A Generalization of the Characteristic Polynomial of a Graph

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Abstract

Given a graph G with its adjacency matrix A, consider the matrix $A(x,y)$ in which the 1s are replaced by the indeterminate x and 0s (other than the diagonals) are replaced by y. The $\mathcal{L}$-polynomial of G is defined as:

$$\mathcal{L}_G(x,y,\lambda) := \det(A(x,y) - \lambda I).$$

This polynomial is a natural generalization of the standard *characteristic polynomial* of a graph.

In this note we characterize graphs which have the same $\mathcal{L}$-polynomial. The answer is rather simple: Two graphs G and H have the same $\mathcal{L}$-polynomial if and only if - G and H are co-spectral and G_c and H_c are co-spectral. (Here G_c (resp. H_c) is the complement of G (resp. H).)

1 Introduction

Given two undirected simple graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ on n vertices, G and H are said to be isomorphic if there is a permutation $\pi \in S_n$ such that for all edges $\{u, v\} \in E_1$ if and only if $\{\pi(u), \pi(v)\} \in E_2$.

Deciding efficiently whether two graphs are isomorphic (**GI**) is a notoriously hard problem in Computer Science and has numerous practical and theoretical applications, see [2]. The complexity of this problem has puzzled researchers for decades. It is unlikely that this problem is **NP**-Complete as that would imply $\mathbf{PH} = \mathbf{\Sigma_2}$, see [2]. Neither is it known to be in **P**.

One way to do establish that **GI** is easy would be to find an efficiently computable graph invariant and show that it separates graphs up to their automorphism classes. Lot of research has been devoted to this and finding such an invariant is still open.

In this note we present a new graph polynomial: Given a graph G with its adjacency matrix A, consider the matrix $A(x,y)$ in which the 1s are replaced by the indeterminate x and 0s (other than the diagonals) are replaced by y. The $\mathcal{L}$-polynomial of G is defined as:

$$\mathcal{L}_G(x,y,\lambda) := \det(A(x,y) - \lambda I).$$

This is a generalization of the standard *characteristic polynomial* of a graph. We prove a simple characterization of graphs which have the same $\mathcal{L}$-polynomial: Two graphs G and H have the same $\mathcal{L}$-polynomial if and only if - G and H are co-spectral and G_c and H_c are co-spectral.

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(Here G_c (resp. H_c) is the complement of G (H).) Moreover we can exactly write down the $\mathcal{L}$-polynomial of a graph in terms of its and its complements characteristic polynomial. It is easy to see that there exist non-isomorphic graphs G and H, such that G and H are co-spectral and G_c and H_c are also co-spectral. Hence this invariant is insufficient to resolve **GI**!

2 The $\mathcal{L}$-polynomial of a graph

All graphs in this note will be simple and undirected. For a graph $G = (V, E)$, let A_G denote its adjacency matrix. Usually n, the number of vertices in the graph will be implicit in the context. Let J_n denote the $n \times n$ matrix all of whose entries are 1, and I_n be the identity matrix of order n. We will drop the subscript n when it is clear from the context.

The characteristic polynomial of a matrix A is defined to be a polynomial in λ as:

$$p_A(\lambda) := \det(A - \lambda I).$$

When A arises as the adjacency matrix of a graph G, we denote the characteristic polynomial of G as $p_G(\lambda) := p_{A_G}(\lambda)$. (For a comprehensive discussion on the characteristic polynomial of a graph see [1].)

For a graph G, its complement G_c is defined to be the graph with adjacency matrix $A_c := J - A_G - I$.

We say $A \sim B$ if $p_A(\lambda) = p_B(\lambda)$. Graphs G and H on n vertices are said to be **cospectral** if $p_G = p_H$. If $p_G = p_H$ and $p_{G_c} = p_{H_c}$, then call them **strongly cospectral.**

For a graph G, and indeterminates x, y denote the following $\mathcal{L}$-polynomial:

$$\mathcal{L}_G(x, y, \lambda) := \det(xA_G + yA_{G_c} - \lambda I).$$

Notice that this polynomial is well defined.

Proposition 2.1. *If $G \cong H$, then $\mathcal{L}_G(x, y, \lambda) = \mathcal{L}_H(x, y, \lambda)$.*

Proof. If $G \cong H$, then there is a permutation matrix Π such that $\Pi^T A_G \Pi = A_H$. Also $\Pi^T A_{G_c} \Pi = A_{H_c}$. Hence $\mathcal{L}_G(x, y, \lambda) = \det(xA_G + yA_{G_c} - \lambda I) = \det(\Pi^T(xA_G + yA_{G_c} - \lambda I)\Pi) = \det(xA_H + yA_{H_c} - \lambda I) = \mathcal{L}_H(x, y, \lambda)$. □

3 Main Result

In this Section we sketch the proof of the main result of this note. Full technical details will be included in a longer version.

Theorem 3.1. *Let G, H be two graphs on n vertices. Then $\mathcal{L}_G = \mathcal{L}_H$ if and only if G and H are strongly cospectral.*

We need the following technical Lemmata:

Lemma 3.2. *If A, B are two $n \times n$ matrices such that $A \sim B$ then for all $\mu \in \mathbb{R}$, $A + \mu I \sim B + \mu I$.*

Proof. Trivial. □

Lemma 3.3. *Let k be a field. Let B be a matrix with entries from k, and A be a matrix with entries in $k[\lambda]$ (here λ is an indeterminate). If the rank of B over k is at most r, then*

$$\det(A + \alpha B) = c_0 + c_1\alpha + \cdots + c_r\alpha^r,$$

where $c_0, \ldots, c_r \in k[\lambda]$.

Proof. Omitted. □

Corollary 3.4. *If A, B are two $n \times n$ matrices such that $A \sim B$ and $J - A \sim J - B$, then for all $\gamma \in \mathbb{R}$,*

$$A + \gamma J \sim B + \gamma J.$$

Proof. Notice that the rank of J is one and use Lemma 3.3. □

Now we can proceed to the proof of the main Theorem.

Proof of Theorem 3.1. The simple direction follows trivially by substituting in $\mathcal{L}_G$ and $\mathcal{L}_H$, $(x, y) = (1, 0)$ and $(0, 1)$ combined with Lemma 3.2.

To prove the other direction, first notice that if for all (x', y') such that $x' = y'$, $\mathcal{L}_G(x', y', \lambda) = \mathcal{L}_H(x', y', \lambda)$. Assume on the contrary that $\mathcal{L}_G \neq \mathcal{L}_H$. Hence there is a point (x_0, y_0), with $x_0 \neq y_0$, such that the polynomials $\mathcal{L}_G(x_0, y_0, \lambda) \neq \mathcal{L}_H(x_0, y_0, \lambda)$. But this combined with Lemma 3.2 contradicts Corollary 3.4 for $\gamma = \frac{y_0}{x_0 - y_0}$. Hence the proof is completed. □

Corollary 3.5. *Given a graph G, let $p := p_{A_G}$ and $\overline{p} := p_{A_{G_c}}$, then*

$$\mathcal{L}_G(x, y, \lambda) = (x - y)^{n-1} \left[xp\left(\frac{\lambda + y}{x - y}\right) + y\overline{p}\left(-\frac{\lambda + x}{x - y}\right) \right].$$

Here the right hand side is to be interpreted as a polynomial rather than a rational function.

Proof. Use Lemma 3.2 along with Corollary 3.4 and Theorem 3.1. □

References

[1] N. Biggs, **Algebraic Graph Theory**. Cambridge, U.K. Cambridge Univ. Press, 1993.

[2] J. Köbler, U. Schöning, J. Torán. **The Graph Isomorphism Problem: Its Structural Complexity.** Progress in Theoretical Computer Science, Birkhäuser, 1993.