On the Structure of Counterexamples to the Coloring Conjecture of Hajós

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On the Structure of Counterexamples to the Coloring Conjecture of Hajós

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In memory of Torsten Küther.

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TABLE OF CONTENTS

LIST OF FIGURES

SUMMARY

Hajós conjectured that, for any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. This is true when $k \leq 3$, and false when $k \geq 6$. Hajós' conjecture remains open for $k = 4, 5$.

We will first present some known results on Hajós' conjecture. Then we derive a result on the structure of 2-connected graphs with no cycle through three specified vertices. This result will then be used for the proof of the main result of this thesis. We show that any possible counterexample to Hajós' conjecture for $k = 4$ with minimum number of vertices must be 4-connected. This is a step in an attempt to reduce Hajós' conjecture for $k = 4$ to the conjecture of Seymour that any 5-connected non-planar graph contains a K_5-subdivision.

CHAPTER I

INTRODUCTION

All graphs considered in this thesis are simple. For terminology not defined here, we refer to [1]. A graph H is a subdivision of a graph G, if H is obtained from G by subdividing some of the edges, that is replacing the edges by internally disjoint paths. An H-subdivision in a graph G is a subgraph of G which is isomorphic to a subdivision of the graph H.

For $m \in \mathbb{N}$, we denote the complete graph on m vertices by K_m. A path P is a graph of the form

$$V(P) = \{x_1, ..., x_n\}, E(P) = \{x_i x_{i+1} | i = 1, ..., n-1\},$$

and x_0, x_n are the endvertices of P, $x_2, ..., x_{n-1}$ its internal vertices. A family of paths $\mathcal{P}$ is said to be internally disjoint if for $P_1, P_2 \in \mathcal{P}$, $v \in V(P_1) \cap V(P_2)$ implies that v is an endvertex of P_1 as well as P_2. A path in a graph G is a branching path if its internal vertices all have degree 2 in G and its endvertices have degree at least 3 in G. The vertices of degree at least 3 in G are called branching vertices of G.

A coloring c of a graph G is a function $c : V(G) \to \{1, ..., k\}$ for some $k \in \mathbb{N}$ such that $c(x) \neq c(y)$ for all $xy \in E(G)$. G is k_0-colorable if there exist a coloring $c : V(G) \to \{1, ..., k_0\}$. The chromatic number $\chi(G)$ is defined as $\chi(G) = \min\{k \in \mathbb{N} | G \text{ is k-colorable}\}$.

Hajós conjectured in [6] that every graph with no K_{k+1}-subdivision is k-colorable. While this is true for $k \leq 3$ and has been disproved by Catlin [2] for $k \geq 6$, it is still open for $k = 4$ and $k = 5$. The case $k = 4$, if true, implies the Four Color Theorem. In 1975, Seymour [10] conjectured that every 5-connected non-planar graph contains a K_5-subdivision. Let us assume for a moment that a counterexample to Hajós' conjecture with minimum number of vertices is 5-connected. Then, Seymour's conjecture implies Hajós' conjecture for $k = 4$. A counterexample to Hajós conjecture is not 4-colorable and therefore non-planar by the Four Color Theorem. Therefore, Seymour's conjecture, if true, implies that the counterexample contains a K_5-subdivision, a contradiction. The main result of this thesis should be seen as a first step in establishing a connection between Hajós' conjecture and Seymour's conjecture.

Let G be a graph with chromatic number at least five, which does not contain a K_5-subdivision and has minimum number of vertices with respect to these properties. We will call such a graph a Hajós graph from now on. Our main result is the following:

Theorem 1 *Every Hajós graph is 4-connected.*

As evidence for Hajós' conjecture for $k = 4$, we mention a related conjecture of Dirac [4]: every simple graph on n vertices with at least $3n - 5$ edges has a K_5-subdivision. In [7] it has been shown that a minor-minimal counterexample to Dirac's conjecture must be 5-connected and that Seymour's conjecture implies Dirac's conjecture. Using the result in [7], Mader proved in [9] that Dirac's conjecture is indeed true. Seymour's conjecture remains open.

In order to show Theorem 1, we need to show that no Hajós graph G admits k-cuts with $k \leq 3$. This is relatively easy to show when $k \leq 2$; the main work here is to show that no Hajós graph admits a 3-cut. We will now introduce the necessary notation to give a short outline of the proof.

A separation of a graph G is a pair (G_1, G_2) of edge disjoint subgraphs of G such that $G = G_1 \cup G_2$, and $V(G_i) - V(G_{3-i}) \neq \emptyset$, for $i = 1, 2$. Note that our definition of a separation is different from the usual one where instead of $V(G_i) - V(G_{3-i}) \neq \emptyset$ one requires $E(G_i) - E(G_{3-i}) \neq \emptyset$. We call (G_1, G_2) a k-separation if $|V(G_1) \cap V(G_2)| = k$. A set $S \subseteq V(G)$ is a k-cut in G, if $|S| = k$ and G has a separation (G_1, G_2) such that $V(G_1) \cap V(G_2) = S$.

Suppose G has a 3-cut. We choose a 3-separation (G_1, G_2) of G that minimizes $|V(G_2)|$. By G'_i, we denote the graph obtained from G_i by adding an edge for every pair of vertices in $V(G_1 \cap G_2)$. To decide whether G'_i has a K_5-subdivision, we need to know whether G_{3-i} contains a cycle through $V(G_1 \cap G_2)$. We characterize graphs which do not have such a cycle. It turned out that this has been done by Watkins and Messner [11] before. However, we obtained our proof independently and it is significantly shorter.

At this point, it becomes clear why proving that every Hajós graph G is 5-connected is much harder then proving it is 4-connected. This next step requires structural information about graphs which have a K_4-subdivision with specified branching vertices. Some work in this direction has been done by Yu [12].

We will now introduce some further notation. Let G be graph. For $U \subseteq V$, we denote by $G[U]$ the subgraph of G with vertex set U and edge set $\{xy \in E(G) | x, y \in U\}$. For $A, B \subseteq V(G)$, an A-B path in G is a path with one endvertex in A and the other one in B, which is internally disjoint from $A \cup B$. If $A = \{x\}$, then we speak of an x-B path, and similarly an x-y path if $B = \{y\}$. We say that a set $S \subseteq V(G)$ separates A and B if there is a separation (G_1, G_2) of G such that $V(G_1 \cap G_2) = S$, $A \subseteq V(G_1)$, $B \subseteq V(G_2)$ and $A - S \neq \emptyset \neq B - S$.

Let H be a subgraph of a graph G, let $v_1, \ldots, v_k \in V(G)$, and (u_i, w_i), $i = 1, \ldots, m$ denote pairs of distinct vertices in G. Then we let $H + \{v_1, \ldots, v_k, u_1w_1, \ldots, u_mw_m\}$ denote the graph with vertex set $V(H) \cup \{v_1, \ldots, v_k\}$ and edge set $E(H) \cup \{u_1w_2, \ldots, u_kw_k\}$.

Let $U \subseteq V(G)$ and $F \subseteq E(G)$. Then, $G - F$ denotes the spanning subgraph of G with edge set $E(G) - F$, and $G - U$ denotes the subgraph of G with vertex set $V(G) - U$ and edge set $\{xy \in E(G) | \{x, y\} \cap U = \emptyset\}$.

We will state one of the fundamental theorems in Graph Theory without proof, since we use it frequently throughout the thesis:

Theorem 2 *(Menger 1927) Let G be a graph and $A, B \subseteq V(G)$. Then the minimum number of vertices separating A from B is equal to the maximum number of disjoint A-B paths in G.*

The rest of this thesis is organized as follows. In Section 2, we present a proof for Hajós' conjecture in the case $k = 3$ and counterexamples for $k \geq 6$. In Section 3 we give a proof of a result of Lovász. This result will be used in Section 4 to characterize 2-connected graphs with no cycle through 3 fixed vertices. In Sections 5 and 6, we prove our main result.

CHAPTER II

KNOWN RESULTS ON HAJÓS' CONJECTURE

In this section we will review some known results on Hajós' conjecture that every graph with no K_{k+1}-subdivision is k-colorable.

First of all, the conjecture is trivially true for $k = 1$. For $k = 2$ it is also easy to see that the conjecture is true. Graphs containing no K_3-subdivision are forests and hence are 2-colorable.

We will now show that the conjecture is true for $k = 3$, a result first shown by Dirac [3]. We will use the following notation. Let P be a path and $x, y \in V(P)$. We write $P[x, y]$ to denote the subpath $P[\{v_0, ..., v_k\}]$ where $x = v_0$ and $y = v_k$.

Theorem 3 *A graph G with $\chi(G) \geq 4$ contains a K_4-subdivision.*

Proof: Suppose the claim is not true. Let G be a graph such that $\chi(G) \geq 4$, G contains no K_4-subdivision and subject to these $|V(G)|$ is minimum.
Obviously, G is connected. We may also assume that $\chi(G) = 4$, for otherwise we may remove edges until we obtain a 4-chromatic subgraph G' of G. If G' contains a K_4-subdivision, so does G.

Also note that G is 2-connected. Otherwise, suppose there is a 1-separation (G_1, G_2) and $\{v_0\} = V(G_1 \cap G_2)$. Then, G_1 and G_2 are 3-colorable as $|V(G_i)| < |V(G)|$ for $i = 1, 2$. Let c_i be a 3-coloring of G_i, where c_1 and c_2 use the same set of colors. We may assume $c_2(v_0) = c_1(v_0)$ by permuting the colors of vertices of G_2. Now we define a 3-coloring c of G where $c(v) = c_i(v)$ for $v \in V(G_i)$, $i = 1, 2$. This contradicts that $\chi(G) = 4$.

G is not 3-connected. We will show a little more generally, that any 3-connected graph contains a K_4-subdivision. Any 3-connected graph G contains a cycle C and a vertex $v \in V(G) - V(C)$ as $G - \{v\}$ is 2-connected. As G is 3-connected, there exist three $C - v$ paths P_1, P_2, P_3 such that $P_i \cap P_j = \{v\}$ for $i, j \in \{1, 2, 3\}$, $i \neq j$. Then, $C \cup P_1 \cup P_2 \cup P_3$ is a K_4-subdivision.

We conclude that there exists a 2-cut $S = \{x, y\}$ in G. Let (G_1, G_2) be a 2-separation of G such that $V(G_1) \cap V(G_2) = S$. Then, G_1 and G_2 are both 3-colorable by the choice of G. Suppose G_1 and G_2 have 3-colorings c_1 respectively c_2 such that $c_i(x) \neq c_i(y)$. We may assume that both colorings use the same set of colors. By permuting the colors of vertices in G_1 we may assume that $c_1(x) = c_2(y)$ and $c_1(y) = c_2(y)$. Then, we define a 3-coloring c of G by setting $c(v) = c_i(v)$ for all $v \in V(G_i)$, a contradiction. Similarly, it is not possible that G_1 and G_2 are both 3-colorable such that x and y receive the same color. Therefore, we may assume that x and y receive different colors in every 3-coloring of G_1 and every 3-coloring of G_2 assigns the same color to x and y. Then, $G_2 + xy$ is 4-chromatic and has fewer vertices then G.

By the choice of G, $G_2 + xy$ has a K_4-subdivision Σ. If $xy \notin E(\Sigma)$, Σ is a K_4-subdivision in G. If $xy \in E(\Sigma)$, let P_{xy} be an $x - y$ path in G_1, which exists as G is 2-connected. Then, $(\Sigma \cup P_{xy}) - xy$ is a K_4-subdivision of G, a contradiction. □

In 1979 Catlin [2] found a counterexample disproving Hajós' conjecture for $k \geq 6$. He uses the notion of a line graph. Let G be a graph. The line graph $L(G)$ has vertex set $E(G)$ and two vertices $e, f \in V(L(G)) = E(G)$ are adjacent if, and only if, they are incident in G. We will also use the following fact.

Lemma 1 *If S is a cut in a K_r-subdivision, $r \geq 2$, separating two branching vertices then* $|S| \geq r - 1$.

Proof. This is an immediate consequence of Menger's Theorem, as there are $r - 1$ internally disjoint paths between two branching vertices a, b in a K_r-subdivision: the a-b branching path, and the union of the c-a and c-b branching paths for each branching vertex $c \notin \{a, b\}$.

□

Example: Let C_5 be a cycle of length 5 and C_5^3 be obtained from C_5 by replacing each $xy \in E(C_5)$ by three edges joining x and y. Let e and f be non-adjacent edges of C_5^3 and G the line graph of $C_5^3 - \{e, f\}$.

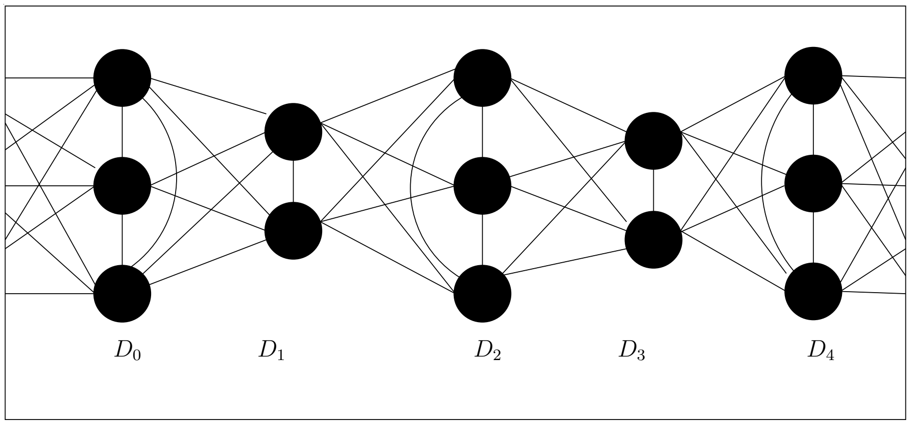

Figure 1: Catlin's Counterexample

We claim that G does not contain a K_7-subdivision and $\chi(G) = 7$.

We first show that G has no K_7-subdivision. We can represent G in the following way: let D_0, D_2, D_4 be disjoint copies of K_3 and D_1, D_3 disjoint copies of K_2 in G that are also disjoint from the D_0, D_2, D_4. Then

$$G = \bigcup_{i=0}^{4} D_i + \{xy | x \in V(D_i), y \in V(D_{(i+1) \bmod 5})\}$$

For any $W \subset V(G)$ with $|W| = 7$, we want to show that W cannot be the set of branching vertices of a K_7-subdivision in G.

If $W = V(D_1) \cup V(D_2) \cup V(D_3)$, then $V(D_0)$ is a 3-cut in $G - V(D_2)$ separating $V(D_1)$ and $V(D_3)$. Hence, there are no four disjoint $V(D_1) - V(D_3)$ paths in $G - V(D_2)$ and W cannot be the set of branching vertices of a K_7-subdivision of G.

So assume $W \neq V(D_1) \cup V(D_2) \cup V(D_3)$, let and $w \in W \cap V(D_0 \cup D_4)$. There exists $v \in W \cap V(D_1 \cup D_2 \cup D_3)$ as $|V(D_0 \cup D_4)| = 6$. If $v \in D_2$, then $D_1 \cup D_3$ is a 4-cut in G separating v and w, and W cannot be the set of branching vertices of a K_7-subdivision of G. Therefore, without loss of generality, let $v \in V(D_3)$. If w can be chosen in $V(D_0)$ then $V(D_1) \cup V(D_4)$ is a 5-cut in G separating v and w, and W can not be the set of branching vertices of a K_7-subdivision in G. Otherwise, $W = V(D_1) \cup V(D_3) \cup V(D_4)$ and $V(D_0) \cup V(D_3)$ is a 5-cut in G separating $V(D_1)$ and $V(D_4)$. Again, W cannot be the set of branching vertices of a K_7-subdivision of G.

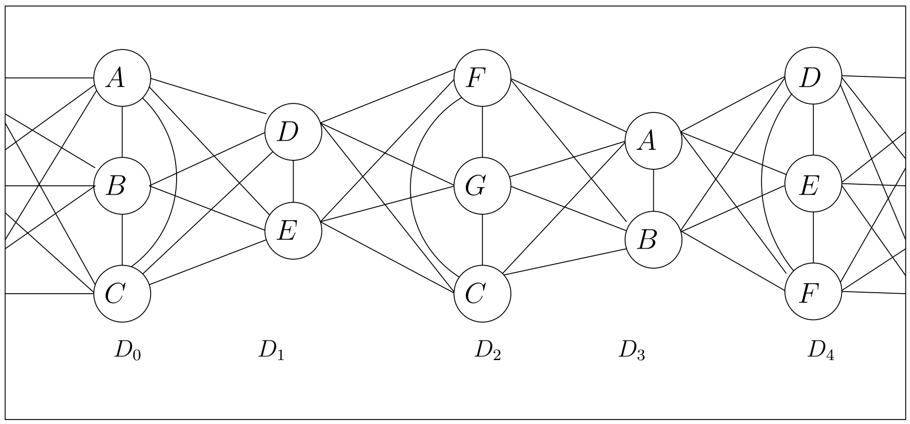

Figure 2: A 7-Coloring of Catlin's Example

As the maximum size of an independent set in G is 2, and $|V(G)| = 13$, the chromatic number of G is at least 7 and Figure 2 shows a 7-coloring of G. □

Using the following construction this example can be extended to a counterexample for any $k \geq 6$: Let $G + v$ be the graph with vertices $V(G) \cup \{v\}$ and edge set $E(G) \cup \{vw | w \in V(G)\}$. Then $\chi(G + v) = \chi(G) + 1$ and if G has no subdivided K_r, then $G + v$ cannot have a subdivided K_{r+1}.

We mention a result by Erdös and Fajtlowicz [5] without proof.

Theorem 4 *Let the topological clique number of a graph G be defined by*

$$tcl(G) = \max\{r | G \textit{ has a } K_r\textit{-subdivision}\}.$$

Then $tcl(G) < \chi(G)$ for almost every graph G.

CHAPTER III

CYCLES THROUGH THREE INDEPENDENT EDGES

In this section we present a result by Lovász and include his proof. We will then use Lemma 2 to prove Theorem 5 in Chapter 4:

Lemma 2 *[Lovász [8, Ex.6.67]] Every set L of three independent edges in a 3-connected graph G lies on a cycle in G if and only if $G - L$ is connected.*

Proof: Every cycle intersects an edge cut in an even number of edges. As G is 3-connected this implies that, if L lies on a cycle, $G - L$ is connected.

Suppose e_1, e_2, e_3 are three independent edges in G which do not lie on a cycle.

(1) There exist a cycle C in G containing two edges out of $\{e_1, e_2, e_3\}$, such that $V(C)$ is disjoint from the endpoints of the third edge e.

Let C_1 be a cycle through e_1 and e_2, which exists as G is 3-connected and let R_1, R_2 be the components of $C_1 - \{e_1, e_2\}$. Let $e_3 = xy$. If $\{x, y\} \subseteq V(C_1)$, then we may assume that $x \in V(R_1)$ and $y \in V(R_2)$. Hence one of the x-y path on C_1 and e_3 form the desired cycle C. So assume by symmetry that $x \in V(R_1)$, and $y \notin V(C_1)$. Then, let P be an y-$V(C_1 - x)$ path, which must end on R_2 as otherwise there is a cycle containing all the edges in L. We may assume that $V(P) \cap V(C_1 - x) = \{z\}$. As e_1, e_2 are independent, we may assume that there is an x-z path Q on C_1 which is disjoint from the endpoints of e_1. Now, $Q \cup P \cup \{e_3\}$ forms the desired cycle C.

We may assume that e_1 and e_2 lie on the cycle C which exits by (1) (and $e = e_3$).

(2) Let $e_3 = xy$. We may choose the notation such that all $\{x, y\}$-$V(C)$ paths are either x-$V(P_1)$ or y-$V(P_2)$ paths.

No two $\{x, y\}$-$V(P_1)$ paths in $G - V(P_2)$ are disjoint, therefore there exist $u \in V(G)$ meeting all of them and similarly there exists $v \in V(G)$ meeting all $\{x, y\}$-P_2 paths in $G - V(P_1)$. Then, $\{u, v\}$ meets all $\{x, y\}$-$V(C)$ paths and as G is 3-connected $\{u, v\} = \{x, y\}$.

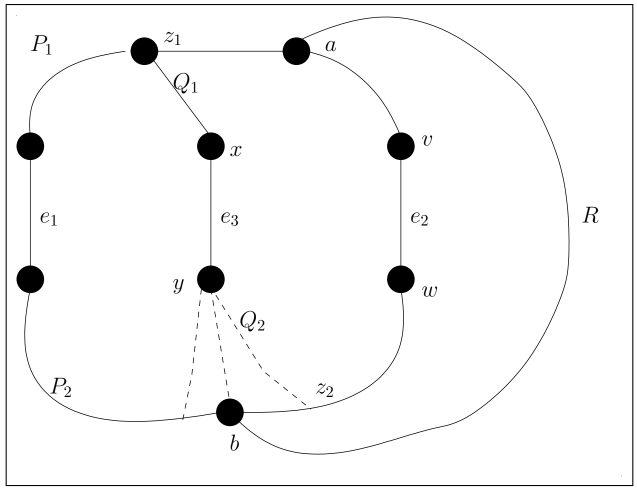

Figure 3: Graphs with no cycle through e_1, e_2, e_3

(3) There is no x-y path in $G - \{e_1, e_2, e_3\}$.

Suppose the assertion is false and let T be an x-y path in $G - \{e_1, e_2, e_3\}$. Suppose $V(P_1) \cap V(T) = \emptyset$. As G is 3-connected, there exists a $V(T)$-$V(P_1)$ path in $G - \{x, y\}$, implying that there is y-$V(P_1)$ path in $G - \{x\}$, contradicting (2) which implies $V(P_1) \cap V(T) \neq \emptyset$. By symmetry, $V(P_2) \cap V(T) \neq \emptyset$. Let R denote the $V(P_1)$-$V(P_2)$ subpath of T, and a its endpoint on P_1, b its endpoint on P_2.

As G is 3-connected, there exists an $\{x, y\}$-$V(C - \{a, b\})$ path Q_1 in $G - \{a, b\}$, say the endpoints of Q_1 are x and $z_1 \in V(P_1)$. We denote the endpoint of e_2 on P_1 by v and its endpoint on P_2 by w. We may assume that a lies on the z_1-v subpath of P_1.

Let Q_2 be an y-P_2 path in $G - \{x, w\}$ and denote its other endpoint by z_2. $Q_1 \cap Q_2 = \emptyset$, for otherwise there is an x-y path in $G - \{e_1, e_2, e_3\}$ which is disjoint from C, a contradiction. Denote the z_1-z_2 path on C not containing e_2 by S and note that $e_1 \in E(S)$. Then, $S \cup Q_1 \cup Q_2 \cup \{e_3\}$ is a cycle C_0 disjoint from the endpoints of e_2. But there exist an v-$V(P_1)$ path in $(G - \{e_1, e_2, e_3\}) - \{w\}$ as well as an v-$V(P_2)$ path, contradicting (2).

□

CHAPTER IV

CYCLES THROUGH THREE FIXED VERTICES

In this section we characterize all 2-connected graphs in which there are three vertices not contained in any cycle. This has been done by Watkins and Mesner [11, Theorem 2] before. Our proof was developed without knowing about this result. We give an altenative proof which makes use of Lemma 2. Our proof is significantly shorter than [11], even including the proof of Lovász result.

The main result of this section is the following theorem:

Theorem 5 *Let G be a 2-connected graph and x, y, z be three distinct vertices of G. Then, there is no cycle through x, y and z in G if and only if one of the following statements holds.*

(i) There exists a 2-cut S in G and there exist three distinct components D_x, D_y, D_z of $G - S$ such that $u \in V(D_u)$ for each $u \in \{x, y, z\}$.

(ii) There exist a vertex v of G, 2-cuts S_x, S_y, S_z in G, and components D_u of $G - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

(iii) There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G and components D_u of $G - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u, for all $u \in \{x, y, z\}$.

Proof: It is straightforward that if one of (i),(ii) or (iii) holds, then G has no cycle through x, y and z. Now assume G contains no cycle through x, y, z.

Suppose that $\{x, y, z\}$ is not an independent set in G. Without loss of generality let $xy \in E(G)$. Then, as G is 2-connected, there exists a z-x path P_x and a z-y path P_y such

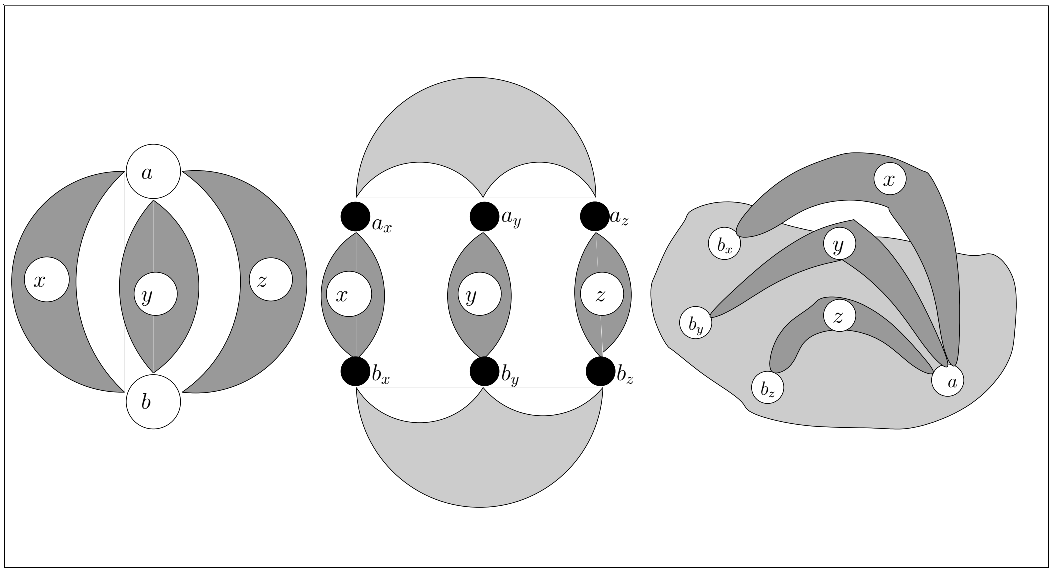

Figure 4: Graphs with no cycle through x, y, z

that $V(P_x \cap P_y) = \{z\}$. Hence, $P_x \cup P_y + xy$ is a cycle through x, y, z, a contradiction. Therefore,

(1) $\{x, y, z\}$ is an independent set in G.

Next we show that,

(2) for any $u \in \{x, y, z\}$, u is not contained in any 2-cut in G separating the two vertices in $\{x, y, z\} - \{u\}$.

For otherwise, we may assume that there is a 2-separation (G_1, G_2) of G such that $x \in V(G_1 \cap G_2)$, $y \in V(G_1) - V(G_1 \cap G_2)$, and $z \in V(G_2) - V(G_1 \cap G_2)$. Since G is 2-connected, G_1 (respectively, G_2) contains two internally disjoint paths from y (respectively, z) to $V(G_1 \cap G_2)$. These four paths form a cycle containing $\{x, y, z\}$, a contradiction.

(3) For any $u \in \{x, y, z\}$ there is a 2-cut $S_u = \{a_u, b_u\}$ in G separating u from $\{x, y, z\} - \{u\}$.

Suppose the assertion is false. Without loss of generality, assume G has no 2-cut separating x from $\{y, z\}$. By Menger's Theorem, there are two internally disjoint y-z paths P_1 and P_2. Let $C = P_1 \cup P_2$ be a cycle through y, z, which by assumption does not contain x. By

Menger's Theorem, there must exist three paths R_1, R_2, R_3 from x to C sharing only x. We may assume two of these paths, say R_1 and R_2, end on P_1. Thus $C \cup R_1 \cup R_2$ contains a cycle through x, y and z, a contradiction.

For each $u \in \{x, y, z\}$ let D_u denote the component of $G - S_u$ containing u.

(4) We may choose S_x, S_y, S_z so that D_x, D_y, D_z are pairwise disjoint.

Since G is 2-connected, there must exist a cycle C through x and z, and by (2), $S_x \cup S_z \subseteq V(C)$ and $y \notin V(C)$ as x, y, z are not contained in any cycle. If there exist three y-$V(C)$ paths sharing only y then two of its paths must end on the same x-y paths in C, yielding a cycle through x, y, z in G, a contradiction. Hence we may choose S_y to separate y from $V(C)$. By (2), $x, z \notin S_y$. Thus, D_x and D_y are disjoint, and D_z and D_y are disjoint.

Again since G is 2-connected, there must be a cycle D through x and y in G, and $S_x \cup S_y \subseteq V(D)$ by (2). By a similar argument as above, we may choose S_z separating z from $V(D)$. By (2), $x, y \notin S_z$, and hence, D_z is disjoint from both D_x and D_y. So we have (4).

Case 1. For some choice of S_x, S_y, S_z satisfying (2) and (3), S_x, S_y, S_z are not pairwise disjoint. Without loss of generality, let $S_x \cap S_y \neq \emptyset$. If $S_x = S_y$ we can also choose $S_z = S_x$ so that x, y, z belong (pairwise) to different components of $G - S_x$. Then, (i) holds.

So let $\{v\} = S_x \cap S_y$. Then, there do not exist two paths from z to $(S_x \cup S_y) - v$ sharing only z in $G - v$; for otherwise, G would contain a cycle through x, y, z. Hence, there is vertex w in $V(G - v) - V(D_x \cup D_y)$ separating $(S_x \cup S_y) \setminus \{v\}$ from S_z in $G - v$. Then, by choosing $S_z = \{v, w\}$ we see that (ii) holds.

Case 2. For any choice of S_x, S_y, S_z satisfying (2) and (3), S_x, S_y, S_z are pairwise disjoint. Choose $S_u = \{a_u, b_u\}$ for all $u \in \{x, y, z\}$ such that, subject to (3) and (4), D_x, D_y, D_z are maximal. Let $G' := (G - V(D_x \cup D_y \cup D_z)) + \{a_x b_x, a_y b_y, a_z b_z\}$. By the maximality of D_x, D_y, D_z, if G' is not 3-connected then, for every 2-separation (G_1, G_2) of G', either $\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_1)$ or $\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_2)$. Suppose $\{a_x b_x, a_y b_y, a_z b_z\} \subseteq E(G_1)$. Replace G_2 by an edge between the vertices in $V(G_1 \cap G_2)$. Repeating this operation until no such 2-separation exists, we obtain a 3-connected graph G'' in which a_xb_x, a_yb_y and a_zb_z are independent edges.

Suppose $G'' - \{a_xb_x, a_yb_y, a_zb_z\}$ is connected. Then by Lemma 2, there exists a cycle C'' in G'' through a_xb_x, a_yb_y and a_zb_z. From C'' we may produce a cycle C through x, y, z in G by replacing the edges in $E(C'') - E(G)$ with paths in G, a contradiction. So assume that $G'' - \{a_xb_x, a_yb_y, c_zb_z\}$ is not connected, and hence, it has exactly two components. Then we see that $G - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from S_u for all $u \in \{x, y, z\}$. Hence (iii) holds. $\square$

CHAPTER V

3-SEPARATIONS

The goal of this section is to show that every Hajós graph is 3-connected, and if a Hajós graph admits a 3-separation (G_1, G_2) chosen to minimize G_2, then G_1 and G_2 admit special 4-colorings.

Lemma 3 *Every Hajós graph is 3-connected.*

Proof: Let G be a Hajós graph. Obviously, G must be connected. Suppose G is not 2-connected. Then, there exists a 1-separation (G_1, G_2) of G and G_1, G_2 are proper subgraphs of G, $\{v\} = V(G_1 \cap G_2)$. Since G has no K_5-subdivision neither G_1 nor G_2 contain a K_5-subdivision. Hence, G_1 and G_2 are 4-colorable. Let c_i denote 4-colorings of G_i for $i = 1, 2$ using the same set of four colors. We may assume $c_1(v) = c_2(v)$ by permuting the colors of vertices in G_1. We obtain a proper 4-coloring c of G by defining $c(u) = c_i(u)$ for $u \in V(G_i)$, a contradiction. Therefore, G is 2-connected.

Now, suppose G is not 3-connected. Then, there exists a 2-separation (G_1, G_2) of G. Let $V(G_1 \cap G_2) = \{x, y\}$. Consider $G_1' = G_1 + xy$ and $G_2' = G_2 + xy$. We claim that G_i', $i = 1, 2$, has no K_5-subdivision. For otherwise, let Σ be a K_5 subdivision in G_i'. Then, $xy \in E(\Sigma)$, or else Σ is also a K_5-subdivision in G, a contradiction. As G is 2-connected, there exists an x-y path P in G_{3-i}', and we may replace xy to obtain a K_5-subdivision in G, a contradiction.

Hence, since $|V(G_i')| < |V(G)|$, both G_1' and G_2' are 4-colorable. Let c_i be a 4-coloring of G_i' for $i = 1, 2$, using the same set of colors. Since $c_i(x) \neq c_i(y)$ for $i = 1, 2$ we can permute the colors of the vertices of G_2' such that $c_1(x) = c_2(x)$ and $c_1(y) = c_2(y)$. Again, this yields a 4-coloring of G by defining $c(u) = c_i(u)$ for $u \in G_i'$, a contradiction. $\square$

Suppose now that G is not 4-connected. Then G has a 3-separation (G_1, G_2) and let $V(G_1) \cap V(G_2) = \{x, y, z\}$. For the remainder of this section, we choose (G_1, G_2) so that

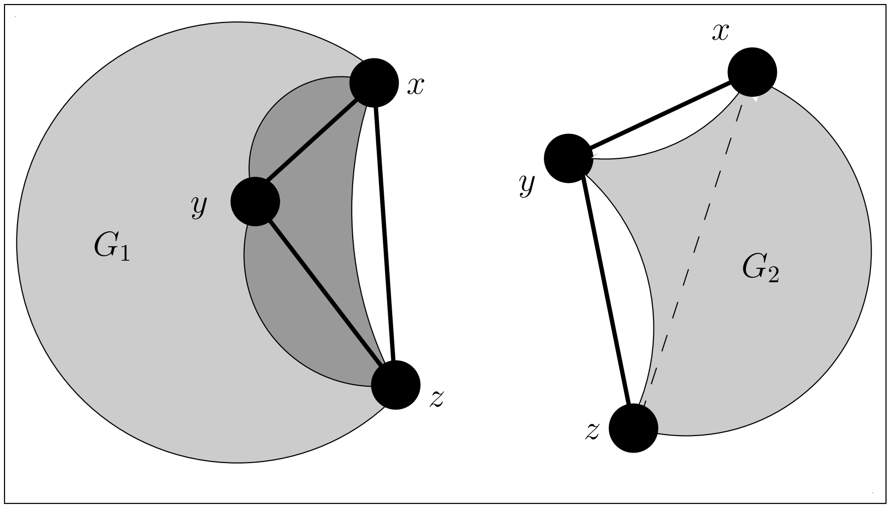

Figure 5: A 3-separation of G

G_2 is minimal. We shall show that G_1 and G_2 admit certain 4-colorings. First, we need some structural information about G_2.

Lemma 4 *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then*

(i) $|V(G_2)| \geq 5$,

(ii) $G_2 - V(G_1 \cap G_2)$ *is connected, and*

(iii) G_2 *is 2-connected.*

Proof: (i) If $|V(G_2)| \leq 4$, then $|V(G_2)| = 4$. Let $v \in V(G_2) \setminus \{x, y, z\}$. Then v has degree at most 3 in G, and as $G - v$ does not contain a K_5-subdivision it is 4-colorable by the choice of G. Since the degree of v in G is at most 3, G is 4-colorable, a contradiction. Hence, $|V(G_2)| \geq 5$.

(ii) Suppose $G_2 - V(G_1 \cap G_2)$ is not connected. Let D denote a component of $G_2 - V(G_1 \cap G_2)$. Then there is a 3-separation (G_1', G_2') with $V(G_1') \cap V(G_2') = V(G_1 \cap G_2)$ and $G_2' - V(G_1 \cap G_2) = D$. This contradicts the choice of (G_1, G_2) since G_2' is properly contained in G_2.

(iii) By (ii), $G_2 - V(G_1 \cap G_2)$ is connected and as G is 3-connected, every vertex in $V(G_1 \cap G_2)$ has a neighbor in $G_2 - V(G_1 \cap G_2)$, so G_2 is connected. Suppose there is a cut vertex

$v \in V(G_2) - V(G_1 \cap G_2)$ in G_2. Then, $V(G_1 \cap G_2)$ cannot be contained in one component of $G_2 - v$, for otherwise v would be a cut vertex in G. We may assume some vertex $x \in V(G_1 \cap G_2)$ is separated from $V(G_1 \cap G_2) - \{x\}$ by $\{v\}$ in G_2. Then, since G is 3-connected, $xv \in E(G)$, and since $|V(G_2)| \geq 5$, $(V(G_1 \cap G_2) - \{x\}) \cup \{v\}$ is a cut in G yielding a separation (G'_1, G'_2) such that G'_2 is a proper subgraph of G_2, a contradiction. □

Proposition 1 *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Then there is a 4-coloring c_1 of G_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct.*

Proof: Suppose this is not true, that is $G'_1 = G_1 + \{xy, xz, yz\}$ is not 4-colorable. By the choice of G, G'_1 contains a K_5-subdivision, say Σ.

First we claim that x, y, z are branching vertices of Σ. If $\{xy, xz, yz\} \subseteq E(\Sigma)$ then we see that x, y, z are branching vertices of Σ. So we may assume by symmetry, that $yz \notin E(\Sigma)$. As G_2 is 2-connected by (iii) of Lemma 4, there exist internally disjoint paths Y from x to y and Z x to z in G_2. Then, $(\Sigma - \{xy, xz\}) \cup Y \cup Z$ is a K_5-subdivision in G, a contradiction.

Therefore, if G_2 contains a cycle C through x, y, z, $(\Sigma - \{xy, xz, yz\}) \cup C$ (and hence G) contains K_5-subdivision, a contradiction. Hence there cannot be a cycle through x, y, z in G_2. By applying Theorem 5 to G_2, it suffices to consider the following three cases.

Case 1 There exist a 2-cut S in G_2 and 3 distinct components D_x, D_y, D_z in $G_2 - S$ such that $u \in V(D_u)$, for $u \in \{x, y, z\}$. Let $S = \{a, b\}$. If, $|V(D_x)| \geq 2$ then $G - \{x, a, b\}$, has a component properly contained in $G_2 - \{x, y, z\}$ contradicting the choice of (G_1, G_2). Thus, $V(D_x) = \{x\}$, and similarly $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$. Hence a, b are the only vertices of G not in G_1.

By the choice of G, G_1 is 4-colorable. Let c_1 be a 4-coloring of G_1. If $c_1(x), c_1(y), c_1(z)$ do all receive distinct colors, then c_1 is also a 4-coloring of G'_1. Otherwise, a coloring c'_1 of G such that $c'_1(a)$ and $c'_1(b)$ are two colors not in $\{c_1(x), c_1(y), c_1(z)\}$, and $c'_1(u) = c_1(u)$ for all $u \in G_1$. Then, c'_1 is a 4-coloring, of G, a contradiction.

Case 2 There exist a vertex $\{v\}$ of G_2, 2-cuts S_x, S_y, S_z in G_2 and components D_u of $G_2 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$

are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

As in Case 1, we conclude $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$. Since G has no K_5-subdivision we see that $G_1 + xy$ has no subdivision of K_5. For otherwise, as G_2 is 2-connected, there exists an x-y paths in $G_2 - z$ and we may produce a K_5-subdivision in G.

Hence, by the choice of G, we know that $G_1 + xy$ admits a 4-coloring c_1. Then $c_1(x) \neq c_1(y)$ and if $c_1(z) \neq c_1(x)$ as well as $c_1(z) \neq c_1(y)$, then c_1 is a 4-coloring of G_1'. We may assume that $c_1(z) = c_1(y)$ by the symmetry between x and y.

Next we extend this coloring of G_1 to a 4-coloring of G. By the choice of G there exists a 4-coloring c_2 of G_2 using the same set of colors as c_1. As y and z only have three neighbors in G_2, we may choose c_2 such that $c_2(y) = c_2(z)$. Since x has only two neighbors in G_2, we may assume that $c_2(x) \neq c_2(y)$. Now, by permuting the colors of vertices of G_2 we may assume that $c_1(u) = c_2(u)$ for $u \in \{x, y, z\}$. Then, c defined by $c(u) = c_i(u)$ for $u \in V(G_i)$ is a 4-coloring of G, a contradiction.

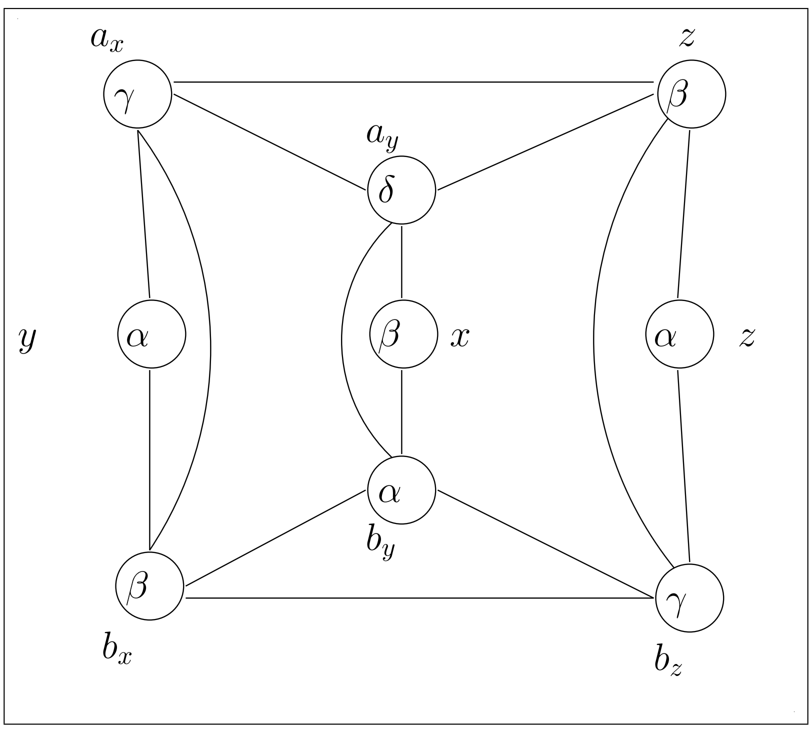

Figure 6: A coloring of G_2 in Case 3.

Case 3 There exist disjoint 2-cuts S_x, S_y, S_z in G_2 and components D_u of $G_2 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint. Moreover, $G_2 - V(D_x \cup D_y \cup D_z)$ has exactly two connected components, each containing exactly one vertex of S_u, for all $u \in \{x, y, z\}$.

As in Case 1, we conclude $V(D_x) = \{x\}$, $V(D_y) = \{y\}$ and $V(D_z) = \{z\}$.

Let $S_x := \{a_x, b_x\}$, $S_y := \{a_y, b_y\}$, and $S_z := \{a_z, b_z\}$, and assume that $\{a_x, a_y, a_z\}$ (respectively, $\{b_x, b_y, b_z\}$) is contained in the component A (respectively, B) of $G - V(D_x \cup D_y \cup D_z)$. Then $|V(A)| = 3 = |V(B)|$; for otherwise, $G - \{a_x, a_y, a_z\}$ or $G - \{b_x, b_y, b_z\}$ has a component which is properly contained in $G_2 - \{x, y, z\}$, contradicting the choice of (G_1, G_2).

$G_1 + \{xy, yz\}$ does not contain a K_5-subdivision. Suppose $G_1 + \{xy, yz\}$ contains a K_5-subdivision Σ. By (iii) of Lemma 4, G_2 has two internally disjoint paths X from y to x and Z from y to z. Now, $(\Sigma - \{xy, yz\}) \cup X \cup Z \subseteq G$ contains a K_5-subdivision, a contradiction.

Since $|V(G_1 + \{xy, yz\})| < |V(G)|$, $G_1 + \{xy, yz\}$ is 4-colorable. Let c_1 be a 4-coloring of $G_1 + \{xy, yz\}$. Then $c_1(x) \neq c_1(y) \neq c_1(z)$. If $c_1(x) \neq c_1(z)$, then G_1' is 4-colorable, a contradiction. So assume that $c_1(x) = c_1(z)$. For convenience, assume that the colors we use are $\{\alpha, \beta, \gamma, \delta\}$ and $c_1(x) = \alpha$ and $c_1(y) = \beta$. Let c be a coloring of G such that $c(u) = c_1(u)$ for all $u \in V(G_1)$, $c(a_x) = c(b_z) = \gamma$, $c(b_x) = c(a_z) = \beta$, $c(a_y) = \delta$ and $c(b_y) = \alpha$. It is easy to check that c is a 4-coloring of G, a contradiction. $\square$

G_1 does not only admit a 4-coloring such that x, y, z receive all different colors. Under additional assumptions, we can also prove that the following special colorings of G_1 exist.

Lemma 5 *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Suppose there is a vertex $x' \in V(G_1) - \{x, y, z\}$ separating x from $\{y, z\}$ in G_1. Then there exist 4-colorings c_1 and c_2 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$ and $c_2(x) = c_2(z) \neq c_2(y)$.*

Proof: Note that $xy, xz \notin E(G)$ for otherwise x' does not separate x from $\{y, z\}$ in G_1. As G is 3-connected, $xx' \in E(G)$, for otherwise $\{x, x'\}$ would be a 2-cut in G. Let

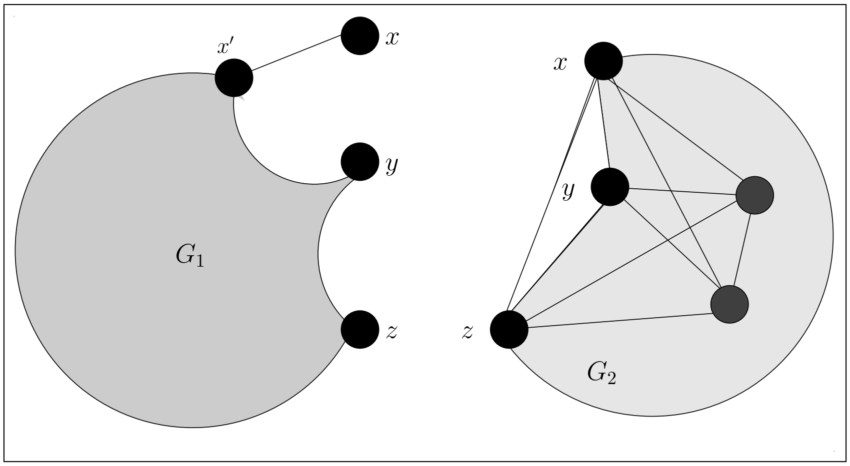

Figure 7: x' separating x from y and z.

$G_1^* := (G_1 - x) + \{x'y, yz\}$. We claim that G_1^* has no K_5-subdivision. Suppose G_1^* has a K_5-subdivision Σ. Since G does not have a K_5-subdivision in G_1', clearly $\{x'y, yz\} \cap E(\Sigma) \neq \emptyset$. Since G_2 is 2-connected, it contains two internally disjoint paths X, Z from y to x, z, respectively. Now $(\Sigma - \{x'y, yz\}) \cup (X + \{x', xx'\}) \cup Z$, and hence G, contains a K_5-subdivision, a contradiction.

Therefore, since $|V(G_1^*)| < |V(G)|$, G_1^* is 4-colorable. Let c_1^* be a 4-coloring of G_1^*. Then $c_1^*(x') \neq c_1^*(y) \neq c_1^*(z)$. Define a coloring c_1 of G_1 by letting $c_1(x) = c_1^*(y)$ and $c_1(u) = c_1^*(u)$ for all $u \in V(G_1) - \{x\}$. It is easy to see that c_1 gives the desired 4-coloring of G_1.

Similarly, by defining $G_1^* := (G_1 - x) + \{x'z, yz\}$, we can show that G_1 has the desired 4-coloring c_2. □

Next, we want to prove that G_2 admits certain 4-colorings. The following Lemma will be needed to do prove this and allow us to apply Theorem 5 to G_1.

Lemma 6 *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then G_1 is 2-connected.*

Proof: Suppose G_1 is not 2-connected. Since G is 3-connected (by Lemma 3), there must exist vertices $x \in V(G_1 \cap G_2)$ and $x' \in V(G_1) - V(G_1 \cap G_2)$ such that x' separates x from $V(G_1 \cap G_2) - \{x\}$. Let y, z denote the other two vertices in $V(G_1 \cap G_2) - \{x\}$. By

Lemma 5, there exists a 4-coloring c_1 of G_1 such that $c_1(x) = c_1(y) \neq c_1(z)$, and there exists a 4-coloring c_1' of G_1 such that $c_1'(x) = c_1'(z) \neq c_1'(y)$.

Note that $G_2 + yz$ contains no K_5-subdivision. For otherwise, let Σ be a K_5-subdivision in $G_2 + yz$. By Lemma 3, G is 3-connected. Hence, if $G_1 - x$ has no y-z path then $xy, xz \in E(G)$ and $V(G_1) = \{x, y, z\}$ contradicting the assumption that (G_1, G_2) is a separation. So we may assume that $G_1 - x$ has a y-z path P. Now $(\Sigma - yz) \cup P \subseteq G$ contains a K_5-subdivision, a contradiction. Since $|V(G_2 + yz)| < |V(G)|$, $G_2 + yz$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + yz$. Then $c_2(y) \neq c_2(z)$.

First, assume that $c_2(y) \neq c_2(x) \neq c_2(z)$. Then c_2 is a 4-coloring of $G_2 + \{xy, xz, yz\}$. By Proposition 1, G_1 has a 4-coloring c_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. We may assume c_1 and c_2 use the same set of four colors, and by permuting colors of vertices in G_1, we have $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Now define a coloring c of G by letting $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. This shows that G is 4-colorable, a contradiction.

Now by the symmetry between y and z, we may assume that $c_2(x) = c_2(y) \neq c_2(z)$. We may assume that c_1 and c_2 use the same set of four colors, and by permuting colors if necessary, $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Define $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then it is easy to see that c is a 4-coloring of G, a contradiction. □

Proposition 2 *For $F \subseteq \{xy, xz, yz\}$ $G_2 + F$ is 4-colorable if and only if $|F| \leq 2$.*

Proof: Suppose that $|F| = 3$ and $G_2 + F = G_2 + \{xy, xz, yz\}$ is 4-colorable. Then there is a 4-coloring c_2 of G_2 such that $c_2(x), c_2(y)$ and $c_2(z)$ are all distinct. By Proposition 1, let c_1 be a 4-coloring of G_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. Assume that c_1 and c_2 use the same set of four colors. By permuting colors if necessary, we may assume that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then we see that c is a 4-coloring of G, a contradiction. Hence $G_2 + F$ is not 4-colorable when $|F| = 3$.

Now assume $|F| = 1$. By symmetry, consider $F = \{xy\}$. If $G_2 + xy$ has no K_5-subdivision, then by the choice of G, we see that $G_2 + xy$ is 4-colorable. So assume that

$G_2 + xy$ has a K_5-subdivision, say Σ. By Lemma 6, we see that $G_1 - z$ has an x-y path P. Now $(\Sigma - xy) \cup P \subseteq G$ contains a K_5-subdivision, a contradiction.

Finally, assume $|F| = 2$. By symmetry, we consider $F = \{xy, xz\}$. If $G_2 + \{xy, xz\}$ contains no K_5-subdivision then, by the choice of G, we see that $G_2+\{xy, xz\}$ is 4-colorable. So we may assume that $G_2 + \{xy, xz\}$ does contain a K_5-subdivision, denoted by Σ. By Lemma 6, G_1 contains internally disjoint paths Y, Z from x to y, z, respectively. Hence $(\Sigma - \{xy, yz\}) \cup Y \cup Z \subseteq G$ contains a K_5-subdivision, a contradiction. $\square$

We conclude this section with a useful observation.

Lemma 7 *Let G be a Hajós graph, and let (G_1, G_2) be a 3-separation of G chosen to minimize G_2. Then there is no cycle in G_1 containing $V(G_1 \cap G_2)$, and $V(G_1 \cap G_2)$ is an independent set in G_1.*

Proof: Let $V(G_1 \cap G_2) = \{x, y, z\}$. By Proposition 2, $G_2 + \{xy, xz, yz\}$ is not 4-colorable. Hence by the choice of G, $G_2 + \{xy, xz, yz\}$ has a K_5-subdivision Σ. If there is a cycle C in G_1 through x, y, z, then $(\Sigma - \{xy, yz, zx\}) \cup C$ (and hence G) contains a K_5-subdivision, a contradiction. So G_1 contains no cycle through x, y, z. Therefore, by Lemma 6 {x,y,z} must be independent in G_1. $\square$

CHAPTER VI

4-CONNECTIVITY

In this section we prove Theorem 1. In order to do so, we need the following lemma.

Lemma 8 *Let G be a Hajós graph, let (G_1, G_2) be a 3-separation of G chosen to minimize G_2, and let $V(G_1 \cap G_2) = \{x, y, z\}$. Let E_x (respectively, E_y) denote the set of edges of G_1 incident with x (respectively, y), and let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^* . Then, $E_x \cap E_y = \emptyset$, G_1^* contains a K_5-subdivision, and for any K_5-subdivision Σ in G_1^*,*

(i) x^* is a branching vertex of Σ,

(ii) $yz \notin E(\Sigma)$,

(iii) $|E_x \cap E(\Sigma)| = 2 = |E_y \cap E(\Sigma)|$, and

(iv) for any two branching vertices u, v of Σ, there are four internally disjoint u-v paths in Σ.

Proof: For convenience, vertices and edges of G_1 are also viewed as vertices and edges of G_1^*, except for x and y. By Lemma 7 and Lemma 6, $E_x \cap E_y = \emptyset$, using Theorem 5.

Suppose G_1^* contains no K_5-subdivision. Then by the choice of G, G_1^* is 4-colorable. Then G_1 has a 4-coloring c_1 such that $c_1(x) = c_1(y) \neq c_1(z)$. By Proposition 2, $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$. Then $c_2(x) \neq c_2(z) \neq c_2(y)$. If $c_2(x) \neq c_2(y)$ then $G_2 + \{xy, yz, zx\}$ is 4-colorable, contradicting Proposition 2. So $c_2(x) = c_2(y)$. We may assume that c_1 and c_2 use the same set of four colors. Then we may permute colors so that $c_1(u) = c_2(u)$ for all $u \in \{x, y, z\}$. Let $c(u) = c_i(u)$ for all $u \in V(G_i)$, $i = 1, 2$. Then c is a 4-coloring of G, a contradiction.

Now let Σ be a K_5-subdivision in G_1^*. By (iii) of Lemma 4, let P_{yz} denote a y-z path in $G_2 - x$, P_{xz} an x-z path in $G_2 - y$, and P_{xy} an x-y path $G_2 - z$. For the same reason, G_2

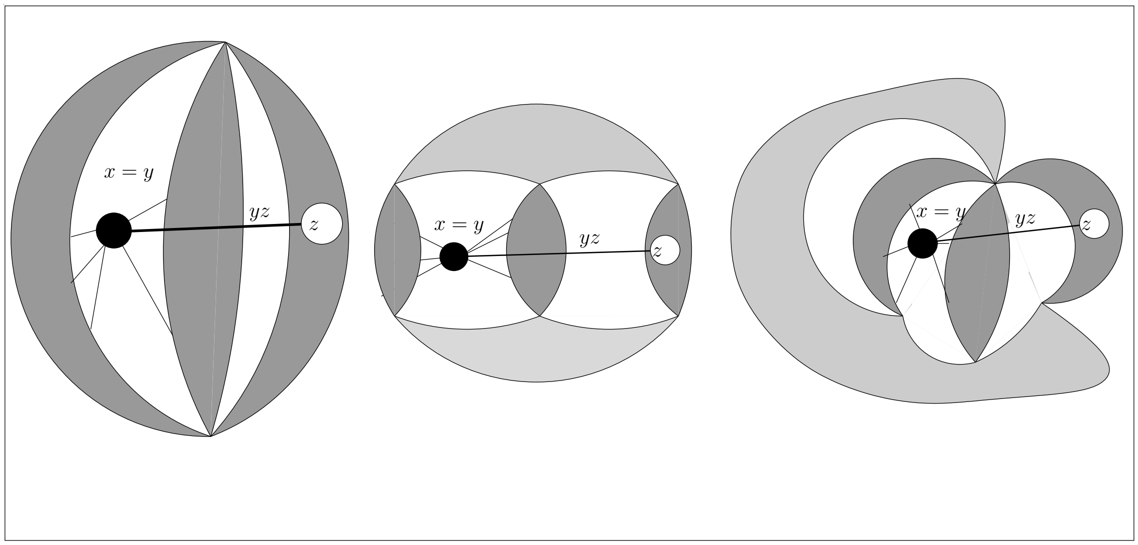

Figure 8: The structure of G_1^*

contains internally disjoint paths X_y, X_z from x to y, z, respectively, and internally disjoint paths Y_x, Y_z from y to x, z, respectively.

Proof of (i). Suppose x^* is not a branching vertex of Σ. Then since G_1 has no K_5-subdivision, exactly one branching path of Σ, say R, uses x^*. Let q, r be the neighbors of x^* in R. First assume that $z \in \{q, r\}$, say $z = r$. If $qy \in E(G_1)$ then $((\Sigma - x^*) + \{y, qy\}) \cup P_{yz}$ is a K_5-subdivision in G, a contradiction. So assume $qx \in E(G_1)$ then $((\Sigma - x^*) + \{x, qx\}) \cup P_{xz}$ is a K_5-subdivision in G, a contradiction. So assume that $z \notin \{q, r\}$. If $qx, rx \in E(G_1)$ then $(\Sigma - x^*) + \{x, qx, rx\}$ is a K_5-subdivision in G_1, a contradiction. If $qy, ry \in E(G_1)$ then $(\Sigma - x^*) + \{y, qy, ry\}$ is a K_5-subdivision in G_1, a contradiction. So assume by symmetry $qx, ry \in E(G_1)$. Then $((\Sigma - x^*) + \{x, y, qx, ry\}) \cup P_{xy}$ is a K_5-subdivision in G, a contradiction. Thus x^* is a branching vertex in Σ, and (i) holds.

Proof of (ii). Suppose $yz \in E(\Sigma)$. Then either $|E_x \cap E(\Sigma)| \leq 1$ or $|E_y \cap E(\Sigma)| \leq 1$. By symmetry, assume that $|E_x \cap E(\Sigma)| \leq 1$. If $|E_x \cap E(\Sigma)| = 0$ then let $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$, and we see that $((\Sigma - x^*) + \{y, yy_1, yy_2, yy_3\}) \cup P_x$ is a K_5-subdivision in G, a contradiction. So assume $|E_x \cap E(\Sigma)| = 1$ then let $yy_1, yy_2 \in E_y \cap E(\Sigma)$ and $xx_1 \in E_x \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, yy_1, yy_2, xx_1\}) \cup Y_x \cup Y_z$ is a K_5-subdivision in G, a contradiction. So $yz \notin E(\Sigma)$, and (ii) holds.

Proof of (iii). If $|E_x \cap E(\Sigma)| = 0$ or $|E_y \cap E(\Sigma)| = 0$, then by (ii), Σ gives a K_5-subdivision in G (by simply renaming x^* as y or x), a contradiction. Suppose (iii) fails and assume by symmetry that $|E_x \cap E(\Sigma)| = 1$ and $|E_y \cap E(\Sigma)| = 3$. Let $xx_1 \in E_x \cap E(\Sigma)$, $yy_1, yy_2, yy_3 \in E_y \cap E(\Sigma)$. Then $((\Sigma - x^*) + \{x, y, xx_1, yy_1, yy_2, yy_3\}) \cup P_{xy}$ is a K_5-subdivision in G, a contradiction. So (iii) must hold.

(iv) is a special case of Lemma 1. □

Proof of Theorem 1. Suppose the assertion of Theorem 1 is not true. Let G be a Hajós' graph and assume that G is not 4-connected. By Lemma 3, G is 3-connected. Let (G_1, G_2) be a 3-separation of G such that $|V(G_i)| \geq 4$ and subject to this, $|V(G_2)|$ is minimum. Let $V(G_1 \cap G_2) = \{x, y, z\}$.

By Lemma 7, $\{x, y, z\}$ is not contained in any cycle in G_1, and $\{x, y, z\}$ is an independent set in G_1 (see claim (1) in the proof of Theorem 5). Let E_x (respectively, E_y) denote the set of edges in G_1 incident with x (respectively, y). Let G_1^* denote the graph obtained from G_1 by adding the edge yz and identifying x and y as x^*. Then by Lemma 8, $E_x \cap E_y = \emptyset$ and G_1^* contains a K_5-subdivision, say Σ. Note that Σ satisfies (i)–(iv) of Lemma 8.

Note that G_1 is 2 connected (by Lemma 6) and contains no cycle through x, y, z (by Lemma 7). By applying Theorem 5 to G_1 and x, y, z, we consider the following three cases.

Case 1. There exists a 2-cut S in G_1 such that x, y, z are in pairwise different components D_x, D_y, D_z of $G_1 - S$, respectively.

Let $S := \{a, b\}$. By (i) of Lemma 8, x^* is a branching vertex of Σ. Therefore, D_z contains no branching vertex of Σ since S and the edge zx^* show that G_1^* contains at most three internally disjoint paths between x^* and D_z. Similarly, either $D_x - x$ or $D_y - y$ has no branching vertex of Σ since $S \cup \{x^*\}$ is a 3-cut in G_1^* separating $D_x - x$ from $D_y - y$.

Therefore, we may assume that all branching vertices of Σ are in $D_x \cup S \cup \{x^*\}$. Then, there is at most one a-b path contained in Σ, and we denote it by P if it exists. We shall derive a contradiction by either constructing a K_5-subdivision in G or giving a 4-coloring of G.

By (iii) of Lemma 8, there let P_a be the path between y and a contained in Σ and P_b the path between y and b contained in Σ. If $G[V(D_y) \cup S]$ contains internally disjoint path Y, B from a to y, b, respectively, then we can produce a K_5-subdivision in G as follows: replace P_a, P by Y, B, respectively, replace P_b by a path in $G[V(D_z) \cup \{b\}]$ from z to b, and add two internally disjoint paths from x to $\{y, z\}$ in G_2 (which exist by (iii) of Lemma 4). This gives a contradiction. So we may assume that such paths Y, B do not exist in $G[V(D_y) \cup S]$. Then there is a cut vertex a_y of $G[V(D_y) \cup S]$ separating a from $\{y, b\}$. Since G is 3-connected, we see that a_y is the only neighbor of a in $G[V(D_y) \cup S]$.

Similarly, we conclude that b has only one neighbor b_y in $G[V(D_y) \cup S]$, a has only one neighbor a_z in $G[V(D_z) \cup S]$, and b has only one neighbor b_z in $G[V(D_z) \cup S]$.

Next we use the above structural information to color vertices of G.

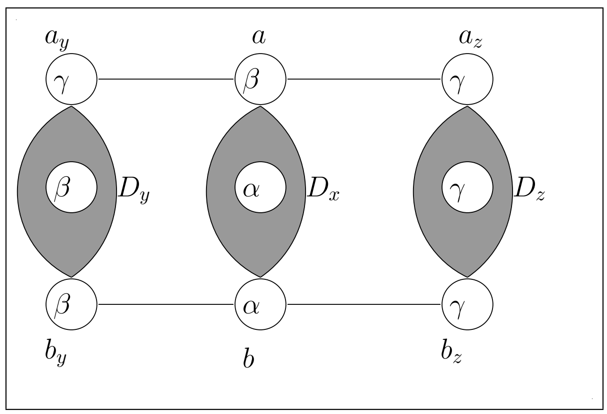

Figure 9: A Coloring of G_1 in Case 1

By Proposition 1, G_1 has a 4-coloring c_1 such that $c_1(x), c_1(y)$ and $c_1(z)$ are all distinct. We shall obtain a new 4-coloring c'_1 of G_1 such that x, y, z use exactly two colors. For convenience, let $\{\alpha, \beta, \gamma, \delta\}$ denote the four colors used by c_1, and let H_{ij} denote the subgraph of G_1 induced by vertices of color i or j, for all $\{i, j\} \subseteq \{\alpha, \beta, \gamma, \delta\}$. Let $c_1(x) = \alpha$, $c_1(y) = \beta$, and $c_1(z) = \gamma$. Note that $\{y, z\}$ must be contained in a component of $H_{\beta\gamma}$, as otherwise we could switch colors in the component of $H_{\beta\gamma}$ containing y, yielding the desired 4-coloring c'_1 of G_1. Therefore by symmetry between a and b, we may assume that $c_1(a_y) = \beta = c_1(a_z)$ and $c_1(a) = \gamma$ or $c_1(a_y) = \gamma = c_1(a_z)$ and $c_1(a) = \beta$. By the same argument, $\{x, z\}$ must be contained in a component of $H_{\alpha\gamma}$, and $\{x, y\}$ must be contained in a component of $H_{\alpha\beta}$.

Therefore, $c_1(b_y) = \beta$, $c_1(b) = \alpha$, and $c_1(b_z) = \gamma$. But then, neither x nor z can be in the component of $H_{\beta\delta}$ containing y, and neither y nor z is in the component of $H_{\alpha\delta}$ containing x. Thus we can switch the colors in the component of $H_{\beta\delta}$ containing y and in the component of $H_{\alpha\delta}$ containing x. This yields the desired 4-coloring c_1' of G_1, with $c_1'(x) = c_1'(y) = \delta$ and $c_1'(z) = \gamma$.

Now by symmetry, assume that $c_1'(x) = c_1'(y) \neq c_1'(z)$. By Proposition 2, $G_2 + \{xz, yz\}$ is 4-colorable. Let c_2 be a 4-coloring of $G_2 + \{xz, yz\}$ using the colors from $\{\alpha, \beta, \gamma, \delta\}$. If $c_2(x) \neq c_2(y)$ then c_2 is a 4-coloring of $G_2 + \{xy, yz, zx\}$, contradicting Proposition 2. So $c_2(x) = c_2(y)$. By permuting colors if necessary, we may assume that $c_2(u) = c_1'(u)$ for all $u \in \{x, y, z\}$. Now let $c(u) = c_1'(u)$ for all $u \in V(G_1)$ and $c(u) = c_2(u)$ for all $u \in V(G_2)$. Then c is a 4-coloring of G, a contradiction.

Case 2 There exist a vertex v of G_1, 2-cuts S_x, S_y, S_z in G_1, and components D_u of $G_1 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that $S_x \cap S_y \cap S_z = \{v\}$, $S_x - \{v\}, S_y - \{v\}, S_z - \{v\}$ are pairwise disjoint, and D_x, D_y, D_z are pairwise disjoint.

Our goal is to show that G_1^* does not admit the K_5-subdivision Σ. By (i) of Lemma 8, x^* is a branching vertex of Σ. Therefore, D_z contains no branching vertex of Σ since S_z and the edge zx^* shows that G_1^* contains at most three internally disjoint paths between x^* and D_z, contradicting (iv) of Lemma 8. In fact, all branching vertices of Σ must be contained in $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. For otherwise, Σ has a branching vertex $v \notin R$, and Σ must have four disjoint path leaving $R \cup \{x^*\}$. This means that $yz \in E(\Sigma)$, contradicting (ii) of Lemma 8.

We claim that, for each $u \in \{x, y\}$, not all branching vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, suppose by symmetry that all branching vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma 8, let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let B_q, B_r be the branching paths in Σ containing x^*q, x^*r, respectively. Since $yz \notin E(\Sigma)$ (by (ii) of Lemma 8), both B_q and B_r have an x^*-S_y subpath whose internal vertices are all contained in D_y. Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z, respectively, which exist by (iii) of Lemma 4. Note that there exists an $(S_z - \{v\})$-$(S_x - \{v\})$ path Q_{xz} in $(G_1 - v) - V(D_x \cup D_y \cup D_z)$;

for otherwise, one of $\{v, x\}, \{v, z\}$ is a 2-cut in G, contradicting Lemma 3. Let Y be a y-v path in $G[V(D_y) \cup \{v\}]$ and let Z be a z-$(S_z - \{v\})$ path in $G[V(D_z) \cup (S_z - \{v\})]$. Then

$$(((\Sigma - x^*) + \{x, xs, xt\}) - (V(B_q \cup B_r) - (V(D_x) \cup S_x))) \cup (P_{xy} \cup Y) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5-subdivision in G, a contradiction.

Since $|\{x^*\} \cup S_x \cup S_y| = 4$, there must exist a branching vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may assume that $x' \in V(D_x - x)$. Hence by the above claim, there is also a branching vertex y' of Σ such that $y' \in V(D_y - y) \cup (S_y - \{v\})$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y', contradicting (iv) of Lemma 8.

Case 3 There exist pairwise disjoint 2-cuts S_x, S_y, S_z in G_1 and components D_u of $G_1 - S_u$ containing u, for all $u \in \{x, y, z\}$, such that D_x, D_y, D_z are pairwise disjoint and $G_1 - V(D_x \cup D_y \cup D_z)$ has exactly two components, each containing exactly one vertex from each of S_u, for all $u \in \{x, y, z\}$.

Let $S_x = \{a_x, b_x\}$, $S_y = \{a_y, b_y\}$, and $S_z = \{a_z, b_z\}$ such that $\{a_x, a_y, a_z\}$ is contained in a component A of $G_1 - V(D_x \cup D_y \cup D_z)$, and $\{b_x, b_y, b_z\}$ is contained in the component B of $G_1 - V(D_x \cup D_y \cup D_z)$.

As in Cases 1 and 2, we can show that all branching vertices of Σ are in $R \cup S_z$, where $R := V(D_x - x) \cup V(D_y - y) \cup S_x \cup S_y \cup \{x^*\}$. In fact, all branching vertices of Σ must be in R. For otherwise, assume by symmetry that a_z is a branching vertex of Σ. Then, since $yz \notin E(\Sigma)$ (by (ii) of Lemma 8), $\{b_z, a_x, a_y\}$ show that Σ cannot contain four internally disjoint paths between a_z and x^*, a contradiction.

We claim that, for each $u \in \{x, y\}$, not all branching vertices of Σ are contained in $V(D_u) \cup S_u \cup \{x^*\}$. For otherwise, we may assume that all branching vertices of Σ are contained in $V(D_x) \cup S_x \cup \{x^*\}$. By (iii) of Lemma 8, let x^*s, x^*t be the two edges in $E(\Sigma) \cap E_x$, let x^*q, x^*r be the two edges in $E(\Sigma) \cap E_y$, and let A_q, B_r be the branching paths in Σ containing x^*q, x^*r, respectively. Since $yz \notin E(\Sigma)$, both A_q and B_r have an x^*-S_y subpath whose internal vertices are all contained in D_y. Let P_{xy}, P_{xz} be two internally disjoint paths in G_2 from x to y, z, respectively, which exist by (iii) of Lemma 4. Note that there exists an a_y-a_x path Q_{xy} in A (since A is connected) and there exists a b_z-b_x path

Q_{xz} in B (since B is connected). Let Y be an y-a_y path in $G[V(D_y) \cup \{a_y\}]$ and let Z be an z-b_z path in $G[V(D_z) \cup \{b_z\}]$. Then,

$$((\Sigma - x^* + \{x, xs, xt\}) - (V(A_q \cup B_r) - (V(D_x) \cup S_x))) \cup (P_{xy} \cup Y \cup Q_{xy}) \cup (P_{xz} \cup Z \cup Q_{xz})$$

is a K_5-subdivision in G, a contradiction.

We claim that the set of branching vertices of Σ is $S_x \cup S_y \cup \{x^*\}$. For otherwise, there must be a branching vertex x' of Σ such that $x' \in V(D_x - x) \cup V(D_y - y)$. By symmetry, we may assume that $x' \in V(D_x - x)$. Then by the above claim, there is a branching vertex y' of Σ such that $y' \in V(D_y - y) \cup S_y$. Now $S_x \cup \{x^*\}$ is a 3-cut in Σ separating x' from y', contradicting (iv) of Lemma 8.

Since $yz \notin E(\Sigma)$, we see that Σ must contain two branching paths from $\{a_x, a_y\}$ to $\{b_y, b_z\}$ which are also contained in $G_1 - V(D_x \cup D_y)$. But this is impossible, because a_z separates $\{a_x, a_y\}$ from $\{b_y, b_z\}$ in $G_1 - V(D_x \cup D_y)$, a contradiction. $\square$

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