\frac{r_3}{\mu_3^2}\right) \left[-k_3^2 W_{3,X}(1) + W_{3,XXX}(1)\right] = 0$$

$$-\left(\frac{r_3}{\mu_3^2}\right) k_3^2 + \left[k_1^2 W_{1,X}(1) + W_{1,XXX}(1)\right] = 0 \quad \text{(tension)} \tag{11}$$

Joint 4

$$U_2(1) = -W_3(0) \quad ; \quad U_3(0) = W_1(1) \quad ; \quad W_{2,X}(1) = W_{3,X}(0) \tag{12}$$

$$\bar{e}_2\bar{Q}_2 + \left(\frac{r_3}{\mu_3}\right) W_{3,XX}(0) - \left(\frac{r_2}{\mu_2}\right) W_{2,XX}(1) = 0$$

$$\left(\frac{r_2^2}{\mu_2}\right) k_2^2 - \bar{Q}_2 + \left(\frac{r_3^2}{\mu_3}\right)\left[k_3^2 W_{3,X}(0) + W_{3,XXX}(0)\right] = 0$$

$$\left(\frac{r_3^2}{\mu_3}\right) k_3^2 - \left(\frac{r_2^2}{\mu_2}\right)\left[k_2^2 W_{2,X}(1) + W_{2,XXX}(1)\right] = 0 \qquad \text{(Compression)} \tag{13}$$

$$\bar{e}_2\bar{Q}_2 + \left(\frac{r_3}{\mu_3}\right) W_{3,XX}(0) - \left(\frac{r_2}{\mu_2}\right) W_{2,XX}(1) = 0$$

$$\left(\frac{r_2^2}{\mu_2}\right) k_2^2 - \bar{Q}_2 + \left(\frac{r_3^2}{\mu_3}\right)\left[-k_3^2 W_{3,X}(0) + W_{3,XXX}(0)\right] = 0$$

$$-\left(\frac{r_3^2}{\mu_3}\right) k_3^2 - \left(\frac{r_2^2}{\mu_2}\right)\left[k_2^2 W_{2,X}(1) + W_{2,XXX}(1)\right] = 0 \qquad \text{(tension)} \tag{14}$$

The solution to the differential equations, Eqs. (7), is characterized by

$$W_k(X) = A_{k1} \sin k_k X + A_{k2} \cos k_k X + A_{k3} X + A_{k4} \tag{15}$$

(for k = 1,2 and 3 when bar 3 is in compression) and

$$W_3(X) = A_{31}\sinh k_3X + A_{32}\cosh k_3X + A_{33}X + A_{34} \tag{16}$$

(when bar 3 is in tension)

The solution for $U_k(X)$ is obtained from Eqs (6)

$$U_k(X) = U_{k0} - \left(\frac{k_k^2}{\lambda_k^2}\right) X - \frac{1}{2}\int_0^X W_{k,Y}^2\, dY \tag{17}$$

from which

$$\begin{aligned} U_k(1) = U_{k0} &- \left(\frac{k_k^2}{\lambda_k^2}\right) - \left(\frac{k_k}{4}\right) A_{k1}^2 \left(k_k + \sin k_k \cos k_k\right) \\ &- \left(\frac{k_k}{4}\right) A_{k2}^2 \left(k_k - \sin k_k \cos k_k\right) - \frac{1}{2} A_{k3}^2 \\ &+ \left(\frac{k_k}{4}\right) A_{k1} A_{k2} \left(1 - \cos 2k_k\right) - A_{k1}A_{k3} \sin k_k \\ &+ A_{k2}A_{k3}\left(1 - \cos k_k\right) \;;\quad k = 1,2,3. \end{aligned} \tag{18}$$

(for compression)

and

$$U_3(X) = U_{30} + \left(\frac{k_3^2}{\lambda_3^2}\right) X - \frac{1}{2}\int_0^X W_{3,Y}^2\, dY \tag{19}$$

from which

$$U_3(1) = U_{30} + \frac{k_3^2}{\lambda_3^2} - \left(\frac{k_3}{4}\right) A_{31}^2 \left(k_3 + \sinh k_3 \cosh k_3\right)$$

$$+ \left(\frac{k_3}{4}\right) A_{32}^2 \left(k_3 - \sinh k_3 \cosh k_3\right) - \frac{1}{2} A_{33}^2$$

$$+ \left(\frac{k_3}{4}\right) A_{31}A_{32} \left(1 - \cosh 2k_3\right) - A_{31}A_{33} \sinh k_3 \qquad (20)$$

$$+ A_{32}A_{33} \left(1 - \cosh k_3\right) .$$

(for tension)

Note that for a frame of given structural geometry, μ_k, λ_k, ℓ_1, $(EI)_1$, ρ_k, $\bar{\beta}$ and of given loading condition $\bar{e}_1$, $\bar{e}_2$, $\bar{Q}_1$, $\bar{Q}_2$ the response is characterized by the expression of equations (15),(16), (17) and (19) for both tension or compression in the horizontal bar, provided that the appropriate constants are evaluated. These constants are: U_{k0} $(k = 1,2,3)$, k_k $(k = 1,2,3)$, A_{ki} $(k = 1,2,3)$ and $(i = 1,2,3,4)$. The total number of these constants is 18. These constants are evaluated by using the following 18 boundary and joint conditions: three boundary conditions for each of the two boundaries, Eqs. 8; six joint 3 conditions, Eqs. (9) and (10) for compression in bar 3 or Eqs. (9) and (11) for tension in bar 3; and six joint 4 conditions, Eqs. (12) and (13) for compression in bar 3, or Eqs. (12) and (14) for tension in bar 3.

SOLUTION

Regardless of whether the axial force developed in bar 3 is tensile or compressive, the solution procedure is the same. Substitution of the expressions for $W_k(X)$ and $U_k(X)$ into the boundary and joint conditions yields a system of 18 nonlinear equations in 18 constants

$$(U_{k0},\ k_k,\ \text{and}\ A_{ki} \quad k = 1,2,3,\ \text{and}\ i = 1,2,3,4).$$

Out of the 18 nonlinear equations, 15 are linear in A_{ki} and U_{ko}. Those equations are then used to express U_{ko} and A_{ki} as functions (nonlinear) of the structural geometry μ_k, λ_k, r_k, loading $\bar{e}_1$, $\bar{e}_2$, $\bar{Q}_1$, $\bar{Q}_2$ and k_k (axial force parameters in the three bars). The remaining three equations comprise a system of three highly nonlinear equations in k_k $(k = 1,2,3)$.

The above steps are not shown herein for the sake of brevity. Only the three nonlinear equations, for each of the cases of tension or compression in the horizontal bar are shown because they are used directly in the solution scheme. These equations are:

(a) Compression in bar 3

$$k_1^2 + \left(\frac{r_2^2}{\mu_2^2}\right) k_2^2 = \bar{Q}_1 + \bar{Q}_2 \tag{21}$$

$$\begin{aligned}
&D_9 \sin k_1 - D_{11}(1 - \cos k_1) - (r_3/\mu_3^2)(k_3/k_1)^2 + (k_3/\lambda_3)^2 \\
&+ \frac{1}{4} D_7^2 \left[1 + \left(\frac{\sin k_3}{k_3}\right) \cos k_3\right] + \frac{1}{4k_3^2} D_8^2 \left[1 - \left(\frac{\sin k_3}{k_3}\right) \cos k_3\right] \\
&+ \frac{1}{2} D_{13}^2 - \frac{1}{2} D_7 D_8 \left(\frac{\sin k_3}{k_3}\right)^2 + D_7 D_{13} \left(\frac{\sin k_3}{k_3}\right) - D_8 D_{13} \left(\frac{1 - \cos k_3}{k_3^2}\right) \\
&- D_{10} \sin k_2 + D_{12}(1 - \cos k_2) - \left(\frac{r_3}{r_2}\right) \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2 = 0
\end{aligned} \tag{22}$$

$$\left(\frac{k_2}{\lambda_2}\right)^2 + \frac{k_2^2}{4} D_{10}^2 \left[1 + \left(\frac{\sin k_2}{k_2}\right) \cos k_2\right] + \frac{k_2^2}{4}\left[1 - \left(\frac{\sin k_2}{k_2}\right) \cos k_2\right] D_{12}^2$$

$$+ \frac{1}{2}\left[\left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\right]^2 - \frac{k_2}{2} D_{10}D_{12} \sin^2 k_2 - \frac{D_8}{k_3^2} - D_{14}$$

$$\left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2 \left[D_{10} \sin k_2 - D_{12} (1 - \cos k_2)\right] = 0 \tag{23}$$

where

$$D_1 = \left(\frac{r_2}{\mu_2}\right) k_2 \left(k_2 \sin k_2 - \bar{\beta} \cos k_2\right)/\left(k_2 \cos k_2 + \bar{\beta} \sin k_2\right)$$

$$D_2 = \left(\frac{r_3}{\mu_3}\right)$$

$$D_3 = \left(\frac{r_2}{\mu_2}\right) \bar{\beta} \cos k_2 \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2 - \bar{e}_2\bar{Q}_2 - \left(\frac{r_2}{\mu_2}\right) k_2 \left(k_2 \sin k_2 - \bar{\beta} \cos k_2\right) \times$$

$$\left[\frac{\mu_3^2}{r_3}\left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right) - \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\left(1 + \bar{\beta}\,\frac{\sin k_2}{k_2}\right)\right] / \left(k_2 \cos k_2 + \bar{\beta} \sin k_2\right)$$

$$D_4 = - \cos k_3 - \left(\frac{r_3}{\mu_3}\right) k_3 \sin k_3 \,\frac{(k_1 \cos k_1 + \bar{\beta} \sin k_1)}{(k_1^2 \sin k_1 - \bar{\beta}k_1 \cos k_1)}$$

$$D_5 = \frac{\sin k_3}{k_3} - \left(\frac{r_3}{\mu_3}\right) \cos k_3 \,\frac{(k_1 \cos k_1 + \bar{\beta} \sin k_1)}{(k_1^2 \sin k_1 - k_1\bar{\beta} \cos k_1)}$$

$$D_6 = \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \bar{\beta}\left(\frac{\sin k_1}{k_1}\right) + \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right) + \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 +$$

$$+ \left[\bar{e}_1\bar{Q}_1 + \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \bar{\beta} \cos k_1\right] \frac{(k_1 \cos k_1 + \bar{\beta} \sin k_1)}{(k_1^2 \sin k_1 - k_1\bar{\beta} \cos k_1)}$$

$$D_7 = (D_3D_5 + D_2D_6)/(D_1D_5 + D_2D_4)$$

$$D_8 = (D_1D_6 - D_3D_4)/(D_1D_5 + D_2D_4)$$

$$D_9 = -\Big[\bar{e}_1\bar{Q}_1 + \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \bar{\beta}\cos k_1 + \left(\frac{r_3}{\mu_3}\right)\Big(D_7 k_3 \sin k_3$$

$$+ D_8 \cos k_3\Big] / \left(k_1^2 \sin k_1 - k_1\bar{\beta}\cos k_1\right)$$

$$D_{10} = \Big[\left(\frac{\mu_3^2}{r^3}\right)\left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right) - \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\left(1 + \frac{\sin k_2}{k_2}\bar{\beta}\right) + D_7\Big]/(k_2 \cos k_2 + \bar{\beta} \sin k$$

$$D_{11} = \left(\frac{\bar{\beta}}{k_1^2}\right)\left[\left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 - k_1 D_9\right]$$

$$D_{12} = -\left(\frac{\bar{\beta}}{k_2^2}\right)\left[k_2 D_{10}\left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\right]$$

$$D_{13} = \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2 - Q_1}{k_3^2}\right)$$

$$D_{14} = \frac{k_1^2}{\lambda_1^2} + \frac{k_1}{4} D_9^2 (k_1 + \sin k_1 \cos k_1) + \frac{k_1}{4} D_{11}^2 (k_1 - \sin k_1 \cos k_1)$$

$$+ \frac{1}{2}\left[\left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2\right]^2 - \frac{k_1}{2} D_9 D_{11} \sin^2 k_1 - D_7 \frac{\sin k_3}{k_3} - D_{13}$$

$$- D_8 \frac{\cos k_3}{k_3^2} - \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \Big[D_9 \sin k_1 - D_{11}(1 - \cos k_1)\Big]$$

(b) <u>Tension in bar 3</u>

$$k_1^2 + \left(\frac{r_2}{\mu_2^2}\right)k_2^2 = \bar{Q}_1 + \bar{Q}_2 \tag{21}$$

$$E_9 \sin k_1 - E_{11} (1 - \cos k_1) + \left(\frac{r_3}{\mu_3^2}\right) \left(\frac{k_3}{k_1}\right)^2 - \left(\frac{k_3}{\lambda_3}\right)^2 + \frac{1}{2} E_{13}^2$$

$$+ \frac{E_7^2}{4} \left[1 + \left(\frac{\sinh k_3}{k_3}\right) \cosh k_3\right] - \frac{E_8^2}{4k_3^2} \left[1 - \left(\frac{\sinh k_3}{k_3}\right) \cosh k_3\right] + \frac{1}{2} E_7 E_8 \left(\frac{\sinh k_3}{k_3}\right)^2$$

$$+E_7E_{13} \left(\frac{\sinh k_3}{k_3}\right) + E_8E_{13}\left(\frac{\cosh k_3 - 1}{k_3^2}\right) - E_{10} \sin k_2$$

$$+E_{12}(1 - \cos k_2) + \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2 = 0 \qquad (24)$$

$$\left(\frac{k_2}{\lambda_2}\right)^2 + \frac{k_2^2}{4} E_{10}^2\left[1 + \left(\frac{\sin k_2}{k_2}\right) \cos k_2\right] + \frac{k_2^2}{4} E_{12}^2\left[1 - \left(\frac{\sin k_2}{k_2}\right) \cos k_2\right]$$

$$+ \frac{1}{2} \left[\left(\frac{r_3}{r_2}\right) \left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\right]^2 - \frac{k_2}{2} E_{10}E_{12} \sin^2 k_2 - \frac{E_8}{k_3^2} - E_{14}$$

$$- \left(\frac{r_3}{r_2}\right) \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2 \left[E_{10} \sin k_2 - E_{12} (1 - \cos k_2)\right] = 0 \qquad (25)$$

where

$$E_1 = D_1$$

$$E_2 = D_2$$

$$E_3 = -\left(\frac{r_2}{\mu_2}\right) \bar{\beta} \cos k_2\left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2 - \bar{c}_2\bar{Q}_2 + \left(\frac{r_2k_2}{\mu_2}\right) \left[\left(\frac{\mu_3^2}{r_3}\right) \left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right)\right.$$

$$\left. -\left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2 \left(1 + \bar{\beta} \frac{\sin k_2}{k_2}\right)\right] \frac{(k_2 \sin k_2 - \bar{\beta} \cos k_2)}{(k_2 \cos k_2 + \bar{\beta} \sin k_2)}$$

$$E_4 = (k_1^2 \sin k_1 - k_1\bar{\beta} \cos k_1) \cosh k_3/(k_1 \cos k_1 + \bar{\beta} \sin k_1) - (r_3k_3/\mu_3) \sinh k_3$$

$$E_5 = (k_1^2 \sin k_1 - k_1\bar{\beta} \cos k_1) (\sinh k_3/k_3)/(k_1 \cos k_1 + \bar{\beta} \sin k_1) - (r_3/\mu_3) \cosh k_3$$

$$E_6 = \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \bar{\beta} \cos k_1 - \bar{e}_1\bar{Q}_1 + \left[\left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \left(1+\bar{\beta}\ \frac{\sin k_1}{k_1}\right) + \left(\frac{r_3^2}{r_3}\right)\left(\frac{k_1^2-\bar{Q}_1}{k_3^2}\right)\right] \times$$

$$\frac{(k_1^2 \sin k_1 - \bar{\beta}k_1 \cos k_1)}{(k_1 \cos k_1 + \beta \sin k_1)}$$

$$E_7 = (E_3E_5 - E_2E_6)/(E_1E_5 - E_2E_4)$$

$$E_8 = (E_1E_6 - E_3E_4)/(E_1E_5 - E_2E_4)$$

$$E_9 = \left[E_7 \cosh k_3 + E_8\frac{\sinh k_3}{k_3} - \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2-Q_1}{k_3^2}\right) - \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 \left(1+\bar{\beta}\ \frac{\sin k_1}{k_1}\right)\right]/(k_1\cos k_1 + \beta \sin k_1)$$

$$E_{10} = \left[E_7 + \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{k_3}{k_2}\right)^2\left(1+\bar{\beta}\frac{\sin k_2}{k_2}\right) - \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2-\bar{Q}_1}{k_3^2}\right)\right] \Big/ (k_2 \cos k_2 + \beta \sin k_2)$$

$$E_{11} = -(\bar{\beta}/k_1^2) \left[k_1E_9 + \left(r_3/\mu_3^2\right)\left(k_3/k_1\right)^2\right]$$

$$E_{12} = \left(\bar{\beta}/k_2^2\right) \left[\left(r_3/r_2\right)\left(\mu_2/\mu_3\right)^2\left(k_3/k_2\right)^2 - k_2E_{10}\right]$$

$$E_{13} = -\left(\mu_3^2/r_3\right) \left(k_1^2 - \bar{Q}_1\right)/k_3^2$$

$$E_{14} = \left(\frac{k_1}{\lambda_1}\right)^2 + \frac{k_1}{4} E_9^2 (k_1 + \sin k_1 \cos k_1) + \frac{k_1}{4} E_{11}^2 (k_1 - \sin k_1 \cos k_1)$$

$$- \frac{1}{2} \left[\left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2\right]^2 - \frac{k_1}{2} E_9E_{11} \sin^2 k_1 + \left(\frac{r_3}{\mu_3^2}\right)\left(\frac{k_3}{k_1}\right)^2 E_9 \sin k_1$$

$$\left(\frac{k_3}{k_1}\right)^2 E_{11}(1 - \cos k_1) - E_7\ \frac{\sinh k_3}{k_3} - E_8\ \frac{\cosh k_3}{k_3^2} - E_{13}$$

The solution to the three nonlinear equations, either Eqs. (21), (22) and (23) or Eqs. (21),(24) and (25), is accomplished as follows: (a) first through the use of Eq. (21) one of the unknowns is eliminated, say k_1, and thus the system is reduced to two nonlinear equations in two unknowns, k_2 and k_3; (b) next, the two equations are identified as

$$f_i(k_2,k_3,\lambda_k,\bar{e}_j,\bar{Q}_j,r_k,\mu_k,\bar{\beta}) = 0 \tag{26}$$

$$i = 1,2; \qquad k = 1,2,3; \quad \text{and} \quad j = 1,2.$$

(c) a new function, F, is obtained through

$$F = f_1^2 + f_2^2 \tag{27}$$

(d) then, it is recognized that the solution (k_2,k_3) to the non-linear equations, Eqs. (26), for a given geometry and loading $(\lambda_k,\bar{e}_j,\bar{Q}_j,r_k,\mu_k,\bar{\beta})$ is the minimum of F in the space of k_2 & k_3. Note that this minimum is zero; (e) this minimum is obtained by employing the simplex method of Nelder and Mead (4). To this end, a computer program is written and the results (equilibrium positions) are presented as plots of load versus some characteristic displacement. In the current study the joint 3 rotation is used.

RESULTS AND DISCUSSION

Numerical solutions are generated by employing the Georgia Tech high-speed digital computer CDC-Cyber 70, Model 74-28. Results are obtained for a square portal frame with equal bending stiffness ($r_4 = \mu_k = 1$)

The effects of slenderness ratio, λ_k of rotational restraint, $\bar{\beta}$, and of load eccentricity, $\bar{e}_i$, are studied for $\bar{Q}_1 = \bar{Q}_2$.

$$\lambda_k = 40, 80, 120, 1000$$

$$\bar{e}_1 = \bar{e}_2 = 0.001,\ 0.01,\ 0.03,\ 0.05$$

$$\bar{\beta} = 0,\ 0.1,\ 1.0,\ 10,\ 100,\ 1000$$

Note that $\bar{\beta} = 0$ corresponds to the simply supported portal frame, while $\bar{\beta}$ = 1000 approximates well the clamped portal frame ($\bar{\beta} \to \infty$).

Results are also generated for $\bar{e}_i = 0$ for one particular geometry ($\bar{\beta} = 0$ and $\lambda_k = 1000$).

The results are presented in both tabular and graphical forms. All of the generated data are not presented herein in order to save space. What is presented though serves to support the conclusions drawn from this investigation.

In Figs. 2 through 5 the effect of load eccentricity is shown for $\lambda_k = 1000$ (very slender portal frame) and various amounts of rotational restraint. These plots of load $\bar{Q}$ versus joint 3 rotation, φ_1, clearly show that sway-buckling of the corresponding perfect configuration ($\bar{e}_i = 0$) is not characterized by unstable equilibrium position as suggested in Ref. 1 (p. 227) and that these frames possess postbuckling strength. As a matter of fact this important conclusion is supported by the fact that such frames are extensively used in civil engineering structures. If they did not possess postbuckling strength, they would be imperfection-sensitive and failures would occur at loads smaller than the linear theory bifurcation load. This result could be expected, if one reasons that a portal frame (for all β) must behave in a manner similar to a cantilever column, a configuration which is imperfection insensitive. Another observation is that the postbuckling equilibrium positions (see Fig. 2) for $\bar{e}_i = 0$ are characterized by compression in the horizontal bar. On the contrary, when a load eccentricity is introduced (one-sided), the equilibrium positions are characterized by tension in the horizontal bar. It is also observed, that the curves corresponding to eccentric loading seem to approach a horizontal asympotote corresponding to the bifurcation load (of the perfect configuration) rather than approaching the postbuckling $\bar{e}_1 = 0$ curve. Finally, the generated data are confined to load and responses which comply with the limitation of the kinematic relations used (moderate rotations; $\varphi \leq 0.2$ so that $\varphi^2 << 1$). It is seen from Figs. 3-5, that the response of the frame for $\beta \neq 0$ is similar to that corresponding to $\bar{\beta} = 0$, but more load can be carried as the amount of rotational restraint increases. Fig. 6 shows a plot of $\bar{Q}_{cr}$ (for $\bar{e}_1 = 0$) versus $\bar{\beta}$, for $\lambda_k = 1000$. As expected the two end values ($\bar{\beta} \to 0$ and $\bar{\beta} \to \infty$) correspond to the critical values obtained from a linear stability analysis ($\bar{Q}_{cr} = 1.82$ and 7.344 respectively; see Ref. 6). When the sign of the eccentricity is taken to be negative, the response is exactly the same, except that the frame bends in the opposite direction (data not shown herein)

Table 1 shows the results corresponding to the postbuckling curve of the perfect configuration (see Fig. 1). For different values of the load $\bar{Q}$, this table shows the corresponding values of the compressive loads in the three bars (k_1, k_2, k_3) and the rotations at joints 3 and 4 (φ_1 and φ_2 respectively). Note that as the applied load increases bar 2 carries more and more of the load. Note also that the roles of bars 1 and 2 can be interchanged provided that the roles of of φ_1 and φ_2 are also interchanged accompanied by a sign change, when both eccentricities are negative.

Table 2 depicts the effect of slenderness ratio λ_k on the frame response for various eccentricities and $\bar{\beta} = 0$. It is clearly seen from this table that this effect is negligibly small. This is true also for all $\bar{\beta}$-values. Therefore the curve of Fig. 6. holds for all slenderness ratios.

CONCLUSIONS

Among the most important conclusions one may list the following:

1. Elastically restrained (against rotation) rigid-jointed portal frames are not sensitive to load eccentricities, when loaded transversely by concentrated loads at or near the rigid joints.

Table 1. Postbuckling Equilibrium Positions

$(\bar{e}_i = 0,\ \lambda_k = 1{,}000;\ \bar{\beta} = 0)$

$\bar{Q}_1 = \bar{Q}_2$	k_1	k_2	k_3	φ_1	φ_2
1.830	1.33246	1.37279	0.000345	-0.00524046	-0.00377101
1.831	1.33165	1.37430	0.000344	-0.00558369	-0.00394151
1.832	1.33092	1.37574	0.000342	-0.00590593	-0.00409505
1.835	1.32902	1.37975	0.000334	-0.00680231	-0.00449089
1.860	1.31996	1.40631	0.000303	-0.0126896	-0.00617815
1.890	1.31510	1.43196	0.000340	-0.0182481	-0.00670515
1.940	1.31197	1.46926	0.000360	-0.0260824	-0.00620550
2.200	1.31983	1.63035	0.000342	-0.0566774	+0.00608078

Table 2. Effect of Slenderness Ratio, λ_k

(Simply Supported Frame, $\bar{\beta} = 0$)

$\varphi_1 \times 10^2$

$\bar{Q}$	$\lambda = 40$				$\lambda = 120$				$\lambda = 1,000$			
	$\bar{e} = .001$	$\bar{e} = .01$	$\bar{e} = .03$	$\bar{e} = .05$	$\bar{e} = .001$	$\bar{e} = .01$	$\bar{e} = .03$	$\bar{e} = .05$	$\bar{e} = .001$	$\bar{e} = .01$	$\bar{e} = .03$	$\bar{e} = .05$
0.2	0.004	0.035	0.105	0.176	0.003	0.035	0.104	0.173	0.003	0.035	0.104	0.173
0.4	0.007	0.073	0.220	0.367	0.007	0.072	0.217	0.362	0.007	0.072	0.217	0.361
0.6	0.012	0.116	0.349	0.582	0.011	0.115	0.344	0.573	0.011	0.114	0.343	0.572
0.8	0.017	0.167	0.500	0.832	0.016	0.165	0.492	0.819	0.016	0.164	0.491	0.817
1.0	0.023	0.230	0.688	1.144	0.022	0.225	0.675	1.124	0.022	0.025	0.674	1.121
1.2	0.032	0.317	0.950	1.560	0.031	0.310	0.925	1.543	0.031	0.310	0.926	1.534
1.4	0.046	0.463	1.382	2.291	0.045	0.452	1.347	2.233	0.045	0.450	1.343	2.225
1.6	0.084	0.834	2.468	4.066	0.080	0.801	2.371	3.904	0.080	0.797	2.359	3.885
1.7	0.142	1.466	4.328	7.230	0.138	1.372	4.150	6.702	0.138	1.361	4.009	6.6[illegible]

2. As expected, the greater the amount of rotational restraint the greater the buckling load (sway buckling) for the perfect configuration.

3. The effect of slenderness ratio (same for all three bars in this study) is negligibly small.

4. When the eccentricities are one-sided (both positive or both negative) the horizontal bar is in tension. When there is no eccentricity the postbuckling curve is characterized by compression in the horizontal bar.

ACKNOWLEDGEMENT

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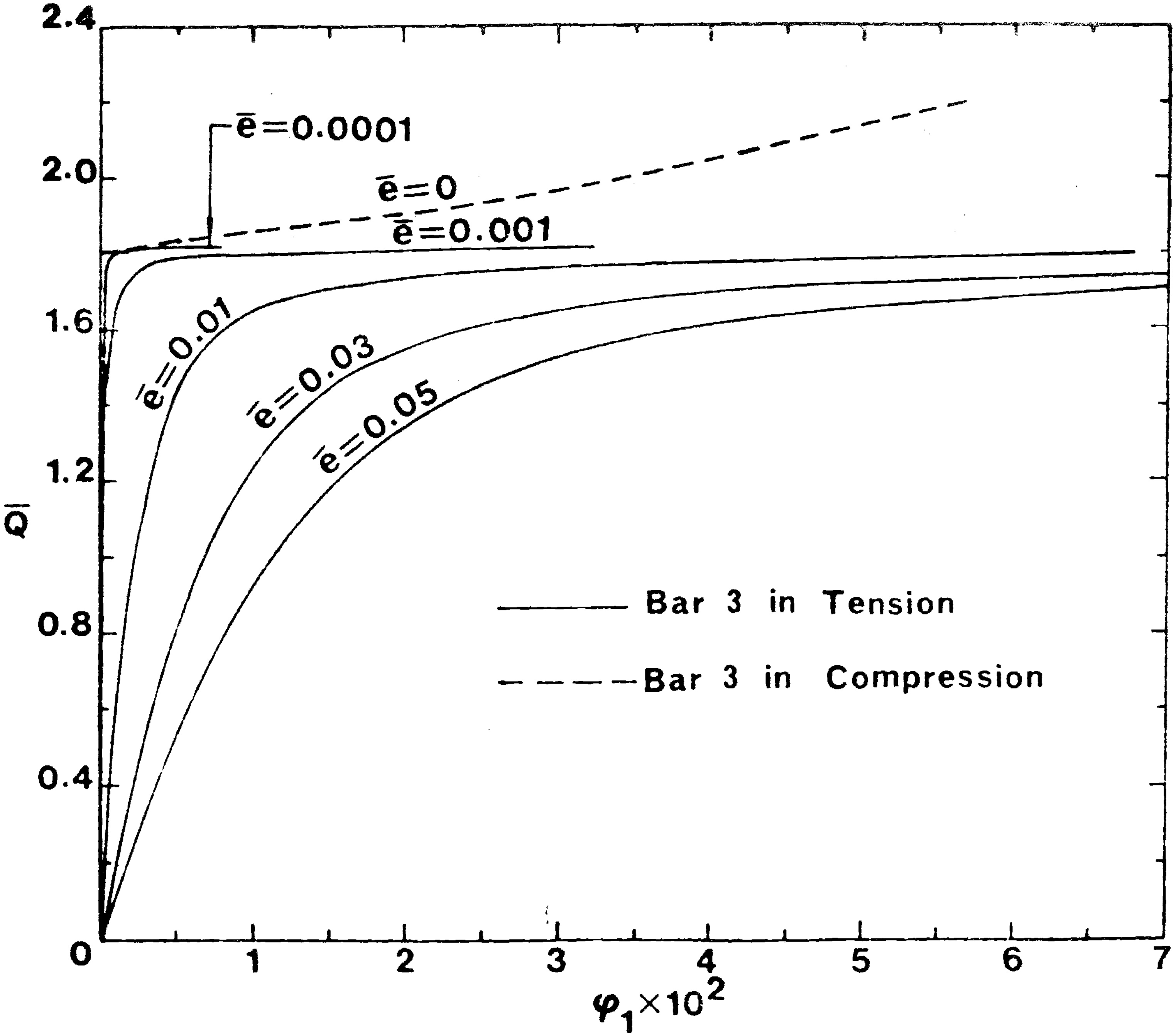
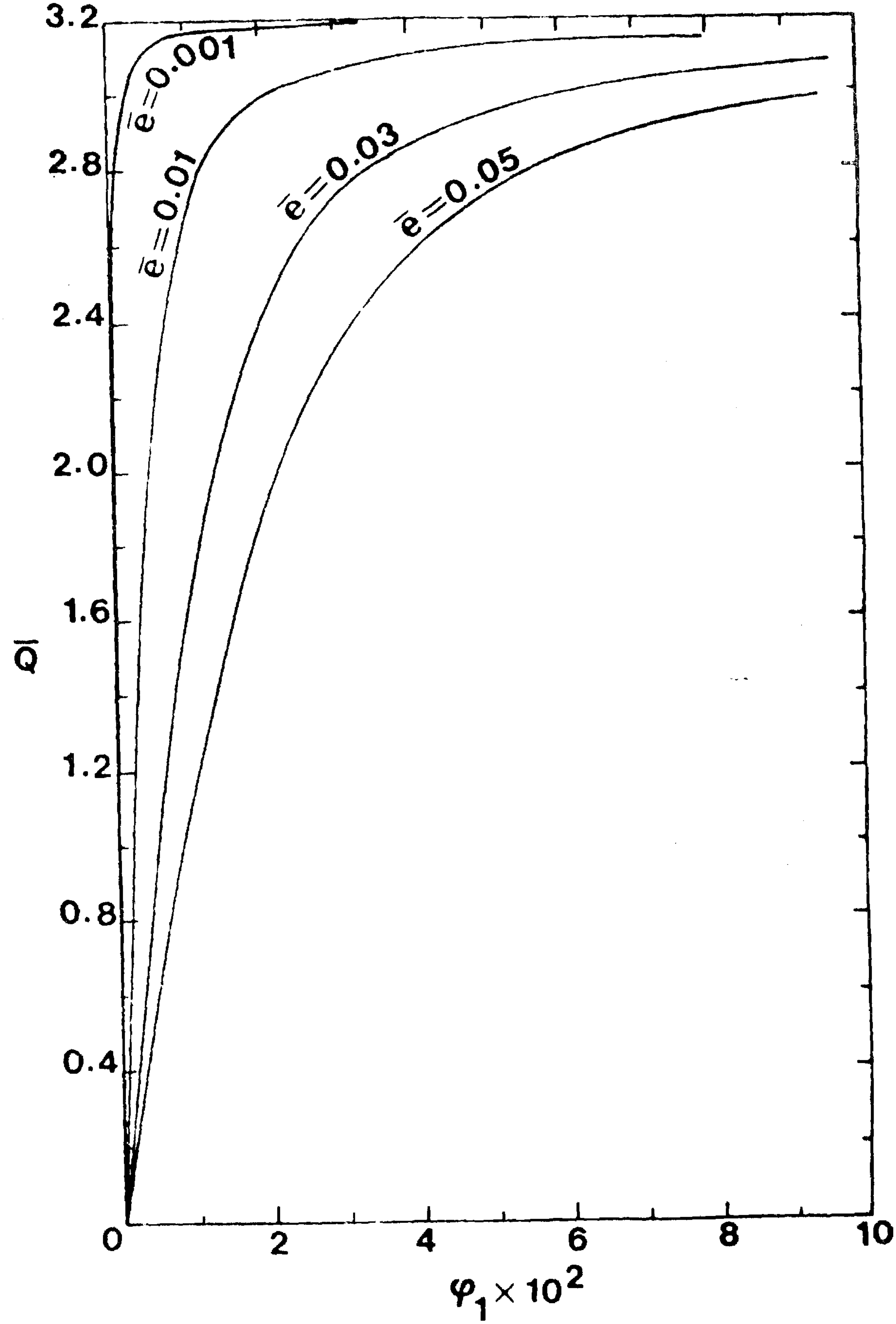
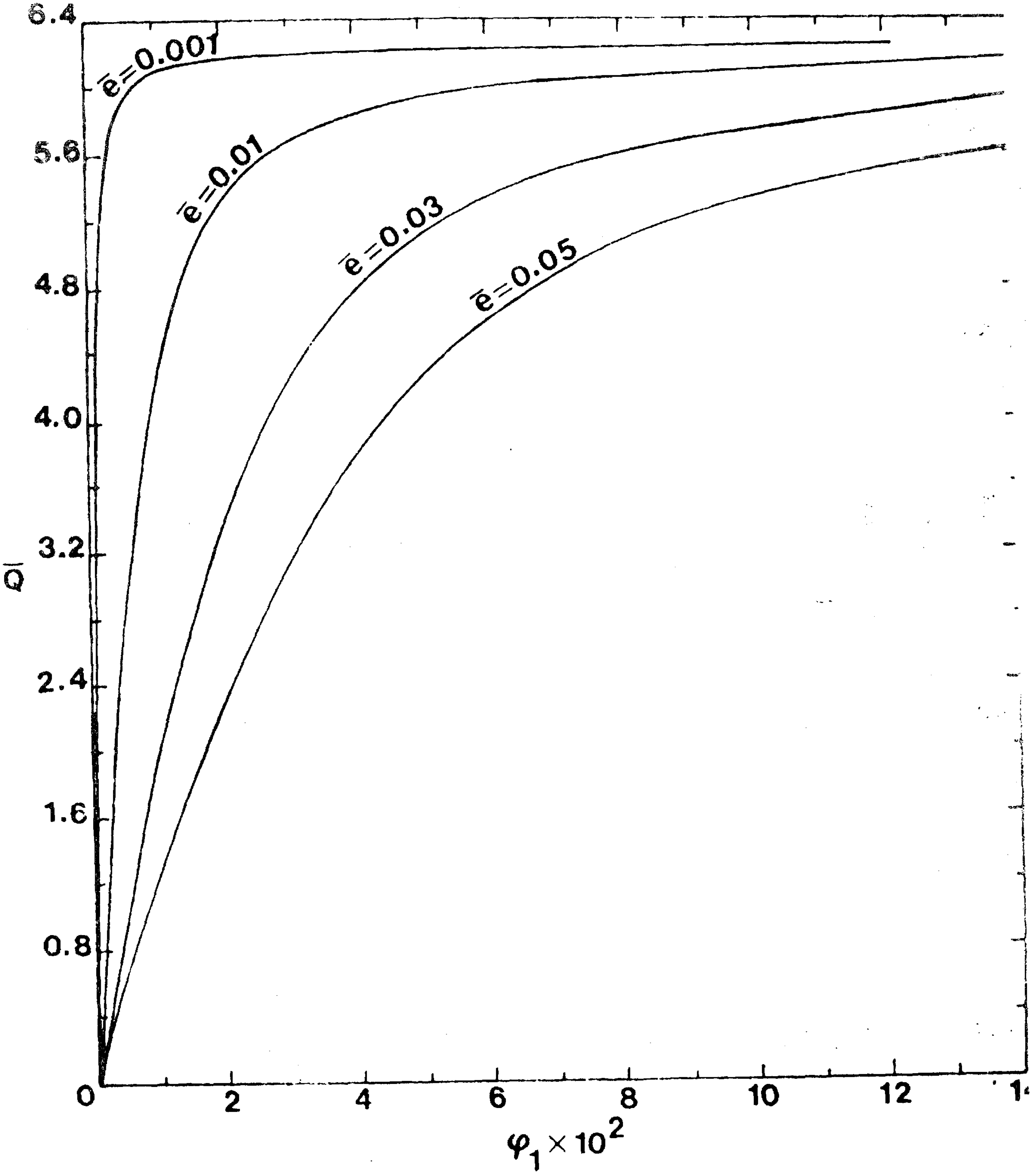


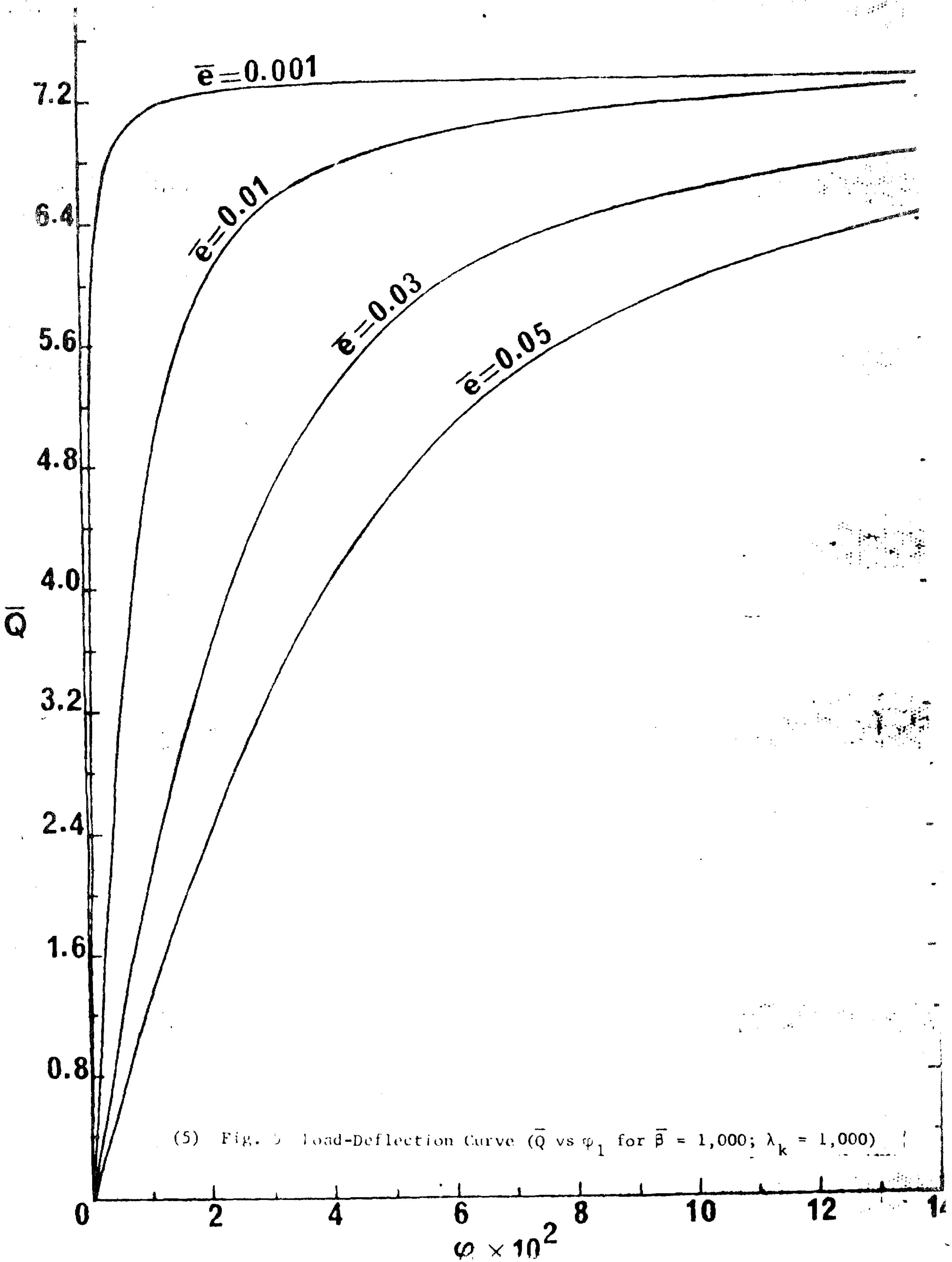
(2) Fig. 2 Load-Deflection Curve ($\overline{Q}$ vs φ_1 for $\bar{\beta} = 0$; λ_k = 1,000)



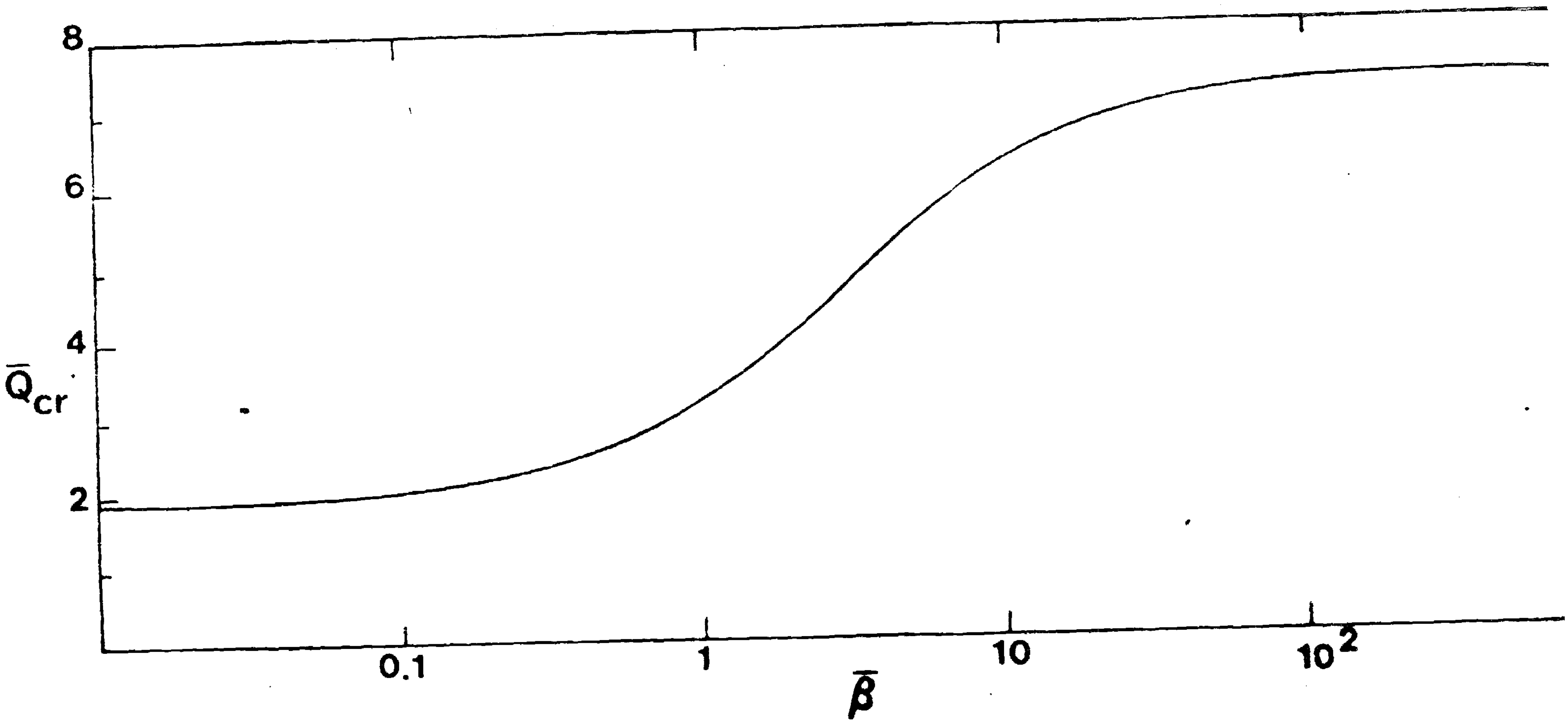
(3) Fig. 3 Load-Deflection Curve ($\bar{Q}$ vs φ_1 for $\bar{\beta}$ = 1; λ_k = 1,000)



(4) Fig. 4 ...d-Deflection Curve ($\bar{Q}$ vs φ_1 for $\bar{\beta}$ =10; λ_k = 1,000)



(5) Fig. 5 Load-Deflection Curve ($\bar{Q}$ vs φ_1 for $\bar{\beta}$ = 1,000; λ_k = 1,000)



(6) Fig. 6 Effect of Rotational Restraint on the Sway Buckling Load

Key Words: Nonlinear Stability Analysis, Buckling of Frames; Eccentrically Loaded Frames; Portal Frames; Rotationally Restrained Frames; Sway-buckling.

Abstract: The problem of sway-buckling of an unbraced, rigid-jointed, eccentrically loaded and elastically restrained portal frame is considered. The analysis is based on nonlinear kinematic relations (moderate rotations) and on linearly elastic material behavior. The effects of load eccentricity, amount of rotational restraint and bar slenderness ratios on the response characteristics of the frame are assessed. Among the most important conclusions of the investigation one may list (a) Portal frames are insensitive to load eccentricities (stable postbuckling branching) (b) the effect of slenderness ratio is negligibly small and (c) the larger the rotational restraint, the greater the buckling load (for the perfect configuration - zero load eccentricity).

NONLINEAR ANALYSIS OF PORTAL FRAMES[+]

By George J. Simitses[1], Jagannath Giri[2], and Anthony N. Kounadis[3]

INTRODUCTION

A kinematically nonlinear analysis of an unbraced portal frame, which is elastically restrained at the base against rotation and loaded through eccentric concentrated and/or uniformly distributed loads, is presented. Through this analysis, it is intended to assess the effect of load eccentricity, member slenderness ratio and amount of rotational restraint on the frame behavior. It is well known that, when portal frames are loaded as stated above, they deform in a symmetric mode and then at some level of the load a bifurecation (smooth buckling) occurs into a sway-buckling mode. Many analyses have been reported in the open literature (see Refs 1, 3 and 6 for a comprehensive historical sketch) which predict the bifurcation load, but through the present analysis the complete postbuckling behavior is obtained. This enhances our understanding of frame behavior with regard to the questions of imperfection sensitivity. Moreover, the present analysis and solution methodology are general so that one can easily study the effects of nonuniform geometry including variable frame bar lengths, extensional and flexunal stiffnesses.

MATHEMATICAL FORMULATION

Consider the frame shown on Fig. 1. Each bar is of length ℓ_k, cross-sectional area A_k, and cross-sectional second moment of area I_k. The in-

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1. Professor of Eng. Sci. and Mech.; Georgia Tech, Atlanta, Georgia
2. Research Engineer; Georgia Tech, Atlanta, Georgia
3. Associate Prof. of Civil Eng.; National Technical University of Athens, Athens, Greece.

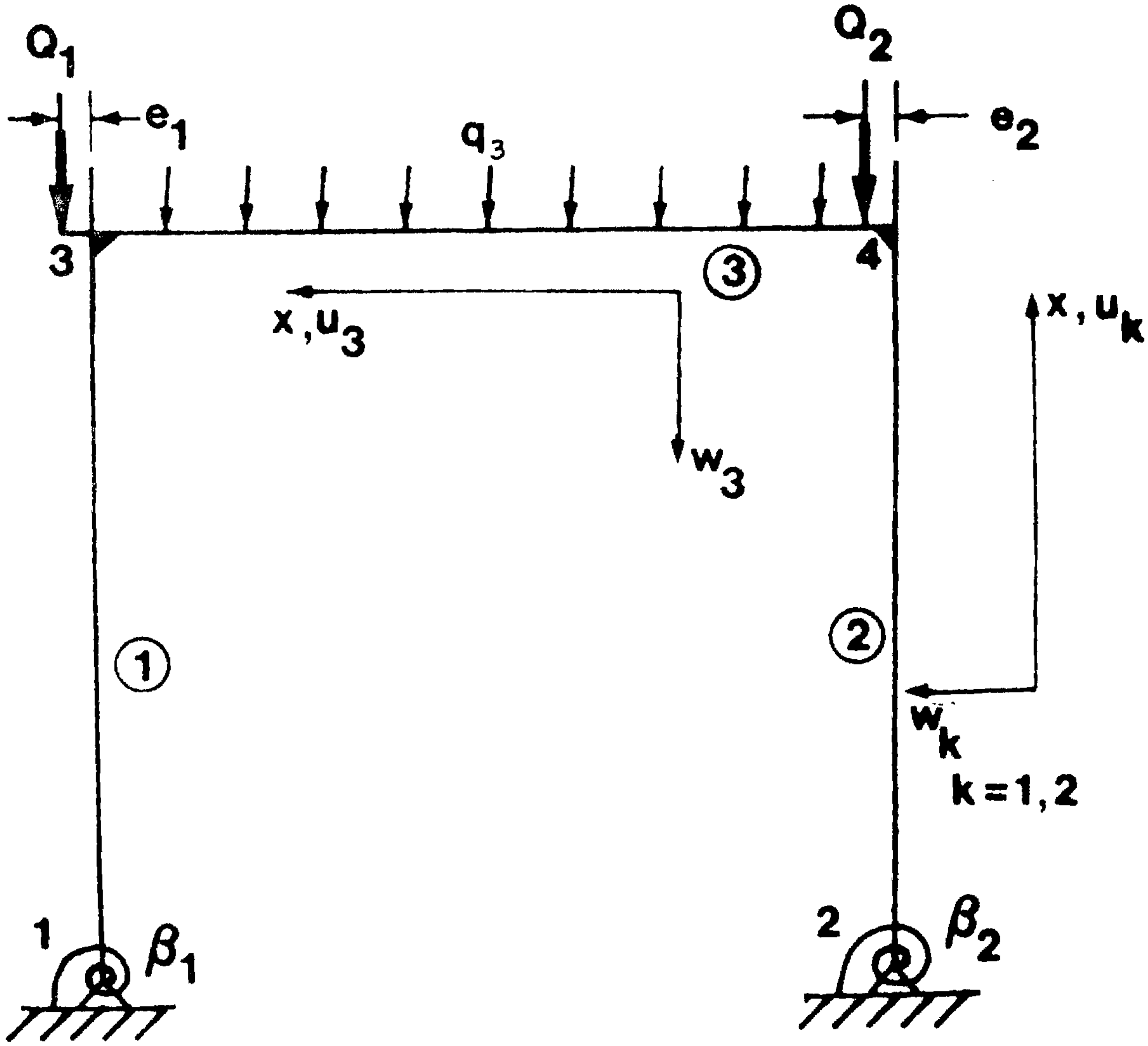


Figure 1 Geometry and Sign Convention

plane and transverse displacement components are denoted by u_k and w_k, and the sign convention is shown on the figure. The loading system consists of a uniformly distributed load q_3 and two concentrated loads Q_1 and Q_2 applied eccentrically as shown. The eccentricities, e_i, are shown in the positive direction on Fig. 1. The rotational restraints at the supports are denoted by β_1 and β_2 and the joints by 1, 2, 3 and 4. The material behavior is assumed to be linearly elastic.

First, the following nondimensionalized parameters are introduced

$$r_k = (EI)_k/(EI)_1; \; \mu_k = \ell_k/\ell_1; \; X = x/\ell_k;$$

$$U_k = u_k/\rho_k; \; W_k = w_k/\ell_k; \; \rho_k^2 = I_k/A_k; \; \lambda_k = \ell_k/\rho_k;$$

$k_k^2 = \pm (P_k \ell_k^2)/(EI)_k$ (+ for tension; P_k positive; - for compression, (1)

P_k negative; P_k force in the bar; $\overline{Q}_i = (Q_i \ell_1^2)/(EI)_1$;

$\overline{e}_i = e_i/\ell_1$; $\overline{\beta}_i = \beta_i \ell_1/(EI)_1$, $i = 1,2.$; $q_k^* = q_k \ell_k^3/(EI)_k$

(Note $q_1 = q_2 = 0$).

Equilibrium Equations

Considering the possibility that bar "3" can be either in tension or in compression, while bars "1" and "2" are always in compression, the equilibrium equations and the associated boundary and joint conditions are given by

In-Plane Equilibrium

$$U_{k,X} + \frac{1}{2} W_{k,X}^2 = -k_k^2/\lambda_k^2, \; k = 1, 2, 3 \text{ (compression)}$$

(2)

$$U_{3,X} + \tfrac{1}{2} W_{3,X}^2 = k_3^2/\lambda_3^2 \text{ (tension)}$$

Transverse Equilibrium

$$W_{k,XXXX} + k_k^2 W_{k,XX} = q_k^*, \; k = 1,2,3. \quad \text{Compression)}$$

$$W_{3,XXXX} - k_3^2 W_{3,XX} = q_3^* \quad \text{(Tension} \tag{3}$$

Boundary Conditions

1. $U_1(0) = 0; \; W_1(0) = 0; \; W_{1,XX}(0) - \bar{\beta}_1 W_{1,X}(0) = 0$ (4)

2. $U_2(0) = 0; \; W_2(0) = 0; \; W_{2,XX}(0) - \bar{\beta}_2 W_{2,X}(0) = 0$ (5)

Joint Conditions

Joint 3. $U_1(1) + W_3(1) = 0; \; W_1(1) - U_3(1) = 0; \; W_{1,x}(1) - W_{3,x}(1) = 0$

$$\bar{e}_1\bar{Q} - W_{1,XX}(1) - (r_3/\mu_3)\, W_{3,XX}(1) = 0$$

$$k_1^2 - \bar{Q}_1 - (r_3/\mu_3^2 \left[\pm k_3^2\, W_{3,X}(1) + W_{3,XXX}(1)\right] = 0 \tag{6}$$

$$\pm\left(\frac{r_3}{\mu_3^2}\right) k_3^2 + \left[k_1^2 W_{1,X}(1) + W_{1,XXX}(1)\right] = 0$$

$$\begin{pmatrix} + : & \text{compression} \\ - : & \text{tension} \end{pmatrix}$$

Joint 4. $U_2(1) + W_3(0) = 0; \; W_2(1) - U_3(0) = 0; \; W_{2,X}(1) - W_{3,X}(0) = 0$

$$\bar{e}_2\bar{Q}_2 + \left(\frac{r_3}{\mu_3}\right) W_{3,XX}(0) - \left(\frac{r_2}{\mu_2}\right) W_{2,XX}(1) = 0$$

(7)

$$\left(\frac{r_2}{\mu_2^2}\right) k_2^2 - \bar{Q}_2 + \left(\frac{r_3}{\mu_3^2}\right) \left[\pm k_3^2 W_{3,X}(0) + W_{3,XXX}(0)\right] = 0$$

$$\pm \left(\frac{r_3}{\mu_3^2}\right) k_3^2 - \left(\frac{r_2}{\mu_2^2}\right) \left[k_2^2 W_{2,X}(1) + W_{2,XXX}(1)\right] = 0$$

The solution to the equilibrium equations, Eq. (2), is characterized by

<u>bar in compression</u>

$$W_k(X) = A_{k1} \sin k_k X + A_{k2} \cos k_k X + A_{k3} X + A_{k4} + \frac{q_k^* X^2}{2k_k^2}$$

$$U_k(X) = U_{ko} - \frac{k_k^2}{\lambda_k^2} X - \frac{1}{2} \int_0^X W_{k,X}^2 dX \qquad k=1,2,3 \tag{8}$$

<u>bar in tension</u> (only bar "3")

$$U_3(X) = U_{30} + \frac{k_3^2}{\lambda_3^2} X - \frac{1}{2} \int_0^X W_{k,X}^2 dX$$

$$W_3(X) = A_{31} \sinh k_3 X + A_{32} \cosh k_3 X + A_{33} X + A_{34} - \frac{q_3^* X^2}{2k_3^2} \tag{9}$$

Note that when bar "3" is in compression Eqs. (8) characterize the solution (for $k = 1,2,3$), while when bar "3" is in tension the solution is characterized by Eqs. (8) for $k = 1,2$ and Eqs. (9). Furthermore note that the solution contains 18 constants, which for every load level and for a specified geometry must be obtained from the 18 boundary and joint conditions, Eqs. (4) - (7). These 18 constants are: U_{k0} $(k=1,2,3)$, k_k $(k = 1,2,3)$ and A_{ki} ($k= 1,2,3$ and $i = 1,2,3,4$). Moreover, if one is interested in finding all equilibrium positions, for a wide range of load values, we must solve the resulting inhomogeneous nonlinear system of 18 equations in 18 constants, and then plot these positions in a load-deflection curve. In so doing, it is found that the constants U_{ko} and A_{ki} (fifteen) appear in a linear sense and then can be eliminated, thus leaving three nonlinear equations in k_k. (The details are not shown herein for the sake of saving space). Then, one must solve these nonlinear equations in order to completely characterize the response of the frame (the procedure is outlined in a later section).

From an academic point of view, by following the above procedure one should be able to start with low value for the applied loads and obtain the primary response, then at a point of bifurcation, he should be able to solve for both the primary response as well as the branched path. Thus he should be able to obtain the buckling load (bifurcation load) as well as the post-buckling behavior. Unfortunately, because of the nonlinearity of the response this procedure is difficult to implement, unless one can establish the bifurcation point. For this reason one must derive the associated buckling equations, and incorporate their solution into the overall solution scheme.

Buckling Equations

The buckling equations and the associated boundary and joint conditions are obtained from Eqs. (2) - (7) by replacing W_k and U_k by $\bar{W}_k + \tilde{W}_k$ and $\bar{U}_k + \tilde{U}_k$, where $\bar{W}_k$ and $\bar{U}_k$ characterize displacement components on the primary equilibrium path, and $\tilde{W}_k$ and $\tilde{U}_k$ characterize kinematically admissible displacement components (buckling modes from the primary path).

Buckling Equations

$$\tilde{U}_{k,X} + \bar{W}_{k,X}\,\tilde{W}_{k,X} = \tilde{\sigma}_k/\lambda_k^2$$

$$\tilde{W}_{k,XXXX} \pm \bar{k}_k\,\tilde{W}_{k,XX} = \tilde{\sigma}_k\,\bar{W}_{k,XX} \qquad (10)$$

where $\bar{k}_k$ is the axial force parameter in the bar for primary path equilibrium positions; the positive sign is used when there is compression in the bars (primary path), and the negative when there is tension $\tilde{\sigma}_k = (\tilde{\varphi}_k \ell_k^2)/(EI)_k$ and it can be either positive or negative regardless of whether the bar is in tension or compression in the primary path, and $k_k^2 = \bar{k}_k^2 \mp \tilde{\sigma}_k$; negative sign used when the primary path axial force in the bar is compressive and the positive sign when it is tensile; since this affects only bar "3" one can write

$$k_k^2 = \bar{k}_k^2 - \tilde{\sigma}_k \qquad k = 1,2 \text{ and}$$

$$k_3^2 = \bar{k}_3^2 - \tilde{\sigma}_3 \qquad \text{compr. in bar "3' (primary path)}$$

$$k_3^2 = \bar{k}_3^2 + \tilde{\sigma}_3 \qquad \text{tension in bar "3" (primary path)}$$

Boundary Conditions

1. $$\tilde{U}_1(0) = \tilde{W}_1(0) = \tilde{W}_{1,XX}(0) - \bar{\beta}_1 \tilde{W}_{1,X}(0) = 0 \qquad (11)$$

2. $$\tilde{U}_2(0) = \tilde{W}_2(0) = \tilde{W}_{2,XX}(0) - \bar{\beta}_2 \tilde{W}_{2,X}(0) = 0 \qquad (12)$$

Joint Conditions

Joint 3: $$\tilde{U}_1(1) + \tilde{W}_3(1) = \tilde{U}_3(1) - \tilde{W}_1(1) = \tilde{W}_{1,X}(1) - \tilde{W}_{3,X}(1) = 0$$

$$\tilde{W}_{1,XX}(1) + \left(\frac{r_3}{\mu_3}\right) \tilde{W}_{3,XX}(1) = 0$$

$$\tilde{\sigma}_1 - \left(\frac{r_3}{\mu_3^2}\right)\left[\pm\bar{k}_3^2 \tilde{W}_{3,X}(1) - \tilde{\sigma}_1 \bar{W}_{1,X}(1) + \tilde{W}_{3,XX}\right] + 0 \qquad (13)$$

$$-\left(\frac{r_3}{\mu_3^2}\right)\tilde{\sigma}_3 + \bar{k}_1^2 \tilde{W}_{1,X}(1) - \tilde{\sigma}_1 \bar{W}_{1,X}(1) + \tilde{W}_{1,XXX}(1) = 0$$

+ : Compression in bar "3" (primary path)

- : tension in bar "3" (primary path)

Joint 4: $\tilde{U}_2(1) + \tilde{W}_3(0) = \tilde{U}_3(0) - \tilde{W}_2(1) = \tilde{W}_{2,X}(1) - \tilde{W}_{3,}(0) = 0$

$$\left(\frac{r_3}{\gamma_3}\right) \tilde{W}_{3,XX}(0) - \left(\frac{r_2}{\mu_3}\right) \tilde{W}_{2,XX}(1) = 0$$

$$-\left(\frac{r_2}{\mu_2^2}\right) \tilde{\sigma}_2 + \left(\frac{r_3}{\mu_3^2}\right)\left[\pm\bar{k}_3^2 \tilde{W}_{3,X}(0) - \tilde{\sigma}_3 \bar{W}_{3,X}(0) + \tilde{W}_{3,XXX}(0)\right] = 0 \tag{14}$$

$$-\left(\frac{r_3}{\mu_3^2}\right)\tilde{\sigma}_3 - \left(\frac{r_2}{\mu_2^2}\right)\left[\bar{k}_2^{2} \tilde{W}_{2,X}(1) - \tilde{\sigma}_2 \bar{W}_{2,X}(1) + \tilde{W}_{2,XXX}(1)\right] = 0$$

+ : compression in bar "3" (primary path).

- : tension in bar "3" (primary path).

The expressions for the solution to the Buckling equations, Eqs. (10), for each bar and each case of tension or compression is characterized by

bars "1" and "2" (in compression)

$$\tilde{W}_k(X) = \tilde{A}_{k1} \sin \bar{k}_k X + \tilde{A}_{k2} \cos \bar{k}_k X + \tilde{A}_{k3} X + \tilde{A}_{k4}$$

$$+ \frac{\tilde{\sigma}_k X}{2\bar{k}_k} (A_{k2} \sin \bar{k}_k X - A_{k1} \cos \bar{k}_k X) \tag{15}$$

$$\tilde{U}_k(X) = \tilde{U}_{k0} + \frac{\tilde{\sigma}_k X}{\lambda_k^2} - \int_0^X \bar{W}_{k,X} \tilde{W}_{k,X} \, dX; \; k = 1,2$$

bar "3" (in compression)

$$\tilde{W}_3(X) = \tilde{A}_{31} \sin \bar{k}_3 X + \tilde{A}_{32} \cos \bar{k}_3 X + \tilde{A}_{33} X + \tilde{A}_{34}$$

$$+ \frac{\tilde{\sigma} X}{2\bar{k}_3} \left(A_{32} \sin \bar{k}_3 X - A_{31} \cos \bar{k}_3 X + \frac{q_3^* X}{\bar{k}_3^3} \right) \qquad (16)$$

$$\tilde{U}_3(x) = \tilde{U}_{30} + \frac{\tilde{\sigma}_3 X}{\lambda_3^2} - \int_0^X \bar{W}_{3,X} \tilde{W}_{3,X} \, dX \; .$$

bar "3" (in tension)

$$\tilde{W}_3(X) = \tilde{A}_{31} \sinh \bar{k}_3 X + \tilde{A}_{32} \cosh \bar{k}_3 X + \tilde{A}_{33} X + \tilde{A}_{34}$$

$$+ \frac{\tilde{\sigma}_3 X}{2\bar{k}_3} \left(A_{32} \sinh \bar{k}_3 X + A_{31} \cosh \bar{k}_3 X + \frac{q_3^* X}{\bar{k}_3^3} \right. \qquad (17)$$

$$\tilde{U}_3 X = \tilde{U}_{30} + \frac{\tilde{\sigma}_3 X}{\lambda_3^2} - \int_0^X \bar{W}_{3,X} \tilde{W}_{3,X} \, dX \; .$$

where $\bar{k}_k$ denotes the axial force parameter at the primary path (solution to equilibrium equations) and A_{k1} and A_{k2} are the constants of the primary path solution to the equilibrium equations.

Note that the solution to the buckling equations contains 18 constants $\tilde{U}_{ko}$, $\tilde{\sigma}_k$, $\tilde{A}_{ki}$; k = 1, 2, 3 and i = 1, 2, 3, 4). Moreover, the boundary and joint conditions result into a system of 18 linear homogeneous algebraic equations in the 18 constants. For a nontrivial solution to exist, the determinant of the coefficients must vanish. The vanishing of the determinant yields the critical load condition (characteristic equation). The derivation of and the determinant are not shown herein for the sake of brevity.

SOLUTION

Regardless of whether the axial force in bar "3" is tensile or compressive, the solution procedure is the same. This procedure consists of the following steps:

(1) Substitution of the expression for W_k and U_k into the boundary and joint conditions yields a system of 18 nonlinear equations in 18 constants (U_{ko}, k_k, and A_{ki}; k = 1, 2, 3, and i = 1, 2, 3, 4).

(2) Since 15 of these equations are linear in U_{ko}, and A_{ki}, elimination of these constants yields a system of three nonlinear equations in k_k as well as in the structural geometry, μ_k, λ_k, r_k, $\bar{\beta}_j$, and in the loading parameters, $\bar{e}_1$, $\bar{e}_2$, $\bar{Q}_1$, $\bar{Q}_2$, and q^*.

(3) One of the three nonlinear equations contains only k_1^2, k_2^2, the loading parameters, and the geometric parameters. This equation is then used to eliminate one of the k's, thus leaving only two nonlinear equations to solve for the response, say

$$f_i(k_2,k_3,\lambda_k,r_k,\mu_k,\bar{\beta}_j,\bar{e}_j,\bar{Q}_j,q^*) = 0 \qquad k= 1,2,3;\ j = 1,2;\ \text{and } i = 1,2. \tag{18}$$

(4) For every level of the load parameters, Eqs. (18) are solved by finding k_2 and k_3 values for which

$$F = f_1^2 + f_2^2 \tag{19}$$

is a minimum in the space of k_1, k_2. Note that this minimum is zero. The simplex method of Nelder and Mead (5) is employed in obtaining the minimum value of F and the minimizing values of k_2 and k_3.

(5) At each load level, use of the eliminating equations yields the corresponding values for k_1, U_{ko} and A_{ki}. Thus the complete equilibrium response is obtained.

(6) Evaluation of the determinant at each load level establishes the position of the bifurcation point (determinant equal to zero).

(7) Once the bifurcation point is established application of steps 3) -5), with slightly lower or higher values for the applied loading, provides a point on the bifurcation branch (postbuckling equilibrium). Then through small changes in the applied loading the remaining postbuckling equilibrium points are obtained.

(8) The complete behavior then is presented as a plot of load parameter versus some characteristic displacement. In the present work the joint rotations are used for this purpose.

Note that for some load cases, such as eccentric concentrated loads on the same side (both eccentricities the same) there is no bifurcational buckling. In those cases equilibrium behavior is established through steps 1) through 5) plus 8).

RESULTS AND DISCUSSION

Numerical solutions are generated for a square portal frame with equal bending stiffnesses and slenderness ratios ($r_k = \mu_k = 1$; $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$), by employing the Georgia Tech high-speed digital computer CDC-Cyber 70, Model 74-28.

The primary reason for the chosen examples is to enhance our understanding of frame behavior and to assess the effect of certain geometric parameters such as slenderness ratio, λ, amount of rotational restraint ($\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}$), and load eccentricity ($\bar{e}_i$).

The results are presented and discussed separately according to the load cases and amount of rotational restraint.

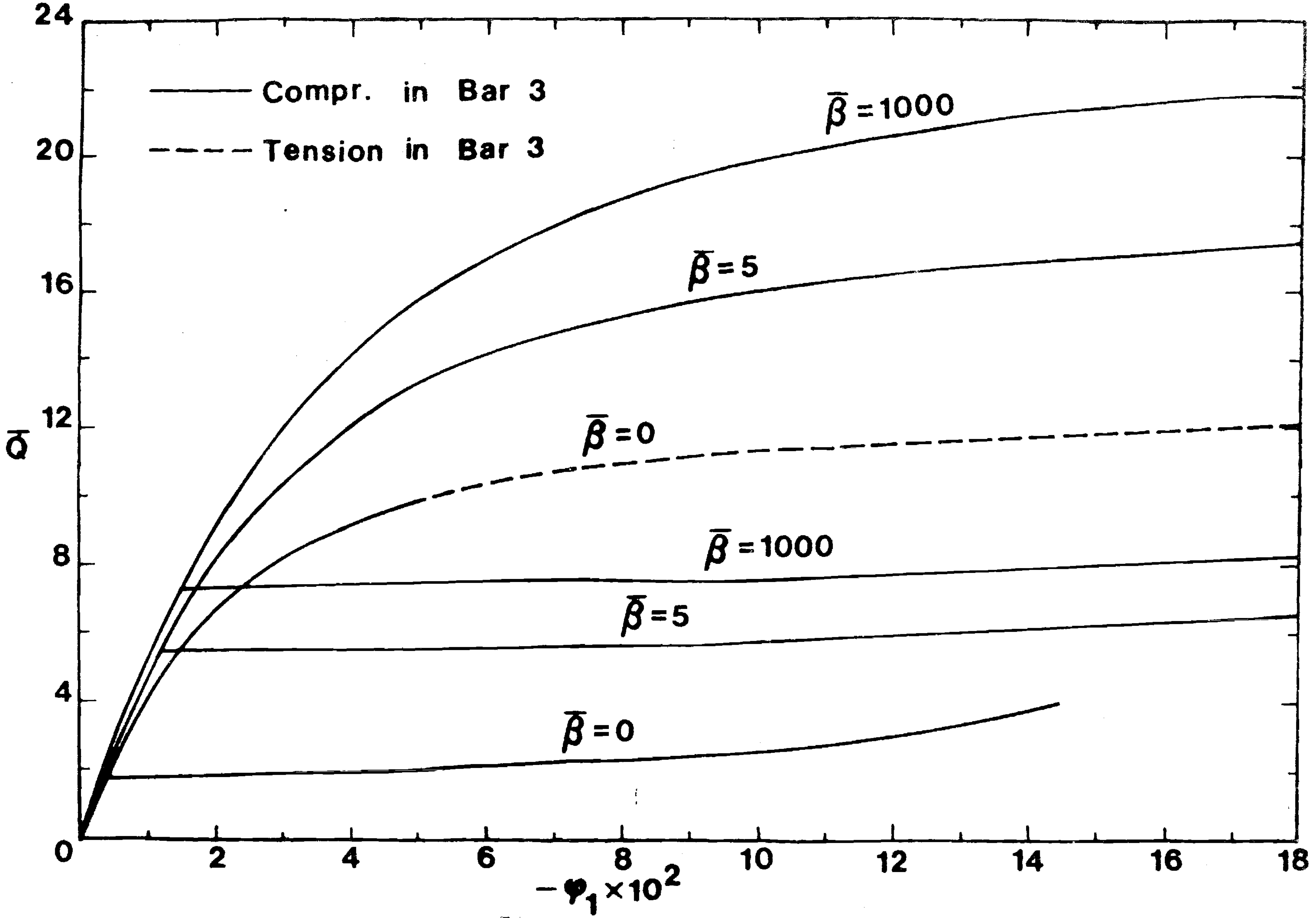


Figure 2 Equilibrium Response of a Symmetrically Loaded Square Frame

($-\bar{e}_1 = \bar{e}_2 = 0.01$; $\lambda_k = 1{,}000$; $\mu_k = 1$; $r_k = 1$)

A) Rotationally Restrained and Symmetrically Loaded Frame Through Eccentric Concentrated Loads.

In this case results are generated for the following parameters.

$$-\bar{e}_1 = \bar{e}_2 = \bar{e} = 0.001,\ 0.005,\ 0.010,\ 0.03,\ 0.05,\ 0.07,\ 0.10.$$

$$\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta} = 0,\ 1,\ 5,\ 10,\ 100,\ 1000.$$

$$\lambda = 40,\ 80,\ 120,\ 1000$$

The results are presented, in part, graphically on Figs. 2, 3, 4 and 5 for this example. The conclusions though are based on all generated data.

For all combinations of rotational restraint, $\bar{\beta}$, and eccentricity $\bar{e}$, it is observed that the effect of slenderness ratio, λ, is insignificant. This means that the nondimensionalized results are not affected by variations in λ-values. Because of this, data are presented only for $\lambda = 1000$ (extremely slender bars), but the results are applicable to all other values for λ.

Figure 2 shows plots of $\bar{Q}$ versus joint "3" rotation, $\varphi_1 \left[\varphi_1 = W_{1,\bar{x}}^{(1)}\right]$, for $\bar{\beta} = 0$, 5, and 1,000, and $\bar{e} = 0.01$. The case of $\bar{\beta} = 0$ corresponds to the simply supported case, while the case of $\bar{\beta} = 1{,}000$ approximates well the clamped case. These plots represent equilibrium positions on the primary path as well as the post Buckling branch (sway-buckling mode). The solid curve characterizes compression in the horizontal bar (bar "3"), while the dashed line curve characterizes tension in the horizontal bar. Moreover, it is seen from this figure that postbuckling behavior suggests that frames are imperfection insensitive, and their postbuckling behavior is similar to that of a cantilever column. Therefore, the sway-buckling load (bifurcation point) is a measure of the load carrying capacity for a symmetrically loaded unbraced portal frame. Similar curves are obtained for the various eccentricities, but are not shown herein, for the sake of saving space.

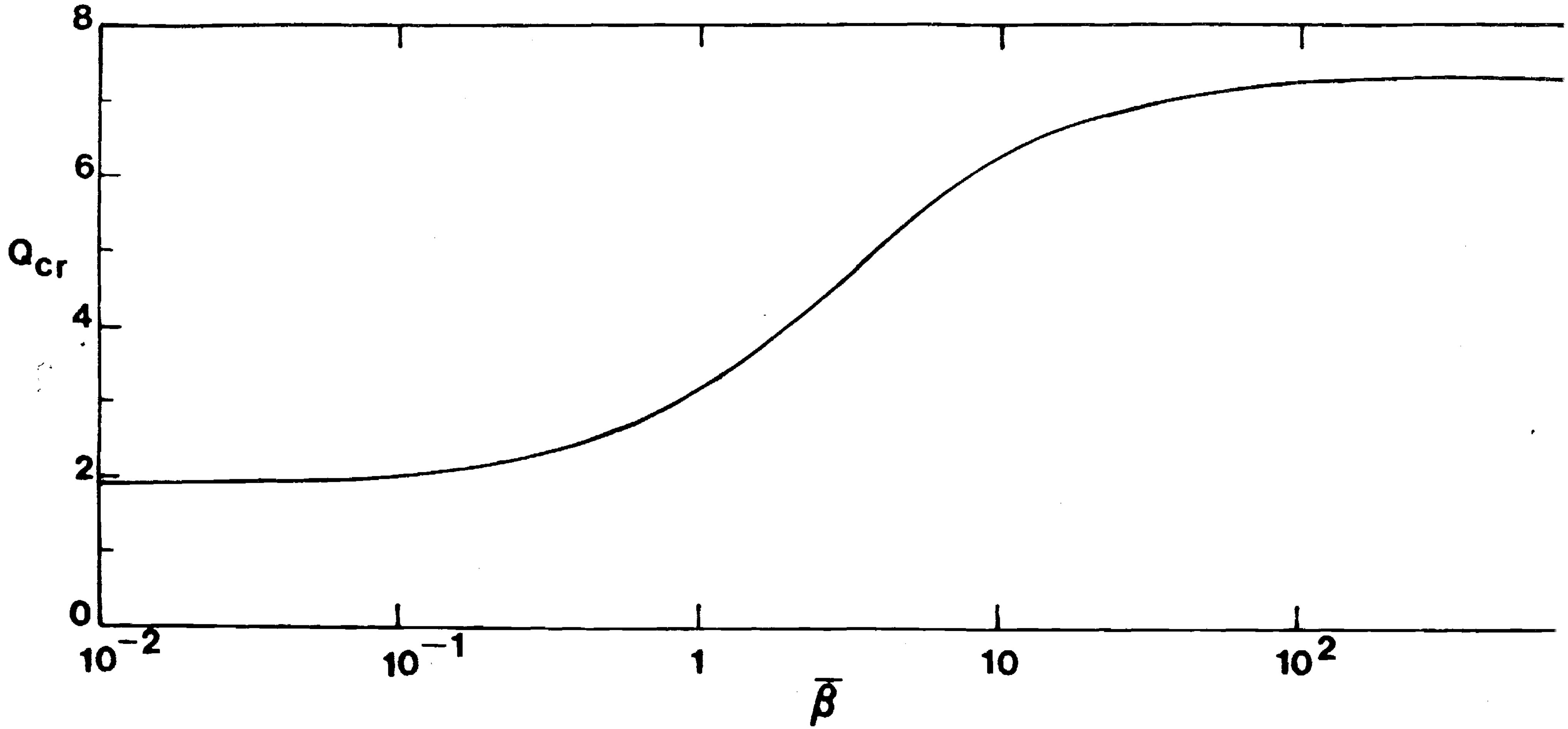


Figure 3 Effect of Rotational Restraint on Critical Load
($-\bar{e}_1 = \bar{e}_2 = 0.01$; $\lambda_k = 1{,}000$; $r_k = 1$; $\mu_k = 1$)

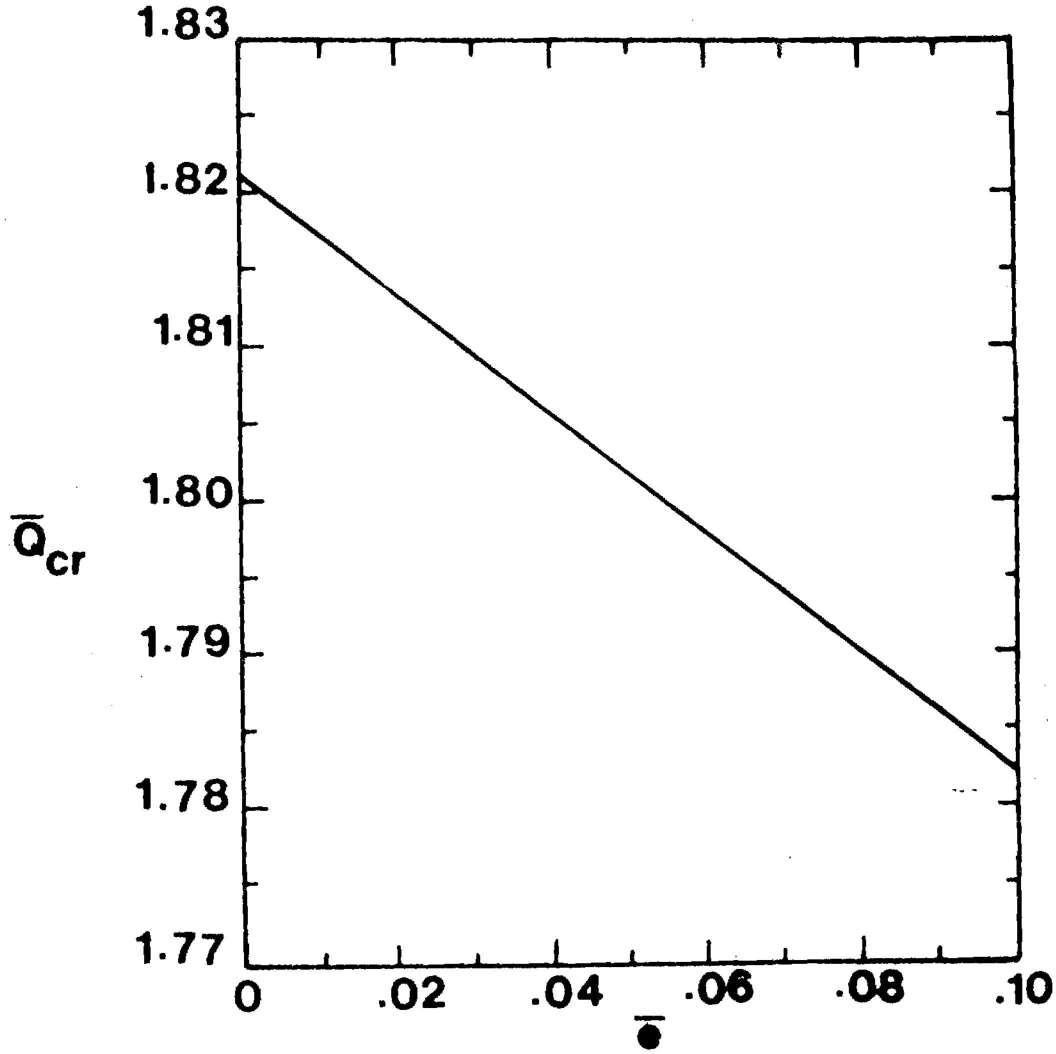
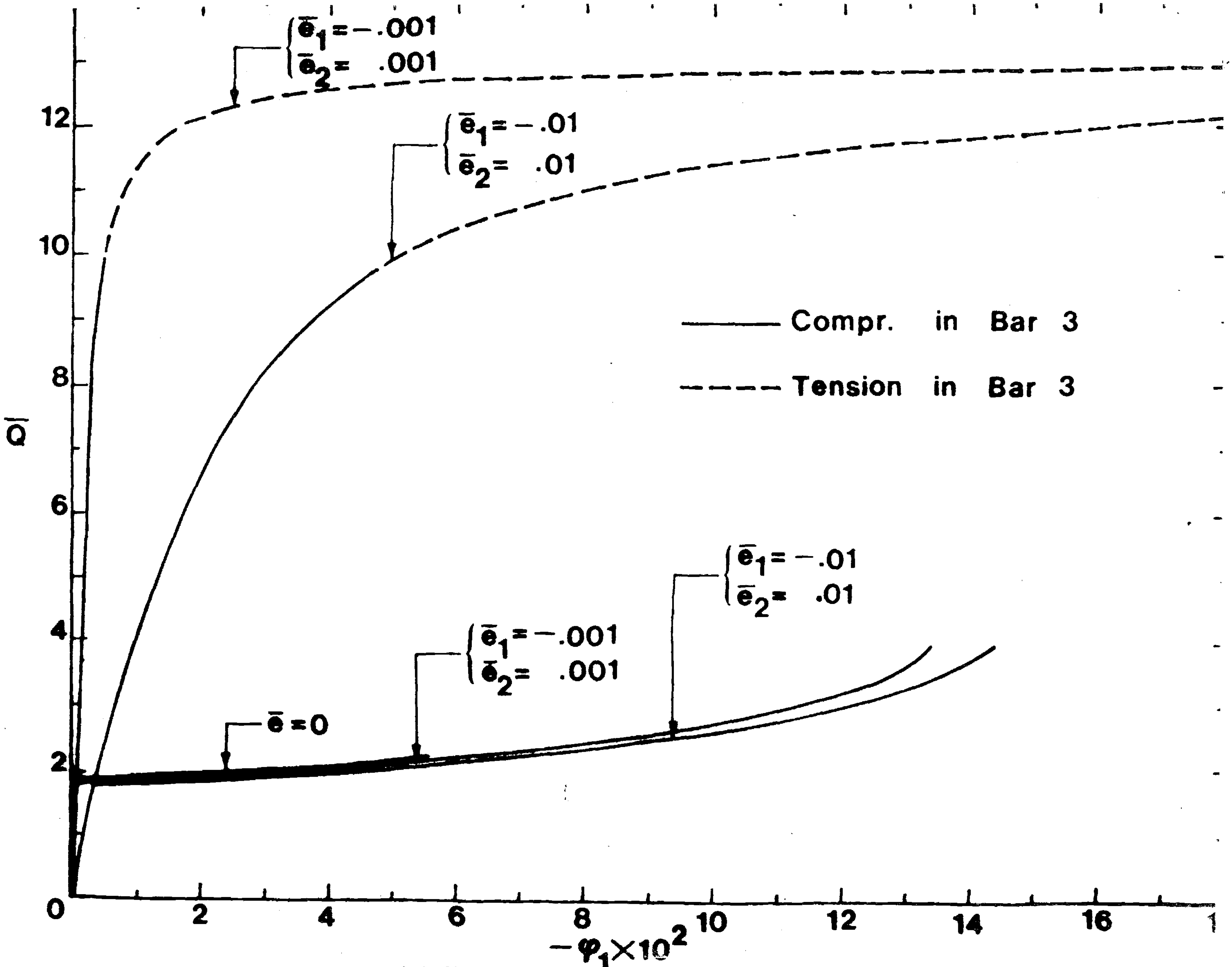


Figure 4 Effect of Load Eccentricity on Critical Load for a Symmetrically Loaded Square Frame (λ_k = 1,000; $r_k = \mu_k$ = 1; β = 0)



Equilibrium Response for a Symmetrically Loaded, Simply Supported

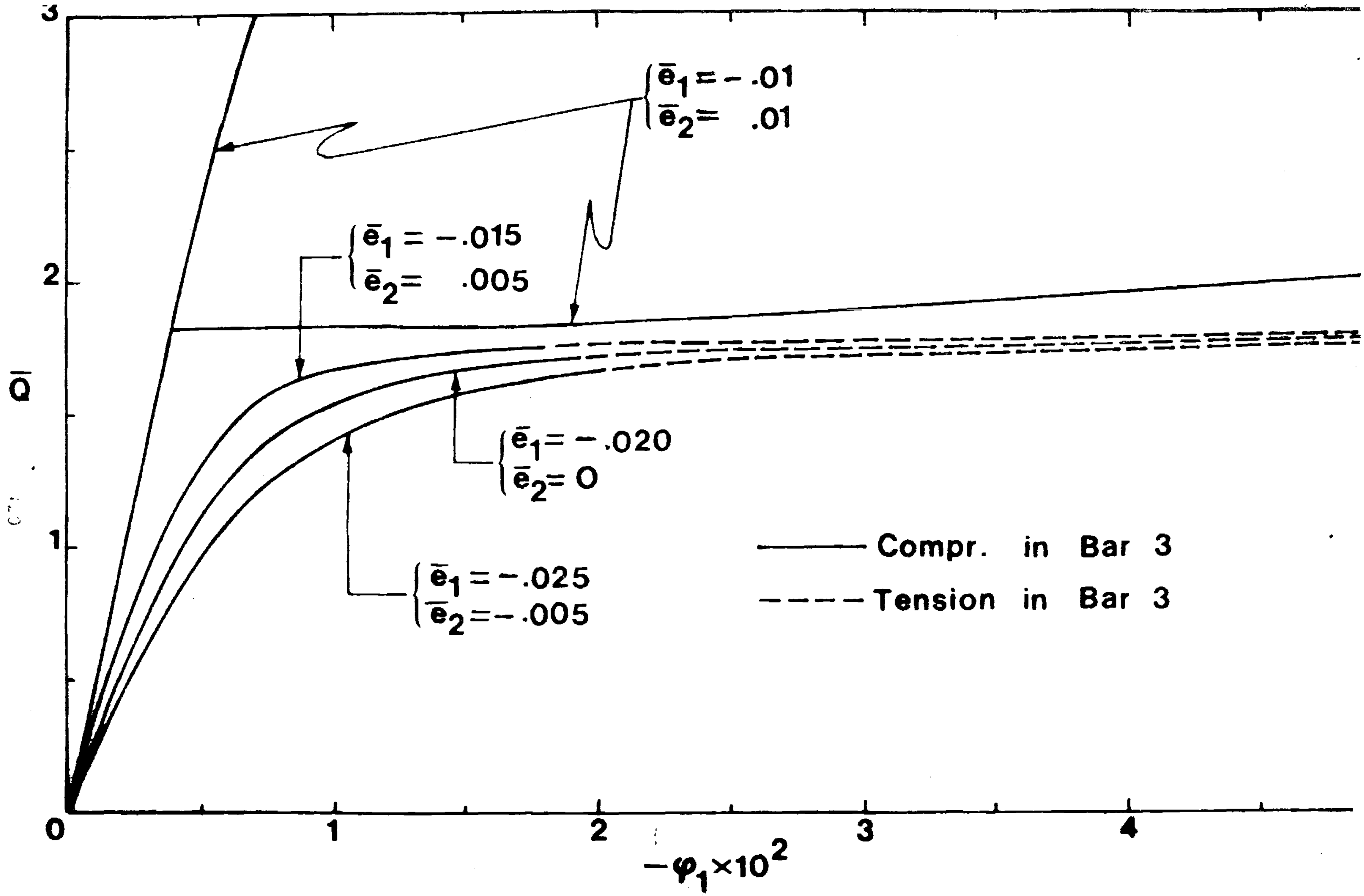


Figure 6 Effect of Nonsymmetrical Load Eccentricity

($\lambda_k = 1{,}000$; $r_k = \mu_k = 1$; $\bar{\beta} = 0$)

B) Eccentrically Loaded Supply Supported Frame with Unequal Eccentricities.

In this case results are generated for the case of two equal magnitude ($\bar{Q}_1 = \bar{Q}_2$) concentrated loads being applied at ($\bar{e}_1 = -0.015$, $\bar{e}_2 = 0.005$), ($\bar{e}_1 = -0.020$, $\bar{e}_2 = 0$), and ($\bar{e}_1 = -0.025$, $\bar{e}_2 = -0.005$). This is done to find the effect of moving both eccentric loads to the right by the same amount, starting from the symmetric load case of $\bar{e}_1 = -\bar{e}_2 = -0.010$. The results are plotted on Fig. 6 as $\bar{Q}$ versus φ_1. As expected, there is no problem of buckling, but the response is such, that the frame cannot carry a load higher than $\bar{Q}_{cr}$ for $\bar{e}_1 = \bar{e}_2 = -0.010$. In all three cases of eccentricities considered as the load is increased quasi-statically from zero, the response is characterized by compression in the horizontal member. As the load approaches the bifurcation load for the symmetric loading ($\bar{e}_1 = -\bar{e}_2 = -0.010$) the response is characterized by tension in the horizontal bar and the curves seem to approach a horizontal asymtote $\bar{Q} = \bar{Q}_{bifurcation}$ rather than the postbuckling curve.

C) Simply supported Frame Loaded by a Uniformly Distributed Load and Two Eccentrically Applied Concentrated Loads.

For this particular example only one eccentricity set is used, $\bar{e}_1 = \bar{e}_2 = -0.010$, and $\lambda = 1{,}000$. Because both eccentricities are of the same sign, there is no buckling problem. The total transverse load is denoted by $2Q'$

where

$$2Q' = 2\bar{Q} + q^*$$

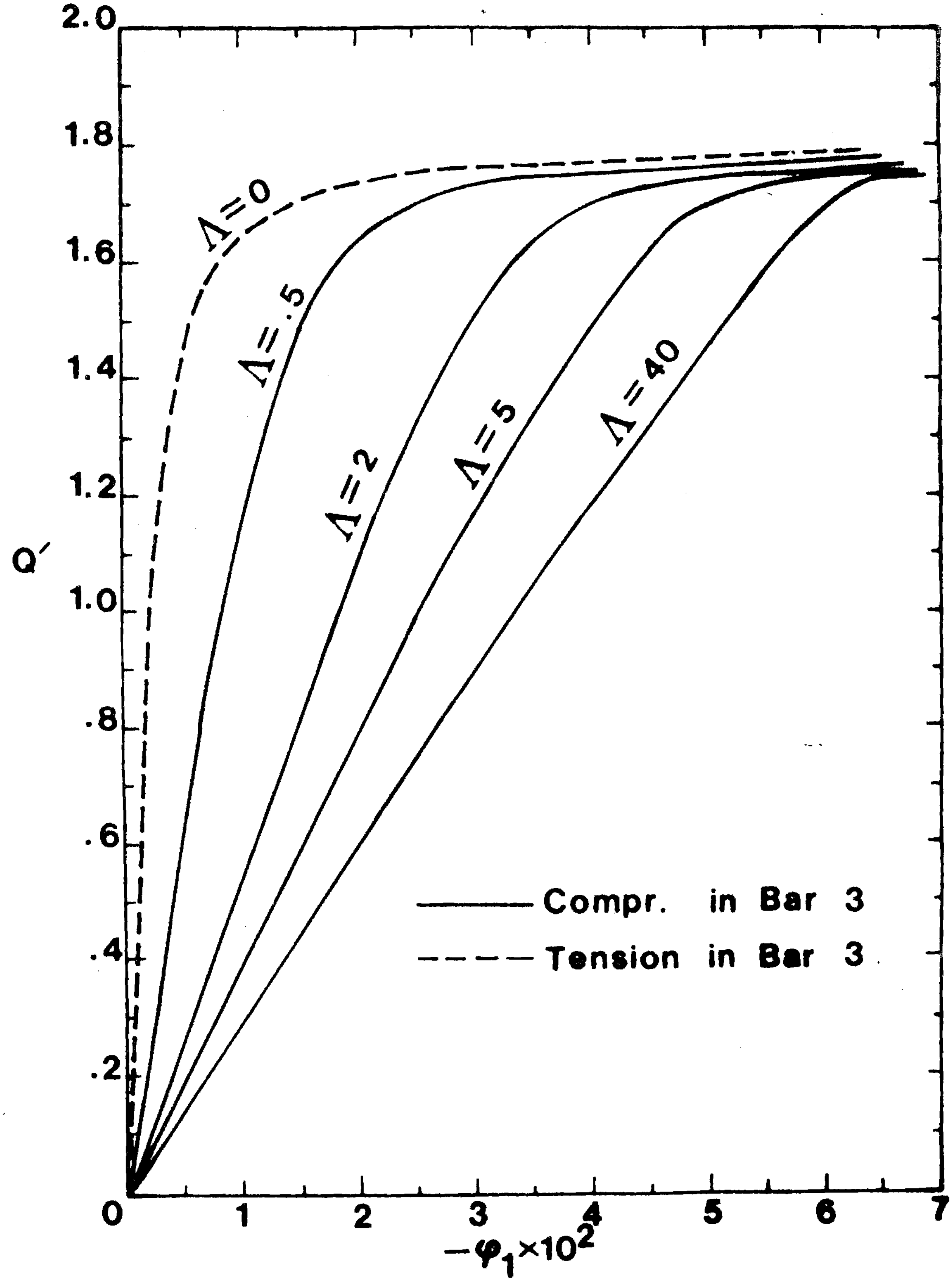


Figure 7 Equilibrium Response of a Simply Supported, Square Frame, under Combined Loads (λ_k = 1,000; $\bar{e}_1 = \bar{e}_2 = 0.01$; $r_k = \mu_k = 1$)

Furthermore q^* is expressed as a multiple of $\bar{Q}$ or $q^* = \Lambda\bar{Q}$. Thus,

$$Q' = (1 + \frac{\Lambda}{2})\ \bar{Q}$$

Different values of Λ are used, in an effort to cover the entire range of combined loads from virtually only concentrated to the case of only distributed. The Λ - value used are

$$\Lambda = 0,\ 0.5,\ 2,\ 5, \text{ and } 40$$

The results are presented graphically on Fig. 7, as plots of Q' versus φ_1. It is clear from this plot that, all five curves tend to approach asymptotically horizontal lines characterized by different but close values for Q'. The higher the value of the concentrated load - smaller Λ - the higher the level of the asymptote. Note also that, when q^* is zero ($\Lambda = 0$) the response of the frame is characterized by tension in the horizontal bar. On the contrary, as q^* increases the horizontal bar is in compression. Finally, when the greatest contribution is provided by the distributed load, ($\Lambda = 40$) the value of the assymptote is $Q' = 1.754$, which agrees well with the value of 1.787 given by Le-Wu Lu (4).

CONCLUSIONS

It is important to continuously be aware of the fact that conclusions are based on the generated data, and therefore they should not be generalized or considered to be applicable to all other situations.

On the basis of the analysis performed and data generated one may list the following as important conclusions.

1) A methodology has been developed and demonstrated for analyzing completely an unbraced, rigid-jointed, portal frame subjected to

eccentric concentrated loads (near the joints) as well as uniformly distributed loads. The method is based on linearly elastic behavior and nonlinear kinematic relations, and provides a complete picture of the frame response including postbuckling behavior.

2) The effect of slenderness ratio, λ_k, of the bars on the nondimensionalized response characteristics (including critical loads) is insignificant.

3) Increase in the amount of rotational restraint $\bar{\beta}$, has a stabilizing effect (the larger the $\bar{\beta}$, the larger the sway-buckling critical load). on the frame.

4) The postbuckling response is stable (similar to that of a cantilever column) and it suggests that the configuration is insensitive to initial imperfections. This is demonstrated for imperfections of the load eccentricity type.

5) The horizontal bar can be either in tension or in compression depending on the type of loading (including eccentricities) as well as on the level of the loading.

6) For symmetrically loaded frames, as the load moves towards the centerline of the frame its critical value for sway-buckling decreases. The amount of decrease is very small though. It is also suggested, from the present results that when the concentrated loads are replaced by a statically equivalent distributed load the critical value is slightly smaller (than $\bar{Q}_{cr}$ with zero eccentricity).

7) For rotationally restrained frame, as the amount of rotational restraint is increased the postbuckling branch becomes flatter (see Fig. 2.).

ACKNOWLEDGEMENTS

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APPENDIX I

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ABSTRACT: A kinematically nonlinear analysis of unbraced, rigid-jointed, portal frames, rotationally restrained at the base and subjected to eccentric concentrated and/or uniformly distributed loads, is presented. Through this analysis the complete behavior, including the primary path, and postbuckling path (whenever it exists), is evaluated. Moreover, through parametric studies, the effects of bar slenderness ratio, load eccentricity, and amount of rotational restraint are assessed. Through this method it is also possible to assess the effect of member lengths and member bending stiffnesses.

Key words: stability of frames; unbraced frames; sway-buckling of frames; postbuckling analysis; bifurcational buckling; portal frames; rotationally restrained frames.

NONLINEAR ANALYSIS OF UNBRACED FRAMES OF VARIABLE GEOMETRY

by G. J. Simitses[+] and J. Giri[++]
School of Engineering Science and Mechanics
Georgia Institute of Technology, Atlanta, Georgia

Abstract

The problem of nonlinear analysis, including sway-buckling of unbraced, rigid-jointed, and elastically restraint (against rotation) portal frames is considered. The analysis is based on linearly elastic material behavior and nonlinear kinematic relations. The analysis considers uniformly distributed loads, eccentric concentrated loads in the transverse direction, as well as horizontal concentrated loads. Results are presented for the uniformly distributed transverse loading and variable geometry for the three bars. The effects of amount of rotational restraint and bar slenderness ratio are fully assessed. The variable geometry includes symmetric and nonsymmetric constructions. Among the most important conclusions of the investigation, one may list the following: (a) symmetric portal frames, loaded symmetrically buckle through a stable bifurcation (sway-buckling) from a bent symmetric equilibrium configuration (b) nonsymmetric portal frames are not subject to instability; their response is similar to that of imperfect colums, and (c) the effect of the bar slenderness ratio on the nondimensionalized response parameters is negligibly small.

I. INTRODUCTION

Buckling of portal frames is of considerable interest to the practicing engineer, and numerous investigations on the problem have been reported in the

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+ Professor of Engineering Science and Mechanics
++ Research Engineer; School of Engineering Science and Mechanics

open literature. For a fairly complete bibliography, the reader is referred to Refs. 1 and 2. In most analyses presented, the emphasis is on finding the bifurcation load. There are virtually no postbuckling analyses and very few dealing with nonuniform geometry. The problem considered herein deals with the nonlinear response of a portal frame of nonuniform geometry (variable bar lengths and stiffnesses constrained by elastic rotational springs at the base and loaded transversely by a uniformly distributed load and eccentric concentrated loads and, horizontally, by a concentrated load. The analysis is based on nonlinear kinematic relations and linearly elastic material behavior. The emphasis in the presented work is to outline the methodology for this nonlinear behavior and to assess the effect of various geometric parameters (structural geometry) on the response characteristics of the frame. Finally, the complete set of the nonlinear governing equations is presented and these may be employed by any interested person to deal with the geometry of his choice.

II. MATHEMATICAL FORMULATION

Consider the frame shown on Fig. 1. Each bar is of length, ℓ_k, cross-sectional area, A_k, and cross-sectional second moment of area, I_k. The sign convention associated with the bar in-plane and normal displacement components u_k and w_k, is given on Fig. 1. At the base, the frame is supported against translation and constrained by elastic (linear) rotational springs. The loading system consists of a uniformly distributed load on bar 3, q_3, eccentric concentrated loads Q_i, (the eccentricity is positive in the positive direction of the coordinate system) and a horizontal concentrated load F_1.

First, the following nondimensionalized parameters are introduced.

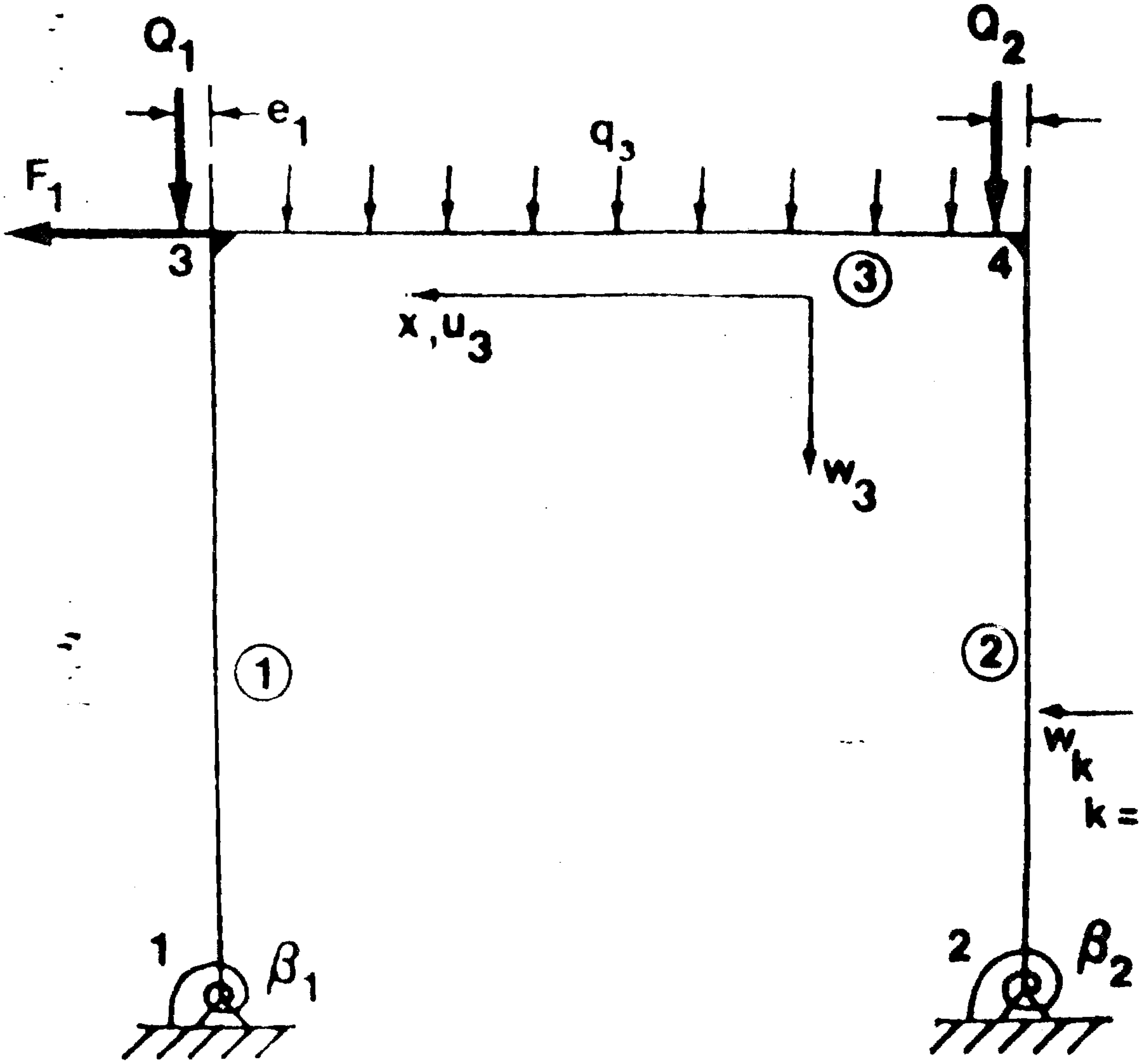

Fig. 1. Geometry and Sign Convention

$r_k = (EI)_k/(EI)_1$; $\mu_k = \ell_k/\ell_1$, $X = x/\ell_k$,

$U_k = u_k/\ell_k$; $W_k = w_k/\ell_k$; $\rho_k^2 = I_k/A_k$, $\lambda_k = \ell_k/\rho_k$,

$k_k^2 = \pm P_k \ell_k^2/(EI)_k$ (+ for tension ; P_k positive ;

- for compression; P_k negative where P_k is the axial force (1)

in the kth bar), $\bar{Q}_i = Q_i \ell_1^2/(EI)_1$,

$\bar{e}_i = e_i/\ell_1$, $\bar{\beta}_i = \beta_i \ell_1/(EI)_1$, $i = 1,2,$;

$\bar{q}_3 = q_3 \ell_3^3/(EI)_3$, $q_3^* = q_3 \ell_1^3/(EI)_1$; $\bar{F}_1 = F_1 \ell_1^2/(EI)_1$

II - 1 Equilibrium Equation

The equilibrium equations for each bar are given below (note that it is possible for bar 3 to be in tension, therefore equations for both possibilities are given). They are obtained by employing the principle of the stationary value of the total potential (see Ref. 2).

In-plane Equilibrium

$$U_{k,X} + \frac{1}{2} W_{k,X}^2 = - k_k^2/\lambda_k^2 \quad , \quad k = 1,\ 2,\ 3 \quad \text{(Compression)}$$

(2)

$$U_{3,X} + \frac{1}{2} W_{3,X}^2 = k_3^2/\lambda_3^2 \qquad \text{(Tension in bar 3)}$$

Transverse Equilibrium

$$W_{k,XXXX} + k_k^2 W_{k,XX} = \bar{q}_k \quad , \quad k = 1, 2, 3 \text{ (Compression)}$$

$$W_{3,XXXX} - k_3^2 W_{3,XX} = \bar{q}_3 \qquad \text{(Tension in bar 3)} \tag{3}$$

The associated boundary and joint conditions are

Boundary Conditions

$$1. \quad U_1(0) = W_1 \ (0) = W_{1,XX}(0) \ - \bar{\beta}_1 W_{1,X}(0) = 0$$

$$2. \quad U_2(0) = W_2 \ (0) = W_{2,XX} \ - \bar{\beta}_2 W_{2,X}(0) = 0 \tag{4}$$

Joint Conditions

Joint 3. $U_1(1) = -W_3(1) \quad ; \quad W_1(1) = U_3 \ (1) \quad ; \quad W_{1,X}(1) = W_{3,X}(1)$

$$\bar{e}_1 \bar{Q}_1 - W_{1,XX}(1) - \left(\frac{r_3}{\mu_3}\right) W_{3,XX}(1) = 0$$

$$k_1^2 - \bar{Q}_1 - \left(\frac{r_3}{\mu_3^2}\right) \left[\pm k_3^2 W_{3,X}(1) + W_{3,XXX}(1)\right] = 0 \tag{5}$$

$$\pm \left(\frac{r_3}{\mu_3^2}\right) k_3^2 + \bar{F}_1 + [k_1^2 W_{1,X} + W_{1,XXX}(1)] = 0$$

(+ : compression, - : tension)

Joint 4

$$U_2(1) = -W_3(0) \quad ; \quad U_3(0) = W_2(1) \quad ; \quad W_{2,X}(1) = W_{3,X}(0)$$

$$\bar{e}_2\bar{Q}_2 + \left(\frac{r_3}{\mu_3}\right) W_{3,XX}(0) - \left(\frac{r_2}{\mu_2}\right) W_{2,XX}(1) = 0$$

$$\left(\frac{r_2^2}{\mu_2^2}\right) k_2^2 - \bar{Q}_2 + \left(\frac{r_3^2}{r_3^2}\right) \left[\pm k_3^2 W_{3,X}(0) + W_{3,XXX}(0)\right] = 0 \qquad (6)$$

$$\pm \left(\frac{r_3^2}{\mu_3^2}\right) k_3^2 - \left(\frac{r_2^2}{\mu_2^2}\right) \left[k_2^2 W_{2,X}(1) + W_{2,XXX}(1)\right] = 0$$

The solution to the equilibrium equations is given by

bar in compassion

$$W_k(X) = A_{k1} \sin k_kX + A_{k2} \cos k_kX + A_{k3}X + A_{k4} + \frac{\bar{q}_kX^2}{2k_k^2}$$

$$U_k(x) = U_{k0} - \left(\frac{k_k}{\lambda_k}\right)^2 X - \frac{1}{2}\int_0^X W_{k,X}^2 \, dX \quad ; \quad k = 1, 2, 3 \qquad (7)$$

bar in tension (only bar 3)

$$W_3(X) = A_{31} \sinh k_3X + A_{32} \cosh k_3X + A_{33}X + A_{34} - \frac{\bar{q}_3X^2}{2k_3^2}$$

$$U_3(X) = U_{30} + \left(\frac{k_3}{\lambda_3}\right)^2 X - \frac{1}{2}\int_0^X W_{3,X}^2 \, dX \qquad (8)$$

Regardless of tension or compression in bar 3, the solution to the equilibrium equations contain 18 constants (U_{ko}, A_{ki}, k_k, $k = 1, 2, 3$ and $i = 1, 2, 3, 4$). These constants are evaluated from the six boundary condition, Eqs. (4), and the twelve joint conditions, Eqs. (5) and (6). Elimination of all constants that appear in a linear manner (U_{ko} and A_{ki}) yield a system of three nonlinear equations in k_k. These are next given for the cases of compression and tension in bar 3. From one of them k_1 can be expressed in terms of k_2 and thus the governing equations become two.

Compression in bar 3

$$D_9 \sin k_1 - D_{11}(1 - \cos k_1) - [F_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2]/k_1^2 + k_3^2/\lambda_3^2$$

$$+ \frac{D_7^2}{4} [1 + \frac{\sin k_3}{k_3} \cos k_3] + \frac{D_8^2}{4k_3^2} [1 - \frac{\sin k_3}{k_3} \cos k_3] + \frac{1}{2} [D_{13} \left(D_{13} + \frac{\bar{q}_3}{k_3^2}\right)$$

$$+ \frac{1}{3} \left(\frac{\bar{q}_3}{k_3^2}\right)^2] - \frac{1}{2} D_7 D_8 \left(\frac{\sin k_3}{k_3}\right)^2 + D_7 D_{13} \left(\frac{\sin k_3}{k_3}\right) - D_8 D_{13} \left(\frac{1 - \cos k_3}{k_3^2}\right)$$

$$+ \left(\frac{\bar{q}_3}{k_3^2}\right) D_7 \left[\frac{\sin k_3}{k_3} - \frac{1}{2} \left\{\frac{\sin (k_3/2)}{(k_3/2)}\right\}^2 \right] + \frac{D_8}{k_3^2} \left(\frac{\bar{q}_3}{k_3^2}\right) \left(\frac{\sin k_3}{k_3} - \cos k_3\right)$$

$$-D_{10} \sin k_2 + D_{12}(1-\cos k_2) - \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3}{k_3}\right)^2 = 0 \qquad (9)$$

$$\left(\frac{k_2}{\lambda_2}\right)^2 + \frac{k_2^2}{4} D_{10}^2 \left(1 + \frac{\sin k_2}{k_2} \cos k_2\right) + \frac{k_2^2}{4} D_{12}^2 \left(1 - \frac{\sin k_2}{k_2} \cos k_2\right)$$

$$+ \frac{1}{2} \left[\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3}{k_2}\right)^2 \right]^2 - \frac{k_2}{2} D_{10} D_{12} \sin^2 k_2 - \frac{D_8}{k_3^2} - D_{14} +$$

$$[D_{10} \sin k_2 - D_{12} (1 - \cos k_2)] \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3}{k_2}\right)^2 = 0 \qquad (10)$$

where

$$D_1 = \left(\frac{r_2 k_2}{\mu_2}\right) (k_2 \sin k_2 - \bar{\beta}_2 \cos k_2)/(k_2 \cos k_2 + \bar{\beta}_2 \sin k_2)$$

$$D_2 = \bar{r}_3/\mu_3$$

$$D_3 = \frac{r_2}{\mu_2} \bar{\beta}_2 (\cos k_2) \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3^2}{k_2}\right)^2 - \bar{e}_2 \bar{Q}_2 k_3^2 - \frac{r_3}{\mu_3} \bar{q}_3$$

$$- \left(\frac{r_2}{\mu_2}\right) k_2 (k_2 \sin k_2 - \bar{\beta}_2 \cos k_2)\left[\frac{\mu_3^2}{r_3} (k_1^2 - \bar{Q}_1) - \bar{q}_3 -\right.$$

$$\left.\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3^2}{k_2}\right)^2 \left(1 + \bar{\beta}_2 \frac{\sin k_2}{k_2}\right)\right] / (k_2 \cos k_2 + \bar{\beta}_2 \sin k_2)$$

$$D_4 = \bar{-} \left[\cos k_3 + \left(\frac{r_3}{\mu_3}\right) k_3 \sin k_3 (k_1 \cos k_1 + \bar{\beta}_1 \sin k_1)/(k_1^2 \sin k_1 - k_1 \bar{\beta}_1 \cos k_1)\right]$$

$$D_5 = \frac{\sin k_3}{k_3} - \left(\frac{r_3}{\mu_3}\right) \cos k_3 (k_1 \cos k_1 + \bar{\beta}_1 \sin k_1)/(k_1^2 \sin k_1 - k_1 \bar{\beta}_1 \cos k_1)$$

$$D_6 = \bar{\beta}_1 \left(\frac{k_3}{k_1}\right)^2 \frac{\sin k_1}{k_1} \left[\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right] + \frac{\mu_3^2}{r_3} (k_1^2 - \bar{Q}_1) + \frac{r_3}{\mu_3^2} \left(\frac{k_3^2}{k_1}\right)^2 + \bar{F}_1 \left(\frac{k_3}{k_1}\right)^2$$

$$+ (k_1 \cos k_1 + \bar{\beta}_1 \sin k_1)\left[\bar{e}_1 \bar{Q}_1 k_3^2 - \left(\frac{r_3}{\mu_3}\right) \bar{q}_3 + \bar{\beta}_1 \left(\frac{k_3}{k_1}\right)^2 \cos k_1 \left\{\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right\}\right] /$$

$$(k_1^2 \sin k_1 - k_1 \bar{\beta}_1 \cos k_1).$$

$$D_7 = \frac{1}{k_3^2} (D_3 D_5 + D_2 D_6)/(D_1 D_5 + D_2 D_4)$$

$$D_8 = \frac{1}{k_3^2} \left(D_1 D_6 - D_3 D_4\right) / \left(D_1 D_5 + D_2 D_4\right)$$

$$D_9 = -\left[\left(\frac{r_3}{\mu_3}\right)\left(\frac{\bar{q}_3}{k_3^2}\right) - \bar{e}_1 \bar{Q}_1 - \left(\frac{\bar{\beta}_1}{k_1^2}\right) \cos k_1 \left[\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right] - \left(\frac{r_3}{\mu_3}\right)\left(k_3 D_7 \sin k_3\right.\right.$$

$$\left.\left. + D_8 \cos k_3\right)\right] / \left(k_1^2 \sin k_1 - k_1 \bar{\beta}_1 \cos k_1\right)$$

$$D_{10} = \left[\left(\frac{\mu_3}{r_3}\right)^2 \left(\frac{k_1^2 - \bar{Q}_I}{k_3^2}\right) - \left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 \left\{1 + \bar{\beta}_2\left(\frac{\sin k_2}{k_2}\right)\right\} - \frac{\bar{q}_3}{k_3^2} + D_7\right] / \left(k_2 \cos k_2\right.$$

$$\left. + \bar{\beta}_2 \sin k_2\right)$$

$$D_{11} = -\frac{\bar{\beta}_1}{k_1} \left[\left\{\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right\} / k_1^3 - D_9\right]$$

$$D_{12} = -\frac{\bar{\beta}_2}{k_2^2} \left[D_{10} k_2 + \left(\frac{r_3}{r_2}\right)\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{k_3}{k_2}\right)^2\right]$$

$$D_{13} = \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2 - \bar{Q}_I}{k_3^2}\right) - \frac{\bar{q}_3}{k_3^2}$$

$$D_{14} = \left(\frac{k_1}{\lambda_1}\right)^2 + \frac{1}{4} k_1 D_9^2 \left(k_1 + \sin k_1 \cos k_1\right) + \frac{1}{4} k_1 D_{11}^2 \left(k_1 - \sin k_1 \cos k_1\right)$$

$$+ \left[\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right]^2 / \left(2k_1^4\right) - \frac{1}{2} k_1 D_9 D_{11} \sin^2 k_1 - \frac{1}{2}\left(\frac{\bar{q}_3}{k_3^2}\right)$$

$$- \left(\frac{D_9}{k_1}\right) \frac{\sin k_1}{k_1} \left[\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right] + \dot{D}_{11} \left(\frac{1-\cos k_1}{k_1^2}\right) \left[\bar{F}_1 + r_3 \left(\frac{k_3}{\mu_3}\right)^2\right]$$

$$- D_7 \frac{\sin k_3}{k_3} - D_8 \frac{\cos k_3}{k_3^2} - D_{13}$$

Moreover, the expressions for k_1, $U_{\kappa o}$, and $A_{\kappa i}$ in terms of k_2 and k_3 are given by:

$$k_1^2 = \bar{Q}_1 + \bar{Q}_2 + \frac{r_3}{\mu_3^2} \bar{q}_3 - \frac{r_2}{\mu_2^2} k_2^2 \tag{11}$$

$$U_{10} = U_{20} = 0$$

$$U_{30} = D_9 \sin k_1 + D_{11} \cos k_1 - \frac{\bar{F}_1}{k_1^2} - \left(\frac{r_3}{\mu_3^2}\right) \left(\frac{k_3}{k_1}\right)^2 - D_{11}$$

$$+ \left(\frac{k_3}{\lambda_3}\right)^2 + \frac{1}{4} D_7^2 \left(1 + \frac{\sin k_3}{k_3} \cos k_3\right) + \frac{1}{4} \frac{D_8^2}{k_3^2} \left(1 - \frac{\sin k_3}{k_3} \cos k_3\right)$$

$$+ \frac{1}{2} D_{13}^2 + \frac{1}{6} \left(\frac{\bar{q}_3}{k_3^2}\right)^2 - \frac{1}{2} D_7 D_8 \left(\frac{\sin k_3}{k_3}\right)^2 + D_7 D_{13} \frac{\sin k_3}{k_3}$$

$$- D_8 D_{13} \left(\frac{1 - \cos k_3}{k_3^2}\right) + D_7 \left(\frac{\bar{q}_3}{k_3^2}\right) \left[\frac{\sin k_3}{k_3} - \frac{1}{2} \left\{\frac{\sin (k_3/2)}{(k_3/2)}\right\}^2\right]$$

$$- \left(\frac{\bar{q}_3}{k_3^4}\right) D_8 \left(\frac{\sin k_3}{k_3} - \cos k_3\right) + \frac{1}{2} D_{13} \left(\frac{\bar{q}_3}{k_3^2}\right) \tag{12}$$

$$A_{11} = D_9 \;;\; A_{12} = D_{11} \;;\; A_{13} = -\left[\bar{F}_1 + r_3\left(\frac{k_3}{\mu_3}\right)^2\right]/k_1^2 \;;$$

$$A_{14} = -D_{11} \;;\; A_{21} = D_{10} \;;\; A_{22} = D_{12} \;;\; A_{23} = \left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 \;; \tag{13}$$

$$A_{24} = -D_{12} \;;\; A_{31} = D_7/k_3 \;;\; A_{32} = D_8/k_3^2$$

$A_{33} = D_{13}$; and $A_{34} = D_{14}$.

<u>Tension in bar 3</u>

$$E_9 \sin k_1 - E_{11}(1 - \cos k_1) - \left[\bar{F}_1 - r_3\left(\frac{k_3}{\mu_3}\right)^2\right] k_1^2 - \left(\frac{k_3}{\lambda_3}\right)^2$$

$$+ \frac{E_7^2}{4}\left(1 + \frac{\sinh k_3}{k_3}\cosh k_3\right) - \frac{E_8^2}{4k_3^2}\left(1 - \frac{\sinh k_3}{k_3}\cosh k_3\right)$$

$$+ \frac{1}{2}\left[E_{13}\left(E_{13} - \frac{\bar{q}_3}{k_3^2}\right) + \frac{1}{3}\left(\frac{\bar{q}_3}{k_3^2}\right)^2\right] + \frac{1}{2}E_7E_8\left(\frac{\sinh k_3}{k_3}\right)^2$$

$$+ E_7E_{13}\left(\frac{\sinh k_3}{k_3}\right) + E_8E_{13}\left(\frac{\cosh k_3 - 1}{k_3^2}\right) - E_7\left(\frac{\bar{q}_3}{k_3^2}\right)\left[\frac{\sinh k_3}{k_3} - \left(\frac{\cosh k_3 - 1}{k_3^2}\right)\right]$$

$$- \frac{E_8}{k_3^2}\left(\frac{\bar{q}_3}{k_3^2}\right)\left(\cosh k_3 - \frac{\sinh k_3}{k_3}\right) - E_{10}\sin k_2 + E_{12}(1 - \cos k_2) + \left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 = 0$$

$$\left(\frac{k_2}{\lambda_2}\right)^2 + \frac{k_2^2}{4}E_{10}^2\left(1 + \frac{\sin k_2}{k_2}\cos k_2\right) + \frac{k_2^2}{4}E_{12}^2\left(1 - \frac{\sin k_2}{k_2}\cos k_2\right)$$

$$- \frac{1}{2}\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 \left[E_{10}\sin k_2 - E_{12}(1 - \cos k_2)\right] - \frac{E_8}{k_3^2} - E_{14}$$

$$+ \frac{1}{2}\left[\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2\right]^2 - \frac{k_2}{2}E_{10}E_{12}\sin^2 k_2 = 0 \tag{15}$$

where

$$E_1 = \left(\frac{r_2}{\mu_2}\right) k_2(k_2 \sin k_2 - \bar{\beta}_2 \cos k_2)/(k_2 \cos k_2 + \bar{\beta}_2 \sin k_2) = D_1$$

$$E_2 = r_3/\mu_3 = D_2$$

$$E_3 = \left(\frac{r_3}{\mu_3}\right) \bar{q}_3 - \bar{e}_2\bar{Q}_2 k_3^2 - \left(\frac{r_2}{\mu_2}\right) \bar{\beta}_2 \ (\cos k_2)\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3^2}{k_2}\right)^2$$

$$- \left(\frac{r_2}{\mu_2}\right) k_2(k_2 \sin k_2 - \bar{\beta}_2 \cos k_2)\left[\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right) \left(\frac{k_3^2}{k_2}\right) \left(1 + \beta_2 \frac{\sin k_2}{k_2}\right)\right.$$

$$\left. + \bar{q}_3 - \left(\frac{\mu_3^2}{r_3}\right)\left(k_1^2 - \bar{Q}_1\right)\right] / (k_2 \cos k_2 + \bar{\beta}_2 \sin k_2)$$

$$E_4 = -\left(\frac{r_3}{\mu_3}\right) k_3 \sinh k_3 + k_1 \cosh k_3 \left(k_1 \sin k_1 - \bar{\beta}_1 \cos k_1\right) / (k_1 \cos k_1 + \bar{\beta}_1 \sin k$$

$$E_5 = \left(\frac{\sinh k_3}{k_3}\right) \left(k_1^2 \sin k_1 - k_1\bar{\beta}_1 \cos k_1\right)/(k_1 \cos k_1 + \bar{\beta}_1 \sin k_1) - \left(\frac{r_3}{\mu_3}\right) \cosh k$$

$$E_6 = -\bar{\beta}_1\left(\frac{k_3}{k_1}\right)^2 \cos k_1 \left[\bar{F}_1 - r_3 \left(\frac{k_3}{\mu_3}\right)^2\right] - \bar{e}_1 \bar{Q}_1 k_3^2 - \left(\frac{r_3}{\mu_3}\right) \bar{q}_3$$

$$- \left(k_1^2 \sin k_1 - k_1\bar{\beta}_1 \cos k_1\right) \left[\bar{F}_1 - r_3 \left(\frac{k_3^2}{\mu_3}\right)\right]\left(\frac{k_3}{k_1}\right)^2 \left(1 + \bar{\beta}_1 \frac{\sin k_1}{k_1}\right)$$

$$- \left(\frac{\mu_3^2}{r_3}\right) \left(k_1^2 - \bar{Q}_1\right)\Big] / \left(k_1 \cos k_1 + \bar{\beta}_1 \sin k_1\right)$$

$$E_7 = \frac{1}{k_3^2} (E_3E_5 - E_2E_6) / (E_1E_5 - E_2E_4)$$

$$E_8 = \frac{1}{k_3^2} (E_1E_6 - E_3E_4)/ (E_1E_5 - E_2E_4)$$

$$E_9 = \left[E_7 \cosh k_3 + E_8 \frac{\sinh k_3}{k_3} - \left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right) + \left\{\bar{F}_1 - r_3\left(\frac{k_3}{r_3}\right)^2\right\}\frac{1}{k_1^2}\left(1\right.\right.$$

$$\left.\left. + \bar{\beta}_1 \frac{\sin k_1}{k_1}\right)\right] / (k_1 \cos k_1 + \bar{\beta}_1 \sin k_1)$$

$$E_{10} = \left[E_7 + \left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 \left(1 + \bar{\beta}_2 \frac{\sin k_2}{k_2}\right) + \frac{\bar{q}_3}{k_3^2} - \right.$$

$$\left.\left(\frac{\mu_3^2}{r_3}\right)\left(\frac{k_1^2 - \bar{Q}_1}{k_3^2}\right)\right] / (k_2 \cos k_2 + \bar{\beta}_2 \sin k_2)$$

$$E_{11} = -\frac{\bar{\beta}_1}{k_1^4}\left[k_1^3 E_9 - \bar{F}_1 + r_3\left(\frac{k_3}{\mu_3}\right)^2\right]$$

$$E_{12} = \frac{\bar{\beta}_2}{k_2^2}\left[\left(\frac{\mu_2}{\mu_3}\right)^2\left(\frac{r_3}{r_2}\right)\left(\frac{k_3}{k_2}\right)^2 - k_2 E_{10}\right]$$

$$E_{13} = \bar{q}_3/k_3^2 + (\mu_3^2/r_3)(k_1^2 - \bar{Q}_1)/k_3^2$$

$$E_{14} = \left(\frac{k_1}{\lambda_1}\right)^2 + \frac{k_1}{4} E_9^2 (k_1 + \sin k_1 \cos k_1) + \frac{k_1}{4} E_{11}^2 (k_1 - \sin k_1 \cos k_1)$$

$$+ \left[\bar{F}_1 - r_3\left(\frac{k_3}{\mu_3}\right)^2\right]/(2k_1^4) - \tfrac{1}{2} k_1 E_9 E_{11} \sin^2 k_1 - E_7 \frac{\sinh k_3}{k_3}$$

$$- \frac{E_9}{k_1}\left(\frac{\sin k_1}{k_1}\right)\left[\bar{F}_1 - r_3\left(\frac{k_3}{\mu_3}\right)^2\right] + E_{11}\left(\frac{1 - \cos k_1}{k_1^2}\right)\left[\bar{F}_1 - r_3\left(\frac{k_3}{\mu_3}\right)^2\right]$$

$$- E_8 \frac{\cosh k_3}{k_3^2} - E_{13} + \tfrac{1}{2}(\bar{q}/k_3^2)$$

Finally, the expressions for k_1, U_{ko} and A_{ki}, for this case, are:

$$k_1^2 = \bar{Q}_1 + \bar{Q}_2 + \frac{r_3}{\mu_3^2}\bar{q}_3 - \left(\frac{r_2}{\mu_2^2}\right) k_2^2 \tag{16}$$

$$U_{10} = U_{20} = 0$$

$$U_{30} = E_9 \sin k_1 + E_{11} \cos k_1 - E_{11} + \left[\left(\frac{r_3}{\lambda_3^2}\right) \left(\frac{k_3}{k_1}\right)^2 - \frac{\bar{F}_1}{k_1^2} \right]$$

$$- \left(\frac{k_3}{\lambda_3}\right)^2 + \frac{E_7^2}{4} \left(1 + \frac{\sinh k_3}{k_3} \cosh k_3\right) - \frac{E_8^2}{4k_3^2} \left(1 - \frac{\sinh k_3}{k_3} \cosh k_3\right)$$

$$+ \frac{1}{2} E_{13}^2 + \frac{1}{6} \left(\frac{\bar{q}_3}{k_3^2}\right)^2 + \frac{1}{2} E_7 E_8 \left(\frac{\sinh k_3}{k_3}\right)^2 + E_7 E_{13} \frac{\sinh k_3}{k_3}$$

$$+ E_8 E_{13} \left(\frac{\cosh k_3 - 1}{k_3^2}\right) - \left(\frac{\bar{q}_3}{k_3^2}\right) \left(\frac{\sinh k_3}{k_3} + \frac{1 - \cosh k_3}{k_3^2}\right) E_7$$

$$- \left(\frac{\bar{q}_3}{k_3^4}\right) E_8 \left(\cosh k_3 - \frac{\sinh k_3}{k_3}\right) - \frac{1}{2} E_{13} \left(\frac{\bar{q}_3}{k_3^2}\right) \qquad (17)$$

$$A_{11} = E_9 \quad ; \quad A_{12} = E_{11} \quad ; \quad A_{13} = -\left[\bar{F}_1 - r_3 \left(\frac{k_3}{\mu_3}\right)^2\right] / k_1^2$$

$$A_{14} = -E_{11} \quad ; \quad A_{21} = E_{10} \quad ; \quad A_{22} = E_{12} ; \quad A_{23} = -\left(\frac{\mu_2}{\mu_3}\right)^2 \left(\frac{r_3}{r_2}\right) \left(\frac{k_3}{k_2}\right)^2 \quad ; \qquad (18)$$

$$A_{24} = -E_{12} \; ; \; A_{31} = E_7/k_3 \quad ; \quad A_{32} = E_8/k_3^2$$

$$A_{33} = E_{13} \quad ; \quad A_{34} = E_{14}$$

II-2. Buckling Equations

The buckling equations and the associated boundary conditions can be obtained by the Trefftz criterion (see Ref. 3). Equivalently, they are obtained from Eqs. (2) - (6) by replacing W_k and U_k by $\bar{W}_k + \tilde{W}_k$ and $\bar{U}_k + \tilde{U}_k$ respectively. The parameters $\bar{W}_k$ and $\bar{U}_k$ characterize displacement components on the primary equilibrium path, while $\tilde{W}_k$ and $\tilde{U}_k$ characterize kinematically admissible displacement components (buckling modes from the primary path). Again here, the distinction between the cases of compression or tension on bar 3, at the instant of buckling, must be made. These equations are:

Buckling Equations

$$\tilde{U}_{k,X} + \bar{W}_{k,X}\,\tilde{W}_{k,X} = \tilde{\sigma}_k / \lambda_k^2$$

$$\tilde{W}_{k,XXXX} \pm \tilde{W}_{k,XX} = \tilde{\sigma}_k \bar{W}_{k,XX} \tag{19}$$

Boundary Conditions

1. $\tilde{U}_1(0) = \tilde{W}_1(0) = \tilde{W}_{1,XX}(0) - \bar{\beta}_1 \tilde{W}_{1,X}(0) = 0$

$$\tilde{U}_2(0) = \tilde{W}_2(0) = \tilde{W}_{2,XX}(0) - \bar{\beta}_2 \tilde{W}_{2,X}(0) = 0 \tag{20}$$

Joint Conditions

Joint 3.

$$\tilde{U}_1(1) = -\tilde{W}_3(1) \quad ; \quad \tilde{U}_3(1) = \tilde{W}_1(1) \quad ; \quad \tilde{W}_{1,X}(1) = \tilde{W}_{3,X}(1)$$

$$\tilde{W}_{1,XX}(1) + \left(\frac{r_3}{\mu_3}\right) \tilde{W}_{3,XX}(1) = 0$$

$$-\tilde{\sigma}_1 - \left(\frac{r_3}{\mu_3^2}\right) \left[\pm \bar{k}_3^2\, \tilde{W}_{3,X}(1) - \tilde{\sigma}_3 \bar{W}_{3,X}(1) + \tilde{W}_{3,XXX}(1) \right] = 0 \tag{21}$$

$$-\left(\frac{r_3}{\mu_3^2}\right) \tilde{\sigma}_3 + \bar{k}_1^2 \tilde{W}_{1,X}(1) - \tilde{\sigma}_1 \bar{W}_{1,X}(1) + \tilde{W}_{1,XXX}(1) = 0$$

Joint 4

$$\tilde{U}_2(1) = -\tilde{W}_3(0) \quad ; \quad \tilde{U}_3(0) = \tilde{W}_2(1) \quad ; \quad \tilde{W}_{2,X}(1) = \tilde{W}_3(0) \quad ;$$

$$\left(\frac{r_3}{\mu_3}\right) \tilde{W}_{3,XX}(0) - \left(\frac{r_2}{\mu_2}\right) \tilde{W}_{2,XX}(1) = 0$$

$$
\begin{aligned}
&-\left(\frac{r_2}{\mu_2^2}\right)\tilde{\sigma}_2+\left(\frac{r_3}{\mu_3^2}\right)\left[\pm\,\bar{k}_3^2\,\tilde{W}_{3,X}(0)-\tilde{\sigma}_3\bar{W}_{3,X}(0)+\tilde{W}_{3,XXX}(0)\right]=0\\
&-\left(\frac{r_3}{\mu_3^2}\right)\tilde{\sigma}_3-\left(\frac{r_2}{\mu_2^2}\right)\left[\bar{k}_2^2\,\tilde{W}_{2,X}(1)-\tilde{\sigma}_2\bar{W}_{2,X}(1)+\tilde{W}_{2,XXX}(1)\right]=0
\end{aligned}
\tag{22}
$$

where + : compression in bar 3 (on the primary path),

- : tension in bar 3 (on the primary path),

$\tilde{\sigma} = (\tilde{P}_k \ell_k^2)/(EI)_k$ and it can be either positive or negative

$\tilde{P}_k$ is the additional axial force in the kth bar corresponding to the kinematically admissible displacements $\tilde{U}_k$ and $\tilde{W}_k$, and

$$k_k^2 = \bar{k}_k^2 - \tilde{\sigma}_k \quad ; \quad k = 1, 2, 3 \quad \text{(compression in kth bar)}$$

$$\bar{\bar{k}}_3^2 = \bar{k}_3^2 + \tilde{\sigma}_3 \qquad \text{(tension in bar 3).}$$

The solution to the buckling equations is given by

(a) bars 1 and 2 ($k = 1, 2$)

$$
\begin{aligned}
\tilde{U}_k(X) &= \tilde{U}_{ko} + \frac{\tilde{\sigma}_k X}{\lambda_k^2} - \int_0^X \bar{W}_{k,X}\,\tilde{W}_{k,X}\,dX\\
\tilde{W}_k(X) &= \tilde{A}_{k1}\sin\bar{k}_k X + \tilde{A}_{k2}\cos\bar{k}_k X + \tilde{A}_{k3}X + \tilde{A}_{k4}\\
&\quad + \frac{\tilde{\sigma}_k X}{2\bar{k}_k}\left(A_{k2}\sin\bar{k}_k X - A_{k1}\cos\bar{k}_k X\right)
\end{aligned}
\tag{23}
$$

(b) bar in compression

$$\tilde{U}_3(X) = \tilde{U}_{30} + \frac{\tilde{\sigma}_3 X}{\lambda_3^2} - \int_0^X \bar{W}_{3,X}\tilde{W}_{3,X}\,dX$$

$$\tilde{W}_3(X) = \tilde{A}_{31} \sin \bar{k}_3 X + \tilde{A}_{32} \cos \bar{k}_3 X + \tilde{A}_{33} X + \tilde{A}_{34}$$

$$+ \frac{\tilde{\sigma}_3 X}{2\bar{k}_3} \left(A_{32} \sin \bar{k}_3 X - A_{31} \cos \bar{k}_3 X + \frac{\bar{q}_3 X}{\bar{k}_3^3} \right) \tag{24}$$

(c) bar 3 in tension

$$\tilde{U}_3(X) = \tilde{U}_{30} + \frac{\tilde{\sigma}_3 X}{\lambda_3^2} - \int_0^X \bar{W}_{3,X} \tilde{W}_{3,X} \, dX$$

$$\tilde{W}_3(X) = \tilde{A}_{31} \sinh \bar{k}_3 X + \tilde{A}_{32} \cosh \bar{k}_3 X + \tilde{A}_{33} X + \tilde{A}_{34} \tag{25}$$

$$+ \frac{\tilde{\sigma}_3 X}{2\bar{k}_3} \left(A_{32} \sinh \bar{k}_3 X + A_{31} \cosh \bar{k}_3 X + \frac{\bar{q}_3 X}{\bar{k}_3^3} \right)$$

Note that $\bar{k}_k$ denotes the axial force parameter at the primary equilibrium path, at the instant of buckling, and A_{k1} and A_{k2} are the values of the constant to the solution of the equilibrium equations, Eqs. (7) and (8), on the primary path at buckling.

There are 18 constants in the solution to the buckling equations, $\tilde{U}_{ko}$, $\tilde{A}_{ki}$, and $\tilde{\sigma}_k$ ($k = 1, 2, 3$, and $i = 1, 2, 3, 4$). The number of boundary and joint conditions is also 18. Moreover, when the solutions, Eqs. (23) and (24) or Eqs. (23) and (25), are substituted into the boundary and joint conditions, a system of 18 linear homogeneous algebraic equations in the 18 constants is obtained (actually 16 because two constants are zero; $U_{10} = U_{20} = 0$). For a nontrivial solution to exist, the determinant of the coefficients must vanish. This yields the characteristic equation. The solution of the characteristic equation leads to the critical load condition.

Instead of defining the elements of the 16 x 16 determinant, the 16 linear homogeneous equations are presented, which lead to the construction of the deter nant.

$$\widetilde{A}_{12} + \widetilde{A}_{14} = 0 \tag{26}$$

$$-\widetilde{A}_{11} + (\bar{k}_1\bar{\beta}_1) + \widetilde{A}_{12}(\bar{k}_1^2) + \widetilde{A}_{13}\ (\bar{\beta}_1) - \widetilde{\sigma}_1\left(A_{12} + A_{11}\frac{\bar{\beta}_1}{2\bar{k}_1}\right) = 0 \tag{27}$$

$$\widetilde{A}_{22} + \widetilde{A}_{24} = 0 \tag{28}$$

$$\widetilde{A}_{21}(\bar{k}_2\bar{\beta}_2) + \widetilde{A}_{22}(\bar{k}_2^2) + \widetilde{A}_{23}(\bar{\beta}_2) - \widetilde{\sigma}_2\ \left(A_{22} + A_{21}\frac{\bar{\beta}_2}{2\bar{k}_2}\right) = 0 \tag{29}$$

$$-\widetilde{A}_{11}\left[\frac{A_{11}\bar{k}_1^2}{2}\left(1 + \frac{\sin 2\bar{k}_1}{2\bar{k}_1}\right) - \frac{\bar{k}_1A_{12}}{2}\sin^2\bar{k}_1 + A_{13}\sin\bar{k}_1\right] +$$

$$\widetilde{A}_{12}\left[\frac{\bar{k}_1A_{11}}{2}\sin^2\bar{k}_1 - \frac{\bar{k}_1^2}{2}A_{12}\left(1 - \frac{\sin 2\bar{k}_1}{2\bar{k}_1}\right) + A_{13}\ (1-\cos\bar{k}_1)\right] -$$

$$-\widetilde{A}_{13}\left[A_{11}\ \sin\bar{k}_1 - A_{12}\ (1 - \cos\bar{k}_1) + A_{13}\right] - \widetilde{\sigma}_1\left[\frac{A_{11}A_{12}\bar{k}_1}{4}\{\sin^2\bar{k}_1/\bar{k}_1^2 + \right.$$

$$\sin 2\bar{k}_1/\bar{k}_1) - \frac{A_{12}^2}{8}\left(1 + 2\sin^2\bar{k}_1 - \frac{\sin 2\bar{k}_1}{2\bar{k}_1}\right) - \frac{A_{11}^2}{8}\left(\frac{\sin 2\bar{k}_1}{2\bar{k}_1} - 2\sin^2\bar{k}_1 + 3\right)$$

$$\left. + \frac{A_{12}A_{13}}{2\bar{k}_1}\sin\bar{k}_1 - \frac{A_{11}A_{13}}{2\bar{k}}\ \cos\bar{k}_1 - \frac{1}{\lambda_1^2}\right] + \widetilde{A}_{31}\begin{pmatrix}\sin\bar{k}_3\\ \sinh\bar{k}_3\end{pmatrix}$$

$$+ \widetilde{A}_{32}\begin{pmatrix}\cos\bar{k}_3\\ \cosh\bar{k}_3\end{pmatrix} + \widetilde{A}_{33} + \widetilde{A}_{34} + \frac{\widetilde{\sigma}_3}{2\bar{k}_3}\ \left[A_{32}\begin{pmatrix}\sin\bar{k}_3\\ \sinh\bar{k}_3\end{pmatrix}\right.$$

$$\left. - A_{31}\begin{pmatrix}\cos\bar{k}_3\\ -\cosh\bar{k}_3\end{pmatrix} + \frac{\bar{q}_3}{\bar{k}_3^2}\right] = 0 \tag{30}$$

$$\widetilde{U}_{30} - \widetilde{A}_{31}\left[\frac{\bar{k}_3^2A_{31}}{2}\begin{pmatrix}1 + \sin 2\bar{k}_3/2\bar{k}_3\\ 1 + \sinh 2\bar{k}_3/2\bar{k}_3\end{pmatrix} - \frac{\bar{k}_3A_{32}}{2}\begin{pmatrix}\sin^2\bar{k}_3\\ -\sinh^2\bar{k}_3\end{pmatrix}\right.$$

$$+ A_{33}\begin{pmatrix}\sin\bar{k}_3\\ \sinh\bar{k}_3\end{pmatrix} + \frac{\bar{q}_3}{\bar{k}_3^3}\begin{pmatrix}\cos\bar{k}_3 - 1 + \bar{k}_3\sin\bar{k}_3\\ \cosh\bar{k}_3 - 1 - \bar{k}_3\sinh\bar{k}_3\end{pmatrix} +$$

$$\tilde{A}_{32}\left[\frac{\bar{k}_3 A_{31}}{2}\begin{pmatrix}\sin^2\bar{k}_3\\ -\sinh^2\bar{k}_3\end{pmatrix} - \frac{\bar{k}_3^2 A_{32}}{2}\begin{pmatrix}1 - \sin 2\bar{k}_3/2\bar{k}_3\\ -1 + \sinh 2\bar{k}_3/2\bar{k}_3\end{pmatrix} +\right.$$

$$\left. A_{33}\begin{pmatrix}1 - \cos\bar{k}_3\\ 1 - \cosh\bar{k}_3\end{pmatrix} + \frac{\bar{q}_3}{\bar{k}_3^3}\begin{pmatrix}\sin\bar{k}_3 - \bar{k}_3\cos\bar{k}_3\\ -\sinh\bar{k}_3 + \bar{k}_3\cosh\bar{k}_3\end{pmatrix}\right] -$$

$$\tilde{A}_{33}\left[A_{31}\begin{pmatrix}\sin\bar{k}_3\\ \sinh\bar{k}_3\end{pmatrix} - A_{32}\begin{pmatrix}1 - \cos\bar{k}_3\\ 1 - \cosh\bar{k}_3\end{pmatrix} + A_{33} + \frac{\bar{q}_3}{2\bar{k}_3^2}\begin{pmatrix}1\\ -1\end{pmatrix}\right] -$$

$$\tilde{\sigma}_3\left[\frac{A_{31}A_{32}\bar{k}_3}{4}\begin{pmatrix}\sin^2\bar{k}_3/\bar{k}_3^2 + \sin 2\bar{k}_3/\bar{k}_3\\ \sinh^2\bar{k}_3/\bar{k}_3^2 + \sinh 2\bar{k}_3/\bar{k}_3\end{pmatrix} + \frac{A_{32}A_{33}}{2}\begin{pmatrix}\sin\bar{k}_3/\bar{k}_3\\ \sinh\bar{k}_3/\bar{k}_3\end{pmatrix}\right.$$

$$+\frac{A_{33}A_{31}}{2\bar{k}_3^2}\begin{pmatrix}-\bar{k}_3\cos\bar{k}_3\\ \bar{k}_3\cosh\bar{k}_3\end{pmatrix} + \frac{A_{31}^2}{8}\begin{pmatrix}\sin 2\bar{k}_3/2\bar{k}_3 + 2\sin^2\bar{k}_3 - 3\\ \sinh 2\bar{k}_3/2\bar{k}_3 + 2\sinh^2\bar{k}_3 + 3\end{pmatrix}$$

$$-\frac{A_{32}^2}{8}\begin{pmatrix}1 - \sin 2\bar{k}_3/2\bar{k}_3 + 2\sin^2\bar{k}_3\\ 1 - \sinh 2\bar{k}_3/2\bar{k}_3 - 2\sinh^2\bar{k}_3\end{pmatrix} + \frac{A_{31}\bar{q}_3}{2\bar{k}_3^5}\begin{pmatrix}3\bar{k}_3\sin\bar{k}_3 + 3\cos\bar{k}_3 - \bar{k}_3^2\cos\bar{k}_3 - 3\\ 3\bar{k}_3\sinh\bar{k}_3 - 3\cosh\bar{k}_3 - \bar{k}_3^2\cosh\bar{k}_3 +\end{pmatrix}$$

$$\left. + \frac{A_{32}\bar{q}_3}{2\bar{k}_3^5}\begin{pmatrix}3k_3\cos\bar{k}_3 - 3\sin\bar{k}_3 + \bar{k}_3^2\sin\bar{k}_3\\ 3\bar{k}_3\cosh\bar{k}_3 - 3\sinh\bar{k}_3 - \bar{k}_3^2\sinh\bar{k}_3\end{pmatrix} + \frac{A_{33}\bar{q}_3}{2\bar{k}_3^4} + \frac{(\bar{q}_3)^2}{3\bar{k}_3^6}\begin{pmatrix}1\\ -1\end{pmatrix} - \frac{1}{\lambda_3^2}\right]$$

$$-\tilde{A}_{11}\sin\bar{k}_1 - \tilde{A}_{12}\cos\bar{k}_1 - \tilde{A}_{13} - \tilde{A}_{14} - \frac{\tilde{\sigma}_1}{2\bar{k}_1}(A_{12}\sin\bar{k}_1 - A_{11}\cos\bar{k}_1) = 0 \quad (31)$$

$$\tilde{A}_{11}(\bar{k}_1\cos\bar{k}_1) - \tilde{A}_{12}(\bar{k}_1\sin\bar{k}_1) + \tilde{A}_{13} + \tilde{\sigma}_1\left(\frac{A_{12}}{2\bar{k}_1}\sin\bar{k}_1 - \frac{A_{11}}{2\bar{k}_1}\cos\bar{k}_1\right.$$

$$\left. + \frac{A_{12}}{2}\cos\bar{k}_1 + \frac{A_{11}}{2}\sin\bar{k}_1\right) - \tilde{A}_{31}\bar{k}_3\begin{pmatrix}\cos\bar{k}_3\\ \cosh\bar{k}_3\end{pmatrix} + \tilde{A}_{32}\bar{k}_3\begin{pmatrix}\sin\bar{k}_3\\ -\sinh\bar{k}_3\end{pmatrix}$$

$$- \tilde{A}_{33} - \tilde{\sigma}_3 \left[\frac{A_{32}}{2\bar{k}_3} \begin{pmatrix} \sin \bar{k}_3 \\ \sinh \bar{k}_3 \end{pmatrix} - \frac{A_{31}}{2\bar{k}_3} \begin{pmatrix} \cos \bar{k}_3 \\ -\cosh \bar{k}_3 \end{pmatrix} + \frac{A_{32}}{2} \begin{pmatrix} \cos \bar{k}_3 \\ \cosh \bar{k}_3 \end{pmatrix} \right.$$

$$\left. + \frac{A_{31}}{2} \begin{pmatrix} \sin \bar{k}_3 \\ \sinh \bar{k}_3 \end{pmatrix} + \frac{\bar{q}_3}{\bar{k}_3^4} \right] = 0 \tag{32}$$

$$-\tilde{A}_{11}(\bar{k}_1^2 \sin \bar{k}_1) - \tilde{A}_{12}(\bar{k}_1^2 \cos \bar{k}_1) + \tilde{\sigma}_1 (A_{12} \cos \bar{k}_1 + A_{11} \sin \bar{k}_1$$

$$+ \frac{A_{11}\bar{k}_1}{2} \cos \bar{k}_1 - \frac{A_{12}\bar{k}_1}{2} \sin \bar{k}_1) - \tilde{A}_{31}\left(\frac{r_3}{\mu_3}\right) \bar{k}_3^2 \begin{pmatrix} \sin \bar{k}_3 \\ -\sinh \bar{k}_3 \end{pmatrix}$$

$$-\tilde{A}_{32}\left(\frac{r_3}{\mu_3}\right) \bar{k}_3^2 \begin{pmatrix} \cos \bar{k}_3 \\ -\cosh \bar{k}_3 \end{pmatrix} + \tilde{\sigma}_3 \left(\frac{r_3}{\mu_3}\right) \left[A_{32} \begin{pmatrix} \cos \bar{k}_3 \\ \cosh \bar{k}_3 \end{pmatrix} \right.$$

$$\left. + A_{31} \begin{pmatrix} \sin \bar{k}_3 \\ \sinh \bar{k}_3 \end{pmatrix} + \frac{A_{31}\bar{k}_3}{2} \begin{pmatrix} \cos \bar{k}_3 \\ \cosh \bar{k}_3 \end{pmatrix} - \frac{A_{32}\bar{k}_3}{2} \begin{pmatrix} \sin \bar{k}_3 \\ -\sinh \bar{k}_3 \end{pmatrix} + \frac{q_3}{\bar{k}_3^4} \right] = 0 \tag{33}$$

$$-\tilde{\sigma}_1 - \tilde{A}_{33} \left(\frac{r_3}{\mu_3^2}\right) \bar{k}_3^2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tilde{\sigma}_3 \left(\frac{r_3}{\mu_3^2}\right) A_{33} = 0 \tag{34}$$

$$\tilde{A}_{13}(\bar{k}_1^2) - \tilde{\sigma}_1 A_{13} - \tilde{\sigma}_3 \left(\frac{r_3}{\mu_3^2}\right) = 0 \tag{35}$$

$$- \tilde{A}_{21} \left[\frac{A_{21}\bar{k}_2^2}{2} \left(1 + \frac{\sin 2\bar{k}_2}{2\bar{k}_2}\right) - \frac{A_{22}\bar{k}_2}{2} \sin^2 \bar{k}_2 + A_{23} \sin \bar{k}_2 \right]$$

$$- \tilde{A}_{22} \left[- \frac{A_{21}\bar{k}_2}{2} \sin^2 \bar{k}_2 + \frac{A_{22}\bar{k}_2^2}{2} \left(1 - \frac{\sin 2\bar{k}_2}{2\bar{k}_2}\right) - A_{23} \left(1 - \cos \bar{k}_2\right) \right]$$

$$- \tilde{A}_{23} \left[A_{21} \sin \bar{k}_2 - A_{22}(1 - \cos \bar{k}_2) + A_{23} \right] -$$

$$\tilde{\sigma}_2 \left[- \frac{A_{21}^2}{8} \left(\frac{\sin 2\bar{k}_2}{2\bar{k}_2} - 2 \sin^2 \bar{k}_2 + 3 \right) + \frac{A_{22}^2}{8} \left(\frac{\sin 2\bar{k}_2}{2\bar{k}_2} - 1 - 2\sin^2 \bar{k}_2 \right) \right.$$

$$\left. + \frac{A_{21}A_{22}}{4} \bar{k}_2 (\sin^2 \bar{k}_2 / \bar{k}_2^2 + \sin 2\bar{k}_2 / \bar{k}_2) - \frac{A_{21}A_{23}}{2\bar{k}_2} \cos \bar{k}_2 + \frac{A_{22}A_{23}}{2\bar{k}_2} \sin \bar{k}_2 - \frac{1}{\lambda_2^2} \right]$$

$$+ \tilde{A}_{32} + \tilde{A}_{34} = 0 \tag{36}$$

$$\tilde{U}_{30} - \tilde{A}_{21}(\sin \bar{k}_2) - \tilde{A}_{22}(\cos \bar{k}_2) - \tilde{A}_{23} - \tilde{A}_{24} - \frac{\tilde{\sigma}_2}{2\bar{k}_2}(A_{22} \sin \bar{k}_2 - A_{21} \cos \bar{k}_2) = 0 \tag{37}$$

$$\tilde{A}_{21}(\bar{k}_2 \cos \bar{k}_2) - \tilde{A}_{22}(\bar{k}_2 \sin \bar{k}_2) + \tilde{A}_{23} + \tilde{\sigma}_2 \left(\frac{A_{22}}{2\bar{k}_2} \sin \bar{k}_2 - \frac{A_{21}}{2\bar{k}_2} \cos \bar{k}_2 + \frac{A_{22}}{2} \cos \bar{k}_2 + \frac{A_{21}}{2} \sin \bar{k}_2\right) - \tilde{A}_{31}(\bar{k}_3) - \tilde{A}_{33} + \tilde{\sigma}_3 \left(\frac{A_{31}}{2\bar{k}_3}\right)\begin{pmatrix}1\\-1\end{pmatrix} = 0 \tag{38}$$

$$- \tilde{A}_{32}\left(\frac{r_3}{\mu_3}\right)\bar{k}_3^2 + \tilde{\sigma}_3\left[\left(\frac{r_3}{\mu_3}\right)\left(A_{32} + \frac{\bar{q}_3}{\bar{k}_3^4}\right)\right] + \tilde{A}_{21}\left(\frac{r_2}{\mu_2}\right)\bar{k}_2^2 \sin \bar{k}_2 + \tilde{A}_{22}\left(\frac{r_2}{\mu_2}\right)\bar{k}_2^2 \cos \bar{k}_2 - \tilde{\sigma}_2\left(\frac{r_2}{\mu_2}\right)\left(A_{22} \cos \bar{k}_2 + A_{21} \sin \bar{k}_2 + \frac{A_{21}\bar{k}_2}{2} \cos \bar{k}_2 - \frac{A_{22}\bar{k}_2}{2} \sin \bar{k}_2\right) = 0 \tag{39}$$

$$- \left(\frac{r_2}{\mu_2^2}\right)\tilde{\sigma}_2 + \tilde{A}_{33}\left(\frac{r_3}{\mu_3^2}\right)\bar{k}_3^2 - \tilde{\sigma}_3\left(\frac{r_3}{\mu_3^2}\right)A_{33} = 0 \tag{40}$$

$$- \tilde{A}_{23}\left(\frac{r_2}{\mu_2^2}\right)\bar{k}_2^2 + \tilde{\sigma}_2\left(\frac{r_2}{\mu_2^2}\right)A_{23} - \tilde{\sigma}_3\left(\frac{r_3}{\mu_3^2}\right) = 0 \tag{41}$$

When the upper term (in parenthesis) is used, the equation corresponds to compression in bar 3, and the lower to tension. Clearly, then, if either a bifurcation point or a limit point exists, the critical condition and the corresponding system response can be obtained from the simultaneous solution of the determinant (characteristic equation) and Eqs. (9) and (10) or Eqs. (14) and (15), for a given load condition. For example, if $\bar{Q}_i = \bar{F}_i = 0$ then the solution yields $\bar{q}_{3_{cr}}$, $\bar{k}_2$ and $\bar{k}_3$. Once these quantities are known, one can solve for all the remaining constants. Moreover, if one is interested in the shape of the buckling mode, 15 of the 16 dependent equations, Eqs. (26)-(41) can be used to solve for all buckling mode constants in terms of one of them. On the contrary, if there is no possibility of instability, the stable response can be obtained from either Eqs. (9) and (10) or Eqs. (14) and (15) for any level of the applied load. The key, though, to obtaining a solution, for either case, is the capability of solving a system of two or three non-linear equations.

III. SOLUTION

Regardless of the case, the solution to the system of nonlinear equations is obtained as follows: Let the three (at most) equations be denoted by

$$\begin{aligned} f_1(\bar{k}_2, \bar{k}_3, \Lambda, \text{geom.}) &= 0 \\ f_2(\bar{k}_2, \bar{k}_3, \Lambda, \text{geom.}) &= 0 \\ f_3(\bar{k}_2, \bar{k}_3, \Lambda, \text{geom.}) &= 0 \end{aligned} \tag{42}$$

where Λ is some load parameter (for the case of $\bar{Q}_i = \bar{F}_1 = 0$, $\Lambda = \bar{q}_3$).

Then, construct a new function, F, defined by

$$F = \sum_{i=1}^{3} f_i^2 \tag{43}$$

If a solution exist, for Eqs. (42), then it corresponds to the minimum of F in the space of $\bar{k}_2$, $\bar{k}_3$ and Λ_{cr}. The minimizing values for F, which also represent the solution to Eqs. (42), yield $F_{min.} = 0$. The simplex method of Nelder and Mead (Ref. 4) is employed in obtaining the minimum value of F and the minimizing values of $\bar{k}_2$, $\bar{k}_3$ and Λ_{cr}. Because of the nonlinear character of Eqs. (42), it is not unusual to have more than one solution for the system. The solution, then obtained by the simplex method depends upon the starting point in the minimization procedure, and therefore, one is never certain of the correctness of his solution. Because of this difficulty the following procedure is employed:

(1) Assign a small value for the load parameter and solve the equilibrium equations for k_2 and k_3, through the simplex method.

(2) Use the expression for the constants and solve for the complete response of the system for this load value.

(3) Choose some characteristic displacement, and obtain its value. The one chosen in this investigation is the rotation at joint 3, φ_1.

$$\varphi_1 = W_{3_X}(1) \tag{44}$$

(4) Increase the load and repeat steps (1) through (3). Use as initial values in the simplex method the values of $\bar{k}_2$ and $\bar{k}_3$ exactly or near the solution obtained for the previous load value.

(5) At each step check the value of the determinant. If there is a sign change, then there is a bifurcation load between the two loads at which the sign change took place.

(6) By adjusting the load increments (load steps) find the value of Λ_{cr}.

(7) Use the same equilibrium equations and obtain, as in steps (1) through (3), a point on the postbuckling branch.

(8) If the postbuckling point corresponds to a load level higher than Λ_{cr} (this is the case in the present investigation), then by small increments in Λ obtain the remaining postbuckling curve.

Thus, the complete response of the system is known (primary path as well as postbuckling path). Note that in steps (1) through (8) both sets of equations are checked (compression and tension in bar 3). It just happens that in the generated data, bar 3 is always in compression. One should not expect this to be always true.

IV. RESULTS AND DISCUSSION

Numerical solutions are generated for a frame acted on by a uniformly distributed load applied transversly on bar 3, and of various geometric parameters. Each case is described and discussed separately.

The Georgia Tech high speed digital computer CDC-Cyber 70, Model 74-28, is employed for data generation.

The first geometry consists of a square frame of uniform geometry and equal amounts of rotational restraint ($r_k = \mu_k = 1$, $\lambda_k = \lambda$, $\bar{\beta}_1 = \bar{\beta}_2 = \bar{\beta}$). The results are presented graphically on Figs. 2 and 3. On Fig. 2 the response of the frame is shown as plots of $\bar{q}$ versus "joint 3" rotation, for three values of $\bar{\beta}$. Both the primary path as well as the frame post-buckling behavior are shown. Note that $\bar{\beta} = o$ corresponds to simple supports, while $\bar{\beta} = 1000$ is a good approximation for the clamped support case. The bar slenderness ratio values used are, 40, 80, 120, and 1000. The results reveal that the effect of bar slenderness ratio, λ, on the nondimensionalized response characteristics is negligibly small. Thus, the data shown on Fig. 2, is applicable to all λ, as long as the material behavior is linearly elastic. On Fig. 3, the bifurcation load (sway-buckling load) is plotted versus the amount of rotational restraint.

The second case consists of a symmetric simply supported portal frame ($r_2 = \mu_2 = 1$) in which the length, as well as, the flexural stiffness of the horizontal bar are varied (μ_3 = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0; and r_3 = 0.5, 1.0, 2.0, 3.0, 10.0, 100.0). The slenderness ratio of the three bars is assumed to be the same. Since r_3 and μ_3 are varied, this assumption requires variation in the bar 3 cross-sectional area. In this case, also, it is found that the effect of slenderness ratio ($\lambda_k = \lambda$ = 40, 80, 120, 1,000) is negligibly small. The results are presented in tabular form on Table 1. This table

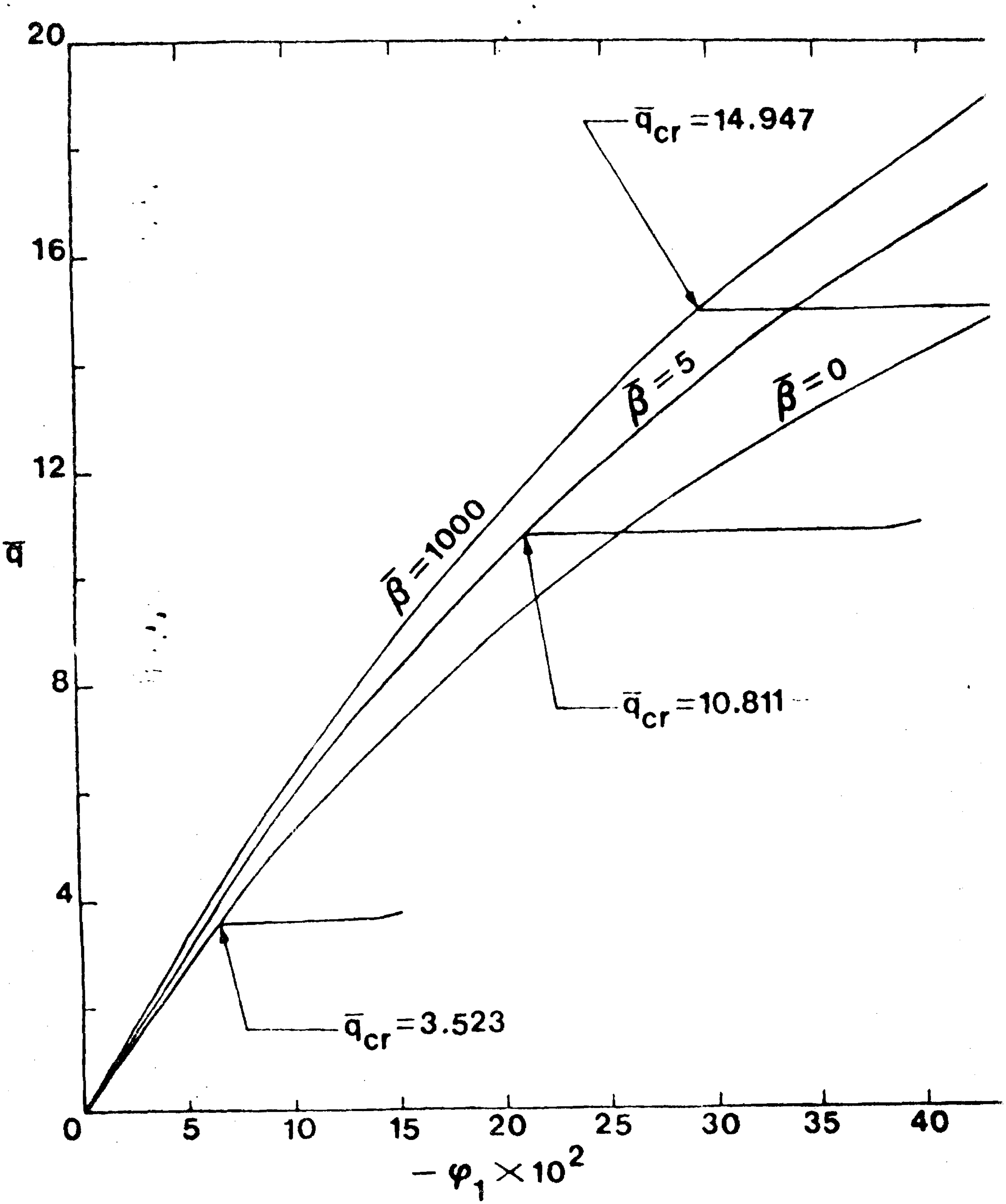

Fig. 2. Prebuckling and Postbuckling Equilibrium States for a Rotationally Restrained Symmetric Frame ($\cancel{\mu_k} = \mu_k = 1$).

γ_k

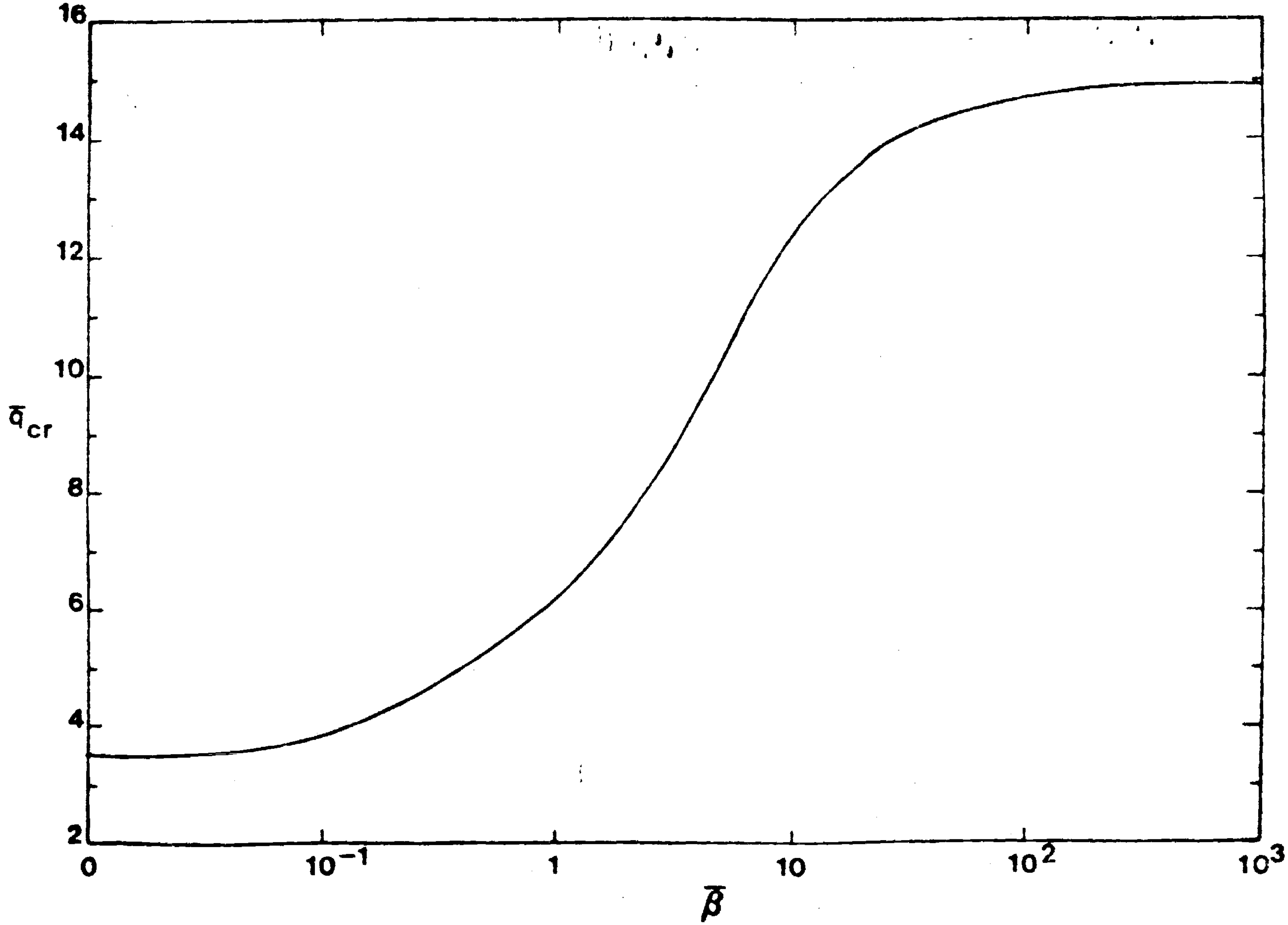

Fig. 3. Effect of Rotational Restraint on the Sway-Buckling Load, $\bar{q}_{cr}$ ($\eta_k = \mu_k = 1$).

shows the values of q^*_{cr} (bifurcational load) for various combinations of r_3, μ_3. The corresponding values of $\bar{k}_2$ and $\bar{k}_3$ are shown in parenthesis. Note that $r_3 = 100$ corresponds to the case where the horizontal bar is extremely stiff. In this case the load q_{cr} can be thought of as applied at the joints ($P_1 = P_2 = q\, \ell/2$) and the result should be the same as the one reported in Ref. 3. Indeed this is the case. The postbuckling behavior for these frames, not shown herein, is similar to that characterized by the data of Fig. 2.

TABLE 1. BIFURCATION LOADS FOR A SYMMETRIC SIMPLY-SUPPORTED FRAME

	$q^*_{cr} = q_3 \ell_1^3/(EI)_1$					
r_3 \ μ_3	0.5	1.0	1.5	2.0	2.5	3.0
0.5	3.550 $\begin{pmatrix}\bar{k}_2=1.3300\\ \bar{k}_3=0.2053\end{pmatrix}$	2.773 $\begin{pmatrix}1.1770\\ 0.5862\end{pmatrix}$	2.261 $\begin{pmatrix}1.0630\\ 1.0346\end{pmatrix}$	1.894 $\begin{pmatrix}0.9730\\ 1.5251\end{pmatrix}$	1.612 $\begin{pmatrix}0.8977\\ 2.0580\end{pmatrix}$	1.381 $\begin{pmatrix}0.8300\\ 2.6451\end{pmatrix}$
1.0	4.108 $\begin{pmatrix}1.4332\\ 0.1294\end{pmatrix}$	3.523 $\begin{pmatrix}1.3260\\ 0.4111\end{pmatrix}$	3.083 $\begin{pmatrix}1.2416\\ 0.7730\end{pmatrix}$	2.736 $\begin{pmatrix}1.1696\\ 1.1939\end{pmatrix}$	2.449 $\begin{pmatrix}1.1065\\ 1.6681\end{pmatrix}$	2.201 $\begin{pmatrix}1.0490\\ 2.2090\end{pmatrix}$
2.0	4.464 $\begin{pmatrix}1.4940\\ 0.0747\end{pmatrix}$	4.074 $\begin{pmatrix}1.4263\\ 0.2583\end{pmatrix}$	3.748 $\begin{pmatrix}1.3701\\ 0.5152\end{pmatrix}$	3.472 $\begin{pmatrix}1.3171\\ 0.8285\end{pmatrix}$	3.232 $\begin{pmatrix}1.2750\\ 1.1920\end{pmatrix}$	3.021 $\begin{pmatrix}1.2270\\ 1.6145\end{pmatrix}$
3.0	4.596 $\begin{pmatrix}1.5150\\ 0.0525\end{pmatrix}$	4.311 $\begin{pmatrix}1.4680\\ 0.1891\end{pmatrix}$	4.047 $\begin{pmatrix}1.4210\\ 0.3876\end{pmatrix}$	3.814 $\begin{pmatrix}1.3840\\ 0.6362\end{pmatrix}$	3.609 $\begin{pmatrix}1.3430\\ 0.9301\end{pmatrix}$	3.425 $\begin{pmatrix}1.3087\\ 1.2702\end{pmatrix}$
10.0	4.840 $\begin{pmatrix}1.5402\\ 0.0161\end{pmatrix}$	4.720 $\begin{pmatrix}1.5350\\ 0.0661\end{pmatrix}$	4.600 $\begin{pmatrix}1.5160\\ 0.1434\end{pmatrix}$	4.483 $\begin{pmatrix}1.4971\\ 0.2462\end{pmatrix}$	4.370 $\begin{pmatrix}1.4780\\ 0.3721\end{pmatrix}$	4.263 $\begin{pmatrix}1.4591\\ 0.5208\end{pmatrix}$
100.0	5.080 $\begin{pmatrix}1.5801\\ 0.0030\end{pmatrix}$	4.976 $\begin{pmatrix}1.5752\\ 0.0075\end{pmatrix}$	4.933 $\begin{pmatrix}1.5702\\ 0.0159\end{pmatrix}$	4.875 $\begin{pmatrix}1.5631\\ 0.0281\end{pmatrix}$	4.848 $\begin{pmatrix}1.5603\\ 0.0435\end{pmatrix}$	4.844 $\begin{pmatrix}1.5564\\ 0.0625\end{pmatrix}$

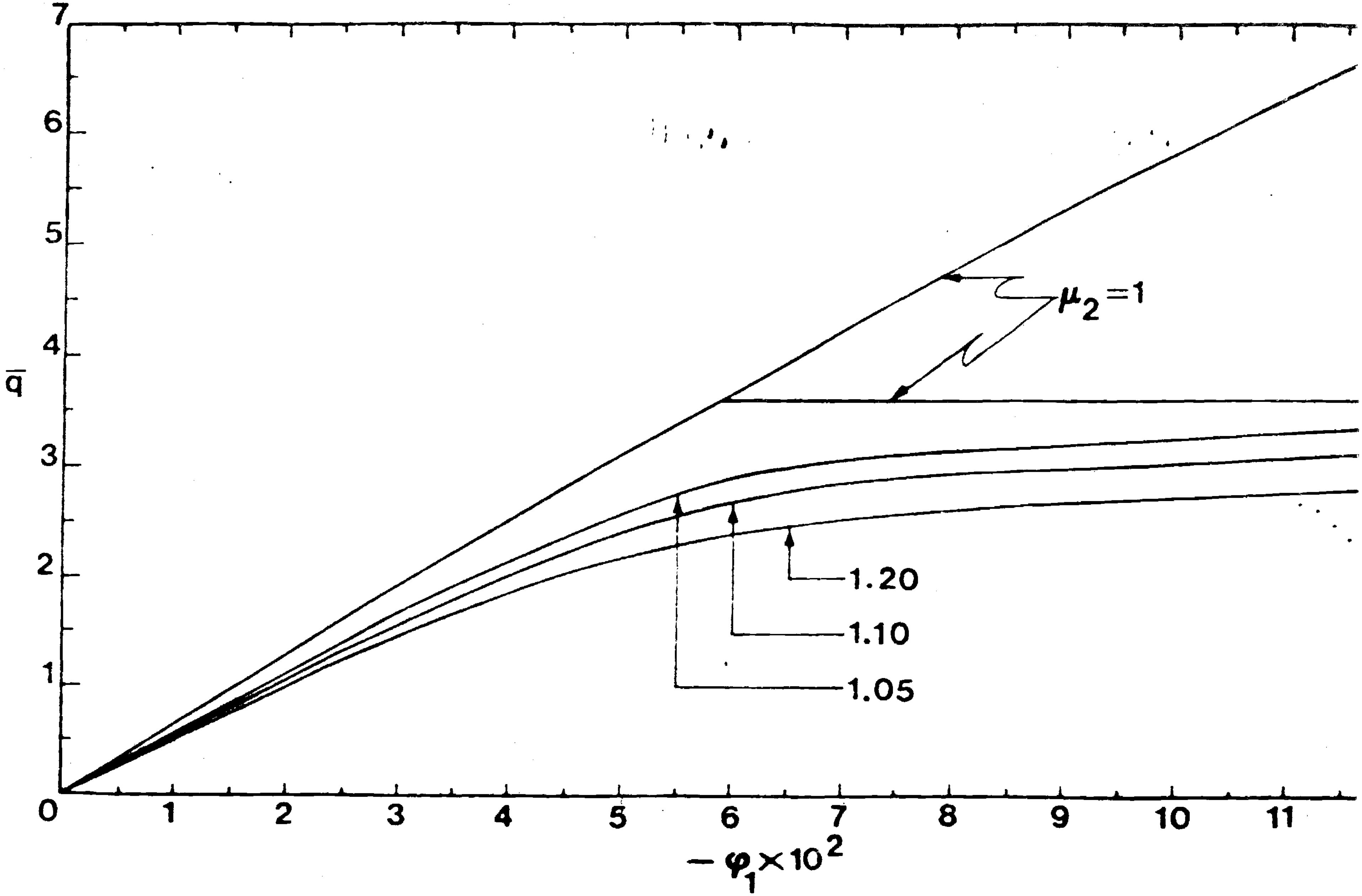

Fig. 4. Effect of Variable Vertical Bar Lengths on the Frame Response Characteristics ($r_k = 1$).

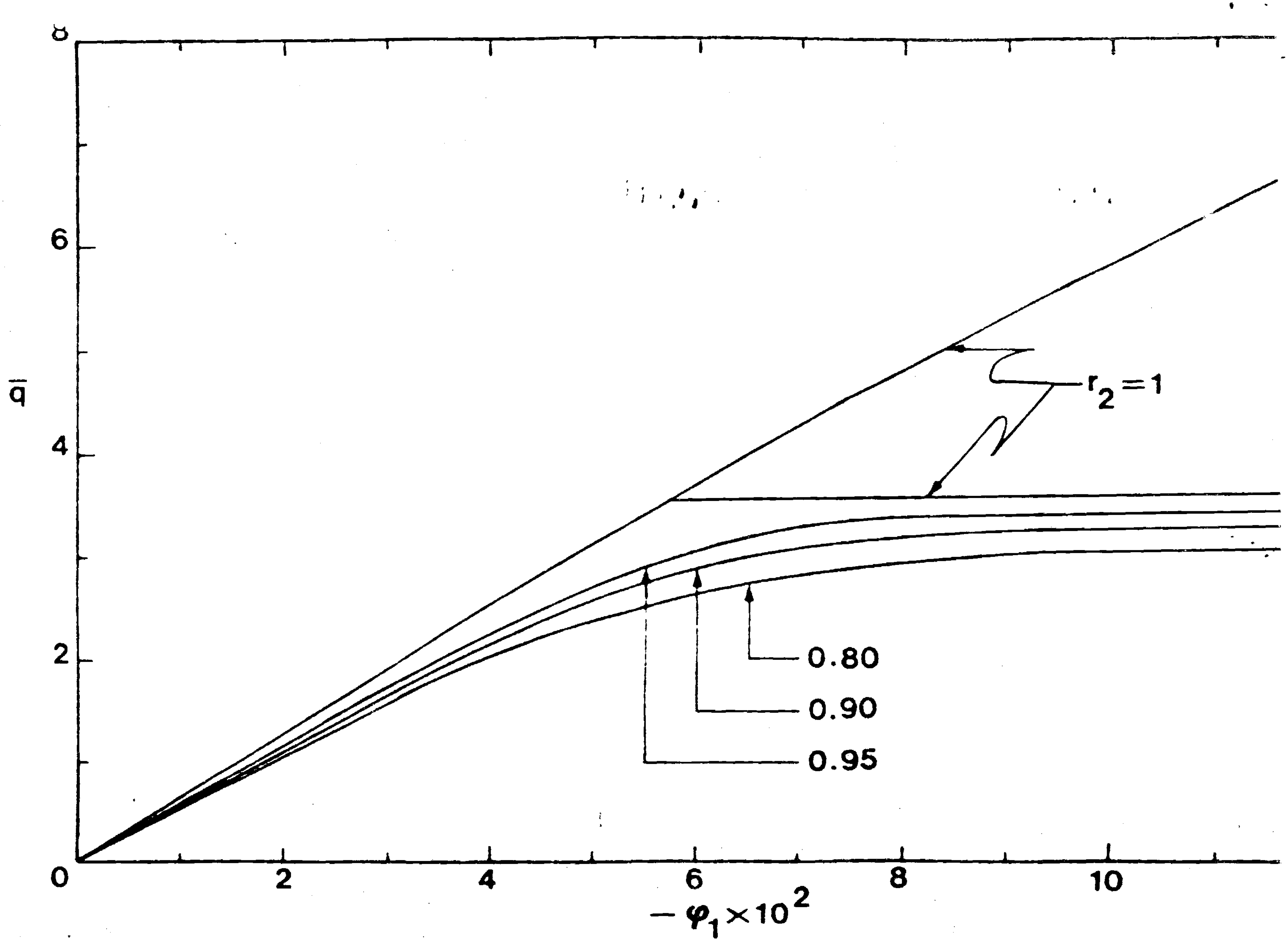

Fig. 5. Effects of Variable Vertical Bar Flexural Stiffness on the Frame Response Characteristics ($\mu_k = 1$).

Note that, in this case, as expected, the load carrying capacity of the frame decreases with increasing length of the loaded bar (for the same flexural stiffness). Similarly, for a constant length of the loaded ba the load carrying capacity of the frame increases with increasing flexu stiffness.

The last two cases considered, deal individually, with the effect nonuniformity. In one case all geometric parameters are the same (r_k = 1, μ_3 = 1, $\lambda_k = \lambda$) except that μ varies (= 1.05, 1.10, 1.20). This mea that the length of bar 2 is longer than that of bar 1. In the second c all geometric parameters are the same, except that the flexural stiffne of bar 2 is smaller than that of bar 1. The results for these cases ar presented graphically in Figs. 4 and 5. In both of these cases the fol observations are made. The effect of slenderness ratio is negligibly s The response is characterized by stable bent equilibrium positions and curves approach asymptotically the corresponding perfect and uniform ge curve. There is always compression in the horizontal bar. Note also t the curves corresponding to r_2 = 0.95, 0.90, 0.80 are very similar to t corresponding to μ_2 = 1.05, 1.10, 1.20. This is reasonable because an crease in length ℓ_2 or a decrease in flexural stiffness $(EI)_2$ yield a mo flexible member. Values of $r_2 > 1$ and correspondingly $\mu_2 < 1$ are not c sidered because the response characteristics would be similar to the on obtained except that the role of bar 1 and 2 would be interchanged.

V. CONCLUSIONS

On the basis of the analysis and the generated data one may list t following as important conclusions:

(1) A methodology has been developed and demonstrated for finding the complete response (including postbuckling, if it exists) of an unbraced, rigid-jointed, elastic portal frame subjected to transverse loads.

(2) The effect of bar slenderness ratio on the nondimensionalized response characteristics is negligibly small.

(3) Portal frames exhibit stable postbuckling behavior, and thus cannot be expected to be sensitive to imperfections. If variation in bar 2 length and flexural stiffness are thought of as geometric imperfections - this point is well proven. As a matter of fact, in many respects, the frame response is similar to that of an axially-loaded cantilever column.

(4) Increase in the amount of rotational restraint, $\bar{\beta}$, increases the bifurcation load.

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