Robin Thomas is 50!! Atlanta, May 10, 2012

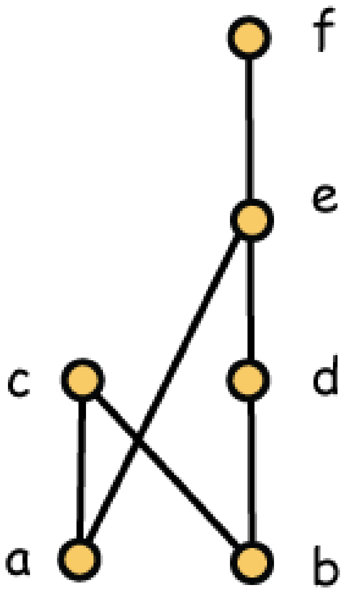
Planarity and Dimension for Graphs and Posets

William T. Trotter
trotter@math.gatech.edu

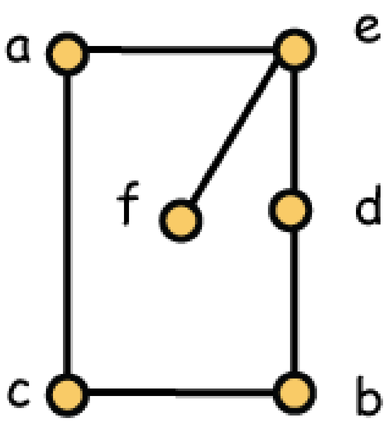
Robin Thomas and WTT - ???

WTT in Prague with Nešetřil and Rődl, 1983

Order Diagrams and Cover Graphs

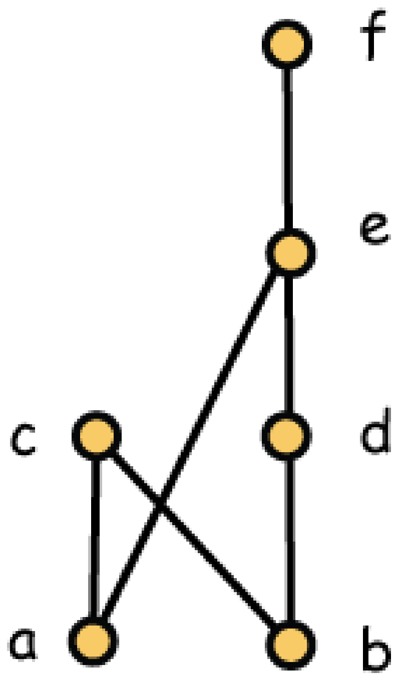


Order Diagram

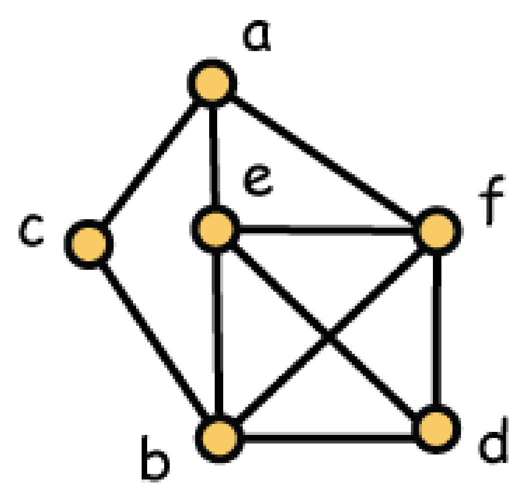


Cover Graph

Diagrams and Comparability Graphs

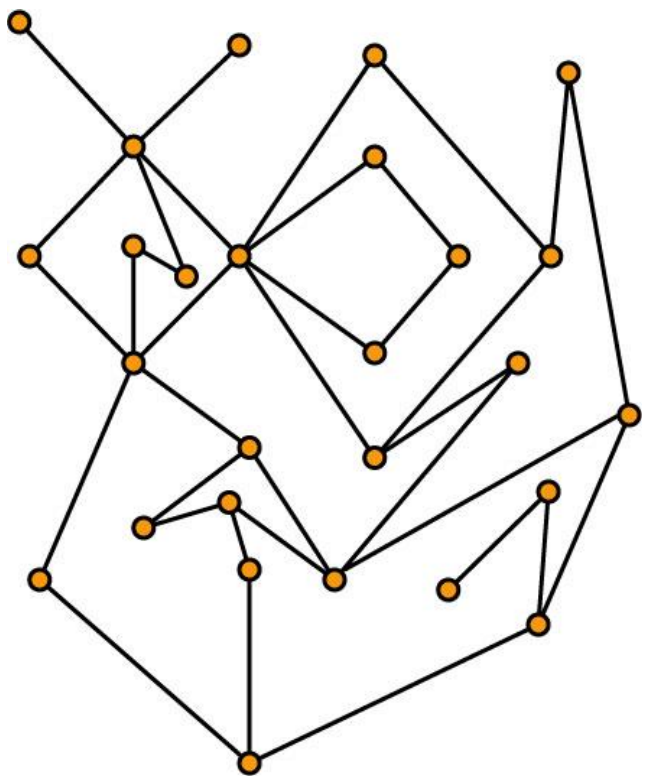


Poset



Comparability Graph

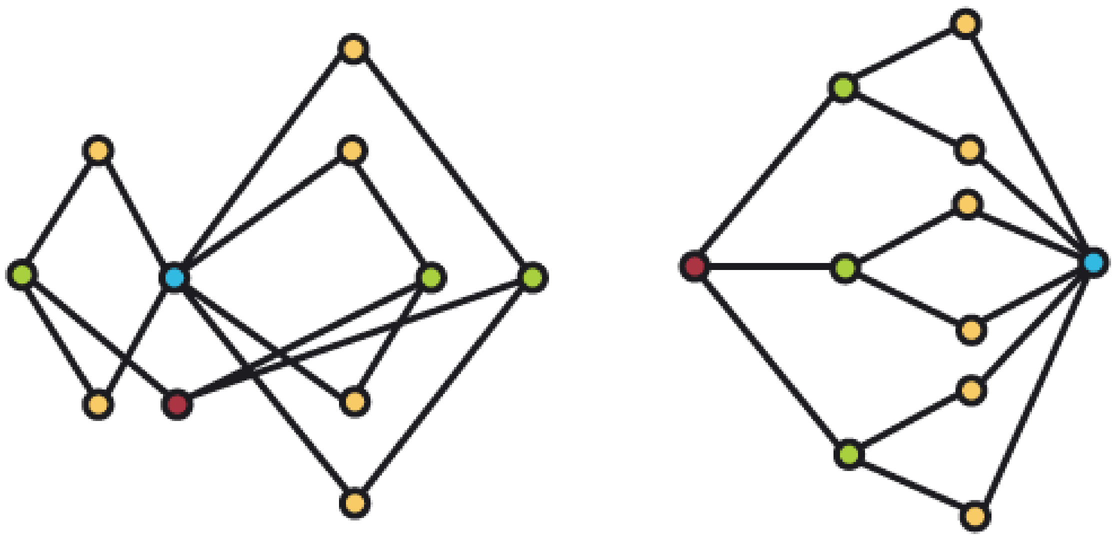
Planar Posets



Definition A poset P is planar when it has an order diagram with no edge crossings.

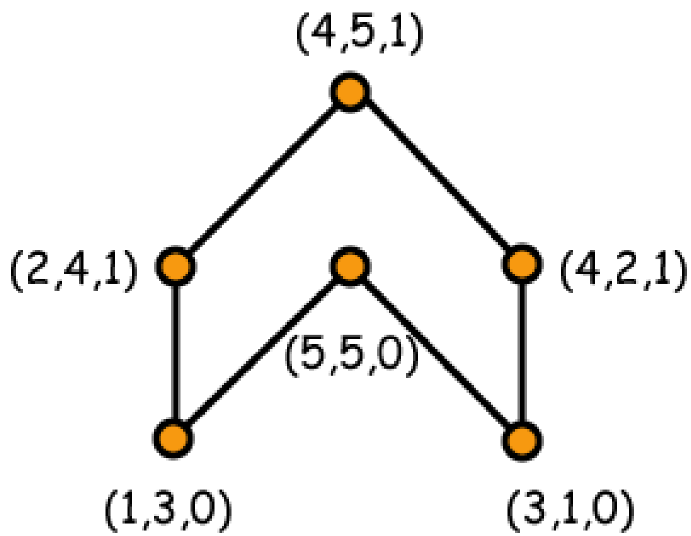
Fact If P is planar, then it has an order diagram with straight line edges and no crossings.

A Non-planar Poset



This height 3 non-planar poset has a planar cover graph.

Definition of Dimension



The dimension of a poset P is the least integer n for which P is a subposet of $\mathbf{R}^n$. This embedding shows that $\dim(P) \leq 3$. In fact,

$$\dim(P) = 3$$

Dimension ~ Chromatic Number

Problem Let $f(k)$ be the maximum chromatic number of a graph G with $\Delta(G) = k$.

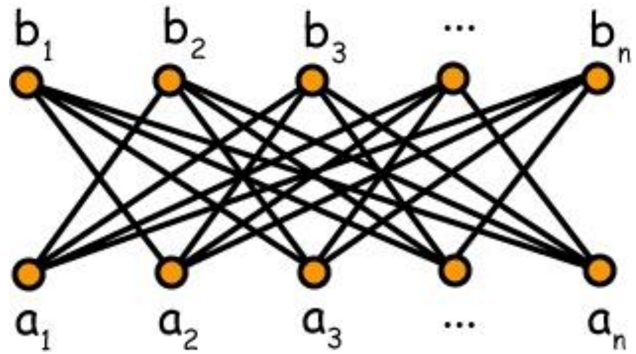
Solution (Brooks Theorem) $f(k) = k + 1$.

Problem Let $f(k)$ be the maximum dimension of a poset P with $\Delta(P) = k$.

Solution (Erdős, Kierstead and WTT; Füredi and Kahn)

$$c\, k \log k < f(k) < c'\, k \log^2 k$$

Standard Examples



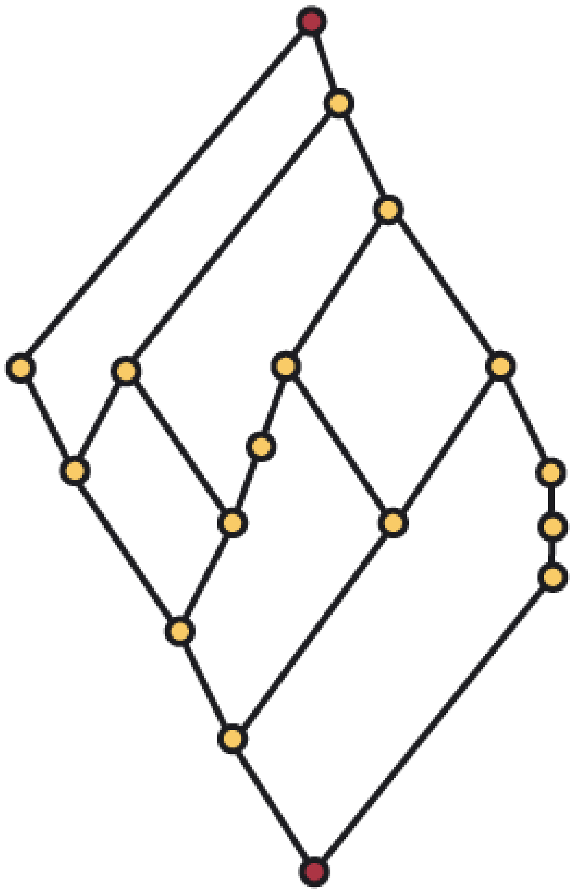
S_n

Fact For $n \geq 2$, the standard example S_n is a poset of dimension n.

Planar Posets with Zero and One

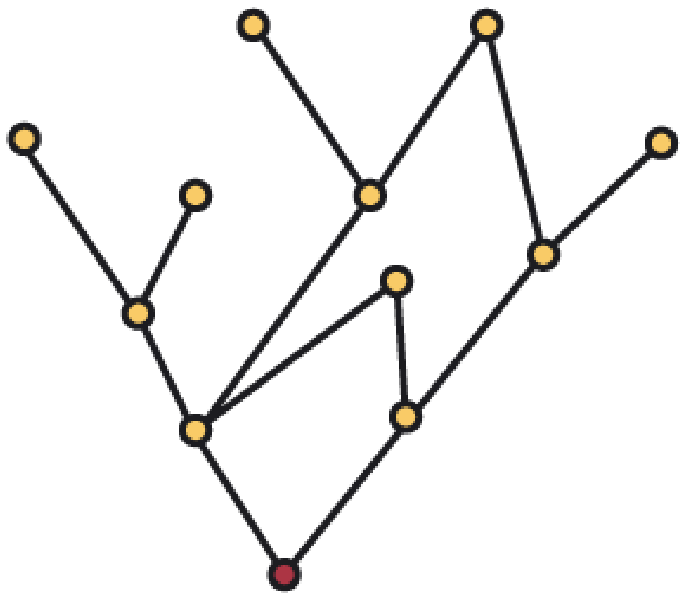
Theorem (Baker, Fishburn and Roberts, 1971 + Folklore)

If P has both a 0 and a 1, then P is planar if and only if it is a lattice and has dimension at most 2.



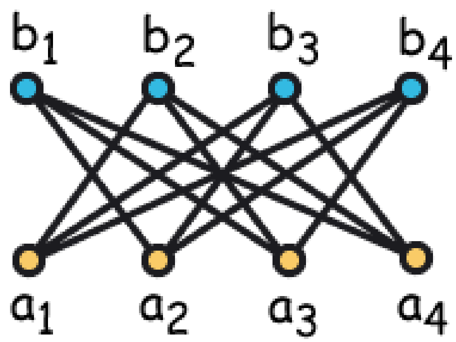
Dimension of Planar Posets with a Zero

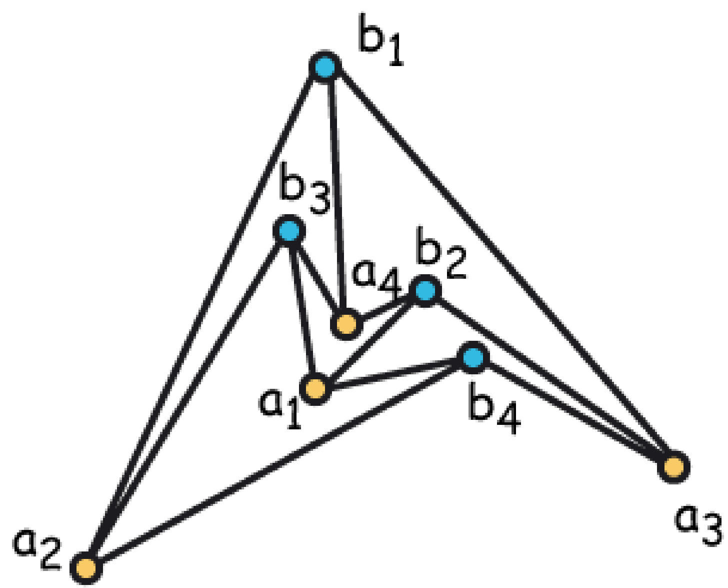
Theorem (WTT and Moore, 1977) If P has a 0 and the diagram of P is planar, then $\dim(P) \leq 3$.



A 4-dimensional planar poset

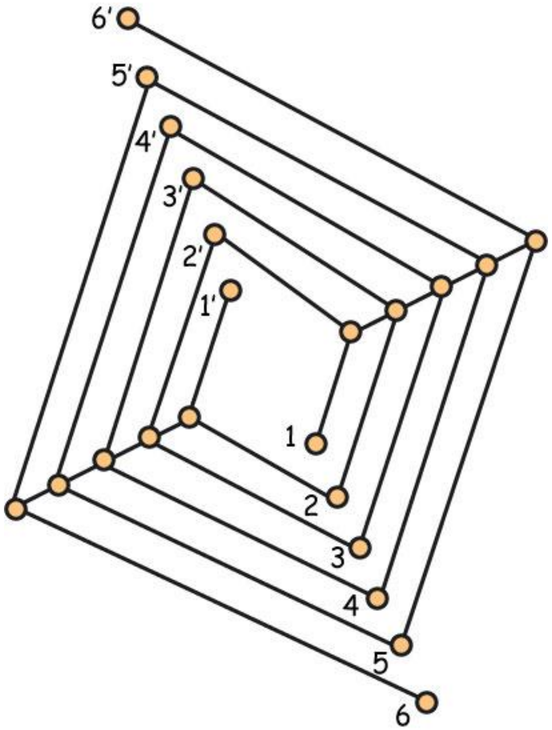
Fact The standard example S_4 is planar!



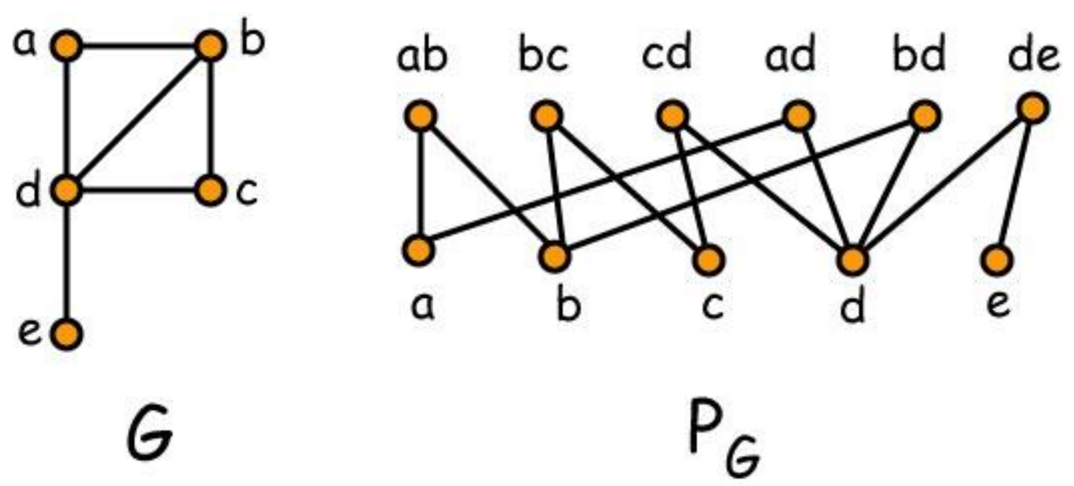


Kelly's Construction

Theorem (Kelly, 1981) For every $n \geq 5$, the standard example S_n is non-planar, but it is a subposet of a planar poset.



The Vertex-Edge Poset of a Graph



The vertex-edge poset of a graph is also called the incidence poset of the graph.

Schnyder's Theorem

Theorem (Schnyder + Babai and Duffus, 1989) *A* graph is planar if and only if the dimension of its vertex-edge poset is at most 3.

Note Testing graph planarity is linear in the number of edges while testing for dimension at most 3 is NP-complete!!!

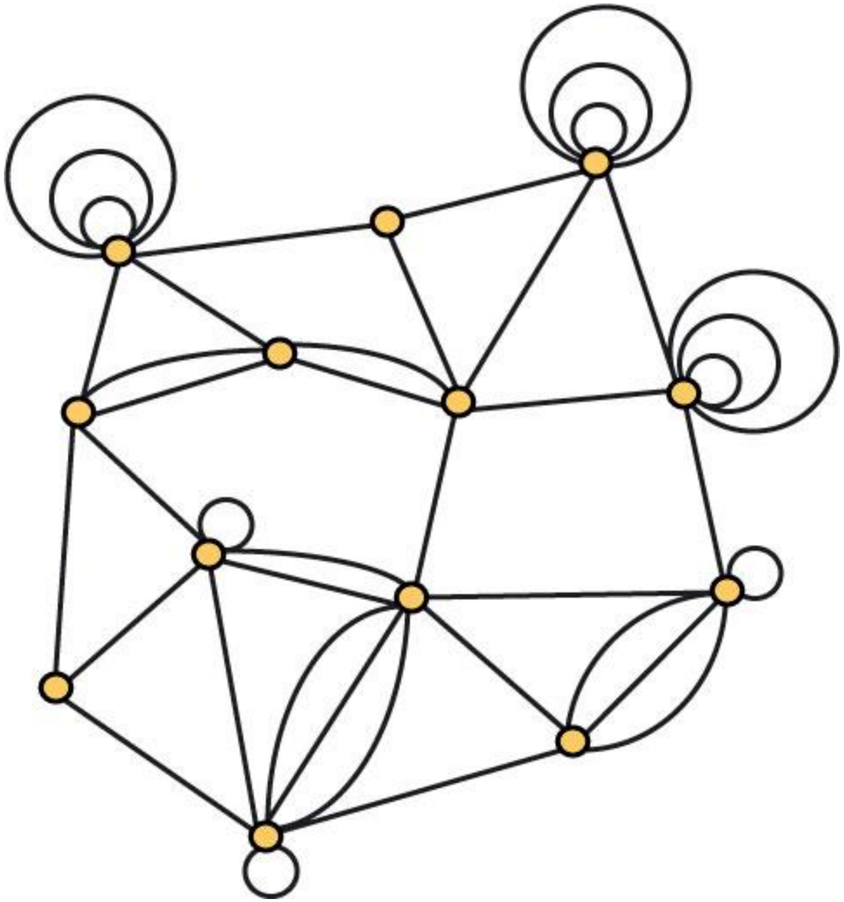
Structure and Schnyder

Schnyder's proof is a classic, elegant and rich in structure.

His principal motivation was to find an efficient layout of a planar graph on a small grid.

Recently, Haxell and Barrera-Cruz (2011) have found a direct – and very compact - proof, sans the structure, but the value of Schnyder's original approach remains intact.

Planar Multigraphs



Planar Multigraphs and Dimension

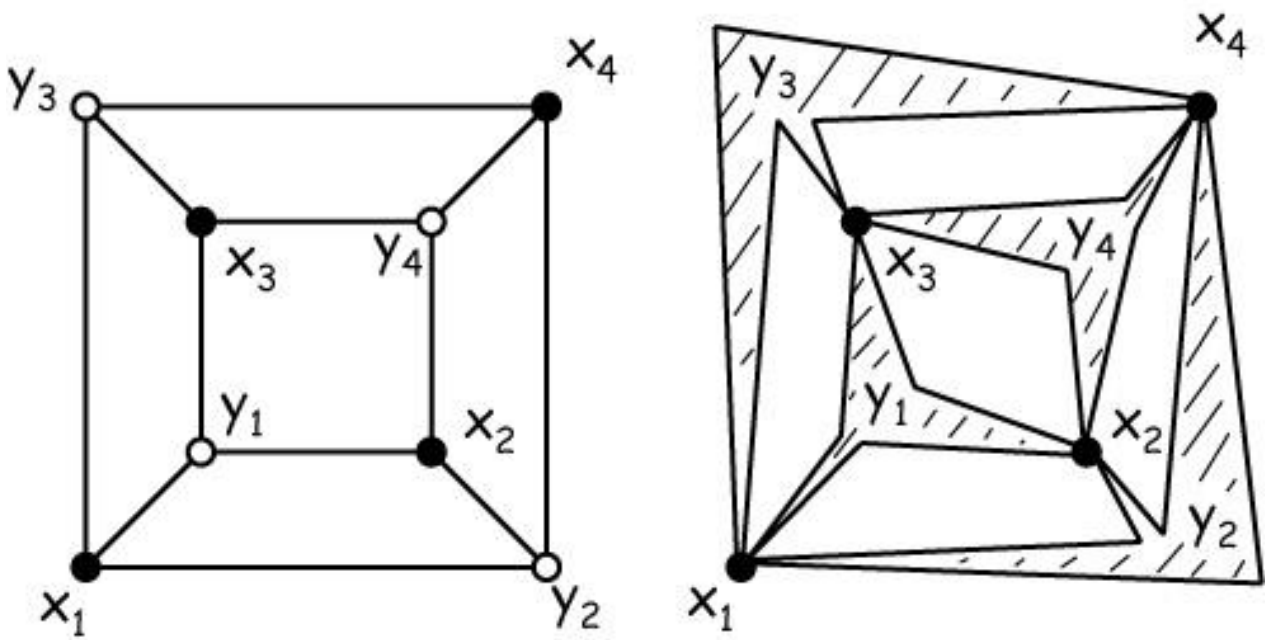
Theorem (Brightwell and WTT, 1996): Let D be a non-crossing drawing of a planar multigraph G, and let P be the vertex-edge-face poset determined by D. Then $\dim(P) \leq 4$.

Note Inductive proof with planar 3-connected graphs as the base case. Done by GRB and WTT four years earlier.

Fact Different drawings may determine posets with different dimensions.

Bipartite Planar Graphs

Theorem (Felsner, Li, WTT, 2010) If P has height 2 and the cover graph of P is planar, then $\dim(P) \leq 4$.



Planar Cover Graphs, Dimension and Height

Conjecture (Felsner, Li and WTT, 2010) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Observation The conjecture holds trivially for $h = 1$ and $c_1 = 2$. Although very non-trivial, the conjecture also holds for $h = 2$, and $c_2 = 4$.

Fact Kelly's construction shows that c_h - if it exists - must be at least $h + 1$.

Conjecture Resolved

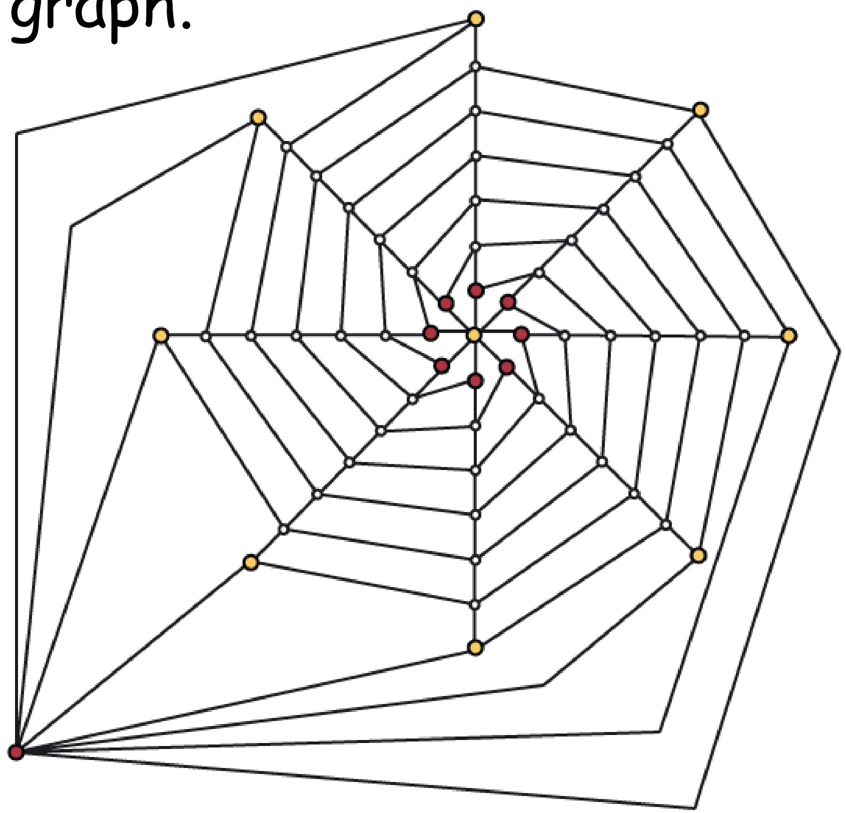
Theorem (Streib and WTT, 2012) For every integer h, there exists a constant c_h so that if P is a poset of height h and the cover graph of P is planar, then $\dim(P) \leq c_h$.

Fact A straightforward modification to Kelly's construction shows that c_h must be at least $h + 2$.

However, our proof uses Ramsey theory at several key places and the bound we obtain is **very** large in terms of h.

A Modest Improvement

Fact For every $h \geq 2$, the standard example S_{h+2} is contained in a poset of height h having a planar cover graph.



Planarity and Dimension

Theorem (Felsner, WTT and Wiechert, 2012)

Let P be a poset.

1. If the comparability graph of P is planar, then $\dim(P) \leq 4$.
2. If the cover graph of P is outerplanar, then $\dim(P) \leq 4$.
3. If the cover graph of P is outerplanar, and P has height at most 3, then $\dim(P) \leq 3$.

Some Open Questions

1. For each $t \geq 4$, what is the smallest planar poset having dimension t?
2. Improve the bounds for the constant c_h in the Streib-WTT theorem.
3. What is the maximum dimension of a poset with a planar incomparability graph?

Robin Thomas is 50!!

Maple told me that

50! =
304140932017133780436126081660647\
68844377641568960512000000000000

But when I asked for 50!!, Maple replied

"Kernel connection has been lost."