CAPILLARY GRAVITY WATER WAVE LINEARIZED AT MONOTONE SHEAR FLOWS: EIGENVALUE AND INVISCID DAMPING

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CAPILLARY GRAVITY WATER WAVE LINEARIZED AT MONOTONE SHEAR FLOWS: EIGENVALUE AND INVISCID DAMPING

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What I think I've been able to do well over the years is play with pain, play with problems, play in all sorts of conditions.

Roger Federer

For my husband Cheng and our son Henry

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SUMMARY

This work is concerned with the two dimensional capillary gravity water waves of finite depth $x_2 \in (-h, 0)$ linearized at a uniformly monotonic shear flow $U(x_2)$. We focus on the eigenvalue distribution and linear inviscid damping. Unlike the linearized Euler equation in a fixed channel at a shear flow where eigenvalues exist only in low wave numbers k of the horizontal variable x_1, we first prove that the linearized capillary gravity wave has two branches of eigenvalues $-ikc^{\pm}(k)$, where the wave speeds $c^{\pm}(k) = O(\sqrt{|k|})$ for $|k| \gg 1$ have the same asymptotics as the those of the linear irrotational capillary gravity waves. Under the additional assumption of $U'' \neq 0$, we obtain the complete continuation of these two branches, which are all the eigenvalues of the linearized capillary gravity waves in this (and some other) case(s). In particular, $-ikc^{-}(k)$ could bifurcate into unstable eigenvalues at $c^{-}(k) = U(-h)$. In general the bifurcation of unstable eigenvalues from inflection values of U is also obtained. Assuming there are no singular modes, i.e. no embedded eigenvalues for any horizontal wave number k, linear solutions $(v(t,x), \eta(t,x_1))$ are considered in both periodic-in-x_1 and $x_1 \in \mathbb{R}$ cases, where v is the velocity and η the surface profile. Each solution can be split into (v^p, η^p) and (v^c, η^c) whose k-th Fourier modes in x_1 correspond to the eigenvalues and the continuous spectra of the wave number k, respectively. The component (v^p, η^p) is governed by a (possibly unstable) dispersion relation given by the eigenvalues, which is simply $k \to kc^{\pm}(k)$ in the case of $x_1 \in \mathbb{R}$ and is conjugate to the linear irrotational capillary gravity waves under certain conditions. The other component (v^c, η^c) satisfies the linear inviscid damping as fast as $|v_1^c|_{L_x^2}, |\eta^c|_{L_2^x} = O(\frac{1}{|t|})$ and $|v_2^c|_{L_x^2} = O(\frac{1}{t^2})$ as $|t| \to \infty$. Furthermore, additional decay of $tv_1^c, t^2 v_2^c$ in $L_x^2 L_t^q$, $q \in (2, \infty]$, is obtained after leading asymptotic terms are singled out, which are in the forms of t-dependent translations in x_1 of certain functions of x. The proof is based on detailed analysis of the Rayleigh equation.

CHAPTER 1

INTRODUCTION AND BACKGROUND

The chief purpose of this thesis is to study the long-time asymptotic behavior of incompressible inviscid gravity-capillary water wave in $\mathbb{R}^2$. Particularly, the focus is on monotone shear flows. The results in this thesis include the eigenvalue distribution and inviscid damping. In this chapter, we present our frame work, some background about the two-dimensional incompressible Euler equation, and some classical results about the linear instability of shear flows.

1.1 2d incompressible inviscid capillary gravity water wave

In our frame work, the fluid is assumed to be incompressible, inviscid and has finite depth under the influence of gravity and surface tension. The density of the fluid is constant. It is normalized to be 1.

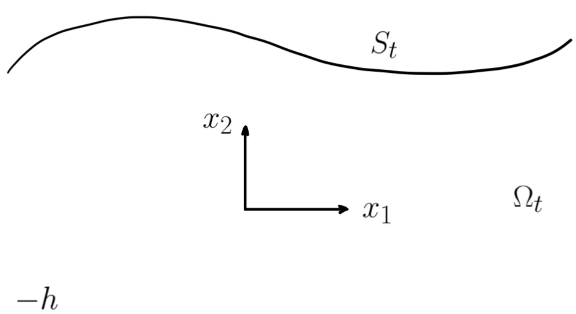

Figure 1.1: Capillary gravity water wave

In this two dimensional capillary gravity water wave problem, we let

$$\mathcal{U}_t = \{(x_1, x_2) \in \mathbb{T}_L \times \mathbb{R} \mid -h < x_2 < \eta(t, x)\}, \quad \mathbb{T}_L := \mathbb{R}/L\mathbb{Z},\ L > 0,$$

or

$$\mathcal{U}_t = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R} \mid -h < x_2 < \eta(t, x)\}.$$

be the fluid domain at time $t \geq 0$. Here $\mathbb{T}_L := \mathbb{R} \setminus L\mathbb{Z}$, L is the period. We consider the case where the free surface S_t is given as the graph of a smooth function η.

$$S_t = \{(t, x) | x_2 = \eta(t, x_1)\}.$$

The constant $h > 0$ is fixed and describes the location of water bed. For $x \in \mathcal{U}_t$, let $v = (v_1(t, x), v_2(t, x)) \in \mathbb{R}^2$ denote the fluid velocity and $p = p(t, x) \in \mathbb{R}$ be the pressure. By the Newtonian's second law, they satisfy the following 2d Euler equations:

$$\partial_t v + (v \cdot \nabla)v + \nabla p + g\vec{e}_2 = 0, \qquad x \in \mathcal{U}_t, \tag{1.1.1a}$$

where $g > 0$ is the gravitational acceleration. By the incompressibility of the fluid, v satisfies divergence free condition.

$$\nabla \cdot v = 0, \qquad x \in \mathcal{U}_t, \tag{1.1.1b}$$

The motion of the surface satisfies the kinematic boundary condition which means that v restricted to the free surface S_t is a boundary velocity.

$$\partial_t \eta(t, x_1) = v(t, x) \cdot (-\partial_{x_1}\eta(t, x_1), 1)^T, \qquad x \in S_t. \tag{1.1.1c}$$

We also impose the dynamic boundary condition.

$$p = \sigma\kappa(t, x), \qquad x \in S_t, \tag{1.1.1d}$$

where $\sigma > 0$ is a fixed material constant, $\kappa(t, x) = -\frac{\eta_{x_1 x_1}}{(1+\eta_{x_1}^2)^{\frac{3}{2}}}$ is the mean curvature of S_t

at x which corresponds to the surface tension. The rigidity of the water bed implies the slip boundary condition.

$$v_2(t, x_1, -h) = 0, \qquad x_2 = -h. \tag{1.1.1e}$$

The local well-posedness theory of the water wave system has been studied extensively. Nalimov [58] proved a local well-posedness result for irrotational gravity waves in the case of infinite depth with small initial data in Sobolev spaces. Without surface tension, Ebin [17] proved that the problem is ill-posed without the Rayleigh-Taylor stability condition. In the breakthrough works of Wu [65, 66], she established the well-posedness for irrotational gravity wave without surface tension in Sobolev spaces locally in time in both two and three dimensions. Later, Lindblad [45] proved the local existence of solutions for the rotational water wave equations in the absence of surface tension. Coutand and Shkoller [15] proved the local well-posedness of rotational fluids with surface tension and without surface tension. Other references include [8, 1, 2, 14, 54, 55, 36, 53, 69, 49, 37]. Wu [67] and Germain-Masmoudi-Shatah [21] proved the global well-posedness without surface tension for irrotational small data. Ionescu-Pusateri [33] proved the global regularity for the 2d irrotational water waves problem with surface tension for irrotational small data. Other references about global well-posedness for irrotational small data include [32, 3, 25, 28, 59, 33, 60].

1.2 Linearization near shear flows

It is well known that shear flow is a class of steady state of incompressible Euler equation. In a shearing flow, adjacent layers of the fluid move parallel to each other. Particularly, we consider the following parallel shear flows in x_1-direction.

$$v_* := \big(U(x_2), 0\big)^T, \quad S_* := \{(t,x) | x_2 = \eta_*(x_1) \equiv 0\}, \quad \nabla p_* = -g\vec{e}_2. \tag{1.2.1}$$

Our primary goal is to analyze the capillary gravity water wave system linearized at a monotone shear flow satisfying

$$U \in C^{l_0}([-h, 0]), \quad l_0 \geq 3, \qquad U'(x_2) > 0, \quad \forall x_2 \in [-h, 0]. \tag{\textbf{H}}$$

The two aspects of this linearized system are the eigenvalue distribution and inviscid damping. We first derive the linearized system of (1.1.1) at the shear flow $(v_* = (U(x_2), 0)^T, \eta_* = 0)$ given in (1.2.1) satisfied by the linearized solutions which we denote by (v, η, p). Let $(S_t^\epsilon, v^\epsilon(t,x), p^\epsilon(t,x))$ be a one-parameter family of solutions of (1.1.1) with

$$(S_t^0, v^0(t,x), p^0(t,x)) = (S_*, v_*, p_*).$$

Differentiating the Euler equation (1.1.1a) and (1.1.1b) with respect to ϵ and then evaluating it at $\epsilon = 0$ yield

$$\partial_t v + U(x_2)\partial_{x_1} v + (U'(x_2)v_2, 0)^T + \nabla p = 0, \qquad \nabla \cdot v = 0, \qquad x_2 \in (-h, 0). \tag{1.2.2a}$$

Taking its divergence and also evaluating the above linearized Euler equation at $x_2 = -h$, we obtain

$$-\triangle p = 2U'(x_2)\partial_{x_1} v_2, \quad x_2 \in (-h, 0), \text{ and } \partial_{x_2} p|_{x_2=-h} = 0. \tag{1.2.2b}$$

From the kinematic boundary condition (1.1.1c), we have

$$\partial_t \eta = v_2|_{x_2=0} - U(0)\partial_{x_1}\eta. \tag{1.2.2c}$$

Finally differentiating (1.1.1d), where the left side is $p^\epsilon(t, x_1, \eta^\epsilon(t, x_1))$, and using $\partial_{x_2} p_* = -g$, we obtain

$$p = g\eta - \sigma\partial_{x_1}^2 \eta, \qquad \text{at } x_2 = 0. \tag{1.2.2d}$$

The above ((1.2.2a) – (1.2.2d)) form the linearization of the capillary gravity water wave problem (1.1.1) at the shear flow (v_*, S_*, p_*) with initial values $(v_{10}(x), v_{20}(x), \eta_0(x_1))$. In fact it can be reduced to an evolutionary problem of the unknowns (v, η), while p can be recovered by the boundary value problem of the elliptic system (1.2.2b) and (1.2.2d).

1.3 Backgrounds and motivations

Due to its physical and mathematical significance there have been extensive studies of the Euler equation linearized at shear currents. Many of these works were done for a finite channel flow which satisfies

$$\text{(1.1.1a)–(1.1.1b) with } \; g = 0, \tag{1.3.1a}$$

and slip boundary conditions

$$v_2(t, x_1, 0) = v_2(t, x_1, -h) = 0, \tag{1.3.1b}$$

and some of the results have been extended to free boundary problems such as the gravity waves. The spectral analysis is naturally a crucial part of such linear systems. Eigenvalues yield linear solutions exponential or oscillated in time, while the continuous spectra often lead to algebraic decay of solutions, the so-called inviscid damping due to the lack of a priori dissipation mechanism of the Euler equation.

• *Eigenvalues.* Since the variable coefficients in the linearized Euler system depend only on x_2, the subspace of the k-th Fourier mode is invariant under the linear evolution for any $k \in \mathbb{R}$. Hence it is a common practice to seek eigenvalues and eigenfunctions in the form of

$$v(t, x) = e^{ik(x_1 - ct)}\big(v_{10}(x_2), v_{20}(x_2)\big), \;\; \eta(t, x_1) = e^{ik(x_1 - ct)}\eta_0(x_1), \tag{1.3.2}$$

in the free boundary case, where apparently the eigenvalues take the form $\lambda = -ikc$ with the wave speed $c = c_R + ic_I \in \mathbb{C}$. The linear system is spectrally unstable if there exist such c, which appear in conjugate pairs, with $c_I > 0$ and $k > 0$. Solutions in the above form with $c \in U([-h, 0])$ are in a subtle situation and are referred to as singular modes (see Definition 2.2.1 and Remark 4.1.1 for singular and non-singular modes). In seeking solutions in the form of (1.3.2), the wave number $k \in \mathbb{R}$ is often treated as a parameter.

Classical results on the spectra of the Euler equation (1.3.1) in a channel linearized at a shear flow include:

- Unstable eigenvalues are isolated for any wave number $k \in \mathbb{R}$ and do not exist for $|k| \gg 1$.

- Rayleigh's necessary condition of instability [51]: unstable eigenvalues do not exist for any k if $U'' \neq 0$ on $[-h, 0]$ (see also [18]).

- Howard's Semicircle Theorem [24]: for any $k \neq 0$, eigenvalues exist only with c in the disk

$$\left(c_R - \tfrac{1}{2}(U_{max} + U_{min})\right)^2 + c_I^2 \leq \tfrac{1}{4}(U_{max} - U_{min})^2. \tag{1.3.3}$$

  Particularly, the unstable wave speed must lie in an upper semicircle.

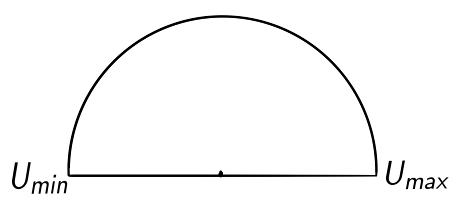

Figure 1.2: Howard's Semicircle Theorem

- Unstable eigenvalues may exist with c near inflection values of U (Tollmien [57] formally, also [38]).

Many classical results can be found in books such as [16, 46] *etc.* For a class of shear flows, the rigorous bifurcation of unstable eigenvalues was proved, e.g., in [19, 39, 41, 42]. In particular, Friedlander and Howard demonstrated the instability of Kolmogorov flow $U(x_2) = cos(mx_2)$. This is also an example of a nonlinearly unstable inviscid flow according to [20]. Lin showed that for a certain class of shear flows, the neutral limiting wave speed must be an inflection value of the velocity profile and proved the global bifurcation of unstable modes from neutral modes rigorously. He also provided some sufficient conditions of linear instability for several classes of shear flows.

It has been extended to the linearized free boundary problem of gravity waves (i.e. $g > 0$ and $\sigma = 0$ in (1.1.1)) at shear flows (see [68, 26, 52, 27] *etc.*) that: a.) assuming $U' > 0$ and $U'' \neq 0$ on $[-h, 0]$, there are no singular neutral modes in $U\big((-h, 0)\big)$ (i.e. solution in the form of (1.3.2) with $c \in U\big((-h, 0)\big)$); b.) the semicircle theorem still holds; and c.) for a class of shear flows, singular neutral modes may exist at inflection values of U and the bifurcation and continuation of branches of unstable eigenvalues were also obtained. Compared to channel flows with fixed boundaries, new phenomena of the linearized gravity waves include: a.) in addition, critical values of U, where $U' = 0$, and $c = U(-h)$ may be limiting singular neutral modes; and b.) there are non-singular neutral modes, i.e. $c \in \mathbb{R} \backslash U([-h, 0])$. Another related result is Miles' critical layer theory [48, 11] on the instability of shear flows in two-phase fluid interface problem due to the resonance between the temporal frequency of the linear irrotational capillary gravity waves at the completely stationary water and the shear flow in the air in the above.

• *Inviscid damping.* The analysis of the inviscid damping phenomenon started with the Euler equation in a fixed periodic channel (1.3.1) linearized at the Couette flow $U(x_2) = x_2$. In 1907, Orr [50] observed that the linearized vertical velocity $v_2(t, x)$ tends to zero as $t \to \infty$. Some explicit calculations were done by some mathematicians(see, e.g., [12, 43]). It has been proven that under the assumption $\int_{-h}^{0} v_{10}(x_1, x_2) dx_1 = 0$, which means that the shear flow component of the linear solutions is removed, as $t \to \infty$,

$$\begin{aligned} \omega_0 \in L^2 &\implies |v|_{L^2} = o(1), \\ \omega_0 \in H^1 &\implies |v|_{L^2} = O(\tfrac{1}{|t|}), \\ \omega_0 \in H^2 &\implies |v_2|_{L^2} = O(\tfrac{1}{|t|^2}), \end{aligned} \tag{1.3.4}$$

where ω_0 denotes the initial vorticity. More general shear flows in a fixed channel have also been studied extensively. For a class of general stable shear flows, Bouchet and Morita [9] predicted similar decay estimates of the linearized velocity as well as the vorticity depletion phenomenon. For monotone shear flows without infection points, an $O(|t|^{-\nu})$ decay of the stream function was proved in [56] and then the (1.3.4) type decay in [70, 71] under a smallness assumption of LU'' (also $\omega_0|_{x_2=-h,0} = 0$ in order for the $O(t^{-2})$ decay of v_2). A significant contribution is [61] by Wei-Zhang-Zhao where the (1.3.4) type estimates were obtained for general monotone shear flows without singular modes. In the follow-up works [62, 63, 64], vorticity depletion and velocity decay (as well as an L_t^2 decay if $\omega_0 \in L^2$ only) were also obtained for a class of non-monotone shear flows. As the decay rates in (1.3.4) are basically optimal, some leading order effects from both the interior and the boundary were identified for the stream function and velocity in [70, 35]. In the absence of boundary impact, for compactly supported initial vorticity, linear inviscid damping near a class of monotone shear flows was also obtained in Gevrey spaces [34]. In [23], a different approach using methods from the study of Schrödinger operators was successfully adopted to analyze inviscid damping. See also [6, 31] for important developments for the linear inviscid damping at circular flows in $\mathbb{R}^2$.

While we focus on the linearized capillary gravity waves at shear flows, among the rich literatures on the related nonlinear dynamics of the 2-d Euler equation on fixed domains we refer the readers to [4] for nonlinear Lyapunov stability of steady states based on energy-Casimir functions by Arnold. In recent years, there are some results on the nonlinear instability of steady states. Under a spectral condition, Friedlander-Strauss-Vishik

[20] proved that linear instability in L^2 implies nonlinear instability in H^s, $s > \frac{d}{2} + 1$ in $d-$dimensional space. For a class of shear flows, of which the maximal Lyapunov exponent is zero, Grenier [22] proved that in two dimension, linear instability implies nonlinear instability with growth in L^2 and L^∞. Bardos-Guo-Strauss [5] proved nonlinear instability reflected in growth of the vorticity in L^2 if the linear growth rate is higher than the maximal Lyapunov exponent of the steady flows defined on bounded domains in two dimension. Lin improved this result and showed nonlinear instability in L^p-norm of velocity ($p > 1$) without any restriction on the growth rate [40, 42] in 2D. Lin-Zeng [44] proved, in any dimensions, the existence of the unique local unstable manifold of a steady state under certain conditions and thus its nonlinear instability. There are also some results on the remarkable asymptotic stability of shear flows in Gevrey class [7, 29]. Recently, Ionescu–Jia [30] and Masmoudi–Zhao [47] independently proved nonlinear inviscid damping for more general monotonic shear flows in two dimenson, with compactly supported vorticity, in Gevrey class based on the linear inviscid damping.

• *Intuitions and goals on linearized capillary gravity waves.* Whether the Euler equation is in a fixed domain or with free boundaries, the vorticity is transported by the fluid flow in the interior of the domain, hence it is natural to expect linear inviscid damping of the linearized free boundary problem at a shear flow. In contrast to the linearized Euler equation on a fixed domain where non-singular modes do not exist for large wave number k, in the linear free boundary problems they exist for all $|k| \gg 1$, which can be seen in the linear irrotational case – a dispersive problem. Therefore in the linearized free boundary problems it is less reasonable to ignore these eigenfunctions of infinite dimensions to focus on the inviscid damping only. For the linearized capillary gravity wave (1.2.2) at a shear flow $U(x_2)$, our main goals are to obtain both the eigenvalue distribution and the linear inviscid damping of the solutions after projected to the components corresponding to the continuous spectra.

For an illustration, some explicit computations of the linearized capillary gravity wave

(1.2.2) at the Couette flow $U(x_2) = x_2$ are given in Section subsection 2.1.2. There it is easy to see that, on the one hand, the linear inviscid damping (1.3.4) holds for the rotational part of the solutions. On the other hand, there exist two branches of neutral modes $c^{\pm}(k)$ (see (2.1.4)) approaching infinity at the same rate as (2.1.5) of the linear irrotational capillary gravity wave. They form two branches of the dispersion relations of irrotational waves contained in the linearized water wave system at the Couette flow, which is linearly stable. At a general shear flow $U(x_2)$, natural questions are a.) linear inviscid damping, b.) what happens to these branches of non-singular modes, c.) where spectral instability could occur, *etc*. Compared to purely gravity waves, it is natural to expect that spectral properties with surface tension may be a.) similar if such properties are local in the wave number k; and b.) different if large wave numbers are involved. Motivated by the results on the linearized gravity waves [26, 52, 27], for monotone shear flows one may imagine unstable modes arising from $c = U(-h)$ and inflection values of U. The possible bifurcation of unstable modes at the end point value $U(-h)$ is particularly subtle due to the regularity issues of the bifurcation equation.

CHAPTER 2
MAIN RESULTS AND PRELIMINARIES

2.1 Main results

2.1.1 Eigenvalue distribution

We first give the main theorem on the eigenvalue distribution. The results on the linear inviscid damping are somewhat more technical and only roughly outlined here. Their more precise statements are given in Theorem 2.1.2 and Theorem 2.1.3 in subsection 2.1.3. See Definition 2.2.1, Lemma 4.1.1(5), and Remark 4.1.1 for what are referred to as singular and non-singular modes. Particularly, by slightly adjusting the same argument as in [24, 68], the Semi-circle Theorem still holds for the linearized system (1.2.2) of the capillary gravity water waves at shear flows. We shall take this as granted in the rest of the paper.

Theorem 2.1.1. *(***Eigenvalues.***) Suppose $U \in C^3$ and $U' > 0$ on $[-h, 0]$, then the following hold.*

1. *There exists $k_0 > 0$ such that for any $k \in \mathbb{R}$ with $|k| \geq k_0$, there are no singular modes and exactly two non-singular modes $c^+(k) \in (U(0), +\infty)$ and $c^-(k) \in (-\infty, U(-h))$ which correspond to semi-simple eigenvalues $-ikc^{\pm}(k)$. Moreover,*
   - *(a) $c^{\pm}(k)$ are even and analytic in k and $c^+(k)$ can be extended for all $k \in \mathbb{R}$ with $c^+(k) > U(0)$;*
   - *(b) $\lim_{|k|\to\infty} c^{\pm}(k)/\sqrt{\sigma|k|} = \pm 1$;*
   - *(c) if $U(-h)$ is not a singular mode for any $k \in \mathbb{R}$, then $c^-(k)$ can also be extended to be even and analytic in all $k \in \mathbb{R}$ with $c^-(k) < U(-h)$; and*
   - *(d) if singular modes do not exist $\forall\, k \in \mathbb{R}$, then $c^{\pm}(k)$ are the only non-singular modes of (1.2.2) which is linearly stable.*

(e) $c^+(k)$ (and $c^-(k) < U(-h)$ as well if it can be extended for all $k \in \mathbb{R}$) has either none or exactly one non-degenerate critical point for $k > 0$ under conditions (4.1.14) and (4.1.15), respectively. Consequently, if $c^-(k)$ can also be extended for all $k \in \mathbb{R}$ and (4.1.14) holds for both $c^\pm(k)$, then on the closed invariant subspace in $\{(v, \eta) \in H^n \times H^{n+1}\}$, $0 \le n \le l_0 - 1$, generated by the eigenfunctions of $-ikc^\pm(k)$ for all $k \in \mathbb{R}$, through an isomorphism, the linear system (1.2.2) is conjugate to the irrotational capillary gravity waves linearized at zero.

2. *There exists $g_\# \ge 0$ depending only on U and σ such that the following hold.*

   (a) If $g > g_\#$, then the non-singular modes $c^-(k) < U(-h)$ can also be extended to be even and analytic in all $k \in \mathbb{R}$ and $\pm(c^\pm(k))' > 0$ for $k > 0$;

   (b) $g_\# = 0$ if and only if

$$\sigma \ge \int_{-h}^{0} \big(U(x_2) - U(-h)\big)^2 dx_2. \tag{2.1.1}$$

3. *If $U'' \neq 0$ on $[-h, 0]$ is also satisfied, then there exists $g_\# \ge 0$ such that the following hold.*

   (a) The only possible singular mode is $c = U(-h)$.

   (b) If $g > g_\#$ then there are no singular modes and $c^-(k)$ can be extended as an even analytic function such that $c^-(k) < U(-h)$ for all $k \in \mathbb{R}$. Moreover $c^\pm(k)$ are the only non-singular modes and thus (1.2.2) is spectrally stable.

   (c) If $g = g_\#$ and $U \in C^6$, then there exists $k_\# > 0$ such that $c^-(k)$ can be extended as an even $C^{1,\alpha}$ function (for any $\alpha \in [0, 1)$) for all $k \in \mathbb{R}$. Moreover $c^-(k) < U(-h)$ is analytic for all $k \neq \pm k_\#$, and $c^-(\pm k_\#) = U(-h)$. For each $k \in \mathbb{R}$, $c^\pm(k)$ are the only singular or non-singular modes and thus (1.2.2) is spectrally stable.

(d) *If* $g < g_\#$ *and* $U \in C^6$*, then there exist* $k_\#^+ > k_\#^- > 0$ *such that we have the following.*

i. *Assume* $U'' > 0$ *on* $[-h, 0]$*, then* $c^-(k)$ *can be extended as an even* $C^{1,\alpha}$ *function (for any* $\alpha \in [0,1)$*) for all* $k \in \mathbb{R}$ *and analytic except at* $k = \pm k_\#^\pm$ *such that*

$$c^-(\pm k_\#^\pm) = U(-h),$$
$$c^-(k) < U(-h), \ \forall |k| \notin [k_\#^-, k_\#^+], \quad c_I^-(k) > 0, \ \forall |k| \in (k_\#^-, k_\#^+).$$

Moreover, for each k*, all singular and non-singular modes are exactly* $c^+(k)$*,* $c^-(k)$*, as well as* $\overline{c^-(k)}$ *if* $|k| \in (k_\#^-, k_\#^+)$*. Consequently, (1.2.2) is spectrally unstable iff A.)* $x_1 \in \mathbb{R}$ *or B.)* $x_1 \in \mathbb{T}_L$ *and there exists* $m \in \mathbb{Z}$ *such that* $\frac{2\pi m}{L} \in (k_\#^-, k_\#^+)$*.*

ii. *Assume* $U'' < 0$ *on* $[-h, 0]$*, then* $c^-(k)$ *can be extended as an even* $C^{1,\alpha}$ *real valued function (for any* $\alpha \in [0,1)$*) for* $|k| \notin (k_\#^-, k_\#^+)$*, analytic in* k *if* $|k| \notin [k_\#^-, k_\#^+]$*, and* $c^-(\pm k_\#^\pm) = U(-h)$*. Moreover, all singular and non-singular modes are exactly* $c^+(k)$ *and* $c^-(k)$*, if* $|k| \notin (k_\#^-, k_\#^+)$*, and (1.2.2) is spectrally stable.*

(e) $g_\# = 0$ *if (2.1.1) holds and consequently the above (3b) holds.*

4. *If* $U \in C^6$ *and* $U''(x_{20}) = 0$ *for some* $x_{20} \in (-h, 0)$*. Let* $c_0 = U(x_{20})$*.*

(a) *There exists* $\sigma_0 > 0$ *such that for any* $\sigma \in (0, \sigma_0)$*, there exists* $k > 0$*, unique among large* k*'s, such that* c_0 *is a singular neutral mode for* $\pm k$*.*

(b) *If* $U'''(x_{20}) \neq 0$ *and* c_0 *is a singular neutral mode for* $k_0 > 0$*, then, under a non-degenerate condition (verified by the one obtained in (4a) for small* σ*), there exist unstable modes near* c_0 *for* k *close to* k_0 *on one side of* k_0*.*

Remark 2.1.1. *Due to symmetry, the case of* $U' < 0$ *is completely identical except the* *signs of U'' in (3d) should be reversed. Theorem 2.1.2 and Theorem 2.1.3 on linear inviscid damping also hold under $U' \neq 0$ on* $[-h, 0]$.

The existence of the unbounded branches of non-singular neutral modes $c^{\pm}(k)$ are in contrast to the gravity waves or the Euler equation on fixed channels. In fact, the geometric multiplicity of $-ikc^{\pm}(k)$ occurs only among different k. These temporal frequencies $-kc^{\pm}(k)$ are asymptotic to those (see (2.1.5)) of the irrotational capillary gravity waves linearized at zero. Moreover, after normalizing the $L^2_{x_2}$ of the v component of the eigenfunction to be 1, the L^2 and H^1 differences in the v and η components, respectively, between the eigenfunctions of (2.2.1) and the linearized irrotational waves are of order $O(|k|^{-\frac{3}{2}})$ as $|k| \to \infty$ (see Remark 6.1.1). While the strong surface tension condition (2.1.1) ensures the branch $c^-(k)$ staying in $(-\infty, U(-h))$, it might reach $U(-h)$ otherwise. Subtle bifurcation of $c^-(k)$ occurs at $c = U(-h)$, the boundary of the domain of regularity of the bifurcation equation. In particular, the sign of U'' determines whether $c^(k)$ becomes unstable or disappears at $U(-h)$.

The spectral stability in the case $U'' < 0$ can also be obtained by directly modifying the usual proof of the Rayleigh theorem in the fixed channel flow case, as done in [68] for the gravity wave. Our proof provides a complete picture of the eigenvalue distribution as in the above theorem, however.

While $U(0)$ is never a singular mode, just like the Rayleigh's theorem in the channel flow case the change of sign of U'' turns out to be necessary for the existence of *interior* singular modes, which is also sufficient if $\sigma \ll 1$. In the contrast this may not be sufficient if the stabilizing gravity g and surface tension σ are strong, see Remark 4.3.1.

Outline of the Proofs of Theorem 2.1.1

In the preliminary analysis in section 2.2, we first apply the Fourier transform in x_1 to (1.2.2), resulting in decoupled systems for each wave number k. The problem can be further reduced to the evolution of $\hat{v}_2(t, k, x_2)$, the Fourier transform of v_2. The Laplacian transform $V_2(k, c, x_2)$ of $\hat{v}_2(t, k, x_2)$, where $s = -ikc$ is the Laplace transform variable, satisfies a non-homogeneous boundary value problem (2.2.6) of the Rayleigh equation, solutions to the associated homogeneous problem of which correspond to eigenvalues and eigenfunctions.

A careful analysis of the homogeneous Rayleigh equation (3.0.1), carried out in chapter 3, lays the foundation of the study of both the eigenvalue distribution and the inviscid damping. We first study the Rayleigh equation away from the singularity for $|U(x_2)-c| \geq O(\mu)$ where $\mu = \frac{1}{\langle k \rangle} = (1+k^2)^{-\frac{1}{2}}$. Near the singularity for $|U(x_2) - c| \leq O(\mu)$, different from those in, e.g., [61, 35], our approach is an improved version of the one in [11] based on the ODE blow-up and invariant manifold method [13]. Solutions to the homogeneous Rayleigh equation are expressed pointwisely through a transformation involving an explicit $\log(U - c)$ and depending on (k, c_R, x_2) smoothly. We focus on a pair of fundamental solutions $y_\pm(k, c, x_2)$ to the homogeneous Rayleigh equation which satisfy the corresponding homogeneous boundary conditions (2.2.6b)-(2.2.6c) in (2.2.6) at $x_2 = 0, -h$, respectively (boundary condition (2.2.6c) reflects *the free boundary* setting). For $y_\pm$, we establish a.) their a priori bounds; b.) the convergence to their limits $y_{0\pm}(k, c_R, x_2)$ as $c_I \to 0+$; and c.) the smoothness of $y_{0\pm}$, particularly, in c_R. Recall $U \in C^{l_0}$, we prove $y_{0\pm}$ is C^{l_0-3} in c_R except at $c_R = U(-h), U(0)$. Due to the analyticity of $y_\pm$ in c with $c_I > 0$, the estimates of $y_{0\pm}$ also yield those of $y_\pm$ for $c_I > 0$. Eventually general solutions to the non-homogeneous boundary value problem (2.2.6) of the Rayleigh equation are expressed using $y_\pm$. Finally, the quantity $Y(k, c) = \partial_{x_2} y_-(k, c, 0)/y_-(k, c, 0)$ related to the Reynolds stress is carefully studied, which plays an important role in the analysis of the Rayleigh equation. This chapter is a little lengthy, but we believe the studies on the Rayleigh equation could be widely useful for various purposes.

In chapter 4 we prove the results on the eigenvalue distribution based on the detailed analysis in chapter 3. We first obtain $c^\pm(k)$ for $|k| \gg 1$, followed by an argument based on analytic continuation and index calculation. Bifurcations may occur at inflection values of U and particularly subtle at $c = U(-h)$, which are on the boundary of the analyticity of the bifurcation equation $F(k,c) = 0$. The regularity obtained in chapter 3 implies, when restricted to $c_I \geq 0$, $F \in C^{l_0-3}$ at $c \in U\big((-h,0)\big)$ and $F \in C^{1,\alpha}$ at $c = U(-h)$. This makes the bifurcation analysis possible near $c = U(-h)$ and much easier even in the relatively classical case near inflection values of U.

Among the results in Theorem 2.1.1, in statement (1), $c^\pm(k)$ are obtained for large $|k|$ in Lemma 4.1.2(3) with more detailed estimates, the extension of $c^\pm(k)$ in Corollary 4.1.3.1, and the semi-simplicity of the eigenvalues $-ikc^\pm(k)$ in Lemma 4.1.2(3), Proposition 4.1.4, Corollary 4.1.3.1, and Corollary 6.1.2.1. Under the additional assumption of non-existence of singular modes, the non-existence of other non-singular modes is proved in Proposition 4.1.4. The analysis of the critical points of $c^\pm(k)$ is given in Lemma 4.1.7. The conjugacy to the linearized irrotational waves is proved in Proposition 6.3.2. See also Remark 6.3.1. With the strong surface tension assumption (2.1.1) in statement (2), the existence of $g_\#$ is proved in Lemma 4.1.6, along with the existence of $k_\#$ and/or $k_\#^\pm$ in statement (3). The rest of statement (3) is proved at the end of section 4.2 after a series of lemmas. Statement (4) is proved in section 4.3 with more details.

2.1.2 Motivation from the Couette flow

Before stating the main theorems we describe two main relevant properties using the Couette flow $U(x_2) = x_2$ as an illustration. The linearized velocity can be decomposed uniquely into the rotational and irrotational/potential parts (see e.g. [54])

$$v = v^{ir} + v^{rot}, \text{ where } \nabla \cdot v^{ir,rot} = 0,$$

where

$$v^{ir} = \nabla\varphi, \quad \Delta\varphi = 0, \ \ x_2 \in (-h,0), \text{ and } \partial_{x_2}\varphi|_{x_2=-h} = 0,$$

and v^{rot} satisfies

$$\nabla \cdot v^{rot} = 0, \quad v_2^{rot}|_{x_2=-h,0} = 0.$$

In particular, the rotational part can almost be determined by the vorticity ω in the *same* way as in the Euler equation (1.3.1) in the fixed channel $x_2 \in (-h, 0)$ with slip boundary condition

$$v^{rot} = (-\partial_{x_2}, \partial_{x_1})^T \Delta^{-1}\omega + (a, 0)^T, \text{ and } \omega = \nabla \times v = \partial_{x_1} v_2 - \partial_{x_2} v_1, \tag{2.1.2}$$

where a is a constant and Δ^{-1} is the inverse Lapacian in the 2-d region $x_2 \in (-h, 0)$ (L-periodic in x_1 or $x_1 \in \mathbb{R}$) under the zero Dirichlet boundary condition along $x_2 = -h, 0$. In the periodic-in-x_1 case, the constant a may be non-zero and is determined by the physical quantity circulation.

I. *Inviscid damping.* From the 2-d Euler equation (1.1.1a), one often also consider the corresponding vorticity formulation

$$\partial_t \omega + v \cdot \omega = 0. \tag{2.1.3}$$

Linearizing it at $\omega_* = -1$ which is the vorticity of the Couette flow yields the linearized vorticity

$$\omega(t, x) = \omega_0(x_1 - x_2 t, x_2)$$

expressed in term of its initial value ω_0. Since v^{rot} component of the linearized capillary gravity waves (1.2.2) at the Couette flow corresponds to the divergence free velocity field determined by its vorticity ω by (2.1.2) which is the same way as in the fixed boundary problem of the channel flow, the inviscid damping (1.3.4) of the latter (in the periodic-in-x_1 case) implies

$$\Big|v^{rot} - \frac{1}{L}\Big(\int_{-\frac{L}{2}}^{\frac{L}{2}} v_1 dx_1\Big)\vec{e}_1\Big|_{L^2} \le C(1+|t|)^{-1}|\omega_0|_{H^2}, \quad |v_2^{rot}|_{L^2} \le C(1+|t|)^{-2}|\omega_0|_{H^2}.$$

II. *Singular and non-singular modes.* Unlike the Euler equation in a fixed channel, there is the additional surface profile η coupled to the irrotational part v^{ir} of the velocity, which may not decay. In fact, for any $k \in \mathbb{R}$, let

$$v(t,x) = (1+k^2)^{\frac{1}{4}} e^{ik(x_1 - c^{\pm}(k)t)) - |k|h} \big(i \cosh k(x_2+h), \sinh k(x_2+h)\big) + c.\,c.$$

$$\eta(t,x_1) = i(1+k^2)^{\frac{1}{4}} e^{ik(x_1 - c^{\pm}(k)t)) - |k|h} \sinh kh/(kc^{\pm}(k)) + c.\,c.\,,$$

$$\begin{aligned}
p(t,x) =& i(1+k^2)^{\frac{1}{4}} e^{ik(x_1 - c^{\pm}(k)t)) - |k|h} \Big((g+\sigma k^2) \frac{\sinh kh}{kc^{\pm}(k)} \\
& - k \int_0^{x_2} (x_2' - c^{\pm}(k)) \sinh k(x_2'+h) dx_2' \Big) + c.\,c. \\
=& i(1+k^2)^{\frac{1}{4}} e^{ik(x_1 - c^{\pm}(k)t)) - |k|h} \Big((g+\sigma k^2) \frac{\sinh kh}{kc^{\pm}(k)} - (x_2 - c^{\pm}(k)) \cosh k(x_2+h) \\
& - c^{\pm}(k) \cosh kh + k^{-1} (\sinh k(x_2+h) - \sinh kh) \Big) + c.\,c.
\end{aligned}$$

where "c.c." denotes "complex conjugates" and

$$\begin{aligned}
& c^{\pm}(k) = \frac{-1 \pm \sqrt{1 + 4k(g+\sigma k^2)\coth kh}}{2k \coth kh} \\
\Longrightarrow\ & F(k,c) = c^2 k \coth kh + c - (g + \sigma k^2) = 0.
\end{aligned} \tag{2.1.4}$$

Even though we write down these formulas based on Lemma 2.2.1 in the below, it is straight forward to verify that they are solutions to ((1.2.2a)–(1.2.2d)) for the Couette flow. Therefore $-ikc^{\pm}(k)$ are eigenvalues of the linearized systems associated with the above eigenfunctions. These solutions do not grow or decay as $t \to \infty$, often referred to as neutral modes.

It is worth paying slightly closer attention to the wave speed $c^{\pm}(k)$ and the function $F(k,c)$, all of which are even in k. We make the following observations.

1. $\lim_{k\to\infty} c^{\pm}(k)/(\sigma|k|)^{\frac{1}{2}} = \pm 1$, so for $|k| \gg 1$ the dispersion relation $kc^{\pm}(k)$ is asymptotic to those of the irrotational capillary gravity waves linearized at zero so-

lution (system (1.2.2) with $U \equiv 0$ and $\nabla \times v \equiv 0$) given by $-kc_{ir}^{\pm}$ with

$$c_{ir}^{\pm}(k) = \pm\sqrt{k^{-1}(g+\sigma k^2)\tanh kh}, \quad C^{-1} \le |c_{ir}^{\pm}(k)| \le C(1+k^2)^{-\frac{1}{4}}, \quad (2.1.5)$$

which can be obtained through direct calculation based on Fourier transform.

2. $c^+(k) > 0$ for all $k \in \mathbb{R}$, so it is a branch of non-singular neutral modes, namely, wave speeds outside $[-h, 0]$, the range of U.

3. While $c^-(k) < -h$ in (2.1.4) as seen in the above observation (1) for large k, it can be happen $c^-(k) \in [-h, 0]$ for $0 < g, \sigma \ll 1$ and thus becomes singular modes (those in the range of U).

4. Since $k \coth kh \ge h^{-1}$ with "=" achieved at $k = 0$, for $g, \sigma \gg 1$, $c^{\pm}(k) \sim c_{ir}^{\pm}(k) = \sqrt{\frac{g+\sigma k^2}{k \coth kh}}$ and thus both $c^{\pm}(k) \notin [-h, 0]$ are non-singular modes. Moreover, one may verify $\frac{d}{dk}|c^{\pm}(k)| > 0$ for all $k > 0$ if $\sigma \gg g \gg 1$. In particular, in the case of $x_1 \in \mathbb{R}$, this implies that a.) the dispersion relations $k \to -kc^{\pm}(k)$ determine a linear dispersive wave system formed by the superposition of these non-singular modes and b.) this dispersive system is conjugate to the irrotational capillary gravity waves linearized at zero, whose the wave speed is given by (2.1.5). The conjugacy isomorphism can be constructed by associating the modes $k_1^{\pm}$ of (2.1.4) and $k_2^{\pm}$ of (2.1.5) if they have the same temporal frequency $k_1^{\pm}c^{\pm}(k_1^{\pm}) = k_2^{\pm}c_{ir}^{\pm}(k_2^{\pm})$. Moreover, $-ikc^{\pm}(k)$ would turn out to the only eigenvalues for the linearization at the Couette flow for $g, \sigma \gg 1$ (see Proposition 4.1.4(2)).

Generalization to general shear flow $U(x_2)$? From the above discussion, one sees that solutions to the capillary gravity water waves linearized at the Couette flow exhibit inviscid damping in their rotational parts while there are infinite non-singular modes with irrotational eigenfunctions determined by two branches of dispersion relations. However, several complications arise in the linearization at a general shear flow $U(x_2)$ including at least the following.

- The crucial function $F(k,c)$ defined in (2.1.4) which determines the wave speed c and consequently the dispersion relations, while analytic for all $c \in \mathbb{C} \setminus U([-h,0])$, may become rather singular for c approaching $U([-h,0])$. What regularity of $F(k,c)$ can one expect?

- Consequently, if a branch of non-singular modes approaches $U([-h,0])$, possibly very subtle bifurcations may occur at the boundary of analyticity of F. Can instability be generated?

- The inviscid damping (still of the rotational parts?) becomes much more involved, even in the case of the channel flow (see e.g. [61] Zillinger, Jia, ...).

In this thesis, we address these issues, with some results even more explicit and detailed than the above, through careful analysis starting at rather fundamental level under reasonable assumptions.

2.1.3 Linear inviscid damping

In this subsection, assuming there are no singular modes, we present the theorems on the splitting and linear inviscid damping of linearized system (1.2.2) of the capillary-gravity water wave problem (1.1.1) at the shear flow (v_*, S_*, p_*). See Definition 2.2.1 Lemma 4.1.1(5), (4.1.5), and Remark 4.1.1 for singular and non-singular modes. According to Theorem 2.1.1(3e), (2.1.1) combined with $U'' \neq 0$ is sufficient to rule out singular modes. In this case, we shall prove that any linear solution (v, η) to (1.2.2) can be decomposed into the parts (v^p, η^p) corresponding to the non-singular modes and (v^c, η^c) to the essential spectra due to $U([-h,0])$. This splitting is invariant under (1.2.2) and (v^c, η^c) is of the order $O(|t|^{-1})$ (and the vertical component $v_2^c = O(t^{-2})$) as $|t| \to \infty$. In fact, we identify their asymptotic leading order terms so that the remainders decays even faster. These leading order terms are in the form of horizontal translations of three functions Ω^c, Λ_B, and

Λ_T, which represent the contributions from the interior vorticity and the bottom and top boundary conditions. Their Fourier transforms are given explicitly in (6.1.9), (6.1.16), and (6.1.15), respectively, using the initial vorticity ω_0, the fundamental solutions $y_\pm(k, c, x_2)$ to the homogeneous Rayleigh equation, and Ω^c also by the Laplace transform of v_2. The results are stated for the cases of $x_1 \in \mathbb{T}_L$ and $x_1 \in \mathbb{R}$ separately in the following.

Theorem 2.1.2. *(***Inviscid damping: periodic-in-**x_1 **case***) Suppose $x_1 \in \mathbb{T}_L$. Assume $U \in C^{l_0}$, $l_0 \geq 3$, $U' > 0$ on $[-h, 0]$, and there are no singular modes (see (4.1.5) and Lemma 4.1.1(5)) for any $k \in \frac{2\pi}{L}\mathbb{N}$. For any $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$, $\epsilon > 0$, $n_1 \in \mathbb{R}$, and integers $n_0 \geq 0$, there exists $C > 0$ depending only on q_1, q_2, ϵ, and U, such that, for any solution $(v(t, x), \eta(t, x_1))$ of (1.2.2) with initial value $(v_0(x), \eta_0(x_1))$ and the corresponding initial vorticity $\omega_0(x)$, there exist unique solutions $(v^\dagger(t, x), \eta^\dagger(t, x_1))$, $\dagger = p, c$, to (1.2.2) and functions $\Omega^c(x)$, $\Lambda_B(x)$, and $\Lambda_T(x)$ determined by (v_0, η_0) linearly (depending on U as well) such that*

$$(v, \eta) = (v^c, \eta^c) + (v^p, \eta^p)$$

and the following hold.

1. *Assume $U \in C^4$, then (v^c, η^c) satisfy the following estimates*

$$\begin{aligned}|\partial_t^{n_0} v^c|_{H_{x_1}^{n_1} L_{x_2}^2 L_t^{q_1}(\mathbb{R})} \leq & C\big(|\eta_0|_{H_{x_1}^{n_0+n_1+\frac{1}{2}-\frac{1}{q_1}}} + |v_{10}(\cdot, 0)|_{H_{x_1}^{n_0+n_1-\frac{3}{2}-\frac{1}{q_1}}} \\ & + |\omega_0|_{H_{x_1}^{n_0+n_1-\frac{1}{2}-\frac{1}{q_1}+\epsilon} L_{x_2}^2}\big),\end{aligned}$$

$$\begin{aligned}|\partial_t^{n_0} \eta^c|_{H_{x_1}^{n_1} L_t^{q_1}(\mathbb{R})} \leq & C\big(|\eta_0|_{H_{x_1}^{n_0+n_1-1-\frac{1}{q_1}}} + |v_{10}(\cdot, 0)|_{H_{x_1}^{n_0+n_1-2-\frac{1}{q_1}}} \\ & + |\omega_0|_{H_{x_1}^{n_0+n_1-2-\frac{1}{q_1}+\epsilon} L_{x_2}^2}\big),\end{aligned}$$

$$\big|t\partial_t^{n_0} v_2^c\big|_{H_{x_1}^{n_1-\frac{3}{2}} L_{x_2}^2 L_t^{q_1}(\mathbb{R})} + |t\partial_t^{n_0} \eta^c|_{H_{x_1}^{n_1} L_t^{q_1}(\mathbb{R})}$$

$$\leq C\big(|\eta_0|_{H_{x_1}^{n_0+n_1-1-\frac{1}{q_1}}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_0+n_1-3-\frac{1}{q_1}}} + |\omega_0|_{H_{x_1}^{n_0+n_1-2-\frac{1}{q_1}+\epsilon}L^2_{x_2}}$$
$$+ |\partial_{x_2}\omega_0|_{H_{x_1}^{n_0+n_1-3-\frac{1}{q_1}+\epsilon}L^2_{x_2}}\big),$$

$$\big|\partial_t^{n_0}\big(tv_1^c - U'(x_2)^{-1}\partial_{x_1}^{-1}\Omega^c(x_1-U(x_2)t,x_2)\big)\big|_{H_{x_1}^{n_1}L^2_{x_2}L_t^{q_2}(\mathbb{R})}$$
$$+ \big|\partial_t^{n_0}\big(\omega^c - \Omega^c(x_1-U(x_2)t,x_2)\big)\big|_{H_{x_1}^{n_1-1}L^2_{x_2}L_t^{q_2}(\mathbb{R})}$$
$$+ \big|\partial_t^{n_0}\big(\partial_{x_2}^2 v_2^c - \partial_{x_1}\Omega^c(x_1-U(x_2)t,x_2)\big)\big|_{H_{x_1}^{n_1-2}L^2_{x_2}L_t^{q_2}(\mathbb{R})}$$
$$\leq C\big(|\eta_0|_{H_{x_1}^{n_0+n_1+\frac{1}{2}-\frac{1}{q_2}}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_0+n_1-\frac{3}{2}-\frac{1}{q_2}}} + |\omega_0|_{H_{x_1}^{n_0+n_1-\frac{1}{2}-\frac{1}{q_2}+\epsilon}L^2_{x_2}}$$
$$+ |\partial_{x_2}\omega_0|_{H_{x_1}^{n_0+n_1-\frac{3}{2}-\frac{1}{q_2}+\epsilon}L^2_{x_2}}\big),$$

and if, in addition, $U \in C^6$, then

$$\big|\partial_t^{n_0}\big(t^2 v_2^c - U'(x_2)^{-2}\partial_{x_1}^{-1}\Omega^c(x_1-U(x_2)t,x_2) - \Lambda_B(x_1-U(-h)t,x_2)$$
$$- \Lambda_T(x_1-U(0)t,x_2)\big)\big|_{H_{x_1}^{n_1}L^2_{x_2}L_t^{q_2}(\mathbb{R})}$$
$$\leq C\big(|\eta_0|_{H_{x_1}^{n_0+n_1+\frac{1}{2}-\frac{1}{q_2}}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_0+n_1-\frac{3}{2}-\frac{1}{q_2}}} + |\omega_0|_{H_{x_1}^{n_0+n_1-\frac{1}{2}-\frac{1}{q_2}+\epsilon}L^2_{x_2}}$$
$$+ |\partial_{x_2}\omega_0|_{H_{x_1}^{n_0+n_1-\frac{3}{2}-\frac{1}{q_2}+\epsilon}L^2_{x_2}} + |\partial_{x_2}^2\omega_0|_{H_{x_1}^{n_0+n_1-\frac{5}{2}-\frac{1}{q_2}+\epsilon}L^2_{x_2}}\big).$$

2. *Assume $U \in C^4$, then for any $n_2 = 0, 1$, and $q \in [1,\infty)$, it holds*

$$|\Omega^c - \omega_0|_{H_{x_1}^{n_1}L^2_{x_2}} \leq C\big(|\eta_0|_{H_{x_1}^{n_1}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_1-2}} + |\omega_0|_{H_{x_1}^{n_1-1+\epsilon}L^2_{x_2}}\big),$$

$$|\partial_{x_2}\Omega^c - \partial_{x_2}\omega_0|_{H_{x_1}^{n_1}L^2_{x_2}} \leq C\big(|\eta_0|_{H_{x_1}^{n_1+1}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_1-1}} + |\omega_0|_{H_{x_1}^{n_1+\epsilon}L^2_{x_2}}$$
$$+ |\partial_{x_2}\omega_0|_{H_{x_1}^{n_1-1+\epsilon}L^2_{x_2}}\big),$$

$$||k|^{n_1}\partial_{x_2}^{n_2}\hat{\Lambda}_B|_{l_k^2 L_{x_2}^q} \le C|\omega_0(\cdot,-h)|_{H_{x_1}^{n_1+n_2-1-\frac{1}{q}}}, \quad ||k|^{n_1}\partial_{x_2}^{n_2}\hat{\Lambda}_T|_{l_k^2 L_{x_2}^q}$$
$$\le C(|\omega_0(\cdot,0)|_{H_{x_1}^{n_1+n_2-1-\frac{1}{q}}} + |\eta_0|_{H_{x_1}^{n_1+n_2-1-\frac{1}{q}}}),$$

where $\hat{f}(k, x_2)$ denotes the Fourier transform of a function $f(x_1, x_2)$ with respect to x_1. Moreover, $\Lambda_\dagger$, $\dagger = B, T$, satisfy $\hat{\Lambda}_\dagger(k = 0, x_2) = 0$ and

$$\begin{cases} -(U - U(0))\Delta\Lambda_T + U''\Lambda_T = 0, & x_2 \in (-h, 0), \\ \Lambda_T(x_1, -h) = 0, \qquad \partial_{x_1}\Lambda_T(x_1, 0) = U'(0)^{-2}\big(U''(0)\eta_0(x_1, 0) - \omega_0(x_1, 0)\big); \end{cases} \tag{2.1.6a}$$

$$\begin{cases} -(U - U(-h))\Delta\Lambda_B + U''\Lambda_B = 0, & x_2 \in (-h, 0), \\ \partial_{x_1}\Lambda_B(\cdot, -h) = -U'(-h)^{-2}\omega_0(x_1, -h), \\ \big(U(0) - U(-h)\big)\partial_{x_2}\Lambda_B(x_1, 0) - \big(U'(0)(U(0) - U(-h)) + g - \sigma\partial_{x_1}^2\big)\Lambda_B(x_1, 0) \\ \qquad = 0. \end{cases} \tag{2.1.6b}$$

3. *There exist $\lambda_0 \ge 0$ and integer $N \ge 0$ (given in (6.2.3)) such that, for any $n_1 \in \mathbb{R}$ and $n_2 \in [1, l_0]$,*

$$\big|\partial_{x_1}^{n_1+1}\big(v_1^p(t,\cdot) - \hat{v}_{10}(k=0,\cdot)\big)\big|_{L_{x_1}^2 H_{x_2}^{n_2-1}} + |\partial_{x_1}^{n_1} v_2^p(t,\cdot)|_{L_{x_1}^2 H_{x_2}^{n_2}}$$
$$\le C e^{\lambda_0|t|}(1 + |t|^{N-1})\big(|\eta_0|_{H_{x_1}^{n_1+n_2+1}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_1+n_2-\frac{1}{2}}} + |\omega_0|_{H_{x_1}^{n_1+n_2-1}L_{x_2}^2}\big),$$

$$|\eta^p(t,\cdot) - \eta_0(0)|_{H_{x_1}^{n_1}} \le C e^{\lambda_0|t|}(1 + |t|^{N-1})\big(|\eta_0|_{H_{x_1}^{n_1}} + |v_{10}(\cdot,0)|_{H_{x_1}^{n_1-\frac{3}{2}}} + |\omega_0|_{H_{x_1}^{n_1-2}L_{x_2}^2}\big).$$

4. *Let*

$$\mathbf{X}^\dagger = \{(v^\dagger, \eta^\dagger)|_{t=0} \mid \textit{ all } (v_0, \eta_0)\} \subset H^1\big(\mathbb{T}_L \times (-h, 0)\big) \times H^2(\mathbb{T}_L), \quad \dagger = c, p,$$

then they are invariant closed subspaces of $H^1\big(\mathbb{T}_L \times (-h,0)\big) \times H^2(\mathbb{T}_L)$ under (1.2.2). Moreover (1.2.2) is also well-posed in the $L^2 \times H^1$ completion of $\mathbf{X}^p$.

Remark 2.1.2. *1.) The above estimates also imply pointwise-in-t decay of v and η as $t \to \infty$. 2.) The function $\Omega^c(x)$ is referred to as the scattering limit of the vorticity in [70, 61, 35]. 3.) The assumption of non-existence of singular modes is satisfied if the horizontal period L is small (by Theorem 2.1.1(1)) or if $U'' \neq 0$ and (2.1.1) (by Theorem 2.1.1(2b)).*

In the above results, the assumption of the non-existence of singular modes, which is equivalent to the absence of embedded eigenvalues of (1.2.2) for each wave number k, turns out to yield the spectral decomposition of the phase space of (1.2.2) into the invariant subspaces corresponding to the non-singular modes/point spectra and the continuous spectra $-ikU([-h,0])$ for each $k \in \mathbb{R}$.

The component (v^c, η^c) corresponds to the continuous spectra and enjoys algebraic decay as in the case (1.3.1) of the Euler equation in a fixed channel. Additional to the above L_t^q bounds, derivatives-in-t estimates are also given in Theorem 2.1.2 and Theorem 2.1.3 which also imply pointwise-in-t decay. Compared with (1.3.4), these additional L_t^q estimates represent an improvement of roughly an order of $O(t^{-\frac{1}{q}})$ (after appropriate time-dependent translations in x_1 of some asymptotic leading terms are identified and singled out in the cases of tv_1^c, $t^2v_2^c$, *etc.*). For the Euler equation in a fixed channel (1.3.1), a.) when $\omega_0 \in L^2$, the $|v|_{L_t^2}$ estimates was also obtained in [62, 64]; b.) comparable asymptotic leading terms were identified for the vorticity $\omega(t,x)$ in Lemma 3 of [70]; and c.) asymptotic leading terms were obtained for $v(t,x)$ in Lemma 5.1 in [35]. The Fourier transforms (in x_1) of these leading terms Ω^c, Λ_T, and Λ_B are given explicitly in (6.1.9), (6.1.16), and (6.1.15), which represent the impact of the interior flow and the top and bottom boundaries, respectively. See also (2.1.6) for singular elliptic boundary value problems satisfied by Λ_T and Λ_B. In particular, the free boundary effect is explicitly reflected in the boundary conditions (2.2.6c) of the corresponding Rayleigh equation (2.2.6) and the form of Λ_T.

From (6.1.16) and (6.1.15), (4.0.1), and Lemma 3.6.1(2), $\hat{\Lambda}_{B,T}(0,x_2) = 0$ and the elliptic boundary value problem (2.1.6b) has a unique solution Λ_B, while (2.1.6a) has a unique solution Λ_T under the assumption of the non-existence of singular modes. Moreover, according to the definitions (6.1.15), (4.0.1), (3.3.1), (3.5.1), and Lemma 3.4.1, $\partial_{x_2}\Lambda_B$ and $\partial_{x_2}\Lambda_T$ exhibit logarithmic singularity at $x_2 = -h$ and 0, respectively. In particular, $\Lambda_B = 0$ vanishes if the initial vorticity $\omega_0|_{x_2=-h} = 0$, while $\Lambda_T = 0$ if $U''(0)\eta_0 - \omega_0|_{x_2=0} = 0$. The error estimates in addition to these leading asymptotic terms also justify that the estimates of tv_1 and t^2v_2 in (1.3.4) are optimal. Moreover, the precise asymptotic leading terms could also be useful for further anaysis.

It is also interesting to observe that, on the one hand, in the upper bounds of the damping estimates the regularity assumption on the initial surface η_0 and surface velocity $v_{10}|_{x_2=0}$ remains unchanged while faster decay requires more regularity on the initial vorticity ω_0. On the other hand, in the higher regularity estimates of (v^p, η^p), the requirement on the regularity of ω_0 in x_2 remains in $L^2_{x_2}$. Compared with the above example of the linearization at the Couette flow, conceptually this phenomenon is due to the fact that the component (v^c, η^c) is mainly the rotational part of the solution which depends on the vorticity more heavily, while (v^p, η^p) more like the irrotational part. In this paper as we focus on the damping estimates with additional L^q_t decay of (v, η) after the leading order terms are singled out, we adopted L^2_x based norms to somewhat simplify the calculations. If the decay in other L^r_x or L^∞_x based norms is necessary, some basic estimates in these norms are also given in section 5.1 and one may make an attempt following the procedure as in chapter 5 and chapter 6. To avoid more technicality, the assumptions on the regularity of ω_0 in x_1 in the theorem may not be close to optimal, particularly when q_1 and q_2 are away from 2, see Remark 6.1.2(b). Moreover, the small ϵ may not be necessary, see e.g. [62, 64] in the fixed boundary case. The assumptions on the more essential regularity of ω_0 in x_2 are optimal even in the existing results in the fixed boundary case.

The component (v^p, η^p) are given by superpositions of the eigenfunctions of those non-singular modes, which is governed by a (possibly unstable) multi-branched dispersion relation given by $k \to kc$ for all non-singular modes c of the k-th Fourier modes in x_1. According to the above eigenvalue analysis, this dispersion relation is asymptotic to that of the linear irrotational capillary gravity wave for $|k| \gg 1$. In the case of $x_1 \in \mathbb{R}$, in the absence of singular modes, all non-singular modes are given by $c^{\pm}(k)$ which are neutral/stable. Under the additional assumptions (2.1.1) and $0 \in U([-h, 0])$, the conjugacy of (v^p, η^p) to the linear irrotational capillary gravity waves implies that it decays at a slower rate. Hence the dynamics of (1.2.2) has two layers: *faster inviscid decay of* (v^c, η^c) *leaves the remaining* (v^p, η^p) *decaying at a slower rate due to the dispersion like a linear irrotational wave.*

In the periodic-in-x_1 case, as the non-existence of singular modes is assumed only for $k \in \frac{2\pi}{L}\mathbb{N}$, there can still be other non-singular modes besides $c^{\pm}(k)$ which may have bifurcated from $U(-h)$ or inflection values of $U([-h, 0])$ at some $k \notin \frac{2\pi}{L}\mathbb{N}$. In particular instability may appear in finitely many dimensions in low wave numbers. In the estimates of the component (v^p, η^p), the possible exponential growth (if $\lambda_0 > 0$) is caused by unstable modes, where λ_0 is the maximum real parts of the eigenvalues and N is the maximum multiplicity of those eigenvalues of the maximal real parts. Due to Theorem 2.1.1(1), growth does not occur for $|k| \gg 1$. It is also worth pointing out that the the regularity of η^p is $\frac{3}{2}$ order better than that of v^p restricted to the surface $x_2 = 0$, which is consistent with the regularity results of capillary gravity waves in the existing literature.

The estimate in statement (3) at $t = 0$ implies the boundedness of the projection onto $\mathbf{X}^p$, whose kernal is $\mathbf{X}^c$. Some more detailed information of this projection can be found in Lemma 6.1.2 and Theorem 6.2.1. In fact the subspace $\mathbf{X}^p$ is generated by the eigenfunction of all non-singular modes for all $k \in \mathbb{R}$.

The inviscid decay estimates in the case of $x_1 \in \mathbb{R}$ is slightly subtle due to the presence of small wave number $|k| \ll 1$. Certain stronger decay for $|k| \ll 1$ (for long waves) is assumed on the initial values, see Remark 2.1.3. We use similar notations in the following theorem.

Theorem 2.1.3. *(***Inviscid damping:** $x_1 \in \mathbb{R}$ **case***) Suppose $x_1 \in \mathbb{R}$. Assume $U \in C^{l_0}$, $l_0 \geq$*

3, $U' > 0$ on $[-h, 0]$, and there are no singular modes (see (4.1.5) and Lemma 4.1.1(5)) for any $k \in \mathbb{R}$. For any $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$, $\epsilon > 0$, $n_1 \in \mathbb{R}$, and integers $n_0 \geq 0$, there exists $C > 0$ depending only on q_1, q_2, ϵ, and U, such that, for any solution $(v(t,x), \eta(t,x_1))$ of (1.2.2) with initial value $(v_0(x), \eta_0(x_1))$, there exist solutions $(v^\dagger(t,x), \eta^\dagger(t,x_1))$, $\dagger = p, c$, to (1.2.2) and functions $\Omega^c(x)$, $\Lambda_B(x)$, and $\Lambda_T(x)$ determined by (v_0, η_0) linearly (depending on U as well) such that

$$(v, \eta) = (v^c, \eta^c) + (v^p, \eta^p)$$

and the following hold.

1. *Assume $U \in C^4$, then (v^c, η^c) satisfy the following estimates*

$$\begin{aligned}
&|\partial_t^{n_0}\partial_{x_1}^{n_1} v_1^c|_{L_x^2 L_t^{q_1}(\mathbb{R})} + |\partial_t^{n_0}\partial_{x_1}^{n_1-1}(1-\partial_{x_1}^2)^{\frac{1}{2}} v_2^c|_{L_x^2 L_t^{q_1}(\mathbb{R})} \\
&\leq C\big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\eta_0\big|_{H_{x_1}^{\frac{1}{2}}} + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}} v_{10}(\cdot,0)\big|_{H_{x_1}^{-\frac{3}{2}}} \\
&+ \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\omega_0\big|_{H_{x_1}^{\epsilon-\frac{1}{2}} L_{x_2}^2}\big)
\end{aligned}$$

$$\begin{aligned}
|\partial_t^{n_0}\partial_{x_1}^{n_1}\eta^c|_{L_{x_1}^2 L_t^{q_1}(\mathbb{R})} \leq & C\big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\eta_0\big|_{H_{x_1}^{-1}} + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}} v_{10}(\cdot,0)\big|_{H_{x_1}^{-2}} \\
&+ \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\omega_0\big|_{H_{x_1}^{\epsilon-2} L_{x_2}^2}\big)
\end{aligned}$$

$$\begin{aligned}
&\big|t\partial_t^{n_0}\partial_{x_1}^{n_1}(1-\partial_{x_1}^2)^{\frac{1}{2}} v_2^c\big|_{L_x^2 L_t^{q_1}(\mathbb{R})} + |t\partial_t^{n_0}\partial_{x_1}^{n_1+1}(1-\partial_{x_1}^2)^{\frac{3}{4}}\eta^c|_{L_{x_1}^2 L_t^{q_1}(\mathbb{R})} \\
\leq & C\big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\eta_0\big|_{H_{x_1}^{\frac{3}{2}}} + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}} v_{10}(\cdot,0)\big|_{H_{x_1}^{-\frac{1}{2}}} \\
&+ \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\omega_0\big|_{H_{x_1}^{\epsilon+\frac{1}{2}} L_{x_2}^2} + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\partial_{x_2}\omega_0\big|_{H_{x_1}^{\epsilon-\frac{1}{2}} L_{x_2}^2}\big)
\end{aligned}$$

$$\begin{aligned}
&\big|\partial_t^{n_0}\partial_{x_1}^{n_1+1}\big(tv_1^c - U'(x_2)^{-1}\partial_{x_1}^{-1}\Omega^c(x_1 - U(x_2)t, x_2)\big)\big|_{L_x^2 L_t^{q_2}(\mathbb{R})} \\
&+ \big|\partial_t^{n_0}\partial_{x_1}^{n_1}\big(\omega^c - \Omega^c(x_1 - U(x_2)t, x_2)\big)\big|_{L_x^2 L_t^{q_2}(\mathbb{R})}
\end{aligned}$$

$$
\begin{aligned}
&+\big|\partial_t^{n_0}\partial_{x_1}^{n_1-1}\big(\partial_{x_2}^2 v_2^c-\partial_{x_1}\Omega^c(x_1-U(x_2)t,x_2)\big)\big|_{L^2_xL^{q_2}_t(\mathbb{R})}\\
\le& C\Big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\eta_0\big|_{H^{\frac32}_{x_1}}+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}v_{10}(\cdot,0)\big|_{H^{-\frac12}_{x_1}}\\
&+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\omega_0\big|_{H^{\epsilon+\frac12}_{x_1}L^2_{x_2}}+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\partial_{x_2}\omega_0\big|_{H^{\epsilon-\frac12}_{x_1}L^2_{x_2}}\Big)
\end{aligned}
$$

and if, in addition, $U\in C^6$, then

$$
\begin{aligned}
&\big|\partial_t^{n_0}\partial_{x_1}^{n_1+1}\big(t^2v_2^c-U'(x_2)^{-2}\partial_{x_1}^{-1}\Omega^c(x_1-U(x_2)t,x_2)-\Lambda_B(x_1-U(-h)t,x_2)\\
&\qquad\qquad-\Lambda_T(x_1-U(0)t,x_2)\big)\big|_{L^2_xL^{q_2}_t(\mathbb{R})}\\
\le& C\Big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\eta_0\big|_{H^{\frac32}_{x_1}}+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}v_{10}(\cdot,0)\big|_{H^{-\frac12}_{x_1}}\\
&+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\omega_0\big|_{H^{\epsilon+\frac12}_{x_1}L^2_{x_2}}+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\partial_{x_2}\omega_0\big|_{H^{\epsilon-\frac12}_{x_1}L^2_{x_2}}\\
&+\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_2}}\partial_{x_2}^2\omega_0\big|_{H^{\epsilon-\frac32}_{x_1}L^2_{x_2}}\Big).
\end{aligned}
$$

2. *Assume $U\in C^4$, then for any $n_2=0,1$, and $q\in[1,\infty)$, it holds*

$$
|\Omega^c-\omega_0|_{H^{n_1}_{x_1}L^2_{x_2}}\le C\big(|\eta_0|_{H^{n_1}_{x_1}}+|v_{10}(\cdot,0)|_{H^{n_1-2}_{x_1}}+|\omega_0|_{H^{n_1-1+\epsilon}_{x_1}L^2_{x_2}}\big),
$$

$$
\begin{aligned}
|\partial_{x_2}\Omega^c-\partial_{x_2}\omega_0|_{H^{n_1}_{x_1}L^2_{x_2}}\le& C\big(|\eta_0|_{H^{n_1+1}_{x_1}}+|v_{10}(\cdot,0)|_{H^{n_1-1}_{x_1}}+|\omega_0|_{H^{n_1+\epsilon}_{x_1}L^2_{x_2}}\\
&+|\partial_{x_2}\omega_0|_{H^{n_1-1+\epsilon}_{x_1}L^2_{x_2}}\big),
\end{aligned}
$$

$$
||k|^{n_1}\partial_{x_2}^{n_2}\hat\Lambda_B|_{L^2_kL^q_{x_2}}\le C|\partial_{x_1}^{n_1-1}\omega_0(\cdot,-h)|_{H^{n_2-\frac1q}_{x_1}},
$$

$$
||k|^{n_1}\partial_{x_2}^{n_2}\hat\Lambda_T|_{L^2_kL^q_{x_2}}\le C(|\partial_{x_1}^{n_1-1}\omega_0(\cdot,0)|_{H^{n_2-\frac1q}_{x_1}}+|\partial_{x_1}^{n_1-1}\eta_0|_{H^{n_2-\frac1q}_{x_1}}),
$$

and Λ_T and Λ_B satisfy (2.1.6).

3. *For any $n_1\in\mathbb{R}$,*

$\forall n_2 \in [0, l_0 - 1]$,

$$|\partial_{x_1}^{n_1}\partial_{x_2}^{n_2} v_1^p(t,\cdot)|_{L_x^2}^2 \le C\big(|\partial_{x_1}^{n_1}\eta_0|_{H_{x_1}^{n_2+1}}^2 + |\partial_{x_1}^{n_1} v_{10}(\cdot,0)|_{H_{x_1}^{n_2-\frac{1}{2}}}^2 + |\partial_{x_1}^{n_1}\omega_0|_{H_{x_1}^{n_2-1}L_{x_2}^2}^2\big),$$

$\forall n_2 \in [0, l_0]$,

$$\begin{aligned}|\partial_{x_1}^{n_1}\partial_{x_2}^{n_2} v_2^p(t,\cdot)|_{L_x^2}^2 \le & C\big(|\partial_{x_1}^{n_1+1}\eta_0|_{H_{x_1}^{n_2}}^2 + |\partial_{x_1}^{n_1+1} v_{10}(\cdot,0)|_{H_{x_1}^{n_2-\frac{3}{2}}}^2 \\ & + |\partial_{x_1}^{n_1+1}\omega_0|_{H_{x_1}^{n_2-2}L_{x_2}^2}^2\big),\end{aligned}$$

$$|\eta^p(t,\cdot)|_{\dot H_{x_1}^{n_1}}^2 \le C\big(|\eta_0|_{\dot H_{x_1}^{n_1}}^2 + |\partial_{x_1}^{n_1} v_{10}(\cdot,0)|_{H_{x_1}^{-\frac{3}{2}}}^2 + |\partial_{x_1}^{n_1}\omega_0|_{H_{x_1}^{-2}L_{x_2}^2}^2\big).$$

4. *Let*

$$\mathbf{X}^\dagger = \{(v^\dagger, \eta^\dagger)|_{t=0} \mid \textit{ all } (v_0, \eta_0)\} \subset H^1\big(\mathbb{R}\times(-h,0)\big) \times H^2(\mathbb{R}),\ \dagger = c, p,$$

then they are invariant closed subspaces of $H^1\big(\mathbb{R}\times(-h,0)\big)\times H^2(\mathbb{R})$ under (1.2.2). Moreover (1.2.2) is also well-posed in the $L^2\times H^1$ completion of $\mathbf{X}^p$. If, in addition, (2.1.1) holds and $0 \in U([-h,0])$, then (1.2.2) restricted to the $L^2\times H^1$ completion of $\mathbf{X}^p$, or $\mathbf{X}^p \cap (H^n \times H^{n+1})$ with $n \le l_0 - 1$, is conjugate through an isomorphism to the irrotational capillary gravity waves linearized at zero (characterized by its wave speed (2.1.5)).

Remark 2.1.3. *Taking $n_1 = n_2 = 0$ in the above estimates, the $|\partial_{x_1}|^{-\frac{1}{q}}$ applied to the initial values indicates some stronger decay assumptions for wave number $|k| \ll 1$.*

Outline of the Proofs of Theorem 2.1.2 and Theorem 2.1.3

The proof of Theorem 2.1.2 is completed in section 6.2. Under the assumption of the absence of singular modes, general solutions $y_B(k, c, x_2)$ to the non-homogeneous boundary value problem (2.2.6) of the Rayleigh equation are studied in chapter 5, which are expressed in the variation of parameter formula using $y_\pm$ obtained in chapter 3. We establish the basic a priori and convergence (as $c_I \to 0+$) estimates in section 5.1. The latter is often referred to as the limiting absorption principle (e.g. [62, 35]). For the inviscid damping estimates, it is crucial to obtain the smoothness of y_B in c (in section 5.2). Since singularity occurs along $c = U(x_2)$, $\partial_c^j y_B$, $j = 1, 2$, behaves badly there. Instead we apply a differential operator D_c to the Rayleigh system (2.2.6) which differentiates along the direction of $c = U(x_2)$, hence $D_c^j y_B$ satisfies another boundary value problem of the Rayleigh equation in the form of (2.2.6) and enjoys better estimates. Essentially this approach is similar to those used in [34, 64] for the Euler equation on fixed channels. The main results of section 5.2 are the estimates of $\partial_c^{j_1} \partial_{x_2} y_B$, $j_1 = 1, 2$ and $j_2 = 0, 1$, with the most singular terms identified.

The splitting and the linear inviscid damping estimates of solutions (v, η) to the linearized capillary gravity waves (1.2.2) are obtained in chapter 6. While the vorticity ω is not sufficient to recover the whole solution (as in e.g. [71, 61]), the solutions are expressed in terms of the inverse Laplace transform of $V_2(k, c, x_2)$, where V_2 is the Laplace transform of $\hat{v}_2(t, k, x_2)$, which is estimated in chapter 5. We use the following Mellin's inverse formula to compute the inverse Laplace transform (see [10]).

$$f(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} F(s) ds,$$

where γ is greater than the real part of all singularities of F. Unlike e.g. [61, 35], technically we do not immediately push the contour integral (in c) of the inverse Laplace transform to the limit spectra set $U([-h, 0])$, but first keep it along the boundary of a small neighborhood of it in the complex plane. This allows easy integration by parts in c to establish the decay estimates in t after the leading asymptotic terms are obtained by applying the Cauchy integral theorem to the most singular terms of $\partial_c^j V_2$, $j = 1, 2$. In fact, in deriving the decay estimates of v, η, tv_2, and $t\eta$ where the leading asymptotic terms were not involved, a priori estimates, but not the limiting absorption principle, is sufficient.

The above approach to obtain the inviscid decay also applies to the Euler equation in a fixed channel linearized at a shear flow $U(x_2)$. Similarly, while the asymptotic leading order terms of tv_1, ω, and $\partial_{x_2}^2 v_2$ are all generated by the asymptotic vorticity Ω^c, that of $t^2 v_2$ involves two additional functions Ω_T and Ω_B due to the contributions from the top and bottom boundaries. We give a brief summary of the results for the channel flow in section 6.4 and see also Remark 6.4.1.

The proof of Theorem 2.1.3 is completed in section 6.3. Most of the remarks after Theorem 2.1.2 are also valid. In particular, there are only two branches of non-singular modes corresponding to eigenvalues $ikc^{\pm}(k)$ of both algebraic and geometric multiplicity two, hence there is no growth at all. The conjugacy of the dynamics of (v^p, η^p) to the linear irrotational capillary gravity waves is basically a restatement of Theorem 2.1.1(2b).

2.2 Preliminary linear analysis

To analyze the linear system (1.2.2), we first reduce it to an evolution problem of the Fourier transform of v_2 in x_1, which in turn determines v_1, η, and p. We then apply the Laplace transform in t to obtain a non-homogeneous boundary value problem of the well-known Rayleigh equation in $x_2 \in (-h, 0)$ with a non-homogeneous Robin type boundary condition at $x_2 = 0$ due the boundary conditions at the free boundary. The main analysis will focus on the Rayleigh equation.

Consider the Fourier transforms of the unknowns $(v(t,x), \eta(t,x_1), p(t,x))$ in x_1

$$v(x) = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \hat{v}(k, x_2) e^{ikx_1}, \quad \eta(x_1) = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \hat{\eta}(k) e^{ikx_1}, \quad p(x) = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}} \hat{p}(k, x_2) e^{ikx_1},$$

in the case of $x_1 \in \mathbb{T}_L$ and

$$v(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{v}(k, x_2) e^{ikx_1} dk,$$
$$\eta(x_1) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{\eta}(k) e^{ikx_1} dk,$$

$$p(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{p}(k, x_2) e^{ikx_1} dk,$$

in the case of $x_1 \in \mathbb{R}$, where we skipped the variable t. The Fourier transform of the linearized system (1.2.2) takes the form

$$\begin{cases} \partial_t \hat{v} + ikU(x_2)\hat{v} + (ik\hat{p}, \hat{p}')^T = 0, \qquad ik\hat{v}_1 + \hat{v}_2' = 0, & x_2 \in (-h, 0) \\ (k^2 - \partial_{x_2}^2) p = 2ikU'(x_2)\hat{v}_2, & x_2 \in (-h, 0) \\ \partial_t \hat{\eta} = -ikU(0)\hat{\eta} + \hat{v}_2(t, k, x_2 = 0), \\ \hat{p}(t, k, 0) = (g + \sigma k^2)\hat{\eta}, \\ \hat{v}_2(t, k, -h) = 0, \quad \hat{p}'(t, k, -h) = 0, \end{cases} \tag{2.2.1}$$

where $'$ denotes the derivative with respect to x_2 as in the rest of the paper. Due to the divergence free condition on v and the boundary conditions, it is easy to see

$$\hat{v}_2(t, 0, x_2) = 0, \quad \hat{p}(t, 0, x_2) = g, \quad \hat{v}_1(t, 0, x_2) = v_{10}(0, x_2), \quad \hat{\eta}(t, 0) = \hat{\eta}_0(0). \tag{2.2.2}$$

For $k \neq 0$, $\hat{v}_1$ can also be determined by $\hat{v}_2$ using the divergence free condition, $\hat{\eta}$ by the third equation of (2.2.1), while $\hat{p}$ by $\hat{v}_2$ and $\hat{\eta}$ by solving the elliptic boundary value problem. So we shall mainly focus on $\hat{v}_2$.

Combining the equation of $\hat{v}_2$ acted by $k^2 - \partial_{x_2}^2$ and the one of $\hat{p}$ acted by ∂_{x_2}, we obtain

$$(\partial_t + ikU)(k^2 - \partial_{x_2}^2)\hat{v}_2 + ikU''\hat{v}_2 = 0, \quad x_2 \in (-h, 0), \tag{2.2.3a}$$

which is the linearized transport equation of the vorticity (as defined in (2.1.2))

$$\hat{\omega} = ik\hat{v}_2 - \hat{v}_1' = \tfrac{i}{k}(k^2 - \partial_{x_2}^2)\hat{v}_2$$

in its Fourier transform. In addition to the above equation, we need its boundary information to completely determine $\hat{v}_2$. Applying ∂_{x_2} to the first equation of (2.2.1), then evaluating at $x_2 = 0$, and using the equation of $\hat{p}$, we have

$$(\partial_t + ikU(0))\hat{v}_2'(t,k,0) - ikU'(0)\hat{v}_2(t,k,0) + k^2(g+\sigma k^2)\hat{\eta}(t,k) = 0.$$

Finally applying $\partial_t + ikU(0)$ to the above equation and using the third equation of ((2.2.1)), we obtain

$$\big((\partial_t + ikU)^2\hat{v}_2' - ikU'(\partial_t + ikU)\hat{v}_2 + k^2(g+\sigma k^2)\hat{v}_2\big)\big|_{x_2=0} = 0, \quad \hat{v}_2|_{x_2=-h} = 0, \tag{2.2.3b}$$

where we also included the boundary value of $\hat{v}_2$ at $x_2 = -h$.

To analyze the evolutionary problem, we apply the Laplace transform $\mathcal{L}$ to the unknowns

$$V(s) = (V_1(s), V_2(s)) := \mathcal{L}\{\hat{v}\}(s), \quad P(s) := \mathcal{L}\{p\}(s), \quad \tilde{\eta} := \mathcal{L}\{\hat{\eta}\}(s). \tag{2.2.4}$$

An often used change of variable for $k \neq 0$ is

$$c := is/k = c_R + ic_I \tag{2.2.5}$$

with c_R and c_I being the real and imaginary parts. From (2.2.3), our main unknown $V_2(k, c, x_2)$ satisfies the following non-homogeneous Rayleigh equation

$$-V_2'' + (k^2 + \frac{U''}{U-c})V_2 = \frac{(k^2 - \partial_{x_2}^2)\hat{v}_{20}}{ik(U-c)} = -\frac{\hat{\omega}_0}{U-c}, \qquad x_2 \in (-h, 0), \tag{2.2.6a}$$

with boundary condition

$$V_2(-h) = 0, \tag{2.2.6b}$$

where $\hat{\omega} = \hat{\omega}_0(k, x_2)$ is the Fourier transform of the initial vorticity and we skipped the k

and c variables of V_2. Similarly, the Laplace transform applied to the boundary equation (2.2.3b) and evaluated at $x_2 = 0$ imply

$$\begin{aligned}&\big((U-c)^2 V_2' - (U'(U-c) + (g+\sigma k^2))V_2\big)\big|_{x_2=0}\\ =&-\tfrac{1}{k^2}\big(\partial_t \hat{v}_2' - ick\hat{v}_2' + 2ikU\hat{v}_2' - ikU'\hat{v}_2\big)\big|_{t=x_2=0}\\ =&-\tfrac{1}{k^2}\big((\partial_t + ikU)\hat{v}_2' - ikU'\hat{v}_2 + ik(U-c)\hat{v}_2'\big)\big|_{t=x_2=0}.\end{aligned}$$

Therefore we obtain

$$\big((U-c)^2 V_2' - (U'(U-c)+(g+\sigma k^2))V_2\big)\big|_{x_2=0} = (g+\sigma k^2)\hat{\eta}_0 - \tfrac{i}{k}(U(0)-c)\hat{v}_{20}'(0), \tag{2.2.6c}$$

The last boundary condition can be viewed as the dispersion relation which is highly non-local. The Laplace transforms of V_1 and $\tilde{\eta}$ of $\hat{v}_1$ and $\hat{\eta}$ can be recovered from the divergence free condition and the third equation of (2.2.1)

$$V_1 = \tfrac{i}{k}V_2', \quad \tilde{\eta}(c,k) = \frac{V_2(c,k,0) + \hat{\eta}_0(k)}{ik\big(U(0)-c\big)}. \tag{2.2.7}$$

Hence in the most of the paper we shall focus on the non-homogeneous boundary value problem (2.2.6) of the Rayleigh equation. The main goals of the analysis are the eigenvalue distribution of linear system (1.2.2) and the inviscid damping of its solutions.

System (2.2.6) is a boundary value problem of a non-homogeneous second order ODE with coefficients analytic in $k \in \mathbb{R}$ and $c \in \mathbb{C} \setminus U([-h,0])$, so it has a unique solution analytic in k and c except for those (k,c) for which the corresponding homogeneous system of (2.2.6), where $\hat{v}_{20} = 0$ and $\hat{\eta}_0 = 0$, has non-trivial solutions. Such singular (k,c) also give the eigenvalues of (2.2.6) in the form of $-ick$. In fact we have the following lemma.

Lemma 2.2.1. *For $k \in \mathbb{R}\backslash\{0\}$, there exists a non-trivial solution $\big(c, V_2(x_2)\big)$ with $c \notin U([-h,0])$ to the corresponding homogeneous problem of (2.2.6) (namely, with $\hat{v}_{20} = 0$ and $\hat{\eta}_0 = 0$) if and only if $-ikc$ is an eigenvalue of the linearized capillary-gravity wave* *system (1.2.2) associated with the linear solution in the form of (1.3.2) given by*

$$
\begin{aligned}
&v_1(t,x) = \frac{i}{k}e^{ik(x_1-ct)}V_2'(x_2), \quad v_2(t,x) = e^{ik(x_1-ct)}V_2(x_2), \\
&\eta(t,x_1) = e^{ik(x_1-ct)}\frac{V_2(0)}{ik(U(0)-c)}, \\
&p(t,x) = e^{ik(x_1-ct)}\Big(\frac{g+\sigma k^2}{ik(U(0)-c)}V_2(0) - ik\int_0^{x_2}(U-c)V_2 dx_2'\Big).
\end{aligned}
$$

Proof. On the one hand, it is straight forward to verify that the above v, η, and p satisfy (1.2.2c), (1.2.2d), $\partial_{x_2}p|_{x_2=-h} = 0$, and $\nabla \cdot v = 0$. The Poisson equation of p in (1.2.2b) is a consequence of the linearized Euler equation in (1.2.2a), the v_2 equation of which is also easily verified. Hence we only need to consider the v_1 equation in (1.2.2a). In fact, that equation holds for the above (v, η, p) if

$$
-(U-c)V_2' + U'V_2 + \frac{g+\sigma k^2}{U(0)-c}V_2(0) + k^2\int_0^{x_2}(U-c)V_2 dx_2' = 0.
$$

The x_2-derivative of this function is equal to 0 due to the Rayleigh equation (2.2.6a) and its boundary value equal is to 0 at $x_2 = 0$ due to the boundary condition (2.2.6c).

On the other hand, suppose $\big(k, c, v_2(t,x), \eta(t,x_1), p(t,x)\big)$ is a solution to (1.2.2) in the form of (1.3.2) with $k \neq 0$ and $c \notin U([-h,0])$. Equation (2.2.3a) implies that V_2 must be a solution to the corresponding homogeneous equation of (2.2.6a), while (2.2.3b) yields the homogeneous boundary conditions of the types of ((2.2.6b)-(2.2.6c)). Therefore $(c, V_2(x_2))$ have to be homogeneous solutions to (2.2.6). Subsequently, v_1 is obtained from $\nabla \cdot v = 0$, η from the third equation in (2.2.1), and p from the v_2 equation in (2.2.1) along with its boundary value at $x_2 = 0$. □

Definition 2.2.1. *(k,c) is a non-singular mode if $c \in \mathbb{C} \setminus U([-h,0])$ and there exists a non-trivial solution $V_2(x_2)$ to the corresponding homogeneous problem of (2.2.6) (thus also yields a solution to (1.2.2) in the form of (1.3.2)). (k,c) is a singular mode if $c \in U([-h,0])$*

and there exists a $H^2_{x_2}$ solution $y(x_2)$ to

$$(U - c)(-y'' + k^2 y) + U'' y = 0 \tag{2.2.8}$$

along with the corresponding homogeneous boundary conditions of ((2.2.6b)–(2.2.6c)). (See also Remark 4.1.1.)

After acquiring good understanding on the homogeneous problem of the Rayleigh equation (2.2.6) (chapter 3) and its eigenvalues (chapter 4), we proceed to analyze the general non-homogeneous problem of (2.2.6) (chapter 5), in particular, the dependence of solutions on c. Finally in chapter 6 we apply the inverse Laplace transform to estimate the solution to the linear system (1.2.2). Recall the inverse Laplace transform

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} F(s) ds = \frac{|k|}{2\pi} \int_{-\infty + \frac{i\gamma}{k}}^{+\infty + \frac{i\gamma}{k}} e^{-ikct} F(-ikc) dc, \tag{2.2.9}$$

where γ is a real number so that $F(s)$ is analytic in the region Re$s > \gamma$ and the change of variable (2.2.5) was used in the second equality. Due to the analyticity, the integral can be eventually carried out along contours enclosing $U([-h, 0]) \subset \mathbb{C}$ and the non-singular modes of (1.2.2). Assuming there is neither singular modes in $U([-h, 0])$ nor nearby non-singular modes, we shall eventually obtain the decay in t of the component of the linear solution corresponding to the integral along the contour surrounding $U([-h, 0])$ by integration by parts in c.

CHAPTER 3

ANALYSIS OF RAYLEIGH EQUATION

In this chapter, we shall thoroughly analyze the homogeneous Rayleigh equation

$$-y''(x_2) + \big(k^2 + \tfrac{U''(x_2)}{U(x_2)-c}\big) y(x_2) = 0, \quad x_2 \in [-h, 0], \tag{3.0.1}$$

where

$$k \in \mathbb{R}, \quad c = c_R + ic_I \in \mathbb{C}, \quad {}' = \partial_{x_2}.$$

Throughout this chapter (except for some lemmas in section 3.6), we assume

$$U'(x_2) > 0, \quad \forall x_2 \in [-h, 0]. \tag{3.0.2}$$

As pointed out in the introduction, due to the symmetry of the reflection in x_1 variable, the case of $U' < 0$ can be reduced to the above one. Hence all results under (3.0.2) hold for all uniformly monotonic $U(x_2)$, namely those U satisfying $U' \neq 0$ on $[-h, 0]$.

To some extent, we will also consider the non-homogeneous Rayleigh equation

$$-y''(x_2) + \big(k^2 + \tfrac{U''(x_2)}{U(x_2)-c}\big) y(x_2) = \phi\big(k, c, x_2\big), \quad x_2 \in [-h, 0]. \tag{3.0.3}$$

More detailed forms and conditions of $\phi(k, c, x_2)$ will be specified when we obtained detailed estimates in chapters chapter 5 and chapter 6.As in typical problems of linear estimates based on density argument, we shall mostly work on ϕ with sufficient regularity, but carefully tracking its norms involved in the estimates.

The solutions to the Rayleigh equation (3.0.1) are obviously even in k and thus $k \geq 0$ will be assumed mostly. Similarly complex conjugate of solutions also solve (3.0.1) with

c replaced by $\bar{c}$, so we will restrict our consideration to $c_I \geq 0$. We have to consider the cases of $c \in \mathbb{C}$ away from $U([-h,0])$, near $U([-h,0])$, and then finally $c \in U([-h,0])$, separately. Due to small scales in x_2 created by $k \gg 1$, the dependence of the estimates on $k \gg 1$ will be carefully tracked. Recall $U \in C^{l_0}$. For technical convenience we extend U to be a C^{l_0} function on a neighborhood $[-h_0 - h, h_0]$ of $[-h,0]$, where

$$h_0 = \min\left\{\frac{h}{2}, \frac{\inf_{[-h,0]} U'}{4|U''|_{C^0([-h,0])}}\right\} > 0, \tag{3.0.4}$$

such that, on $[-h_0 - h, h_0]$,

$$U' \geq \tfrac{1}{2} \inf_{[-h,0]} U'(x_2), \quad |U'|_{C^{l_0-1}([-h_0-h,h_0])} \leq 2|U'|_{C^{l_0-1}([-h,0])}. \tag{3.0.5}$$

In the analysis of the most singular case of c close to the range $U([-h,0])$, we let x_2^c be such that

$$c_R = U(x_2^c), \ \text{ if } c_R \in U([-h_0 - h, h_0]). \tag{3.0.6}$$

We also extend the non-homogeneous term $\phi(k,c,x_2)$ for $x_2 \in [-h_0 - h, h_0]$ while keeping its relevant bounds comparable.

3.1 Rayleigh equation in the regular region

In the initial step we consider the rather regular case where $k^2|U-c|$ is bounded from below. For not so small k, we first transform the homogeneous Rayleigh equation (3.0.1) into a system of first order (complex valued) ODEs. Let

$$z_\pm = y' \pm |k| y,$$

and then (3.0.1) takes the form of the coupled equations

$$z'_\pm = \pm|k| z_\pm + \tfrac{1}{2}\beta(k,c,x_2)(z_+ - z_-), \qquad \beta(k,c,x_2) = \tfrac{U''}{|k|(U-c)}. \tag{3.1.1}$$

Lemma 3.1.1. *There exists $C > 0$ depending only on $|U'|_{C^1}$, and $|(U')^{-1}|_{C^0}$, such that for any $\rho \in (0,1]$, $k \neq 0$, and $\mathcal{I} = [x_{2l}, x_{2r}] \subset [-h_0 - h, h_0]$ satisfying*

$$\left|\frac{1}{U-c}\right| \leq \rho k^2 (1 + |U''|_{C^0([-h_0-h,h_0])})^{-1}, \quad \forall\, x_2 \in \mathcal{I}, \tag{3.1.2}$$

and any solution $z = (z_+, z_-)^T$ to (3.1.1) with

$$|z_+(x_{2l})| \geq |z_-(x_{2l})|, \tag{3.1.3}$$

it holds, for $x_2 \in \mathcal{I}$, $|z_+(x_2)| \geq |z_-(x_2)|$ and

$$\begin{aligned} &\big|z_+(x_2) - e^{|k|(x_2 - x_{2l})} z_+(x_{2l})\big| + \big|z_-(x_2) - e^{-|k|(x_2-x_{2l})} z_-(x_{2l})\big| \\ \leq & C|k|^{-1} \log\big(1 + C\rho k^2 (x_2 - x_{2l})\big) e^{|k|(x_2 - x_{2l})} |z_+(x_{2l})|. \end{aligned} \tag{3.1.4}$$

Moreover, for any solution with

$$|z_+(x_{2r})| \leq |z_-(x_{2r})|, \tag{3.1.5}$$

we have, for $x_2 \in \mathcal{I}$, $|z_+(x_2)| \leq |z_-(x_2)|$ and

$$\begin{aligned} &\big|z_+(x_2) - e^{|k|(x_2 - x_{2r})} z_+(x_{2r})\big| + \big|z_-(x_2) - e^{-|k|(x_2-x_{2r})} z_-(x_{2r})\big| \\ \leq & C|k|^{-1} \log\big(1 + C\rho k^2 (x_{2r} - x_2)\big) e^{|k|(x_{2r} - x_2)} |z_-(x_{2r})|. \end{aligned} \tag{3.1.6}$$

While (3.1.3) provides some technical convenience, indeed some assumption of this type on the initial values is needed to ensure estimates of solutions such as (3.1.4). For example, if $|\beta| \ll k$, the standard ODE theory implies that there are two solutions behaving like $e^{\pm k(x_2 - x_{2l})}$ corresponding to the Lyapunov exponents close to $\pm k$, then the decaying solution may not satisfy (3.1.4) with C uniform in $k \gg 1$.

Proof. We start with the observation of a simple consequence of (3.1.2). Namely, one may compute straight forwardly

$$(|z_+|^2 - |z_-|^2)' = 2|k|(|z_+|^2 + |z_-|^2) + \mathrm{Re}\beta|z_+ - z_-|^2 \geq 0. \tag{3.1.7}$$

This monotonicity along with boundary conditions yields an order relation between $|z_\pm|$ which can be used to control terms in (3.1.1).

We shall focus on the case under assumption (3.1.3), which ensures

$$|z_+| \geq |z_-|, \quad \forall\, x_2 \in \mathcal{I}. \tag{3.1.8}$$

By factorizing z_+ on the right side of (3.1.1), its solutions satisfy

$$z_+(x_2) - e^{|k|(x_2 - x_{2l})} z_+(x_{2l}) = \big(e^{\frac{1}{2}\int_{x_{2l}}^{x_2} \beta(k,c,x_2')\left(1 - \frac{z_-(x_2')}{z_+(x_2')}\right)dx_2'} - 1\big) e^{|k|(x_2 - x_{2l})} z_+(x_{2l}). \tag{3.1.9}$$

If $c_R \in U([-h_0 - h, h_0])$, let x_2^c be defined as in (3.0.6) and we use (3.1.2) to estimate

$$\begin{aligned} \int_{x_{2l}}^{x_2} |\beta(k,c,x_2')|dx_2' \leq & \frac{C}{|k|} \int_{x_{2l}}^{x_2} (|x_2' - x_2^c|^2 + c_I^2)^{-\frac{1}{2}} dx_2' \\ = & \frac{C}{|k|} \left| \log \frac{x_2 - x_2^c + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}{x_{2l} - x_2^c + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}} \right|, \end{aligned}$$

where the last equality is the exact integral. If $x_2^c \leq x_{2l} \leq x_2$, then the numerator in the logarithm is greater than the denominator. Applying the triangle inequality to x_2, x_{2l} and c, we obtain

$$\begin{aligned} \left| \log \frac{x_2 - x_2^c + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}{x_{2l} - x_2^c + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}} \right| \leq & \left| \log \left(1 + \frac{C(x_2 - x_{2l})}{x_{2l} - x_2^c + |U(x_{2l}) - c|} \right) \right| \\ \leq & \log\big(1 + C\rho k^2 (x_2 - x_{2l})\big). \end{aligned}$$

If $x_{2l} \leq x_2 \leq x_2^c$, multiplying the top and bottom of the quotient by their conjugates and proceeding much as in the previous case, we have

$$\left|\log \frac{x_2 - x_2^c + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}{x_{2l} - x_2^c + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}}\right| = \left|\log \frac{x_2^c - x_{2l} + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}}{x_2^c - x_2 + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}\right|$$
$$\leq \log\big(1 + C\rho k^2(x_2 - x_{2l})\big).$$

Finally, in the case $x_{2l} < x_2^c < x_2$, by splitting the interval at x_2^c and applying the above estimates on the two subintervals, we obtain

$$\begin{aligned}
&\left|\log \frac{x_2 - x_2^c + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}{x_{2l} - x_2^c + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}}\right| \\
=&\left|\log \frac{x_2 - x_2^c + \sqrt{(x_2 - x_2^c)^2 + c_I^2}}{|c_I|} + \log \frac{|c_I|}{x_{2l} - x_2^c + \sqrt{(x_{2l} - x_2^c)^2 + c_I^2}}\right| \\
\leq&\log\big(1 + C\rho k^2(x_2 - x_2^c)\big) + \log\big(1 + C\rho k^2(x_2^c - x_{2l})\big) \leq 2\log\big(1 + C\rho k^2(x_2 - x_{2l})\big).
\end{aligned}$$

Therefore the desired estimate (3.1.4) on z_+ follows from (3.1.9) and (3.1.8) and

$$\begin{aligned}
\left|e^{\frac{1}{2}\int_{x_{2l}}^{x_2} \beta(k,c,x_2')\left(1 - \frac{z_-(x_2')}{z_+(x_2')}\right)dx_2'} - 1\right| \leq& C\int_{x_{2l}}^{x_2} |\beta(k, c, x_2')|dx_2' \\
\leq& C|k|^{-1}\log\big(1 + C\rho k^2(x_2 - x_{2l})\big),
\end{aligned}$$

as $C|k|^{-1}\log\big(1 + C\rho k^2(x_2 - x_{2l})\big)$ is bounded uniformly in all $k \neq 0$. If $c_R \notin U([-h_0 - h, h_0])$, one can bound $|\beta|$ by $\frac{C}{|k|}\min\{1, \rho k^2\}$ which is also bounded for all $k \neq 0$. If $\rho k^2 \leq 1$, then $\rho k^2(x_2 - x_{2l})$ is bounded by $C\log\big(1 + \rho k^2(x_2 - x_{2l})\big)$. If $1 \leq \rho k^2$, then

$$x_2 - x_{2l} \leq C\log\big(1 + x_2 - x_{2l}\big) \leq C\log\big(1 + \rho k^2(x_2 - x_{2l})\big).$$

Therefore in both cases we have

$$\int_{x_{2l}}^{x_2} |\beta(k, c, x_2')|dx_2' \leq \tfrac{C}{|k|}\min\{1, \rho k^2\}(x_2 - x_{2l}) \leq \tfrac{C}{|k|}\log\big(1 + \rho k^2(x_2 - x_{2l})\big)$$

and thus (3.1.4) for z_+ follows from (3.1.9) and (3.1.8).

Turning attention to z_-, from the variation of parameter formula, we have

$$z_-(x_2) - e^{-|k|(x_2-x_{2l})} z_-(x_{2l}) = \frac{1}{2}\int_{x_{2l}}^{x_2} e^{-|k|(x_2-x_2')}\beta(k,c,x_2')\big(z_+(x_2') - z_-(x_2')\big)dx_2', \tag{3.1.10}$$

which along with (3.1.2), (3.1.4) for z_+, and (3.1.8), implies

$$|z_-(x_2) - e^{-|k|(x_2-x_{2l})} z_-(x_{2l})| \le Ce^{|k|(x_2-x_{2l})}|z_+(x_{2l})| \int_{x_{2l}}^{x_2} |\beta(k,c,x_2')|dx_2'.$$

The desired estimate on z_- follows from the above inequality on $\int|\beta|$. The estimates on $z_\pm(x_2)$ with initial condition $z_\pm(x_{2r})$ satisfying (3.1.5) can be derived in exactly the same fashion. □

In the following we use the above lemma to analyze some solutions to the homogeneous and non-homogeneous Rayleigh equations (3.0.1) and (3.0.3).

Lemma 3.1.2. *Consider*

$$(\Theta_1, \Theta_2) \in \{\sinh, \cosh\}^2 \setminus \{(\cosh, \sinh)\}.$$

There exists $C > 0$ depending only on $|U'|_{C^1}$ and $|(U')^{-1}|_{C^0}$, such that, for any $k \neq 0$, $\rho \in (0, 1]$, $C_0 \geq 0$, and interval $\mathcal{I} = [x_{2l}, x_{2r}] \subset [-h, 0]$ satisfying (3.1.2),

1. *if a solution $y(x_2)$ to (3.0.1) satisfies*

$$\big||k|y(x_{2l}) - \sinh|k|s\big| \le C_0\Theta_1(|k|s), \quad |y'(x_{2l}) - \cosh ks| \le C_0\Theta_2(|k|s), \quad s \ge 0, \tag{3.1.11}$$

then it holds that, for all $x_2 \in \mathcal{I}$,

$$\big||k|y(x_2) - \sinh|k|(x_2 - x_{2l} + s)|$$

$$\leq C\big(C_0 + (1+C_0)\big(\rho + |k|^{-1}\log(1+C\rho k^2)\big)\big)\Theta_1(|k|(x_2 - x_{2l} + s)),$$

$$|y'(x_2) - \cosh k(x_2 - x_{2l} + s)|$$

$$\leq C\big(C_0 + (1+C_0)\big(\rho + |k|^{-1}\log(1+C\rho k^2)\big)\big)\Theta_2(|k|(x_2 - x_{2l} + s));$$

2. *if a solution $y(x_2)$ to (3.0.1) satisfies*

$$\big||k|y(x_{2r}) - \sinh|k|s\big| \leq C_0\Theta_1(|ks|), \quad |y'(x_{2r}) - \cosh ks| \leq C_0\Theta_2(|ks|), \quad s \leq 0, \tag{3.1.12}$$

then it holds that, for all $x_2 \in \mathcal{I}$,

$$\big||k|y(x_2) - \sinh|k|(x_2 - x_{2r} + s)\big|$$

$$\leq C\big(C_0 + (1+C_0)\big(\rho + |k|^{-1}\log(1+C\rho k^2)\big)\big)\Theta_1(|k(x_2 - x_{2r} + s)|),$$

$$|y'(x_2) - \cosh k(x_2 - x_{2r} + s)|$$

$$\leq C\big(C_0 + (1+C_0)\big(\rho + |k|^{-1}\log(1+C\rho k^2)\big)\big)\Theta_2(|k(x_2 - x_{2r} + s)|).$$

3. *Moreover, the solution $y(x_2)$ to (3.0.3) with $y(x_{20}) = y'(x_{20}) = 0$ for some $x_{20} \in \mathcal{I}$ satisfies*

$$\begin{aligned}
&\Big||k|y(x_2) - \int_{x_{20}}^{x_2}\phi(k,c,x_2')\sinh|k(x_2 - x_2')|dx_2'\Big| \\
&+\Big|y'(x_2) - \int_{x_{20}}^{x_2}\phi(k,c,x_2')\cosh k(x_2 - x_2')dx_2'\Big| \\
\leq& C\big(\rho + |k|^{-1}\log(1+C\rho k^2)\big)\Big|\int_{x_{20}}^{x_2}\phi\big(k,c,x_2'\big)\sinh|k(x_2 - x_2')|dx_2'\Big|.
\end{aligned} \tag{3.1.13}$$

Proof. We first consider the special solution $y(x_2)$ to the homogeneous (3.0.1) satisfying (3.1.11) with $C_0 = 0$, namely, with the initial values

$$y(x_{2l}) = |k|^{-1}\sinh|k|s, \quad y'(x_{2l}) = \cosh|k|s, \quad s \geq 0,$$

whose corresponding form in terms of $z_\pm$ with initial values $z_\pm(x_{2l}) = e^{\pm|k|s}$ satisfies the assumptions of Lemma 3.1.1. On the one hand, for $|k|(x_2 - x_{2l}) \le 1$, it holds

$$|k|^{-1}e^{|k|(x_2-x_{2l})}\log\big(1 + C\rho k^2(x_2 - x_{2l})\big) \le C\rho|k|(x_2 - x_{2l}) \le C\rho\sinh|k|(x_2 - x_{2l}),$$

while, for $|k|(x_2 - x_{2l}) \ge 1$, we have

$$|k|^{-1}e^{|k|(x_2-x_{2l})}\log\big(1 + C\rho k^2(x_2 - x_{2l})\big) \le C|k|^{-1}\log(1 + C\rho k^2)\sinh|k|(x_2 - x_{2l}).$$

Therefore Lemma 3.1.1 and $\rho \in (0, 1]$ imply

$$\begin{aligned}&|z_+(x_2) - e^{|k|(x_2-x_{2l}+s)}| + |z_-(x_2) - e^{-|k|(x_2-x_{2l}+s)}|\\ &\qquad\le C\big(\rho + |k|^{-1}\log(1 + C\rho k^2)\big)e^{|k|s}\sinh|k|(x_2 - x_{2l}).\end{aligned}$$

Recovering $y(x_2)$ and $y'(x_2)$ from $z_\pm(x_2)$, we obtain the desired estimates in the case of $\Theta_1 = \Theta_2 = \sinh$ under the additional assumption $C_0 = 0$.

In the following we prove the estimates for a homogeneous solution $y(x_2)$ to (3.0.1) under (3.1.11) with general $C_0 \ge 0$. Let $Y_1(x_2)$ and $Y_2(x_2)$ be solution to (3.0.1) with initial values

$$Y_1(x_{2l}) = |k|^{-1}\sinh 1, \quad Y_1'(x_{2l}) = \cosh 1; \quad\; Y_2(x_{2l}) = 0, \quad Y_2'(x_{2l}) = 1.$$

Clearly Y_1 and Y_2 satisfy the above estimates with $s = |k|^{-1}$ and $s = 0$, respectively, and

$$y(x_2) = |k|(\sinh 1)^{-1}y(x_{2l})Y_1(x_2) + \big(y'(x_{2l}) - |k|(\coth 1)y(x_{2l})\big)Y_2(x_2).$$

Therefore, for $x_2 \in \mathcal{I}$,

$$\big||k|y(x_2) - \sinh|k|(x_2 - x_{2l} + s)\big|$$

$$
\begin{aligned}
=&\Big|(\sinh 1)^{-1}\Big(|k|y(x_{2l})\big(|k|Y_1(x_2)-\sinh(|k|(x_2-x_{2l})+1)\big)\\
&+(|k|y(x_{2l})-\sinh|k|s)\sinh(|k|(x_2-x_{2l})+1)\Big)\\
&+(\sinh 1)^{-1}\sinh|k|s\sinh(|k|(x_2-x_{2l})+1)\\
&+\big(y'(x_{2l})-(\coth 1)|k|y(x_{2l})\big)\big(|k|Y_2(x_2)-\sinh|k|(x_2-x_{2l})\big)\\
&+\big(y'(x_{2l})-\cosh|k|s-(\coth 1)(|k|y(x_{2l})-\sinh|k|s)\big)\sinh|k|(x_2-x_{2l})\\
&+(\cosh|k|s-(\coth 1)\sinh|k|s)\sinh|k|(x_2-x_{2l})-\sinh|k|(x_2-x_{2l}+s)\Big|.
\end{aligned}
$$

In the above summation, all the hyperbolic trigonometric combinations without $y(x_{2l})$ or $Y_{1,2}(x_2)$ are eventually cancelled and the remaining terms can be estimated by the using the assumptions on the initial values and the already obtained estimates on Y_1 and Y_2. We have

$$
\begin{aligned}
&\big||k|y(x_2)-\sinh|k|(x_2-x_{2l}+s)\big|\\
\le&\big((1+C_0)\big(\rho+|k|^{-1}\log(1+C\rho k^2)\big)+C_0\big)\big(\Theta_1(|k|s)\sinh(|k|(x_2-x_{2l})+1)\\
&+\cosh|k|s\sinh|k|(x_2-x_{2l})\big)\\
\le&\big((1+C_0)\big(\rho+|k|^{-1}\log(1+C\rho k^2)\big)+C_0\big)\Theta_1|k|(x_2-x_{2l}+s),
\end{aligned}
$$

where the last inequality was obtained by considering the two possible cases of Θ_1 spearately. The inequality on $y'(x_2)$ can be obtained similarly as

$$
\begin{aligned}
&\big|y'(x_2)-\cosh|k|(x_2-x_{2l}+s)\big|\\
\le&\Big|(\sinh 1)^{-1}\Big(|k|y(x_{2l})\big(|k|Y_1'(x_2)-\cosh(|k|(x_2-x_{2l})+1)\big)\\
&+(|k|y(x_{2l})-\sinh|k|s)\cosh(|k|(x_2-x_{2l})+1)\Big)\\
&+(\sinh 1)^{-1}\sinh|k|s\cosh(|k|(x_2-x_{2l})+1)\\
&+\big(y'(x_{2l})-(\coth 1)|k|y(x_{2l})\big)\big(|k|Y_2'(x_2)-\cosh|k|(x_2-x_{2l})\big)\\
&+\big(y'(x_{2l})-\cosh|k|s-(\coth 1)(|k|y(x_{2l})-\sinh|k|s)\big)\cosh|k|(x_2-x_{2l})\\
&+(\cosh|k|s-(\coth 1)\sinh|k|s)\cosh|k|(x_2-x_{2l})-\cosh|k|(x_2-x_{2l}+s)\Big|
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\big|y'(x_2) - \cosh|k|(x_2 - x_{2l} + s)\big| \\
\leq&\big((1 + C_0)\big(\rho + |k|^{-1}\log(1 + C\rho k^2)\big) + C_0\big)\big(\Theta_1(|k|s)\cosh(|k|(x_2 - x_{2l}) + 1) \\
&+ \cosh|k|s\sinh|k|(x_2 - x_{2l}) + \Theta_2(|k|s)\cosh|k|(x_2 - x_{2l})\big) \\
\leq&\big((1 + C_0)\big(\rho + |k|^{-1}\log(1 + C\rho k^2)\big) + C_0\big)\Theta_2|k|(x_2 - x_{2l} + s).
\end{aligned}
$$

This proves the desired estimates under the assumption (3.1.11). The proofs of the inequalities under assumption (3.1.12) are similar and we omit the details.

Using the variation of parameter formula, we can write the solution $y(x_2)$ with $y(x_{20}) = y'(x_{20}) = 0$ to the non-homogeneous Rayleigh equation (3.0.3) as

$$
\begin{pmatrix} y \\ y' \end{pmatrix}(x_2) = \int_{x_{20}}^{x_2} \phi\big(k, c, x_2'\big) S(x_2, x_2') \begin{pmatrix} 0 \\ 1 \end{pmatrix} dx_2'
$$

where $S(x_2, x_2')$ is the 2×2 fundamental matrix of the homogeneous equation (3.0.1) with initial value $S(x_2', x_2') = I$. Therefore,

$$
S(x_2, x_2') \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{y}(x_2, x_2') \\ \tilde{y}'(x_2, x_2') \end{pmatrix}
$$

where $\tilde{y}(\cdot, x_2')$ is the solution to (3.0.1) whose initial value is given by $\tilde{y}(x_2', x_2') = 0$ and $\tilde{y}(x_2', x_2') = 1$. The desired estimates follow from applying the above estimates in the homogeneous case with $s = 0 = C_0$ and $\Theta_1 = \Theta_2 = \sinh$. $\square$

Practically the above estimates are more effective for k bounded from below. To end this section, we give the following simple estimate of the Rayleigh equation for k bounded from above, which compares $y(x_2)$ to the free solution (where the U term is removed)

$$y_F(x_2) = \big(\cosh k(x_2 - x_{20})\big) y(x_{20}) + k^{-1}\big(\sinh k(x_2 - x_{20})\big) y'(x_{20}).$$

Here $k^{-1} \sinh ks|_{k=0} = s$ is understood.

Lemma 3.1.3. *For any $k^*, M > 0$, there exists $C > 0$ depending only on h, k^*, and M such that for any $|k| \le k^*$, $C_0 > 0$, $x_{20} \in \mathcal{I} = [x_{2l}, x_{2r}] \subset [-h, 0]$ satisfying*

$$\left|\frac{1}{U - c}\right| \le C_0 \le M, \quad \forall\, x_2 \in \mathcal{I},$$

and any solution $y(x_2)$ to (3.0.3), it holds

$$\begin{aligned}|y(x_2) - y_F(x_2)| + |y'(x_2) - y_F'(x_2)| \le C\Big(&C_0\big(|y(x_{20})||x_2 - x_{20}| + |y'(x_{20})||x_2 - x_{20}|^2\big) \\ &+ \Big|\int_{x_{20}}^{x_2} |\phi(k, c, x_2')| dx_2'\Big|\Big).\end{aligned}$$

Proof. The proof is based on some straight forward elementary argument and we shall only outline it. Let $\tilde{y} = y - y_F$. We can write the solution $y(x_2)$ using the variation of constant formula

$$\begin{aligned}\begin{pmatrix} \tilde{y}(x_2) \\ \tilde{y}'(x_2) \end{pmatrix} =& \int_{x_{20}}^{x_2} \left(\frac{U'' y_F}{U - c} - \phi\right)(x_2') \begin{pmatrix} k^{-1}\sinh k(x_2 - x_2') \\ \cosh k(x_2 - x_2') \end{pmatrix} dx_2' \\ &+ \int_{x_{20}}^{x_2} \left(\frac{U''\tilde{y}}{U - c}\right)(x_2') \begin{pmatrix} k^{-1}\sinh k(x_2 - x_2') \\ \cosh k(x_2 - x_2') \end{pmatrix} dx_2'.\end{aligned}$$

It implies

$$\begin{aligned}|\tilde{y}(x_2)| + |\tilde{y}'(x_2)| \le& C\Big(C_0\big(|y(x_{20})||x_2 - x_{20}| + |y'(x_{20})||x_2 - x_{20}|^2\big) \\ &+ \Big|\int_{x_{20}}^{x_2} |\phi(k, c, x_2')| dx_2'\Big|\Big) + CC_0\left|\int_{x_{20}}^{x_2} |\tilde{y}(x_2')| dx_2'\right|\end{aligned}$$

and the estimates on $y - y_F$ and $y' - y'_F$ follow immediately from the Gronwall inequality.

$\square$

3.2 Rayleigh equation near singularity and its convergence as $c_I \to 0+$

In the rest of the chapter, we shall mostly focus on the case when $(1+k^2)^{\frac{1}{2}}|U - c|$ is small, so

$$c_R = U(x_2^c), \quad x_2^c \in [-\tfrac{1}{2}h_0 - h, \tfrac{1}{2}h_0], \tag{3.2.1}$$

will always be assumed, while the domains of U and ϕ have been extended to $[-h_0 - h, h_0]$. Due to complex conjugacy, we only need to consider $c_I \geq 0$. In particular, if $x_2^c \in (-h, x_2)$, the strong singularity in (3.0.1) will lead to $y\big(c_R + i(0+), k, x_2\big) \notin \mathbb{R}$ even if $y(-h), y'(-h) \in \mathbb{R}$. Even though some estimates are stated for $c_I > 0$, most of the inequalities are mostly uniform as $c_I \to 0+$ and thus hold for the limits.

In order to obtain estimates uniform in $k \in \mathbb{R}$, rescale

$$\mu = \langle k \rangle^{-1} = \tfrac{1}{\sqrt{k^2+1}}, \quad x_2 = x_2^c + \mu\tau, \quad c_I = \mu\epsilon, \quad w = (w_1, w_2)^T = (\mu^{-1}y, y')^T \in \mathbb{C}^2, \tag{3.2.2}$$

where x_2^c satisfies (3.2.1) as well as in the above. Equation (3.0.1) becomes

$$w_\tau = \begin{pmatrix} 0 & 1 \\ 1 - \mu^2 + \frac{\mu^2 U''(x_2^c + \mu\tau)}{U(x_2^c+\mu\tau) - c} & 0 \end{pmatrix} w - \begin{pmatrix} 0 \\ \tilde{\phi}(\mu, c, \tau) \end{pmatrix}, \tag{3.2.3}$$

where

$$\tilde{\phi}(\mu, c, \tau) = \mu\phi\big(k, c, x_2^c + \mu\tau\big).$$

We shall consider this ODE on intervals $\tau \in [-M, M]$ such that

$$[x_2^c - \mu M, x_2^c + \mu M] \subset [-h_0 - h, h_0], \tag{3.2.4}$$

is that U is well-defined when $|\tau| \leq M$. As $c_I \to 0+$, one would naturally expect $w(\tau)$ to converge to solutions to

$$W_\tau = \begin{pmatrix} 0 & 1 \\ 1 - \mu^2 + \frac{\mu^2 U''(x_2^c + \mu\tau)}{U(x_2^c + \mu\tau) - c_R} & 0 \end{pmatrix} W - \begin{pmatrix} 0 \\ \tilde{\phi}\big(\mu, c_R, \tau\big) \end{pmatrix}. \tag{3.2.5}$$

However, this limit equation becomes singular at $\tau = 0$ and conditions have to be specified there.

• **Fundamental matrix of the homogeneous Rayleigh equation.** Its construction is adapted from the one used in [11]. Let

$$\Gamma(\mu, c_R, \epsilon, \tau) = (1-\mu^2)\tau + \frac{\mu U''(x_2^c)}{2U'(x_2^c)} \log(\tilde{U}^2 + \epsilon^2) + \gamma(\mu, c_R, \epsilon, \tau) + \int_{-M}^{\tau} \frac{i\mu\epsilon U_2(\tau')}{\tilde{U}(\tau')^2 + \epsilon^2} d\tau', \tag{3.2.6}$$

where, for $j = 1, 2$,

$$\begin{aligned} U_j(c_R, \mu, \tau) =& \big(\tfrac{d^j}{dx_2^j} U\big)(x_2^c + \mu\tau), \tilde{U}(c_R, \mu, \tau) = \tfrac{1}{\mu}\big(U(x_2^c + \mu\tau) - c_R\big) \\ =& \tfrac{1}{\mu}\big(U(x_2^c + \mu\tau) - U(x_2^c)\big), \end{aligned} \tag{3.2.7}$$

and the remainder γ of Γ is given by

$$\gamma(\mu, c_R, \epsilon, 0) = 0, \quad \gamma_\tau = \frac{\mu\big(U_1(0)U_2 - U_2(0)U_1\big)\tilde{U}}{U_1(0)(\tilde{U}^2 + \epsilon^2)}, \implies \Gamma_\tau = 1 - \mu^2 + \frac{\mu U_2}{\tilde{U} - i\epsilon}. \tag{3.2.8}$$

It is not hard to see that $\gamma(\mu, c_R, 0, \tau)$ is C^{l_0-2} in τ and μ and C^{l_0-3} in c_R. We often skip writing the explicit dependence on those variables other than τ. Denote

$$\begin{aligned}\Gamma_0(\mu, c_R, \tau) = &\lim_{\epsilon\to 0+} \Gamma(\mu, c_R, \epsilon, \tau)\\ &=(1-\mu^2)\tau + \frac{\mu U''(x_2^c)}{U'(x_2^c)} \log|\tilde{U}(\tau)| + \gamma(\mu, c_R, 0, \tau)\\ &\quad + \frac{i\pi\mu U''(x_2^c)}{2U'(x_2^c)}\big(sgn(\tau)+1\big),\end{aligned} \tag{3.2.9}$$

where we note that the integrand of the imaginary part of Γ converges to a delta mass as $\epsilon \to 0+$ and produces a jump in Γ_0 at $\tau = 0$ (see Lemma 3.2.1 in the below). Let $\tilde{B}(\mu, c_R, \epsilon, \tau)$ be a 2×2 matrix given by

$$\tilde{B}_\tau = \begin{pmatrix} \Gamma(\mu, c_R, \epsilon, \tau) & 1 \\ -\Gamma(\mu, c_R, \epsilon, \tau)^2 & -\Gamma(\mu, c_R, \epsilon, \tau)\end{pmatrix} \tilde{B}, \quad \tilde{B}(\mu, c_R, \epsilon, 0) = I_{2\times 2}, \tag{3.2.10}$$

and

$$\tilde{\Phi}(\mu, c, \tau) = \begin{pmatrix} \tilde{\Phi}_1(\mu, c, \tau) \\ \tilde{\Phi}_2(\mu, c, \tau)\end{pmatrix} = \int_{-M}^{\tau} \tilde{\phi}(\mu, c, \tau')\tilde{B}(\mu, c_R, \tfrac{c_I}{\mu}, \tau')^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d\tau'. \tag{3.2.11}$$

It is worth pointing out that Γ_0 is real for $\tau < 0$ and imaginary for $\tau > 0$. To keep the notations simple we often skip the arguments other than τ. In the following lemma we collect some basic estimates of Γ and $\tilde{B}$ where we often bound the $\log|\tau|$ singularity in Γ by $|\tau|^{-\alpha}$, $\alpha > 0$, for simplicity.

Lemma 3.2.1. *For any $M > 0$ satisfying (3.2.4) and $\alpha, \alpha' \in (0,1)$ with $\alpha + \alpha' < 1$, there exists $C > 0$ depending only on M, α, α', $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that, for any $0 < \epsilon < M$, the following hold for $|\tau| \le M$,*

$$\det\tilde{B} = 1, \quad |\tilde{B} - I| \le e^{|\tau| + C(|\tau|^3 + \mu^2|\tau|^\alpha)} - 1, \quad |\tilde{B}^{-1} - I| \le 4(e^{|\tau| + C(|\tau|^3+\mu^2|\tau|^\alpha)} - 1). \tag{3.2.12}$$

$$|\Gamma(\epsilon,\tau)-\Gamma_0(\tau)|\leq C\mu\big(\mu\epsilon|\log\epsilon|+\tfrac{\epsilon}{\epsilon+|\tau|}+\log(1+\tfrac{C\epsilon^2}{\tau^2})\big) \tag{3.2.13}$$

$$\left|\tilde{B}(\epsilon,\tau)-\tilde{B}_0(\tau)\right|\leq C\mu\min\{\epsilon^{\alpha}(|\tau|^{1-\alpha}+\mu|\tau|^{\alpha'}),\ \epsilon(1+|\log\epsilon|+\mu\log^2\epsilon)\} \tag{3.2.14}$$

where $\tilde{B}_0(\mu,c_R,\tau)=\lim_{\epsilon\to0+}\tilde{B}(\mu,c_R,\epsilon,\tau)$*. Moreover, general solutions of (3.2.3) with* $c_I>0$ *is given by*

$$\begin{aligned}&\begin{pmatrix}\mu^{-1}y(x_2)\\ y'(x_2)\end{pmatrix}=w(\tau)=\begin{pmatrix}1 & 0\\ \Gamma(\mu,c_R,\epsilon,\tau) & 1\end{pmatrix}\tilde{B}(\mu,c_R,\epsilon,\tau)\big(b-\tilde{\Phi}(\mu,c,\tau)\big),\\ &b=\begin{pmatrix}b_1\\ b_2\end{pmatrix}\in\mathbb{C}^2.\end{aligned} \tag{3.2.15}$$

Remark 3.2.1. *Even though* $\epsilon>0$ *is assumed in the above and the remaining statements in this and the next sections, as* $C>0$ *is independent of* $\epsilon=\langle k\rangle c_I\in(0,M]$ *in a priori estimates and thus they hold even as* $\epsilon\to0+$*.*

Expression (3.2.15) essentially is the variation of parameter formula including the fundamental matrix of the Rayleigh equation. Due to $\det\tilde{B}=1$, it is possible to extend the definition of $\tilde{B}$ to include all $x_2\in[-h_0-h,h_0]$, but its bound would be non-uniform in $k\gg1$ for $|x_2-x_2^c|\gg\mu$.

Proof. Since Γ has a logarithmic singularity at the worst even if $\epsilon=0$, $\tilde{B}$ is obviously well-defined. The zero trace value of the coefficient matrix in (3.2.10) yields $\det B=1$. The form (3.2.15) of general solutions of (3.2.3) for $c_I>0$ follows from straightforward verifications.

Equation (3.2.10) implies

$$|(\tilde{B}-I)_\tau| \le 1+\Gamma^2+(1+\Gamma^2)|\tilde{B}-I|,$$

where $1+\Gamma^2$ is the operator norm of the coefficient matrix. From Gronwall inequality, we obtain

$$|\tilde{B}-I| \le e^{|\tau+\int_0^\tau \Gamma^2 d\tau'|}-1.$$

It is clear from the definition of γ that

$$|\gamma_\tau| \le C\mu^2, \quad \left|\int_{-M}^{\tau} \frac{i\mu\epsilon U_2(\tau')}{\tilde{U}(\tau')^2+\epsilon^2} d\tau'\right| \le C\mu.$$

The definition of Γ, the boundedness of $|\tilde{U}|$, and the estimate on γ_τ imply that, for $\tau \in [-M, M]$,

$$\left|\int_0^\tau \Gamma^2 d\tau'\right| \le C(|\tau|^2+\mu^2)|\tau| + C\mu^2 \left|\int_0^\tau \log^2(|\tau'|+\epsilon) d\tau'\right| \le C(|\tau|^3+\mu^2|\tau|^\alpha),$$

where C is a generic constant determined by M and k_* and the Hölder inequality was used to obtain $|\tau|^\alpha$, for any $\alpha \in (0,1)$. The desired estimate in (3.2.12) on $\tilde{B}-I$ follows immediately which along with $\det B = 1$ in turn yields the estimate on $\tilde{B}^{-1}-I$.

The definition of γ implies

$$|\gamma(\epsilon,\tau)-\gamma(0,\tau)| \le \left|\int_0^\tau \frac{C\mu^2\epsilon^2}{(\tau')^2+\epsilon^2} d\tau'\right| = C\mu^2\epsilon \tan^{-1}\frac{|\tau|}{\epsilon}.$$

Regarding the imaginary part of Γ, we observe

$$\begin{aligned}
&\int_{-M}^{\tau} \left| \frac{U_2(\tau')}{\tilde{U}(\tau')^2+\epsilon^2} - \frac{U_2(0)}{U_1(0)^2(\tau')^2+\epsilon^2} \right| d\tau' \\
\le &\int_{-M}^{\tau} \frac{\left|U_2(\tau')\big(U_1(0)^2(\tau')^2+\epsilon^2\big) - U_2(0)\big(\tilde{U}(\tau')^2+\epsilon^2\big)\right|}{\big(\tilde{U}(\tau')^2+\epsilon^2\big)\big(U_1(0)^2(\tau')^2+\epsilon^2\big)} d\tau'
\end{aligned}$$

$$\leq C\mu \int_{-M}^{\tau} \frac{|\tau'|}{(\tau')^2+\epsilon^2} d\tau' \leq C\mu(1+|\log\epsilon|),$$

where we used the smoothness of U_1 and U_2 in $\mu\tau$. It implies

$$\Big|\int_{-M}^{\tau} \frac{\mu\epsilon U_2(\tau')}{\tilde{U}(\tau')^2+\epsilon^2} d\tau' - \frac{\mu U_2(0)}{U_1(0)}\Big(\tan^{-1}\frac{U_1(0)\tau}{\epsilon}+\frac{\pi}{2}\Big)\Big| \leq C\mu\epsilon(1+\mu|\log\epsilon|), \quad (3.2.16)$$

and thus

$$\begin{aligned}\Big|\int_{-M}^{\tau} \frac{\mu\epsilon U_2(\tau')}{\tilde{U}(\tau')^2+\epsilon^2} d\tau' - \frac{\pi\mu U_2(0)}{2U_1(0)}\big(sgn(\tau)+1\big)\Big| \leq & C\mu\big(\mu\epsilon|\log\epsilon| + \min\{1,\, \epsilon|\tau|^{-1}\}\big) \\ \leq & C\mu\big(\mu\epsilon|\log\epsilon| + \frac{1}{1+\frac{|\tau|}{\epsilon}}\big).\end{aligned}$$

The error estimate (3.2.13) follows consequently.

Proceeding to consider $\tilde{B}(\epsilon,\tau)-\tilde{B}_0(\tau)$ where $\tilde{B}_0(\mu,c_R,\tau)=\tilde{B}(\mu,c_R,0,\tau)$, we have

$$\begin{aligned}&\partial_\tau\big(\tilde{B}(\epsilon,\tau)-\tilde{B}_0(\tau)\big) - \begin{pmatrix}\Gamma_0(\tau) & 1 \\ -\Gamma_0(\tau)^2 & -\Gamma_0(\tau)\end{pmatrix}\big(\tilde{B}(\epsilon,\tau)-\tilde{B}_0(\tau)\big) \\ =&\begin{pmatrix}\Gamma(\epsilon,\tau)-\Gamma_0(\tau) & 0 \\ -\Gamma(\epsilon,\tau)^2+\Gamma_0(\tau)^2 & -\Gamma(\epsilon,\tau)+\Gamma_0(\tau)\end{pmatrix}\tilde{B}(\epsilon,\tau).\end{aligned}$$

Recalling that $\tilde{B}_0(\tau)$ is the elementary fundamental matrix of the above corresponding homogeneous ODE system, the variation of parameter formula implies

$$\begin{aligned}&\Big|\tilde{B}(\epsilon,\tau)-\tilde{B}_0(\tau)\Big| \\ =&\Big|\int_0^{\tau} \tilde{B}_0(\tau)\tilde{B}_0(\tau')^{-1}\begin{pmatrix}\Gamma(\epsilon,\tau')-\Gamma_0(\tau') & 0 \\ -\Gamma(\epsilon,\tau')^2+\Gamma_0(\tau')^2 & -\Gamma(\epsilon,\tau')+\Gamma_0(\tau')\end{pmatrix}\tilde{B}(\epsilon,\tau')d\tau'\Big| \\ \leq & C\Big|\int_0^{\tau}\big(1+|\Gamma(\epsilon,\tau')|+|\Gamma_0(\tau')|\big)|\Gamma(\epsilon,\tau')-\Gamma_0(\tau')|d\tau'\Big| \\ \leq & C\Big|\int_0^{\tau}\big(1+\mu\big|\log|\tau'|\big|\big)|\Gamma(\epsilon,\tau')-\Gamma_0(\tau')|d\tau'\Big|\end{aligned}$$

$$\le C\big|1+\mu|\log(\cdot)|\big|_{L^{\frac{1}{1-\alpha}}}|\Gamma(\epsilon,\cdot)-\Gamma_0(\cdot)|_{L^{\frac{1}{\alpha}}}.$$

The second desired upper bound in (3.2.14) of $\tilde{B}-\tilde{B}_0$ follows from direct estimating the above integral without using the Hölder inequality. For the first upper bound there we use, for any $|\tau_1|, |\tau_2| \le M$,

$$|\Gamma(\epsilon,\cdot)-\Gamma_0(\cdot)|_{L^\rho[\tau_1,\tau_2]} \le C\mu\epsilon^{\frac{1}{\rho}},\ \rho\in(1,+\infty);\ |\Gamma(\epsilon,\cdot)-\Gamma_0(\cdot)|_{L^1[\tau_1,\tau_2]} \le C\mu\epsilon(1+|\log\epsilon|), \tag{3.2.17}$$

which can be verified by straight forward computation. The proof of the lemma is complete.

□

• **A priori estimates.** A direct corollary of the form (3.2.15) of the general solution to the Rayleigh equation (3.2.3) is an estimate of $w(\tau)$ in terms of b and $\tilde{\Phi}$. Let $\tilde{\Gamma}(\tau)$ denote

$$\tilde{\Gamma}(\tau) = \frac{\mu U''(x_2^c)}{U'(x_2^c)}\big(\frac{1}{2}\log(\tilde{U}(\tau)^2+\epsilon^2)+\tan^{-1}\frac{U'(x_2^c)\tau}{\epsilon}+\frac{\pi}{2}\big).$$

Corollary 3.2.1.1. *For $b\in\mathbb{C}^2$ and $|\tau|\le M$, let*

$$\tilde{b}(\tau) = \begin{pmatrix} 1 & 0 \\ \Gamma(\tau) & 1 \end{pmatrix}\tilde{B}(\tau)b, \quad \tilde{b}_0(\tau) = \begin{pmatrix} 1 & 0 \\ \Gamma_0(\tau) & 1 \end{pmatrix}\tilde{B}_0(\tau)b,$$

then under the same assumptions of Lemma 3.2.1, it holds, for any $\alpha_1\in[0,1-\alpha)$,

$$|\tilde{b}_1(\tau)-b_1| \le C(|\tau|+\mu^2|\tau|^\alpha)|b|, \quad \big|\tilde{b}_2(\tau)-\big(b_2+b_1\tilde{\Gamma}(\tau)\big)\big| \le C\big(|\tau|+\mu(|\tau|^\alpha+\epsilon^\alpha)\big)|b|$$

$$|\tilde{b}_1(\tau)-\tilde{b}_{01}(\tau)| \le C\mu\epsilon^\alpha\big(|\tau||b|+\min\{|\tau|^{1-\alpha},\ \epsilon^{1-\alpha}(1+|\log\epsilon|)\}|b_1|\big),$$

$$|\tilde{b}_2(\tau)-\tilde{b}_{02}(\tau)| \le C\mu\big(\epsilon^\alpha|\tau|^{\alpha_1}|b|+\big(\tfrac{\epsilon}{\epsilon+|\tau|}+\log\big(1+\tfrac{C\epsilon^2}{\tau^2}\big)\big)|b_1|\big).$$

Proof. The estimates on $\tilde{b}$ follows from straight forward calculation based on (3.2.16) and the bound on $\tilde{B} - I$ given in Lemma 3.2.1 and we omit the details.

Regrading $\tilde{b}(\tau) - \tilde{b}_0(\tau)$, let $\tilde{B}_{jl}$ denote the entries of $\tilde{B}$. Using Lemma 3.2.1 where the estimates are uniform in $\epsilon > 0$, we have

$$
\begin{aligned}
&|\tilde{b}_2(\tau) - \tilde{b}_{02}(\tau)| \leq (1 + |\Gamma_0|)|\tilde{B} - \tilde{B}_0||b| + |\Gamma - \Gamma_0|(|\tilde{B}_{11}||b_1| + |\tilde{B}_{12}||b_2|) \\
\leq & C\big((1 + \mu\big|\log|\tau|\big|)|\tilde{B} - \tilde{B}_0||b| + |\Gamma - \Gamma_0|(|b_1| + (|\tau| + \mu|\tau|^{\alpha'})|b_2|)\big) \\
\leq & C\mu\big(\epsilon^{\alpha}|\tau|^{\alpha_1}|b| + \big(\tfrac{\epsilon}{\epsilon+|\tau|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big)(|b_1| + |\tau|^{\alpha'}|b_2|)\big).
\end{aligned}
$$

Since

$$
|\tau|^{\beta}\big(\tfrac{\epsilon}{\epsilon+|\tau|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big) \leq C\epsilon^{\beta}, \quad \beta \in (0, 1], \tag{3.2.18}
$$

the upper on $\tilde{b}_2(\tau) - \tilde{b}_{02}(\tau)$ follows accordingly.

To derive the estimate on $\tilde{b}_1(\tau) - \tilde{b}_{01}(\tau)$, we notice $\tilde{b}_1(0) = \tilde{b}_{01}(0) = b_1$ and the desired estimate follows from integrating $\partial_\tau(\tilde{b}_1 - \tilde{b}_{01}) = \tilde{b}_2 - \tilde{b}_{02}$ using (3.2.17). $\square$

Remark 3.2.2. *The above estimates imply, that for any solution $w(\tau)$ to (3.2.3)*

$$
\big|w_1(\tau) - \big(b_1 - \tilde{\Phi}_1(\tau)\big)\big| \leq C(|\tau| + \mu^2|\tau|^{\alpha})\big|b - \tilde{\Phi}(\tau)\big|, \tag{3.2.19}
$$

$$
\Big|w_2(\tau) - \Big(b_2 - \tilde{\Phi}_2(\tau) + \tilde{\Gamma}(\tau)\big(b_1 - \tilde{\Phi}_1(\tau)\big)\Big)\Big| \leq C\big(|\tau| + \mu(|\tau|^{\alpha} + \epsilon^{\alpha})\big)\big|b - \tilde{\Phi}(\tau)\big|. \tag{3.2.20}
$$

The following lemma gives another estimate of $w(\tau)$ in terms of some initial value $w(\tau_0)$ which we shall used mainly for τ_0 away from 0.

Lemma 3.2.2. *For any $M > 0$ satisfying (3.2.4) and $\alpha \in (0, 1)$, there exists $C > 0$ depending only on M, α, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that, for any $0 < \epsilon < M$, and*

$\tau_0, \tau \in [-M, M]$*, the following hold for any solution* $w(\tau)$ *to (3.2.3):*

$$\begin{aligned}\big|w_1(\tau) - w_1(\tau_0)\big| \leq & C|\tau - \tau_0|\big(|w(\tau_0)| + \mu\big|\log\big(\tau_0^2 + \epsilon^2\big)\big||w_1(\tau_0)|\big) \\ & + C\mu|\tau - \tau_0|^\alpha\big(|w(\tau_0)| + \big|\tilde{\Phi}_1(\cdot) - \tilde{\Phi}_1(\tau_0)\big|_{L^\infty[\tau_0,\tau]}\big) \\ & + C\big|\tilde{\Phi}(\cdot) - \tilde{\Phi}(\tau_0)\big|_{L^1[\tau_0,\tau]},\end{aligned} \tag{3.2.21a}$$

$$\begin{aligned}&\big|w_2(\tau) - \big(w_2(\tau_0) + \tilde{\Phi}_2(\tau_0) - \tilde{\Phi}_2(\tau) - \tilde{\Gamma}(\tau_0)w_1(\tau_0) + \tilde{\Gamma}(\tau)(w_1(\tau_0) + \tilde{\Phi}_1(\tau_0) - \tilde{\Phi}_1(\tau))\big)\big| \\ &\leq C\Big((|\tau|^\alpha + \mu\epsilon^\alpha)\big|\tilde{\Phi}(\tau_0) - \tilde{\Phi}(\tau)\big| + \big(\mu\epsilon^\alpha + |\tau|^\alpha + |\tau_0|^\alpha(1 + \mu|\log(\tau^2 + \epsilon^2)|)\big)|w(\tau_0)| \\ &\quad + \mu|\tau|^\alpha|\log(\tau_0^2 + \epsilon^2)||w_1(\tau_0)|\Big).\end{aligned} \tag{3.2.21b}$$

Proof. We shall first estimate $b - \tilde{\Phi}(\tau_0)$ based on $w(\tau_0)$ and then apply Corollary 3.2.1.1. From (3.2.15) and $\det \tilde{B} = 1$ which allows us to write $\tilde{B}^{-1}$ explicitly, we have

$$b - \tilde{\Phi}(\tau_0) = \begin{pmatrix} \tilde{B}_{22} & -\tilde{B}_{12} \\ -\tilde{B}_{21} & \tilde{B}_{11} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\Gamma & 1 \end{pmatrix} w\Big|_{\tau_0} = \begin{pmatrix} \tilde{B}_{22} + \Gamma\tilde{B}_{12} & -\tilde{B}_{12} \\ -\tilde{B}_{21} - \Gamma\tilde{B}_{11} & \tilde{B}_{11} \end{pmatrix} w\Big|_{\tau_0}. \tag{3.2.22}$$

Using Lemma 3.2.1, one may estimate

$$|b_1 - \tilde{\Phi}_1(\tau_0) - w_1(\tau_0)| \leq C(|\tau_0| + \mu|\tau_0|^\alpha)|w(\tau_0)|, \tag{3.2.23}$$

$$|b_2 - \tilde{\Phi}_2(\tau_0) + \tilde{\Gamma}(\tau_0)w_1(\tau_0) - w_2(\tau_0)| \leq C\big(|\tau_0| + \mu(|\tau_0|^\alpha + \epsilon^\alpha)\big)|w(\tau_0)|, \tag{3.2.24}$$

where we also used (3.2.16). Combining these inequalities and Corollary 3.2.1.1, we obtain

$$\big|w_2(\tau) - \big(w_2(\tau_0) + \tilde{\Phi}_2(\tau_0) - \tilde{\Phi}_2(\tau) - \tilde{\Gamma}(\tau_0)w_1(\tau_0) + \tilde{\Gamma}(\tau)(w_1(\tau_0) + \tilde{\Phi}_1(\tau_0) - \tilde{\Phi}_1(\tau))\big)\big|$$

$$
\begin{aligned}
\leq & C\big(|\tau| + \mu(|\tau|^{\alpha'} + \epsilon^{\alpha'})\big)\big|b - \tilde{\Phi}(\tau)\big| + C\big(|\tau_0| + \mu(|\tau_0|^{\alpha} + \epsilon^{\alpha})\\
& + |\tilde{\Gamma}(\tau)|(|\tau_0| + \mu|\tau_0|^{\alpha})\big)|w(\tau_0)|\\
\leq & C(|\tau|^{\alpha'} + \mu\epsilon^{\alpha'})\big(\big|\tilde{\Phi}(\tau_0) - \tilde{\Phi}(\tau)\big| + |w(\tau_0)| + |\tilde{\Gamma}(\tau_0)||w_1(\tau_0)|\big)\\
& + C\big(\mu\epsilon^{\alpha} + |\tau_0|^{\alpha}(1 + \mu|\log(\tau^2 + \epsilon^2)|)\big)|w(\tau_0)|\\
\leq & C\Big((|\tau|^{\alpha} + \mu\epsilon^{\alpha})\big|\tilde{\Phi}(\tau_0) - \tilde{\Phi}(\tau)\big| + \big(\mu\epsilon^{\alpha} + |\tau|^{\alpha} + |\tau_0|^{\alpha}(1 + \mu|\log(\tau^2 + \epsilon^2)|)\big)|w(\tau_0)|\\
& + \mu|\tau|^{\alpha}|\log(\tau_0^2 + \epsilon^2)||w_1(\tau_0)|\Big),
\end{aligned}
$$

where $\alpha' \in (\alpha, 1)$. This yields inequality (3.2.21b) of $w_2(\tau)$. The estimate of $w_1(\tau)$ is obtained through integrating that of $w_2(\tau) = w_{1\tau}(\tau)$. $\square$

• **Convergence estimates as $c_I \to 0+$.** As $\epsilon = \mu^{-1}c_I = \langle k\rangle c_I \to 0+$, from Lemma 3.2.1, it is natural to expect that the limit of solutions to the non-homogenous Rayleigh equation (3.0.3) is also given by formula (3.2.15) with Γ, $\tilde{B}$, and $\tilde{\Phi}$ replaced by Γ_0, $\tilde{B}_0$, and $\tilde{\Phi}_0 = \lim_{\epsilon\to 0+}\tilde{\Phi}$.

With the above preparations, we are ready to obtain the convergence and error estimates of solutions to the Rayleigh equation (3.2.5). While the limits of non-homogeneous Rayleigh equation under appropriate assumptions on $\phi(k, c, x_2)$ can be studied in the framework in this chapter, we shall just focus on the homogeneous case, i.e. with $\phi \equiv 0+$, and leave the non-homogeneous one to chapter 5. In fact, (3.2.9) and Lemma 3.2.1 imply that, as $c_I \to 0$, $w_1(\tau)$ would converge to a Hölder continuous limit, while $w_2(\tau)$ develops a jump proportional to $w_1(0)$ and a logarithmic singularity at $\tau = 0$. More precisely, the limit $W(\tau)$ of solutions should (see the proposition in the below) satisfy the Rayleigh equation (3.2.5) with $c \in \mathbb{R}$ for $\tau \neq 0$ and satisfy at $\tau = 0$,

$$
\begin{cases}
W_1 \in C^0\big([-M, M]\big), \quad W_2 \in C^0\big([-M, M]\backslash\{0\}\big),\\
\lim_{\tau\to 0+}\big(W_2(\tau) - W_2(-\tau)\big) = \frac{i\pi\mu U''(x_2^c)}{U'(x_2^c)}W_1(0).
\end{cases}
\tag{3.2.25}
$$

It is worth pointing out that the existence of the limit of $W_2(\tau) - W_2(-\tau)$ does not imply a simple jump discontinuity of W_2, which actually has a symmetric logarithmic singularity. In the distribution sense, the limit homogeneous Rayleigh equation (3.2.5) (with $\phi = 0$) along with (3.2.25) can be written as

$$W_\tau = (P.V.)_\tau \begin{pmatrix} 0 & 1 \\ 1 - \mu^2 + \frac{\mu^2 U''(x_2^c+\mu\tau)}{U(x_2^c+\mu\tau)-c} & 0 \end{pmatrix} W + \begin{pmatrix} 0 \\ \frac{i\pi\mu U''(x_2^c)}{U'(x_2^c)} W_1(0) \end{pmatrix} \delta(\tau). \quad (3.2.26)$$

Here $\delta(\tau)$ denotes the delta function of τ and "$(P.V.)_\tau$" indicate the principle value when the corresponding distributions are applied to test functions of τ. They occur in $W_{2\tau}$ only. In terms of the original unknown $y(x_2)$, the limit of (3.0.3) as $c_I \to 0+$ is

$$-y'' + k^2 y + (P.V.)_{x_2}\left(\frac{U''y}{U-c}\right) = -\frac{i\pi U''(x_2^c)}{U'(x_2^c)} y(x_2^c)\delta_{x_2}(x_2 - x_2^c), \quad (3.2.27)$$

where the subscript $\cdot_{x_2}$ indicates the distributions as generalized functions of x_2. For $c_I \to 0-$, the parallel results hold except with the complex conjugate. It also means that homogeneous Rayleigh equation takes different limit as $c_I \to \pm 0$.

Lemma 3.2.3. *General solutions of homogeneous (3.2.5) (with $\phi = 0$) along with (3.2.25) are*

$$W(\tau) = \begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c_R, \tau) & 1 \end{pmatrix} \tilde{B}_0(\mu, c_R, \tau) b_0, \quad b_0 = \begin{pmatrix} b_{01} \\ b_{02} \end{pmatrix} \in \mathbb{C}^2. \quad (3.2.28)$$

Moreover, $W(\tau) \in C^0$ if $W_1(0) = 0$.

Proof. On $[-M, 0)$ and $(0, M]$, (3.2.5) is regular and thus Lemma 3.2.1, in particular the form (3.2.15) of the general solutions implies the above (3.2.28) with parameters $b_0^\pm = (b_{01}^\pm, b_{02}^\pm)^T \in \mathbb{C}^2$. The continuity of $W_1(\tau)$ and the estimates of Γ and $\tilde{B}$ in Lemma 3.2.1 immediately yields $b_{01}^+ = b_{01}^-$. Finally $b_{02}^+ = b_{02}^-$ follows from the jump condition of $W_2(\tau)$ at $\tau = 0$ after writing $b_{02}^\pm$ using (3.2.22) and again using the estimates of Γ and $\tilde{B}$.

Finally, the continuity of $W(\tau)$ under the assumptions $W_1(0) = 0$ follows from (3.2.28), the Hölder continuity of $\tilde{B}$, and the logarithmic upper bound of Γ_0. $\square$

The following proposition provides the convergence estimates.

Proposition 3.2.4. *For any $M > 0$ satisfying (3.2.4) and $\alpha, \alpha' \in (0, 1)$, there exists $C > 0$ depending only on M, α, α', $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that, for any $0 < \epsilon < M$, $\tau \in [-M, M]$, and solutions $w(\tau)$ and $W(\tau)$ to (3.2.3) and (3.2.5) (with $\phi = 0$) in the forms (3.2.15) and (3.2.28) with parameters $b, b_0 \in \mathbb{C}^2$, respectively, the following hold:*

$$
\begin{aligned}
|w_1(\tau) - W_1(\tau) - (w_1(0) - W_1(0))| \leq C\big(&|\tau|(|b_2 - b_{02}| + \mu\epsilon^\alpha |b_{02}|) \\
&+ (|\tau| + \mu|\tau|^\alpha)|w_1(0) - W_1(0)| + \epsilon^{\alpha'}\mu|\tau|^{1-\alpha'}|W_1(0)|\big),
\end{aligned}
\tag{3.2.29}
$$

$$
\begin{aligned}
&|w_2(\tau) - W_2(\tau)| \\
\leq &C\Big(\mu\epsilon^\alpha|\tau|^{\frac{1-\alpha}{2}}(|W_1(0)| + |b_{02}|) + (1 + \mu\big|\log(|\tau| + \epsilon)\big|)|w_1(0) - W_1(0)| \\
&+ |b_2 - b_{02}| + \mu\big(\tfrac{\epsilon}{\epsilon+|\tau|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big)|W_1(0)|\Big).
\end{aligned}
\tag{3.2.30}
$$

Moreover, for any $\tau, \tau_0 \in [-M, M]$, let $\tau_* = \min\{|\tau|, |\tau_0|\} > 0$, we have

$$
\begin{aligned}
&|w_1(\tau) - W_1(\tau) - (w_1(\tau_0) - W_1(\tau_0))| \\
\leq &C\mu\epsilon^\alpha|\tau - \tau_0||W(\tau_0)| + C|\tau - \tau_0|^\alpha|w(\tau_0) - W(\tau_0)| \\
&+ C\mu\Big(|\tau - \tau_0|\big(\tfrac{\epsilon}{\epsilon+|\tau_0|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau_0^2}\big)\big) + \epsilon^{\alpha'}|\tau - \tau_0|^{1-\alpha'}\Big) \\
&\times \big(|W_1(\tau_0)| + |\tau_0|^\alpha|W(\tau_0)|\big),
\end{aligned}
\tag{3.2.31}
$$

$$
\begin{aligned}
|w_2(\tau) - W_2(\tau)| \leq & C\Big(\big(1 + \mu |\log(\epsilon + |\tau|)|\big) |w(\tau_0) - W(\tau_0)| \\
& + \mu \epsilon^\alpha |W(\tau_0)| + \mu\big(\tfrac{\epsilon}{\epsilon+\tau_*} + \log\big(1 + \tfrac{C\epsilon^2}{\tau_*^2}\big)\big)\big(|W_1(\tau_0)| + |\tau_0|^\alpha |W(\tau_0)|\big)\Big).
\end{aligned}
\tag{3.2.32}
$$

Remark 3.2.3. *When the above convergence estimate is applied in the rest of the manuscript, it always holds that $|W_1(\tau_0)| \leq M|\tau_0|^{\alpha_0}$ for some $\alpha_0 > 0$ which makes the right sides of (3.2.31) and (3.2.32) converging to 0 as $\epsilon \to 0$ locally uniformly in $\tau \neq 0$.*

Proof. We first work on the error estimates in terms of $W_1(0)$ and b_2. Let

$$
\tilde{w}(\tau) = \begin{pmatrix} 1 & 0 \\ \Gamma(\mu, c_R, \epsilon, \tau) & 1 \end{pmatrix} \tilde{B}(\mu, c_R, \epsilon, \tau) b_0.
$$

Controlling $w_2 - \tilde{w}_2$ and $\tilde{w}_2 - W_2$ by Corollary 3.2.1.1 ($w_2 - \tilde{w}_2$ by (3.2.19) and (3.2.20) in particular), where we recall the estimates are uniform in $\epsilon > 0$, we have

$$
\begin{aligned}
|w_2(\tau) - W_2(\tau)| \leq & |w_2(\tau) - \tilde{w}_2(\tau)| + |\tilde{w}_2(\tau) - W_2(\tau)| \\
\leq & C\Big(|b - b_0| + |\tilde{\Gamma}(\tau)||b_1 - b_{01}| + \mu\epsilon^\alpha |\tau|^{\alpha_1}|b_0| + \mu\big(\tfrac{\epsilon}{|\tau|+\epsilon} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big)|b_{01}|\Big) \\
\leq & C\Big(|b_2 - b_{02}| + (1 + \mu|\log(|\tau| + \epsilon)|)|w_1(0) - W_1(0)| \\
& + \mu\big(\tfrac{\epsilon}{\epsilon+|\tau|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big)|W_1(0)| + \mu\epsilon^\alpha|\tau|^{\alpha_1}\big(|W_1(0)| + |b_{02}|\big)\Big).
\end{aligned}
\tag{3.2.33}
$$

where $\alpha_1 \in [0, 1 - \alpha)$ and we also used

$$
W_1(0) = b_{01}, \quad w_1(0) = b_1.
$$

This completes the proof of inequality (3.2.30). The estimate (3.2.29) on $w_1 - W_1$ is derived by integrating $\partial_\tau(w_1 - W_1) = w_2 - W_2$ and using (3.2.17) and (3.2.13).

In the following, based on (3.2.33) we establish the error estimates in terms of initial values given at some $\tau_0 \neq 0$. From formula (3.2.22) we have

$$
\begin{aligned}
& b - b_0 - \big(\Gamma_0(\tau_0)W_1(\tau_0) - \Gamma(\tau_0)w_1(\tau_0)\big)(0,\,1)^T \\
=& \Big(\tilde{B}^{-1}(w - W) + (\tilde{B}^{-1} - \tilde{B}_0^{-1})W + \Big(\Gamma_0(W_1 - w_1)(\tilde{B}_0^{-1} - I) + \Gamma_0 w_1(\tilde{B}_0^{-1} - \tilde{B}^{-1}) \\
& + (\Gamma_0 - \Gamma)w_1(\tilde{B}^{-1} - I)\Big)(0,\,1)^T\Big)\Big|_{\tau_0}.
\end{aligned}
$$

From (3.2.18) and Lemma 3.2.1, one may estimate

$$
|\Gamma_0(\tau_0)||\tilde{B}(\tau_0)^{-1} - I| \leq C\big(1 + \mu\big|\log|\tau_0|\big|\big)(|\tau_0| + \mu^2|\tau_0|^{\alpha'}) \leq C(|\tau_0| + \mu|\tau_0|^{\alpha}),
$$

$$
\begin{aligned}
|\Gamma_0||\tilde{B}_0^{-1} - \tilde{B}^{-1}|\big|_{\tau_0} &\leq C\big(1 + \mu\big|\log|\tau_0|\big|\big)\mu\epsilon^{\alpha}(|\tau_0|^{1-\alpha} + \mu|\tau_0|^{\alpha_1}) \leq C\mu\epsilon^{\alpha}|\tau_0|^{\alpha_1}, \\
|\Gamma_0 - \Gamma||\tilde{B}^{-1} - I|\big|_{\tau_0} &\leq C\mu(|\tau_0| + \mu^2|\tau_0|^{\alpha'})\big(\mu\epsilon|\log\epsilon| + \tfrac{\epsilon}{\epsilon + |\tau_0|} + \log(1 + \tfrac{C\epsilon^2}{\tau_0^2})\big) \\
&\leq C\mu\epsilon^{\alpha}|\tau_0|^{\alpha_1},
\end{aligned}
$$

where $\alpha_1 \in [0, 1-\alpha)$. Therefore we obtain

$$
\begin{aligned}
& \big|b_2 - b_{02} - \big(\Gamma_0(\tau_0)W_1(\tau_0) - \Gamma(\tau_0)w_1(\tau_0)\big)\big| + \big|b_1 - b_{01}\big| \\
\leq & C\big(|(w - W)(\tau_0)| + \mu\epsilon^{\alpha}|\tau_0|^{\alpha_1}|W(\tau_0)|\big).
\end{aligned}
$$

Applying (3.2.23) and (3.2.24) to control b_0 in (3.2.33), we can estimate

$$
\begin{aligned}
|w_2(\tau) - W_2(\tau)| \leq & C\Big(\big(1 + \mu\big|\log(\epsilon + |\tau|)\big|\big)\big(|w(\tau_0) - W(\tau_0)| + \mu\epsilon^{\alpha}|\tau_0|^{\alpha_1}|W(\tau_0)|\big) \\
& + |(\Gamma_0 W_1 - \Gamma w_1)(\tau_0)| + \mu\epsilon^{\alpha}\big(|W(\tau_0)| + \mu\big|\log|\tau_0|\big||W_1(\tau_0)|\big) \\
& + \mu\big(\tfrac{\epsilon}{\epsilon + |\tau|} + \log\big(1 + \tfrac{C\epsilon^2}{\tau^2}\big)\big)\big(|W_1(\tau_0)| + |\tau_0|^{\alpha}|W(\tau_0)|\big)\Big)
\end{aligned}
$$

Inequality (3.2.32) is obtained by simplifying the above. In particular, we used

$$\epsilon^\alpha |\log \tau_*| \le C\epsilon^{\alpha'} \text{ if } \tau_* \ge \min\{1, \epsilon^2\} \text{ and}$$
$$\log\big(1 + \tfrac{C\epsilon^2}{\tau_*^2}\big) \ge \log\big(\tfrac{\epsilon^2}{\tau_*}\tfrac{1}{\tau_*}\big) \ge |\log \tau_*| \text{ if } \tau_* \le \min\{1, \epsilon^2\},$$

to absorb the term $\mu^2\epsilon^\alpha \big|\log|\tau_0|\big||W_1(\tau_0)|$.

Again we integrate (3.2.32) to derive (3.2.31). The only non-trivial terms are those involving τ_*

$$\begin{aligned}
&\Big|\int_{\tau_0}^{\tau} \tfrac{\epsilon}{\epsilon+\min\{|\tau'|,|\tau_0|\}} + \log\big(1 + \tfrac{C\epsilon^2}{\min\{|\tau'|,|\tau_0|\}^2}\big) d\tau'\Big| \\
\le & C\epsilon^\alpha |\tau-\tau_0|^{1-\alpha} + \big||\tau| - |\tau_0|\big|\big(\tfrac{\epsilon}{\epsilon+|\tau_0|} + \log\big(1+\tfrac{C\epsilon^2}{\tau_0^2}\big)\big) \\
\le & C\epsilon^\alpha |\tau-\tau_0|^{1-\alpha} + |\tau-\tau_0|\big(\tfrac{\epsilon}{\epsilon+|\tau_0|} + \log\big(1+\tfrac{C\epsilon^2}{\tau_0^2}\big)\big),
\end{aligned}$$

which are obtained by considering whether $|\tau'| \ge |\tau_0|$ and using (3.2.17). □

3.3 Two fundamental solutions to the homogeneous Rayleigh equation

In this section, we analyze and derive the basic estimates of two fixed solutions $y_\pm(k, c, x_2)$ to the homogeneous equation (3.0.1) with initial values

$$\begin{aligned}
& y_-(-h) = 0, \quad y_-'(-h) = 1, \\
\text{and} \quad y_+(0) = \frac{(U(0)-c)^2}{g+\sigma k^2}, &\quad y_+'(0) = 1 + \frac{U'(0)(U(0)-c)}{g+\sigma k^2},
\end{aligned} \tag{3.3.1}$$

which also depend on parameters k and $c \in \mathbb{C}$. The initial condition of y_+ at $x_2 = 0$ is motivated by the linearized capillary gravity water wave problem (2.2.6). (If it had been the linearized Euler equation at a shear flow in the channel, then naturally the boundary condition would be $y_+(0) = 0$ and $y_+'(0) = 1$.) As throughout this chapter, we often skip the arguments rather than x_2. Particularly when working near $x_2^c = U^{-1}(c_R)$, we shall continue using the notations introduced in section 3.2, like c_R, μ, ϵ, *etc.* The following lemma is standard. Due to conjugacy, we only consider $c_I \geq 0$.

Lemma 3.3.1. *For $c \notin U([-h,0])$ and $x_2 \in [-h,0]$, the solutions $y_\pm(k,c,x_2)$ are even in k, analytic in k^2 and c, and is C^{l_0+2} in x_2. Moreover $y_\pm(k,\bar{c},x_2) = \overline{y_\pm(k,c,x_2)}$.*

In the next step we give a priori estimates of $y_\pm(k,c,x_2)$. In particular, we consider up to three subintervals,

$$\mathcal{I}_2 := (x_{2l}, x_{2r}) = \left\{ x_2 \in [-h,0] \,:\, \frac{1}{|U(x_2)-c|} > \rho_0 \mu^{-\frac{3}{2}} \right\}, \quad \rho_0 = \frac{4}{h_0 \inf_{[-h_0-h,h_0]} U'}, \tag{3.3.2}$$

$$\mathcal{I}_1 = [-h, x_{2l}), \quad \mathcal{I}_3 = (x_{2r}, 0]. \tag{3.3.3}$$

Here $\mu = \langle k \rangle^{-1}$ as in (3.2.2). Clearly $[-h,0] = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ and any of these subintervals may be empty. If $\mathcal{I}_2 = \emptyset$, then $[-h,0]$ is considered as $\mathcal{I}_1$ for y_- and as $\mathcal{I}_3$ for y_+ in the statement of the following lemma. The choice of the above constant ρ_0 and the fact $0 \leq \mu \leq 1$ ensure

$$c_R \in U([-\tfrac{1}{4}h_0 - h, \tfrac{1}{4}h_0]) \ \text{ if } \ \mathcal{I}_2 \neq \emptyset. \tag{3.3.4}$$

Lemma 3.3.2. *For any $\alpha \in (0, \frac{1}{2})$, there exists $C > 0$ depending only on α, M, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that, for any $c \in \mathbb{C}\backslash U([-h,0])$, the following hold:*

$$|\mu^{-1} y_-(x_2) - \sinh(\mu^{-1}(x_2+h))| \leq C\mu^\alpha \sinh(\mu^{-1}(x_2+h)), \tag{3.3.5}$$

$$|\mu^{-1} y_+(x_2) - \sinh(\mu^{-1}x_2)| \leq C\big(\mu^\alpha + \mu|c|^2\big)\cosh(\mu^{-1}x_2), \tag{3.3.6}$$

for all $x_2 \in [-h,0]$. Moreover, if $\mathcal{I}_2 = \emptyset$, then for all $x_2 \in [-h,0]$,

$$|y_-'(x_2) - \cosh(\mu^{-1}(x_2+h))| \leq C\mu^\alpha \sinh(\mu^{-1}(x_2+h)), \tag{3.3.7}$$

$$|y'_+(x_2) - \cosh(\mu^{-1}x_2)| \le C\big(\mu^\alpha + \mu|c|^2\big)\cosh(\mu^{-1}x_2). \tag{3.3.8}$$

If otherwise $\mathcal{I}_2 \neq \emptyset$, then

$$|y'_-(x_2) - \cosh(\mu^{-1}(x_2+h))| \le \begin{cases} C\mu^\alpha \sinh(\mu^{-1}(x_2+h)), & x_2 \in \mathcal{I}_1 \\ C\mu^\alpha \cosh(\mu^{-1}(x_2+h)), & x_2 \in \mathcal{I}_3 \end{cases} \tag{3.3.9}$$

$$|y'_+(x_2) - \cosh(\mu^{-1}x_2)| \le C\mu^\alpha \cosh(\mu^{-1}x_2), \qquad x_2 \in \mathcal{I}_1 \cup \mathcal{I}_3, \tag{3.3.10}$$

and for $x_2 \in \mathcal{I}_2$,

$$\begin{aligned} \Big|y'_-(x_2) - \cosh(\mu^{-1}(x_2+h)) - \frac{U''(x_2^c)}{U'(x_2^c)} y_-(x_{2l}) \log|U(x_2)-c|\Big| \\ \le C\mu^\alpha\big(1 + \mu\big|\log|U(x_2)-c|\big|\big)\cosh(\mu^{-1}(x_2+h)), \end{aligned} \tag{3.3.11}$$

$$\begin{aligned} \Big|y'_+(x_2) - \cosh(\mu^{-1}x_2) - \frac{U''(x_2^c)}{U'(x_2^c)} y_+(x_{2r}) \log|U(x_2)-c|\Big| \\ \le C\mu^\alpha\big(1 + \mu\big|\log|U(x_2)-c|\big|\big)\cosh(\mu^{-1}x_2). \end{aligned} \tag{3.3.12}$$

Remark 3.3.1. *Even though the lemma assumes $c \in \mathbb{C}\backslash U([-h,0])$, the estimates are uniform in c and thus they also hold for the limits of solutions as $c_I \to 0+$, while the limits as $c_I \to 0-$ are the conjugates of those as $c_I \to 0+$.*

It is possible that $x_2^c \notin [-h,0]$ as the domain of U has been extended. However, the constant C in (3.3.7), (3.3.8), (3.3.9), and (3.3.10) are independent of the extensions of U satisfying (3.0.5).

Proof. The estimates of $y_\pm$ can be derived in exactly the same procedure by reversing the direction of the variable x_2. We shall focus on $y_-(k,c,x_2)$ and give a brief description on the argument for y_+ afterwards. The cases of x_2 close to and away from x_2^c will be considered differently based on Lemma 3.1.2 and Proposition 3.2.4, respectively.

Step 1. Assume $\mathcal{I}_1 \neq \emptyset$. We consider k in two cases. The first on is for those larger $|k|$ such that

$$\rho := \rho_0 \mu^{-\frac{3}{2}} k^{-2} (1 + |U''|_{C^0([-h_0-h,h_0])}) \leq \min\{1, C\mu^{\frac{1}{2}}\}, \tag{3.3.13}$$

where (3.1.2) is satisfied and Lemma 3.1.2 is applicable. Observe

$$\mu^{-1} - k = \sqrt{1+k^2} - k = \tfrac{1}{\mu^{-1}+k} \in (0, \mu), \tag{3.3.14}$$

and

$$\begin{aligned} |\sinh(\mu^{-1}(x_2+h)) - \sinh(k(x_2+h))| =& 2\sinh \tfrac{x_2+h}{\mu^{-1}+k} \cosh(\tfrac{1}{2}(\mu^{-1}+k)(x_2+h)) \\ \leq& C\mu \sinh(\mu^{-1}(x_2+h)), \end{aligned} \tag{3.3.15}$$

where the last inequality could be derived by considering whether $\mu^{-1}(x_2+h) \geq 1$. The same upper bound also holds for $\cosh$. Therefore applying Lemma 3.1.2 on $\mathcal{I}_1$ with $s=0$ and $C_0 = 0$, we immediately obtain the desired estimates (3.3.5), (3.3.7), (3.3.9) on y_- and y_-' on $\mathcal{I}_1$, respectively. Otherwise in the case of smaller $|k|$, the desired estimates follows from Lemma 3.1.3 with $\phi = 0$.

Step 2. Assume $\mathcal{I}_2 \neq \emptyset$ and $x_{2r} > x_{2l}$ otherwise step 1 has completed the proof. In this case, $x_2^c \in [-\frac{h_0}{4} - h, \frac{h_0}{4}]$ due to (3.3.4). Let

$$M = \rho_0^{-1} |(U')^{-1}|_{C^0} = \tfrac{1}{4} h_0, \tag{3.3.16}$$

which implies

$$\mathcal{I}_2 \subset [x_2^c - \mu M, x_2^c + \mu M] \subset [x_2^c - 2\mu M, x_2^c + 2\mu M] \subset [-h_0 - h, h_0]. \tag{3.3.17}$$

Therefore results in section 3.2 in the corresponding rescaled variables $w_{1,2}(\tau)$ and $x_2 = x_2^c + \mu\tau$ given in (3.2.2) are applicable. Moreover the definition of $\mathcal{I}_2$ further yields

$$|\tau| = \mu^{-1}|x_2 - x_2^c| \le C\mu^{\frac{1}{2}}, \quad \forall\, x_2 \in \mathcal{I}_2.$$

Let

$$\tau_0 = \mu^{-1}(x_{2l} - x_2^c).$$

Lemma 3.2.2 (with $\phi = 0$) implies that, for any $x_2 \in \mathcal{I}_2$

$$\begin{aligned}
&\Big| y_-'(x_2) + \tfrac{U''(x_2^c)}{U'(x_2^c)} y_-(x_{2l}) \log \tfrac{|U(x_{2l})-c|}{|U(x_2)-c|} - y_-'(x_{2l})\Big| \\
\le & C\big(1 + \mu^{\frac{\alpha'}{2}}\big|\log|U(x_{2l}) - c|\big|\big)|y_-(x_{2l})| \\
&\quad + C\big(\mu^{\frac{\alpha'}{2}} + \mu|\tau_0|^{\alpha'}\big|\log|U(x_2) - c|\big|\big)\big(\mu^{-1}|y_-(x_{2l})| + |y_-'(x_{2l})|\big),
\end{aligned}$$

for any $\alpha' \in (0,1)$. Moving the $\log|U(x_{2l}) - c|$ term to the right side, we obtain

$$\begin{aligned}
&\Big| y_-'(x_2) - \tfrac{U''(x_2^c)}{U'(x_2^c)} y_-(x_{2l}) \log|U(x_2) - c| - y_-'(x_{2l})\Big| \\
\le & C\big(1 + \big|\log|U(x_{2l}) - c|\big|\big)|y_-(x_{2l})| \\
&\quad + C\big(\mu^{\frac{\alpha'}{2}} + \mu|\tau_0|^{\alpha'}\big|\log|U(x_2) - c|\big|\big)\big(\mu^{-1}|y_-(x_{2l})| + |y_-'(x_{2l})|\big).
\end{aligned} \tag{3.3.18}$$

Notice that, no matter whether $\mathcal{I}_1 = \emptyset$ or not, (3.3.5) and (3.3.7) are satisfied at x_{2l} due to either the initial condition of $y_-(x_2)$ or the above step 1. On the one hand, regarding the above first term on the right side, it holds that either $y_-(x_{2l}) = 0$ if $x_{2l} = -h$ or $\mu^{\frac{3}{2}} \le C|x_2^c - x_{2l}|$ if $x_{2l} > -h$, hence this term would only contribute an error term of at most $O(\mu^{-\alpha''}|y_-(x_{2l})|)$, for any $\alpha'' > 0$, in the upper bounds. On the other hand, $0 \le x_2 - x_{2l} \le C\mu^{\frac{3}{2}}$ implies that replacing the above $\mu^{-1}y_-(x_{2l})$, $y_-'(x_{2l})$ and $\cosh(\mu^{-1}(x_{2l}+h))$ by $\cosh(\mu^{-1}(x_2+h))$ would also only produce an error terms of at most $O\big(\mu^{\frac{\alpha'}{2}}\cosh(\mu^{-1}(x_2+$

$h)))$ in the upper bounds. Therefore we have

$$\big|y_-'(x_2) - \cosh(\mu^{-1}(x_2+h)) - \tfrac{U''(x_2^c)}{U'(x_2^c)} y_-(x_{2l}) \log|U(x_2)-c|\big| \\ \leq C\big(\mu^{\frac{\alpha'}{2}} + \mu|\tau_0|^{\alpha'}\big|\log|U(x_2)-c|\big|\big)\cosh(\mu^{-1}(x_2+h)), \tag{3.3.19}$$

and thus (3.3.11) follows by letting $\alpha' = 2\alpha$.

Integrating (3.3.19) over $[x_{2l}, x_2] \subset \mathcal{I}_2$, we have, for $\alpha' \in (2\alpha, 1)$,

$$\begin{aligned} &|\mu^{-1}y_-(x_2) - \sinh(\mu^{-1}(x_2+h))| \\ \leq& C\mu^\alpha \sinh(\mu^{-1}(x_{2l}+h)) + \frac{C}{\mu}\int_{x_{2l}}^{x_2} |y_-(x_{2l})|\big(1 + \big|\log|x_2' - x_2^c|\big|\big) \\ &\qquad + \big(\mu^{\frac{\alpha'}{2}} + \mu|\tau_0|^{\alpha'}\big|\log|x_2'-x_2^c|\big|\big)\cosh(\mu^{-1}(x_{2l}+h))dx_2' \\ \leq& C\mu^\alpha \sinh(\mu^{-1}(x_2+h)) + C|\tau_0|^{\alpha'}\cosh(\mu^{-1}(x_{2l}+h))\int_{x_{2l}}^{x_2}\big|\log|x_2'-x_2^c|\big|dx_2'. \end{aligned}$$

where we used (3.3.5), $|x_2 - x_{2l}| \leq C\mu^{\frac{3}{2}}$, and and the first term of the right side of (3.3.18) was incorporated into others as remarked just below (3.3.18). For $|x_2 - x_{2l}| \leq \frac{1}{2}|x_{2l} - x_2^c|$, we have

$$\begin{aligned} |\tau_0|^{\alpha'}\int_{x_{2l}}^{x_2}\big|\log|x_2'-x_2^c|\big|dx_2' \leq& \mu^{-\alpha'}|x_{2l}-x_2^c|^{\alpha'}|x_2-x_{2l}|\big(1+\big|\log|x_{2l}-x_2^c|\big|\big) \\ \leq& C\mu^\alpha|x_2-x_{2l}|, \end{aligned}$$

while for $|x_2 - x_{2l}| \geq \frac{1}{2}|x_{2l} - x_2^c|$,

$$|\tau_0|^{\alpha'}\int_{x_{2l}}^{x_2}\big|\log|x_2'-x_2^c|\big|dx_2' \leq C\mu^{-\alpha'}|x_{2l}-x_2^c|^{\alpha'}|x_2-x_{2l}|^{1-\frac{1}{3}(\alpha'-2\alpha)} \leq C\mu^\alpha|x_2-x_{2l}|.$$

Therefore we obtain

$$|\mu^{-1}y_-(x_2) - \sinh(\mu^{-1}(x_2+h))|$$

$$\leq C\mu^\alpha\big(\sinh(\mu^{-1}(x_2+h)) + |x_2 - x_{2l}|\cosh(\mu^{-1}(x_{2l}+h))\big)$$
$$\leq C\mu^\alpha \sinh(\mu^{-1}(x_2+h))$$

which proves (3.3.5) on $\mathcal{I}_2$.

Step 3. Assume $\mathcal{I}_3 = [x_{2r}, 0] \neq \emptyset$, which implies $x_{2r} > -h$. In this case, surely $\mathcal{I}_2 \neq \emptyset$ either and $\mu^{\frac{3}{2}} \leq C|U(x_{2r}) - c|$. With (3.3.5) for y_- and (3.3.11) for y_-' established at $x_2 = x_{2r}$, $y_-(x_2)$ satisfies assumption (3.1.11) for the interval $\mathcal{I}_3$ with $\Theta_1 = \sinh$, $\Theta_2 = \cosh$, and $C_0 = C\mu^\alpha$.

As in the step 1, for larger $|k|$ so that (3.3.13) holds, the desired estimates (3.3.5) and (3.3.11) in $\mathcal{I}_3$ follow directly from (3.3.14), (3.3.15), and Lemma 3.1.2.

For smaller k, say, $|k| \leq k_1$, we express $y_-(x_2)$ and $y_-'(x_2)$ in terms of $w(\tau)$, $\tau \in [\mu^{-1}(-h - x_2^c), -\mu^{-1}x_2^c]$, as in (3.2.2). Let

$$M = (1+k_1^2)^{\frac{1}{2}}(2h_0 + h), \quad \tau_0 = \mu^{-1}(-h - x_2^c).$$

Since $\mathcal{I}_2 \neq \emptyset$, otherwise $[-h, 0] = \mathcal{I}_1$ for $y_-(x_2)$, it along with (3.3.4) and $|k| \leq k_1$ implies

$$x_2^c \in [-h_0 - h, h_0] \Longrightarrow |h + x_2^c|, \, |x_2^c| \leq 2h_0 + h \Longrightarrow [\mu^{-1}(-h - x_2^c), -\mu^{-1}x_2^c] \subset [-M, M].$$

Namely, the domain of $w(\tau)$ is contained in $[-M, M]$. Applying (3.2.21b) (with $\phi = 0$), using $w_1(\tau_0) = 0$, $w_2(\tau_0) = 1$, and

$$\mathcal{I}_3 \neq \emptyset \Longrightarrow \rho_0^{-1}\mu^{\frac{3}{2}} = |U(x_{2r}) - c| \leq |U(x_2) - c|, \quad \forall x_2 \in \mathcal{I}_3,$$

we obtain $|y_-'(x_2)| \leq C$ on $\mathcal{I}_3$. It in turn implies

$$|\mu^{-1}y_-(x_2) - \sinh(\mu^{-1}(x_2+h))|$$
$$\leq \mu^{-1}|y_-(x_2) - y_-(x_{2r})| + |\mu^{-1}y_-(x_{2r}) - \sinh(\mu^{-1}(x_{2r}+h))|$$

$$
\begin{aligned}
&+ |\sinh(\mu^{-1}(x_{2r}+h)) - \sinh(\mu^{-1}(x_2+h))| \\
\le & C\big(|x_2 - x_{2r}| + \sinh(\mu^{-1}(x_{2r}+h))\big) \le C\sinh(\mu^{-1}(x_2+h)).
\end{aligned}
$$

Therefore (3.3.5) and (3.3.9) hold on $\mathcal{I}_3$ due to $|k| \le k_1$.

Estimating y_+ Finally, we give a brief sketch of the argument for y_+, for which we proceed from $\mathcal{I}_3$ to $\mathcal{I}_1$.

Suppose $\mathcal{I}_3 \neq \emptyset$. The initial values of y_+ at $x_2 = 0$ satisfy (3.1.12) with $\Theta_1 = \Theta_2 = \cosh$ and $C_0 = C(1+|c|^2)\mu$. For larger $|k|$ so that (3.3.13) holds, the desired estimates (3.3.6) and (3.3.8) in $\mathcal{I}_3$ follow directly from (3.3.14), (3.3.15), and Lemma 3.1.2. The estimates for smaller k is again a consequence of Lemma 3.1.3.

Suppose $\mathcal{I}_2 \neq \emptyset$ which implies $|c| \le C$. Inequality (3.3.18) with x_{2l} replaced by x_{2r} still follows from exactly the same argument, namely, for $x_2 \in \mathcal{I}_2$ and any $\alpha' \in [0,1)$,

$$
\begin{aligned}
&\Big| y_+'(x_2) - \tfrac{U''(x_2^c)}{U'(x_2^c)} y_+(x_{2r}) \log |U(x_2) - c| - y_+'(x_{2r}) \Big| \\
\le & C\big(1 + \big|\log|U(x_{2r}) - c|\big|\big)|y_+(x_{2r})| \\
&+ C\big(\mu^{\frac{\alpha'}{2}} + \mu|\tau_0|^{\alpha'}\big|\log|U(x_2) - c|\big|\big)\big(\mu^{-1}|y_+(x_{2r})| + |y_+'(x_{2r})|\big).
\end{aligned}
$$

If $x_{2r} = 0$, then

$$
\big|\log|U(x_{2r}) - c|\big||y_+(x_{2r})| = \big|\log|U(0) - c|\big||y_+(0)| \le C\mu^2 \cosh \mu^{-1} x_2.
$$

Otherwise, $x_{2r} < 0$ and thus, for any $\alpha' \in (0,1)$,

$$
|U(x_{2r}) - c| = \rho_0^{-1}\mu^{\frac{3}{2}} \implies \big|\log|U(x_{2r}) - c|\big||y_+(x_{2r})| \le \mu^{\alpha'} \cosh \mu^{-1} x_2,
$$

where (3.3.6) at $x_2 = x_{2r}$ was also used. These estimates, along with (3.3.6) and (3.3.8) at $x_2 = x_{2r}$ yield (3.3.12) on $\mathcal{I}_2$. Inequality (3.3.6) follows from direct integrating the estimate on y_+', actually without going through the technical argument at the end of step 2 for y_- since the $\cosh$, instead of $\sinh$, is in the upper bound in (3.3.6).

Suppose $\mathcal{I}_1 \neq \emptyset$ where it must hold $\mathcal{I}_2 \neq \emptyset$ and $|c| \leq C$. From step 2, $y_+(x_2)$ satisfies assumption (3.1.12) for the interval $\mathcal{I}_1$ with $\Theta_1 = \Theta_2 = \cosh$, and $C_0 = C\mu^\alpha$. For larger $|k|$, the desired estimates (3.3.6) and (3.3.10) follow from Lemma 3.1.2 and for smaller $|k|$ from Lemma 3.1.3. $\square$

3.4 Limits of solutions to the homogeneous Rayleigh equation with $c_I = 0+$

Now that the convergence of solutions of the Rayleigh equation as $c_I \to 0+$ has been established in Proposition 3.2.4, in this section, we shall focus on the analysis of the limit equation (3.2.5) along with the jump condition (3.2.25) at the singularity $\tau = 0$. In this section we consider $c = U(x_2^c) \in U\big([-\frac{1}{2}h_0 - h, \frac{1}{2}h_0]\big)$ unless otherwise specified. As transformation (3.2.15) was rather helpful in the proof of Proposition 3.2.4, its limit would also turn out to be an effective tool in the study of (3.2.5). However $\tilde{B}(\tau)$ as well as $\tilde{B}_0(\tau)$ appears only Hölder in τ, or equivalently in x_2. In the notations given in (3.2.2) in section 3.2, we first prove the following lemma to isolate the singularity in $\tilde{B}_0$. Recall $U \in C^{l_0}$, x_2^c and c_R correspond to each other via (3.0.6), $\tilde{U}, U_1 \in C^{l_0-1}$, and $U_2 \in C^{l_0-2}$ are defined in (3.2.7), and $\Gamma_0(\mu, c, \tau) = \Gamma(\mu, c, \epsilon = 0, \tau)$ in (3.2.9).

Lemma 3.4.1. *There exists a unique continuous real 2×2 matrix valued $B(\mu, c, \tau)$ satisfying*

$$B_\tau = \begin{pmatrix} 0 & 1 \\ 1 - \mu^2 + \frac{\mu U_2}{\tilde{U}} & 0 \end{pmatrix} B - B \begin{pmatrix} 0 & 0 \\ 1 - \mu^2 + \frac{\mu U_2}{\tilde{U}} & 0 \end{pmatrix}, \quad B(\mu, c, 0) = I_{2\times 2}. \quad (3.4.1)$$

Moreover the following hold.

1. *The matrix $B(\mu, c, \tau)$ is C^{l_0-2} in $c_R \in U\big([-\frac{1}{2}h_0 - h, \frac{1}{2}h_0]\big)$, τ, and μ and*

$$\det B = 1, \quad B_\tau(\mu, c, 0) = \begin{pmatrix} -\frac{\mu U''(x_2^c)}{U'(x_2^c)} & 1 \\ -2\frac{\mu^2 U''(x_2^c)^2}{U'(x_2^c)^2} & \frac{\mu U''(x_2^c)}{U'(x_2^c)} \end{pmatrix},$$

$$B(0, c, \tau) = \begin{pmatrix} \cosh\tau - \tau\sinh\tau & \sinh\tau \\ \sinh\tau - \tau\cosh\tau & \cosh\tau \end{pmatrix} = \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix}.$$

Moreover for any $M > 0$ satisfying (3.2.4), there exists $C > 0$ depending only on $|U'|_{C^{l_0-1}}$ and $|(U')^{-1}|_{C^0}$, such that $|B|_{C^{l_0-2}} \le C$.

2. *B and $\tilde{B}_0$ are conjugate, namely,*

$$B(\mu, c, \tau) = \begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c, \tau) & 1 \end{pmatrix} \tilde{B}_0(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ -\Gamma_0(\mu, c, \tau) & 1 \end{pmatrix}. \tag{3.4.2}$$

3. *General solutions to (3.2.5) satisfying (3.2.25) are*

$$\begin{aligned} W(\tau) = \begin{pmatrix} W_1(\tau) \\ W_2(\tau) \end{pmatrix} &= B(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c, \tau) & 1 \end{pmatrix} \big(b - \tilde{\Phi}_0(\mu, c, \tau)\big), \\ &= \begin{pmatrix} (B_{11} + \Gamma_0 B_{12})(b_1 - \tilde{\Phi}_{01}) + B_{12}(b_2 - \tilde{\Phi}_{02}) \\ (B_{21} + \Gamma_0 B_{22})(b_1 - \tilde{\Phi}_{01}) + B_{22}(b_2 - \tilde{\Phi}_{02}) \end{pmatrix}, \\ b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} &\in \mathbb{C}^2, \end{aligned} \tag{3.4.3}$$

where $B_{j_1 j_2}$ are the entries of B and $\tilde{\Phi}_0 = (\tilde{\Phi}_{01}, \tilde{\Phi}_{02})^T = \lim_{\epsilon\to 0+} \tilde{\Phi}$ with $\tilde{\Phi}$ given in (3.2.11).

4. *If $\phi \equiv 0$, the general solution $W(\tau)$ to (3.2.5)–(3.2.25) with $b \in \mathbb{C}^2$ as in (3.4.3)*

satisfies $W_1 \in C^{\alpha'}_{loc}$ *for any* $\alpha' \in (0,1)$*,* $W_1(0) = b_1$*, and*

$$\lim_{\tau \to 0} \Big(W_2(\tau) - b_2 - W_1(0) \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log(U'(x_2^c)|\tau|) + \frac{i\pi}{2}(sgn(\tau)+1) \big) \Big) = 0.$$

5. *Finally,* $W(\tau)$ *are* C^{l_0-2} *in* μ*,* c_R*, and* τ *if* $\phi \equiv 0$ *and* $b_1 = W_1(0) = 0$*.*

Remark 3.4.1. *If needed, higher order Taylor expansions of* B *can be obtained based on (3.4.4) through rather standard calculations in the analysis of local invariant manifolds.*

One is reminded that both $\Gamma_0(\tau)$ has a logarithmic singularity and a jump at $\tau = 0$ which leads to such singularities of $W_2(\tau)$ there even in the homogeneous case. Since $\Gamma_0 \notin \mathbb{R}$ for $\tau > 0$, $\tilde{B}_0$ should not be real for $\tau > 0$. Hence it is a non-obvious statement that this conjugate matrix B is real. The above lemma isolates the singularity of $\tilde{B}_0$ into the explicit Γ_0 along with the smooth B. Conceptually, the smoothness of B in c_R is related to the smoothness of the spectral resolution of the identity with respect to the spectral parameter, and thus would play crucial role in proving the partial inviscid damping to the linearized Euler equation at the shear flow $U(x_2)$.

Proof. The construction of $B(\mu, c, \tau)$ is adapted from the one in [11], where the main issue is to handle the singularity caused by $\tilde{U}(\mu, c, 0) = 0$. We first make (3.4.1) autonomous by changing the independent variable an auxiliary one s such that $\tau_s = \tau$ and thus we have

$$\begin{cases} B_s = \begin{pmatrix} 0 & \tau \\ (1-\mu^2)\tau + \frac{\mu \tau U_2}{\tilde{U}} & 0 \end{pmatrix} B - B \begin{pmatrix} 0 & 0 \\ (1-\mu^2)\tau + \frac{\mu \tau U_2}{\tilde{U}} & 0 \end{pmatrix}, \\ \tau_s = \tau. \end{cases} \tag{3.4.4}$$

Obviously solutions to (3.4.1) corresponds (up to a translation in s) to those to the C^{l_0-2} ODE system (3.4.4) of 5-dim which converge to $(I_{2\times 2}, 0)$ as $s \to -\infty$, namely those on the unstable manifold of the steady state $(I_{2\times 2}, 0)$. The linearized system of (3.4.4) is given by

$$\begin{cases} B_s = \frac{\mu U''(x_2^c)}{U'(x_2^c)}\mathcal{A}B + \tau \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{where } \mathcal{A}B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} B - B \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \\ \tau_s = \tau. \end{cases}$$

It is easy to compute that, on the one hand, an eigenvector associated to the eigenvalue 1 is

$$(B_1, 1), \quad B_1 = \begin{pmatrix} -\frac{\mu U''(x_2^c)}{U'(x_2^c)} & 1 \\ -2\frac{\mu^2 U''(x_2^c)^2}{U'(x_2^c)^2} & \frac{\mu U''(x_2^c)}{U'(x_2^c)} \end{pmatrix}.$$

On the other hand, one may verify

$$e^{s\mathcal{A}}B = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$$

which implies that in the 4-dim center subspace $\{\tau = 0\}$ there is no any decay backward in s. Therefore there exists a unique C^{l_0-2} unstable manifold of 1-dim which corresponds a unique solution $B(\mu, c, \tau)$ satisfying $B(\mu, c, 0) = I$ and $B_\tau(\mu, c, 0) = B_1$ and C^{l_0-2} in all its variables. In fact, the 4-dim center subspace $\{\tau = 0\}$ is also invariant under the nonlinear system (3.4.4), where the flow is given by the above non-decaying linear flow of conjugation. Therefore this $B(\mu, c_R, \tau)$ is the only solution to (3.4.1) decaying to I as $s \to -\infty$, or equivalently $\tau \to 0+$. Even though this construction is local in τ, the domain of B can be extended due to the linearity of equation (3.4.1). The property $\det B = 1$ follows directly from its equation (3.4.1).

With the existence of the C^{l_0-2} solution $B(\mu, c, \tau)$ to (3.4.1) established, letting $\mu = 0$

in (3.4.4) and then transforming back to (3.4.1), we have

$$B_\tau(0,c,\tau) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B(0,c,\tau) - B(0,c,\tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B(0,c,0) = I_{2\times 2}.$$

This equation can be solved explicitly to yield

$$B(0,c,\tau) = \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tau & 1 \end{pmatrix} = \begin{pmatrix} \cosh\tau - \tau\sinh\tau & \sinh\tau \\ \sinh\tau - \tau\cosh\tau & \cosh\tau \end{pmatrix}.$$

The conjugation relation is the consequence of the facts that both B and the right side of (3.4.2) a.) both are equal to I at $\tau = 0$, b.) both satisfy the same ODE system (3.4.1) for $\tau \neq 0$, c.) are both continuous in τ due to the construction of B and (3.2.12) in Lemma 3.2.1, and d.) the uniqueness of solutions to (3.4.1) satisfying a.)–c.), which is obtained in the above construction based on the local invariant manifold theory.

Formula (3.4.3) of the general solutions follows from the conjugacy relation (3.4.2) and Lemma 3.2.3. Under the assumption $\phi \equiv 0$, since $W_2(\tau)$ has at most logarithmic singularity at $\tau = 0$ and $W_{1\tau} = W_2$, the Hölder continuity of W_1 in τ follows. From formula (3.4.3) and $|B(\mu,c,0) - I| = O(|\tau|)$, we obtain $W_1(0) = b_1$. The limit property of $W_2(\tau) - b_2$ also follows from similar calculation. Finally, the C^{l_0-3} smoothness of $W(\tau)$ under the assumptions $\phi \equiv 0$ and $b_1 = W_1(0) = 0$ is again obvious from the representation of the solution (3.4.3). The proof of the lemma is complete. $\square$

For $c \in U([-\frac{h_0}{2} - h, \frac{h_0}{2}])$, with the help of $B(\mu,c,\tau)$ and Lemma 3.4.1 we shall analyze the 2×2 fundamental matrices in two different forms of the homogeneous problem (3.2.5) with the condition (3.2.25) at $\tau = 0$

$$\begin{aligned} S^0(\mu, c, \tau) &= B(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c, \tau) & 1 \end{pmatrix}, \\ S(\mu, c, \tau, \tau_0) &= B(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c, \tau) - \Gamma_0(\mu, c, \tau_0) & 1 \end{pmatrix} B(\mu, c, \tau_0)^{-1}, \end{aligned} \tag{3.4.5}$$

where τ_0 in S is the initial value of the independent variable and hence $S(\mu, c, \tau_0, \tau_0) = I$. To analyze S^0 and S, let

$$\begin{aligned} \tilde{S}^0(\mu, c, \tau) =& B(\mu, c, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} B_{12}(\mu, c, \tau) & 0 \\ B_{22}(\mu, c, \tau) & 0 \end{pmatrix}, \\ \tilde{S}(\mu, c, \tau, \tau_0) =& B(\mu, c, \tau) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} B(\mu, c, \tau_0)^{-1} \\ =& \begin{pmatrix} B_{12}(\mu, c, \tau) B_{22}(\mu, c, \tau_0) & -B_{12}(\mu, c, \tau) B_{12}(\mu, c, \tau_0) \\ B_{22}(\mu, c, \tau) B_{22}(\mu, c, \tau_0) & -B_{22}(\mu, c, \tau) B_{12}(\mu, c, \tau_0) \end{pmatrix}, \end{aligned} \tag{3.4.6}$$

where $\det B = 1$ was used to compute the more explicit form of $\tilde{S}$ in the above, and

$$\begin{aligned} S^0_{err} &= S^0 - \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix} - \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tau| + \tfrac{i\pi}{2}\big(sgn(\tau) + 1\big)\big) \tilde{S}^0, \\ S_{err} &= S - \begin{pmatrix} \cosh(\tau - \tau_0) & \sinh(\tau - \tau_0) \\ \sinh(\tau - \tau_0) & \cosh(\tau - \tau_0) \end{pmatrix} \\ &\quad - \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}\big(sgn(\tau) - sgn(\tau_0)\big)\big) \tilde{S}. \end{aligned} \tag{3.4.7}$$

The following lemma provides some very basic estimates on S. More detailed ones on S_{jl} will be derived when needed.

Lemma 3.4.2. *Assume $U \in C^{l_0}$, $l_0 \geq 3$. The fundamental matrices $S^0(\mu, c, \tau)$ and*

$S(\mu, c, \tau, \tau_0)$ and their entries S^0_{jl} and $S_{j_1 j_2}$ satisfy the following.

1. *S^0 is C^{l_0-3} in its variables if $\tau \neq 0$ and S is C^{l_0-3} in its variables if $\tau \neq 0$ and $\tau_0 \neq 0$.*

2. *S^0_{11}, S^0_{12}, and S^0_{22} are C^α in τ and C^{l_0-3} in μ and c. If $\tau_0 \neq 0$, then S_{11} and S_{12} are C^α in τ and C^{l_0-3} in μ, c, and τ_0.*

3. *If $\tau \neq 0$, then S_{12} and S_{22} are C^α in τ_0 and C^{l_0-3} in μ, c, and τ.*

4. *S_{12} and $\tau_0 S_{11}$ are C^α in τ_0 and τ and C^{l_0-3} in μ and c.*

5. *S_{err} and S^0_{err} are C^{l_0-3} in their arguments.*

6. *Assume $l_0 \geq 5$. For any M satisfying (3.2.4), there exists $C > 0$ depending only on M, $|U'|_{C^{l_0-1}}$, and $|(U')^{-1}|_{C^0}$ such that for any $\tau, \tau_0 \in [-M, M]$,*

$$|\partial_\mu^{j_1} D_1 \ldots D_{j_2} S_{err}| \leq C|\tau - \tau_0|, \quad |\partial_\mu^{j_1} \partial_c^{j_2} S^0_{err}| \leq C|\tau|,$$

for $1 \leq j_1 \leq l_0 - 4$, $0 \leq j_2 \leq l_0 - 4 - j_1$, and $D_1, \ldots D_j \in \{\partial_c, \frac{1}{U'(x_2^c)}(\partial_\tau + \partial_{\tau_0})\}$, and

$$|D_1 \ldots D_{j_2} S_{err}| \leq C\mu|\tau - \tau_0|, \quad |\partial_c^{j_2} S^0_{err}| \leq C\mu|\tau|,$$

for $1 \leq j_2 \leq l_0 - 5$.

The reason we consider $\partial_\tau + \partial_{\tau_0}$ of S instead of individual ∂_τ or ∂_{τ_0} is not only that it yields better estimate. Recall the change of variables $\tau = \mu^{-1}(x_2 - x_2^c)$. The above fundamental matrix is in the form of $S\big(\mu, c, \mu^{-1}(x_2 - x_2^c), \mu^{-1}(x_{20} - x_2^c)\big)$. Therefore $\partial_c - \frac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}$ corresponds to the partial differentiation with respect to c in the (c, x_2) coordinates. Here we also used

$$\partial_c x_2^c = \tfrac{1}{U'(x_2^c)}. \tag{3.4.8}$$

Proof. The argument for S^0 and S are very similar and we shall mainly focus on S. Let

$$S^{\#}(\mu, c, \tau, \tau_0) = B(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ g_{\#}(\mu, c, \tau, \tau_0) & 1 \end{pmatrix} B(\mu, c, \tau_0)^{-1}, \tag{3.4.9}$$

where

$$\begin{aligned} g_{\#}(\mu, c, \tau, \tau_0) =& \Gamma_0(\mu, c, \tau) - \Gamma_0(\mu, c, \tau_0) - \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \Big(\log \big|\tfrac{\tau}{\tau_0}\big| + \tfrac{i\pi}{2}\big(sgn(\tau) - sgn(\tau_0)\big)\Big) \\ =& (1-\mu^2)(\tau - \tau_0) + \gamma(\mu, c, 0, \tau) - \gamma(\mu, c, 0, \tau_0) \\ & + \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \log \Big| \tfrac{\tau_0}{\tau} \tfrac{U(x_2^c + \mu\tau) - U(x_2^c)}{U(x_2^c + \mu\tau_0) - U(x_2^c)} \Big|. \end{aligned}$$

Clearly we have

$$S = S^{\#} + \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \Big(\log \big|\tfrac{\tau}{\tau_0}\big| + \tfrac{i\pi}{2}\big(sgn(\tau) - sgn(\tau_0)\big)\Big) \tilde{S}. \tag{3.4.10}$$

The C^{l_0-3} (and C^{l_0-2} in μ) smoothness follows from that of Γ_0 and B. The C^α Hölder regularity in statements (2)–(4) is due to $B_{12}(\mu, c, 0) = 0$.

From the definition (3.2.8) of γ, we have $\gamma \equiv 0$ if $\mu = 0$. Straight forward computation based on Lemma 3.4.1 yields

$$S^{\#}(0, c, \tau, \tau_0) = \begin{pmatrix} \cosh(\tau - \tau_0) & \sinh(\tau - \tau_0) \\ \sinh(\tau - \tau_0) & \cosh(\tau - \tau_0) \end{pmatrix}, \quad S^{\#}(\mu, c, \tau_0, \tau_0) = I.$$

Therefore

$$S_{err}(\mu, c, \tau, \tau_0) = S^{\#}(\mu, c, \tau, \tau_0) - S^{\#}(0, c, \tau, \tau_0).$$

By mimicking $f(\mu, s) = f(0, s) + \mu \int_0^1 f_\mu(\theta_1 \mu, 0) + s \int_0^1 f_{\mu s}(\theta_1 \mu, \theta_2 s) d\theta_2 d\theta_1$, we have

$$|S^{\#}(\mu, c, \tau, \tau_0) - S^{\#}(0, c, \tau, \tau_0)|$$

$$
\begin{aligned}
&= \mu|\tau-\tau_0| \left| \int_0^1 \int_0^1 \partial_\mu \partial_\tau S^{\#}(\theta_1\mu, c, \tau_0+\theta_2(\tau-\tau_0), \tau_0) d\theta_2 d\theta_1 \right| \\
&\leq C\mu|\tau-\tau_0|.
\end{aligned}
$$

Moreover, for $1 \leq j_2 \leq l_0 - 5$ and $D_1, \dots D_{j_2} \in \{\partial_c, \frac{1}{U'(x_2^c)}(\partial_\tau + \partial_{\tau_0})\}$, we have

$$
D_1 \dots D_{j_2} S^{\#} = 0, \text{ if } \mu = 0, \qquad \partial_\mu D_1 \dots D_{j_2} S^{\#} = 0, \text{ if } \tau = \tau_0.
$$

A similar procedure yields

$$
\begin{aligned}
&|D_1 \dots D_{j_2} S^{\#}(\mu, c, \tau, \tau_0)| \\
=&\mu|\tau-\tau_0| \left| \int_0^1 \int_0^1 \partial_\mu \partial_\tau D_1 \dots D_{j_2} S^{\#}(\theta_1\mu, c, \tau_0+\theta_2(\tau-\tau_0), \tau_0) d\theta_2 d\theta_1 \right| \\
\leq& C\mu|\tau-\tau_0|.
\end{aligned}
$$

Finally, since $S^{\#}$ is C^{l_0-3} in all variables, for $l_0 \geq 5$, $1 \leq j_1 \leq l_0 - 4$, and $0 \leq j_2 \leq l_0 - j_1 - 4$, the estimate on $\partial_\mu^{j_1} D_1 \dots D_{j_2} S_{err}$ follows from its C^1 smoothness and vanishing at $\tau = \tau_0$.

To analyze S^0, parallelly we consider

$$
S_0^{\#}(\mu, c, \tau) = B(\mu, c, \tau) \begin{pmatrix} 1 & 0 \\ g_{0\#}(\mu, c, \tau) & 1 \end{pmatrix},
$$

where

$$
\begin{aligned}
g_{0\#}(\mu, c, \tau) =& \Gamma_0(\mu, c, \tau) - \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tau| + \tfrac{i\pi}{2}(sgn(\tau)+1) \big) \\
=& (1-\mu^2)\tau + \gamma(\mu, c, 0, \tau) + \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \log \left| \tfrac{U(x_2^c+\mu\tau) - U(x_2^c)}{\mu U'(x_2^c)\tau} \right|.
\end{aligned}
$$

Subsequently we have

$$S^0 = S_0^{\#} + \tfrac{\mu U''(x_2^c)}{U'(x_2^c)}\big(\log|U'(x_2^c)\tau| + \tfrac{i\pi}{2}(sgn(\tau)+1)\big)\tilde{S},$$
$$S_0^{\#}|_{\mu=0} = \begin{pmatrix} \cosh\tau & \sinh\tau \\ \sinh\tau & \cosh\tau \end{pmatrix}, \quad S_0^{\#}|_{\tau=0} = I.$$

The rest of the proof follows exactly as in the case of S. □

Recall the expressions (3.4.3) of a solution $W(\tau)$ to the non-homogeneous Rayleigh equation (3.2.5) along with (3.2.25) at $\tau = 0$. We can use this formula to solve for the parameter b from $W(\tau_0)$ for some $\tau_0 \in [-M, M]$ and then rewrite $W(\tau)$ using the fundamental matrix $S(\mu, c, \tau, \tau_0)$ as

$$W(\tau) = S(\mu, c, \tau, \tau_0)W(\tau_0) - B(\mu, c, \tau)\begin{pmatrix} 1 & 0 \\ \Gamma_0(\mu, c, \tau) & 1 \end{pmatrix}\big(\tilde{\Phi}_0(\mu, c, \tau) - \tilde{\Phi}_0(\mu, c, \tau_0)\big). \tag{3.4.11}$$

3.5 Dependence in c and k of the fundamental solutions with $c_I = 0+$

In this section we revisit the two fundamental solutions to (3.0.1)

$$y_{0\pm}(k, c, x_2) = \lim_{c_I \to 0+} y_\pm(k, c + ic_I, x_2), \quad x_2 \in [-h, 0], \tag{3.5.1}$$

of the homogeneous Rayleigh equation (3.0.1) for $c \in U([-\frac{h_0}{2} - h, \frac{h_0}{2}])$ satisfying initial conditions (3.3.1). We often skip the dependence on c and k (or equivalently, on $\mu = (1 + k^2)^{-\frac{1}{2}}$) when there is no confusion. The following lemma is a summary of results from Proposition 3.2.4, Lemma 3.4.1, and Remark 3.2.2, where x_2^c is defined in (3.0.6).

Lemma 3.5.1. *Assume $U \in C^{l_0}$, $l_0 \geq 3$. For $c \in U([-\frac{h_0}{2} - h, \frac{h_0}{2}])$ and $x_2 \in [-h, 0]$, the following hold.*

1. *As* $c_I \to 0+$*,* $y_\pm(k, c+ic_I, x_2) \to y_{0\pm}(k, c, x_2)$ *uniformly in* x_2 *and* c*.*

2. *As* $c_I \to 0+$*,* $y'_\pm \to y'_{0\pm}$ *locally uniformly in* $\{U(x_2) \neq c\}$ *and also in* $L^\infty_c L^r_{x_2}$ *and* $L^\infty_{x_2} L^r_c$ *for any* $r \in [1, \infty)$*.*

3. *For each* c*,* $y_{0-}(x_2) \in \mathbb{R}$ *if* $U(x_2) \leq c$*,* $y_{0+}(x_2) \in \mathbb{R}$ *if* $U(x_2) \geq c$*,* $y_{0\pm} \in C^\alpha([-\frac{h_0}{2} - h, \frac{h_0}{2}])$ *for any* $\alpha \in [0,1)$ *and* C^{l_0} *in* $x_2 \neq x_2^c$*.*

4. *Moreover,*

$$\begin{pmatrix} \frac{1}{\mu} y_{0\pm}(x_2) \\ y'_{0\pm}(x_2) \end{pmatrix} = B\big(\mu, c, \tfrac{1}{\mu}(x_2 - x_2^c)\big) \begin{pmatrix} 1 & 0 \\ \Gamma_0\big(\mu, c, \frac{1}{\mu}(x_2 - x_2^c)\big) & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\mu} y_{0\pm}(x_2^c) \\ b_{2\pm} \end{pmatrix},$$

   where

$$b_{2\pm} = \lim_{x_2 \to x_2^c} \Big(y'_{0\pm}(x_2) - \tfrac{U''(x_2^c)}{U'(x_2^c)} y_{0\pm}(x_2^c) \big(\log\big(\tfrac{U'(x_2^c)}{\mu} |x_2 - x_2^c| \big) + \tfrac{i\pi}{2} (sgn(x_2 - x_2^c) + 1) \big) \Big),$$

   exists.

Remark 3.5.1. *When* c *takes the end point values* $U(-h)$*, according to the above representation formula and the smoothness of* B*, actually* $y_{0-} \in C^{l_0}([-h_0 - h, h_0])$*.*

Remark 3.5.2. *Suppose* $c \in U\big((-h, 0)\big)$ *and* $y(k, c, x_2) = \lim_{\epsilon \to 0+} y(k, c+i\epsilon, x_2)$ *where* $y(k, c+i\epsilon, x_2)$ *is a solution to the homogeneous Rayleigh equation (3.0.1) with* $y(-h)$*,* $y'(-h) \in \mathbb{R}$*. The above analysis in section 3.2 implies that a.)* $y(k, c, x_2) \in \mathbb{R}$ *for* $x_2 \in [-h, x_2^c]$*; and b.) if* $U''(x_2^c) \neq 0$*, an imaginary part Im* $y(k, c, x_2)$ *occurs for* $x_2 > x_2^c$ *which satisfies the homogeneous Rayleigh equation (3.0.1) for* $x_2 \in [x_2^c, 0]$ *with initial condition*

$$Im\, y(x_2^c) = 0, \quad Im\, y'(x_2^c) = \tfrac{\pi U''(x_2^c)}{U'(x_2^c)} y(x_2^c).$$

The main goal of this section is to analyze the differentiation of y_{0-} in c. Even though most of the results also hold for y_{0+}, the proof is slightly more technical. We shall skip those analysis of y_+ as they are not necessary for the rest of the paper. See Remark 3.5.3.

The proof of the following lemma would be embedded in those of the three subsequent lemmas, actually mainly Lemma 3.5.4.

Lemma 3.5.2. *Assume $U \in C^{l_0}$, $l_0 \geq 4$. For $k, c \in \mathbb{R}$, it holds that*

a.) y_{0-} is locally C^α in both k and c for any $\alpha \in [0, 1)$;

b.) (y_{0-}, y'_{0-}) are locally C^α in both k and c for any $\alpha \in [0, 1)$ at any (k, c, x_2) satisfying $U(x_2) \neq c$;

c.) (y_{0-}, y'_{0-}) are C^{l_0-3} in both k and c at any (k, c, x_2) satisfying $U(x_2) \neq c$ and $c \neq U(-h)$;

d.) $y_{0-}(k, c, x_2^c)$ is C^{l_0-3} in c and k if $c \in U([-h, 0])$;

e.) (y_{0-}, y'_{0-}) are C^{l_0-2} in k, at any (k, c, x_2) except for y'_{0-} at $c = U(x_2)$;

f.) assume $l_0 \geq 5$, then, for any $l = 0, 1$, $j_1, j_2 \geq 0$, $j_1 + j_2 \leq l_0 - 4$, $r \in [1, \infty)$, and $x_2 \in [-h, 0]$,

$$(U(-h) - c)^{j_2} \partial_k^{j_1} \partial_c^{j_2} \partial_{x_2}^l y_{0-}(k, c, x_2),$$

are locally $L_k^\infty W_c^{1,r}$ in c for c near $U(-h)$.

To obtain the estimates, for fixed $c \in \mathbb{R}$ near $U([-h, 0])$, as in Lemma 3.3.2, we divide $[-h, 0]$ into subintervals

$$\mathcal{I}_2 := (x_{2l}, x_{2r}) = \left\{ x_2 \in [-h, 0] \, : \, \tfrac{1}{|U(x_2) - c|} > \tfrac{\rho_0}{\mu} \right\}, \quad \mathcal{I}_1 = [-h, x_{2l}], \quad \mathcal{I}_3 = [x_{2r}, 0], \tag{3.5.2}$$

where ρ_0 is defined as in (3.3.2). $\mathcal{I}_2$ is an interval due to the monotonic assumption of U. Clearly $[-h, 0] = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ and any of these subintervals may be empty. If $\mathcal{I}_2 = \emptyset$, then $[-h, 0]$ is considered as $\mathcal{I}_1$ for y_{0-} and as $\mathcal{I}_3$ for y_{0+}. If $\mathcal{I}_2 \neq \emptyset$, then (3.3.4) holds and $x_2^c \in [-\frac{1}{2}h_0 - h, \frac{1}{2}h_0]$ is well defined. In the next three lemmas, we obtain the estimates on y_{0-} on subintervals in the order of $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$. The proof of Lemma 3.5.2 is mainly contained in that of Lemma 3.5.4 as the smooth dependence of solutions to the Rayleigh equation on k and c and the initial values is trivial on $\mathcal{I}_1$ and $\mathcal{I}_3$. While we mainly focus on y_{0-} in the following lemmas, we shall also just outline the estimates on $\partial_c y_{0+}$, which would be enough for the rest of the paper.

Lemma 3.5.3. *Assume $l_0 \geq 3$ and $\mathcal{I}_1 \neq \emptyset$. For any $k \in \mathbb{R}$ and any $c \in \mathbb{R}$, the following estimates hold for $x_2 \in \mathcal{I}_1$ and $j_1, j_2 \geq 0$ with $j_1 + j_2 > 0$,*

$$\begin{aligned} \mu^{-1}|\partial_k^{j_1}\partial_c^{j_2} y_{0-}(x_2)| + |\partial_k^{j_1}\partial_c^{j_2} y_{0-}'(x_2)| \leq & C_{j_1,j_2}\mu\big(|U(x_2)-c|^{-j_2} + |U(-h)-c|^{-j_2}\big) \\ & \times \big(1+\mu^{-j_1}(x_2+h)^{j_1}\big)\sinh(\mu^{-1}(x_2+h)) \\ \leq & C_{j_1,j_2}\mu^{1-j_1-j_2}\sinh(\mu^{-1}(x_2+h)), \end{aligned} \tag{3.5.3}$$

where $C_{j_1,j_2} > 0$ depends only on j_1, j_2, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$. Moreover, it also holds, for any $x_2 \in \mathcal{I}_3$

$$\mu^{-1}|\partial_c y_{0+}(x_2)| + |\partial_c y_{0+}'(x_2)| \leq C(\sinh\mu^{-1}|x_2| + \mu(1+|c|)\cosh\mu^{-1}x_2).$$

The above estimate holds in a neighborhood of $\mathcal{I}_1$ actually.

Proof. It is obvious that, for $x_2 \in \mathcal{I}_1$, y_{0-} is analytic in c and k. Let $K = k^2 = \mu^{-2} - 1$. One may compute that $\partial_K^{j_1}\partial_c^{j_2} y_{0-}(x_2)$ satisfies the non-homogeneous Rayleigh equation (3.0.3) in the form of

$$-\partial_K^{j_1}\partial_c^{j_2} y_{0-}'' + \big(K + \frac{U''}{U-c}\big)\partial_K^{j_1}\partial_c^{j_2} y_{0-} = -j_1\partial_K^{j_1-1}\partial_c^{j_2} y_{0-} - \sum_{j'=0}^{j_2-1}\frac{m_{j_2,j'}U''}{(U-c)^{j_2+1-j'}}\partial_K^{j_1}\partial_c^{j'} y_{0-}, \tag{3.5.4}$$

with some constants $m_{j_2,j'}$. Note that the definition of $\mathcal{I}_2$ implies that (3.1.2) is satisfied on $\mathcal{I}_1$ with

$$\rho = \rho_0\mu^{-1}k^{-2}(1+|U''|_{C^0([-h_0-h,h_0])}) = \rho_0 k^{-1}\sqrt{1+k^{-2}}(1+|U''|_{C^0([-h_0-h,h_0])}). \tag{3.5.5}$$

We shall estimate the derivatives of y_{0-} with respect to c and k for large k and small k separately.

For any $k_* \geq 1$ sufficiently large so that $\rho \leq 1$, we shall apply (3.1.13) with $x_{02} = -h$ to prove

$$\begin{aligned}
&\mu^{-1}|\partial_K^{j_1}\partial_c^{j_2} y_{0-}(x_2)| + |\partial_K^{j_1}\partial_c^{j_2} y'_{0-}(x_2)| \\
\leq & C_{j_1,j_2}\mu(x_2+h)^{j_1}\big(|U(x_2)-c|^{-j_2} + |U(-h)-c|^{-j_2}\big)\sinh(\mu^{-1}(x_2+h)),
\end{aligned} \tag{3.5.6}$$

for any $|k| \geq k_*$, $j_1, j_2 \geq 0$ with $j_1 + j_2 > 0$. The proof is a simple mathematical induction in $j_1 + j_2$.

Since (3.5.6) does not include the case $j_1 = j_2 = 0$, there are two base cases $(j_1, j_2) = (0, 1)$ and $(j_1, j_2) = (1, 0)$, which we have to consider separately. For $\partial_c y_{0-}$, from (3.5.4), (3.1.13), Lemma 3.3.2, and the definition of $\mathcal{I}_2$, we have, for any $x_2 \in \mathcal{I}_1$,

$$\begin{aligned}
k|\partial_c y_{0-}(x_2)| + |\partial_c y'_{0-}(x_2)| \leq & C\int_{-h}^{x_2} \cosh(\mu^{-1}(x_2 - x_2'))\frac{\mu\sinh(\mu^{-1}(x_2'+h))}{(U(x_2')-c)^2}dx_2' \\
\leq & C\mu\sinh(\mu^{-1}(x_2+h))\int_{-h}^{x_2}\frac{1}{(U(x_2')-c)^2}dx_2' \\
\leq & C\mu\big(|U(x_2)-c|^{-1} + |U(-h)-c|^{-1}\big)\sinh(\mu^{-1}(x_2+h)),
\end{aligned}$$

where (3.3.14) and (3.3.15) are also used for $k \geq k_*$ to convert the estimates in terms of k into those in terms of μ. Similarly, $\partial_K y_{0-}$ satisfies

$$\begin{aligned}
k|\partial_K y_{0-}(x_2)| + |\partial_K y'_{0-}(x_2)| \leq & C\mu\int_{-h}^{x_2}\cosh(\mu^{-1}(x_2-x_2'))\sinh(\mu^{-1}(x_2'+h))dx_2' \\
\leq & C\mu(x_2+h)\sinh(\mu^{-1}(x_2+h)).
\end{aligned}$$

With the estimates in the base cases established, for $j_1 + j_2 > 1$, using the induction assumption (and Lemma 3.3.2 for $j_1 = j' = 0$ in (3.5.4)) and proceeding much as in the above, we obtain

$$
\begin{aligned}
&k|\partial_K^{j_1}\partial_c^{j_2} y_{0-}(x_2)| + |\partial_K^{j_1}\partial_c^{j_2} y_{0-}'(x_2)| \\
\leq & C\mu \sinh(\mu^{-1}(x_2+h)) \int_{-h}^{x_2} j_1(x_2'+h)^{j_1-1}\big(|U(x_2')-c|^{-j_2} + |U(-h)-c|^{-j_2}\big) \\
&+ (x_2'+h)^{j_1}|U(x_2')-c|^{-2}\big(|U(x_2')-c|^{1-j_2} + |U(-h)-c|^{1-j_2}\big) dx_2',
\end{aligned}
$$

and (3.5.6) follows consequently.

For $|k| \leq k_*$, as $\mu \sim 1$, we apply Lemma 3.1.3 to (3.5.4) on $[-h, x_2]$ with

$$
C_0 = \max\{(U(-h)-c)^{-1},\ (U(x_2)-c)^{-1}\} \leq \rho_0 \mu^{-1} \leq C.
$$

Following a similar induction procedure and using Lemma 3.1.3, we obtain, for $x_2 \in \mathcal{I}_1$, $l = 0, 1$, and $j_1, j_2 \geq 0$ with $j_1 + j_2 > 0$,

$$
|\partial_K^{j_1}\partial_c^{j_2}\partial_{x_2}^l y_{0-}(x_2)| \leq C_{j_1,j_2}(x_2+h)^{j_1}\big(|U(x_2)-c|^{-j_2} + |U(-h)-c|^{-j_2}\big).
$$

Therefore (3.5.6) holds for all $k \in \mathbb{R}$.

Since

$$
\partial_k = 2k\partial_K \Longrightarrow \partial_k^j = \sum_{0 \leq l \leq \frac{j}{2}} \tilde{m}_{j,l} k^{j-2l} \partial_K^{j-l}
$$

for some constants $\tilde{m}_{j,l}$, (3.5.6) implies (3.5.3) on $\mathcal{I}_1$ (actually in a neighborhood of $\mathcal{I}_1$).

Estimating $\partial_c y_{0+}$ on $\mathcal{I}_3$. Let $y_1(x_2)$ be solutions to the homogeneous Rayleigh equation (3.0.1) with initial values

$$
y_1(0) = -2(U(0)-c)/(g+\sigma k^2), \quad y_1'(0) = -U'(0)/(g+\sigma k^2),
$$

and $y_2(x_2)$ be the solution to the initial value problem of the non-homogeneous Rayleigh equation

$$-y_2'' + \big(k^2 + \tfrac{U''}{U-c}\big) y_{0+} = -\tfrac{U''}{(U-c)^2} y_{0+}, \quad y_2(0) = y_2'(0) = 0.$$

On $\mathcal{I}_3$, y_2 can be estimated much as y_{0-} on $\mathcal{I}_1$, while y_1 much as in the proof of Lemma 3.3.2. When Lemma 3.1.2 is used to estimate y_1 for large $|k$, we set $s = 0$, $\Theta_1 = \Theta_2 = \cosh$, and $C_0 = C\mu(1 + |c|)$. The desired inequality on $\partial_c y_{0+}$ follows from $\partial_c y_{0+} = y_1 + y_2$. $\square$

Lemma 3.5.4. *Assume $U \in C^{l_0}$, $l_0 \geq 4$, and $\mathcal{I}_2 \neq \emptyset$, then Lemma 3.5.2a.)–e.) hold for $x_2 \in \mathcal{I}_2$. Moreover, if $l_0 \geq 6$, then there exists $C > 0$ depending only on $|U'|_{C^{l_0-1}}$, and $|(U')^{-1}|_{C^0}$, such that, for any $k \in \mathbb{R}$ and any $c \in \mathbb{R}$, the following estimates hold.*

1. *For any $x_2 \in \mathcal{I}_2$, we have*

$$\mu^{-1}|\partial_c y_{0-}(x_2)| \leq C\left(1 + \left|\log \frac{|U(x_2) - c|}{|U(x_{2l}) - c|}\right|\right) \sinh(\mu^{-1}(x_2 + h)), \tag{3.5.7}$$

$$\begin{aligned}&\Big|\partial_c y_{0-}'(x_2) + \frac{U''(x_2^c)}{U'(x_2^c)}\big((P.V.)_c(\frac{1}{U(x_2) - c}) + i\pi\delta_c(U(x_2) - c)\big) y_{0-}(x_2^c)\Big| \\ \leq & C\Big(1 + \Big|\log \frac{|U(x_2) - c|}{|U(x_{2l}) - c|}\Big|\Big) \cosh(\mu^{-1}(x_2 + h)),\end{aligned} \tag{3.5.8}$$

$$\mu^{-1}|\partial_c y_{0+}(x_2)| \leq C\left(1 + \left|\log \frac{|U(x_2) - c|}{|U(x_{2r}) - c|}\right|\right) \cosh(\mu^{-1} x_2),$$

$$\begin{aligned}&\Big|\partial_c y_{0+}'(x_2) + \frac{U''(x_2^c)}{U'(x_2^c)}\big((P.V.)_c(\frac{1}{U(x_2) - c}) + i\pi\delta_c(U(x_2) - c)\big) y_{0+}(x_2^c)\Big| \\ \leq & C\Big(1 + \Big|\log \frac{|U(x_2) - c|}{|U(x_{2r}) - c|}\Big|\Big) \cosh(\mu^{-1} x_2),\end{aligned}$$

and for $2 \leq j \leq l_0 - 5$ and $c \neq U(x_2)$ and $c \neq U(-h)$,

$$\mu^{-1}|\partial_c^j y_{0-}(x_2)| \leq C(|U(x_2) - c|^{1-j} + |U(-h) - c|^{1-j}) \sin(\mu^{-1}(x_2 + h)), \tag{3.5.9}$$

$$|\partial_c^j y'_{0-}(x_2)| \leq C\mu|U(x_2)-c|^{-1}(|U(x_2)-c|^{1-j} + |U(-h)-c|^{1-j})\sinh(\mu^{-1}(x_2+h)). \tag{3.5.10}$$

2. *For* $1 \leq j \leq l_0 - 5$,

$$|\partial_c^j\big(y_{0-}(k,c,x_2^c)\big)| \leq C\mu^{1-j}\cosh(\mu^{-1}(x_2^c+h)), \ \ \textit{if}\ x_2^c \in [-h,0]. \tag{3.5.11}$$

In the above lemma $\delta(\cdot)$ denotes the delta mass supported at 0 and $(P.V.)_c$ and δ_c emphasize them as distributions of the variable c. Near $U(x_2)=c$ or $U(-h)=c$, singular distributions of $\partial_c^j y_{0-}$ and $\partial_c^j y'_{0-}$ at the level comparable to those negative exponents in (3.5.9) and (3.5.10) would occur. The quantities with log upper bounds are L^p functions for any $p \in [1,\infty)$.

Proof. Since $\mathcal{I}_2 \neq \emptyset$, it is easy to prove that (3.3.4) holds and $x_2^c \in [-\frac{1}{4}h_0 - h, \frac{1}{4}h_0]$ is well defined. Let M be defined as in (3.3.16) and (3.3.17) still holds. This allows us to work in the $\tau = \mu^{-1}(x_2 - x_2^c)$ coordinate and apply Lemma 3.4.1, Lemma 3.4.2, and Lemma 3.5.1. It is natural to express y_{0-} using the fundamental matrix $S(\mu,c,\tau,\tau_0)$ defined in (3.4.5). One is reminded that x_{2l} depends on c. To study the regularity of y_{0-} and y'_{0-} with respect to c at some $c_* \in U([-\frac{1}{2}h_0 - h, \frac{1}{2}h_0])$, we fix some $x_{20} \in [-h, x_{2l}(c_*)]$ in a $O(\mu)$ neighborhood of $x_{2l}(c_*)$. For c near c_*, $x_2 \in \mathcal{I}_2$, we can write

$$\begin{pmatrix} \mu^{-1}y_{0-}(k,c,x_2) \\ y'_{0-}(k,c,x_2) \end{pmatrix} = S\big(\mu,c,\tau,\tau_0\big)\begin{pmatrix} \mu^{-1}y_{0-}(k,c,x_{20}) \\ y'_{0-}(k,c,x_{20}) \end{pmatrix}, \quad \tau = \frac{x_2 - x_2^c}{\mu}, \ \ \tau_0 = \frac{x_{20}-x_2^c}{\mu}. \tag{3.5.12}$$

Note that $\tau = \mu^{-1}(x_2 - x_2^c) = 0$ iff $U(x_2) = c$ and $\tau_0 = \mu^{-1}(x_{20} - x_2^c) = 0$ iff $U(x_{20}) = c$, the latter of which happens iff $U(-h) = c_*$. Clearly $y_-(x_{20})$ and $y'_-(x_{20})$ are smooth in c

and k either due to the initial conditions or due to the smoothness of the Rayleigh equation on $\mathcal{I}_1$. Hence the regularity statement (c) of Lemma 3.5.2 follows from statement (1) in Lemma 3.4.2. If $c \neq U(x_2)$ is close to $U(-h)$, then we could fix $x_{20} = -h$. In this case, y_{0-} and y_{0-}' involve only S_{12} and S_{22} due to $y_{0-}(-h) = 0$, and thus statement (b) follows from statement (3) in Lemma 3.4.2. When c is close to $U(x_2)$, the C^α regularity of y_{0-} in k and c is a consequence of statement (2) in Lemma 3.4.2, unless $c = U(x_2) = U(x_{20}) = U(-h)$. Near the last exceptional case, the C^α regularity of y_- in k and c is due to (4) of Lemma 3.4.2. Statement (e) of the C^{l_0-2} in k of (y_{0-}, y_{0-}') also following from the properties of S given in Lemma 3.4.2.

We shall derive the estimates of the differentiation by ∂_c at c_* in two cases.

* *Case 1:* $x_{2l}(c_*) \geq \mu - h$. In this case, let $x_{20} = x_{2l}(c_*)$ which implies $-C\tau_0 \geq 1$. Hence $sgn(\tau_0) = -1$ and $\log|\tau_0|$ as well as its derivatives are of order $O(1)$ when c varies slightly. Therefore the τ_0 related terms can be estimated easily. From the estimate at x_{20} derived in Lemma 3.5.3 (or from the initial condition at $x_2 = -h$), (3.4.8), and Lemma 3.4.2, for $1 \leq j \leq l_0 - 5$, it holds on $\mathcal{I}_2$,

$$\begin{aligned}
&\partial_c^j \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y_{0-}'(x_2) \end{pmatrix} \\
=&\sum_{j'=0}^{j} \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)^{j'} \Big(\tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau) - sgn(\tau_0))\big)\tilde{S}(\tau, \tau_0)\Big) \\
&\times \partial_c^{j-j'} \begin{pmatrix} \mu^{-1} y_{0-}(x_{20}) \\ y_{0-}'(x_{20}) \end{pmatrix} + O\big(\mu^{1-j}\sinh(\mu^{-1}(x_2+h))\big) \\
=&\sum_{j'=0}^{j} \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)^{j'} \Big(\tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tau| + \tfrac{i\pi}{2} sgn(\tau)\big)\tilde{S}(\tau, \tau_0)\Big) \partial_c^{j-j'} \begin{pmatrix} \mu^{-1} y_{0-}(x_{20}) \\ y_{0-}'(x_{20}) \end{pmatrix} \\
&+ O\big(\mu^{1-j}\sinh(\mu^{-1}(x_2+h))\big),
\end{aligned}$$

where $\tilde{S}$ is given in (3.4.6) and the constant C in the $O(\cdot)$ terms depends only on $|U'|_{C^6}$

and $|(U')^{-1}|_{C^0}$. We also used that $\sinh$ and $\cosh$ are comparable at $\frac{x_2+h}{\mu}$ for $x_2 \in \mathcal{I}_2$ in this case.

For $j = 1$, keeping the most singular terms arising from the derivatives of $\log$ and sgn in the distribution sense, we have

$$\begin{aligned}\partial_c \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y_{0-}'(x_2) \end{pmatrix} =& -\tfrac{U''(x_2^c)}{U'(x_2^c)^2}\big(P.V.(\tfrac{1}{\tau}) + i\pi\delta(\tau)\big)\tilde{S}(\tau, \tau_0) \begin{pmatrix} \mu^{-1} y_{0-}(x_{20}) \\ y_{0-}'(x_{20}) \end{pmatrix} \\ &+ O\Big(\big(1 + \big|\log \tfrac{|U(x_2)-c|}{\mu}\big|\big)\sinh(\mu^{-1}(x_2+h))\Big).\end{aligned}$$

Using (3.4.6), (3.4.3), the smoothness of B and $B(\mu, c, 0) = I$, one may compute

$$\big(B_{22}(\tau_0), -B_{12}(\tau_0)\big)\big(\mu^{-1}y_{0-}(x_{20}), y_{0-}'(x_{20})\big)^T = \mu^{-1}y_{0-}(x_2^c), \tag{3.5.13}$$

$$\begin{aligned}\tilde{S}(\tau, \tau_0) \begin{pmatrix} \mu^{-1} y_{0-}(x_{20}) \\ y_{0-}'(x_{20}) \end{pmatrix} =& \mu^{-1} y_{0-}(x_2^c) \begin{pmatrix} B_{12}(\mu, c, \tau) \\ B_{22}(\mu, c, \tau) \end{pmatrix} \\ =& \mu^{-1} y_{0-}(x_2^c) \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + O(|\tau|) \right),\end{aligned} \tag{3.5.14}$$

Due to the Hölder continuity of y_{0-} in x_2 and

$$\delta(\tau) = \delta_c\big(\tfrac{x_2 - x_2^c}{\mu}\big) = \delta_c\big(\tfrac{U(x_2)-c}{\mu U'(x_2^c)}\big) = \mu U'(x_2^c)\delta_c(U(x_2) - c), \tag{3.5.15}$$

where $\delta_c(\cdot)$ emphasizes the delta function with the variable c, we obtain the desired estimates for $j = 1$ in this case.

Similarly, at $x_2 \neq x_2^c$ for $2 \leq j \leq l_0 - 5$, keeping the worst term and using (3.5.14), we have

$$\partial_c^j \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y_{0-}'(x_2) \end{pmatrix} = \big(\big(- \tfrac{\partial_\tau}{\mu U'(x_2^c)} \big)^j \log|\tau| \big) \tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \tilde{S}(\tau, \tau_0) \begin{pmatrix} \mu^{-1} y_{0-}(x_{20}) \\ y_{0-}'(x_{20}) \end{pmatrix}$$
$$+ O\big(\mu^{1-j}|\tau|^{1-j} \sinh(\mu^{-1}(x_2+h))\big)$$
$$= \mu^{1-j} \sinh(\mu^{-1}(x_2+h)) \begin{pmatrix} O(|\tau|^{1-j}) \\ O(|\tau|^{-j}) \end{pmatrix}.$$

The desired inequality (3.5.9) in case 1 follows.

To finish the analysis in this case, we consider $y_{0-}(k, c, x_2^c)$. From (3.5.13) we obtain C^{l_0-2} smoothness in k and c. Differentiating (3.5.13) in c and using Lemma 3.4.1 and Lemma 3.5.3, one may estimate, for $1 \le j \le l_0 - 5$,

$$\big|\partial_c^j \big(y(c, x_2^c)\big)\big| \le \Bigg| \sum_{j'=0}^{j} \Big(\big((\partial_c - \tfrac{\partial_\tau}{\mu U'(x_2^c)})^{j-j'} B_{22}\big)(c, \tau_0) \partial_c^{j'} y_{0-}(c, x_{20})$$
$$- \mu \big((\partial_c - \tfrac{\partial_\tau}{\mu U'(x_2^c)})^{j-j'} B_{12}\big)(c, \tau_0) \partial_c^{j'} y_{0-}'(c, x_{20}) \Big) \Bigg|$$
$$\le C \mu^{1-j} \cosh(\mu^{-1}(x_2^c + h)),$$

which proves (3.5.11) in case 1.

* *Case 2:* $-h \le x_{2l}(c_*) \le \mu - h$. In this case, let $x_{20} = -h$. While we have to deal with possibly very small τ_0 in (3.5.12), the initial values of $(y_{0-}(x_{20}), y_{0-}'(x_{20})) = (0, 1)$. Hence from Lemma 3.4.2 we obtain, for $0 \le j \le l_0 - 5$, $x_2 \in \mathcal{I}_2$,

$$\begin{aligned} \partial_c^j \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y_{0-}'(x_2) \end{pmatrix} &= \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)^j \begin{pmatrix} S_{12}(\tau, \tau_0) \\ S_{22}(\tau, \tau_0) \end{pmatrix} \\ &= \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)^j \Big(\tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau) - sgn(\tau_0)) \big) \begin{pmatrix} \tilde{S}_{12}(\tau, \tau_0) \\ \tilde{S}_{22}(\tau, \tau_0) \end{pmatrix} \Big) \\ &\quad + O(\mu^{1-j}|\tau - \tau_0|). \end{aligned} \tag{3.5.16}$$

From (3.4.6) and Lemma 3.4.1, $\frac{\tilde{S}_{12}}{\tau\tau_0}$ and $\frac{\tilde{S}_{22}}{\tau_0}$ are C^{l_0-4} and C^{l_0-3} functions, which could be used to reduce some singularity. As $\mu|\tau-\tau_0| = |x_2+h|$, one can compute for $j=1$,

$$\begin{aligned}\mu^{-1}\partial_c y_{0-}(x_2) = &- \tfrac{U''(x_2^c)}{U'(x_2^c)^2}\tfrac{\tilde{S}_{12}}{\tau\tau_0}(\partial_\tau+\partial_{\tau_0})\Big(\tau\tau_0\big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau)-sgn(\tau_0))\big)\Big)\\ &+ O\Big(\big|\tau\tau_0\big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau)-sgn(\tau_0))\big)\big|\Big) + O(\mu^{-1}|x_2+h|).\end{aligned}$$

We use the following elementary inequalities to handle the above $\log$ terms:

$$\big|\log|\tfrac{\tau}{\tau_0}|\big| = \left|\int_{|\tau|}^{|\tau_0|}\tfrac{1}{\tau'}d\tau'\right| \le \frac{\big||\tau|-|\tau_0|\big|}{\min\{|\tau|,|\tau_0|\}}, \quad |\tau|+|\tau_0| \le |\tau-\tau_0| + 2\min\{|\tau|,|\tau_0|\},$$

which also imply

$$\big|\tau\tau_0\log|\tfrac{\tau}{\tau_0}|\big| \le C|\tau-\tau_0|, \quad (|\tau|+|\tau_0|)\big|\log|\tfrac{\tau}{\tau_0}|\big| \le |\tau-\tau_0|\big(2+\big|\log|\tfrac{\tau}{\tau_0}|\big|\big).$$

The delta functions produced by differentiating sgn are cancelled by $\tau\tau_0$. Finally $sgn(\tau)-sgn(\tau_0)\neq 0$ only when $-h\le x_2^c \le x_2$ which implies $\mu(|\tau_0|+|\tau|) = x_2+h$. Summarizing these estimates we obtain

$$|\mu^{-1}\partial_c y_{0-}(x_2)| \le C\mu^{-1}|x_2+h|\big(1+\big|\log|\tfrac{\tau}{\tau_0}|\big|\big).$$

If $\mu - h \ge x_{2l}(c_*) > -h$, then $\frac{1}{C}\mu \le x_2^c(c_*) - x_{2l}(c_*) \le -\mu\tau_0 \le C\mu$, while $\mu|\tau_0| = |x_{2l}(c_*) - x_2^c(c_*)|$ if $x_{2l}(c_*) = -h$. Hence $\big|\log|\mu\tau_0| - \log|x_{2l}(c_*)-x_2^c(c_*)|\big| \le C$, which along with the estimate in case 1 yields (3.5.7).

Much as in the above, we estimate $\partial_c y'_{0-}(x_2)$ in case 2 using (3.4.6) and Lemma 3.4.1

$$\begin{aligned}\partial_c y'_{0-}(x_2) = &- \tfrac{U''(x_2^c)}{U'(x_2^c)^2}\tfrac{\tilde{S}_{22}}{\tau_0}\tau_0(\partial_\tau+\partial_{\tau_0})\big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau)-sgn(\tau_0))\big)\\ &+ O\big(1+\big|\log|\tfrac{\tau}{\tau_0}|\big|\big)\\ = &- \tfrac{U''(x_2^c)}{U'(x_2^c)^2}\tilde{S}_{22}\big(P.V.(\tfrac{1}{\tau}) + i\pi\delta(\tau)\big) + O\big(1+\big|\log|\tfrac{\tau}{\tau_0}|\big|\big)\end{aligned}$$

$$= \frac{U''(x_2^c)}{U'(x_2^c)^2} B_{12}(\tau_0)\big(P.V.(\tfrac{1}{\tau}) + i\pi\delta(\tau)\big) + O\big(1 + \big|\log|\tfrac{\tau}{\tau_0}|\big|\big).$$

It along with (3.5.15) and (3.5.13) implies (3.5.8).

Similarly, for $\tau \neq 0$, $\tau_0 \neq 0$, and $2 \leq j \leq l_0 - 5$, one may compute

$$\begin{aligned}
&\partial_c^j \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y'_{0-}(x_2) \end{pmatrix} \\
\leq & C\mu^{1-j}\Big(\sum_{j'=0}^{j} \left|(\partial_\tau + \partial_{\tau_0})^{j'} \Big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau) - sgn(\tau_0))\begin{pmatrix} \tau\tau_0 \\ \tau_0 \end{pmatrix}\Big)\right| + |\tau - \tau_0|\Big).
\end{aligned}$$

For $j \geq 3$, we have

$$\begin{aligned}
|(\partial_\tau + \partial_{\tau_0})^j \big(\tau\tau_0 \log|\tfrac{\tau}{\tau_0}|\big) \leq & C(|\tau^{2-j} - \tau_0^{2-j}| + |\tau + \tau_0||\tau^{1-j} - \tau_0^{1-j}| + |\tau\tau_0||\tau^{-j} - \tau_0^{-j}|) \\
\leq & C|\tau\tau_0||\tau^{-1} - \tau_0^{-1}|(|\tau|^{1-j} + |\tau|^{2-j}|\tau_0|^{-1} + \ldots + |\tau_0|^{1-j}) \\
\leq & C|\tau - \tau_0|(|\tau|^{1-j} + |\tau_0|^{1-j}).
\end{aligned}$$

If $j = 2$, the first term on the right side of the first inequality would be $\log|\frac{\tau}{\tau_0}|$ which as shown previously also satisfies the above final estimate. Similarly, one can also calculate, for $\tau \neq 0$, $\tau_0 \neq 0$, and $j \geq 2$,

$$\begin{aligned}
|(\partial_\tau + \partial_{\tau_0})^j \big(\tau_0 \log|\tfrac{\tau}{\tau_0}|\big) \leq & C(|\tau^{1-j} - \tau_0^{1-j}| + |\tau_0||\tau^{-j} - \tau_0^{-j}|) \\
\leq & C|\tau_0||\tau^{-1} - \tau_0^{-1}|(|\tau|^{1-j} + |\tau|^{2-j}|\tau_0|^{-1} + \ldots + |\tau_0|^{1-j}) \\
\leq & C|\tau|^{-1}|\tau - \tau_0|(|\tau|^{1-j} + |\tau_0|^{1-j}).
\end{aligned}$$

The cases of $j = 0, 1$ have been considered earlier and would only make minor contributions. Therefore (3.5.9) and (3.5.10) are satisfied in case 2 as well.

Regarding $y_{0-}(k, c, x_2^c)$, much as in case 1, but with much simpler initial value at $\tau_0 =$

$\frac{-h-x_2^c}{\mu}$, we have

$$y(k,c,x_2^c) = -\mu B_{12}(\mu,c,\tau_0)$$

which also yields its C^{l_0-2} smoothness. Differentiating in c and using Lemma 3.4.1 and Lemma 3.5.3, one may estimate, for $1 \le j \le l_0 - 5$,

$$\left|\partial_c^j\big(y(k,c,x_2^c)\big)\right| = \mu\left|\big((\partial_c - \tfrac{\partial_\tau}{\mu U'(x_2^c)})^j B_{12}\big)(\mu,c,\tau_0)\right| \le C\mu^{1-j}.$$

which proves (3.5.11) in case 2.

Estimating $\partial_c y_{0+}$ on $\mathcal{I}_2$. In this case $\mathcal{I}_2 \neq \emptyset$ implies $|c| \le C$. Much as in the above argument for y_{0-}, we consider the estimates related to $\partial_c y_{0+}$ at some $c_* \in [-\frac{1}{2}h_0 - h, \frac{1}{2}h_0]$. Observe that, as an expression of solution to the homogeneous Rayleigh equation, (3.5.12) also applies to y_{0+} on $\mathcal{I}_2$ with x_{20} chosen near $x_{2r}(c_*)$. In the case of $x_{2r}(c_*) \le -\mu$, the same arguments yields the desired estimates of $\partial_c y_+$.

In the case of $x_{2r}(c_*) \in [-\mu, 0]$, we take $x_{20} = 0$ and proceed roughly as in the above case 2. Due to the initial condition (3.3.1), equation (3.5.16) is replaced by

$$\begin{aligned}
&\partial_c \begin{pmatrix} \mu^{-1} y_{0+}(x_2) \\ y'_{0+}(x_2) \end{pmatrix} = \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)\left(S(\tau,\tau_0)\begin{pmatrix} \mu^{-1} y_{0+}(0) \\ y'_{0+}(0) \end{pmatrix}\right) \\
=&\big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)\left(\tfrac{\mu U''(x_2^c)}{U'(x_2^c)}\big(\log|\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau) - sgn(\tau_0))\big)\tilde{S}(\tau,\tau_0)\begin{pmatrix} \mu^{-1} y_{0+}(0) \\ y'_{0+}(0) \end{pmatrix}\right) \\
&+ O(|\tau - \tau_0|),
\end{aligned}$$

where (3.4.7) and Lemma 3.4.2 are used. Let

$$W(\tau,\tau_0) = (W_1, W_2)^T = S(\tau,\tau_0)\big(\mu^{-1} y_{0+}(0), y'_{0+}(0)\big)^T.$$

Recall from initial condition (3.3.1)

$$|y_{0+}(0)| \leq C\mu^2\tau_0^2, \quad |y_{0+}'(0) - 1| \leq C\mu^2|\tau_0|.$$

On the one hand, from (3.4.6),, and Lemma 3.4.1, we have that

$$\Big(\frac{W_1}{\tau\tau_0}, \frac{W_2}{\tau_0}\Big) = \Big(\frac{B_{12}(\tau)}{\tau}, B_{22}(\mu)\Big)\Big(B_{22}(\tau_0)\frac{y_{0+}(0)}{\mu\tau_0} - \frac{B_{12}(\tau_0)}{\tau_0}y_{0-}'(0)\Big)$$

are smooth function with bounds uniform in c and μ. Hence the estimate on $\partial_c y_{0+}'(x_2)$ is obtained much as that of $\partial_c y_{0+}'(0)$. On the other hand, as (3.5.14) and (3.5.13) also apply to y_{0+}, it holds

$$W_2 = \mu^{-1}y_{0+}(x_2^c)B_{22}(\tau) = (1 + O(|\tau|))\mu^{-1}y_{0+}(x_2^c).$$

With these estimates, the desired estimate on $\partial_c y_{0+}'(x_2)$ follows much as that of $\partial_c y_{0+}'(x_2)$. This completes the proof. □

Lemma 3.5.5. *Assume $l_0 \geq 4$, $\mathcal{I}_2 \neq \emptyset$, and $\mathcal{I}_3 \neq \emptyset$, then Lemma 3.5.2a.)–e.) hold for $x_2 \in \mathcal{I}_3$. Moreover, if $l_0 \geq 6$, then there exists $C > 0$ depending only on $|U'|_{C^{l_0-1}}$ and $|(U')^{-1}|_{C^0}$, such that, for any $k \in \mathbb{R}$ and any $c \in \mathbb{R}$, the following estimates hold for $x_2 \in \mathcal{I}_3$*

$$\mu^{-1}|\partial_c y_{0-}(x_2)| + |\partial_c y_{0-}'(x_2)| \leq C\Big(1 + \log\frac{\mu}{\min\{\mu, |U(-h) - c|\}}\Big)\cosh(\mu^{-1}(x_2 + h)), \tag{3.5.17}$$

and for $2 \leq j \leq l_0 - 5$,

$$\begin{aligned}&\mu^{-1}|\partial_c^j y_{0-}(x_2)| + |\partial_c^j y_{0-}'(x_2)| \\ \leq &C\big(\mu^{1-j} + \mu^{-1}|U(-h) - c|^{2-j} + \mu^{2-j}|U(-h) - c|^{-1}\big)\cosh(\mu^{-1}(x_2 + h)).\end{aligned}$$

Moreover, if $\mathcal{I}_2 \neq \emptyset$ and $\mathcal{I}_1 \neq \emptyset$, then it also holds for $x_2 \in \mathcal{I}_1$,

$$\mu^{-1}|\partial_c y_{0+}(x_2)| + |\partial_c y'_{0+}(x_2)| \le C\Big(1 + \log \frac{\mu}{\min\{\mu, |U(0) - c|\}}\Big) \cosh(\mu^{-1}x_2).$$

Proof. The assumption $\mathcal{I}_2 \neq \emptyset$ and $\mathcal{I}_3 = [x_{2r}, 0] \neq \emptyset$ imply $x_{2r} > -h$ and $\mu^{-1}\big(U(x_{2r}) - c\big)$ is uniformly bounded from above and below away from 0. The regularity of y_{0-} and y'_{0-} in c and k for $x_2 \in \mathcal{I}_3$ follow directly from such smoothness at x_{2r} obtained in Lemma 3.5.4. Their estimates at x_{2r} can be summarized into

$$\mu^{-1}|\partial_c y_{0-}(x_{2r})| + |\partial_c y'_{0-}(x_{2r})| \le C\Big(1 + \log \frac{\mu}{\min\{\mu, |U(-h) - c|\}}\Big) \cosh(\mu^{-1}(x_{2r} + h)),$$

and for $2 \le j \le l_0 - 5$

$$\mu^{-1}|\partial_c^j y_{0-}(x_{2r})| + |\partial_c^j y'_{0-}(x_{2r})| \le C\big(\mu^{1-j} + |U(-h) - c|^{1-j}\big) \cosh(\mu^{-1}(x_{2r} + h))$$

where we also used $1 \le \mu^{-1}(x_{2r} + h)$ as $x_{2l} > -h$. Much as the proof of Lemma 3.5.3, we shall obtain the estimates inductively in j by considering the cases of small and large k separately.

As $\rho_0 > 0$, we take $k_* \ge 1$ such that $\rho < 1$ (defined in (3.5.5)) for $|k| \ge k_*$ and thus (3.1.2) is satisfied on $\mathcal{I}_3$ with $\rho < \min\{1, C\mu\}$. We shall obtain the estimates for this case of $|k| \ge k_*$ by splitting $\partial_c y_{0-}$ into homogeneous and non-homogeneous parts. For $j \ge 1$, let $y_1(x_2)$ be the solution to the homogeneous Rayleigh equation (3.0.1) with initial condition

$$y_1(x_{2r}) = \partial_c^j y_{0-}(x_{2r}), \quad y'_1(x_{2r}) = \partial_c^j y'_{0-}(x_{2r}),$$

and $y_2(x_2)$ be the solution to the non-homogeneous Rayleigh equation (3.0.3) with the zero initial conditions at $x_2 = x_{2r}$ and the non-homogeneous term given by the right side of

(3.5.4) (with $j_1 = 0$ and $j_2 = j$). Clearly it holds

$$\partial_c^j y_{0-} = y_1 + y_2, \quad \text{on } \mathcal{I}_3. \tag{3.5.18}$$

Using the the above estimates on $\partial_c y_{0-}$ at x_{2r}, we apply Lemma 3.1.2 to y_1 with

$$\Theta_1 = \Theta_2 = \cosh, \quad s = 0, \quad C_0 = \mu^{-1}|\partial_c^j y_{0-}(x_{2r})| + |\partial_c^j y_{0-}'(x_{2r})| + 1$$

to obtain, for $x_2 \in \mathcal{I}_3$,

$$\mu^{-1}|y_1(x_2)| + |y_1'(x_2)| \leq C\big(\mu^{-1}|\partial_c^j y_{0-}(x_{2r})| + |\partial_c^j y_{0-}'(x_{2r})| + 1\big) \cosh \mu^{-1}(x_2 - x_{2r}).$$

Concerning $y_2(x_2)$, Lemma 3.1.2 and the same computation as in the proof of Lemma 3.5.3 implies, for any $x_2 \in \mathcal{I}_3$,

$$|\mu^{-1} y_2(x_2)| + |y_2'(x_2)| \leq C \sum_{j'=0}^{j-1} \int_{x_{2r}}^{x_2} \frac{\cosh(\mu^{-1}(x_2 - x_2'))}{|U(x_2') - c|^{j+1-j'}} |\partial_c^{j'} y_{0-}(x_2')| dx_2'.$$

The desired estimate for $j = 1$ follows from (3.5.18), Lemma 3.3.2, and direct integration. For $j \geq 2$, one may compute inductively using the above estimates and (3.5.12),

$$\begin{aligned} |\mu^{-1} y_2(x_2)| + |y_2'(x_2)| \leq & C \sum_{j'=0}^{j-1} (U(x_{2r}) - c)^{j'-j} \frac{|\partial_c^{j'} y_{0-}(x_{2r})|}{\cosh \mu^{-1}(x_{2r} + h)} \cosh \mu^{-1}(x_2 + h) \\ \leq & C\Big(\mu^{1-j} + \mu^{-1}|U(-h) - c|^{2-j} + \mu^{1-j} \log \frac{\mu}{\min\{\mu, |U(-h) - c|\}}\Big) \cosh \mu^{-1}(x_2 + h). \end{aligned}$$

If $|U(-h) - c| \geq \mu$, the desired estimate follows immediately, otherwise it follows from the fact $\log x \leq x$ for any $x \geq 1$.

In the case $k \leq k_*$, $\mu \sim 1$ and Lemma 3.1.3 yields the estimates through a similar induction. The estimates on $\partial_c y_{0+}$ is also obtained much as $\partial_c y_{0-}$ using Lemma 3.1.2 and Lemma 3.1.3 based on the estimates of $\partial_c y_{0+}$ at x_{2l} obtained in Lemma 3.5.4. In particular, the fact that $\mathcal{I}_2 \neq \emptyset$ also implies $|c| < C$ is also used. We skip the details. $\square$

The following lemma proves Lemma 3.5.2(f) and (the case of $x_2 = 0$) will be used in analyzing the eigenvalues.

Lemma 3.5.6. *Assume $U \in C^{l_0}$. For any $k \in \mathbb{R}$ and $x_2 \in [-h, 0]$, there exists $R, \tilde{C} > 0$ such that*

$$|U(-h) - c|^j |\partial_k^{j_1} \partial_c^{j_2} \partial_{x_2}^l y_{0-}(k, c, x_2)| \leq \tilde{C}\big(1 + \big|\log |U(-h) - c|\big|\big), \quad j = \max\{0, j_2 - 1\}, \tag{3.5.19}$$

for any $|c - U(-h)| \leq R$, $l = 0, 1$, $j_1, j_2 \geq 0$, $j_1 + j_2 \leq l_0 - 3$. Here $\tilde{C}$ can be taken independent of k for k in any bounded set.

Unlike in most other lemmas, the constants R and $\tilde{C}$ may depend on k and x_2.

Proof. The lemma is trivial if $x_2 = -h$, so we assume $x_2 > -h$. Since the lemma is concerned with c close to $U(-h)$ where R and $\tilde{C}$ may depend on x_2 and k, we consider

$$c = U(x_2^c), \;\; x_2^c \in [-h_0 - h, (-h + x_2)/2] \implies \tau = \mu^{-1}(x_2 - x_2^c) \geq \mu^{-1}(x_2 + h)/2 > 0.$$

Let $\tau_0 = -\mu^{-1}(x_2^c + h)$. From formula (3.5.16) and the C^{l_0-3} smoothness of S_{err} due to Lemma 3.4.2, we have

$$\begin{aligned}
&\tau_0^j \partial_k^{j_1} \partial_c^{j_2} \begin{pmatrix} \mu^{-1} y_{0-}(x_2) \\ y_{0-}'(x_2) \end{pmatrix} \\
=&\tau_0^j \big(\partial_c - \tfrac{\partial_\tau + \partial_{\tau_0}}{\mu U'(x_2^c)}\big)^{j_2} \left[\big(\log |\tfrac{\tau}{\tau_0}| + \tfrac{i\pi}{2}(sgn(\tau) - sgn(\tau_0))\big) \partial_k^{j_1} \Big(\tfrac{\mu U''(x_2^c)}{U'(x_2^c)} \begin{pmatrix} \tilde{S}_{12}(\tau, \tau_0) \\ \tilde{S}_{22}(\tau, \tau_0) \end{pmatrix} \Big) \right] \\
&+ O(1)
\end{aligned}$$

where we also used that $\log |\frac{\tau}{\tau_0}| + \frac{i\pi}{2}(sgn(\tau) - sgn(\tau_0))$ is independent of k. The desired inequality (3.5.19) follows from straight forward calculations using the C^{l_0-3} smoothness of $\frac{\tilde{S}_{12}}{\tau_0}$, $\frac{\tilde{S}_{22}}{\tau_0}$, and $\frac{U(-h)-c}{\tau_0}$. $\square$

Remark 3.5.3. *Most of the above regularity results and estimates also hold for $y_{0+}(k, c, x_2)$. Since y_+ plays a less substantial role as y_- in the rest of the paper, we only gave the basic estimates on y_+.*

In the above $\partial_c y_\pm$ was considered only for $c \in U([-\frac{h_0}{2} - h, \frac{h_0}{2}])$. To end this section, we extend some estimate for $c \in \mathbb{C}$ using the analyticity of $y_\pm$ in c in the following lemma.

Lemma 3.5.7. *Assume $U \in C^6$. The following hold.*

1. *For any $c \in \mathbb{C}$ with $c_I > 0$, it holds*

$$\partial_c y_-(k, c, x_2) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_c y_{0-}(k, c', x_2)}{c' - c} dc',$$

$$(U(x_2) - c)\partial_c y'_-(k, c, x_2) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(U(x_2) - c')\partial_c y'_{0-}(k, c', x_2)}{c' - c} dc'.$$

$$\frac{\partial_c y_+(k, c, x_2)}{(U(x_2) - c + i)^2} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_c y_{0+}(k, c', x_2)}{(U(x_2) - c' + i)^2 (c' - c)} dc',$$

$$\frac{(U(x_2) - c)\partial_c y'_+(k, c, x_2)}{(U(x_2) - c + i)^3} = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(U(x_2) - c')\partial_c y_{0+}(k, c', x_2)}{(U(x_2) - c' + i)^3 (c' - c)} dc'.$$

2. *For any $r \in (1, \infty)$,*

   (a) *there exists $C > 0$ depending only on r, $|U'|_{C^5}$, and $|(U')^{-1}|_{C^0}$, such that for any $k \in \mathbb{R}$, $x_2 \in [-h, 0]$, $c_I > 0$,*

$$\begin{aligned} &\mu^{-1} |\partial_c y_-(k, c, x_2)|_{L^r_{c_R}(\mathbb{R})} + |(U(x_2) - c)\partial_c y'_-(k, c, x_2)|_{L^r_{c_R}(\mathbb{R})} \\ \leq& C \cosh \mu^{-1}(x_2 + h); \end{aligned}$$

   (b) *as $c_I \to 0+$, $\partial_c y_-$ and $(U - c)\partial_c y'_-$ converge to $\partial_c y_{0-}$ and $(U - c_R)\partial_c y'_{0-}$ in $L^r_{c_R}(\mathbb{R})$, respectively, for any $x_2 \in [-h, 0]$. Moreover, the convergence also holds in $L^r_{c_R, x_2}(\mathbb{R} \times [-h, 0])$.*

3. *For any $r \in (1, \infty)$ and compact interval $\mathcal{I} \subset \mathbb{R}$,*

   (a) *there exists $C > 0$ depending only on r, $\mathcal{I}$, $|U'|_{C^5}$, and $|(U')^{-1}|_{C^0}$, such that for any $k \in \mathbb{R}$, $x_2 \in [-h, 0]$, $c_I > 0$,*

$$\begin{aligned}&\mu^{-1}|\partial_c y_+(k, c, x_2)|_{L^r_{c_R}(\mathcal{I})} + |(U(x_2) - c)\partial_c y'_+(k, c, x_2)|_{L^r_{c_R}(\mathcal{I})}\\ \leq& C \cosh \mu^{-1}(x_2 + h);\end{aligned}$$

   (b) *as $c_I \to 0+$, $\partial_c y_+$ and $(U - c)\partial_c y'_+$ converge to $\partial_c y_{0+}$ and $(U - c_R)\partial_c y'_{0+}$ in $L^r_{c_R}(\mathcal{I})$, respectively, for any $x_2 \in [-h, 0]$. Moreover, the convergence also holds in $L^r_{c_R, x_2}(\mathcal{I} \times [-h, 0])$.*

Proof. Due to the conjugacy of y_- in c, we only consider $c_I > 0$. Let $B_{h_0} \subset \mathbb{C}$ be the open disk with diameter segment $U([-\frac{h_0}{2} - h, \frac{h_0}{2}])$. For any $c \notin B_{h_0}$, let

$$\rho = k^{-2}(1 + |U''|_{C^0}) \max_{[-h,0]} |U - c|^{-1} \leq C k^{-2}(1 + |c|)^{-1}.$$

There exists $k_* > 0$ such that $\rho < 1$ for any $|k| \geq k_*$. Lemma 3.1.2 (with $x_{2l} = -h$, $\mathcal{I} = [-h, 0]$, $C_0 = 0$, and $\Theta_1 = \Theta_2 = \sinh$) implies, for $|k| \geq k_*$ and $c \notin B_{h_0}$,

$$\begin{aligned}&\big|y_-(k, c, x_2) - |k|^{-1} \sinh |k|(x_2 + h)\big| + \mu\big|y'_-(k, c, x_2) - \cosh |k|(x_2 + h)\big|\\ &\qquad\qquad\leq C\mu k^{-1}(1 + |c|)^{-1} \sinh |k|(x_2 + h).\end{aligned} \tag{3.5.20}$$

For $|k| < k_*$, Lemma 3.1.3 implies that the above inequality still holds for $c \notin B_{h_0}$.

From equation (3.5.4) ($j_1 = 0$ and $j_2 = 1$) of $\partial_c y_-$, applying (3.1.13) with $\phi = -\frac{U''}{(u-c)^2} y_-$ and using Lemma 3.3.2, we have for $|k| \geq k_*$ and $c \notin B_{h_0}$,

$$\mu^{-1}|\partial_c y_-(k, c, x_2)| + |\partial_c y'_-(k, c, x_2)| \leq C\mu(1 + |c|)^{-2} \sinh \mu^{-1}(x_2 + h). \tag{3.5.21}$$

For $|k| \le k_*$, Lemma 3.1.3 implies that the above inequality still holds for $c \notin B_{h_0}$.

For any $c \in \mathbb{C}$ with $c_I > 0$, the analyticity of $\partial_c y_-$ and its $O(|c|^{-2})$ decay as $|c| \to \infty$ imply, for any $\beta \in (0, c_I)$,

$$\partial_c y_-(k, c, x_2) = \frac{1}{2\pi i} \int_{\mathbb{R}+i\beta} \frac{\partial_c y_-(k, c', x_2)}{c' - c} dc' = \frac{1}{2\pi i} \int_{\mathbb{R}+i\beta} \frac{y_-(k, c', x_2)}{(c' - c)^2} dc',$$

where the boundary terms at infinity in the above integration by parts vanish due to the uniform-in-c bound on $|y_-|$ given in (3.5.20). Letting $\beta \to 0+$, the same bound and Lemma 3.5.1 yield

$$\begin{aligned}\partial_c y_-(k, c, x_2) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{y_{0-}(k, c', x_2)}{(c' - c)^2} dc' =& \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_c y_{0-}(k, c', x_2)}{c' - c} dc' \\ =& -\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\partial_c y_{0-}(k, c', x_2)}{(c_R - c') + i c_I} dc',\end{aligned}$$

where we integrated by parts again. The desired estimate on $|\partial_c y_-|_{L^r_{c_R}}$ follows from the boundedness of the convolution kernel $\frac{1}{c' + i c_I}$ on $L^r(\mathbb{R})$, (3.5.21) for $|c| \gg 1$, and Lemma 3.5.3–Lemma 3.5.5.

The results for $\partial_c y'_-$ are derived in the same manner. In fact

$$\begin{aligned}(U(x_2) - c)\partial_c y'_-(k, c, x_2) =& \frac{1}{2\pi i} \int_{\mathbb{R}+i\beta} \frac{(U(x_2) - c')\partial_c y'_-(k, c', x_2)}{c' - c} dc' \\ =& \frac{1}{2\pi i} \int_{\mathbb{R}+i\beta} \frac{(U(x_2) - c) y'_-(k, c', x_2)}{(c' - c)^2} dc' = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(U(x_2) - c) y'_{0-}(k, c', x_2)}{(c' - c)^2} dc' \\ =& \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{(U(x_2) - c')\partial_c y'_{0-}(k, c', x_2)}{c' - c} dc',\end{aligned}$$

where we used (3.5.20) to cancel the two boundary terms at infinity in the above both integrations by parts and also used the integrability of $(U(x_2) - c)\partial_c y'_-(k, c, x_2)$ near $U(x_2) = c$ given in Lemma 3.5.4. The latter also yields the estimate on $(U - c)\partial_c y'_-$.

In statement (2b), the pointwise-in-x_2 convergence in $L^r_{c_R}$ is standard due to the convergence of the convolution kernel $\frac{1}{c' + i c_I}$ on $L^r(\mathbb{R})$ as $c_I \to 0+$, as well as the analyticity of y_- for $c_I > 0$. The convergence in $L^r_{c_R,x_2}$ follows from the pointwise-in-x_2 convergence in $L^r_{c_R}$, the $L^\infty_{x_2} L^r_{c_R}$ bounds in statement (2a), and the dominant convergence theorem.

Finally, $\partial_c y_+$ can be analyzed similar. However, the initial values (3.3.1) induce an $O(|c|^2)$ growth in y_+ and y_+', and an $O(|c|)$ growth of $\partial_c y_+$ and $\partial_c y_+'$, for $|c| \gg 1$ (Lemma 3.1.3). Instead we consider, for $c_I > 0$,

$$\frac{\partial_c y_+(k,c,x_2)}{(U(x_2)-c+i)^2} = \frac{1}{2\pi i} \int_{\mathbb{R}+i\beta} \frac{\partial_c y_+(k,c',x_2)}{(U(x_2)-c'+i)^2(c'-c)} dc',$$

which holds for any $\beta \in (0, c_I)$. From this Cauchy integral formula we proceed much as in the above and obtain the integral representation in term of $\partial_c y_{0+}$. The derivation of the corresponding formula of $\partial_c y_+'$ is also similar. The desired convergence and estimates of $\partial_c y_+$ and $\partial_c y_+'$ in $L^r_{c_R}(\mathcal{I})$ on a compact interval $\mathcal{I}$ again follow from the properties of the convolution by the kernel $\frac{1}{c'+ic_I}$. $\square$

3.6 An important quantity Y

To end this chapter, we analyze a quantity related to the Reynolds stress, which is crucial for the linearized water wave problem:

$$\begin{aligned} &Y(k,c) = Y_R(k,c) + iY_I(k,c) := \frac{y_-'(k,c,0)}{y_-(k,c,0)}, \quad c = c_R + ic_I \in \mathbb{C} \setminus U([-h,0]), \\ &Y(k,c) = \lim_{\epsilon \to 0+} Y(k, c+i\epsilon) = \frac{y_{0-}'(k,c,0)}{y_{0-}(k,c,0)}, \quad c \in U\big([-h,0)\big), \end{aligned} \tag{3.6.1}$$

where $y_-(k,c,x_2)$ is the solution to homogeneous Rayleigh equation (3.0.1) satisfying $y_-(-h) = 0$ and $y_-'(-h) = 1$ defined in section 3.3 and $y_{0-}(k,c,x_2) = \lim_{\epsilon \to 0+} y_-(k, c+i\epsilon, x_2)$ for $c \in \mathbb{R}$. Due to Remark 3.3.1, $y_-(k, c_R + ic_I, x_2)$ satisfies estimates uniform in $0 < \epsilon \ll 1$. With slight abuse of notations, we would not distinguish y_{0-} from y_- in the rest of this chapter. Apparently the domain of $Y(k,c)$ is given by

$$D(Y) = \{(k,c) \in \mathbb{R} \times \mathbb{C} \mid c \neq U(0),\ y_-(k,c,0) \neq 0\},$$

and those excluded points (except $c = U(0)$) exactly are the eigenvalues of of the linearized Euler equation in the fixed channel $x_2 \in [-h, 0]$ at the shear flow $(U(x_2), 0)$. Y is not defined at $c = U(0)$ since $y'_-(x_2)$ has singularity at $x_2 = 0$. We first summarize some basic or standard properties of $y_-(k, c, 0)$ in the following lemma.

Lemma 3.6.1. *Assume $U \in C^3$. The following hold.*

1. *For any $k \in \mathbb{R}$, $y_-(k, c, x_2) > 0$ for any $x_2 \in (-h, 0]$ and $c \in \mathbb{R} \setminus U\big((-h, 0)\big)$.*

2. *There exists $C > 0$ depending on U such that, for any $k \in \mathbb{R}$ and $c \in U\big((-h, 0]\big)$, it holds*

$$(Ck)^{-1} \sinh k(x_2 + h) \le y_-(k, c, x_2) \le Ck^{-1} \sinh k(x_2 + h), \quad \forall x_2 \in [-h, x_2^c]. \tag{3.6.2}$$

3. *There exists $C > 0$ depending only on U such that, for any $c = U(x_2^c)$, $x_2^c \in [-h, 0)$, it holds, for any $k \in \mathbb{R}$,*

$$\begin{aligned}&C^{-1}\mu^2 |U''(x_2^c)| \sinh \mu^{-1}(x_2^c + h) \sinh \mu^{-1}|x_2^c| \\ &\le |\mathit{Im}\, y_-(k, c, 0)| \le C\mu^2 |U''(x_2^c)| \sinh \mu^{-1}(x_2^c + h) \sinh \mu^{-1}|x_2^c|.\end{aligned}$$

4. *There exists $k_* > 0$ and $C > 0$ depending only on $M > 0$, $|U'|_{C^2}$ and $|(U')^{-1}|_{C^0}$ such that, if $|k| \ge k_*$ or $|c - (U(-h) + U(0))/2| \ge M + (U(0) - U(-h))/2$ then*

$$|y_-(k, c, 0)| \ge (Ck)^{-1} \sinh kh. \tag{3.6.3}$$

5. *Suppose a closed subset $S \subset \mathbb{C}$ satisfies $y_-(k, c, 0) \neq 0$ for all $c \in S$ and $k \in \mathbf{K}$ where $\mathbf{K} = \mathbb{R}$ or $\frac{2\pi}{L}\mathbb{Z}$, then there exists $C > 0$ depending only on S and U such that (3.6.3) holds for all $k \in \mathbf{K}$ and $c \in S$.*

Remark 3.6.1. *According to Lemma 3.1.3 and Lemma 3.3.2, the assumption $y_-(k, c, 0) \neq 0$ on S in Statement (5) is automatically satisfied except possibly a compact set of $(k, c) \subset$*

$\mathbf{K} \times S$. *In particular, due to statement (3), it is satisfied for* $S = \mathbb{C}$ *if* $U'' \neq 0$ *on* $[-h, 0]$. *We also recall* $y_-(k, c, 0) = 0$ *is equivalent to that* $-ikc$ *is an eigenvalue of the linearized Euler equation at the shear flow* U *on the fixed channel* $x_2 \in (-h, 0)$ *associated with an eigenfunction* $v_2(x_2) = e^{ikx_1} y_-(k, c, x_2)$.

Proof. We first claim the following standard result.

Claim. Let $y(x_2)$ is a solution to the homogeneous Rayleigh equation (3.0.1) on an interval $\mathcal{I} = (x_{2l}, x_{2r}) \subset [-h, 0]$ with $c \in \mathbb{R} \setminus U(\mathcal{I})$ such that $(y(x_{20}), y'(x_{20})) \in \{0\} \times (\mathbb{R} \setminus \{0\})$ at some $x_{20} \in \overline{\mathcal{I}}$, then $y(x_2) \in \mathbb{R} \setminus \{0\}$ at any $x_2 \in \overline{\mathcal{I}} \setminus \{x_{20}\}$.

If $U(x_{20}) \neq c$, then the claim $y(x_2) \in \mathbb{R}$ and y is in C^α on $\overline{\mathcal{I}}$ are obvious since the coefficients of (3.0.1) are real. If $U(x_{20}) = c \notin U(\mathcal{I})$, then it must hold $x_{20} \in \{x_{2l}, x_{2r}\}$ and Lemma 3.4.1 implies that $y \in C^1(\overline{\mathcal{I}})$ and $W = (\mu^{-1} y, y')(\cdot + x_{20})$ satisfies (3.4.3) with $W_1(0) = 0$ and $\tilde{\Phi}_0 \equiv 0$. This formula yields $y \in \mathbb{R}$. Finally, suppose $y(x_{21}) = 0$ at some $x_{21} \in \overline{\mathcal{I}} \setminus \{x_{20}\}$. Let $y = (U - c)\xi$. Again $\xi \in C^0(\overline{\mathcal{I}})$ due to Lemma 3.4.1 and it is standard to verify

$$-\big((U-c)^2 \xi'\big)' + k^2 (U-c)^2 \xi = 0, \quad x_2 \in \mathcal{I}. \tag{3.6.4}$$

Multiplying it by ξ and integrating it between x_{20} and x_{21} leads to a contradiction. Hence the claim is proved.

For $c \in \mathbb{R}$, applying the above claim to y_- on the interval $[-h, 0]$ if $c \notin U\big((-h, 0)\big)$ and on $[-h, x_2^c]$ if $c \in U\big((-h, 0)\big)$, respectively, implies that $y_-(x_2) \in \mathbb{R}$ does not change signs on these intervals. Hence we obtain statement (1) and $y_-(x_2) > 0$ for $x_2 \in [-h, x_2^c]$ if $c \in U\big((-h, 0)\big)$. The latter along with (3.3.5) and the continuity of $\frac{y_-(k,c,x_2)}{\mu \sinh \mu^{-1}(x_2+h)}$ also yields Statement (2).

In the view of Lemma 3.5.1, Remark 3.5.2, and statement (1), $y(x_2) = \operatorname{Im} y_-(x_2)$ is also a solution on $[x_2^c, 0]$ satisfying $y(x_2^c) = 0$ and $y'(x_2^c) = \frac{\pi U''(x_2^c)}{U'(x_2^c)} y_-(x_2^c)$. Statement (3) follows from statement (2) applied to y_- on $[-h, x_2^c]$ and to $\operatorname{Im} y_-$ on $[x_2^c, 0]$.

From (3.3.5) and Remark 3.3.1, there exists $k_* > 0$ such that (3.6.3) holds for all $|k| \geq k_*$ and $c \in \mathbb{C}$. For $|k| \leq k_*$, the restriction on c involving $M > 0$ ensures $y_-(k, c, 0) \neq 0$ due to the semicircle theorem and thus we obtain (3.6.3) from Lemma 3.1.3, which completes the proof of statement (4).

Finally assume $y_-(k, c, 0) \neq 0$ for all $k \in \mathbf{K}$ and $c \in S$. Recalling the convergence estimates (3.2.31) and the locally Hölder continuity of y_- in $c \in \mathbb{R}$ (Lemma 3.5.2), we obtain the continuity of y_- in $c \in \mathbb{C}$ for $c_I \geq 0$. Lemma 3.1.3 and Lemma 3.3.2 along with the continuity of $y_-(k, c, 0)$ and the non-vanishing assumption imply that (3.6.3) holds for all $k \in \mathbf{K}$ and $c \in S$ with $c_I \geq 0$. As $y_-(k, \bar{c}, x_2) = \overline{y_-(k, c, x_2)}$, statement (5) follows and it completes the proof of the lemma. $\square$

In the following we give some basic properties of $Y(k, c)$.

Lemma 3.6.2. *Assume $U \in C^{l_0}$, $l_0 \geq 4$. It holds that $Y(k, \bar{c}) = \overline{Y(k, c)}$ and Y is a.) analytic in both $(k, c) \in D(Y) \setminus (\mathbb{R} \times U([-h, 0]))$, and, when restricted to $c_I \geq 0$, b.) C^{l_0-3} in $(k, c) \in D(Y) \setminus (\mathbb{R} \times \{U(-h), U(0)\})$, and c.) C^{l_0-3} in k and locally C^α in $(k, c) \in D(Y)$ for any $\alpha \in [0, 1)$. Moreover,*

1. *$Y(k, U(-h)) \in \mathbb{R}$ and $Y(0, U(-h)) = \frac{U'(0)}{U(0) - U(-h)}$.*

2. *There exists $C, \rho > 0$ depending only on α and U such that*

$$|Y(k, c)| \leq C\big(\mu^{-1} + \big|\log \min\big\{1, |U(0) - c|\big\}\big|\big), \quad \forall k \in \mathbb{R}, \ |c - U(0)| \leq \rho.$$

3. *For any $\alpha \in (0, \frac{1}{2})$, there exist $k_0 > 0$ and $C > 0$ depending only on α, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$ such that,*

$$|Y(k, c) - k \coth kh| \leq C(\mu^{\alpha-1} + |\log \min\{1, \, |U(0) - c|\}|), \quad \forall |k| \geq k_0, \ c \neq U(0).$$

4. *For any $M > 0$ and $k_* > 0$, there exists $C > 0$ depending only on k_* and M such*

that

$$|Y(k,c) - k\coth kh| \leq \frac{C}{dist(c, U([-h,0]))}, \quad \forall |k| \leq k_*,$$
$$\Big|c - \frac{U(-h)+U(0)}{2}\Big| \geq M + \frac{U(0)-U(-h)}{2}.$$

Proof. The analyticity and the conjugacy property of Y are obvious from its definition. The property $Y(k, U(-h)) \in \mathbb{R}$ is a direct corollary of Lemma 3.6.1(1). The C^{l_0-3} smoothness of Y follows from Lemma 3.5.2. The Hölder continuity of Y is again a corollary of Lemma 3.5.2 for c varying along $\mathbb{R}$ and Proposition 3.2.4 for c varying along $i\mathbb{R}$. The explicit form of $Y\big(0, U(-h)\big)$ is a direct consequence of the observation

$$y_-\big(0, U(-h), x_2\big) = \big(U(x_2) - U(-h)\big)/U'(-h). \tag{3.6.5}$$

To end the proof of the lemma, we obtain the quantitive estimate on $Y(k,c)$. From Lemma 3.6.1, $y_-(k, U(0), 0) \neq 0$ for any $k \in \mathbb{R}$. Along with Lemma 3.3.2, it implies that (3.6.3) holds for $|c - U(0)| \leq \rho$ for some $\rho > 0$ depending only on U. Statement (2) follows from the upper bound of $|y_-'(k,c,0)|$ given in Lemma 3.3.2. Statement (3) is also a direct consequence of Lemma 3.3.2 where k_0 is involved to ensure $y_-(k,c,0) \neq 0$. In statement (4), the restriction on c guarantees $y_-(k,c,0) \neq 0$ due to the semicircle theorem and the desired inequality follows Lemma 3.1.3. □

The analyticity of Y in c allows us to use the Cauchy integral to analyze $Y(k,c)$. For $r > 0$, let

$$\mathcal{D}_r = B\big(U([-h,0]), r\big) \subset \mathbb{C} \tag{3.6.6}$$

be the r-neighborhood of $U([-h,0]) \subset \mathbb{C}$.

Lemma 3.6.3. *Assume $U \in C^3$. There exists $k_0 > 0$ depending only on $|U'|_{C^2}$ and $|(U')^{-1}|_{C^0}$ such that for any $|k| \geq k_0$, $0 < r < dist\big(c, U([-h,0])\big)$, and $n \geq 1$ such* *that*

$$Y(k,c) = k\coth kh - \frac{1}{2\pi i}\oint_{\partial\mathcal{D}_r} \frac{Y(k,c')}{c'-c}dc', \quad \partial_c^n Y(k,c) = -\frac{n!}{2\pi i}\oint_{\partial\mathcal{D}_r} \frac{Y(k,c')}{(c'-c)^{n+1}}dc', \tag{3.6.7}$$

where $\oint$ *denote the integral along the contours counterclockwisely.*

Here $\partial_c Y = \frac{1}{2}(\partial_{c_R} - i\partial_{c_I})Y$ denotes the derivative of Y as a function of the complex variable c and thus $\partial_c Y = \partial_{c_R} Y = -i\partial_{c_I} Y$ due to its analyticity.

Proof. According to Lemma 3.3.2 and Lemma 3.6.1, there exist k_0 and $C > 0$ such that, for any $|k| \geq k_0$, the domain $D(Y(k,\cdot)) = \mathbb{C}$. For any $r' \gg 1$, the analyticity of Y in $c \notin U([-h,0])$ implies

$$Y(k,c) = \frac{1}{2\pi i}\left(\oint_{\partial\mathcal{D}_{r'}} - \oint_{\partial\mathcal{D}_r}\right)\frac{Y(k,c')}{c'-c}dc'. \tag{3.6.8}$$

For $|r'| \gg 1$, applying Lemma 3.1.3 on $\mathcal{I} = [-h,0]$, $x_{20} = -h$, with $k^* = k$ and $C_0 \ll 1$ for any $c \in \partial\mathcal{D}_{r'}$ uniformly, we obtain

$$|y_-(k,c,0) - k^{-1}\sinh kh| + |y_-'(k,c,0) - \cosh kh| \leq \tilde{C}C_0,$$

$\tilde{C}$ may depend on k, but independent of c, which implies

$$|Y(k,c) - k\coth kh| \leq \tilde{C}C_0, \quad \forall c \in \partial\mathcal{D}_{r'}.$$

Since $C_0 \to 0$ as $r' \to +\infty$, we have

$$\lim_{r'\to+\infty}\left(\frac{1}{2\pi i}\oint_{\partial\mathcal{D}_{r'}}\frac{Y(k,c')}{c'-c}dc' - k\coth kh\right) = 0$$

and thus the desired integral formula of $Y(k,c)$ follows. The representation of $\partial_c^n Y$ simply follows from direct differentiation. $\square$

Remark 3.6.2. *Though not needed in the rest of the paper, this lemma could be modified for general k and $c \notin U([-h,0])$. In this case, $0 < r < dist\big(c, U([-h,0])\big)$ should be chosen so that $y_-(k,c,0) \neq 0$ at any point along $\partial\mathcal{D}_r$. The integral representation formula would involve the residue at those roots of $y_-(k,\cdot,0)$ outside $\mathcal{D}_r$. The estimates should also be modified accordingly.*

To analyze the remaining integral in (3.6.7) (actually also for general $k \in \mathbb{R}$ instead of just large $|k|$),in the rest of the section, we assume $y_-(k,c,0) \neq 0$ so that Y is well-defined. We start with the imaginary part Y_I of Y.

Lemma 3.6.4. *$Y_I(k,c) = 0$ for $c \in \mathbb{R}\backslash U\big((-h,0]\big)$. Assume $U \in C^3$, $c = U(x_2^c) \in U\big((-h,0)\big)$, and $y_-(k,c,0) \neq 0$, then*

$$Y_I(k,c) = \frac{\pi U''(x_2^c) y_-(k,c,x_2^c)^2}{U'(x_2^c)|y_-(k,c,0)|^2}, \quad c \in U\big((-h,0)\big).$$

Proof. The vanishing of $Y_I(k,c)$ for $c \in \mathbb{R}\backslash U([-h,0])$ is obvious from its definition and Lemma 3.6.1(1). To derive the expression of $Y_I(k,c)$ for $c = U(x_2^c)$ with $x_2^c \in (-h,0)$ and $y_-(k,c,0) \neq 0$, we may consider $y(\epsilon, x_2) = \frac{y_-(k,c+i\epsilon,x_2)}{y_-(k,c+i\epsilon,0)}$, $\epsilon > 0$, which is also a solution to the homogeneous Rayleigh equation with $y(\epsilon,-h) = 0$ and $y(\epsilon,0) = 1$. It is straight forward to calculate

$$\operatorname{Im} y'(\epsilon,0) = \frac{1}{2i}\int_{-h}^{0} \partial_{x_2}(y'\bar{y} - y\bar{y}')dx_2 = \int_{-h}^{0} \frac{\epsilon U''|y|^2}{|U-c|^2+\epsilon^2}dx_2.$$

Applying the convergence estimates (3.2.31) and the Hölder continuity of $y_{0-}(k,c,x_2) \in \mathbb{R}$ in x_2, we obtain the desired

$$Y_I(k,c) = \lim_{\epsilon\to 0+} \operatorname{Im} y'(\epsilon,0) = \frac{\pi U''(x_2^c) y_-(k,c,x_2^c)^2}{U'(x_2^c)|y_-(k,c,0)|^2}.$$

This completes the proof of the lemma. $\square$

The above formula yields some refined estimates of Y_I for $c \in U([-h,0])$.

Lemma 3.6.5. *Assume $U \in C^{l_0}$, $l_0 \geq 4$, $k_0 \in \mathbb{R}$, and $y_-(k_0, c, 0) \neq 0$ for all $c \in U([-h, 0])$, then the following hold for $Y_I(k, c)$ and k in a neighborhood of k_0.*

1. *$Y_I(k, c)$ is C^{l_0-3} in k and $c \in U\big((-h, 0)\big)$. Moreover, for any $q \in [1, \infty)$, $j_1, j_2 \geq 0$, $j_2 \leq 2$, and $j_1 + j_2 \leq l_0 - 4$, $\partial_k^{j_1} \partial_{c_R}^{j_2} Y_I$ is $L_k^\infty W_c^{1,q}$ locally in $k \in \mathbb{R}$ and $c \in U\big([-h, 0)\big)$.*

2. *Moreover, assume $U \in C^6$, then there exists $C > 0$ depending only on $|U'|_{C^5}$ and $|(U')^{-1}|_{C^0}$ such that, for any $k \in \mathbb{R}$ and $c \in U\big((-h, 0)\big)$, we have*

$$\lim_{c \to U(0)-} Y_I(k, c) = \frac{\pi U''(0)}{U'(0)}, \quad \lim_{c \to U(-h)+} \partial_{c_R}^2 Y_I(k, c) = \frac{2\pi U''(-h)}{U'(-h)^3 |y_-(k, U(-h), 0)|^2},$$

$$\frac{\sinh^2(\mu^{-1}(x_2^c + h))}{C \sinh^2(\mu^{-1} h)} \leq Y_I\big(k, c\big) \leq \frac{C \sinh^2(\mu^{-1}(x_2^c + h))}{\sinh^2(\mu^{-1} h)},$$

$$\begin{aligned} |\partial_{c_R} Y_I(k, c)| \leq & C \frac{\sinh(2\mu^{-1}(x_2^c + h))}{\mu \sinh^2(\mu^{-1} h)} \\ & + C \frac{\sinh^2(\mu^{-1}(x_2^c + h))}{\sinh^2(\mu^{-1} h)} \big| \log \min\{1, |\mu^{-1}(U(0) - c)|\} \big|, \end{aligned}$$

where $\mu = \langle k \rangle^{-1} = (1 + k^2)^{-\frac{1}{2}}$.

Proof. Lemma 3.5.2 implies the C^{l_0-3} smoothness of $y_-(k, c, 0)$ in k and $c \in U\big((-h, 0)\big)$ and that of $y_-(k, c, x_2^c)$ in $c \in U([-h, 0])$, which also yields $y_-(k, c, x_2^c) = O(|c - U(-h)|)$ for $0 \leq c - U(-h) \ll 1$. Hence Y_I is C^{l_0-3} in k and $c \in U\big((-h, 0)\big)$. Despite the logarithmic singularity of $\partial_c y_-(k, c, 0)$ at $c = U(-h)$, we obtain the regularity of Y_I for c near $U(-h)$ from the vanishing of $y_-(k, c, x_2^c)$ and Lemma 3.5.2(f) which also leads to the regularity of Y_I for $c \neq U(0)$.

The upper bound estimate of Y_I and its limits as c approaches $U(0)-$ and $U(-h)+$ are direct corollaries of Lemma 3.3.2 and Lemma 3.6.1(5) and Remark 3.3.1, as well as (3.6.2) and (3.6.3). In particular,

$$\partial_{c_R}(y_-(k, c, x_2^c)) = \partial_{c_R} y_-(k, c, x_2) + U'(x_2^c)^{-1} y_-'(k, c, x_2^c) \to U'(-h)^{-1},$$

as $c \to U(-h)+$, implies the limit of $\partial_{c_R}^2 Y_I$ as $c \to U(-h)+$. In

$$\begin{aligned}\partial_{c_R} Y_I(k,c) =& \partial_c \left(\frac{\pi U''(x_2^c)}{U'(x_2^c)}\right) \frac{y_-(k,c,x_2^c)^2}{|y_-(k,c,0)|^2} + \frac{2\pi U''(x_2^c) y_-(k,c,x_2^c) \partial_c \big(y_-(k,c,x_2^c)\big)}{U'(x_2^c)|y_-(k,c,0)|^2} \\ & - \frac{2\pi U''(x_2^c) y_-(k,c,x_2^c)^2 \big(y_-(k,c,0) \cdot \partial_c y_-(k,c,0)\big)}{U'(x_2^c)|y_-(k,c,0)|^4},\end{aligned}$$

$\partial_c\big(y_-(k,c,x_2^c)\big)$ is estimated by (3.5.11). The other key term $\partial_c y_-(k,c,0)$ will be considered in three possible cases of $c \in U([-h,0])$ according to the division of $[-h,0] = \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3$ defined in (3.5.2) in section 3.5. Observing $c \in U([-h,0])$ implies $\mathcal{I}_2 \neq \emptyset$ and $x_2 = 0 \in \mathcal{I}_2 \cup \mathcal{I}_3$.

* ***Case 1***: $x_2 = 0 \in \mathcal{I}_3$ and $x_{2l} = -h$. The former happens if and only if $c \le U(0) - \rho_0^{-1}\mu$, while $x_{2l} = -h$ if and only if $c \le U(-h) + \rho_0^{-1}\mu$. Lemma 3.5.5 implies

$$|\partial_c y_-(k,c,0)| \le C\mu\big(1 + \big|\log\big(\mu^{-1}(c - U(-h))\big)\big|\big)\cosh \mu^{-1} h.$$

* ***Case 2***: $x_2 = 0 \in \mathcal{I}_3$ and $x_{2l} > -h$ which occurs if and only if $U(-h) + \rho_0^{-1}\mu \le c \le U(0) - \rho_0^{-1}\mu$. Also from Lemma 3.5.5, we have

$$|\partial_c y_-(k,c,0)| \le C\mu \cosh \mu^{-1} h.$$

* ***Case 3***: $x_2 = 0 \in \mathcal{I}_2$ which happens iff $U(0) - c \le \rho_0^{-1}\mu$ and $x_{2r} = 0$. From the definitions (3.5.2) of $\mathcal{I}_2$, (3.3.2) of ρ_0, and (3.0.4) of h_0, it holds

$$0 \le U(x_{2r}) - U(x_{2l}) \le 2\rho_0^{-1}\mu \le \tfrac{1}{2} h_0 \inf U' \Longrightarrow -x_{2l} = x_{2r} - x_{2l} \le \tfrac{1}{2} h_0 \Longrightarrow x_{2l} > -h.$$

This in turn implies $c - U(x_{2l}) = \rho_0^{-1}\mu$ and thus Lemma 3.5.4 yields

$$|\partial_c y_-(k,c,0)| \le C\mu\big(1 + \big|\log\big(\mu^{-1}(U(0) - c)\big)\big|\big)\cosh \mu^{-1} h.$$

The desired estimates on $\partial_{c_R} Y_I$ follow from (3.5.11), Lemma 3.6.1 and Lemma 3.3.2, and the above estimates. In particular, in the above case 1, we also used

$$\mu \sinh |\mu^{-1}(x_2^c + h)| \big| \log |\mu^{-1}\big(c - U(-h)\big)| \big| \le C\mu \cosh(\mu^{-1}(x_2^c + h)),$$

which can be shown by considering whether $|\mu^{-1}(c - U(-h))| \le 1$ separately. $\square$

In the following we analyze $Y(k, c)$ by writing it as a Cauchy integral of Y_I.

Lemma 3.6.6. *Assume $U \in C^{l_0}$, $l_0 \ge 6$, and $k \in \mathbb{R}$ satisfy that $y_-(k, c, 0) \ne 0$ for all $c \in U([-h, 0])$, then $Y(k, c)$ and $\partial_k^{j_1} \partial_{c_R}^{j_2} Y(k, c)$ is $L_k^\infty L_{c_R}^q$ locally in $k \in \mathbb{R}$ and c_R in the domain $D(Y)$ for any $q \in (1, \infty)$, $0 \le j_2 \le 2$, and $0 \le j_1 \le l_0 - 4 - j_2$. Assume, in addition, $y_-(k, c, 0) \ne 0$ for all $c \in \mathbb{C}$, then, for any $c \notin U([-h, 0])$,*

$$Y(k, c) = \frac{1}{\pi} \int_{U(-h)}^{U(0)} \frac{Y_I(k, c')}{c' - c} dc' + k \coth kh, \tag{3.6.9}$$

and for $c \in U\big([-h, 0)\big)$,

$$Y(k, c) = -\mathcal{H}\big(Y_I(k, \cdot)\big)(c) + iY_I(k, c) + k \coth kh. \tag{3.6.10}$$

Here $\mathcal{H}$ denotes the Hilbert transform in $c \in \mathbb{R}$, namely,

$$\mathcal{H}\big(Y_I(k, \cdot)\big)(c) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{Y_I(k, c')}{c - c'} dc' = \frac{1}{\pi} \text{P.V.} \int_{U(-h)}^{U(0)} \frac{Y_I(k, c')}{c - c'} dc',$$

where P.V.$\int$ represent the principle value of the singular integral. We also recall $Y(k, c) = Y(k, c + i0)$ and $Y(k, c - i0) = \overline{Y(k, c + i0)}$ for $c \in \mathbb{R}$.

Proof. Let us first assume $y_-(k, c, 0) \ne 0$ for all $c \in \mathbb{C}$, then $Y(k, c)$ is well-defined for all $c \ne U(0)$. The same argument as in the proof of Lemma 3.6.3 yields (3.6.7) for all $k \in \mathbb{R}$, $c \ne U([-h, 0])$, and $0 < r < dist\big(c, U([-h, 0])\big)$. The contour $\partial \mathcal{D}_r$ is the union of two segments $[U(-h), U(0)] \pm ir$, the left half circle centered at $U(-h)$ with radius r, and the right half circle centered at $U(0)$ with radius r. As $r \to 0+$, due to the continuity of Y at $c \neq U(0)$ and its logarithmic upper bound near $U(0)$ given in Lemma 3.6.2, the Cauchy integrals along the two half circles converge to zero as $r \to 0+$. Hence the integral form (3.6.9) of $Y(k,c)$ follows from taking the limit of (3.6.7) as $r \to 0+$ and the conjugacy $Y(k,\overline{c'}) = \overline{Y(k,c')}$.

For $c_I \neq 0$, the integral form (3.6.9) can be rewritten as

$$Y(k,c) = \frac{1}{\pi}\int_{U(-h)}^{U(0)} \frac{(c'-c_R+ic_I)}{(c'-c_R)^2+c_I^2} Y_I(k,c')dc'.$$

A standard treatment of the above singular integral as $c_I \to 0+$, along with the regularity of $Y_I(k,c')$ in $c' \in U\big([-h,0)\big)$ given in Lemma 3.6.4 and Lemma 3.6.5, yields (3.6.10).

From Lemma 3.6.5, even though Y_I is $W^{3,q}$ locally in $c \in U\big([-h,0)\big)$ if $l_0 \geq 6$, when viewed as a function of $k \in \mathbb{R}$ and $c \in \mathbb{R} \setminus \{U(0)\}$, we only have $\partial_k^{j_1}\partial_{c_R}^2 Y_I \in L^\infty$ due to its jump at $c = U(-h)$. The regularity of Y follows from that of Y_I and the boundedness in L^q of the convolution by $\frac{1}{c'+ic_I}$ with the parameter $c_I \geq 0$. Here the singularity of Y_I near $c = U(0)$ does not affect the regularity of Y away from $U(0)$ due to the localization property of this convolution operator.

Finally, if we only assume $y_-(k,c,0) \neq 0$ for $c \in U([-h,0])$, then there would be additional contour integrals of Y in (3.6.9) along contours in $\mathbb{C}$ enclosing the roots of $y_-(k,\cdot,0)$ outside $U([-h,0])$. Those integrals in the analytic region of Y would not affect the regularity of Y. The proof of the lemma is complete. $\square$

With the representation of Y in terms of Cauchy integrals, we may also calculate its derivatives in more details.

Corollary 3.6.6.1. *It holds, for $c \notin U([-h,0])$,*

$$\partial_c Y(k,c) = \frac{1}{\pi}\int_{U(-h)}^{U(0)} \frac{Y_I(k,c')}{(c'-c)^2}dc' = \frac{1}{\pi}\int_{U(-h)}^{U(0)} \frac{\partial_{c_R} Y_I(k,c')}{c'-c}dc' - \frac{U''(0)}{U'(0)\big(U(0)-c\big)}, \tag{3.6.11}$$

and for $c \in U\big([-h,0)\big)$,

$$\partial_c Y(k,c) = -\mathcal{H}\big(\partial_{c_R} Y_I(k,\cdot)\big)(c) + i\partial_{c_R} Y_I(k,c) - \frac{U''(0)}{U'(0)\big(U(0)-c\big)}. \tag{3.6.12}$$

Using the regularity of and estimates on Y_I and $\partial_{c_R} Y_I$ given in Lemma 3.6.5 , (3.6.11) follows from direct differentiation and integration by parts, along with the explicit form of $Y_I\big(k, U(0)-\big)$. Equality (3.6.12) is obtained by taking the limit of (3.6.11) as $c_I \to 0+$. We omit the details of these straight forward calculations.

CHAPTER 4

EIGENVALUES OF THE LINEARIZATION OF THE WATER WAVE AT SHEAR FLOWS

In this chapter, we shall discuss the distribution of eigenvalues of the linearized gravity-capillary water wave system (1.2.2) at the shear flow $\big(U(x_2), 0\big)^T$. As (1.2.2) preserves Fourier mode e^{ikx_1} for any k, the wave number $k \in \mathbb{R}$ would be treated as a parameter in this chapter. According to Lemma 2.2.1, $-ikc \in \mathbb{C}$, $c \in \mathbb{C} \setminus U([-h, 0])$, is an eigenvalue of (1.2.2) with parameter k if and only if

$$
\begin{aligned}
&\mathbf{F}(k,c) = \mathbf{F}_R + i\mathbf{F}_I := (g+\sigma k^2)y_+(k,c,-h) = (g+\sigma k^2)(y_+y_-' - y_+'y_-)(k,c,0)\\
=&\big(U(0)-c\big)^2 y_-'(k,c,0) - \big(U'(0)\big(U(0)-c\big) + g + \sigma k^2\big)y_-(k,c,0) = 0,
\end{aligned}
\tag{4.0.1}
$$

where the last equal sign in the first row is due to the Wronskian structure of $\mathbf{F}$. Let

$$
\mathbf{F}(k,c) = \lim_{\epsilon\to 0+} \mathbf{F}(k, c+i\epsilon) = \overline{\lim_{\epsilon\to 0+} \mathbf{F}(k, c-i\epsilon)}, \quad c \in U\big([-h,0]\big).
$$

It is easy to see that, if $\mathbf{F}(k,c) = 0$, then $y_-(k,c,x_2)$ also generates the associated eigenfunction of (1.2.2). In the literatures, those zero point c of $\mathbf{F}$ with $c_I > 0$ are often referred to as unstable modes, while those zero point $c \in \mathbb{R}$ as neutral modes. We recall that Yih proved that the semicircle theorem also holds for free boundary problem [68], namely, (1.3.3) holds for all unstable modes.

From the analysis in section 3.5, it is not clear whether $\mathbf{F}$ is C^1 at $c = U(-h)$ which would be crucial for the bifurcation analysis of eigenvalues. We also consider an almost equivalent quantity

$$\begin{aligned} F(k,c) =& y_-(k,c,0)^{-1}\mathbf{F} = F_R + iF_I \\ &= Y(k,c)\big(U(0)-c\big)^2 - U'(0)\big(U(0)-c\big) - (g+\sigma k^2) = 0, \end{aligned} \tag{4.0.2}$$

where $Y(k,c)$ is defined in (3.6.1), and

$$F(k,c) = \lim_{\epsilon\to 0+} F(k, c+i\epsilon) = \overline{\lim_{\epsilon\to 0+} F(k, c-i\epsilon)}, \quad \forall c \in U\big([-h,0]\big).$$

Apparently $\mathbf{F}$ and F satisfy

$$\mathbf{F}(-k,c) = \mathbf{F}(k,c) = \overline{\mathbf{F}(k,\bar{c})},\ \forall c \notin U\big((-h,0)\big); \tag{4.0.3}$$

$$F(-k,c) = F(k,c) = \overline{F(k,\bar{c})},\ c \in D(Y) \setminus U\big((-h,0)\big). \tag{4.0.4}$$

From Lemma 3.6.6 F is $C^{1,\alpha}$ near $c_0 = U(-h)$ if $y_-(k,c,0) \neq 0$ for all $c \in U([-h,0])$, which is crucial for the bifurcation analysis.

4.1 Basic properties of eigenvalues

Apparently it holds that

$$\mathbf{F} \text{ is analytic in } k \in \mathbb{R}\ \&\ c \notin U([-h,0]) \text{ and } F \text{ analytic in } k \in \mathbb{R}\ \&\ c \in D(Y)\backslash U([-h,0]), \tag{4.1.1}$$

$$\mathbf{F}(k,c) = 0 \Longleftrightarrow c \text{ is a non-singular or singular mode of (2.2.6).}$$

In the following we first give some basic properties of $\mathbf{F}$ under minimal assumptions.

Lemma 4.1.1. *Assume $U \in C^{l_0}$, $l_0 \geq 3$, then for any $k \in \mathbb{R}$, the following hold.*

1. *$\mathbf{F}$ is well defined for all $k \in \mathbb{R}$ and $c \in \mathbb{C}$. When restricted to $c_I \geq 0$, $\mathbf{F}$ is C^{l_0-3} in k*

and $c \notin \{U(-h), U(0)\}$ and if, in addition $l_0 \geq 4$, then $\mathbf{F}$ is also C^α in both k and c.

2. *$F(k,c)$ is well-defined for c close to $U(-h)$ and $U(0)$, C^1 near $c = U(0)$, and*

$$F(k, U(-h)) \in \mathbb{R}, \quad F\big(0, U(-h)\big) = -g,$$
$$F\big(k, U(0)\big) = -g - \sigma k^2, \quad \partial_c F\big(k, U(0)\big) = U'(0).$$

3. *Assume $l_0 \geq 6$, then for any $r \in (1, \infty)$, there exists $C > 0$ determined only by r, $|U'|_{C^5}$, and $|(U')^{-1}|_{C^0}$, such that, for any $c_I \geq 0$ and $k \in \mathbb{R}$,*

$$|\partial_c \mathbf{F}(k, \cdot + ic_I)|_{L^r_{c_R}} \leq C\mu^{-1} e^{\mu^{-1} h}, \quad \lim_{c_I \to 0+} |\partial_c \mathbf{F}(k, \cdot + ic_I) - \partial_c \mathbf{F}(k, \cdot)|_{W^{1,r}_{c_R}} = 0,$$

where the norm is taken on $c_R \in [-\frac{1}{2}h_0 - h, \frac{1}{2}h_0]$ and we recall $\mu = (1+k^2)^{-\frac{1}{2}}$.

4. *$\mathbf{F}(k,c) \neq 0$ if $y_-(k,c,0) = 0$. Hence $\{c \mid \mathbf{F}(k,c) = 0\} = \{c \mid F(k,c) = 0\}$ for any $k \in \mathbb{R}$.*

5. *$\mathbf{F}(k,c) = 0$ iff there exists a C^2 solution $y(x_2)$ to (2.2.8) satisfying the corresponding homogeneous boundary conditions of ((2.2.6b)-(2.2.6c)).*

6. *For any $x_2 \in (0, -h)$, $\mathbf{F}_I(k, U(x_2)) \neq 0$ if $U''(x_2) \neq 0$.*

Proof. For $c_R \in U\big([-h, 0)\big)$, the convergence of $\mathbf{F}(k, c_R + ic_I)$ as $c_I \to 0+$ follows from the convergence estimates given in Proposition 3.2.4. For c near $U(0)$, the logarithmic singularity in $y_-'(k,c,0)$ is cancelled by $(U(0)-c)^2$ and thus the convergence of $\mathbf{F}(k, U(0)+ic_I)$ and the continuity of $\mathbf{F}$ at $c = U(0)$ follow. The C^α and C^{l_0-3} smoothness of $\mathbf{F}$ is obtained from those of $y_-(k,c,0)$ and $y_-'(k,c,0)$ (Lemma 3.5.2 and Lemma 3.5.4) as well as using the factor $(U(0)-c)^2$ multiplied to $y_-'(k,c,0)$.

From Lemma 3.6.1(1), $y_-(k, U(-h), 0), y_-(k, U(0), 0) > 0$ and thus F is well-defined near $c = U(-h), U(0)$. The property $F(k, U(-h)) \in \mathbb{R}$ and the value of $F(0, U(-h))$ are due to those of Y given in Lemma 3.6.2(1). The C^1 smoothness of F for c near $U(0)$

follows from Lemma 3.6.2(2) and the definition of F. The values of F and $\partial_c F$ at $(k, U(0))$ is obtained by direct computation.

Statement (3) is a corollary of Proposition 3.2.4, Lemma 3.5.7, and the definition of $\mathbf{F}$.

Suppose $y_-(k,c,0) = 0$. Lemma 3.6.1(1) implies $c \neq U(0)$. As a non-trivial solution to the homogeneous Rayleigh equation (3.0.1), it must hold $y_-'(k,c,0) \neq 0$. Therefore $\mathbf{F}(k,c) \neq 0$.

To prove statement (5), we first observe that $\mathbf{F}(k,c) = 0$ iff y_- satisfies the corresponding homogeneous boundary conditions of (2.2.6c), which happens only if $y_-(k,c,0) \neq 0$ and thus $Y(k,c)$ and $F(k,c)$ are well-defined. Moreover the statement is obvious for $c \notin U\big([-h,0]\big)$ and also for $c = U(-h)$ due to the smoothness of y_- (Lemma 3.4.1), while $F(k, U(0)) \neq 0$ due to statement (2). Hence we focus on $c \in U\big((-h,0)\big)$ only. "$\Longrightarrow$": As $c \in U\big((-h,0)\big)$, $F(k,c) = 0$ implies $Y_I(k,c) = 0$ and consequently $U''(x_2^c) = 0$ according to Lemma 3.6.4. Consequently Lemma 3.4.1, particularly formula (3.4.3), and the definition of Γ_0 yield the smoothness of y_- which apparently satisfies (2.2.8). "$\Longleftarrow$": This solution $y(x_2)$ has to be proportional to y_- on $[-h, x_2^c]$ which yields $y(x_2^c) \neq 0$ due to Lemma 3.6.1(2). Hence the smoothness of $y(x_2)$ and equation (2.2.8) imply $U''(x_2^c) = 0$. Consequently both (2.2.8) and the homogeneous Rayleigh equation (3.0.1) are regular on $[-h,0]$ and are equivalent to each other. Therefore $y_-(x_2)$ and $y(x_2)$ are proportional on $[-h,0]$ and thus y_- satisfies the boundary condition at $x_2 = 0$.

To prove the last statement, let $c = U(x_2)$, $x_2 \in (-h,0)$. According to Lemma 3.6.1, $\operatorname{Im} y_-(k,c,0) \neq 0$ if $U''(x_2) \neq 0$ and thus $Y(k,c)$ is well-defined. Lemma 3.6.4 yields

$$F_I(k,c) = \big(U(0)-c\big)^2 Y_I(k,c) = \frac{\pi (U(0)-c)^2 U''(x_2) y_-(k,c,x_2)^2}{U'(x_2)|y_-(k,c,0)|^2} \neq 0, \qquad (4.1.2)$$

which prove statement (5). This is the same argument as in [68] in the case of gravity waves. □

Remark 4.1.1. *The monotonicity assumption on U is used in the above proof of statement*

(5). If U is not monotonic, $U^{-1}(c)$ may contain several points in $[-h, 0]$ for a root of $F(k, \cdot)$ and the corresponding solution $y_-(k, c, x_2)$ may not be in $H^2_{x_2}$. Therefore the set of roots of $F(k, \cdot)$, which is what really matters, may be larger than those defined as singular modes in Definition 2.2.1.

In the next step, we consider $\mathbf{F}$ for $|k| \gg 1$. Unlike the linearized Euler equation on a fixed channel where no eigenvalues exist for large k. Eigenvalues do exist for each large k for the linearized water wave system. According to Lemma 4.1.2(2), we often consider $F(k, c)$ as well.

Lemma 4.1.2. *Assume $U \in C^3$, then the following hold for any $\alpha \in (0, \frac{1}{2})$.*

1. *There exists $C > 0$ depending only on α, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that*

$$|\mathbf{F} + \sigma k^2 \mu \sinh \mu^{-1} h - (U(0) - c)^2 \cosh \mu^{-1} h| \leq C\big(\mu^{\alpha - 1} + |c|^2 \mu^{\alpha}\big) \cosh \mu^{-1} h,$$

   where we recall $\mu = (1 + k^2)^{-\frac{1}{2}}$.

2. *For any $k_*, M > 0$, there exists $C > 0$ depending only on M, k_*, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that, for any $|k| \leq k_*$ and c satisfying $dist(c, U(-h, 0]) \geq M$,*

$$|\mathbf{F} - (U(0) - c)^2 \cosh kh| \leq C\big(1 + |c| + |U(0) - c|^2 dist(c, U([-h, 0])^{-1}\big).$$

3. *There exist $k_0 > 0$ and $C > 0$ depending only on $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that for any $|k| \geq k_0$, (4.0.1) has exactly two solutions $c^{\pm}(k) \in \mathbb{C} \setminus U([-h, 0])$ depending on k analytically. Moreover they satisfy*

$$c^{\pm}(k) \in \mathbb{R}, \quad c^{\pm}(-k) = c^{\pm}(k),$$
$$\Big|c^{\pm}(k) \mp \sqrt{\sigma |k|} - U(0)\Big| \leq C, \big|\partial_c F\big(k, c^{\pm}(k)\big) \mp 2\sqrt{\sigma} |k|^{\frac{3}{2}}\big| \leq C|k|.$$

Proof. The first statement follows directly from Lemma 3.3.2, where the factor $(U(0) -$

$c)^2$ is used to cancel the logarithmic singularity in the estimate of y'_-, and the second from Lemma 3.1.3 with $C_0 = dist(c, U([-h,0])^{-1}$. We focus on the roots of $\mathbf{F}$. From Lemma 3.3.2,

$$\exists k_0 > 0, \text{ s .t. } |y_-(k,c,0)| \geq (1/2)\mu \sinh \mu^{-1} k > 0, \quad \forall |k| \geq k_0,\, c \in \mathbb{C},$$

and thus we can work with $F(k,c)$ and $Y(k,c)$. Let

$$S_k = \{c \in \mathbb{C} \mid |c| \geq \sqrt{\sigma|k|}/2\}.$$

From statement (1) and Lemma 3.6.2(3), it holds that there exist $k_0 > 0$ such that, for any $|k| \geq k_0$, $F(k,c) = 0$ only if $c \in S_k$. We may take larger $k_0 > 0$ if necessary such that $dist\big(S_k, U([-h,0])\big) \geq 1$. From Lemma 3.6.6 and Lemma 3.6.5 and Corollary 3.6.6.1, there exists $C > 0$ depending only on U such that, for all $|k| \geq k_0$, $c \in S_k$,

$$|Y(k,c) - k\coth kh| \leq \frac{C}{(1+|c|)\sinh^2 \frac{h}{\mu}} \int_{U(-h)}^{U(0)} \sinh^2 \tfrac{1}{\mu}(U^{-1}(c') + h) dc',$$

$$|\partial_c Y(k,c)| \leq \frac{C}{1+|c|} + \frac{C}{(1+|c|)\mu \sinh^2 \frac{h}{\mu}} \int_{U(-h)}^{U(0)} \sinh \tfrac{2}{\mu}\big(U^{-1}(c') + h\big) dc'.$$

By a substitution $\tau = \frac{1}{\mu}\big(U^{-1}(c') + h\big)$ we obtain

$$|Y(k,c) - k\coth kh| \leq C(|k|+1)^{-1}(1+|c|)^{-1}, \quad |\partial_c Y(k,c)| \leq C(1+|c|)^{-1}, \quad \forall |k| \geq k_0.$$

On the other hand, viewing $F(k,c) = 0$ as a quadratic equation of $U(0) - c$, its roots also satisfy

$$c = f_\pm(k,c), \text{ where } f_\pm(k,c) = U(0) - \frac{U'(0)}{2Y(k,c)} \pm \sqrt{\frac{U'(0)^2}{4Y(k,c)^2} + \frac{g + \sigma k^2}{Y(k,c)}}.$$

Using the above estimates on Y and $\coth s = 1 + \frac{2}{e^{2s}-1}$, it is straight forward to verify that for any $|k| \geq k_0$ and $c \in S_k$,

$$\left| f_\pm(k,c) \mp \sqrt{\sigma |k|} - U(0) \right| \leq C,$$

and

$$|\partial_c f_\pm(k,c)| = \left| \frac{U'(0)}{2Y^2} \mp \frac{1}{2}\left(\frac{U'(0)^2}{4Y^2} + \frac{g+\sigma k^2}{Y} \right)^{-\frac{1}{2}} \left(\frac{U'(0)^2}{2Y^3} + \frac{g+\sigma k^2}{Y^2} \right) \right| |\partial_c Y| \leq \frac{C}{|k|}.$$

Therefore $f_\pm(k,\cdot)$ are contractions acting on S_k. Their fixed points $c^\pm(k)$, analytic in k, are the only solutions to (4.0.1), or equivalently (4.0.2). These $c^\pm(k) \in \mathbb{R}$ since $f_\pm(k,c) \in \mathbb{R}$ for $c \in \mathbb{R}$ which allows the iteration to be taken in $\mathbb{R}$. Finally, one may compute

$$\partial_c F = (U(0)-c)^2 \partial_c Y + 2(c - U(0))Y + U'(0). \tag{4.1.3}$$

Using the above estimates on $Y - |k|$, $\partial_c Y$, and $c^\pm(k)$, one may compute

$$\begin{aligned} &\left|\partial_c F\big(k, c^\pm(k)\big) \mp 2\sqrt{\sigma}|k|^{\frac{3}{2}}\right| \\ =&\left|2Y\big(c - U(0)\big) \mp 2\sqrt{\sigma}|k|^{\frac{3}{2}} + \partial_c Y\big(U(0)-c\big)^2 + U'(0)\right|_{c=c^\pm(k)} \leq C|k|. \end{aligned}$$

The evenness of $c^\pm(k)$ in k is due to that of $\mathbf{F}(k,c)$ and the uniqueness of the fixed points of the above contractions. This completes the proof of the lemma. $\square$

We shall track the two roots $c^\pm(k)$ of the analytic function $F(k,\cdot)$ as $|k|$ decreases, based on a standard analytic continuation argument.

Lemma 4.1.3. *Assume $U \in C^3$. Suppose $k_0 \in \mathbb{R}$ and $c_0 \in \mathbb{C} \setminus U([-h,0])$ satisfy $\mathbf{F}(k_0,c_0) = 0$ and $\partial_c \mathbf{F}(k_0,c_0) \neq 0$, then the following hold.*

1. *There exists an analytic function $c(k) \in \mathbb{C} \setminus U([-h,0])$ defined on an max interval (k_-,k_+) such that $\mathbf{F}\big(k,c(k)\big) = 0$ and $\partial_c \mathbf{F}\big(k,c(k)\big) \neq 0$.*

2. *$c(k) \in \mathbb{R}$ for all $k \in (k_-,k_+)$ if and only if $c_0 \in \mathbb{R}$.*

3. If $k_+ < \infty$ (or $k_- > -\infty$), then

(a) $\lim_{k\to(k_+)-} dist(c(k), U([-h,0])) = 0$ *(or* $\lim_{k\to(k_-)+} dist(c(k), U([-h,0])) = 0$ *if* $k_- > -\infty$*), or*

(b) $\liminf_{k\to(k_+)-} \min\{|c(k) - c| \ : \ \forall c$ *s. t.* $\mathbf{F}(k,c) = 0, c \neq c(k)\} = 0$ *(or* $\liminf_{k\to(k_-)+} \min\{|c(k) - c| \ : \ \mathbf{F}(k,c) = 0, c \neq c(k)\} = 0$ *if* $k_- > -\infty$*).*

Proof. We start the proof with a simple and standard consideration of the index of complex analytic functions. Suppose $\mathbf{F}(k,c) \neq 0$ at any $c \in \partial\Omega$ where $\Omega \subset \mathbb{C} \setminus U([-h,0])$ is a domain with piecewise smooth boundary $\partial\Omega$, then the index

$$\mathrm{Ind}\big(\mathbf{F}(k,\cdot),\Omega\big) := \frac{1}{2\pi i}\oint_{\partial\Omega} \frac{\partial_c \mathbf{F}(k,c)}{\mathbf{F}(k,c)} dc \in \mathbb{N} \cup \{0\} \tag{4.1.4}$$

is equal to the number of zeros of $\mathbf{F}(k,\cdot)$ inside Ω, counting their multiplicities. Therefore the analyticity of F in k and c implies that $\mathrm{Ind}\big(\mathbf{F}(k,\cdot),\Omega\big)$ is a constant in k as long as $\mathbf{F}(k,c) = 0$ does not occur on $\partial\Omega$.

As a consequence, starting with the simple root $c_0 \in \mathbb{C}\setminus U([-h,0])$ of $\mathbf{F}(k_0,\cdot)$, a unique continuation of $c(k) \subset \mathbb{C} \setminus U([-h,0])$ of *simple* roots of $\mathbf{F}(k,\cdot)$ exists and is analytic in k. The simplicity of $c(k)$ is due to the fact $\mathrm{Ind}\big(\mathbf{F}(k,\cdot),\Omega\big) = 1$ for any sufficiently small neighborhood Ω of $c(k)$ in the continuation procedure. For any $c \in \mathbb{R} \setminus U([-h,0])$, we have $\mathbf{F}(k,c) \in \mathbb{R}$ and $\partial_{c_R}\mathbf{F}_R(k,c) = \partial_c \mathbf{F}(k,c) \neq 0$. Therefore if $c(k_1) \in \mathbb{R} \setminus U([-h,0])$ for some k_1 along the continuation curve, then the unique extension $c(k)$ coincides with the (real) root of $\mathbf{F}_R(k,c_R)$ obtained by applying the Implicit Function Theorem to the real function $\mathbf{F}_R(k,c_R)$. Hence $c(k) \in \mathbb{R}$ if and only if $c_0 \in \mathbb{R}$.

Let (k_-, k_+) be the *max* interval of the continuation $c(k) \subset \mathbb{C} \setminus U([-h,0])$ as simple roots of $\mathbf{F}(k,\cdot)$ and we shall prove statement (3). Suppose $k_- > -\infty$, while the other case $k_+ < +\infty$ can be analyzed similarly. As $k \to (k_-)+$, the solution curve $c(k)$ is bounded due to Lemma 4.1.2(2). Therefore there exists a sequence $(k_j)_{j=1}^\infty \subset (k_-, k_+)$ such that $\lim_{j\to\infty} k_j = k_-$ and $c_- = \lim_{j\to\infty} c(k_j) \in \mathbb{C}$ exists. Statement (2) implies that

$c(k)$ stays in the closure of either the upper or lower half of $\mathbb{C}$ and thus $\mathbf{F}(k_-, c_-) = 0$. Assume statement (3)(a) does not hold, then such a subsequence can be chosen such that $c_- \notin U([-h, 0])$. Therefore c_- is a root in the domain of analyticity of $\mathbf{F}(k_-, \cdot)$. Clearly c_- is not a simple zero of $\mathbf{F}(k_-, \cdot)$, otherwise $c(k)$ can be extended beyond k_-. Recall c_- has to be an isolated root of $\mathbf{F}(k_-, \cdot)$ since all roots of non-trivial analytic functions are isolated. Therefore, there exists a small neighborhood Ω of c_- such that, for any $k \geq k_-$ sufficiently close to k_-, it hold $\mathrm{Ind}\big(\mathbf{F}(k, \cdot), \Omega\big) \geq 2$. Consequently, for each k_j close to k_-, there exists at least another root c of $\mathbf{F}(k_j, \cdot)$ in Ω and thus (3)(b) holds. $\square$

The semicircle theorem of Yih [68] states that all imaginary roots c of $\mathbf{F}(k, \cdot)$ are contained in the circle with the diameter segment $U([-h, 0])$, so the only possibility for the branches $c^{\pm}(k)$ of simple roots of $\mathbf{F}(k, \cdot)$ obtained in Lemma 4.1.2 can not be extended for all $k \in \mathbb{R}$ is when they reaches $U(0)$ or $U(-h)$, respectively. As a corollary of $\mathbf{F}\big(k, U(0)\big) \neq 0$ and we have

Corollary 4.1.3.1. *(1) The branch $c^+(k)$ can be extended for all $k \in \mathbb{R}$. Moreover $c^+(k) \in \mathbb{R}$ is even in k, $\partial_c F\big(k, c^+(k)\big) > 0$, and $c^+(k) > U(0) + \rho_0$ for all $k \in \mathbb{R}$, for some $\rho_0 > 0$ independent of k.*

(2) If $\mathbf{F}(k, U(-h)) \neq 0$ for all $k \in \mathbb{R}$, then $c^-(k)$ of simple roots of $\mathbf{F}(k, \cdot)$ obtained in can also be extended for all $k \in \mathbb{R}$. Moreover $c^-(k) \in \mathbb{R}$ is even in k, $\partial_c F\big(k, c^-(k)\big) < 0$, and $c^-(k) < U(-h) - \rho_0$ for all $k \in \mathbb{R}$, for some $\rho_0 > 0$ independent of k.

Proof. Let k_0 be given in Lemma 4.1.2(3) and we only need to focus on $|k| \leq k_0$. We may assume k_0 is sufficiently large such that $c^+(k_0) > U(0)$ and $c^-(k_0) < U(-h)$. From Lemma 4.1.2(2), there exists $R > 0$ such that $\mathbf{F}(k, c) \neq 0$ for all $k \in [-k_0, k_0]$ and $|c| \geq R$. Hence $c^+(k_0) \in (U(0), R)$ and $c^-(k_0) \in (-R, U(-h))$ are the only roots of $\mathbf{F}(\pm k_0, \cdot)$, which are also simple with $\pm\partial_c F(k_0, c^{\pm}(k_0)) > 0$.

We first consider $c^+(k)$. Let

$$\Omega = \{c \in \mathbb{C} \mid c_R \in (U(0), R),\ c_I \in (-1, 1)\}.$$

According to Lemma 4.1.1(2), $\mathbf{F}(k, U(0)) \neq 0$ for any k. Hence the semicircle theorem and the choice of R imply that a.) $c^+(k_0) \in \Omega$ and b.) $\mathbf{F}(k, c) \neq 0$ for all $|k| \leq k_0$ and $c \in \partial\Omega$, and thus

$$\text{Ind}(\mathbf{F}(k, \cdot), \Omega) = \text{Ind}(\mathbf{F}(k_0, \cdot), \Omega) = 1, \forall |k| \leq k_0.$$

Therefore none of the possibilities in Lemma 4.1.3(3ab) can happen to the extension $c^+(k) \in \Omega$ starting from $k = k_0$, so this branch of simple root of $\mathbf{F}(k, c)$ can be uniquely extended for all $k \in [-k_0, k_0]$ with $c^+(k) \in (U(0), R)$ as the only root of $\mathbf{F}(k, \cdot)$ in Ω. The value of this extension at $k = -k_0$ has to coincide with $c^+(-k_0) = c^+(k_0)$ as $c^\pm(-k_0)$ are the only roots of $\mathbf{F}(-k_0, \cdot)$ while $c^-(-k_0) < U(-h)$. Therefore the extensions starting from $c^+(\pm k_0)$ have to coincide. The evenness of $c^+(k)$ in $k \in [-k_0, k_0]$ follows from that of $\mathbf{F}$ and the uniqueness of its root in Ω. The sign of $\partial_c F\big(k, c^+(k)\big)$ remains positive from $k = k_0$ as $c^+(k)$ is always simple. The existence of $\rho_0 > 0$ is simple due to the continuity of $\mathbf{F}$. The same argument applies to $c^-(k)$ under the assumption $\mathbf{F}(k, U(-h)) \neq 0$ all for k. The proof is complete. $\square$

Based on the above analysis, we shall conclude that $-ikc^\pm(k)$ are the only eigenvalues of the linearized capillary gravity wave under the additonal assumption of the absence of singular modes

$$\mathbf{F}(k, U(x_2)) \neq 0, \quad \forall k \in \mathbf{K},\ x_2 \in [-h, 0], \tag{4.1.5}$$

where $\mathbf{K} = \mathbb{R}$ or $\frac{2\pi}{L}\mathbb{N}$ and L is the period of the water wave in the x_1 direction.

Proposition 4.1.4. *Assume $U \in C^3$ and (4.1.5) for $\mathbf{K} = \mathbb{R}$ or $\frac{2\pi}{L}\mathbb{N}$, then there exists $\rho > 0$ such that*

1. $F_0 \triangleq \inf\{(1 + k^2)^{-\frac{1}{2}} e^{-\frac{h}{\mu}} |\mathbf{F}(k, c)| \mid k \in \mathbf{K}, c_R \in [U(-h) - \rho, U(0) + \rho], c_I \in [-\rho, \rho]\} > 0.$
2. *Assume $\mathbf{K} = \mathbb{R}$, then $\{c \mid \mathbf{F}(k, c) = 0\} = \{c^\pm(k)\}$.*

Proof. The first statement is a direct corollary of the continuity of $\mathbf{F}$, its analyticity outside $U([-h,0])$, assumption (4.1.5), and Lemma 4.1.2.

Let us consider statement (2). Corollary 4.1.3.1 and (4.1.5) imply that both $c^+(k) \in (U(0), +\infty)$ and $c^-(k) \in (-\infty, U(-h))$ can be extended as even analytic functions of $k \in \mathbb{R}$. Let $k_0, R > 0$ be taken as in the proof of Corollary 4.1.3.1 and we only need to focus on $|k| \leq k_0$. Assumption (4.1.5) also yields $\rho > 0$ such that

$$\mathbf{F}(k,c) \neq 0, \ \ \forall dist\big(c, U([-h,0])\big) = \rho, \ |k| \leq k_0.$$

Let

$$\Omega = \{c \in \mathbb{C} \mid |c| < R, \, dist\big(c, U([-h,0])\big) > \rho\},$$

then we have $\mathbf{F}(k,c) \neq 0$ for all $|k| \leq k_0$ and $c \in \partial\Omega$. Therefore

$$\text{Ind}(\mathbf{F}(k,\cdot), \Omega) = \text{Ind}(\mathbf{F}(k_0,\cdot), \Omega) = 2, \forall |k| \leq k_0,$$

and $\mathbf{F}(k,\cdot)$ does not have any other roots. $\square$

In order to obtain a more complete picture of the eigenvalue distribution we shall derive some sign properties in the following lemma, where F and Y are viewed as function of c and $K = k^2 \geq 0$. According to Lemma 3.6.1(1), F is well-defined for c in a neighborhood of $\mathbb{R} \setminus U\big((-h,0)\big)$.

Lemma 4.1.5. *Assume $U \in C^3$, then we have*

$$\partial_K^2\big(F\big(\sqrt{K}, c\big)\big) < 0, \ \ \forall k \in \mathbb{R}, \ c \in \mathbb{R} \setminus U\big((-h,0]\big),$$

$$\partial_K F(0,c) < -\sigma + \int_{-h}^0 \big(U(x_2) - c\big)^2 dx_2, \quad \forall c \in \mathbb{R} \setminus U([-h,0]),$$

$$\partial_K F\big(0, U(-h)\big) = -\sigma + \int_{-h}^0 \big(U(x_2) - U(-h)\big)^2 dx_2.$$

Proof. For $K \geq 0$ and $c \in \mathbb{C}$ with $y_-(k, c, 0) \neq 0$ and $c_I \geq 0$, let

$$\mathcal{R} = \mathcal{R}(K, c) = -\partial_{x_2}^2 + K + \frac{U''(x_2)}{U(x_2) - c}, \;\; \tilde{y}(K, c, x_2) = \frac{y_-(\sqrt{K}, c, x_2)}{y_-(\sqrt{K}, c, 0)}, \qquad x_2 \in [-h, 0], \tag{4.1.6}$$

be the differential operator in the Rayleigh equation (3.0.1) and the normalization of the fundamental solution y_- defined in (3.3.1) and (3.5.1). Clearly

$$\tilde{y}(-h) = 0, \quad \tilde{y}'(-h) = y_-(\sqrt{K}, c, 0)^{-1}, \quad \tilde{y}(x_2) > 0, \; x_2 \in (-h, 0),$$
$$\tilde{y}(0) = 1, \quad Y(\sqrt{K}, c) = \tilde{y}'(0),$$

where the sign properties follows from Lemma 3.6.1(1). It is straight forward to compute, for $c \in \mathbb{R} \setminus U\big((-h, 0)\big)$ and $x_2 \in (-h, 0)$,

$$\mathcal{R}\partial_K \tilde{y} = -\tilde{y} < 0, \quad \mathcal{R}\partial_K^2 \tilde{y} = -2\partial_K \tilde{y},$$

where the smoothness of $\tilde{y}$ in K is ensured by Lemma 3.4.1. The following claim is used to analyze these and some other functions.

Claim. Suppose $y \in C^0([-h, 0])$ is a solution to $(\mathcal{R}y)(x_2) = f(x_2)$ and $y(-h) = y(0) = 0$ with $c \in \mathbb{R} \setminus U\big((-h, 0)\big)$, where f is C^0 on $[-h, 0]$, then we have the following through direct computations

$$\begin{aligned} &(\tilde{y}'y - \tilde{y}y')' = \tilde{y}f \Rightarrow y'(0) = -\int_{-h}^0 \tilde{y} f dx_2, \\ &\qquad\qquad y(x_2) = \tilde{y}(x_2) \int_{x_2}^0 \frac{1}{\tilde{y}(x_2')^2} \int_{-h}^{x_2'} \tilde{y}(x_2'') f(x_2'') dx_2'' dx_2'. \end{aligned} \tag{4.1.7}$$

Applying this claim to $\partial_K \tilde{y}$ and $\partial_{KK} \tilde{y}$ implies

$$\begin{aligned} &\partial_K Y = \partial_K \tilde{y}'(0) = \int_{-h}^0 \tilde{y}^2 dx_2 > 0, \\ &\partial_K^2 Y = -2 \int_{-h}^0 \tilde{y}(x_2)^2 \int_{x_2}^0 \tilde{y}(x_2')^{-2} \int_{-h}^{x_2'} \tilde{y}(x_2'')^2 dx_2'' dx_2' dx_2 < 0. \end{aligned} \tag{4.1.8}$$

The definition of F implies $\partial_K^2 F < 0$ for $c \in \mathbb{R} \setminus U\big((-h, 0]\big)$.

For $k = 0$, through direct calculation, one may verify, for $c \notin U([-h, 0])$,

$$y_-(0, c, x_2) = (U(x_2) - c) \int_{-h}^{x_2} \frac{U(-h) - c}{(U(x_2') - c)^2} dx_2'. \tag{4.1.9}$$

For $c \in \mathbb{R} \setminus U([-h, 0])$, from (4.1.8), we have

$$\begin{aligned}\partial_K Y(0, c) &= \int_{-h}^{0} \tilde{y}^2 dx_2 \\ &= \int_{-h}^{0} \frac{(U - c)^2}{(U(0) - c)^2} \Big(\int_{-h}^{x_2} \frac{dx_2'}{(U(x_2') - c)^2} \Big)^2 dx_2 \Big(\int_{-h}^{0} \frac{dx_2'}{(U(x_2') - c)^2} \Big)^{-2},\end{aligned}$$

and thus

$$\begin{aligned}\partial_K F(0, c) =&(U(0) - c)^2 \partial_K Y(0, c) - \sigma \\ =& \int_{-h}^{0} (U - c)^2 \Big(\int_{-h}^{x_2} \frac{dx_2'}{(U(x_2') - c)^2} \Big)^2 dx_2 \Big(\int_{-h}^{0} \frac{dx_2'}{(U(x_2') - c)^2} \Big)^{-2} - \sigma \\ <& \int_{-h}^{0} (U - c)^2 dx_2 - \sigma.\end{aligned} \tag{4.1.10}$$

For $k = 0$ and $c = U(-h)$, we can use (3.6.5) to compute

$$\tilde{y}\big(0, U(-h), x_2\big) = \big(U(x_2) - U(-h)\big) / \big(U(0) - U(-h)\big). \tag{4.1.11}$$

Consequently, one obtains explicitly

$$\partial_K Y\big(0, U(-h)\big) = \int_{-h}^{0} \frac{\big(U(x_2) - U(-h)\big)^2}{\big(U(0) - U(-h)\big)^2} dx_2,$$

which in turn yields the desired formula of $\partial_K F\big(0, U(-h)\big)$. $\square$

The information on the derivatives of F leads to the following properties of the roots of F.

Lemma 4.1.6. *Assume $U \in C^3$, the following hold.*

1. *If*

$$\sigma \geq \int_{-h}^{0} \big(U(x_2) - U(-h)\big)^2 dx_2 \Longleftrightarrow \partial_K F\big(0, U(-h)\big) \leq 0, \tag{4.1.12}$$

then $F\big(k, U(-h)\big) \leq -g = F(0, U(-h))$ for all $k \in \mathbb{R}$.

2. *Let*

$$\begin{aligned} g_{\#} = &\max \big\{ Y\big(k, U(-h)\big)\big(U(0) - U(-h)\big)^2 - U'(0)\big(U(0) - U(-h)\big) \\ &\quad - \sigma k^2 \mid k \in \mathbb{R} \big\} \\ = &\max \big\{ F\big(k, U(-h)\big) + g \mid k \in \mathbb{R} \big\}, \end{aligned}$$

then we have

(a) *$g_{\#} \geq F\big(0, U(-h)\big) + g = 0$ and "=" in the "$\leq$" holds if and only if (4.1.12) holds.*

(b) *If $g > g_{\#}$, then $F\big(k, U(-h)\big) < 0$ for all $k \in \mathbb{R}$.*

(c) *If $0 < g = g_{\#}$, then there exists a unique $k_{\#} > 0$ such that $F\big(\pm k_{\#}, U(-h)\big) = 0$ and $F\big(k, U(-h)\big) < 0$ for all $|k| \neq k_{\#}$.*

(d) *If $0 < g < g_{\#}$, then there exist $k_{\#}^{+} > k_{\#}^{-} > 0$ such that*

$$F\big(k, U(-h)\big) < 0, \quad |k| \notin (k_{\#}^{-}, k_{\#}^{+});$$

$$F\big(k, U(-h)\big) > 0, \quad |k| \in (k_{\#}^{-}, k_{\#}^{+}); \quad \mp \partial_k F(k_{\#}^{\pm}, U(-h)) > 0.$$

Proof. Statement (1) is a direct consequence of the concavity of $F\big(k, U(-h)\big)$ in $K = k^2$ and $F\big(0, U(-h)\big) = -g < 0$. Statement (2) is also an immediate implication of this concavity and Lemma Theorem 4.1.2(1). □

Along with statement (2b) and Corollary 4.1.3.1, (Equation 4.1.12) provides an explicit sufficient condition ensuring that the branch $c^-(k)$ does not reach $U([-h,0])$ and thus staying in $(-\infty, U(-h))$ for all $k \in \mathbb{R}$.

To end this subsection we prove the following monotonicity of the even functions $c^\pm(k)$ which will be used in obtaining the conjugacy between the irrotational linearized capillary gravity water waves and the component of the solutions linearized at the shear $U(x_2)$. From the definition of F and (Equation 3.6.5), we first compute, for $c \notin U([-h,0])$,

$$Y(0,c) = \frac{U'(0)\int_{-h}^0 (U-c)^{-2}dx_2 + (U(0)-c)^{-1}}{(U(0)-c)\int_{-h}^0 (U-c)^{-2}dx_2}$$

and thus

$$F(0,c) = (U(0)-c)^2 Y(0,c) - U'(0)(U(0)-c) - g = \frac{1}{\int_{-h}^0 (U-c)^{-2}dx_2} - g,$$

which is uniformly increasing on $(-\infty, U(-h))$ and uniformly decreasing on $(U(0), +\infty)$. Therefore $F(0,\cdot)$ has two real roots

$$c_0^+ \in (U(0), +\infty), \;\; c_0^- \in (-\infty, U(-h)), \;\; \text{s. t. } \; F(0, c_0^\pm) = \frac{1}{\int_{-h}^0 (U-c_0^\pm)^{-2}dx_2} - g = 0, \tag{4.1.13}$$

which are unique in the above intervals.

Lemma 4.1.7. *Assume $U \in C^3$, then the following hold.*

1. *For $\dagger \in \{+,-\}$, suppose $c^\dagger(k) \in \mathbb{R} \setminus U([-h,0])$ can be extended as simple roots of $F(k,\cdot)$ for all $k \geq k_* \geq 0$, then $(c^\dagger)'(k) = 0$ has most one solution on $(k_*, +\infty)$, where $(c^\dagger)''(k) \neq 0$ is also satisfied.*

2. *For $\dagger \in \{+,-\}$, suppose $c^\dagger(k) \in \mathbb{R} \setminus U([-h,0])$ can be extended as simple roots of $F(k,\cdot)$ for all $k \in \mathbb{R}$. If, in addition*

$$\sigma > g^2 \int_{-h}^0 (U-c_0^\dagger)^2 \Big(\int_{-h}^{x_2} \frac{dx_2'}{(U(x_2')-c_0^\dagger)^2} \Big)^2 dx_2, \tag{4.1.14}$$

with $c_0^\dagger$ defined in (4.1.13), then $(c^\dagger)'(k) \neq 0$ for all $k \neq 0$. If

$$\sigma < g^2 \int_{-h}^{0} (U - c_0^\dagger)^2 \Big(\int_{-h}^{x_2} \frac{dx_2'}{(U(x_2') - c_0^\dagger)^2} \Big)^2 dx_2, \tag{4.1.15}$$

then $c^\dagger(k)$ does have a unique critical point $k_0 > 0$.

Proof. We shall work with $c^-(k)$, while the same proof works for $c^+(k)$. Observe that the evenness of $c^-(k)$ yields $(c^-)'(0) = 0$. Suppose there exists $k_0 > k_* \geq 0$ such that $(c^-)'(k_0) = 0$, then

$$2k_0(\partial_K F)(k_0, c^-(k_0)) = \partial_k F(k_0, c^-(k_0)) = -\partial_c F(k_0, c^-(k_0))(c^-)'(k_0) = 0.$$

Computing the second order derivative at k_0, we have

$$(c^-)''(k_0) = -\frac{\partial_k^2 F(k_0, c^-(k_0))}{\partial_c F(k_0, c^-(k_0))} = -\frac{4k_0^2(\partial_K^2 F)(k_0, c^-(k_0)) + 2(\partial_K F)(k_0, c^-(k_0))}{\partial_c F(k_0, c^-(k_0))},$$

which along with Lemma 4.1.5 and $\partial_c F(k, c^-(k_0)) < 0$ (Corollary 4.1.3.1) implies that $(c^-)''(k_0) < 0$. Hence $k_0 > k_*$ has to be the only positive critical point of $c^-(k)$.

To prove Statement (2) where $k_* = 0$, on the one hand, we first observe that since c_0^- is the unique root of $F(0, \cdot)$ in $(-\infty, U(-h)$ and $c^-(0)$ is also such a root, so $c^-(0) = c_0^-$. Moreover, (4.1.10) implies that (4.1.14) and (4.1.15) are equivalent to $\mp\partial_K F(0, c^-(0)) > 0$, respectively. On the other hand, From the evenness of F and $c^-(k)$ in k, one may compute

$$(\partial_K F)(0, c^-(0)) = \partial_k^2 F(0, c^-(0))/2 = -\partial_c F(0, c^-(0))\big((c^-)''(0)\big)/2.$$

From Lemma 4.1.2(3), $(c^-)'(k) < 0$ for some $k \gg 1$. Hence, on the one hand, (4.1.15), $\partial_c F(k, c^-(0)) < 0$, and the above identity implies $(c^-)''(0) > 0$. Along with $(c^-)'(0) = 0$ due to the evenness of $c^-(k)$, it yields that c^- has a critical point $k_0 > 0$. On the other hand, through the same argument, (4.1.14) yields $(c^-)''(0) < 0$ while $(c^-)'(0) = 0$. Therefore it is impossible that there exists a unique critical point of c^- where $(c^-)'' < 0$. The proof of the lemma is complete. $\square$

4.2 Eigenvalue distribution of convex/concave shear flows

To analyze eigenvalues under less implicit assumptions than (4.1.5), particularly the generation of unstable modes from $c = U(-h)$, we further assume $U'' \neq 0$ on $[-h, 0]$. Due to Lemma 4.1.1(6), this rules out the possibility of roots of $\mathbf{F}$ on $U\big((-h, 0]\big)$ and provides better smoothness of F for the bifurcation analysis.

Lemma 4.2.1. *Assume $U \in C^{l_0}$, $l_0 \geq 6$, and $U'' \neq 0$ on $[-h, 0]$, then $F(k, c)$ is well defined for all $k \in \mathbb{R}$ and $c \in \mathbb{C}$ and*

a.) F is analytic in both $k \in \mathbb{R}$ and $c \notin U([-h, 0])$ and, when restricted to $c_I \geq 0$, is C^{l_0-3} in both $k \in \mathbb{R}$ and $c \notin \{U(-h), U(0)\}$,

b.) F and $\partial_k F$ are locally $C^{1,\alpha}$ in both k and $c \neq U(0)$ with $c_I \geq 0$ for any $\alpha \in [0, 1)$,

c.) F is C^1 in k and c with $c_I \geq 0$.

Remark 4.2.1. *Note that, in the above statement, for fixed $c \in U([-h, 0))$, F is C^{l_0-3} in k. This stronger regularity in k follows from that of (y_{0-}, y_{0-}') and Y (see Lemma 3.5.1 and Lemma 3.6.6). Moreover, one could prove that F and $\partial_k F$ are also $C^{1,\alpha}$ near $c = U(0)$ with $c_I \geq 0$ by estimating $\partial_{c_R}^2 Y_I(k, c) = O\big(|c - U(0)|^{-1}\big)$ using Lemma 3.5.3–Lemma 3.5.5 and Lemma 3.6.6 as well as Corollary 3.6.6.1.*

Proof. The assumption $U'' \neq 0$ implies that $y_-(k, c, 0) \neq 0$ for all k and c (Lemma 3.6.1(5)) and thus F is well defined. The analyticity and the C^{l_0-3} and $C^{1,\alpha}$ (restricted to $c_I \geq 0$ for the latter two) regularity of F follow directly from those of Y given in Lemma 3.6.6 except at $c = U(0)$. Near $c \in U(0)$, the regularity and estimates on Y (Lemma 3.6.2, Lemma 3.6.5, Lemma 3.6.6) and $\partial_c Y$ (Lemma 3.6.5 and Corollary 3.6.6.1) yield the regularity of F. $\square$

As a corollary of the Lemma 4.1.3, Lemma 4.1.6 and Lemma 4.2.1 and the semicircle theorem, we obtain a sufficient condition for (4.1.5) to hold for $\mathbf{K} = \mathbb{R}$.

Corollary 4.2.1.1. *Suppose $U'' \neq 0$ on $[-h, 0]$ and (4.1.12) hold, then (4.1.5) is true for all $k \in \mathbb{R}$.*

Assuming $U'' \neq 0$, in general $c = U(-h)$ is the only point outside the domain of analyticity of $F(k, \cdot)$ which might happen to be a root and also might be the end point of branches of roots of $F(k, \cdot)$, it is a crucial step to analyze zeros of F around $U(-h)$.

Lemma 4.2.2. *Assume $U \in C^6$, then (a) $\partial_c F(k, U(-h)) < 0$ for all $k \in \mathbb{R}$ if $U'' > 0$ on $[-h, 0]$; and (b) if $U'' < 0$ on $[-h, 0]$, then $\partial_c F(k, U(-h)) < 0$ if $F(k, U(-h)) = 0$.*

Proof. We shall use the notations $\mathcal{R}$ and $\tilde{y}$ defined in the proof of Lemma 4.1.5 and F and Y are also viewed as function of c and $K = k^2 \geq 0$. It is straight forward to compute, for $c < U(-h)$ and $x_2 \in (-h, 0)$,

$$\mathcal{R}\partial_c \tilde{y} = -\tfrac{U''}{(U-c)^2}\tilde{y}, \quad \mathcal{R}\partial_{Kc}\tilde{y} = -\partial_c \tilde{y} - \tfrac{U''}{(U-c)^2}\partial_K \tilde{y}.$$

Applying (4.1.7) we obtain that for $c < U(-h)$,

$$U''(0)\partial_c Y = U''(0)\partial_c \tilde{y}'(0) = U''(0)\int_{-h}^{0} \frac{U''\tilde{y}^2}{\big(U-c\big)^2}dx_2 > 0, \;\; U''(0)\partial_{Kc} Y < 0. \tag{4.2.1}$$

These integral representation of $\partial_c Y$ still holds as $c \to U(-h)-$, and thus also its sign. For $k = 0$ and $c = U(-h)$, we can use (4.1.11) to compute

$$\partial_c Y\big(0, U(-h)\big) = \frac{U'(0) - U'(-h)}{\big(U(0) - U(-h)\big)^2} \implies \partial_c F\big(0, U(-h)\big) = -U'(-h) < 0.$$

Finally we obtain the sign of $\partial_c F(k, U(-h))$ in two cases separately, based on the sign of U''. Suppose $U'' > 0$. The above (4.1.8) and (4.2.1) implies that, for $c \leq U(-h)$,

$Y(\sqrt{K}, c)$ is strictly increasing in K and $\partial_c Y(\sqrt{K}, c)$ is strictly deceasing in K, and thus

$$\partial_c F = (U(0) - c)^2 \partial_c Y - 2(U(0) - c)Y + U'(0)$$

is also strictly decreasing in K. Letting $c \to U(-h)-$, this monotonicity yields

$$\partial_c F(k, U(-h)) \leq \partial_c F(0, U(-h)) = -U'(-h) < 0.$$

In the other case of $U'' < 0$, suppose $F(k, U(-h)) = 0$ for some $k \in \mathbb{R}$, which implies

$$Y(k, U(-h)) = \frac{g + \sigma k^2}{(U(0) - U(-h))^2} + \frac{U'(0)}{U(0) - U(-h)}.$$

Therefore

$$\begin{aligned}
\partial_c F(k, U(-h)) =& (U(0) - U(-h))^2 \partial_c Y(k, U(-h)) - 2(U(0) - U(-h))Y(k, U(-h)) \\
&+ U'(0) \\
=& (U(0) - U(-h))^2 \partial_c Y(k, U(-h)) - U'(0) - \frac{2(g + \sigma k^2)}{(U(0) - U(-h))}.
\end{aligned}$$

We also have $\partial_c Y(k, U(-h)) < 0$ from taking the limit of (4.2.1). Hence we obtain $\partial_c F(k, U(-h)) < 0$ and the proof of the lemma is complete. $\square$

In the next step we shall study the roots of $F(k, \cdot)$ near $c = U(-h)$.

Lemma 4.2.3. *Assume $U \in C^6$, and $U'' \neq 0$ on $[-h, 0]$. Suppose $F\big(k_0, U(-h)\big) = 0$, then there exist $\epsilon > 0$, $\rho \in \big(0, U(0) - U(-h)\big)$, and $\mathcal{C} \in C^{1,\alpha}\big([k_0 - \epsilon, k_0 + \epsilon], \mathbb{C}\big)$ for any $\alpha \in [0, 1)$ such that $\mathcal{C}(k_0) = U(-h)$, $\mathcal{C}_I(k) \geq 0$, $\partial_c F(k, \mathcal{C}(k)) \neq 0$ at $k \neq k_0$, and*

$$\begin{aligned}
F(k, c) = 0 \textit{ with } k \in [k_0 - \epsilon, k_0 + \epsilon],\ |c_R - U(-h)| \leq \rho,\ c_I \in [0, \rho], \\
\textit{iff} \quad c = \mathcal{C}(k) = \mathcal{C}_R(k) + i\mathcal{C}_I(k).
\end{aligned}$$

Moreover, without loss of generality assume $k_0 > 0$ (Lemma 4.2.1 implies $k_0 \neq 0$) and this branch of roots of F satisfies

1. *If $\partial_k F\big(k_0, U(-h)\big) = 0$, then $\mathcal{C}'(k_0) = 0$, $\mathcal{C}_I \equiv 0$ and $\mathcal{C}(k) < U(-h)$ for all $0 < |k - k_0| \leq \epsilon$.*

2. *If $\pm\partial_k F\big(k_0, U(-h)\big) > 0$, then $\pm\mathcal{C}_R'(k_0) > 0$ and*

$$\mathcal{C}_R(k) < U(-h), \;\; \mathcal{C}_I(k) = 0, \quad \forall\, 0 < \pm(k_0 - k) \leq \epsilon,$$

and for some $\tilde{C} > 0$ determined by k_0 and U, $\forall\, 0 < \pm(k - k_0) \leq \epsilon$,

$$\mathcal{C}_R(k) > U(-h), \;\; \left| \frac{\mathcal{C}_I(k)}{Y_I\big(k, \mathcal{C}_R(k)\big)} + \frac{\big((U(0) - U(-h)\big)^2}{\partial_c F\big(k_0, U(-h)\big)} \right| \leq \tilde{C}|k - k_0|^\alpha,$$

which implies

$$0 < |\mathcal{C}_I(k)| \leq \tilde{C}(k - k_0)^2, \;\; U''(0)\mathcal{C}_I(k) > 0, \quad \forall\, 0 < \pm(k - k_0) \leq \epsilon.$$

In the generic case $\partial_k F\big(k_0, U(-h)\big) \neq 0$, locally the roots of $F(k,c)$ consists of the intersection of the graph of $\mathcal{C}(k)$ and the closure of the upper half complex plane, along with its complex conjugate. In this case, however, one observes that $d\mathcal{C}_I(k)/d\mathcal{C}_R(k) = 0$ at $k = k_0$. The following proof is based on both the Implicit Function Theorem and the Intermediate Value Theorem.

Proof. According to Lemma 4.2.1, F is $C^{1,\alpha}$ in k and c in the region $c_I \geq 0$. As F_I is not continuous at $c \in U\big((-h, 0]\big) \subset \mathbb{C}$ in general, let $\tilde{F}(k,c) = \tilde{F}_R + i\tilde{F}_I \in \mathbb{C}$ be a $C^{1,\alpha}$ extension of F into a neighborhood of $\big(k_0, U(-h)\big) \in \mathbb{R} \times \mathbb{C}$ which coincides with F for

$c_I \geq 0$. From Lemma 4.2.2, the 2×2 Jacobian matrix of $D_c\tilde{F}$ satisfies

$$D_c\tilde{F}\big(k_0, U(-h)\big) = \begin{pmatrix} \partial_{c_R}\tilde{F}_R & \partial_{c_I}\tilde{F}_R \\ \partial_{c_R}\tilde{F}_I & \partial_{c_I}\tilde{F}_I \end{pmatrix} \Big|_{\big(k_0, U(-h)\big)} = \partial_c F\big(k_0, U(-h)\big) I_{2\times 2},$$

$$\partial_c F\big(k_0, U(-h)\big) < 0,$$

where we used the Cauchy-Riemann equation and the fact $F(k,c) \in \mathbb{R}$ for all $c < U(-h)$. Therefore the Implicit Function Theorem implies that all roots of $\tilde{F}(k,c)$ near $\big(k_0, U(-h)\big)$ form the graph of a $C^{1,\alpha}$ complex-valued function $\mathcal{C}(k)$ which contains $\big(k_0, U(-h)\big)$. To complete the proof of the lemma, we only need to prove that $\mathcal{C}(k)$ satisfies properties (1) and (2).

Firstly we prove $\mathcal{C}(k) \in \mathbb{R}$ if $\mathcal{C}_R(k) \leq U(-h)$ and thus $F\big(k, \mathcal{C}(k)\big) = \tilde{F}\big(k, \mathcal{C}(k)\big) = 0$ as well. As $F_R \in C^1$ and $\partial_{c_R} F_R\big(k_0, U(-h)\big) = \partial_c F(k_0, U(-h)) < 0$, the Implicit Function Theorem yields a C^1 real-valued function $\tilde{\mathcal{C}}(k)$ for k near k_0 such that

$$\tilde{\mathcal{C}}(k_0) = U(-h), \quad F_R\big(k, \tilde{\mathcal{C}}(k)\big) = 0. \tag{4.2.2}$$

Since $F_I(k,c) = 0$ if $c \leq U(-h)$, the uniqueness of solutions ensured by the Implicit Function Theorem implies that $\mathcal{C}(k) = \tilde{\mathcal{C}}(k) \in \mathbb{R}$ if $\tilde{\mathcal{C}}(k) \leq U(-h)$.

Next we consider the case $\partial_k F\big(k_0, U(-h)\big) = 0$. Along with

$$\partial_{c_R} F_R\big(k_0, U(-h)\big),\ \partial_{KK} F_R\big(k_0, U(-h)\big) < 0, \ \text{ where } K = k^2,$$

it implies

$$F_R(k_0, c) > 0,\ \forall 0 < U(-h) - c \ll 1, \quad F_R\big(k, U(-h)\big) < 0,\ \forall k \in \mathbb{R}^+ \setminus \{k_0\}.$$

From the Intermediate Value Theorem, for k near k_0, there exist real roots of $F_R(k, \cdot)$ slightly smaller than $U(-h)$, which must belong to $\tilde{\mathcal{C}}(k)$ due to the uniqueness of solutions ensured by the Implicit Function Theorem. Therefore along with the last step, we conclude $\mathcal{C}(k) = \tilde{\mathcal{C}}(k) < U(-h)$ for $k \neq k_0$ close to k_0.

Finally, we consider the case of $\partial_k F\big(k_0, U(-h)\big) > 0$, while the opposite case can be handled similarly. The fact $\partial_c F\big(k_0, U(-h)\big) < 0$ yields

$$\partial_k \mathcal{C}(k_0) = \partial_k \tilde{\mathcal{C}}(k_0) = -\frac{\partial_k F\big(k_0, U(-h)\big)}{\partial_c F\big(k_0, U(-h)\big)} > 0,$$

where in the calculation of $\tilde{\mathcal{C}}(k_0)$ we also used $F_I(k,c) = 0$ for $c \leq U(-h)$ and the smoothness of F. Hence we obtain $\mathcal{C}(k) = \tilde{\mathcal{C}}(k) < U(-h)$ for k slightly smaller k_0. In the following we shall focus on $k > k_0$ where $\mathcal{C}_R(k) > U(-h)$. From the Mean Value Theorem, there exists θ between 0 and $\mathcal{C}_I(k)$ such that

$$0 = \tilde{F}_I\big(k, \mathcal{C}(k)\big) = F_I\big(k, \mathcal{C}_R(k)\big) + \mathcal{C}_I(k)\partial_{c_I}\tilde{F}_I\big(k, \mathcal{C}_R(k) + i\theta\big),$$

which along with the $C^{1,\alpha}$ regularity of F and $\mathcal{C}(k)$ imply

$$\begin{aligned}
\mathcal{C}_I(k) &= -\frac{F_I\big(k, \mathcal{C}_R(k)\big)}{\partial_{c_I}\tilde{F}_I\big(k, \mathcal{C}_R(k) + i\theta\big)} = -\frac{Y_I\big(k, \mathcal{C}_R(k)\big)\big(U(0) - \mathcal{C}_R(k)\big)^2}{\partial_{c_I} F_I\big(k, \mathcal{C}_R(k)\big) + O\big(|\mathcal{C}_I(k)|^\alpha\big)} \\
&= -\frac{Y_I\big(k, \mathcal{C}_R(k)\big)\big(U(0) - \mathcal{C}_R(k)\big)^2}{\partial_{c_R} F_R\big(k, \mathcal{C}_R(k)\big) + O\big(|\mathcal{C}_I(k)|^\alpha\big)} \\
&= -\frac{Y_I\big(k, \mathcal{C}_R(k)\big)\big(U(0) - U(-h) + O(|k-k_0|)\big)^2}{\partial_c F\big(k_0, U(-h)\big) + O\big(|k-k_0|^\alpha\big)}.
\end{aligned}$$

The proof of the lemma is complete. □

While the branch $c^+(k) \in (U(0), +\infty)$ of neutral modes is global in $k \in \mathbb{R}$ and contained in $\big(U(0), \infty\big)$ as addressed in Corollary 4.1.3.1, in the following we completes the picture of the other branch $c^-(k)$ by combining Lemma 4.1.3 – Lemma 4.2.3 and finish the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1(3).

Let $g_\# \geq 0$, $k_\#$, and/or $k_\#^\pm$ be the thresholds given in Lemma 4.1.6.

Case 1. $g > g_\#$. The desired result follows from Lemma 4.1.6 and Corollary 4.1.3.1 immediately.

We start the rest of the proof much as in that of Corollary 4.1.3.1 and Proposition 4.1.4. Namely, let k_0 be given by Lemma 4.1.2(3) and we only need to focus on $c^-(k)$ for $|k| \leq k_0$. From Lemma 4.1.2(2), there exists $R > 0$ such that $F(k, c) \neq 0$ for all $k \in [-k_0 - 1, k_0 + 1]$ and $|c| \geq R$, which also implies $c^+(k) \in \big(U(0), R\big)$ for all $|k| \leq k_0 + 1$ and $c^-(k) \in \big(-R, U(-h)\big)$ for all $|k| \in [k_0, k_0 + 1]$.

Case 2. $g = g_\#$. One the one hand, for any $k_1 \in (k_\#, k_0]$, Lemma 4.1.2, Lemma 4.2.1, and Lemma 4.1.6 imply that there exists $r_0 > 0$ such that

$$F(k, c) \neq 0, \ \forall k \in [k_1, k_0], \ c \in \partial\Omega_1 \cup \mathcal{D}_{r_0}, \quad \text{where } \Omega_1 = \{c \in \mathbb{C} \mid |c| < R, \ c \notin \overline{\mathcal{D}_{r_0}}\},$$

where the r-neighborhood $\mathcal{D}_r$ of $U([-h, 0])$ (see also (3.6.6)). Hence for all $k \in [k_1, k_0]$, we have $\mathrm{Ind}\big(F(k, \cdot), \Omega_1\big) = \big(F(k_0, \cdot), \Omega_1\big) = 2$, which is equal to the number of roots of $F(k, \cdot)$ in Ω_1. According to Corollary 4.1.3.1, $c^+(k) \in \big(U(0), R\big)$, $\forall |k| \leq k_0 + 1$, is one of them. Therefore neither cases in Lemma 4.1.3(3) can happen to the branch $c^-(k)$ and the simple root $c^-(k) \in \big(-\infty, U(-h)\big)$ can be extended analytically for all $k \in [k_1, k_0]$. Therefore $c^-(k)$ can be extended to at least $(k_\#, \infty)$ which along with $c^+(k)$ are the only roots of $F(k, \cdot)$ for $k \in (k_\#, \infty)$. On the other hand, according to Lemma 4.2.3, there exists a $C^{1,\alpha}$ branch $\mathcal{C}(k)$ of the only roots of $F(k, c)$ for $|k - k_\#|, |c - U(-h)| \ll 1$. Moreover $\mathcal{C}(k) < U(-h)$ for $0 < |k - k_\#| \ll 1$. Therefore $c^-(k) = \mathcal{C}(k)$ for $0 < k - k_\# \ll 1$ as $c^\pm(k)$ are the only roots of $F(k, \cdot)$ for $k > k_\#$. In particular, $c^-(k)$ is thus extended to $|k_\# - k| \ll 1$ as a $C^{1,\alpha}$ function with $c^-(k) \in (-R, U(-h))$ for $0 < |k_\# - k| \ll 1$.

Moreover, on the one hand, $c^-(k)$ is the only root of $F(k, \cdot)$ near $U([-h, 0])$ for k near $k_\#$ and it satisfies $c^-(k_\#) = U(-h)$. On the other hand, the continuity of $c^-(k)$ implies that there exists $\epsilon_1, r_1 > 0$ such that $F(k, c) \neq 0$ for any $|k - k_\#| \leq \epsilon_1$ and

$dist\big(c, U([-h,0])\big) = r_1$. It implies

$$\mathrm{Ind}\big(F(k,\cdot), \Omega_2\big) = \mathrm{Ind}\big(F(k_\# + \epsilon_1, \cdot), \Omega_2\big) = 1, \ \forall |k - k_\#| \le \epsilon_1,$$

due to the root $c^+(k)$, where

$$\Omega_2 = \{c \in \mathbb{C} \mid |c| < R, \, dist\big(c, U([-h,0])\big) > r_1\}.$$

Therefore, $c^\pm(k)$ are the only root of $F(k,\cdot)$ for k near $k_\#$, which are also simple. As $c^-(k) \in (-\infty, U(-h))$ is away from $c^+(k)$, Lemma 4.1.3 implies that the branch $c^-(k)$ of simple roots can be extended at least to $(-k_\#, +\infty)$ and remains in $\big(-\infty, U(-h)\big)$. As F is even in k, we have $c^\pm(k)$ are also the only roots of $F(-k,\cdot)$ for $k \in (-\infty, k_\#)$. Therefore the extension $c^-(k)$ must be even on $(-k_\#, k_\#)$ and we obtain the whole branch $c^-(k)$ for $k \in \mathbb{R}$.

Case 3a. $g < g_\#$ and $U'' > 0$. Following the same arguments as in case 2, we obtain that $c^-(k) = c_R^-(k) + ic_I^-(k)$ can be extended to a $C^{1,\alpha}$ function on $(k_1, +\infty)$ for some $k_1 < k_\#^+$, such that $c^\pm(k)$ and $\overline{c^-(k)}$ are the only roots of $F(k,\cdot)$ for all $k \in (k_1, +\infty)$ and $c_I^-(k) > 0$ for $k \in (k_1, k_\#^+)$. Let $(k_1, k_\#^+)$ also denote the maximal interval of the analytic extension of $c^-(k)$ as a simple root of $F(k,\cdot)$ inside $\mathbb{C} \setminus U([-h,0])$. The same above index based argument (in case 1) applied to $[k, k_\#^+ - \epsilon]$ for any $k \in (\max\{k_1, k_\#^-\}, k_\#^+)$ and $0 \ll \epsilon < k_\#^+ - k$ also implies that $c^\pm(k)$ and $\overline{c^-(k)}$ are the only roots of $F(k,\cdot)$ for all $k \in (\max\{k_1, k_\#^-\}, k_\#^+)$. According to Lemma 4.1.2 we have $k_1 \ge -k_0 > -\infty$. For $k \in (k_1, k_\#^+)$, the semicircle theorem implies that $c^-(k)$ lies in the closed upper semi-disk with the boundary diameter $U([-h,0])$ and thus $|c^-(k) - c^+(k)| > \rho_0$ where $\rho_0 > 0$ is given in Corollary 4.1.3.1. Moreover, since $F(k,c) \ne 0$ for any $c \in U\big((-h,0]\big)$ (Lemma 4.1.1 and Lemma 4.2.1), we obtain from Lemma 4.1.3

$$\lim_{k \to k_1+} c^-(k) = U(-h) \Longrightarrow F\big(k_2, U(-h)\big) = 0 \Longrightarrow k_2 \in \{k_\#^-, \, -k_\#^-, \, -k_\#^+\}.$$

It must hold $k_2 = k_{\#}^-$, otherwise we must have $c^-(k_{\#}^-) \neq U(-h)$, $F\big(k_{\#}^-, U(-h)\big) = 0$, $\partial_k F\big(k_{\#}^-, U(-h)\big) > 0$, and Lemma 4.2.3 imply that there exists the fourth root near $U(-h)$ for $0 < k - k_{\#}^- \ll 1$. This contradicts that $F(k, \cdot)$ has exactly three roots for all $k \in (\max\{k_1, k_{\#}^-\}, k_{\#}^+)$ and thus $k_2 = k_{\#}^-$ and $c^-(k_{\#}^-) = U(-h)$. For $0 < k_{\#}^- - k \ll 1$, Lemma 4.2.3 yields the further extension of $c^-(k)$ back into $\big(-\infty, U(-h)\big)$. From a similar argument, we can extend this branch to $k = -k_{\#}^-$ with $c^-(-k_{\#}^-) = U(-h)$. Finally, the whole branch $c^-(k)$ for $k \in \mathbb{R}$ is obtained by the evenness $c^-(-k) = c^-(-k)$.

Case 3b. $g < g_{\#}$ *and* $U'' < 0$. Following the same arguments as in case 2, we obtain that $c^-(k) = c_R^-(k) + ic_I^-(k)$ can be extended to a $C^{1,\alpha}$ function on $[k_{\#}^+, +\infty)$ and $c^-(k_{\#}^+) = U(-h)$. However, for $0 < k_{\#}^+ - k \ll 1$, Lemma 4.2.3 implies that there does not exist any roots of $F(k, \cdot)$ near $U(-h)$ (as $\mathcal{C}_I < 0$ due to $U'' < 0$). The same index argument further yields that $c^+(k)$ is the only root for $k \in (k_{\#}^-, k_{\#}^+)$. From Lemma 4.2.3, we obtain another branch of roots in $(-\infty, U(-h))$ of $F(k, \cdot)$ for $k \in (-k_{\#}^-, k_{\#}^-)$ which along with the $c^+(k)$ are the only roots. The final conclusion again follows from the even symmetry as in the above cases. □

Remark 4.2.2. *As in [68] for the gravity wave, the spectral stability in the case* $U'' < 0$ *can also be obtained by directly modifying the usual proof of the Rayleigh theorem in the fixed boundary case. Namely, multiplying (3.0.1) by* $\bar{y}$*, integrating on* $[-h, 0]$*, using the homogeneous boundary condition as in (2.2.6b) and (2.2.6c), and the semicircle theorem, a contradiction occurs if an unstable mode* c *exists. Our above proof provides a complete picture of the eigenvalue distribution, however.*

4.3 Singular neutral modes at inflection values

To end this chapter, we discuss the spectrum near inflection values of U, which are the only possible singular neutral modes other than $U(-h)$ according to Lemma 4.1.1(6).

Proposition 4.3.1. *Assume* $U \in C^6$, $x_{20} \in [-h, 0)$*, and* $U''(x_{20}) = 0$*, then the following hold for* $c_0 = U(x_{20})$.

1. *For any $\alpha \in (0, \frac{1}{2})$, there exist $C > 0$ depending only on U, g, and α, such that, with*

$$k_* = C \max\{1, (U(0) - c_0)^{-2}\}, \quad \sigma_0 = (U(0) - c_0)^2/(2k_*),$$

   for any $\sigma \in (0, \sigma_0)$, there exists a unique $k_0 \geq k_*$ such that $F(k_0, c_0) = 0$. Moreover it satisfies

$$|k_0 - (U(0) - c_0)^2/\sigma| \leq C(U(0) - c_0)^{-2},$$
$$|\partial_k F(k_0, c_0) + (U(0) - c_0)^2| \leq C(U(0) - c_0)^{-2\alpha}\sigma^{\alpha}.$$

2. *In addition, suppose $x_{20} \neq -h$ and*

$$\mathbf{F}(k_0, c_0) = 0, \; k_0 > 0, \; \partial_k F(k_0, c_0) \neq 0, \; U'''(x_{20}) \neq 0,$$

   then there exist $\tilde{C} > 0$, $\delta > 0$, and a C^1 function $c(k)$ defined for

$$0 \leq |k - k_0| \leq \delta, \quad (k - k_0)U'''(x_{20})\partial_k F(k_0, c_0) > 0,$$

   such that $c(k_0) = c_0$, $c_I(k) > 0$ for the above $k \neq k_0$, and

$$F(k, c) = 0, \; |k - k_0| \leq \delta \textit{ and } |c - c_0| \leq \tilde{C}\delta \; \textit{ iff } \; c \in \{c(k), \overline{c(k)}\}.$$

In the above statement (2), note that $\mathbf{F}(k_0, c_0) = 0$ and Lemma 4.1.1(4) imply that $y_-(k_0, c_0, 0) \neq 0$ and thus $Y(k, c_0)$ is well-defined which is actually real due to $U''(x_{20}) = 0$ and Lemma 3.6.4. Therefore it makes sense to talk about the sign of $\partial_k F(k_0, c_0)$. Statement (1) also implies that assumptions of statement (2) may be satisfied at inflection values of U with $|k| \gg 1$ if σ is small.

Proof. From Lemma 3.3.2 and Remark 3.3.1, there exists $C_0 > 0$ such that

$$ky_-(k, c, x_2) \geq (1/2) \sinh \mu^{-1}(x_2 + h) \implies y_-(k, c, 0) \neq 0, \quad \forall |k| \geq C_0,\ c \in \mathbb{C},$$

and thus $F(k, c)$ and $Y(k, c)$ are defined for all $|k| \geq C_0$. According to (4.1.2), $F_I(k, c_0) = 0$ for all $k \in \mathbb{R}$ and thus $F(k, c_0) \in \mathbb{R}$. Lemma 3.6.6 and Lemma 3.6.4 imply, for $|k| \geq C_0$ and $c \in U([-h, 0))$,

$$|Y_I(k, c)| \leq C|U''(x_2^c)|e^{2\mu^{-1}x_2^c} \implies \big|Y(k, c_0) - |k|\big| \leq C\mu.$$

Therefore, for $|k| \geq C_0$, it holds

$$\big||k|^{-1}F(k, c_0) - (U(0) - c_0)^2 + \sigma|k|\big| \leq C\mu.$$

Let

$$k_* = \max\{C_0, 3C(U(0) - c_0)^{-2}\} \implies C\langle k_*\rangle^{-1} \leq (U(0) - c_0)^2/3.$$

From the Intermediate Value Theorem, for every $0 < \sigma \leq \sigma_0$, there exists a root $k_0 \in [k_*, +\infty)$ of $F(\cdot, c_0)$ close to $(U(0) - c_0)^2/\sigma$.

To estimate $\partial_k F(k_0, c_0)$ and obtain the uniqueness of k_0, we analyze $\partial_k Y(k_0, c_0)$ using the same standard method used in the proof of Lemma 4.1.5. Let

$$\begin{aligned} y(k, x_2) = {} & \frac{y_{0-}(k, c_0, x_2)}{y_{0-}(k, c_0, 0)} \\ \implies & -y'' + \Big(k^2 + \frac{U''}{U - c_0}\Big)y = 0,\ \ y(-h) = 0,\ y(0) = 1,\ Y(k, c_0) = y'(0), \end{aligned}$$

where $\frac{U''}{U-c_0} \in C^3([-h, 0])$. Differentiating the above equation with respect to k yields

$$
\begin{aligned}
&-\partial_k y'' + \Big(k^2 + \frac{U''}{U-c_0}\Big)\partial_k y = -2ky, \quad \partial_k y(-h) = \partial_k y(0) = 0,\\
&\partial_k Y(k, c_0) = \partial_k y'(0),\\
\Longrightarrow\, &\partial_k Y(k, c_0) = \int_{-h}^{0} (\partial_k y' y - \partial_k y y')' dx_2 = 2k \int_{-h}^{0} y(x_2)^2 dx_2.
\end{aligned}
$$

From Lemma 3.3.2, we can estimate, for any $\alpha \in (0, \frac{1}{2})$ and $|k| > k_*$,

$$
\begin{aligned}
&\Big|\partial_k Y(k, c_0) - 2k \int_{-h}^{0} \Big(\frac{\sinh \mu^{-1}(x_2 + h)}{\sinh \mu^{-1} h}\Big)^2 dx_2\Big|\\
\leq & C\mu^{\alpha-1} \int_{-h}^{0} \Big(\frac{\sinh \mu^{-1}(x_2 + h)}{\sinh \mu^{-1} h}\Big)^2 dx_2\\
\Longrightarrow\, & |\partial_k Y(k, c_0) - sgn(k)| \leq C\mu^{\alpha}.
\end{aligned}
$$

Therefore we obtain

$$
\partial_k F(k, c_0) = (U(0) - c_0)^2 \partial_k Y(k, c_0) - 2\sigma k = (U(0) - c_0)^2 sgn(k) - 2\sigma k + O(|k|^{-\alpha}),
$$

which implies

$$
\partial_k F(k_0, c_0) = -(U(0) - c_0)^2 + O(k_0^{-\alpha}) \ \text{ if } \ k_0 \in (k_*, \infty) \text{ and } F(k_0, c_0) = 0.
$$

The desired estimate on $\partial_k F(k_0, c_0)$ follows immediately, whose always negative sign also implies the uniqueness of such $k_0 \in (k_*, \infty)$.

Under the assumption in statement (2) of the proposition, Lemma 4.1.1(4) implies $y_-(k_0, c_0, 0) \neq 0$ and thus $F(k, c)$ is C^1 in (k, c) near (k_0, c_0) with $c_I \geq 0$. Much as in the proof of Lemma 4.2.3, statement (2) can be proved by applying the Implicit Function Theorem to $\tilde{F}(k, c)$, an extension of $F(k, c)$ which is C^1 in (k, c) in $\mathbb{R} \times \mathbb{C}$ near (k_0, c_0).

The Jacobi matrix of $\tilde{F}$ is

$$D_c\tilde{F}(k_0, c_0) = \begin{pmatrix} \partial_{c_R}\tilde{F}_R & \partial_{c_I}\tilde{F}_R \\ \partial_{c_R}\tilde{F}_I & \partial_{c_I}\tilde{F}_I \end{pmatrix}\Big|_{(k_0,c_0)} = \begin{pmatrix} \partial_{c_R}F_R & -\partial_{c_R}F_I \\ \partial_{c_R}F_I & \partial_{c_R}F_R \end{pmatrix}\Big|_{(k_0,c_0)},$$

where we also used the Cauchy-Riemann equation. According to Lemma 3.6.4, $Y(k, c_0) \in \mathbb{R}$ and

$$\partial_{c_R}F_I(k_0, c_0) = (U(0) - c_0)^2\partial_{c_R}Y_I(k_0, c_0) = (U(0) - c_0)^2\frac{\pi U'''(x_{20})y_{0-}(k_0, c_0, x_{20})^2}{U'(x_{20})^2 y_{0-}(k_0, c_0, 0)^2} \neq 0,$$

and has the same sign as $U'''(x_{20})$. Therefore $D_c\tilde{F}(k_0, c_0)$ is invertible and thus there exist $\delta > 0$ and a C^1 function $c(k) = c_R(k) + ic_I(k)$ defined for all $|k - k_0| \leq \delta$ such that $\tilde{F}(k, c) = 0$ for $(k, c) \in \mathbb{R} \times \mathbb{C}$ iff $c = c(k)$. Consequently $F(k, c) = 0$ for (k, c) near (k_0, c_0) iff $c \in \{c(k), \overline{c(k)}\}$ and $c_I(k) \geq 0$. Identifying complex numbers with 2-d column vectors, since

$$\partial_k c(k_0) = -(D_c\tilde{F}(k_0, c_0))^{-1}\partial_k\tilde{F}(k_0, c_0) = -\partial_k F(k_0, c_0)/\partial_c F(k_0, c_0)$$

implies $c_I(k)(k - k_0)\partial_k F(k_0, c_0)U'''(x_{20}) > 0$ for k near k_0, statement (2) follows readily.

□

Remark 4.3.1. *In part (1) of the proposition, one may also seek k_0 satisfying $\mathbf{F}(k_0, c_0) = 0$ using the Intermediate Value Theorem instead. It is easy to see $\mathbf{F}(k, c_0) \in \mathbb{R}$ approaches $-\infty$ as $k \to \infty$. Therefore such k_0 exists if $\sup_{k\geq 0}\mathbf{F}(k, c_0) > 0$ and only if $\sup_{k\geq 0}\mathbf{F}(k, c_0) \geq 0$, which may not the case if g and σ are sufficiently large. This is different from the gravity waves (i.e. $\sigma = 0$), see [68, 26, 27]. It is also worth pointing out that the smoothness of F for $c_I \geq 0$ based on chapter 3 made the analysis using the Implicit Function Theorem in part (2) easier, compared with, e.g. [26].*

CHAPTER 5

BOUNDARY VALUE PROBLEMS OF THE NON-HOMOGENEOUS RAYLEIGH EQUATION

In this chapter, using the fundamental solutions $y_\pm(k,c,x_2)$ to the homogeneous Rayleigh equation (3.0.1), we study the boundary value problem of the non-homogeneous Rayleigh equation

$$-y'' + \big(k^2 + \frac{U''}{U-c}\big)y = \frac{\psi(c,x_2)}{U-c}, \quad x_2 \in (-h,0); \tag{5.0.1a}$$

$$y(-h) = \zeta_-(c), \quad \big(U(0)-c\big)^2 y'(0) - \big(U'(0)(U(0)-c)+g+\sigma k^2\big)y(0) = \zeta_+(c), \tag{5.0.1b}$$

where the boundary conditions are from the linearized water wave system (2.2.6).

Using the two fundamental solutions $y_\pm$ to the homogeneous equation with zero boundary values, for $c \in \mathbb{C} \setminus U([-h,0])$ it is standard to compute the solution to (5.0.1) in the form

$$y_B(k,c,x_2) = \frac{\zeta_+(c)}{\mathbf{F}(k,c)} y_-(k,c,x_2) + \frac{\zeta_-(c)}{y_+(k,c,-h)} y_+(k,c,x_2) + y_{nh}(k,c,x_2), \tag{5.0.2}$$

where y_{nh} is the solution to (5.0.1a) with zero boundary values in (5.0.1b) given by

$$\begin{aligned} y_{nh}(k,c,x_2) = &\frac{y_+(k,c,x_2)}{y_+(k,c,-h)} \int_{-h}^{x_2} \frac{(y_-\psi)(k,c,x_2')}{U(x_2')-c} dx_2' \\ &+ \frac{y_-(k,c,x_2)}{y_+(k,c,-h)} \int_{x_2}^{0} \frac{(y_+\psi)(k,c,x_2')}{U(x_2')-c} dx_2'. \end{aligned} \tag{5.0.3}$$

Its derivative in x_2 is given by

$$\begin{aligned} y'_{nh}(k,c,x_2) =& \frac{y'_+(k,c,x_2)}{y_+(k,c,-h)} \int_{-h}^{x_2} \frac{(y_-\psi)(k,c,x'_2)}{U(x'_2)-c} dx'_2 \\ &+ \frac{y'_-(k,c,x_2)}{y_+(k,c,-h)} \int_{x_2}^{0} \frac{(y_+\psi)(k,c,x'_2)}{U(x'_2)-c} dx'_2. \end{aligned} \tag{5.0.4}$$

Here the unique solvability condition of (5.0.1) is $\mathbf{F}(k,c) \neq 0$, where $\mathbf{F}$ is defined in (4.0.1), as the Wronskian of the fundamental solutions $y_\pm$, which is a constant in x_2, is given by

$$y_+(k,c,-h) = (g+\sigma k^2)^{-1}\mathbf{F}(k,c) = (y_+y'_- - y'_+y_-)(k,c,x_2). \tag{5.0.5}$$

Throughout this chapter, we consider

$$c = c_R + ic_I, \quad c_R \in \mathcal{I} = U([-h-\rho_0, \rho_0]), \quad |c_I| \leq \rho_0,$$

where $\rho_0 \in [0, h_0]$. By choosing ρ_0 smaller, we also have that, for some $C>0$ depending only on $|U|_{C^1}$ and $|(U')^{-1}|_{C^0}$,

$$\text{Re}\,\big(g + \sigma k^2 + U'(0)(U(0)-c)\big) \geq (1+k^2)/C, \quad \forall k \in \mathbb{R},\ c \in \mathcal{I} + i[-\rho_0, \rho_0]. \tag{5.0.6}$$

This and boundary condition (5.0.1b) imply

$$|y(0)| \leq C\mu^2(|U(0)-c|^2|y'(0)| + |\zeta_+|), \tag{5.0.7}$$

which will be used repeatedly to control $y(0)$ in terms of $y'(0)$.

Throughout this chapter, we assume that, there exists $\rho_0 > 0$ such that

$$\begin{aligned} F_0 =& \inf\{(1+k^2)^{-\frac{1}{2}} e^{-\frac{h}{\mu}} |\mathbf{F}(k,c)| \mid c_R \in \mathcal{I} = U([U(-h)-\rho_0, U(0)+\rho_0]), \\ & |c_I| \in [-\rho_0, \rho_0]\} > 0. \end{aligned} \tag{5.0.8}$$

In this section, mostly we shall not vary $k \in \mathbb{R}$, but carefully track the dependence of the estimates on k, or equivalently $\mu = (1+k^2)^{-\frac{1}{2}}$. From Lemma 3.3.2, it is easy to compute that, for any $r_1 \in [1, \infty]$, $r_2 \in [1, \infty)$, and $|c_I| \le \rho_0$,

$$\begin{aligned}&\mu^{-(1+\frac{1}{r_1})}|y_\pm|_{L^{r_1}_{x_2}L^\infty_{c_R}} + \mu^{-\frac{1}{r_1}}|y'_\pm|_{L^{r_1}_{x_2}L^{r_2}_{c_R}} + \mu^{-\frac{1}{r_2}}|y'_\pm|_{L^\infty_{c_R}L^{r_2}_{x_2}} + |y'_+(-h)|_{L^{r_2}_{c_R}} + |y'_-(0)|_{L^{r_2}_{c_R}}\\ \le& Ce^{\mu^{-1}h},\end{aligned}$$

where $x_2 \in [-h, 0]$ and $c_R \in \mathcal{I}$. This inequality will be used repeatedly in the rest of the paper.

Solutions to this system are rather smooth away from $c \in \{U(x_2), U(0), U(-h)\}$ and their singular behaviors near this set could be analyzed rather detailedly following the approach in chapter 3, based on (3.2.15) and (3.4.3) and the estimates on $\tilde{B}$ and B. However, for the purpose of this paper, it is sufficient just to obtain certain bounds of the solutions based on the properties of the homogeneous solutions $y_\pm$, which is carried out in this chapter.

As a preparation, in section 5.1 we shall first consider (5.0.1) with zero boundary conditions $\zeta_\pm = 0$ in (5.0.1b). Subsequently in section 5.2, we study the non-homogenous Rayleigh system (5.0.1) with $\zeta_\pm$ linear in c, particularly focusing on the derivatives of the solutions on $c \in \mathcal{I} + i[-\rho_0, \rho_0]$. We sometimes skip writing parameters k and c explicitly.

5.1 Non-homogeneous Rayleigh system (5.0.1) with zero boundary conditions $\zeta_\pm = 0$

The formulas (5.0.3) and (5.0.4) of $y_{nh}(k, c, x_2)$ and $y'_{nh}(k, c, x_2)$ are actually consistent with (3.2.15) for x_2 near x_2^c. In fact, (3.2.15) implies that $\begin{pmatrix} 1 & 0 \\ \Gamma & 1 \end{pmatrix} \tilde{B}$ is a fundamental matrix of (3.0.1) and hence $\tilde{B}$ can be rewritten in terms of $y_\pm$ and Γ. A straight forward calculation using (3.2.11) and (3.2.15) also yields (5.0.3). This solution also satisfy

$$\overline{y_{nh}(k, \bar{c}, x_2)} = y_{nh}(k, c, x_2) = y_{nh}(-k, c, x_2),$$

so we mainly focus on $c_I \geq 0$. Assume $\psi(c_R + ic_I, x_2) \to \psi_0(c_R, x_2)$ as $c_I \to 0+$. Due to the singularity of the non-homogeneous term at $x_2 = x_2^c$ (as defined in (3.2.1) by $U(x_2^c) = c_R$) as $c_I \to 0+$, the limits of y_{nh} and y'_{nh} involve $P.V.$ of integrals and delta masses

$$
\begin{aligned}
y_{nh0}(x_2) =& P.V. \int_{-h}^{0} \psi_0(x'_2) \frac{y_{0+}(x_2) y_{0-}(x'_2) \chi_{\{x'_2 < x_2\}} + y_{0-}(x_2) y_{0+}(x'_2) \chi_{\{x'_2 > x_2\}}}{y_{0+}(-h)(U(x'_2) - c_R)} dx'_2 \\
&+ \frac{i\pi \psi_0(x_2^c)}{U'(x_2^c)} \Big(\frac{y_{0+}(x_2) y_{0-}(x_2^c)}{y_{0+}(-h)} \chi_{\{U(x_2) > c_R > U(-h)\}} \\
&+ \frac{y_{0-}(x_2) y_{0+}(x_2^c)}{y_{0+}(-h)} \chi_{\{U(0) > c_R > U(x_2)\}} \Big),
\end{aligned}
\tag{5.1.1}
$$

$$
\begin{aligned}
y'_{nh0}(x_2) =& P.V. \int_{-h}^{0} \psi_0(x'_2) \frac{y'_{0+}(x_2) y_{0-}(x'_2) \chi_{\{x'_2 < x_2\}} + y'_{0-}(x_2) y_{0+}(x'_2) \chi_{\{x'_2 > x_2\}}}{y_{0+}(-h)(U(x'_2) - c_R)} dx'_2 \\
&+ \frac{i\pi \psi_0(x_2^c)}{U'(x_2^c)} \Big(\frac{y'_{0+}(x_2) y_{0-}(x_2^c)}{y_{0+}(-h)} \chi_{\{U(x_2) > c_R > U(-h)\}} \\
&+ \frac{y'_{0-}(x_2) y_{0+}(x_2^c)}{y_{0+}(-h)} \chi_{\{U(0) > c_R > U(x_2)\}} \Big),
\end{aligned}
\tag{5.1.2}
$$

where χ is the characteristic function and we skipped the dependence on c_R of ψ_0, $y_{0\pm}$, and y_{nh0}. Naturally, in the above the $P.V.$ is taken only when there are singularities in the integral.

We consider a priori and convergence estimates of y_{nh} as $c_I \to 0+$ in the following two cases of $\psi(c, x_2)$, motivated by the non-homogenous Rayleigh system (2.2.6) and its differentiation in c.

• **Case 1:** $\psi'(c, \cdot) \in L^r_{x_2},\ r \in (1, \infty)$**.** While this case occurs in the linearized capillary gravity wave (2.2.6) when some regularity is assumed on the initial vorticity, it is also a crucial part of the analysis when (2.2.6) is differentiated in c.

Lemma 5.1.1. *Assume (5.0.8). For any $\epsilon > 0$*[1]*, there exists $C > 0$ depending only on r, ϵ, F_0, ρ_0, $|U'|_{C^2}$ and $|(U')^{-1}|_{C^0}$, such that the following hold.*

1. *For any $k \in \mathbb{R}$, $x_2 \in [-h, 0]$, $c_I \in (0, \rho_0]$, and $c_R \in \mathcal{I}$ it holds*

$$|y_{nh}(k, c, x_2)| \leq C\mu^{1-\frac{1}{r}-\epsilon}(\mu|\psi'|_{L^r_{x_2}} + |\psi|_{L^r_{x_2}}),$$

$$|y'_{nh}(k, c, x_2)| \leq C\mu^{-\frac{1}{r}-\epsilon}\big(1 + \big|\log|U(x_2) - c|\big|\big)(\mu|\psi'|_{L^r_{x_2}} + |\psi|_{L^r_{x_2}}).$$

2. *Assume $\psi(\cdot + ic_I, \cdot) \to \psi(\cdot, \cdot)$ in $L^{r_1}_{c_R} W^{1,r}_{x_2}$ as $c_I \to 0+$ with $r_1 \in (1, \infty)$ and $r \in (1, \infty)$, then*

   (a) *$y_{nh} \to y_{nh0}$ in $L^{q_1}_{c_R} L^\infty_{x_2}$ for any $q_1 \in [1, r_1)$ and $y'_{nh} \to y'_{nh0}$ in $L^{q_1}_{c_R} L^{q_2}_{x_2}$ for any $q_1 \in [1, r_1)$ and $q_2 \in [1, \infty)$;*

   (b) *at $\tilde{x}_2 = -h$ and $\tilde{x}_2 = 0$, $y_{nh}(\cdot + ic_I, \tilde{x}_2) \to y_{nh0}(\cdot, \tilde{x}_2)$ and $y'_{nh}(\cdot + ic_I, \tilde{x}_2) \to y'_{nh0}(\cdot, \tilde{x}_2)$ in $L^{q_1}_{c_R}$ for any $q_1 \in [1, r_1)$. Moreover, and for any $\epsilon > 0$, for any $k \in \mathbb{R}$, $c_I \in [0, 1]$,*

$$|y'_{nh}(c, \tilde{x}_2)| \leq C\big(\mu^{-\frac{1}{r}-\epsilon}(\mu|\psi'|_{L^r_{x_2}} + |\psi|_{L^r_{x_2}}) + \big(1 + \big|\log|U(\tilde{x}_2) - c|\big|\big)|\psi(\tilde{x}_2)|\big).$$

Even though the above formulas of y_{nh0} involve some subtlety at $x_2 = x_2^c$, the regularity of y'_{nh0} in x_2 implies that y_{nh0} is Hölder continuous. In fact, the continuity of y_{nh0} at $x_2 = x_2^c$ can also be seen directly by using the rather precise local form of $y_{0\pm}$ near x_2^c given in Lemma 3.4.1. Moreover, while the convergence is given in the integral norms, one could attempt to obtain more detailed convergence estimates near x_2^c using the tools given in Lemma 3.2.1 and Proposition 3.2.4.

[1] Like the generic upper bound $C > 0$, the small constant $\epsilon > 0$ in this and the next chapter may change from line to line.

Proof. Since $c_I > 0$, no singularity is involved in (5.0.3) and (5.0.4), one can compute via integration by parts

$$\begin{aligned}\int_{-h}^{x_2} \frac{y_-\psi}{U-c} dx_2' = &\int_{-h}^{x_2} \frac{y_-\psi}{U'} \big(\log(U-c)\big)' dx_2' \\ =&\big(\frac{y_-\psi}{U'}\log(U-c)\big)(x_2) - \int_{-h}^{x_2} \big(\frac{y_-\psi}{U'}\big)' \log(U-c) dx_2'.\end{aligned}$$

The other integral can be handled similarly,

$$\int_{x_2}^{0} \frac{y_+\psi}{U-c} dx_2' = \big(\frac{y_+\psi}{U'}\log(U-c)\big)\Big|_{x_2}^{0} - \int_{x_2}^{0} \big(\frac{y_+\psi}{U'}\big)' \log(U-c) dx_2'.$$

Observing that the boundary terms at x_2 are canceled and we have

$$\begin{aligned} y_{nh}(x_2) = &-\frac{y_+(x_2)}{y_+(-h)} \int_{-h}^{x_2} \big(\frac{y_-\psi}{U'}\big)' \log(U-c) dx_2' - \frac{y_-(x_2)}{y_+(-h)} \int_{x_2}^{0} \big(\frac{y_+\psi}{U'}\big)' \log(U-c) dx_2' \\ &+ \frac{y_-(x_2)}{y_+(-h)} \big(\frac{y_+\psi}{U'}\log(U-c)\big)(0). \end{aligned} \tag{5.1.3}$$

The above two integrals can be estimated similarly and we shall focus on the first one only. Lemma 3.3.2 implies

$$\begin{aligned} \big|\big(\frac{y_-\psi}{U'}\big)' \log(U-c)\big| =&|U'|^{-2}\big|(y_-'\psi U' + y_-\psi' U' - y_-\psi U'')\log(U-c)\big| \\ \le& C\cosh(\mu^{-1}(x_2+h))\Big(\mu|\psi'| + \big(1+\mu\big|\log|U-c|\big|\big)|\psi|\Big) \\ &\times \big(1+\big|\log|U-c|\big|\big). \end{aligned}$$

Using the Hölder inequality we obtain

$$\begin{aligned} &\Big|\int_{-h}^{x_2} \big(\frac{y_-\psi}{U'}\big)' \log(U-c) dx_2'\Big| \\ \le& C(\mu|\psi'|_{L^r_{x_2}} + |\psi|_{L^r_{x_2}})\big|\cosh(\mu^{-1}(x_2'+h))\big(1+\big|\log|U(x_2')-c|\big|^2\big)\big|_{L^{\frac{r}{r-1}}_{x_2'}([-h,x_2])} \end{aligned}$$

$$\leq C\mu^{1-\frac{1}{r}-\epsilon}(|\psi|_{L^r_{x_2}} + \mu|\psi'|_{L^r_{x_2}}) \cosh \mu^{-1}(x_2 + h).$$

From the initial condition (3.3.1) (in particular $y_+(0) = O(\mu^2|c - U(0)|^2)$) and (5.0.5), the remaining boundary term can be estimated as

$$\Big|\frac{y_-(x_2)}{y_+(-h)}\big(\frac{y_+\psi}{U'}\log(U-c)\big)(0)\Big| \leq \frac{C\mu}{|\mathbf{F}(k,c)|}|\psi(0)| \sinh \mu^{-1}(x_2+h).$$

The desired estimate on y_{nh} follows from (5.0.8), (5.0.5), Lemma 3.3.2, the above inequalities, and the standard Sobolev inequality

$$|\psi|_{L^\infty_{x_2}} \leq C(\mu^{1-\frac{1}{r}}|\psi'|_{L^r_{x_2}} + \mu^{-\frac{1}{r}}|\psi|_{L^r_{x_2}}). \tag{5.1.4}$$

The estimate of y'_{nh} can be obtained much as in the above. Integrating by parts and using (5.0.5) to handle the boundary terms at x_2, we have

$$\begin{aligned} y'_{nh}(x_2) = &-\frac{y'_+(x_2)}{y_+(-h)}\int_{-h}^{x_2}\big(\frac{y_-\psi}{U'}\big)'\log(U-c)dx_2' - \frac{y'_-(x_2)}{y_+(-h)}\int_{x_2}^{0}\big(\frac{y_+\psi}{U'}\big)'\log(U-c)dx_2' \\ &- \big(\frac{\psi}{U'}\log(U-c)\big)(x_2) + \frac{y'_-(x_2)}{y_+(-h)}\big(\frac{y_+\psi}{U'}\log(U-c)\big)(0). \end{aligned} \tag{5.1.5}$$

The desired estimate on y'_{nh} follows from (5.0.4), (5.1.4), the above estimate on the integrals, and Lemma 3.3.2.

To consider the convergence of y_{nh}, we first note that, for $c_I > 0$, the imaginary part of $\log(U(x_2) - c)$ belongs to $(-\pi, 0)$ and as $c_I \to 0+$,

$$\log(U(x_2)-c) \to \log|U(x_2)-c_R| + \tfrac{i\pi}{2}\big(sgn(U(x_2)-c_R)-1\big) \text{ in } L^\infty_{c_R}L^q_{x_2}, \ \ \forall q \in [1,\infty). \tag{5.1.6}$$

Using expression (5.1.3), the estimates thereafter, bounds on $y_\pm$ in Lemma 3.3.2, and the convergence of $y_\pm$ to $y_{0\pm}$ as $c_I \to 0$ in Lemma 3.5.1, it is straight forward to obtain

$$\begin{aligned}
&y_{nh}(x_2) \to \\
&-\frac{y_{0+}(x_2)}{y_{0+}(-h)}\int_{-h}^{x_2}\big(\frac{y_{0-}\psi_0}{U'}\big)'\log|U-c_R|dx_2' - \frac{y_{0-}(x_2)}{y_{0+}(-h)}\int_{x_2}^{0}\big(\frac{y_{0+}\psi_0}{U'}\big)'\log|U-c_R|dx_2' \\
&+\frac{i\pi\psi_0(x_2^c)}{U'(x_2^c)}\Big(\frac{y_{0+}(x_2)y_{0-}(x_2^c)}{y_{0+}(-h)}\chi_{\{U(x_2)>c_R>U(-h)\}} + \frac{y_{0-}(x_2)y_{0+}(x_2^c)}{y_{0+}(-h)}\chi_{\{U(0)>c_R>U(x_2)\}}\Big) \\
&+\frac{y_{0-}(x_2)}{y_{0+}(-h)}\big(\frac{y_{0+}\psi_0}{U'}\log|U-c_R|\big)(0),
\end{aligned}$$

in $L^{q_1}_{c_R}L^\infty_{x_2}$ for any $q_1 \in [1, r_1)$, where, for $c_R > U(0)$, two other terms involving $sgn(U - c_R)$ (one from upper limit term from the second integral and the other from the boundary term in (5.1.3)) cancelled each other. Here the loss of the integrability in c_R in the convergence is due to the last logarithmic term. Since $(\log|U-c_R|)' = P.V.\frac{U'}{U-c_R}$ in the distribution sense, the above limit is equal to y_{nh0} after integration by parts. The convergence of y'_{nh} is obtained using (5.1.5) along with (5.0.5) in a similar fashion

$$\begin{aligned}
&y'_{nh}(x_2) \to \\
&-\frac{y'_{0+}(x_2)}{y_{0+}(-h)}\int_{-h}^{x_2}\big(\frac{y_{0-}\psi_0}{U'}\big)'\log|U-c_R|dx_2' - \frac{y'_{0-}(x_2)}{y_{0+}(-h)}\int_{x_2}^{0}\big(\frac{y_{0+}\psi_0}{U'}\big)'\log|U-c_R|dx_2' \\
&+i\pi\Big(\frac{y'_{0+}(x_2)y_{0-}(x_2^c)\psi_0(x_2^c)}{y_{0+}(-h)U'(x_2^c)}\chi_{\{U(x_2)>c_R>U(-h)\}} - \frac{\psi_0(x_2)}{U'(x_2)}\chi_{\{c_R>U(x_2)\}} \\
&+\frac{y'_{0-}(x_2)y_{0+}(x_2^c)\psi_0(x_2^c)}{y_{0+}(-h)U'(x_2^c)}\chi_{\{U(0)>c_R>U(x_2)\}}\Big) \\
&-\Big(\big(\frac{\psi_0}{U'}\log|U-c_R|\big)(x_2) - i\pi\frac{\psi_0(x_2)}{U'(x_2)}\chi_{\{c_R>U(x_2)\}}\Big) \\
&+\frac{y'_{0-}(x_2)}{y_{0+}(-h)}\big(\frac{y_{0+}\psi_0}{U'}\log|U-c_R|\big)(0)
\end{aligned}$$

where again two other terms involving $sgn(U - c_R)$ cancelled each other for $c_R > U(0)$. Here the convergence in the slightly weaker norm $L^{q_1}_{c_R}L^{q_2}_{x_2}$, for any $q_1 \in [1, r_1)$ and $q_2 \in [1, \infty)$ is due to the logarithmic singularity both explicitly outside the integrals and in $y'_\pm$

(see also Lemma 3.5.1). The limit can be simplified to

$$
\begin{aligned}
&-\frac{y_{0+}'(x_2)}{y_{0+}(-h)}\int_{-h}^{x_2}\big(\frac{y_{0-}\psi_0}{U'}\big)'\log|U-c_R|dx_2' - \frac{y_{0-}'(x_2)}{y_{0+}(-h)}\int_{x_2}^{0}\big(\frac{y_{0+}\psi_0}{U'}\big)'\log|U-c_R|dx_2'\\
&+\frac{i\pi\psi_0(x_2^c)}{U'(x_2^c)}\Big(\frac{y_{0+}'(x_2)y_{0-}(x_2^c)}{y_{0+}(-h)}\chi_{\{U(x_2)>c_R>U(-h)\}} + \frac{y_{0-}'(x_2)y_{0+}(x_2^c)}{y_{0+}(-h)}\chi_{\{U(0)>c_R>U(x_2)\}}\Big)\\
&-\big(\frac{\psi_0}{U'}\log|U-c_R|\big)(x_2) + \frac{y_{0-}'(x_2)}{y_{0+}(-h)}\big(\frac{y_{0+}\psi_0}{U'}\log|U-c_R|\big)(0),
\end{aligned}
$$

which is equal to y_{nh0}' after an integration by parts.

At the end point $x_2=-h,0$, $y_{nh}(\tilde{x}_2)$ and $y_{nh}'(\tilde{x}_2)$ have only one integrals and, unlike for general $x_2\in(-h,0)$, the terms $y_+(0)$, $y_+'(0)$ and $y_-'(-h)$ outside the integrals are prescribed in (3.3.1) without any singularity. Hence the same above argument yields slightly better estimates and convergence. One may make the following computations using (5.0.3) and (5.0.4),

$$
\begin{aligned}
y_{nh}'(0) =& \frac{y_+'(0)}{y_+(-h)}\int_{-h}^{0}\frac{y_-\psi}{U-c}dx_2'\\
=& -\frac{y_+'(0)}{y_+(-h)}\int_{-h}^{0}\big(\frac{y_-\psi}{U'}\big)'\log(U-c)dx_2' + \frac{(y_+'y_-\psi)(0)}{U'(0)y_+(-h)}\log(U(0)-c),
\end{aligned}
$$

$$
\begin{aligned}
y_{nh}'(h) = \frac{1}{y_+(-h)}\int_{-h}^{0}\frac{y_+\psi}{U-c}dx_2' =& -\frac{1}{y_+(-h)}\int_{-h}^{0}\big(\frac{y_+\psi}{U'}\big)'\log(U-c)dx_2'\\
&+\frac{1}{y_+(-h)}\Big(\frac{y_+\psi}{U'}\log(U-c)\Big)\Big|_{-h}^{0}.
\end{aligned}
$$

The desired inequalities follow from (3.3.1) and the above estimates, which completes the proof of the lemma. □

Assuming $\psi\in L_{c_R}^2H_{x_2}^1$, in the following we estimate y_{nh} and y_{nh}' as well as their derivatives in x_2 in L_{c_R,x_2}^2, in particular their dependence on k, by an energy estimate approach.

Lemma 5.1.2. *Assume (5.0.8). For any $\epsilon\in(0,1)$, there exists $C>0$ depending only on ϵ,*

F_0, ρ_0, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that for any $c_I \geq 0$ and $k \in \mathbb{R}$, it holds

$$|y'_{nh}|^2_{L^2_{c_R,x_2}} + \mu^{-2}|y_{nh}|^2_{L^2_{c_R,x_2}} \leq C(|\psi|^2_{L^2_{c_R,x_2}} + \mu^{2-\epsilon}|\psi'|^2_{L^2_{c_R,x_2}}), \tag{5.1.7}$$

where the norms are taken for $c_R \in \mathcal{I}$ and $x_2 \in [-h, 0]$.

Proof. We first assume $c_I > 0$ and drop the subscript $\cdot_{nh}$ for notation simplification. Multiplying the Rayleigh equation (5.0.1a) by $\bar{y}$ and integrating in both c_R and x_2, we have

$$\begin{aligned}
&\int_{\mathcal{I}}\int_{-h}^{0} |y'|^2 + k^2|y|^2 dx_2 dc_R = \int_{\mathcal{I}} y'\bar{y} dc_R\Big|_{x_2=0} + \int_{\mathcal{I}}\int_{-h}^{0} \frac{\psi\bar{y} - U''|y|^2}{U-c} dx_2 dc_R \\
=& \int_{\mathcal{I}} y'\bar{y} dc_R\Big|_{x_2=0} \\
&+ \int_{\mathcal{I}}\int_{-h}^{0} \frac{U'}{U-c}\Big(\big(\frac{\psi\bar{y} - U''|y|^2}{U'}\big)(c, x_2) - \big(\frac{\psi\bar{y} - U''|y|^2}{U'}\big)(c, x_2^c)\Big) dx_2 dc_R \\
+& \left(\int_{U(-h-\frac{1}{2}\rho_0)}^{U(-\frac{1}{2}h)} + \int_{U(-\frac{1}{2}h)}^{U(\frac{1}{2}\rho_0)} \right) \big(\frac{\psi\bar{y} - U''|y|^2}{U'}\big)(c, x_2^c) \\
&\qquad\qquad \times \big(\log(U(0)-c) - \log(U(-h)-c)\big) dc_R \triangleq \sum_{j=1}^{4} A_j.
\end{aligned}$$

The first term A_1 of boundary contribution can be estimated by Lemma 5.1.1(2b) and (5.0.7) with $\zeta_\pm = 0$, as well as (5.0.6), (5.0.8) and (5.1.4),

$$\begin{aligned}
|A_1| \leq \Big| \int_{\mathcal{I}} y'\bar{y} dc_R\Big|_{x_2=0}\Big| \leq & C\mu^2 \Big| \int_{\mathcal{I}} |U(0)-c|^2 |y'(0)|^2 dc_R \Big| \\
\leq & C\mu^{1-\epsilon}(\mu|\psi'|_{L^2_{c_R,x_2}} + |\psi|_{L^2_{c_R,x_2}})^2.
\end{aligned}$$

Concerning the last integral A_4, we first split it as

$$\begin{aligned}
|A_4| \leq \int_{U(-\frac{1}{2}h)}^{U(\frac{1}{2}\rho_0)} & \Big(\big|(\psi\bar{y} - U''|y|^2)(c, \cdot)\big|_{C^\alpha_{x_2}} |x_2^c|^\alpha + \big|(\psi\bar{y} - U''|y|^2)(c, 0)\big| \Big) \\
& \times \big(1 + \big|\log|U(0)-c|\big|\big) dc_R.
\end{aligned}$$

The above terms at $x_2 = 0$ can estimated much as A_1 and we obtain

$$
\begin{aligned}
&\int_{U(-\frac{1}{2}h)}^{U(\frac{1}{2}\rho_0)} \big|(\psi\bar{y} - U''|y|^2)(c,0)\big|\big(1 + \big|\log|U(0)-c|\big|\big) dc_R \\
\le & C \int_{U(-\frac{1}{2}h)}^{U(\frac{1}{2}\rho_0)} (\mu^2|\psi|^2 + \mu^2|y'|^2)(c,0)|U(0)-c| dc_R \le C\mu^{1-\epsilon}(\mu|\psi'|_{L^2_{c_R,x_2}} + |\psi|_{L^2_{c_R,x_2}})^2.
\end{aligned}
$$

We shall estimate all the remaining terms using the Hölder norms of $\psi\bar{y}$ and $|y|^2$. For any H^1 function $f(x)$ on an interval, it holds

$$
|f|_{C^\alpha} \le C|f|_{L^2}^{\frac{1}{2}-\alpha}|f|_{H^1}^{\frac{1}{2}+\alpha}, \quad \alpha \in [0, \tfrac{1}{2}], \tag{5.1.8}
$$

which applies to $\psi\bar{y}$ and $|y|^2$. In the $f|_{H^1}$ can be replaced by $|f'|_{L^2}$ if f vanishes somewhere in the interval. Hence for each fixed c with $c_I > 0$ and $c_R \in \mathcal{I}$,

$$
\big||y|^2\big|_{C^\alpha_{x_2}} \le C|y|_{C^\alpha_{x_2}}|\bar{y}|_{C^0_{x_2}} \le C|y|_{L^2_{x_2}}^{1-\alpha}|y'|_{L^2_{x_2}}^{1+\alpha},
$$

$$
|\psi\bar{y}|_{C^\alpha_{x_2}} \le C\big(|\psi|_{L^2_{x_2}}^{\frac{1}{2}-\alpha}|\psi|_{H^1_{x_2}}^{\frac{1}{2}+\alpha}|y|_{L^2_{x_2}}^{\frac{1}{2}}|y'|_{L^2_{x_2}}^{\frac{1}{2}} + |\psi|_{L^2_{x_2}}^{\frac{1}{2}}|\psi|_{H^1_{x_2}}^{\frac{1}{2}}|y|_{L^2_{x_2}}^{\frac{1}{2}-\alpha}|y'|_{L^2_{x_2}}^{\frac{1}{2}+\alpha}\big).
$$

For any $\alpha \in (0, \frac{1}{2}]$ and $k > 0$, using $y(c, -h) = 0$ and the above estimates, we obtain

$$
\begin{aligned}
&|y'|^2_{L^2_{c_R,x_2}} + k^2|y|^2_{L^2_{c_R,x_2}} \\
\le & C\int_{\mathcal{I}}\int_{-h}^{0} \big|(\psi\bar{y} - U''|y|^2)(c,\cdot)\big|_{C^\alpha_{x_2}}|x_2 - x_2^c|^{\alpha-1} dx_2 dc_R \\
&+ C\int_{U(-h-\frac{1}{2}\rho_0)}^{U(-\frac{1}{2}h)} \big|(\psi\bar{y} - U''|y|^2)(c,\cdot)\big|_{C^\alpha_{x_2}}|x_2^c + h|^\alpha\big(1 + \big|\log|U(-h)-c|\big|\big) dc_R \\
&+ C\int_{U(-\frac{1}{2}h)}^{U(\frac{1}{2}\rho_0)} \big|(\psi\bar{y} - U''|y|^2)(c,\cdot)\big|_{C^\alpha_{x_2}}|x_2^c|^\alpha\big(1 + \big|\log|U(0)-c|\big|\big) dc_R \\
&+ C\mu^{1-\epsilon}(\mu|\psi'|_{L^2_{c_R,x_2}} + |\psi|_{L^2_{c_R,x_2}})^2 \\
\le & C\int_{\mathcal{I}}\big(|\psi|_{L^2_{x_2}}^{\frac{1}{2}-\alpha}|\psi|_{H^1_{x_2}}^{\frac{1}{2}+\alpha}|y|_{L^2_{x_2}}^{\frac{1}{2}}|y'|_{L^2_{x_2}}^{\frac{1}{2}} + |\psi|_{L^2_{x_2}}^{\frac{1}{2}}|\psi|_{H^1_{x_2}}^{\frac{1}{2}}|y|_{L^2_{x_2}}^{\frac{1}{2}-\alpha}|y'|_{L^2_{x_2}}^{\frac{1}{2}+\alpha} + |y|_{L^2_{x_2}}^{1-\alpha}|y'|_{L^2_{x_2}}^{1+\alpha}\big)\big|_c dc_R \\
&+ C\mu^{1-\epsilon}(\mu|\psi'|_{L^2_{c_R,x_2}} + |\psi|_{L^2_{c_R,x_2}})^2
\end{aligned}
$$

$$\le C\big(|\psi|_{L^2_{c_R,x_2}}^{\frac12-\alpha}|\psi|_{L^2_{c_R}H^1_{x_2}}^{\frac12+\alpha}|y|_{L^2_{c_R,x_2}}^{\frac12}|y'|_{L^2_{c_R,x_2}}^{\frac12}+|\psi|_{L^2_{c_R,x_2}}^{\frac12}|\psi|_{L^2_{c_R}H^1_{x_2}}^{\frac12}|y|_{L^2_{c_R,x_2}}^{\frac12-\alpha}|y'|_{L^2_{c_R,x_2}}^{\frac12+\alpha}$$
$$+|y|_{L^2_{c_R,x_2}}^{1-\alpha}|y'|_{L^2_{c_R,x_2}}^{1+\alpha}\big)+C\mu^{1-\epsilon}(\mu|\psi'|_{L^2_{c_R,x_2}}+|\psi|_{L^2_{c_R,x_2}})^2$$
$$\le\tfrac12|y'|^2_{L^2_{c_R,x_2}}+(C+\tfrac12k^2)|y|^2_{L^2_{c_R,x_2}}+C\big(|\psi|^2_{L^2_{c_R,x_2}}+k^{-2(1-2\alpha)}|\psi'|^2_{L^2_{c_R,x_2}}\big).$$

By choosing $\alpha=\epsilon/2$, we have that, there exists $k_0>0$ such that for any $|k|\ge k_0$ and $c_I>0$, $y(\cdot+ic_I,\cdot)$ satisfies (5.1.7). To obtain the estimates for y_{nh0} and y'_{nh0} in the limiting case $c_I=0+$, for $c_I>0$, let $y(c,x_2)$ and $y'(c,x_2)$ be defined by (5.0.3) and (5.0.4), which satisfy the desired estimates uniform in $c_I>0$. For $|k|\le k_0$ and $c_I>0$, the desired estimates simply follows from the estimates and convergence obtained in Lemma 5.1.1.

Finally we consider the case $c_I=0$. Given $\psi(c_R,x_2)\in L^2_{c_R}H^1_{x_2}$, let $y_{nh}(k,c_R+ic_I,x_2)$ be given by (5.0.3) with $c=c_R+ic_I$ with $1\gg c_I>0$, which solves (5.0.1a). From Lemma 5.1.1, it holds that $y(\cdot+ic_I,\cdot)\to y_{nh0}$ and $y'(\cdot+ic_I,\cdot)\to y'_{nh0}$ in $L^{\frac32}_{c_R}L^2_{x_2}$ as $c_I\to0+$. Therefore y_{nh0} and y'_{nh0} are also the weak limit of y and y' in $L^2_{c_R,x_2}$ as $c_I\to0+$ and thus also satisfy (5.1.7). □

• **Case 2:**

$$\psi(c,x_2)=f(c,x_2)\psi_0(x_2),\ f(\cdot+ic_I,\cdot)\in L^{r_1}_{c_R}C^\alpha_{x_2},\ \psi_0\in L^r,r>1,r_1\in[\tfrac{r}{r-1},\infty],\alpha>0. \tag{5.1.9}$$

Again we start with rough estimates on y_{nh} and y'_{nh}.

Lemma 5.1.3. *Assume (5.0.8) and (5.1.9). For any $q\in[1,\frac{rr_1}{r+r_1})$, the following hold for $x_2\in[-h,0]$ and $c_R\in\mathcal{I}$.*

1. *There exists $C>0$ depending only on r, r_1, q, α, F_0, ρ_0, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that for any $k\in\mathbb{R}$ and $c_I\in(0,\rho_0]$, it holds*

$$|y'_{nh}(k,\cdot+ic_I,\cdot)|_{L^\infty_{x_2}L^q_{c_R}}+\mu^{-1}|y_{nh}(k,\cdot+ic_I,\cdot)|_{L^\infty_{x_2}L^{\frac{rr_1}{r+r_1}}_{c_R}}$$
$$\le C\mu^{-\alpha}|f(\cdot+ic_I,\cdot)|_{L^{r_1}_{c_R}C^\alpha_{x_2}}|\psi|_{L^r}.$$

2. *Assume $f(\cdot + ic_I, \cdot) \to f_0(\cdot, \cdot)$ in $L^{r_1}_{c_R} C^\alpha_{x_2}$ as $c_I \to 0+$, then*

   (a) *$y_{nh} \to y_{nh0}$ in $L^\infty_{x_2} L^{\frac{rr_1}{r+r_1}}_{c_R}$ and $y'_{nh} \to y'_{nh0}$ in $L^\infty_{x_2} L^q_{c_R}$, where y_{nh0} and y'_{nh0} are given by (5.1.1) and (5.1.2) with ψ_0 replaced by $f_0\psi_0$;*

   (b) *at $\tilde{x}_2 = -h, 0$, $y'_{nh}(k, \cdot + ic_I, \tilde{x}_2) \to y'_{nh0}(k, \cdot, \tilde{x}_2)$ in $L^{\frac{rr_1}{r+r_1}}_{c_R}$. Moreover, and for any $\epsilon \in (0, \frac{1}{r})$ with $\epsilon \le \alpha$, for any $k \in \mathbb{R}$, $c_I \ge 0$, it holds*

$$|y'_{nh}(\cdot + ic_I, \tilde{x}_2)|_{L^{\frac{rr_1}{r+r_1}}_{c_R}} \le C\mu^{-\epsilon} |f|_{L^{r_1}_{c_R} C^\alpha_{x_2}} |\psi|_{L^r},$$

   where C also depends on $\epsilon > 0$.

Proof. Since the desired estimates are stronger and with weaker assumptions if $\alpha \in (0,1)$ is smaller (with possibly greater $C > 0$), without loss of generality, we may assume $\alpha < \frac{1}{r}$. In the following we shall need the modification $\tilde{x}^c_2$ determined by $c_R \in \mathcal{I}$:

$$\begin{aligned} \tilde{x}^c_{2-}(c, x_2) &= \begin{cases} \min\{x_2, x^c_2\}, & \text{if } c_R > U(-h), \\ -h, & \text{if } c_R \le U(-h), \end{cases}, \\ \tilde{x}^c_{2+}(c, x_2) &= \begin{cases} \max\{x_2, x^c_2\}, & \text{if } c_R < U(0), \\ 0, & \text{if } c_R \ge U(0). \end{cases} \end{aligned} \tag{5.1.10}$$

For $c_I > 0$, we first split y_{nh} into

$$y_1(x_2) = \frac{y_+(x_2)}{y_+(-h)} \big(\frac{y_- f}{U'}\big)(\tilde{x}^c_{2-}) \int_{-h}^{x_2} \frac{\psi_0 U'}{U - c} dx'_2 + \frac{y_-(x_2)}{y_+(-h)} \big(\frac{y_+ f}{U'}\big)(\tilde{x}^c_{2+}) \int_{x_2}^0 \frac{\psi_0 U'}{U - c} dx'_2,$$

and

$$\begin{aligned} y_2(x_2) = &\frac{y_+(x_2)}{y_+(-h)} \int_{-h}^{x_2} \Big(\big(\frac{y_- f}{U'}\big)(x'_2) - \big(\frac{y_- f}{U'}\big)(\tilde{x}^c_{2-}) \Big) \frac{\psi_0 U'}{U - c} dx'_2 \\ &+ \frac{y_-(x_2)}{y_+(-h)} \int_{x_2}^0 \Big(\big(\frac{y_+ f}{U'}\big)(x'_2) - \big(\frac{y_+ f}{U'}\big)(\tilde{x}^c_{2+}) \Big) \frac{\psi_0 U'}{U - c} dx'_2, \end{aligned}$$

where we skipped all the dependence on c and k. Clearly $y_{nh} = y_1 + y_2$.

To estimate y_1, we can rewrite its integral part as

$$\int_{-h}^{x_2} \frac{\psi_0 U'}{U-c} dx_2' = \int_{\mathbb{R}} \frac{\chi_{U([-h,x_2])}(\psi_0 \circ U^{-1})}{\tau - c_R - ic_I} d\tau = -\Big(\big(\frac{1}{\tau + ic_I}\big) * \tilde{\psi}_-(x_2, \cdot)\Big)(c_R).$$

where

$$\tilde{\psi}_-(x_2, \tau) = \chi_{U([-h,x_2])}(\psi_0 \circ U^{-1})(\tau), \quad \tilde{\psi}_+(x_2, \tau) = \chi_{U([x_2,0])}(\psi_0 \circ U^{-1})(\tau).$$

The operator of convolution by $\frac{1}{\tau+ic_I}$ is bounded on L^r uniformly in $c_I > 0$ and converges to $\pi(\mathcal{H}+iI)$ strongly in L^r as $c_I \to 0+$, where $\mathcal{H}$ is the Hilbert transform and I is the identity. The other integral can be treated similarly and we obtain from (5.0.8) and Lemma 3.3.2

$$\begin{aligned} |y_1|_{L^\infty_{x_2} L^{\frac{rr_1}{r+r_1}}_{c_R}} \leq & C\Big(\big|\tfrac{y_+(x_2) y_-(\tilde{x}^c_{2-})}{y_+(-h)}\big|_{L^\infty_{c_R, x_2}} + \big|\tfrac{y_-(x_2) y_+(\tilde{x}_{2+})}{y_+(-h)}\big|_{L^\infty_{c_R, x_2}}\Big) |f|_{L^{r_1}_{c_R} L^\infty_{x_2}} |\psi_0|_{L^r} \\ \leq & C\mu |f|_{L^{r_1}_{c_R} L^\infty_{x_2}} |\psi_0|_{L^r}. \end{aligned}$$

Moreover, since $x_2 \to \tilde{\psi}_\pm(x_2, \cdot)$ are two uniformly continuous mapping from $[-h, 0]$ to $L^r(\mathbb{R})$ and the above convolution $\big(\frac{1}{\tau+ic_I}\big)*$ is bounded on $L^r_{c_R}(\mathbb{R})$ uniformly in $c_I > 0$, we have that $\big(\frac{1}{\tau+ic_I}\big) * \tilde{\psi}_\pm(x_2, \cdot)$ are two families (with parameter c_I) of equicontinuous functions (of x_2) from $[-h, 0]$ to $L^r_{c_R}$. As $c_I \to 0+$, they converge pointwisely (in x_2) to $\pi(\mathcal{H} + iI)\tilde{\psi}_\pm(x_2, \cdot) \in L^r_{c_R}$ which are also uniformly continuous in x_2. The equicontinuity and the compactness of $[-h, 0]$ imply that the convergence is uniform in x_2. Therefore, along with the $L^\infty_{c_R, x_2}$ convergence of $y_\pm$ as $c_I \to 0+$ (Lemma 3.5.1), we obtain that, as $c_I \to 0+$, in $L^\infty_{x_2} L^{\frac{rr_1}{r+r_1}}_{c_R}$,

$$\begin{aligned} y_1(c_R + ic_I, x_2) \to & \pi \frac{y_{0+}(c_R, x_2)}{y_{0+}(c_R, -h)} \big(\frac{y_{0-} f_0}{U'}\big)(c_R, \tilde{x}^c_{2-}) \big((\mathcal{H} + iI)\tilde{\psi}_-(x_2, \cdot)\big)(c_R) \\ & + \pi \frac{y_{0-}(c_R, x_2)}{y_{0+}(c_R, -h)} \big(\frac{y_{0+} f_0}{U'}\big)(c_R, \tilde{x}^c_{2+}) \big((\mathcal{H} + iI)\tilde{\psi}_+(x_2, \cdot)\big)(c_R). \end{aligned}$$

The other part y_2 can be estimated by the Hölder continuity of f and $y_\pm$ in x_2 as

$$\begin{aligned}|y_2(c,x_2)| \leq & C\Big(\Big|\frac{y_+(x_2)}{y_+(-h)}\Big||y_-f|_{C^\alpha_{x_2'}([-h,x_2])}\int_{-h}^{x_2}\frac{|U-U(\tilde{x}_{2-}^c)|^\alpha}{|U-c|}|\psi_0|U'dx_2' \\ & + \Big|\frac{y_-(x_2)}{y_+(-h)}\Big||y_+f|_{C^\alpha_{x_2'}([x_2,0])}\int_{x_2}^{0}\frac{|U-U(\tilde{x}_{2+}^c)|^\alpha}{|U-c|}|\psi_0|U'dx_2'\Big) \\ \leq & C\mu^{1-\alpha}|f|_{C^\alpha_{x_2}}\int_{\mathbb{R}}\frac{|\tau-c_R|^\alpha}{|\tau-c|}\big(\chi_{U([-h,0])}(|\psi_0|\circ U^{-1})\big)(\tau)d\tau,\end{aligned}$$

where we also used

$$|y_-f|_{C^\alpha_{x_2'}[-h,x_2]} \leq C|y_-|_{C^\alpha_{x_2'}([-h,x_2])}|f|_{C^\alpha_{x_2}} \leq C\mu^{1-\alpha}e^{\mu^{-1}(x_2+h)}|f|_{C^\alpha_{x_2}}.$$

and a similar estimate for $|y_+f|_{C^\alpha_{x_2'}([x_2,0])}$ due to Lemma 3.3.2. Since $\frac{|\tau|^\alpha}{|\tau+ic_I|}$ is a weak-$L^{\frac{1}{1-\alpha}}$ function of τ with norm uniformly bounded in $c_I>0$, the weak Young's inequality yield

$$|y_2|_{L^\infty_{x_2}L^{r_2}_{c_R}} \leq C\mu^{1-\alpha}|f|_{L^{r_1}_{c_R}C^\alpha_{x_2}}|\psi_0|_{L^r}, \text{ where } \tfrac{1}{r_2}=\tfrac{1}{r_1}+\tfrac{1}{r}-\alpha<\tfrac{1}{r_1}+\tfrac{1}{r}.$$

To obtain the convergence of y_2 as $c_I\to 0$, using the $L^\infty_{c_R,x_2}$ convergence of $y_\pm$ and the $L^\infty_{c_R}L^{\tilde q}_{x_2}$ and $L^\infty_{x_2}L^{\tilde q}_{c_R}$, $\forall \tilde q\in(1,\infty)$, convergence of $y_\pm'$ (Lemma 3.5.1), one may easily reduce the problem to the convergence of

$$\begin{aligned}\tilde\Delta = & \Big|\frac{y_{0+}(x_2)}{y_{0+}(-h)}\int_{-h}^{x_2}\Big(\big(\frac{y_{0-}f_0}{U'}\big)(x_2')-\big(\frac{y_{0-}f_0}{U'}\big)(\tilde{x}_{2-}^c)\Big)\big(\frac{1}{U-c}-\frac{1}{U-c_R}\big)\psi_0U'dx_2'\Big|_{L^\infty_{x_2}L^{r_2}_{c_R}} \\ \leq & C\mu^{1-\alpha}\Big||f_0|_{L_{C^\alpha_{x_2}}}\int_{\mathbb{R}}\Big|\frac{|\tau-c_R|^\alpha}{|\tau-c|}-|\tau-c_R|^{\alpha-1}\Big|\big(\chi_{U([-h,0])}(|\psi_0|\circ U^{-1})\big)(\tau)d\tau\Big|_{L^{r_2}_{c_R}}\end{aligned}$$

and that of a similar term of the other integral. It is easy to see via a rescaling that, for $s\in[1,\frac{1}{1-\alpha})$,

$$\Big|\frac{|\tau|^\alpha}{|\tau+ic_I|}-|\tau|^{\alpha-1}\Big|_{L^s}=|c_I|^{\alpha-1}\big|\gamma\big(\frac{\tau}{c_I}\big)\big|_{L^s}=|c_I|^{\frac{1}{s}-1+\alpha}|\gamma|_{L^s},$$

where $\gamma(\tau)=\frac{|\tau|^\alpha}{|\tau+i|}-|\tau|^{\alpha-1}$, while with the weak-$L^{\frac{1}{1-\alpha}}$ norm equal to $|\gamma|_{w-L^{\frac{1}{1-\alpha}}}$. Hence

$$\Big\| \Big| \frac{|\tau|^\alpha}{|\tau + ic_I|} - |\tau|^{\alpha-1} \Big| * \varphi \Big|_{L^{\frac{1}{\frac{1}{r}-\alpha}}} \to 0, \qquad \text{as } c_I \to 0,$$

for any $\varphi \in L^{\tilde r}$ with $\tilde r > r$. Through a standard density argument and using the above uniform bound on the weak-$L^{\frac{1}{1-\alpha}}$ norm of the convolution kernel, this convergence also holds for any $\varphi \in L^r$. Therefore, we obtain $\tilde\Delta \to 0$ and thus

$$\begin{aligned} y_2(c_R + ic_I, x_2) \to & \frac{y_{0+}(c_R, x_2)}{y_{0+}(c_R, -h)} \int_{-h}^{x_2} \Big((\frac{y_{0-} f_0}{U'})(c_R, x_2') - (\frac{y_{0-} f_0}{U'})(c_R, \tilde x_{2-}^c) \Big) \frac{\psi_0 U'}{U - c_R} dx_2' \\ & + \frac{y_{0-}(c_R, x_2)}{y_{0+}(c_R, -h)} \int_{x_2}^{0} \Big((\frac{y_{0+} f_0}{U'})(c_R, x_2') - (\frac{y_{0+} f_)}{U'})(c_R, \tilde x_{2+}^c) \Big) \frac{\psi_0 U'}{U - c_R} dx_2'. \end{aligned}$$

The above estimates of y_1 and y_2 together yield the desired estimates of y_{nh} and its convergence as $c_I \to 0$. The analysis on y_{nh}' also follows from the above estimates with minor modifications, mostly replacing some $|y_\pm|_{L^\infty_{c_R, x_2}}$ by $|y_\pm'|_{L^\infty_{x_2} L^s_{c_R}}$ or $|y_\pm'|_{L^\infty_{c_R} L^s_{x_2}}$ outside the integrals, needed to control its logarithmic singularity caused by $y_\pm'$. We omit the details.

Finally, as in Lemma 5.1.1, stronger estimates and convergence can be obtained at $x_2 = -h, 0$ due to prescribed boundary values (3.3.1).

In fact,

$$\begin{aligned} y_{nh}'(0) =& \frac{y_+'(0)}{y_+(-h)} \int_{-h}^0 \frac{y_- f \psi_0}{U - c} dx_2' \\ =& \frac{y_+'(0)}{y_+(-h)} (\frac{y_- f}{U'})(\tilde x_{2-}^c) \int_{-h}^0 \frac{\psi_0 U'}{U - c} dx_2' \\ & + \frac{y_+'(0)}{y_+(-h)} \int_{-h}^0 \Big((\frac{y_- f}{U'})(x_2') - (\frac{y_- f}{U'})(\tilde x_{2-}^c) \Big) \frac{\psi_0}{U - c} dx_2' \end{aligned}$$

implies

$$|y_{nh}'(0)|_{L_{c_R}^{\frac{rr_1}{r+r_1}}} \le C\Big(|f|_{L^{r_1}_{c_R} L^\infty_{x_2}} \Big| \int_{U(-h)}^{U(0)} \frac{\psi_0}{\tau - c} d\tau \Big|_{L^r_{c_R}}$$

$$+\mu^{-1}e^{-\mu h}|y_-|_{L^{\frac{1}{\epsilon}}_{c_R}C^\epsilon_{x_2}}|f|_{L^{r_1}_{c_R}C^\epsilon_{x_2}}\Big|\int_{U(-h)}^{U(0)}\frac{|\psi_0|}{|\tau-c|^{1-\epsilon}}d\tau\Big|_{L_{c_R}^{\frac{r}{1-\epsilon r}}}\Big).$$

From the same procedure as in estimating y_1 and y_2 in the above, we obtain the desired estimate. Its convergence follows much as that of y_{nh}. The same argument applies to $y'_{nh}(c,-h)$ and the proof of the lemma is complete. $\square$

The following is an estimate y_{nh0} and y'_{nh0} in $L^2_{c_R,x_2}$ and their dependence on k.

Lemma 5.1.4. *In addition to (5.0.8) and (5.1.9), assume $\frac{1}{2}\geq\frac{1}{r}+\frac{1}{r_1}$. For any $\epsilon\in(0,1)$, there exists $C>0$ depending only on ϵ, r, r_1, F_0, ρ_0, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that for any $k\in\mathbb{R}$ and $c_I\geq 0$, it holds*

$$|y'_{nh}|^2_{L^2_{c_R,x_2}}+\mu^{-2}|y_{nh}|^2_{L^2_{c_R,x_2}}\leq C\mu^{1-\epsilon}|f|^2_{L^{r_1}_{c_R}C^\alpha_{x_2}}|\psi_0|^2_{L^r}.$$

where the norms are taken for $c_R\in\mathcal{I}$ and $x_2\in[-h,0]$.

Proof. As in the proof of Lemma 5.1.2, we first consider for $c_I>0$ and drop the subscript $\cdot_{nh}$ for notation simplification. Multiplying the Rayleigh equation (5.0.1a) by $\bar{y}$ and integrating in both c_R and x_2, we have

$$\begin{aligned}&\int_{\mathcal{I}}\int_{-h}^0|y'|^2+k^2|y|^2dx_2dc_R=\int_{\mathcal{I}}\int_{-h}^0\frac{f\psi_0\bar{y}-U''|y|^2}{U-c}dx_2dc_R+\int_{\mathcal{I}}y'\bar{y}dc_R\Big|_{x_2=0}\\ =&\int_{\mathcal{I}}\int_{-h}^0\frac{U'\psi_0}{U-c}\Big(\big(\frac{f\bar{y}}{U'}\big)(c,x_2)-\big(\frac{f\bar{y}}{U'}\big)(c,x_2^c)\Big)dx_2dc_R\\ &+\int_{\mathcal{I}}\big(\frac{f\bar{y}}{U'}\big)(c,x_2^c)\int_{-h}^0\frac{U'\psi_0}{U-c}dx_2'dc_R-\int_{\mathcal{I}}\int_{-h}^0\frac{U''|y|^2}{U-c}dx_2dc_R+\int_{\mathcal{I}}y'\bar{y}dc_R\Big|_{x_2=0}\\ \triangleq& I_1+I_2+I_3+I_4.\end{aligned}$$

The term I_4 can be estimated much as in the proof of Lemma 5.1.2 using Lemma 3.3.2 and Lemma 5.1.3(2b)

$$|I_4|\leq C\mu^2\Big|\int_{\mathcal{I}}|U(0)-c|^2|y'(0)|^2dc_R\Big|\leq C\mu^{2-\epsilon}|f|^2_{L^{r_1}_{c_R}C^\alpha_{x_2}}|\psi_0|^2_{L^r}.$$

Choose α_1 and r_2 such that

$$0 < \alpha_1 \le \max\{\tfrac{\epsilon}{2}, \alpha, \tfrac{1}{r} + \tfrac{1}{r_1}\}, \quad \tfrac{1}{r_2} = 1 + \alpha_1 - \tfrac{1}{r} - \tfrac{1}{r_1} \in (\tfrac{1}{2}, 1],$$

which is possible due to our assumption on α, r, and r_1. The integral I_1 can be controlled by the Hölder continuity of f and y in x_2, the weak Young's inequality, and the (5.1.8) type interpolation inequality as

$$\begin{aligned}
|I_1| \le & C \int_{\mathbb{R}} \int_{\mathbb{R}} \big(\chi_{\mathcal{I}} |(f\bar{y})(c_R, \cdot)|_{C^{\alpha_1}_{x_2}}\big) |\tau - c_R|^{\alpha_1 - 1} |(\chi_{U([-h,0])} \psi_0 \circ U^{-1})(\tau)| d\tau dc_R \\
\le & C |f\bar{y}|_{L_{c_R}^{\frac{1}{1+\alpha_1 - \frac{1}{r}}} C^{\alpha_1}_{x_2}} |\psi_0|_{L^r} \le C |f|_{L^{r_1}_{c_R} C^{\alpha_1}_{x_2}} |y|_{L^{r_2}_{c_R} C^{\alpha_1}_{x_2}} |\psi_0|_{L^r} \\
\le & C |f|_{L^{r_1}_{c_R} C^{\alpha_1}_{x_2}} \big| |y'|^{\frac{1}{2}+\alpha_1}_{L^2_{x_2}} |y|^{\frac{1}{2}-\alpha_1}_{L^2_{x_2}} \big|_{L^{r_2}_{c_R}} |\psi_0|_{L^r} \le C |f|_{L^{r_1}_{c_R} C^{\alpha_1}_{x_2}} |y'|^{\frac{1}{2}+\alpha_1}_{L^2_{c_R, x_2}} |y|^{\frac{1}{2}-\alpha_1}_{L^{r_3}_{c_R} L^2_{x_2}} |\psi_0|_{L^r}
\end{aligned}$$

where $r_3 < 2$ is determined by $\frac{\frac{1}{2}+\alpha_1}{2} + \frac{\frac{1}{2}-\alpha_1}{r_3} = \frac{1}{r_2}$. Therefore we obtain

$$|I_1| \le \tfrac{1}{4}\big(|y'|^2_{L^2_{c_R,x_2}} + k^2 |y|^2_{L^2_{c_R,x_2}}\big) + C k^{-(1-2\alpha_1)} |f|^2_{L^{r_1}_{c_R} C^{\alpha_1}_{x_2}} |\psi_0|^2_{L^r}.$$

The estimate of I_2 is much as in the proof of Lemma 5.1.3 based on the boundedness of the convolution operator on L^r

$$\begin{aligned}
|I_2| \le & C |\psi|_{L^r} |(f\bar{y})(c_R, x_2^c)|_{L_{c_R}^{\frac{r}{r-1}}} \le C |\psi|_{L^r} \big| |f|_{L^\infty_{x_2}} |y|^{\frac{1}{2}}_{L^2_{x_2}} |y'|^{\frac{1}{2}}_{L^2_{x_2}} \big|_{L_{c_R}^{\frac{r}{r-1}}} \\
\le & C |\psi|_{L^r} |f|_{L^{r_4}_{c_R} L^\infty_{x_2}} |y|^{\frac{1}{2}}_{L^2_{c_R,x_2}} |y'|^{\frac{1}{2}}_{L^2_{c_R,x_2}},
\end{aligned}$$

where $r_4 = \frac{2r}{r-2} \le r_1$. Hence

$$|I_2| \le \tfrac{1}{4}\big(|y'|^2_{L^2_{c_R,x_2}} + k^2 |y|^2_{L^2_{c_R,x_2}}\big) + C k^{-1} |f|^2_{L^{r_1}_{c_R} C^{\alpha_1}_{x_2}} |\psi|^2_{L^r}.$$

Finally I_3 can be estimated exactly as in the proof of Lemma 5.1.2 (and also applying

Lemma 5.1.3(2b)) and we have

$$|I_3| \le \tfrac{1}{4}|y'|^2_{L^2_{c_R,x_2}} + C\big(|y|^2_{L^2_{c_R,x_2}} + \mu^{4-\epsilon}|f|^2_{L^{r_1}_{c_R}C^{\alpha}_{x_2}}|\psi_0|^2_{L^r}\big).$$

Therefore, there exists $k_0 > 0$ such that y and y' satisfy the desired estimates for $|k| \ge k_0$ and $c_I > 0$. For those $|k| \le k_0$, the $|y|^2_{L^2_{c_R,x_2}}$ term in the upper bound of I_3 can be controlled by Lemma 5.1.3 directly and thus the desired estimates are also satisfied by y and y'. The estimate in the limiting case of $c_I = 0+$ can be obtained through the same weak convergence argument as in the proof of Lemma 5.1.2. $\square$

Remark 5.1.1. *In some sense the $L^2_{c_R,x_2}$ assumption on ψ and ψ' in the Lemma 5.1.2 is the (unreachable) borderline case of Lemma 5.1.4. In fact, $\psi(c_R, x_2)$ can be written as $\psi \cdot 1$, where the former belongs to $L^2_{c_R}C^{\frac{1}{2}}_{x_2}$ with $r_1 = 2$. As $r < \infty$ and $\frac{1}{r} + \frac{1}{r_1} = \frac{1}{2}$ are assumed in (5.1.9) and Lemma 5.1.4, it does not apply in this case.*

5.2 Differentiation in c of solutions to non-homogeneous Rayleigh system

Based on the analysis of the non-homogeneous Rayleigh equation (5.0.1) with zero boundary conditions, in this section we shall mainly consider (2.2.6c) type non-zero boundary conditions, in particular the estimates of the derivative of solutions $y_B(k, c, x_2)$ given in (5.0.2) with respect to c.

Through straight forward calculations and applying Lemma 3.3.2, we obtain

Lemma 5.2.1. *Assume (5.0.8) and $c \in \mathcal{I} + i[-\rho_0, \rho_0]$. For any $1 < r_1 < r_2 < \infty$, there exists $C > 0$ depending only on r_1, r_2, F_0, ρ_0, $|U'|_{C^2}$, and $|(U')^{-1}|_{C^0}$, such that for any $|c_I| \le \rho_0$, the unique solution $y_B(k, c, x_2)$ to (5.0.1) satisfies*

$$|y_B|_{L^2_{c_R,x_2}} \le C\big(|y_{nh}|_{L^2_{c_R,x_2}} + \mu^{\frac{5}{2}}|\zeta_+|_{L^2_{c_R}} + \mu^{\frac{1}{2}}|\zeta_-|_{L^2_{c_R}}\big),$$

$$|y_B'|_{L^2_{c_R,x_2}} \le C\big(|y_{nh}'|_{L^2_{c_R,x_2}} + \mu^{\frac{3}{2}}|\zeta_+|_{L^2_{c_R}} + \mu^{-\frac{1}{2}}|\zeta_-|_{L^2_{c_R}}\big),$$

$$|y_B'(-h)|_{L^{r_1}_{c_R}} \leq C\big(|y_{nh}'(-h)|_{L^{r_1}_{c_R}} + \mu^{-1}|\zeta_-|_{L^{r_1}_{c_R}} + |\zeta_-|_{L^{r_2}_{c_R}} + \mu e^{-\mu^{-1}h}|\zeta_+|_{L^{r_1}_{c_R}}\big),$$

$$|y_B'(0)|_{L^{r_1}_{c_R}} \leq C\big(|y_{nh}'(0)|_{L^{r_1}_{c_R}} + \mu|\zeta_+|_{L^{r_1}_{c_R}} + \mu^2|\zeta_+|_{L^{r_2}_{c_R}} + \mu^{-1}e^{-\mu^{-1}h}|\zeta_-|_{L^{r_1}_{c_R}})\big),$$

where the norm is taken on $c_R \in \mathcal{I}$ and $x_2 \in [-h, 0]$.

We shall also consider the limit

$$y_{B0} = y_B|_{c_I=0+} = \lim_{c_I \to 0+} y_B = b_{0-}y_{0-} + b_{0+}y_{0+} + y_{nh0}, \tag{5.2.1}$$

which exists for appropriate $\psi(c, x_2)$ and satisfies the same estimates as y_B (see section 5.1).

In the rest of the section, we shall focus on the special case motivated by (2.2.6):

$$\psi = \psi_0(x_2), \quad \zeta_-(c) = \xi_-, \quad \zeta_+(c) = \xi_1 + (U(0) - c)\xi_2, \tag{5.2.2}$$

where ψ_0, ξ_-, ξ_1, and ξ_2 are all independent of c. Our goal is to obtain the estimates of the derivatives of the solution $y_B(k, c, x_2)$ to (5.0.1) in c_R.

Proposition 5.2.2. *Assume $U \in C^4$, (5.0.8), and (5.2.2). For any $\epsilon \in (0, 1)$, $r \in (1, \infty)$, there exists $C > 0$ depending on ϵ, r, F_0, ρ_0, $|U'|_{C^3}$, and $|(U')^{-1}|_{C^0}$ such that the solution $y_B(k, c, x_2)$ to (5.0.1) satisfies that for any $|c_I| \leq \rho_0$ and $k \in \mathbb{R}$,*

$$\begin{aligned}&|y_B|_{L^2_{c_R,x_2}} + \mu|y_B'|_{L^2_{c_R,x_2}} + \mu^{\frac{3}{2}}|y_B'(0)|_{L^2_{c_R}} + \mu^{\frac{1}{2}}|y_B(0)|_{L^2_{c_R}}\\ \leq &C\mu^{\frac{5}{2}}\big(\mu^{-1-\epsilon}|\psi_0|_{L^2} + |\xi_1| + |\xi_2| + \mu^{-2}|\xi_-|\big),\end{aligned}$$

$$\begin{aligned}&|\partial_{c_R}y_B|_{L^2_{c_R,x_2}} + \mu|\partial_{c_R}y_B' + \tfrac{1}{U'(x_2^c)}y_B''|_{L^2_{c_R,x_2}} + \mu^{\frac{3}{2}}|(\partial_{c_R}y_B' + \tfrac{1}{U'(x_2^c)}y_B'')(0)|_{L^2_{c_R}}\\ &+ \mu^{\frac{1}{2}}|\partial_{c_R}y_B(0)|_{L^2_{c_R}}\\ \leq &C\mu^{\frac{3}{2}}\big(\mu^{-1-\epsilon}|\psi_0|_{L^2} + \mu^{-\epsilon}|\psi_0'|_{L^2} + |\xi_1| + |\xi_2| + \mu^{-2}|\xi_-|\big),\end{aligned}$$

and, if $U \in C^5$, then

$$|\tilde{y}_B|_{L^2_{c_R,x_2}} \le C\mu^{\frac{1}{2}}\big(\mu^{-1-\epsilon}|\psi_0|_{L^2} + \mu^{-\epsilon}|\psi_0'|_{L^2} + \mu^{1-\epsilon}|\psi_0''|_{L^2} + |\xi_1| + |\xi_2| + \mu^{-2}|\xi_-|\big),$$

where C also depends on $|U'|_{C^4}$ and

$$\tilde{y}_B = \partial_{c_R}^2 y_B + \frac{1}{U'(x_2^c)^2}\Big(-y_B'' + \frac{g+\sigma k^2}{\mathbf{F}(k,c)}\big(y_B''(-h)y_+ - y_B''(0)y_-\big)\Big),$$

and all the norms are taken on $(c_R, x_2) \in \mathcal{I}\times[-h,0]$. Moreover, as $c_I \to 0+$, the following hold.

1. *Assume $\psi_0 \in L^2$, then for any $r \in [1,2)$, $y_B \to y_{B0}$ in $L^\infty_{x_2}L^2_{c_R}$, $y_B' \to y_{B0}'$ in $L^\infty_{x_2}L^r_{c_R}$, and $y_B'(0) \to y_{B0}'(0)$ in $L^2_{c_R}$.*

2. *Assume $\psi_0 \in H^1$, then for any $r \in [1,2)$ and $q \in [1,\infty)$, $\partial_{c_R}y_B \to \partial_{c_R}y_{B0}$ in $L^\infty_{x_2}L^r_{c_R}$, $\partial_c y_B' + \frac{1}{U'(x_2^c)}y_B'' \to \partial_c y_{B0}' + \frac{1}{U'(x_2^c)}y_{B0}''$ in $L^q_{x_2}L^r_{c_R}$, and $(\partial_c y_B' + \frac{1}{U'(x_2^c)}y_B'')(0) \to (\partial_c y_{B0}' + \frac{1}{U'(x_2^c)}y_{B0}'')(0)$ in $L^r_{c_R}$.*

3. *Assume $\psi_0 \in H^2$, then for any $r \in [1,2)$, $\tilde{y}_B$ also converges in $L^\infty_{x_2}L^r_{c_R}$ to its limit $\tilde{y}_{B0}$.*

Since y_B is holomorphic in $c \notin U([-h,0])$, $\partial_c y_B = \partial_{c_R}y_B$. From the Rayleigh equation, singularity at the level of delta mass appears in y_B'' along $U(x_2) = c_R$, $x_2 \in [-h,0]$, as $c_I \to 0+$. Therefore $\partial_c^2 y_B$ and $\partial_c y_B'$ also display such singularities which are singled out in the above estimates. The y_B'' involved in the singular terms will be substituted by using the Rayleigh equation (5.0.1a) whenever necessary.

Proof. The $L^2_{c_R,x_2}$ estimates on y_B and y_B', as well as the $L^r_{c_R}$ estimate of $y_B'(0)$ with $r \in (1,\infty)$, follow readily from (5.0.8), (5.2.2), Lemma 3.3.2, Lemma 5.1.4, Lemma 5.1.3 (with $r = 2$, $r_1 = \infty$, and $f_0 = 1$), and Lemma 5.2.1. The estimate of $y_B(0)$ is simply obtained from those of y_B and y_B'. Moreover, for the rest of the proof of the proposition we shall also need the following inequality for $r \in (1, \infty)$ which is also derived form Lemma 5.1.3 and Lemma 5.2.1 and uniform in $c_I \in [0, \rho_0]$

$$\begin{aligned} &|y_B'(0)|_{L^r_{c_R}} \leq C(\mu^{-\epsilon}|\psi_0|_{L^r} + \mu(|\xi_1| + |\xi_2|) + \mu^{-1}e^{-\mu^{-1}h}|\xi_-|), \\ &|y_B'(-h)|_{L^r_{c_R}} \leq C(\mu^{-\epsilon}|\psi_0|_{L^r} + \mu e^{-\mu^{-1}h}(|\xi_1| + |\xi_2|) + \mu^{-1}|\xi_-|). \end{aligned} \tag{5.2.3}$$

The convergence of y_B, y_B', and $y_B'(0)$ follows directly from the continuity of function **F** (Lemma 4.1.1) and the convergence of $y_\pm$ and $y_\pm'$ (Lemma 3.5.1) and y_{nh} (Lemma 5.1.3). Moreover, we also have the convergence of $y_B'(-h)$ in $L^2_{c_R}$.

In the following differentiations in c_R are all carried out for $c_I > 0$. The convergence analysis based on the convergence results of $y_\pm$ and those of y_{nh} in section 5.1 ensure that the estimates hold also for $c_I = 0+$. Directly differentiating the Rayleigh equation (5.0.1a) in c_R directly would cause worse singularity in the equation. Instead we first consider

$$D_c = U'(x_2^c)\partial_{c_R} + \partial_{x_2}, \qquad \partial_{c_R} = U'(x_2^c)^{-1}(D_c - \partial_{x_2}), \qquad [D_c, \partial_{x_2}] = 0, \tag{5.2.4}$$

where x_2^c is defined by $U(x_2^c) = c_R$ as in (3.0.6). It satisfies

$$D_c\big(\tfrac{1}{U(x_2)-c}\big) = -\tfrac{U'(x_2)-U'(x_2^c)}{(U(x_2)-c)^2}, \quad D_c^2\big(\tfrac{1}{U(x_2)-c}\big) = \tfrac{2(U'(x_2)-U'(x_2^c))^2}{(U(x_2)-c)^3} - \tfrac{U''(x_2)-U''(x_2^c)}{(U(x_2)-c)^2}, \tag{5.2.5}$$

where the singularity remains at the same level.

• **Estimating $\partial_{c_R} y_B$.** Applying D_c to (5.0.1a) and simplifying, we obtain

$$-(D_c y_B)'' + \big(k^2 + \frac{U''}{U-c}\big)D_c y_B = \frac{\psi_0'(x_2) + f_1(c, x_2)\psi_0(x_2) + \psi_1(c, x_2)}{U-c}; \tag{5.2.6a}$$

where

$$\psi_1 = \big(\frac{U''(U' - U'(x_2^c))}{U-c} - U'''\big)y_B, \quad f_1 = (U-c)D_c\big(\frac{1}{U-c}\big) = -\frac{U' - U'(x_2^c)}{U-c},$$

and boundary conditions

$$D_c y_B(-h) = \zeta_{1-} \triangleq y_B'(-h); \tag{5.2.6b}$$

$$\big(U(0)-c\big)^2 (D_c y_B)'(0) - \big(U'(0)(U(0)-c) + g + \sigma k^2\big) D_c y_B(0) = \zeta_{1+}(c) \tag{5.2.6c}$$

where

$$\begin{aligned}\zeta_{1+} =& - \xi_2 U'(x_2^c) - (U(0)-c)\psi_0(0) + \big((2U'(x_2^c) - U'(0))(U(0)-c) - g - \sigma k^2\big) y_B'(0) \\ &+ \big(k^2 (U(0)-c)^2 + U''(0)(U(0)-c) - U'(x_2^c)U'(0)\big) y_B(0).\end{aligned}$$

Let $\tilde{y}_1(c, x_2)$ and $\tilde{y}_2(c, x_2)$ be the solution to the non-homogeneous Rayleigh equation (5.0.1a), but with zero boundary values in (5.0.1b), with $\psi(c, x_2)$ replaced by ψ_1 and $\psi_0' + f_1\psi_0$, respectively. Both are given by the formula (5.0.3). Using the estimates of y_B derived in the above and apply Lemma 5.1.2, we have

$$\begin{aligned}|\tilde{y}_1|_{L^2_{c_R, x_2}} + \mu |\tilde{y}_1'|_{L^2_{c_R, x_2}} \le & C\mu\big(|y_B|_{L^2_{c_R, x_2}} + \mu^{1-\frac{\epsilon}{4}} |y_B'|_{L^2_{c_R, x_2}}\big) \\ \le & C\mu^{\frac{5}{2}-\epsilon} \big(\mu|\xi_1| + \mu|\xi_2| + |\psi_0|_{L^2} + \mu^{-1}|\xi_-|\big).\end{aligned}$$

Moreover, from Lemma 5.1.1(2b) and (5.0.1b), (5.0.7), and (5.2.2), one can compute

$$\begin{aligned}|\tilde{y}_1'(c, 0)| \le & C\mu^{-\frac{1}{2}(1+\epsilon)} (|y_B|_{L^2_{x_2}} + \mu |y_B'|_{L^2_{x_2}}) \\ &+ C\mu^2 \big(1 + \big|\log|U(0)-c|\big|\big)(|\zeta_+| + |U(0)-c|^2 |y_B'(c, 0)|),\end{aligned}$$

where $y_B(0)$ was substituted by using (5.0.1b). It along with the above estimates on y_B implies

$$|\tilde{y}_1'(0)|_{L^2_{c_R}} \le C\mu^{-\frac{1}{2}(1+\epsilon)} (|y_B|_{L^2_{c_R, x_2}} + \mu |y_B'|_{L^2_{c_R, x_2}}) + C\mu^2 (|\xi_1| + |\xi_2| + |y_B'(0)|_{L^2_{c_R}})$$

$$\leq C\mu^{1-\epsilon}\big(\mu|\xi_1| + \mu|\xi_2| + |\psi_0|_{L^2} + \mu^{-1}|\xi_-|\big).$$

The estimate at $x = -h$ based on Lemma 5.1.1(2b) is similar

$$|\tilde{y}_1'(c,-h)| \leq C\mu^{-\frac{1}{2}(1+\epsilon)}\big(|y_B|_{L^2_{x_2}} + \mu|y_B'|_{L^2_{x_2}}\big) + C\big(1 + \big|\log|U(-h) - c|\big|\big)|\xi_-|,$$

which yields

$$\begin{aligned}|\tilde{y}_1'(-h)|_{L^2_{c_R}} \leq & C\mu^{-\frac{1}{2}(1+\epsilon)}(|y_B|_{L^2_{c_R,x_2}} + \mu|y_B'|_{L^2_{c_R,x_2}}) + C|\xi_-| \\ \leq & C\mu^{1-\epsilon}\big(\mu|\xi_1| + \mu|\xi_2| + |\psi_0|_{L^2} + \mu^{-1}|\xi_-|\big).\end{aligned}$$

From the convergence of y_B and Lemma 5.1.1, as $c_I \to 0+$, we have the convergence of $\tilde{y}_1$ in $L^r_{c_R}L^\infty_{x_2}$, $\tilde{y}_1'$ in $L^r_{c_R}L^q_{x_2}$, and $\tilde{y}_1'(0)$ in $L^r_{c_R}$, for any $r \in [1,2)$ and $q \in [1,\infty)$.

Due to the smoothness of f_1, we apply Lemma 5.1.4 and Lemma 5.1.3 instead to estimate $\tilde{y}_2$

$$|\tilde{y}_2|_{L^2_{c_R,x_2}} + \mu|\tilde{y}_2'|_{L^2_{c_R,x_2}} \leq C\mu^{\frac{3}{2}-\epsilon}|\psi_0|_{H^1}, \quad |\tilde{y}_2'(-h)|_{L^2_{c_R}} + |\tilde{y}_2'(0)|_{L^2_{c_R}} \leq C\mu^{-\epsilon}|\psi_0|_{H^1}.$$

Again from Lemma 5.1.3, as $c_I \to 0+$, we have the convergence of $\tilde{y}_2$ in $L^\infty_{x_2}L^2_{c_R}$, $\tilde{y}_2'$ in $L^\infty_{x_2}L^r_{c_R}$, for any $r \in [1,2)$, and $\tilde{y}_2'(0)$ in $L^2_{c_R}$.

Finally, from (5.2.3) and (5.0.7), we have, for any $r \in (1,\infty)$,

$$|\zeta_{1-}|_{L^r_{c_R}} \leq C\big(\mu^{-\epsilon}|\psi_0|_{L^r} + \mu e^{-\mu^{-1}h}(|\xi_1| + |\xi_2|) + \mu^{-1}|\xi_-|\big),$$

$$|\zeta_{1+}|_{L^r_{c_R}} \leq C\mu^{-1}\big(|\xi_1| + |\xi_2| + \mu|\psi_0(0)| + \mu^{-1-\epsilon}|\psi_0|_{L^r} + \mu^{-2}e^{-\mu^{-1}h}|\xi_-|\big),$$

where again we substituted $y_B(0)$ by (5.0.1b) and (5.0.7). Moreover, from the convergence of y_B and y_B', we have the convergence of $\zeta_{1\pm}$ in $L^2_{c_R}$.

As $\tilde{y}_1+\tilde{y}_2$ plays the role of "y_{nh}" in the representation of $D_c y_B$ as given in Lemma 5.2.1, the above estimates imply

$$|D_c y_B|_{L^2_{c_R,x_2}} + \mu |D_c y_B'|_{L^2_{c_R,x_2}} \le C\big(\mu^{\frac{3}{2}}|\xi_1| + \mu^{\frac{3}{2}}|\xi_2| + \mu^{-\frac{1}{2}}|\xi_-| + \mu^{\frac{1}{2}-\epsilon}|\psi_0|_{L^2} + \mu^{\frac{3}{2}-\epsilon}|\psi_0'|_{L^2}\big), \tag{5.2.7}$$

where the $\psi_0(0)$ term was bounded by the other norms of ψ_0 via interpolation. The desired $L^2_{c_R,x_2}$ estimates on $\partial_{c_R} y_B$ and $\partial_{c_R} y_B'$ follow from that of y_B', (5.2.4), and the above inequality. We also obtain the $L^2_{c_R}$ estimate of $D_c y_B(0)$ from (5.2.7) which in turn yields the $L^2_{c_R}$ bound on $\partial_{c_R} y_B(0)$. The convergence of $\partial_{c_R} y_B$ is a direct consequence of those of $\tilde{y}_1$, $\tilde{y}_2$, $\zeta_{1\pm}$, and the representation formula given in Lemma 5.2.1. Moreover, we also have the convergence of $D_c y_B'|_{x_2=0,-h}$ in $L^r_{c_R}$ for any $r \in [1,2)$.

To complete the estimates on $\partial_{c_R} y_B$ and also for the next step, we also need the following inequalities which are also derived from the above estimates and Lemma 5.2.1

$$\begin{aligned}
|D_c y_B'(-h)|_{L^2_{c_R}} \le & C\big(\mu^{-1}|\psi_0|_{L^2} + |\psi_0|_{L^r} + |\psi_0'|_{L^2} + \mu^{2-\epsilon}(|\xi_1| + |\xi_2|)\big) + C\mu^{-2}|\xi_-| \\
\le & C\mu^{-\epsilon}\big(\mu^2(|\xi_1| + |\xi_2|) + \mu^{-1}|\psi_0|_{L^2} + |\psi_0'|_{L^2}\big) + C\mu^{-2}|\xi_-|,
\end{aligned}$$

$$\begin{aligned}
|D_c y_B'(0)|_{L^2_{c_R}} \le & C\big(\mu^{-1}|\psi_0|_{L^2} + |\psi_0|_{L^r} + \mu|\psi_0(0)| + |\psi_0'|_{L^2} + |\xi_1| + |\xi_2|\big) \\
\le & C\big(|\xi_1| + |\xi_2| + \mu^{-1-\epsilon}|\psi_0|_{L^2} + \mu^{-\epsilon}|\psi_0'|_{L^2} + \mu^{-\epsilon}|\xi_-|\big),
\end{aligned}$$

where the terms involving $|\psi_0(0)|$ and $|\psi_0|_{L^r}$, $r > 2$, are bounded by other norms of ψ_0.

• **Estimating $\partial^2_{c_R} y_B$.** In order to analyze $\partial^2_{c_R} y_B$, we still first apply D_c to (5.2.6). Due to the commutativity (5.2.4) between D_c and ∂_{x_2}, the Rayleigh equation (5.0.1a) and (5.2.2) imply

$$\begin{aligned}
-(D_c^2 y_B)'' + \big(k^2 + \frac{U''}{U-c}\big) D_c^2 y_B = & \frac{\psi_0'' - U^{(4)} y_B - 2U''' D_c y_B}{U-c} \\
& + 2D_c\big(\frac{1}{U-c}\big)\big(\psi_0' - D_c(U'' y_B)\big)
\end{aligned}$$

$$+ D_c^2\big(\frac{1}{U-c}\big)(\psi_0 - U''y_B).$$

We can write

$$-(D_c^2 y_B)'' + \big(k^2 + \frac{U''}{U-c}\big)D_c^2 y_B = \frac{\psi_0''(x_2) + f_2(c,x_2)\psi_0(x_2) + 2f_1(c,x_2)\psi_0'(x_2)}{U-c} + \frac{\psi_2(c,x_2)}{U-c}, \tag{5.2.8a}$$

where f_1 was defined in (5.2.6) and

$$f_2 = (U-c)D_c^2\big(\frac{1}{U-c}\big), \quad \psi_2 = -(2U''' + U''f_1)D_c y_B - (U^{(4)} + U'''f_1 + U''f_2)y_B.$$

From (5.2.5) and the assumption $U \in C^4$, it holds f_2 and f_3 are C^1 in x_2 and c_R with bounds uniform in $|c_I| \le \rho_0$. At $x_2 = -h$, one can compute using (3.4.8),

$$(D_c^2 y_B)(-h) = \big(U'(x_2^c)^2\partial_{c_R}^2 y_B + U''(x_2^c)\partial_{c_R} y_B + U'(x_2^c)\partial_{c_R} y_B' + (D_c y_B)'\big)\big|_{x_2=-h}.$$

From (5.2.2) and (5.0.1a), we can write

$$(D_c^2 y_B)(c,-h) = \zeta_{2-}(c) \triangleq \big(2(D_c y_B)' - y_B''\big)(-h) = 2(D_c y_B)'(-h) + \frac{\psi_0(-h)}{U(-h)-c}. \tag{5.2.8b}$$

At $x_2 = 0$, we write

$$\big(U(0)-c\big)^2(D_c^2 y_B)'(0) - \big(U'(0)(U(0)-c) + g + \sigma k^2\big)D_c^2 y_B(0) = \zeta_{2+}(c). \tag{5.2.8c}$$

One may compute ζ_{2+} using (5.2.4) and (5.2.6c)

$$\begin{aligned}\zeta_{2+} =& U'(x_2^c)\big(\partial_{c_R}\zeta_{1+} + 2(U(0)-c)(D_c y_B)'(0) - U'(0)D_c y_B(0)\big)\\ &+ \big(U(0)-c\big)^2(D_c y_B)''(0) - \big(U'(0)(U(0)-c) + g + \sigma k^2\big)(D_c y_B)'(0).\end{aligned}$$

On the one hand, the $U'(x_2^c)\partial_{c_R}\zeta_{1+}$ turns out to involve some of the most singular terms in ζ_{2+},

$$
\begin{aligned}
U'(x_2^c)\partial_{c_R}\zeta_{1+} = &-\xi_2 U''(x_2^c) + U'(x_2^c)\psi_0(0) \\
&+ \big(2U''(x_2^c)(U(0)-c) - U'(x_2^c)(2U'(x_2^c)-U'(0))\big)y_B'(0) \\
&+ U'(x_2^c)\big((2U'(x_2^c)-U'(0))(U(0)-c) - g - \sigma k^2\big)\partial_{c_R}y_B'(0) \\
&+ U'(x_2^c)\big(k^2(U(0)-c)^2 + U''(0)(U(0)-c) - U'(x_2^c)U'(0)\big)\partial_{c_R}y_B(0) \\
&+ \big(U'(x_2^c)(2k^2(c-U(0)) - U''(0)) - U''(x_2^c)U'(0)\big)y_B(0).
\end{aligned}
$$

We shall use (5.2.4) to replace $\partial_{c_R}y_B$ and $\partial_{c_R}y_B'$ by $D_c y_B$ and $(D_c y_B)'$, the latter of which would produce $y_B''(0)$. All those $y_B''(0)$ multiplied by $U(0)-c$ can be substituted by (5.0.1a), but we keep other $y_B''(0)$ terms in the expression. On the other hand, we use (5.2.6a) to substitute $(D_c y_B)''(0)$ in ζ_{2+}, which turns out to be rather regular due to the multiplier $(U(0)-c)^2$. Finally, we can write

$$
\begin{aligned}
\zeta_{2+} = &-\xi_2 U''(x_2^c) + f_3(c)\psi_0(0) + f_4(c)\psi_0'(0) + f_5(k,c)y_B(0) + f_6(k,c)y_B'(0) \\
&+ f_7(k,c)D_c y_B(0) + f_8(k,c)(D_c y_B)'(0) + (g+\sigma k^2)y_B''(0),
\end{aligned}
$$

where the functions $f_j(k,c,x_2)$, $j = 3,\dots,8$, are

$$
\begin{aligned}
f_3 &= (c-U(0))f_1 + 3U'(x_2^c) - U'(0), \quad f_4 = c - U(0), \\
f_5 &= k^2\big((4U'(x_2^c)-U'(0))(c-U(0))\big) + U'''(0)(U(0)-c) - 2U''(0)U'(x_2^c) \\
&\quad - U'(0)U''(x_2^c), \\
f_6 &= -k^2(U(0)-c)^2 + (2U''(x_2^c)-U''(0))(U(0)-c) - 2U'(x_2^c)(U'(x_2^c)-U'(0)), \\
f_7 &= 2\big(k^2(U(0)-c)^2 + U''(0)(U(0)-c) - U'(x_2^c)U'(0)\big), \\
f_8 &= 2\big((2U'(x_2^c)-U'(0))(U(0)-c) - g - \sigma k^2\big),
\end{aligned}
$$

and are at least C^1 in c_R and x_2.

The terms $y_B''(-h)$ in ζ_{2-} and $y_B''(0)$ in ζ_{2+} generate the most singular part of $D_c^2 y_B$ which, based on Lemma 5.2.1, takes the form

$$\begin{aligned}y_S(x_2) =& -\frac{y_B''(-h)}{y_+(-h)} y_+(x_2) + \frac{(g+\sigma k^2) y_B''(0)}{\mathbf{F}(k,c)} y_-(x_2) \\ =& \frac{g+\sigma k^2}{\mathbf{F}(k,c)} \big(- y_B''(-h) y_+(x_2) + y_B''(0) y_-(x_2) \big).\end{aligned}$$

Let

$$\tilde{y} = D_c^2 y_B - y_S.$$

Clearly, it satisfies the same non-homogeneous Rayleigh equation (5.2.8a) and boundary conditions

$$\tilde{y}(c,-h) = \tilde{\zeta}_{2-}(c) \triangleq 2(D_c y_B)'(c,-h) \tag{5.2.9}$$

$$\big(U(0)-c\big)^2 \tilde{y}'(0) - \big(U'(0)(U(0)-c) + g + \sigma k^2\big)\tilde{y}(0) = \tilde{\zeta}_{2+}(c) \triangleq \zeta_{2+} - (g+\sigma k^2) y_B''(0). \tag{5.2.10}$$

Let $\tilde{y}_3$ and $\tilde{y}_4$ be the solutions to (5.2.8a) with zero boundary values in (5.0.1b) and non-homogeneous terms

$$\frac{\psi_2}{U-c}, \quad \frac{\psi_0'' + f_2\psi_0 + 2f_1\psi_0'}{U-c},$$

respectively. Using the above estimates of y_B and $D_c y_B$ and applying Lemma 5.1.2, we obtain

$$\begin{aligned}|\tilde{y}_3|_{L^2_{c_R,x_2}} + \mu|\tilde{y}_3'|_{L^2_{c_R,x_2}} \le& C\mu\big(|y_B|_{L^2_{c_R,x_2}} + |D_c y_B|_{L^2_{c_R,x_2}} + \mu^{1-\epsilon}|y_B'|_{L^2_{c_R,x_2}} \\ &+ \mu^{1-\epsilon}|(D_c y_B)'|_{L^2_{c_R,x_2}}\big) \\ \le& C\mu^{\frac{5}{2}-\epsilon}\big(|\xi_1| + |\xi_2| + \mu^{-1}|\psi_0|_{L^2} + |\psi_0'|_{L^2} + \mu^{-2}|\xi_-|\big).\end{aligned}$$

As $c_I \to 0+$, the convergence of y_B and $D_c y_B$ implies that of ψ_2 in $L^r_{c_R} W^{1,r}_{x_2}$ for any $r \in [1,2)$. From Lemma 5.1.1(2a), we obtain the convergence of $\tilde{y}_3$ in $L^r_{c_R} L^\infty_{x_2}$.

Again we apply Lemma 5.1.4 to estimate $\tilde{y}_4$

$$|\tilde{y}_4|_{L^2_{c_R,x_2}} + \mu|\tilde{y}_4'|_{L^2_{c_R,x_2}} \leq C\mu^{\frac{3}{2}-\epsilon}\big(|\psi_0|_{L^2} + |\psi_0'|_{L^2} + |\psi_0''|_{L^2}\big).$$

As $c_I \to 0+$, Lemma 5.1.4(2a) implies that $\tilde{y}_4$ converges in $L^\infty_{x_2} L^2_{c_R}$.

The boundary values of $\tilde{y}$ satisfy

$$|\tilde{\zeta}_{2-}|_{L^2_{c_R}} \leq C\mu^{-\epsilon}\big(\mu^2(|\xi_1| + |\xi_2|) + \mu^{-1}|\psi_0|_{L^2} + |\psi_0'|_{L^2}\big) + C\mu^{-2}|\xi_-|,$$

$$\begin{aligned}|\tilde{\zeta}_{2+}|_{L^2_{c_R}} \leq& C\big(|\xi_2| + |\psi_0(0)| + |\psi_0'(0)| + \mu^{-2}\big(|y_B|_{L^2_{c_R}} + |D_c y_B|_{L^2_{c_R}} + |y_B'|_{L^2_{c_R}} \\ &+ |(D_c y_B)'|_{L^2_{c_R}}\big)\big|_{x_2=0}\big) \\ \leq& C\big(|\psi_0'(0)| + \mu^{-2}\big(|\xi_1| + |\xi_2| + \mu^{-1-\epsilon}|\psi_0|_{L^2} + \mu^{-\epsilon}|\psi_0'|_{L^2} + \mu^{-\epsilon}|\xi_-|\big)\big),\end{aligned}$$

where we also used the boundary conditions of y_B and $D_c y_B$ to express them in terms of y_B' and $D_c y_B'$ at $x_2 = 0$. As $c_I \to 0+$, the convergence of y_B and $D_c y_B$ at $x_2 = 0, -h$ implies that of $\xi_\pm$ in $L^r_{c_R}$ for any $r \in [1,2)$.

As $\tilde{y}_3 + \tilde{y}_4$ plays the role of "y_{nh}" in the representation of $D_c^2 y_B$ as given in Lemma 5.2.1, the above estimates and Lemma 5.2.1 imply

$$|\tilde{y}|_{L^2_{c_R,x_2}} + \mu|\tilde{y}'|_{L^2_{c_R,x_2}} \leq C\big(\mu^{\frac{1}{2}}|\xi_1| + \mu^{\frac{1}{2}}|\xi_2| + \mu^{-\frac{1}{2}-\epsilon}|\psi_0|_{L^2} + \mu^{\frac{1}{2}-\epsilon}|\psi_0'|_{L^2} + \mu^{\frac{3}{2}-\epsilon}|\psi_0''|_{L^2}\big),$$

where the $\psi_0'(0)$ term was bounded by the other norms of ψ_0 via interpolation. Finally, using (5.2.4) one can compute

$$\partial_{c_R}^2 = U'(x_2^c)^{-2}(D_c^2 - 2\partial_{x_2} D_c + \partial_{x_2}^2) - (U'(x_2^c))^{-3} U''(x_2^c)(D_c - \partial_{x_2}). \tag{5.2.11}$$

This relationship and the definition of $\tilde{y}_B$ and $\tilde{y}$ yield

$$\begin{aligned}\tilde{y}_B =&\partial_{c_R}^2 y_B + \frac{1}{U'(x_2^c)^2}\Big(- y_B'' + \frac{g+\sigma k^2}{\mathbf{F}(k,c)}\big(y_B''(-h)y_+ - y_B''(0)y_-\big)\Big)\\ =&\frac{1}{U'(x_2^c))^2}\Big(\tilde{y} - 2(D_c y_B)' - \frac{U''(x_2^c)}{U'(x_2^c)}(D_c y_B - y_B')\Big).\end{aligned}$$

Therefore the desired estimate on $\tilde{y}_B$ follows from those of $\tilde{y}$, y_B, and $D_c y_B$. The convergence of $\tilde{y}_B$ is also obtained much as that of $D_c y_B$. $\square$

CHAPTER 6

SOLUTIONS TO THE EULER EQUATION LINEARIZED AT SHEAR FLOWS

In this chapter, we finally return to the linearized flow of the capillary gravity water waves at the shear flow $U(x_2)$ in both the horizontally L-periodic (in x_1) case and the $x_1 \in \mathbb{R}$ case. Under the assumption (4.1.5) of the absence of singular modes for all k, we shall show that a.) inviscid damping occurs to a large component (remotely related to the rotational part) of the solutions and b.) what is left in the solutions are superpositions of non-singular modes (smooth eigenfunctions). The latter is a linear dispersive flow which is asymptotic to the linear irrotational flow for high spatial wave numbers k.

6.1 Estimating each Fourier mode of the linear solutions

Based on (2.2.6) and the formula of the inverse Laplace transform, we first derive some integral representation formulas of the linear solution $\big(\hat{v}(t,k,x_2), \hat{\eta}(t,k,x_2)\big)$ of (2.2.1) for a fixed wave number $k \neq 0$ satisfying (5.0.8). This procedure is essentially obtaining the linear solution group from contour integrals of the resolvents of the linear operator defined by the linearized water wave problem at the shear flow. Subsequently estimates of solutions are obtained using these formulas. Due to the conjugacy relation $\hat{v}(t,-k,x_2) = \overline{\hat{v}(t,k,x_2)}$ and $\hat{\eta}(t,-k) = \overline{\hat{\eta}(t,k)}$, we shall mostly work on estimates for $k > 0$ in this section, unless otherwise specified.

Recall $\mathbf{F}$ defined in (4.0.1). Denote the set of non-singular modes

$$R(k) = \{c \notin U([-h,0]) \mid \mathbf{F}(k,c) = 0\} \tag{6.1.1}$$

Throughout this section, we fix $k \neq 0$ and assume (5.0.8). We shall also use (5.0.6), possibly after choosing smaller ρ_0. The continuity of $\mathbf{F}$ and (5.0.8) imply that $R(k)$ is a finite set, which consists of only simple roots $c^{\pm}(k)$ for large k due to Lemma 4.1.2(3). We shall work on the following type of neighborhoods of $U([-h,0]) \subset \mathbb{C}$

$$\mathcal{D}_{r_1,r_2} = [-r_1 + U(-h), U(0) + r_1] + i[-r_2, r_2] \subset R(k)^c, \quad r_1, r_2 \in (0, \rho_0), \tag{6.1.2}$$

where ρ_0 is given in (5.0.8).

Recall the Laplace transform $V_2(k,c,x_2)$ of $\hat{v}_2(t,k,x_2)$, defined by (2.2.4) and (2.2.5), is the solution of the boundary value problem (2.2.6) of the Rayleigh equation, or equivalently, the solution to (5.0.1) and (5.2.2) with

$$\begin{aligned} \psi = -\hat{\omega}_0(k,x_2) = -ik^{-1}(k^2 - \partial_{x_2}^2)\hat{v}_{20}, \\ \xi_- = 0, \ \ \xi_1 = (g+\sigma k^2)\hat{\eta}_0(k), \ \ \xi_2 = -ik^{-1}\hat{v}_{20}'(k,0), \end{aligned} \tag{6.1.3}$$

and $\hat{\omega}_0(k,x_2)$, $\hat{\eta}_0(k)$ and $\hat{v}_{20}(k,x_2)$ are the Fourier transforms with respect to x_1 of the initial values $\omega_0(x)$, $\eta_0(x_1)$ and $v_{20}(x)$. The solution $V_2(k,c,x_2)$ to (2.2.6) is still given by Lemma 5.2.1 along with (5.2.2) and (6.1.3). More explicitly, if $\mathbf{F}(k,c) \neq 0$, then

$$V_2(k,c,x_2) = \frac{(g+\sigma k^2)\hat{\eta}_0(k) - \frac{i}{k}(U(0)-c)\hat{v}_{20}'(k,0)}{\mathbf{F}(k,c)} y_-(k,c,x_2) + y_{nh}(k,c,x_2), \tag{6.1.4}$$

where $y_\pm$ are solutions to the homogeneous Rayleigh equation (3.2.5) satisfying initial conditions (3.3.1) and y_{nh} the solution to (5.0.1) given by (5.0.3) with $\zeta_\pm = 0$ and $\psi = \hat{\omega}_0(k,x_2)$. The Laplace transform $\tilde{\eta}(k,c)$ of $\hat{\eta}(t,k)$ can be computed by using (2.2.7) and the boundary condition (5.0.1b) along with (5.2.2), (6.1.3), and (5.0.6)

$$\tilde{\eta}(k,c) = \frac{V_2(k,c,0) + \hat{\eta}_0(k)}{ik(U(0)-c)} = \frac{V_2'(k,c,0)(U(0)-c) + U'(0)\hat{\eta}_0(k) + \frac{i}{k}\hat{v}_{20}'(k,0)}{ik\big(U'(0)(U(0)-c) + g + \sigma k^2\big)}, \ k \neq 0. \tag{6.1.5}$$

We shall also need the following quantities

$$\begin{aligned}
&\mathbf{b}(t,k,c_*,x_2) = -(ik)Res\big(V_2 e^{-ik(c-c_*)t}, c_*\big),\\
&\mathbf{b}_S(t,k,c_*) = -(ik)Res\big(\tilde{\eta} e^{-ik(c-c_*)t}, c_*\big) = -Res\big(V_2(k,c,0)e^{-ik(c-c_*)t}/(U(0)-c), c_*\big)
\end{aligned} \tag{6.1.6}$$

where $Res(f(z), z_*)$ is the residue of a meromorphic function $f(z)$ at z_*. Apparently $\mathbf{b} = \mathbf{b}_S = 0$ unless $\mathbf{F}(k, c_*) = 0$, or equivalently $c_* \in R(k)$. The following lemma is obtained from applying the inverse Laplace transform.

Lemma 6.1.1. *Assume $U \in C^3$ and $k > 0$ satisfies (5.0.8), then for any $r_1, r_2 \in (0, \rho_0)$, we have*

$$\hat{v}_2(t,k,x_2) = \hat{v}_2^c + \hat{v}_2^p \triangleq -\frac{k}{2\pi}\oint_{\partial\mathcal{D}_{r_1,r_2}} e^{-ikct} V_2(k,c,x_2)dc + \sum_{c_*\in R(k)} e^{-ic_*kt}\mathbf{b}(t,k,c_*,x_2),$$

$$\hat{\eta}(t,k) = \hat{\eta}^c + \hat{\eta}^p \triangleq -\frac{k}{2\pi}\oint_{\partial\mathcal{D}_{r_1,r_2}} e^{-ikct}\tilde{\eta}(k,c)dc + \sum_{c_*\in R(k)} e^{-ic_*kt}\mathbf{b}_S(k,c_*).$$

From Lemma 4.1.2, $c_* \in R(k)$ implies $y_-(k,c,0) \neq 0$ and thus $F(k,c)$ is well-defined for c near c_*. In part (2), similar types of formula and estimates of $\mathbf{b}_S$ can be obtained from those of $\mathbf{b}$ and (6.1.6). In the subsequent analysis, the limits of the above contour integrals as $\mathcal{D}_{r_1,r_2}$ shrinks to $U([-h,0])$ will be taken and estimated whenever needed.

Proof. From the definition (2.2.4) and the inverse Laplace transform formula (2.2.9), we have

$$\hat{v}_2(t,k,x_2) = \frac{k}{2\pi}\int_{-\infty+i\gamma}^{+\infty+i\gamma} e^{-ikct}V_2(k,c,x_2)dc,$$

where $\gamma > 0$ is chosen such that the above integrand is analytic for $c_I > \gamma$. Apparently V_2 is analytic in $c \notin \big(U([-h,0]) \cup \{\mathbf{F}=0\}\big)$. In order to analyze V_2 for $|c| \gg 1$, we first consider y_+ and then y_{nh} for $|c| \gg 1$. From Lemma 3.1.3 and initial conditions (3.3.1), it holds that

$$0 < \liminf_{|c|\to\infty} |y_+(k,c,x_2)|/(1+|c|^2) \le \limsup_{|c|\to\infty} |y_+(k,c,x_2)|/(1+|c|^2) < \infty.$$

Along with (6.1.3) and Lemma 3.3.2 which yields the boundedness of y_- for $|c| \gg 1$, it implies

$$\limsup_{|c|\to\infty} |c||y_{nh}(k,c,x_2)| < \infty.$$

From Lemma 4.1.2(2) and again Lemma 3.3.2, we obtain[1]

$$\limsup_{|c|\to\infty} |c||V_2(k,c,x_2)| < \infty.$$

As $|e^{-ikct}| = e^{kt \operatorname{Im} c}$, the Cauchy integral theorem yields

$$\int_{-\infty-i\gamma}^{+\infty-i\gamma} e^{-ikct} V_2(k,c,x_2)dc = 0,$$
$$\hat{v}_2(t,k,x_2) = \frac{k}{2\pi}\Big(\int_{-\infty+i\gamma}^{+\infty+i\gamma} - \int_{-\infty-i\gamma}^{+\infty-i\gamma}\Big) e^{-ikct} V_2(k,c,x_2)dc.$$

The desired expression of $\hat{v}_2$ follows immediately from the residue calculation.

Concerning $\hat{\eta}$, one first obtains

$$\hat{\eta}(t,k) = \frac{k}{2\pi}\Big(\int_{-\infty+i\gamma}^{+\infty+i\gamma} - \int_{-\infty-i\gamma}^{+\infty-i\gamma}\Big) e^{-ikct} \tilde{\eta}(k,c)dc.$$

Using the first expression in (6.1.5), the desired formula for $\hat{\eta}$ is derived via the same arguments as in the above. In particular, the $\hat{\eta}_0$ term does not contribute to the residue as $R(k)$ is away from $U([-h,0])$ due to assumption (5.0.8). □

From the divergence free condition on the velocity, it holds that the Fourier transform (in x_1) of the velocity field satisfies $ik\hat{v}_1 = -\hat{v}_2'$. Therefore, we have

[1]Through a more careful analysis we may obtain a Taylor expansion of V_2 in terms of $\frac{1}{c}$ as $|c| \to \infty$.

Corollary 6.1.1.1. *Under the assumptions of Lemma 6.1.1, we have*

$$\hat{v}_1(t,k,x_2) = \hat{v}_1^c + \hat{v}_1^p \triangleq -\frac{i}{2\pi}\oint_{\partial\mathcal{D}_{r_1,r_2}} e^{-ikct}V_2'(k,c,x_2)dc + \sum_{c_*\in R(k)} \frac{i}{k}e^{-ic_*kt}\mathbf{b}'(t,k,c_*,x_2).$$

In the following lemma, we give some basic properties of $\mathbf{b}(t,k,c,x_2)$ and $\mathbf{b}_S(t,k,c)$ at some $c_* \in R(k)$. Since c_* is away from $U([-h,0])$ and $\mathbf{F}(k,\cdot)$ and $F(k,\cdot)$ are analytic in a neighborhood of c_*, the assumption (5.0.8) is not needed.

Lemma 6.1.2. *Assume $U \in C^{l_0}$ and $k > 0$. Let $c_* \in R(k)$ be a root of $\mathbf{F}(k,\cdot)$ (or equivalently, of $F(k,\cdot)$ defined in (4.0.2)) of degree $n \geq 1$, then the following hold.*

1. *$e^{-ikc_*t}\mathbf{b}(t,k,c_*,x_2)$ is a solution to (2.2.3).*

2. *$\mathbf{b}(t,k,c_*,x_2)$ is a linear combination of $t^{l_1}\partial_c^{l_2}y_-(k,c_*,x_2)$, $0 \leq l_1+l_2 \leq l = n-1$, and $\mathbf{b}_S(t,k,c_*)$ a linear combination of t^l, $0 \leq l \leq n-1$, with coefficients depending on k and c_*. The leading terms of $\mathbf{b}(t,k,c_*,x_2)$ with $l_1+l_2 = n-1$ are given by*

$$\frac{(n!)(-ik)^{l_1+1}}{l_1!l_2!\partial_c^n F(k,c_*)}\Big((g+\sigma k^2)\hat{\eta}_0(k) - \frac{i}{k}(U(0)-c_*)\hat{v}_{20}'(k,0) \\ + \frac{(U(0)-c_*)^2}{y_-(k,c_*,0)}\int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c_*,x_2')}{U(x_2')-c_*}dx_2'\Big)\frac{t^{l_1}\partial_c^{l_2}y_-(k,c_*,x_2)}{y_-(k,c_*,0)}, \tag{6.1.7}$$

   and the leading terms of $ik(U(0)-c_*)\mathbf{b}_S(t,k,c_*)$ is given by the above expression evaluated at $x_2 = 0$.

3. *If c_* is a simple root of $\mathbf{F}(k,\cdot)$, i.e., $n = 1$, then $\mathbf{b}$ and $ik(U(0)-c_*)\mathbf{b}_S$ are given by the above expression and there exists $C > 0$ determined only by $|U'|_{C^{l_0-1}}$ and $|(U')^{-1}|_{C^0}$ such that*

$$|\partial_{x_2}^{n_2}\mathbf{b}(k,c_*,x_2)| \leq C|\partial_c F(k,c_*)|^{-1}\Big(|k|\mu^{-2}|\hat{\eta}_0(k)| + (1+|c_*|)|\hat{v}_{20}'(k,0)| \\ + \frac{|k|\mu^{\frac{3}{2}}e^{\mu^{-1}h}(1+|c_*|^2)|\hat{\omega}_0(k)|_{L^2_{x_2}}}{dist(c_*,U([-h,0]))|y_-(k,c_*,0)|}\Big)\Big|\frac{\mu^{1-n_2}e^{\mu^{-1}(x_2+h)}}{y_-(k,c_*,0)}\Big|,$$

for any $n_2 \in [0, l_0]$*, where we recall* $\mu = (1+k^2)^{-\frac{1}{2}}$.

Proof. According to Lemma 4.1.1(4), $\mathbf{F}(k, c_*) = 0$ implies $y_-(k, c_*, 0) \neq 0$ and thus $F(k, c)$ is analytic in c for c near c_* and the degree of c_* as a root of both $F(k, \cdot)$ and $\mathbf{F}(k, \cdot)$ is $n \geq 1$. By the definition of $R(k)$ and the analyticity of $F(k, \cdot)$, $c_* \in R(k)$ is an isolated root of $F(k, \cdot)$. Let $1 \gg R > 0$ such that there are no other roots of $F(k, \cdot)$ in the disk $B(c_*, R)$ centered at c_* with radius R. Using the fact that $V_2(k, c, x_2)$ solves (2.2.6), one may compute

$$\begin{aligned}
&(\partial_t + ikU)(k^2 - \partial_{x_2}^2)(e^{-ikc_*t}\mathbf{b}(t, k, c_*, x_2)) \\
=&\frac{\partial_t + ikU}{2\pi} \oint_{\partial B(c_*, R)} ke^{-ikct}(k^2 - \partial_{x_2}^2)V_2(k, c, x_2)dc \\
=&\frac{ik^2}{2\pi} \oint_{\partial B(c_*, R)} e^{-ikct}(U - c)(k^2 - \partial_{x_2}^2)V_2(k, c, x_2)dc \\
=&-\frac{ik^2U''}{2\pi} \oint_{\partial B(c_*, R)} e^{-ikct}V_2(k, c, x_2)dc \\
=&-ikU''e^{-ikc_*t}\mathbf{b}(t, k, c_*, x_2),
\end{aligned}$$

and thus (2.2.3a) is satisfied. Similar calculation also proves the boundary condition (2.2.3b) at $x_2 = 0$. The zero boundary value at $x_2 = -h$ is obvious from that of V_2 at $x_2 = -h$. Therefore statement (1) is proved.

To analyze $\mathbf{b}$ in more details, let

$$F_1(c) = (c - c_*)^{-n}F(k, c) \implies F_1(c_*) = \partial_c^n F(k, c_*)/(n!) \neq 0,$$

and

$$\tilde{y}(c, x_2) = y_+(k, c, x_2) - \frac{(U(0) - c)^2}{(g + \sigma k^2)y_-(k, c, 0)}y_-(k, c, x_2).$$

From the initial conditions (3.3.1) of $y_\pm$, it is straight forward to verify

$$\tilde{y}(c,0) = 0, \quad \tilde{y}'(c,0) = -\frac{F(k,c)}{g+\sigma k^2} = O(|c-c_*|^n)$$
$$\Longrightarrow \; y_+(x_2) = \frac{(U(0)-c)^2 y_-(x_2)}{(g+\sigma k^2) y_-(0)} + O(|c-c_*|^n).$$

Using the above expression to substitute $y_+(k,c,x_2)$ in the residue (in the definition of $\mathbf{b}$) and observing that the $O(|c-c_*|^n)$ term cancels the singularity of $y_+(k,c,-h)$ for $|c-c_*| \ll 1$ which results in an analytic function contributing nothing to the residue, we have

$$\begin{aligned}
&Res(y_{nh}(k,c,x_2)e^{-ik(c-c_*)t}, c_*) \\
=&Res\Big(\frac{(U(0)-c)^2 y_-(k,c,x_2)e^{-ik(c-c_*)t}}{(g+\sigma k^2)y_-(k,c,0)y_+(k,c,-h)} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c,x_2')}{U(x_2')-c} dx_2', c_*\Big).
\end{aligned}$$

From definitions (5.0.3), (4.0.1), and (4.0.2), of y_{nh}, $\mathbf{F}$, and F, (6.1.3), we have

$$\begin{aligned}
&Res(y_{nh}(k,c,x_2)e^{-ik(c-c_*)t}, c_*) \\
=&Res\Big(\frac{(U(0)-c)^2 y_-(k,c,x_2)e^{-ik(c-c_*)t}}{(c-c_*)^n F_1(c) y_-(k,c,0)^2} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c,x_2')}{U(x_2')-c} dx_2', c_*\Big) \\
=&\frac{1}{(n-1)!}\partial_c^{n-1}\Big(\frac{(U(0)-c)^2 y_-(k,c,x_2)e^{-ik(c-c_*)t}}{F_1(c) y_-(k,c,0)^2} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c,x_2')}{U(x_2')-c} dx_2'\Big)\Big|_{c=c_*} \\
=&\sum_{l=0}^{n-1} \frac{\partial_c^l y_-(k,c_*,x_2)}{l!(n-l-1)!}\partial_c^{n-l-1}\Big(\frac{(U(0)-c)^2 e^{-ik(c-c_*)t}}{F_1(c) y_-(k,c,0)^2} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c,x_2')}{U(x_2')-c} dx_2'\Big)\Big|_{c=c_*}.
\end{aligned}$$

Therefore this residue is a linear combination of $t^{l_1}\partial_c^{l_2} y_-(k,c_*,x-2)$, $0 \le l_1 + l_2 \le n-1$, with coefficients depending on k and c_*. The coefficients for $l_1 + l_2 = n-1$ are given by

$$\begin{aligned}
&\frac{(\partial_c^{l_1} e^{-ik(c-c_*)t})|_{c=c_*}}{l_1! l_2! t^{l_1}} \frac{(U(0)-c_*)^2}{F_1(c_*) y_-(k,c_*,0)^2} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c_*,x_2')}{U(x_2')-c_*} dx_2' \\
=&\frac{(n!)(-ik)^{l_1}(U(0)-c_*)^2}{l_1! l_2! \partial_c^n F(k,c_*) y_-(k,c_*,0)^2} \int_{-h}^0 \frac{(y_-\hat{\omega}_0)(k,c_*,x_2')}{U(x_2')-c_*} dx_2'.
\end{aligned}$$

The contributions of the terms involving $\eta_0(k)$ and $\hat{v}_{20}'(k,0)$ can be analyzed similarly (actually simpler as y_+ is not involved) and we obtain the desired statement (2) on the form of $\mathbf{b}$ and $\mathbf{b}_S$.

If $c_* \in R(k)$ is a simple root of $\mathbf{F}(k, \cdot)$, i.e. $n = 1$, then $\mathbf{b}$ and $\mathbf{b}_s$ have only one term with $l_1 = l_2 = 0$ and are constants in t as given in statement (2). It along with Lemma 3.3.2 readily leads to its estimate. □

Corollary 6.1.2.1. *$\hat{v}_2^c$ is also a solution to (2.2.3). Moreover if c_* is a simple root of $F(k, \cdot)$, then the corresponding eigenvalue $-ikc_*$ is algebraically simple in the subspace of the k-th Fourier modes.*

Based on the above lemmas, it is natural to define

$$\begin{aligned} \mathbf{P}(k, c_*) : (\hat{v}_0, \hat{\eta}_0) \to \big(i\mathbf{b}(0, k, c_*, \cdot)'/k, \mathbf{b}(0, k, c_*, \cdot), \mathbf{b}_S(0, k, c_*)\big), \\ \mathbf{X}(k, c_*) = range(\mathbf{P}(k, c_*)). \end{aligned} \tag{6.1.8}$$

The following lemma gives that $\mathbf{P}(k, c_*)$ defines the invariant spectral projection to the eigenspace $\mathbf{X}(k, c_*)$ of $-ikc_*$ spanned by $\partial_c^l y_-(k, c_*, \cdot)$, $0 \le l \le n-1$.

Lemma 6.1.3. *Assume the same conditions as in Lemma 6.1.2, then*

$$\begin{aligned} \mathbf{X}(k, c_*) = & span\big\{\big(i\partial_c^l y_-'(k, c_*, \cdot)/k, \partial_c^l y_-(k, c_*, \cdot), \partial_c^l y_-(k, c_*, 0)/(ik(U(0) - c_*)\big) \\ & \mid l = 0, \ldots, n-1\big\}, \end{aligned}$$

is an invariant subspace of (2.2.1) and

$$\mathbf{P}(k, c_*) : (\hat{v}_0, \hat{\eta}_0) \to \big(i\mathbf{b}(0, k, c_*, \cdot)'/k, \mathbf{b}(0, k, c_*, \cdot), \mathbf{b}_S(0, k, c_*)\big)$$

is an invariant projection operator of (2.2.1) to $\mathbf{X}(k, c_*)$ with

$$\ker\big(\Sigma_{c_* \in R(k)} \mathbf{P}(k, c_*)\big) = \big\{\big(\hat{v}^c(0, k, \cdot), \hat{\eta}^c(0, k)\big) \mid \textit{ all initial values } \hat{v}_0(k, \cdot), \eta_0(k)\big\}.$$

Proof. The statement of the lemma is rather standard in the operator calculus and Laplace transform, while constructing solutions to (2.2.1) using Laplace transform is equivalent to using contour integrals of the resolvent operators in the complex spectral plane. We shall only outline the proof and skip some details.

Due to the translation invariance in t of solutions to (2.2.1), the $t = 0$ in the definition of $\mathbf{X}(k, c_*)$ can be replaced by any $t \in \mathbb{R}$. From Lemma 6.1.2, all solutions $(i\mathbf{b}'/k, \mathbf{b}, \mathbf{b}_S)$ are polynomials of t of degree no more than $n - 1$. It is standard to show inductively that $\mathbf{X}(k, c_*)$ consists of all possible coefficients of t^l, which can be computed to be generated by $\partial_c^l y_-(k, c_*, \cdot)$, $0 \leq l \leq n - 1$, using (6.1.7) and the relationship between $\mathbf{b}$ and $\mathbf{b}_S$. The invariance of $\mathbf{X}(k, c_*)$ under (2.2.1) is due to the fact that $(i\mathbf{b}/k, \mathbf{b}, \mathbf{b}_s)$ are solutions to (2.2.1). To show $\mathbf{P}(k, c_*)^2 = \mathbf{P}(k, c_*)$, let $(\hat{u}_0, \hat{\nu}_0) = \mathbf{P}(k, c_*)(\hat{v}_0, \hat{\eta}_0) \in \mathbf{X}(k, c_*)$. With this initial value, the solution $(\hat{u}(t), \hat{\nu}(t))$ is simply the $(i\mathbf{b}/k, \mathbf{b}, \mathbf{b}_S)$ component of the solution with the initial value $(\hat{v}_0, \hat{\eta}_0)$. Hence $(\hat{u}(t), \hat{\nu}(t))$ takes the form given in Lemma 6.1.2(2). Its Laplace transform is analytic at all $c \neq c_*$ and thus the $(i\mathbf{b}/k, \mathbf{b}, \mathbf{b}_S)$ component of $(\hat{u}(t), \hat{\nu}(t))$ is equal to itself. Therefore we obtain $\mathbf{P}(k, c_*)(\hat{u}_0, \hat{\nu}_0) = (\hat{u}_0, \hat{\nu}_0)$. Finally the description of the kernel of $\sum_{c_* \in R(k)} \mathbf{P}(k, c_*)$ is obvious due to the fact that both $(i\mathbf{b}/k, \mathbf{b}, \mathbf{b}_S)$ and $(\hat{v}^c(t), \hat{\eta}^c(t))$ are solutions. □

Remark 6.1.1. *In particular, if*

$$\hat{v}_{20}(k, x_2) = y_-(k, c_*, x_2), \quad \hat{\eta}_0 = y_-(k, c_*, 0)/\big(ik(U(0) - c_*\big),$$

then straight forward verification yields

$$V_2(k, c, x_2) = \frac{y_-(k, c_*, x_2)}{ik(c_* - c)}, \quad \hat{v}^c = 0,$$
$$\mathbf{b} = y_-(k, c_*, x_2) = \hat{v}_{20}, \quad \mathbf{b}_S = \frac{y_-(k, c_*, 0)}{ik(U(0) - c_*)} = \hat{\eta}_0.$$

From Lemma 2.2.1, $-ic_*k$ is an eigenvalue (with the above eigenfunctions generated by $y_-(k, c_*, x_2)$) of the linearized capillary gravity water wave at the shear flow, which has to be geometrically simple when restricted to the k-th Fourier mode in x_1. Its algebraic mul-

tiplicity is equal to the degree of the root c_* of $\mathbf{F}(k,\cdot)$. The eigenfunctions of the linearized irrotational capillary gravity wave are generated by $\frac{1}{k}\sinh k(x_2+h)$. From Lemma 3.1.2(1) (with $\rho = O(k^{-\frac{5}{2}})$, $s = 0$, $C_0 = 0$, and $\Theta_1 = \Theta_2 = \sinh$) and Lemma 4.1.2(3), it is straight forward to estimate that, after normalizing the L^2 norm of v_2 to be 1, the L^2 and H^1 differences in the v and η components, respectively, between the eigenfunctions of (2.2.1) and the irrotational capillary gravity waves linearized at zero is of order $O(k^{-\frac{3}{2}})$ as $|k| \to \infty$.

In the rest of this section we consider $\hat{v}^c(t,k,x_2)$ and $\hat{\eta}^c(t,k)$. We shall always work on $c \in [U(-h)-\rho_0, U(0)+\rho_0] + i[-\rho_0,\rho_0]$. We first present some properties of V_2 and $\tilde{\eta}$. Let us keep in mind that for analytic functions, ∂_c and ∂_{c_R} are equivalent.

Lemma 6.1.4. *It holds that V_2 and $\tilde{\eta}$ are analytic in $c \in \mathbb{C} \setminus \big(U([-h,0]) \cup R(k)\big)$ and satisfy*

$$V_2(-k,\bar{c},x_2) = \overline{V_2(k,c,x_2)}, \quad \tilde{\eta}(-k,\bar{c},x_2) = \overline{\tilde{\eta}(k,c,x_2)}.$$

Assume $U \in C^4$ and (5.0.8), then the following hold.

1. *For any $\epsilon > 0$, there exists $C > 0$ determined only by ϵ, F_0, ρ_0, $|U'|_{C^3}$, and $|(U')^{-1}|_{C^0}$ (independent of $k \in \mathbb{R}$) such that for any $c_I \in [0,\rho_0]$,*

$$|\tilde{\eta}|_{L^2_{c_R}} \leq C\big(|k|^{-1}\mu|\hat{\eta}_0(k)| + |k|^{-2}\mu^2|\hat{v}_{20}'(k,0)| + |k|^{-1}\mu^{2-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}\big),$$

$$\begin{aligned}|\partial_{c_R}\tilde{\eta}|_{L^2_{c_R}} \leq & C\big(|k|^{-1}|\hat{\eta}_0(k)| + |k|^{-2}\mu^2|\hat{v}_{20}'(k,0)| + |k|^{-1}\mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} \\ & + |k|^{-1}\mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big),\end{aligned}$$

$$\begin{aligned}&|V_2|_{L^2_{c_R,x_2}} + \mu|V_2'|_{L^2_{c_R,x_2}} + \mu^{\frac{3}{2}}|V_2'(0)|_{L^2_{c_R}} + \mu^{\frac{1}{2}}|V_2(0)|_{L^2_{c_R}} \\ \leq & C\big(\mu^{\frac{1}{2}}|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{5}{2}}|\hat{v}_{20}'(k,0)| + \mu^{\frac{3}{2}-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}\big),\end{aligned}$$

$$
\begin{aligned}
&|\partial_{c_R} V_2|_{L^2_{c_R,x_2}} + \mu|\partial_{c_R} V_2' + \tfrac{V_2''}{U'}|_{L^2_{c_R,x_2}} + \mu^{\frac{3}{2}}\big|\big(\partial_{c_R} V_2' + \tfrac{V_2''}{U'}\big)(0)\big|_{L^2_{c_R}} \\
&+ \mu^{\frac{1}{2}}|\partial_{c_R} V_2(0)|_{L^2_{c_R}} \\
\le& C\big(\mu^{-\frac{1}{2}}|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{3}{2}}|\hat{v}_{20}'(k,0)| + \mu^{\frac{1}{2}-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{\frac{3}{2}-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big),
\end{aligned}
$$

and if $U \in C^5$, then

$$
\begin{aligned}
|\tilde{V}_2|_{L^2_{c_R,x_2}} \le& C\big(\mu^{-\frac{3}{2}}|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{1}{2}}|\hat{v}_{20}'(k,0)| + \mu^{-\frac{1}{2}-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} \\
&+ \mu^{\frac{1}{2}-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}} + \mu^{\frac{3}{2}-\epsilon}|\hat{\omega}_0''(k)|_{L^2_{x_2}}\big),
\end{aligned}
$$

where C also depends on $|U'|_{C^4}$ and

$$
\tilde{V}_2(x_2) = \partial_{c_R}^2 V_2(x_2) - \frac{V_2''(x_2)}{U'(x_2)^2} + \frac{g+\sigma k^2}{\mathbf{F}(k,c)}\Big(\frac{V_2''(-h)}{U'(-h)^2}y_+(x_2) - \frac{V_2''(0)}{U'(0)^2}y_-(x_2)\Big),
$$

and all the norms are taken on $(c_R, x_2) \in [U(-h)-\rho_0, U(0)+\rho_0] \times [-h, 0]$.

2. *As $c_I \to 0+$, on $[-r_1 + U(-h), r_1]$*

$$
V_{20}(k, c_R, x_2) \triangleq \lim_{c_I \to 0+} V_2(k, c_R + ic_I, x_2), \quad \tilde{\eta}_0(k, c_R, x_2) \triangleq \lim_{c_I \to 0+} \tilde{\eta}(k, c_R + ic_I, x_2)
$$

exist and the following hold.

   (a) *Assume $\hat{\omega}_0(k) \in L^2$, then for any $r \in [1,2)$, $V_2 \to V_{20}$ in $L^\infty_{x_2}L^2_{c_R}$, $V_2' \to V_{20}'$ in $L^\infty_{x_2}L^r_{c_R}$, and $V_2'(0) \to V_{20}'(0)$ and $\tilde{\eta} \to \tilde{\eta}_0$ in $L^2_{c_R}$.*

   (b) *Assume $\hat{\omega}_0(k) \in H^1$, then for any $r \in [1,2)$ and $q \in [1,\infty)$, $\partial_{c_R} V_2 \to \partial_{c_R} V_{20}$ in $L^\infty_{x_2}L^r_{c_R}$, $\partial_{c_R} V_2' + \frac{V_2''}{U'} \to \partial_{c_R} V_{20}' + \frac{V_{20}''}{U'(x_2)}$ in $L^q_{x_2}L^r_{c_R}$, and $\big(\partial_{c_R} V_2' + \frac{V_2''}{U'}\big)(0) \to \big(\partial_{c_R} V_{20}' + \frac{V_{20}''}{U'}\big)(0)$ and $\partial_{c_R}\tilde{\eta} \to \partial_{c_R}\tilde{\eta}_0$ in $L^r_{c_R}$.*

   (c) *Assume $\hat{\omega}_0(k) \in H^2$, then for any $r \in [1,2)$, $\tilde{V}_2$ converges to its limit $\tilde{V}_{20}$ in $L^\infty_{x_2}L^r_{c_R}$.*

Compared to Proposition 5.2.2, the modifications in the definition of $\tilde{V}_2$ is to make it analytic in c which will make it more convenient in applying Lemma 6.1.5 in the below.

Proof. The estimates of V_2, V_2', $V_2'(0)$, $\partial_{c_R} V_2$, and their convergences are all direct corollaries of (6.1.3) and Proposition 5.2.2. The estimate of $\tilde{\eta}$ and its convergence follows from the second expression of (6.1.5) and the above properties of V_2.

We also notice that, compared to Proposition 5.2.2, in the definition of $\tilde{V}_2$ as well as in the estimate related to $\partial_{c_R} V_2'$, the $U'(x_2^c)$ in front of V_2'', $V_2''(-h)$, and $V_2''(0)$ had been replaced by $U'(x_2)$, $U'(-h)$, and $U'(0)$, respectively. This modification brings at most minor changes to the upper bounds. In fact,

$$\begin{aligned} \big|\big(U'(x_2^c)^{-n} - U'(x_2)^{-n}\big)V_2''\big|_{L^2_{c_R, x_2}} \le & C\big||U(x_2) - c||V_2''|\big|_{L^2_{c_R, x_2}} \\ \le & C\big(\mu^{-2}|V_2|_{L^2_{c_R, x_2}} + |\hat{\omega}_0(k)|_{L^2_{x_2}}\big), \end{aligned}$$

for $n = 1, 2$, where the Rayleigh equation was also used. This error bound and the estimate on V_2 are then used to obtain the desired inequality on $\partial_{c_R} V_2'$. The term $\frac{V_2''}{U'(x_2)^2}$ in $\tilde{V}_2$ is handled by the same argument. Similarly,

$$\begin{aligned} \big|\big(U'(x_2^c)^{-1} - U'(0)^{-1}\big)V_2''(0)\big|_{L^2_{c_R}} \le & C\big||U(0) - c||V_2''(0)|\big|_{L^2_{c_R}} \\ \le & C\big(\mu^{-2}|V_2(0)|_{L^2_{c_R}} + |\hat{\omega}_0(k, 0)|\big), \end{aligned}$$

and this along with the estimate on $V_2(0)$ yields the estimate on $\big(\partial_{c_R} V_2' + \frac{V_2''}{U'}\big)(0)$. It remain the consider the modifications to the correction terms in $\tilde{V}_2$ at $x_2 = -h$ and $x_2 = 0$. Similarly,

$$\big|\big(U'(x_2^c)^{-2} - U'(0)^{-2}\big)V_2''(0)y_-(x_2)\big|_{L^2_{c_R, x_2}} \le C\big(\mu^{-2}|V_2(0)|_{L^2_{c_R}} + |\hat{\omega}_0(k, 0)|\big)|y_-|_{L^\infty_{c_R} L^2_{x_2}}$$

which is controlled using $|y_-|_{L^\infty_{c_R} L^2_{x_2}} \le C\mu^{\frac{3}{2}} e^{\frac{h}{\mu}}$ due to Lemma 3.3.2. The last remaining modification from $U'(x_2^c)^{-2}$ to $U'(-h)^{-2}$ can be justified by the same argument (even easier as $V_2(-h) = 0$.)

Finally we consider $\partial_{c_R}\tilde{\eta}$ for in $c \in \mathcal{D}_{\rho_0,\rho_0}$. From (6.1.5), one may compute

$$
\begin{aligned}
\partial_{c_R}\tilde{\eta} =& \tfrac{1}{ik}\big(U'(0)(U(0)-c)+g+\sigma k^2\big)^{-2} \\
& \quad \times \Big(\partial_{c_R}V_2'(0)(U(0)-c)\big(U'(0)(U(0)-c)+g+\sigma k^2\big) \\
& - (g+\sigma k^2)V_2'(0) + U'(0)\big(U'(0)\hat{\eta}_0(k) + \tfrac{i}{k}\hat{v}_{20}'(k,0)\big)\Big) \\
=& \tfrac{1}{ik}\big(U'(U-c)+g+\sigma k^2\big)^{-2} \\
& \times \Big(\big(\partial_{c_R}V_2' + \tfrac{V_2''}{U'} - \tfrac{1}{U'}\big(k^2 V_2 + \tfrac{U''V_2+\hat{\omega}_0}{U-c}\big)\big)(U-c)\big(U'(U-c)+g+\sigma k^2\big) \\
& - (g+\sigma k^2)V_2' + U'\big(U'\hat{\eta}_0(k) + \tfrac{i}{k}\hat{v}_{20}'(k,0)\big)\Big)\Big|_{x_2=0}
\end{aligned}
$$

where we used the Rayleigh equation (5.0.1a) in the last step. Therefore from Proposition 5.2.2 we have, for any $k \in \mathbb{R}$ and $c_I \in [0, r_2]$,

$$
\begin{aligned}
|\partial_{c_R}\tilde{\eta}|_{L^2_{c_R}} \le& C\big(|k|^{-1}\mu^4|\hat{\eta}_0(k)| + |k|^{-2}\mu^4|\hat{v}_{20}'(k,0)| + |k|^{-1}\mu^2\big|\big(\partial_{c_R}V_2' + \tfrac{V_2''}{U'}\big)(0)\big|_{L^2_{c_R}} \\
& + |k|^{-1}\mu^2|(k^2(U(0)-c)+U''(0))V_2(0) + \hat{\omega}(k,0)|_{L^2_{c_R}} + |k|^{-1}\mu^2|V_2'(0)|_{L^2_{c_R}}\big) \\
\le& C\big(|k|^{-1}|\hat{\eta}_0(k)| + |k|^{-2}\mu^2|\hat{v}_{20}'(k,0)| + |k|^{-1}\mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} \\
& + |k|^{-1}\mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}} + |k|^{-1}\mu^2|\hat{\omega}_0(k,0)|\big).
\end{aligned}
$$

The last terms can be controlled by the previous two terms, which completes the estimate on $\partial_{c_R}\tilde{\eta}$. The convergence of $\partial_{c_R}\eta$ also follows from those of $V_2(0)$, $V_2'(0)$ and $\big(\partial_{c_R}V_2' + \frac{1}{U'(x_2^c)}V_2''\big)(0)$. $\square$

The following lemma will be used in the decay estimates.

Lemma 6.1.5. *Suppose $n \ge 0$ is an integer, $q \in [2,\infty]$, $f(c)$ and $f_1(c)$ are analytic functions on*

$$\mathcal{D} \setminus \mathcal{I}_0 \subset \mathbb{C}, \ \ \textit{where}\ \ \mathcal{D} = \mathcal{I} + i[-\rho,\rho], \ \ \mathcal{I}_0 \subsetneq \mathcal{I} = [b_1,b_2] \subset \mathbb{R}, \ \ \rho > 0,$$

and there exists $M > 0$ such that $|(f^{(n)} - f_1)(\cdot + ic_I)|_{L^{\frac{q}{q-1}}(\mathcal{I})} \le M$ for all $0 < |c_I| \le \rho$, then there exists $C > 0$ depending only on $b_2 - b_1$ such that, for any $k \neq 0$,

$$\Big| \oint_{\partial\mathcal{D}} e^{-ickt}\big(t^n f(c) - (ik)^{-n} f_1(c)\big)dc\Big|_{L^q_t(\mathbb{R})} \le C|k|^{-n-\frac{1}{q}} M.$$

Proof. Integrating by parts we have, for any $0 < |r| \le \rho$,

$$\begin{aligned}\int_{\mathcal{I}+ir} t^n e^{-ickt} f(c)dc =& e^{rkt}\Big((ik)^{-n}\int_{\mathcal{I}} e^{-ic_R kt} f^{(n)}(c_R + ir)dc_R \\ &- \sum_{l=1}^{n} t^{n-l}(ik)^{-l} f^{(l-1)}(c)\big|_{c=b_1+ir}^{c=b_2+ir}\Big).\end{aligned}$$

For any $T > 0$, the $L^{\frac{q}{q-1}} \to L^q$ boundedness (for $q \in [2, \infty]$) of the Fourier transform implies

$$\begin{aligned}\Big|\int_{\mathcal{I}} e^{-ic_R kt}(f^{(n)} - f_1)(c_R + ir)dc_R\Big|_{L^q_t([-T,T])} \le& C|k|^{-\frac{1}{q}}|(f^{(n)} - f_1)(\cdot + ir)|_{L^{\frac{q}{q-1}}(\mathcal{I})} \\ \le& C|k|^{-\frac{1}{q}} M.\end{aligned}$$

From this inequality and the Cauchy integral theorem, we obtain, for any $r \in (0, \rho]$,

$$\begin{aligned}&\Big| \oint_{\partial\mathcal{D}} e^{-ickt}\big(t^n f(c) - (ik)^{-n} f_1(c)\big)dc\Big|_{L^q_t([-T,T])} \\ =&\Big| \oint_{\partial(\mathcal{I}+i[-r,r])} e^{-ickt}\big(t^n f(c) - (ik)^{-n} f_1(c)\big)dc\Big|_{L^q_t([-T,T])} \\ \le& C|k|^{-n-\frac{1}{q}} e^{r|k|T} M + \Big|\Big(\int_{b_1+ir}^{b_1-ir} + \int_{b_2-ir}^{b_2+ir}\Big) e^{-ickt}\big(t^n f(c) - (ik)^{-n} f_1(c)\big)dc \\ &+ \sum_{l=1}^{n} t^{n-l}(ik)^{-l}\big(e^{rkt} f^{(l-1)}(c)\big|_{c=b_1+ir}^{c=b_2+ir} - e^{-rkt} f^{(l-1)}(c)\big|_{c=b_1-ir}^{c=b_2-ir}\big)\Big|_{L^q_t([-T,T])}.\end{aligned}$$

Letting $r \to 0$, the analyticity assumption of f and f_1 implies all those terms on the vertical boundary of $\mathcal{D}$ vanish and the above estimates on the integrals along the horizontal edges yield

$$\Big| \oint_{\partial\mathcal{D}} t^n e^{-ickt}\big(t^n f(c) - (ik)^{-n} f_1(c)\big)dc\Big|_{L^q_t([-T,T])} \le C|k|^{-n-\frac{1}{q}} M.$$

The lemma follows by letting $T \to +\infty$. $\square$

Remark 6.1.2. *In the following applications of this lemma, we often use the L^2 norm to control the $L^{\frac{q}{q-1}}$ norm. This leads to fact that the regularity requirements in x_1 (i.e. the exponents of k) may not be close to optimal.*

Applying the above lemma, we first obtain the decay of $\hat{v}^c(t, k, x_2)$ and $\hat{\eta}^c(t, k)$.

Lemma 6.1.6. *Assume $U \in C^4$ and (5.0.8), then for any $\epsilon \in (0, 1)$, $q \in [2, \infty]$, and integer $m \geq 0$, there exists $C > 0$ determined only by ϵ, q, m, F_0, ρ_0, $|U'|_{C^3}$, and $|(U')^{-1}|_{C^0}$ (independent of $k \neq 0$) such that*

$$
\begin{aligned}
|\partial_t^m \hat{v}_2^c(k)|_{L^2_{x_2} L^q_t(\mathbb{R})} + |k|\mu|\partial_t^m \hat{v}_1^c(k)|_{L^2_{x_2} L^q_t(\mathbb{R})} + \mu|\partial_t^m (\hat{v}_2^c)'(k)|_{L^2_{x_2} L^q_t(\mathbb{R})}& \\
\leq C|k|^{m+1-\frac{1}{q}} \big(\mu^{\frac{1}{2}}|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{5}{2}}|\hat{v}_{20}'(k, 0)| + \mu^{\frac{3}{2}-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}\big)&,
\end{aligned}
$$

$$
|\partial_t^m \hat{\eta}^c(k)|_{L^q_t(\mathbb{R})} \leq C|k|^{m-1-\frac{1}{q}} \big(|k|\mu|\hat{\eta}_0(k)| + \mu^2|\hat{v}_{20}'(k, 0)| + |k|\mu^{2-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}\big),
$$

$$
\begin{aligned}
|t\partial_t^m \hat{v}_2^c(k)|_{L^2_{x_2} L^q_t(\mathbb{R})} \leq& C|k|^{m-\frac{1}{q}} \big(\mu^{-\frac{1}{2}}|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{3}{2}}|\hat{v}_{20}'(k, 0)| + \mu^{\frac{1}{2}-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} \\
&+ \mu^{\frac{3}{2}-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big),
\end{aligned}
$$

$$
\begin{aligned}
|t\partial_t^m \hat{\eta}^c(k)|_{L^q_t(\mathbb{R})} \leq& C|k|^{m-2-\frac{1}{q}} \big(|k||\hat{\eta}_0(k)| + \mu^2|\hat{v}_{20}'(k, 0)| + |k|\mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} \\
&+ |k|\mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big).
\end{aligned}
$$

Proof. The estimates of $\partial_t^m \hat{v}_2^c$, $t\partial_t^m \hat{v}_2^c$, $\partial_t^m (\hat{v}_2^c)'$, $\partial_t^m \hat{v}_1^c$, $\partial_t^m \hat{\eta}^c$, and $t\partial_t^m \hat{\eta}^c$ are based on the definitions of $\hat{v}^c(t, k, x_2)$ and $\hat{\eta}^c(t, k)$ from direct application of Lemma 6.1.4 and Lemma 6.1.5 on $\mathcal{D}_{\rho_0,\rho_0}$ with $f_1 = 0$ and f being $c^m V_2$ (with $n = 0, 1$), $c^m V_2'$, $c^m \tilde{\eta}$ (with $n = 0, 1$), respectively. We omit the details. $\square$

In the following we shall focus on $t\partial_t^m \hat{v}_1^c(t, k, x_2)$, $t^2\partial_t^m \hat{v}_2^c(t, k, x_2)$, and $\partial_t^m \hat{\omega}^c(t, k, x_2)$, where $\hat{\omega}^c$ is the Fourier transform (in x_1) of the vorticity $\omega^c = \partial_{x_1} v_2^c - \partial_{x_2} v_1^c$ of $v^c(t, x)$. In order to characterize their asymptotic behavior, define

$$\begin{aligned}\hat{\Omega}^c(k, x_2) =&\hat{\omega}_0(k, x_2) + \tfrac{1}{2}U''(x_2)\big((1 + sgn(kt))V_{20}(k, U(x_2), x_2) \\ &+ (1 - sgn(kt))\overline{V_{20}(-k, U(x_2), x_2)}\big).\end{aligned} \tag{6.1.9}$$

In the above expression, exactly one of $1 + sgn(kt)$ and $1 - sgn(kt)$ is equal to 2 and the other equal to 0. The dependence of $\hat{\Omega}^c$ on t is only through its sign, so we skipped specifying the t dependence. We also notice that V_2 may not be C^0 at $c \in U([-h, 0]) \subset \mathbb{C}$. The available conjugacy properties of V_2 are not sufficient to imply $\overline{V_{20}(-k, U(x_2), x_2)} = V_{20}(k, U(x_2), x_2)$. We shall see that $\hat{\Omega}^c$ provides the asymptotic profile of the vorticity $\hat{\omega}^c$. We first give the following some basic properties of $\hat{\Omega}^c$.

Lemma 6.1.7. *Assume $U \in C^4$ and (5.0.8), then $\hat{\Omega}^c(-k, x_2) = \overline{\hat{\Omega}^c(k, x_2)}$ and, for any $\epsilon \in (0, 1)$, there exists $C > 0$ determined only by ϵ, F_0, ρ_0, $|U'|_{C^3}$, and $|(U')^{-1}|_{C^0}$ (independent of $k \neq 0$) such that*

$$|\hat{\Omega}^c - \hat{\omega}_0|_{L^2_{x_2}} \leq C\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k, 0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}),$$

$$|(\hat{\Omega}^c)' - \hat{\omega}_0'|_{L^2_{x_2}} \leq C\mu^{-1}\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k, 0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big).$$

Proof. The conjugacy relation of $\hat{\Omega}^c$ is clear from its definition. According to Lemma 6.1.4, V_{20} satisfies the same estimates as V_2 for $|c_I| \in (0, \rho_0]$. We have, for $x_2 \in [-h, 0]$ and $c \in U([-h, 0])$,

$$\begin{aligned}|V_{20}(k, U(\cdot), \cdot)|_{L^2_{x_2}} \leq& C|V_{20}|^{\frac{1}{2}}_{L^2_{c_R, x_2}}|V_{20}|^{\frac{1}{2}}_{L^2_{c_R}H^1_{x_2}} \\ \leq& C\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k, 0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}),\end{aligned} \tag{6.1.10}$$

which implies the estimate of $\hat{\Omega}^c$. Apparently the estimate of $(\hat{\Omega}^c)'$ depends on that of

$$\partial_{x_2}\big(V_{20}(k, U(x_2), x_2)\big) = (D_cV_{20})(k, U(x_2), x_2),$$

where D_c was defined in (5.2.4). From (5.2.7) and (6.1.3), we have

$$
\begin{aligned}
\big|\partial_{x_2}\big(V_{20}(k,U(\cdot),\cdot)\big)\big|_{L^2_{x_2}} \le& C|D_cV_{20}|^{\frac12}_{L^2_{c_R,x_2}}|D_cV_{20}|^{\frac12}_{L^2_{c_R}H^1_{x_2}} \\
\le& C\mu^{-1}\big(|\hat\eta_0(k)| + |k|^{-1}\mu^2|\hat v_{20}'(k,0)| + \mu^{1-\epsilon}|\hat\omega_0(k)|_{L^2_{x_2}} \\
&+ \mu^{2-\epsilon}|\hat\omega_0'(k)|_{L^2_{x_2}}\big),
\end{aligned}
$$

which yields and completes the proof of the lemma. □

In the following lemma, we obtain the leading order terms of $t\hat v_1^c$, $(\hat v_2^c)''$, and $\hat\omega^c$.

Lemma 6.1.8. *Assume $U \in C^4$ and (5.0.8), then, for any $\epsilon \in (0,1)$, $q \in (2,\infty]$, and integer $m \ge 0$, there exists $C > 0$ determined only by ϵ, q, m, F_0, ρ_0, $|U'|_{C^3}$, and $|(U')^{-1}|_{C^0}$ (independent of $k \ne 0$) such that*

$$
\begin{aligned}
&k^2\big|\partial_t^m\big(t\hat v_1^c(t,k,x_2) + ik^{-1}U'(x_2)^{-1}e^{-ikU(x_2)t}\hat\Omega^c(k,x_2)\big)\big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
&+ |k|\big|\partial_t^m\big(\hat\omega^c(t,k,x_2) - e^{-ikU(x_2)t}\hat\Omega^c(k,x_2)\big)\big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
&+ \big|\partial_t^m\big((\hat v_2^c)''(t,k,x_2) - ike^{-ikU(x_2)t}\hat\Omega^c(k,x_2)\big)\big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
\le& C|k|^{m+1-\frac1q}\mu^{-\frac32}\big(|\hat\eta_0(k)| + |k|^{-1}\mu^2|\hat v_{20}'(k,0)| + \mu^{1-\epsilon}|\hat\omega_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat\omega_0'(k)|_{L^2_{x_2}}\big).
\end{aligned}
$$

Remark 6.1.3. *This lemma also implies, for any integer $m \ge 1$,*

$$
\begin{aligned}
|t\partial_t^m\big(e^{ikU(x_2)t}\hat v_1^c(t,k,x_2)\big)\Big|_{L^2_{x_2}L^q_t(\mathbb{R})} \le& C|k|^{m-1-\frac1q}\mu^{-\frac32}\big(|\hat\eta_0(k)| + |k|^{-1}\mu^2|\hat v_{20}'(k,0)| \\
&+ \mu^{1-\epsilon}|\hat\omega_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat\omega_0'(k)|_{L^2_{x_2}}\big).
\end{aligned}
$$

while there is a limit term as $t \to \infty$ for $m = 0$. The form in the lemma is more consistent with other estimates including that of $t^2\hat v_2^c$ to be given in the following, however.

Proof. The definition of $\hat{v}^c$ implies, for each $x_2 \in [-h, 0]$ and $r_1, r_2 \in (0, \rho_0]$,

$$
\begin{aligned}
t\partial_t^m \hat{v}_1^c(t,k,x_2) &= \frac{-i(-ik)^m}{2\pi} \oint_{\partial\mathcal{D}_{r_1,r_2}} te^{-ikct} c^m V_2'(k,c,x_2)dc, \\
\partial_t^m\big((ik\hat{\omega}^c + k^2\hat{v}_2^c)(t,k,x_2)\big) &= \partial_t^m(\hat{v}_2^c)''(t,k,x_2) \\
&= \frac{-k(-ik)^m}{2\pi} \oint_{\partial\mathcal{D}_{r_1,r_2}} e^{-ikct} c^m V_2''(k,c,x_2)dc.
\end{aligned}
$$

Applying Lemma 6.1.5 with $n = 1$ and $f = c^m V_2'$ and $f_1 = -\frac{c^m}{U'(x_2)} V_2''$ and Lemma 6.1.4, we obtain

$$
\begin{aligned}
&\Big| t\partial_t^m \hat{v}_1^c(t,k,x_2) - \oint_{\partial\mathcal{D}_{r_1,r_2}} \frac{(-i)^m k^{m-1} c^m}{2\pi U'(x_2)} e^{-ikct} V_2''(k,c,x_2)dc \Big|_{L^2_{x_2} L^q_t(\mathbb{R})} \\
\le& C|k|^{m-1-\frac{1}{q}} \sup_{|c_I| \in (0, r_2]} \big(|\partial_{c_R} V_2' + U'(x_2)^{-1} V_2''|_{L^2_{c_R,x_2}} + |V_2'|_{L^2_{c_R,x_2}} \big) \\
\le& C|k|^{m-1-\frac{1}{q}} \mu^{-\frac{3}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2 |\hat{v}_{20}'(k,0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}_0'(k)|_{L^2_{x_2}} \big).
\end{aligned}
\tag{6.1.11}
$$

In the rest of the proof, we shall focus on the integral involving V_2'' which also yields the other desired estimates. Substituting the term V_2'' by the Rayleigh equation (2.2.6a) and applying the Cauchy Integral Theorem yield

$$
\begin{aligned}
&\oint_{\partial\mathcal{D}_{r_1,r_2}} \frac{c^m e^{-ikct}}{U'(x_2)} V_2'' dc = \oint_{\partial\mathcal{D}_{r_1,r_2}} \frac{c^m e^{-ikct}}{U'(x_2)} \big(k^2 V_2 + \frac{U''(x_2)V_2 + \hat{\omega}_0(k,x_2)}{U(x_2) - c} \big) dc \\
=& \oint_{\partial\mathcal{D}_{r_1,r_2}} \frac{e^{-ikct}}{U'(x_2)} \Big(k^2 c^m + \big(\frac{U(x_2)^m}{U(x_2) - c} + \frac{c^m - U(x_2)^m}{U(x_2) - c} \big) U''(x_2) \Big) V_2 dc \\
&- \frac{2\pi i U(x_2)^m \hat{\omega}_0(k,x_2)}{U'(x_2)} e^{-ikU(x_2)t}
\end{aligned}
$$

Since $k^2 c^m + \frac{c^m - U^m}{U - c} U''$ is bounded by $C\mu^{-2}$ on $\mathcal{D}_{r_1,r_2}$, we can control those terms using

Lemma 6.1.5 and obtain

$$\begin{aligned}
&\Big|\oint_{\partial\mathcal{D}_{r_1,r_2}} \frac{(-i)^m k^{m-1} c^m}{2\pi U'(x_2)} e^{-ikct} V_2''(k,c,x_2)dc + \frac{(-ik)^{m-1}U(x_2)^m}{U'(x_2)} e^{-ikU(x_2)t}\Big(\hat{\omega}_0(k,x_2) \\
&- \frac{iU''(x_2)}{2\pi}\oint_{\partial\mathcal{D}^{x_2}_{r_1,r_2}} \frac{1}{c} e^{-ikct} V_2(k,c+U(x_2),x_2)dc\Big)\Big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
\le& C|k|^{m-1-\frac{1}{q}}\mu^{-\frac{3}{2}}\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k,0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\big),
\end{aligned} \tag{6.1.12}$$

where we also changed the variable $c - U(x_2) \to c$ in the last integral and

$$\mathcal{D}^{x_2}_{r_1,r_2} = \mathcal{D}_{r_1,r_2} - U(x_2). \tag{6.1.13}$$

It remains to handle this integral term and we shall identify its leading terms.

Fix $T > 0$. We first let

$$\begin{aligned}
&w(k,c,x_2) = V_2(k,c+U(x_2),x_2) - V_{20}(k,c_R+U(x_2),x_2) \\
\Longrightarrow& \lim_{c_I\to 0+} |w(k,\cdot+ic_I,\cdot)|_{L^\infty_{x_2}W^{1,q_1}_{c_R}} = 0,
\end{aligned}$$

for any $q_1 \in [1,2)$, where $\hat{\omega}_0 \in H^1_{x_2}$, Lemma 6.1.4 was used. In the rest of the proof of this lemma, we use $\partial^\dagger\mathcal{D}^{x_2}_{r_1,r_2}$, $\dagger = L, R, T, B$, to denote the left, right, top, bottom sides of the rectangle $\mathcal{D}^{x_2}_{r_1,r_2}$ with the counterclockwise orientation. For any $r \in (0, r_2]$ and $k \neq 0$, and $1 \le \frac{q}{q-1} < q_1 < 2$, integrating by parts and using the $L^{\frac{q}{q-1}} \to L^q$ boundedness (for

$q \in [2, \infty]$) of the Fourier transform, we obtain

$$
\begin{aligned}
&\Big| \oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{1}{c} e^{-ikct} w dc \Big|_{L^2_{x_2} L^q_t([-T,T])} \\
=&\Big| (e^{-ikct} w \log c)\big|^{U(-h)-U(x_2)-r_1+ir}_{U(0)-U(x_2)+r_1+ir} \\
&- e^{krt} \oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} e^{-ikc_R t}(-iktw + \partial_{c_R} w) \log c \, dc \Big|_{L^2_{x_2} L^q_t([-T,T])} \\
\leq& C e^{|k|rT} \big(T^{\frac{1}{q}} (1 + |\log r_1|) |w(k, \cdot + ir, \cdot)|_{L^2_{x_2} L^\infty_{c_R}} \\
&+ |k|^{-\frac{1}{q}} (1 + |k|T) |w(k, \cdot + ir, \cdot)|_{L^2_{x_2} W^{1,q_1}_{c_R}} \big),
\end{aligned}
$$

where $\log$ is taken along $\partial^T \mathcal{D}^{x_2}_{r_1,r}$ which in the upper half plane. Next from Lemma 6.1.4 we have

$$
\begin{aligned}
&\Big| \oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{1}{c} e^{-ikct} \big(V_{20}(k, c_R + U(x_2), x_2) - V_{20}(k, U(x_2), x_2) \big) dc \Big|_{L^2_{x_2} L^q_t([-T,T])} \\
\leq& C |k|^{-\frac{1}{q}} e^{|k|rT} |\partial_{c_R} V_{20}|_{L^2_{x_2} L^{q_1}_{c_R}} \\
\leq& C |k|^{-\frac{1}{q}} \mu^{-\frac{1}{2}} e^{|k|rT} \big(|\hat{\eta}_0(k)| + |k|^{-1} \mu^2 |\hat{v}'_{20}(k, 0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}'_0(k)|_{L^2_{x_2}} \big).
\end{aligned}
$$

The above error analysis implies that the main contribution of the integral along $\partial^T \mathcal{D}^{x_2}_{r_1,r}$ would come from the product

$$
V_{20}(k, U(x_2), x_2) f(r, x_2, kt), \quad \text{where} \ \ f(r, x_2, \tau) = \oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{e^{-i\tau c}}{c} dc.
$$

For any $r \in (0, r_2]$, on the one hand,

$$
\begin{aligned}
|f(r, x_2, \tau)| =& e^{r\tau} \Big| - (e^{-i\tau c_R} \log c)\big|^{U(0)-U(x_2)+r_1}_{U(-h)-U(x_2)-r_1} + i\tau \oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} e^{-i\tau c_R} \log c \, dc \Big| \\
\leq& C(1 + |\tau|) e^{r\tau},
\end{aligned}
$$

which is useful for $|\tau| \leq 1$. On the other hand,

$$f(r, x_2, \tau) = -\Big(\int_{\mathbb{R}+ir} - \int_{(\mathbb{R}+ir)\backslash \partial^T \mathcal{D}^{x_2}_{r_1,r}} \Big) \frac{1}{c} e^{-i\tau c} dc.$$

The first integral can be evaluated as $i\pi(sgn(\tau)+1)$ by using the Cauchy Integral Theorem. Integrating the second integral (in the way opposite to the above) we obtain

$$\begin{aligned}\Big| \int_{(\mathbb{R}+ir)\backslash \partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{1}{c} e^{-i\tau c} dc \Big| =& \frac{e^{r\tau}}{|\tau|} \Big| \frac{e^{-i\tau c_R}}{c} \Big|^{U(-h)-U(x_2)-r_1}_{U(0)-U(x_2)+r_1} + \int_{(\mathbb{R}+ir)\backslash \partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{e^{-i\tau c_R}}{c^2} dc \Big| \\ \leq & C \frac{e^{r\tau}}{|\tau|}.\end{aligned}$$

Therefore

$$|f(r, x_2, \tau) - i\pi(sgn(\tau) + 1)| \leq C(1 + |\tau|)^{-1} e^{r|\tau|}, \quad \forall \tau \in \mathbb{R}.$$

Along with (6.1.10), we have

$$\begin{aligned}&\Big| V_{20}(k, U(x_2), x_2) \Big(\oint_{\partial^T \mathcal{D}^{x_2}_{r_1,r}} \frac{1}{c} e^{-ikct} dc - i\pi(sgn(kt) + 1) \Big) \Big|_{L^2_{x_2} L^q_t([-T,T])} \\ \leq & C|k|^{-\frac{1}{q}} e^{r|k|T} \big(|\hat{\eta}_0(k)| + |k|^{-1} \mu^2 |\hat{v}'_{20}(k, 0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} \big).\end{aligned}$$

The integrals along the vertical sides of $\partial \mathcal{D}^{x_2}_{r_1,r}$ converge to 0 as $r \to 0+$ as all the integrands are smooth there. The integrals along $\partial^B \mathcal{D}_{r_1,r}$, $r \in (0, r_2]$, can be treated much as in the above. Recall $V_2(k, \bar{c}, x_2) = \overline{V_2(-k, c, x_2)}$. Letting $r \to 0+$, the Cauchy Integral Theorem and the above error analysis imply

$$\begin{aligned}&\Big| \oint_{\partial \mathcal{D}^{x_2}_{r_1,r_2}} \frac{1}{c} e^{-ikct} V_2(k, c + U(x_2), x_2) dc - i\pi \big((1 + sgn(kt)) V_{20}(k, U(x_2), x_2) \\ &+ (1 - sgn(kt)) \overline{V_{20}(-k, U(x_2), x_2)} \big) \Big|_{L^2_{x_2} L^q_t([-T,T])} \\ \leq & C|k|^{-\frac{1}{q}} \mu^{-\frac{1}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1} \mu^2 |\hat{v}'_{20}(k, 0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}'_0(k)|_{L^2_{x_2}} \big).\end{aligned}$$

Taking $T \to \infty$, it follows from the above inequality and (6.1.12)

$$\begin{aligned}
&\Big| \oint_{\partial \mathcal{D}_{r_1, r_2}} \frac{(-i)^m k^{m-1} c^m}{2\pi U'(x_2)} e^{-ikct} V_2''(k, c, x_2) dc \\
&+ \frac{(-ik)^{m-1} U(x_2)^m}{U'(x_2)} e^{-ikU(x_2)t} \hat{\Omega}^c(k, x_2) \Big|_{L^2_{x_2} L^q_t(\mathbb{R})} \\
\le& C|k|^{m-1-\frac{1}{q}} \mu^{-\frac{3}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1} \mu^2 |\hat{v}_{20}'(k, 0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}_0'(k)|_{L^2_{x_2}} \big).
\end{aligned} \tag{6.1.14}$$

Along with (6.1.11) and Lemma 6.1.6 it implies the desired estimate of $\partial_t^m (t \hat{v}_1^c)$. The estimates on $\partial_t^m (\hat{v}_2^c)''$ and $\partial_t^m \hat{\omega}^c$ are also obtained from the above inequality and Lemma 6.1.6.

□

Finally we consider $t^2 \hat{v}_2$.

Lemma 6.1.9. *Assume $U \in C^6$ and (5.0.8), then, for any $\epsilon \in (0,1)$, $q \in (2, \infty]$, and integer $m \ge 0$, there exists $C > 0$ determined only by ϵ, q, m, F_0, $|U'|_{C^5}$, and $|(U')^{-1}|_{C^0}$ (independent of $k \neq 0$) such that*

$$\begin{aligned}
&\Big| \partial_t^m \big(t^2 \hat{v}_2^c(t, k, x_2) \\
&- \Big(- \frac{i e^{-ikU(x_2)t}}{kU'(x_2)^2} \hat{\Omega}^c(k, x_2) + e^{-ikU(0)t} \hat{\Lambda}_T(k, x_2) + e^{-ikU(-h)t} \hat{\Lambda}_B(k, x_2) \Big) \Big) \Big|_{L^2_{x_2} L^q_t(\mathbb{R})} \\
\le& C|k|^{m-1-\frac{1}{q}} \mu^{-\frac{3}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1} \mu^2 |\hat{v}_{20}'(k, 0)| + \mu^{1-\epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}_0'(k)|_{L^2_{x_2}} \\
&+ \mu^{3-\epsilon} |\hat{\omega}_0''(k)|_{L^2_{x_2}} \big),
\end{aligned}$$

where

$$\hat{\Lambda}_T(k, x_2) = -\frac{i}{k} \frac{U''(0) \hat{\eta}_0(k) - \hat{\omega}_0(k, 0)}{U'(0)^2 y_{0-}(k, U(0), 0)} y_{0-}(k, U(0), x_2), \tag{6.1.15}$$

$$\hat{\Lambda}_B(k, x_2) = \frac{i \hat{\omega}_0(k, -h) y_{0+}(k, U(-h), x_2)}{kU'(-h)^2 y_{0+}(k, U(-h), -h)}, \tag{6.1.16}$$

and they satisfy

$$|\hat{\Lambda}_T(k)|_{L^2_{x_2}} \le C|k|^{-1}\mu^{\frac{1}{2}}(|\hat{\eta}_0(k)| + |\hat{\omega}_0(k,0)|), \quad |\hat{\Lambda}_B(k)|_{L^2_{x_2}} \le C|k|^{-1}\mu^{\frac{1}{2}}|\hat{\omega}_0(k,-h)|).$$

Remark 6.1.4. *In the above lemmas, we also notice* $\hat{\Lambda}_\dagger(-k,x_2) = \overline{\hat{\Lambda}_\dagger(k,x_2)}$, $\dagger = T, B$. *The leading order terms* $\hat{\Lambda}_B$ *and* $\hat{\Lambda}_T$ *represent the contribution from the rigid bottom and the water surface, while the asymptotic vorticity* $\hat{\Omega}^c$ *from the fluid interior. In the fixed boundary problem for* $x_2 \in [-h, 0]$ *with slip boundary condition on both horizontal boundaries,* $\hat{\Omega}^c$ *and* $\hat{\Lambda}_B$ *would take similar forms and* $\hat{\Lambda}_T$ *would be similar to* $\hat{\Lambda}_B$. *See section 6.4.*

Proof. The definition of $\hat{v}_2^c$ implies, for each $x_2 \in [-h, 0]$ and $r_1, r_2 \in (0, \rho_0]$,

$$t^2\partial_t^m \hat{v}_2^c(t,k,x_2) = \frac{-(-i)^m k^{m+1}}{2\pi} \oint_{\partial\mathcal{D}_{r_1,r_2}} t^2 e^{-ikct} c^m V_2(k,c,x_2)dc.$$

Let $f = c^m V_2$ and

$$f_1 = c^m\Big(\frac{V_2''(x_2)}{U'(x_2)^2} - \frac{g+\sigma k^2}{\mathbf{F}(k,c)}\Big(\frac{V_2''(-h)}{U'(-h)^2}y_+(x_2) - \frac{V_2''(0)}{U'(0)^2}y_-(x_2)\Big)\Big) = c^m(\partial_{c_R}^2 V_2 - \tilde{V}_2),$$

with $\tilde{V}_2$ defined in Lemma 6.1.4. Applying Lemma 6.1.5 with $n = 2$ and Lemma 6.1.4, we obtain

$$\begin{aligned}
&\Big|t^2\partial_t^m \hat{v}_2^c(t,k,x_2) - \frac{(-i)^m k^{m-1}}{2\pi}\oint_{\partial\mathcal{D}_{r_1,r_2}} e^{-ikct} f_1(k,c,x_2)dc\Big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
\le& C|k|^{m-1-\frac{1}{q}} \sup_{c_I\in(0,r_2]} \big(|\tilde{V}_2|_{L^2_{c_R,x_2}} + |\partial_{c_R}V_2|_{L^2_{c_R,x_2}} + |V_2|_{L^2_{c_R,x_2}}\big) \\
\le& C|k|^{m-1-\frac{1}{q}}\mu^{-\frac{3}{2}}\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k,0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}} \\
&+ \mu^{3-\epsilon}|\hat{\omega}_0''(k)|_{L^2_{x_2}}\big).
\end{aligned}$$

Substituting V_2'' in f_1 by using the Rayleigh equation (2.2.6a) yields

$$f_1 = c^m\Big(\frac{V_2''(x_2)}{U'(x_2)^2} + \frac{f_{1B}}{U(-h)-c} + \frac{f_{1T}}{U(0)-c} + \frac{(g+\sigma k^2)y_-(x_2)}{U'(0)^2\mathbf{F}(k,c)}k^2V_2(0)\Big),$$

where

$$f_{1B} = -\frac{(g+\sigma k^2)\hat{\omega}_0(-h)}{U'(-h)^2\mathbf{F}(k,c)}y_+(x_2) = -\frac{\hat{\omega}_0(-h)y_+(x_2)}{U'(-h)^2y_+(-h)},$$
$$f_{1T} = \frac{(g+\sigma k^2)\big(U''(0)V_2(0)+\hat{\omega}_0(0)\big)}{U'(0)^2\mathbf{F}(k,c)}y_-(x_2).$$

Again the terms involving $k^2V_2(0)$ not being divided by $U-c$ can be estimated by using assumption (5.0.8) and Lemma 6.1.5, Lemma 3.3.2, and Lemma 6.1.4 and we have

$$\begin{aligned}&\Big|t^2\partial_t^m\hat{v}_2^c(t,k,x_2)\\ &\quad-\frac{(-i)^mk^{m-1}}{2\pi}\oint_{\partial\mathcal{D}_{r_1,r_2}}e^{-ikct}c^m\Big(\frac{V_2''(x_2)}{U'(x_2)^2}+\frac{f_{1B}}{U(-h)-c}+\frac{f_{1T}}{U(0)-c}\Big)dc\Big|_{L^2_{x_2}L^q_t(\mathbb{R})}\\ \leq&C|k|^{m+1-\frac{1}{q}}\mu^{\frac{1}{2}}\big(|\hat{\eta}_0(k)|+|k|^{-1}\mu^2|\hat{v}_{20}'(k,0)|+\mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}}+\mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}}\\ &\quad+\mu^{3-\epsilon}|\hat{\omega}_0''(k)|_{L^2_{x_2}}\big).\end{aligned}$$

We shall identify the principle contributions from the terms f_{11}, f_{1B}, and f_{1T} following a similar strategy and use the same notations $\partial^\dagger\partial_{r_1,r_2}$, $\dagger = T, B, L, R$, as in the proof of Lemma 6.1.8, with necessary modifications to treat the contributions from the $x_2 = 0, -h$.

Fix $T>0$. We start with f_{1T} by letting

$$f_{1T}^0(k,c_R,x_2) = \lim_{c_I\to 0+} f_{1T}(k,c,x_2) = \frac{(g+\sigma k^2)\big(U''(0)V_{20}(0)+\hat{\omega}_0(0)\big)}{U'(0)^2\mathbf{F}(k,c_R)}y_{0-}(x_2).$$

From assumption (5.0.8), Lemma 3.5.1, Lemma 3.5.7(2b), Lemma 4.1.2, and Lemma 6.1.4, we have, for any $q_1\in[1,2)$,

$$|(f_{1T}-f_{1T}^0)(k,\cdot+c_I,\cdot)|_{L^\infty_{x_2}W^{1,q_1}_{c_R}} \to 0, \;\text{ as } c_I\to 0+.$$

The next step is the same argument via integrating by parts in c_R as in the proof of Lemma 6.1.8, as $r \to 0+$,

$$
\begin{aligned}
&\Big|\oint_{\partial^T \mathcal{D}_{r_1,r}} e^{-ikct} c^m \frac{f_{1T}(k,c,x_2) - f_{1T}^0(k,c_R,x_2)}{U(0)-c} dc\Big|_{L^2_{x_2} L^q_t([-T,T])} \\
=&\Big|\big(e^{-ikct} c^m (f_{1T} - f_{1T}^0) \log(U(0)-c)\big)\big|_{U(0)+r_1+ir}^{U(-h)-r_1+ir} \\
&- \oint_{\partial^T \mathcal{D}_{r_1,r}} e^{-ikct}(-ikt + \partial_{c_R})\big(c^m (f_{1T} - f_{1T}^0)\big) \log(U(0)-c) dc\Big|_{L^2_{x_2} L^q_t([-T,T])} \to 0.
\end{aligned}
$$

From Lemma 3.3.2, Lemma 3.5.3–Lemma 3.5.5, Lemma 4.1.1(3), Lemma 4.1.2(1), and (5.0.8), one may estimate,

$$
|y_{0-}/\mathbf{F}|_{L^2_{x_2} L^\infty_{c_R}} + |\partial_{c_R}(y_{0-}/\mathbf{F})|_{L^2_{x_2} L^{q_1}_{c_R}} \le C\mu^{\frac{5}{2}}, \ \ \forall q_1 \in [1,\infty).
$$

Along with Lemma 6.1.4, it implies, for any $q_2 \in [1,2)$,

$$
\begin{aligned}
|f_{1T}^0|_{L^2_{x_2} L^\infty_{c_R}} + |\partial_{c_R} f_{1T}^0|_{L^2_{x_2} L^{q_2}_{c_R}} \le & C\mu^{-\frac{1}{2}}\big(|\hat\eta_0(k)| + |k|^{-1}\mu^2 |\hat v_{20}'(k,0)| + |\hat\omega_0(k)|_{L^2_{x_2}} \\
& + \mu^{2-\epsilon} |\hat\omega_0'(k)|_{L^2_{x_2}}|\big),
\end{aligned}
$$

where $|\hat\omega_0(0)|$ and $|V_2(0)|_{L^\infty_{c_R}}$ were bounded by the L^2 norms of $\hat\omega_0(k)$, $\hat\omega_0'(k)$, $|V_2(0)|_{L^2_{c_R}}$, and $|\partial_{c_R} V_2(0)|_{L^2_{c_R}}$. Consequently, for any $r \in (0, r_2]$

$$
\begin{aligned}
&\Big|\oint_{\partial^T \mathcal{D}_{r_1,r}} \frac{e^{-ikct} c^m}{U(0)-c} f_{1T}^0(k,c_R,x_2) dc \\
&- f_{1T}^0(k,U(0),x_2) \oint_{\partial^T \mathcal{D}_{r_1,r}} \frac{e^{-ikct} c^m}{U(0)-c} dc\Big|_{L^2_{x_2} L^q_t([-T,T])} \\
\le & Ce^{r|k|T} |k|^{-\frac{1}{q}} \mu^{-\frac{1}{2}} \big(|\hat\eta_0(k)| + |k|^{-1}\mu^2 |\hat v_{20}'(k,0)| + |\hat\omega_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat\omega_0'(k)|_{L^2_{x_2}}\big).
\end{aligned}
$$

As in the proof of Lemma 6.1.8, by considering contour integrals, we have

$$
\Big|\oint_{\partial^T \mathcal{D}_{r_1,r}} \frac{e^{-ikct}}{U(0)-c} dc + i\pi(1 + sgn(kt)) e^{-ikU(0)t}\Big| \le \frac{Ce^{r|kt|}}{1+|kt|}.
$$

Again, since $\frac{c^m - U(0)^m}{c - U(0)}$ is bounded for $m \geq 1$, the above $|f^0_{1T}|_{L^2_{x_2} L^\infty_{c_R}}$ estimate implies

$$
\begin{aligned}
&\Big| f^0_{1T}(k, U(0), x_2)\Big(\oint_{\partial^T \mathcal{D}_{r_1, r}} \frac{e^{-ikct} c^m}{U(0) - c} dc \\
&\qquad\qquad + i\pi(1 + sgn(kt)) e^{-ikU(0)t} U(0)^m \Big)\Big|_{L^2_{x_2} L^q_t([-T,T])} \\
\leq & C e^{r|k|T} |k|^{-\frac{1}{q}} \mu^{-\frac{1}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2 |\hat{v}'_{20}(k, 0)| + |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^2 |\hat{\omega}'_0(k)|_{L^2_{x_2}} \big).
\end{aligned}
$$

The contributions from the integral along $\partial^B \mathcal{D}_{r_1, r_2}$ can be treated similarly and using the conjugacy relation, while the integrals along the vertical boundaries of $\partial \mathcal{D}_{r_1, r_2}$ vanish as $r \to 0+$. Using the Cauchy Integral Theorem, combining the above analysis, letting $r \to 0+$, and then $T \to 0+$, we obtain

$$
\begin{aligned}
&\Big| \oint_{\partial^T \mathcal{D}_{r_1, r_2}} e^{-ikct} c^m f_{1T}(k, c, x_2) dc - 2\pi k e^{-ikU(0)t} U(0)^m \hat{\Lambda}_T(k, x_2) \Big|_{L^2_{x_2} L^q_t(\mathbb{R})} \\
\leq & C |k|^{-\frac{1}{q}} \mu^{-\frac{1}{2}} \big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2 |\hat{v}'_{20}(k, 0)| + |\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon} |\hat{\omega}'_0(k)|_{L^2_{x_2}} \big),
\end{aligned}
$$

where

$$
\hat{\Lambda}_T(k, x_2) = -\tfrac{i}{2k} \big((1 + sgn(kt)) f^0_{1T}(k, U(0), x_2) + (1 - sgn(kt)) \overline{f^0_{1T}(-k, U(0), x_2)} \big).
$$

We give closer look at $\hat{\Lambda}_T$. From boundary condition (5.0.1b), (5.2.2), and (6.1.3),

$$
V_2(k, U(0), 0) = -\zeta_+(U(0))/(g + \sigma k^2) = -\hat{\eta}_0(k),
$$

and thus

$$
f^0_{1T}(k, U(0), x_2) = \frac{U''(0)\hat{\eta}_0(k) - \hat{\omega}_0(k, 0)}{U'(0)^2 y_{0-}(k, U(0), 0)} y_{0-}(k, U(0), x_2).
$$

Since $y_{0-}(k, U(0), x_2) \in \mathbb{R}$ for $x_2 \in [-h, 0]$, we obtain that

$$
\overline{f^0_{1T}(k, U(0), x_2)} = f^0_{1T}(-k, U(0), x_2),
$$

and hence the desired form (6.1.15) of Λ_T. The term involving f_{1B} can be analyzed similarly (actually slightly simpler due to $V_2(-h) = 0$) using Lemma 3.5.3–Lemma 3.5.5 and Lemma 3.5.7. The term involving f_{11} has been estimated in (6.1.14). Summarizing this estimates we obtain

$$\begin{aligned}
&\Big|t^2\partial_t^m \hat{v}_2^c(t,k,x_2) - (-ik)^m\Big(- \frac{iU(x_2)^m}{kU'(x_2)^2}e^{-ikU(x_2)t}\hat{\Omega}^c(k,x_2) \\
&\quad + U(0)^m e^{-ikU(0)t}\hat{\Lambda}_T(k,x_2) + U(-h)^m e^{-ikU(-h)t}\hat{\Lambda}_B(k,x_2)\Big)\Big|_{L^2_{x_2}L^q_t(\mathbb{R})} \\
\le & C|k|^{m-1-\frac{1}{q}}\mu^{-\frac{3}{2}}\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^2|\hat{v}_{20}'(k,0)| + \mu^{1-\epsilon}|\hat{\omega}_0(k)|_{L^2_{x_2}} + \mu^{2-\epsilon}|\hat{\omega}_0'(k)|_{L^2_{x_2}} \\
&\quad + \mu^{3-\epsilon}|\hat{\omega}_0''(k)|_{L^2_{x_2}}\big).
\end{aligned}$$

Combining it with Lemma 6.1.6, the desired estimate follows. □

6.2 Linearized capillary gravity waves in the horizontally periodic case

In this section, we consider the case where the system is periodic in x_1 with wave length $L > 0$. In this case

$$k \in \tfrac{2\pi}{L}\mathbb{Z}, \quad \hat{v}_2(t, k=0, x_2) = 0,$$

where the latter properties is due to the divergence free condition on v. For $\dagger = c, p$, let

$$v_2^\dagger(t,x) = \sum_{|k|\in\frac{2\pi}{L}\mathbb{N}} \hat{v}_2^\dagger(t,k,x_2)e^{ikx_1}, \quad \eta^c(t,x_1) = \sum_{|k|\in\frac{2\pi}{L}\mathbb{N}} \hat{\eta}^c(t,k)e^{ikx_1},$$

$$v_1^c(t,x) = \sum_{|k|\in\frac{2\pi}{L}\mathbb{N}} \hat{v}_1^c(t,k,x_2)e^{ikx_1},$$

$$\eta^p(t,x_1) = \hat{\eta}_0(0) + \sum_{|k|\in\frac{2\pi}{L}\mathbb{N}} \hat{\eta}^p(t,k)e^{ikx_1}, \quad v_1^p(t,x) = \hat{v}_1(0,x_2) + \sum_{|k|\in\frac{2\pi}{L}\mathbb{N}} \hat{v}_1^p(t,k,x_2)e^{ikx_1}$$

where $v^\dagger = (v_1^\dagger, v_2^\dagger)$, and $\hat{v}_1^\dagger$, $\hat{v}_2^\dagger$, and $\hat{\eta}^\dagger$ are defined in Lemma 6.1.1 and Corollary 6.1.1.1. Here we used (2.2.2) that the zeroth modes $\hat{v}_1(k=0)$ and $\hat{\eta}(0)$ are invariant in t. Throughout this section, we assume (4.1.5) holds for $\mathbf{K} = \frac{2\pi}{L}\mathbb{N}$.

We first give the decay estimates of (v^c, η^c) based on Lemma 6.1.6–Lemma 6.1.9. In particular, for the estimates of tv_1^c and $t^2 v_2^c$, recall $\hat{\Omega}^c(k, x_2)$ and $\hat{\Lambda}_\dagger(k, x_2)$, $\dagger = B, T$ defined in (6.1.9), (6.1.16), and (6.1.15), respective. Let

$$\Omega^c(x_1, x_2) = \sum_{|k| \in \frac{2\pi}{L}\mathbb{N}} \hat{\Omega}^c(k, x_2) e^{ikx_1}, \quad \Lambda_\dagger(x_1, x_2) = \sum_{|k| \in \frac{2\pi}{L}\mathbb{N}} \hat{\Lambda}_\dagger(k, x_2) e^{ikx_1}. \qquad (6.2.1)$$

Proof of Theorem 2.1.2(1–2). The assumption of the non-existence of singular modes is given in the form of (4.1.5). According to Proposition 4.1.4, (4.1.5) for $\mathbf{K} = \frac{2\pi}{L}\mathbb{N}$ implies (5.0.8) holds for all k with constants ρ_0 and F_0 uniform in k. Therefore from the definition of v_2^c and Lemma 6.1.6, it is straight forward to estimate

$$\begin{aligned}
|\partial_t^{n_0} v^c|^2_{H^{n_1}_{x_1} L^2_{x_2} L^{q_1}_t(\mathbb{R})} \leq & C \sum_{|k| \in \frac{2\pi}{L}\mathbb{N}} \mu^{-2n_1} |k|^{2n_0 + 2 - \frac{2}{q_1}} \big(\mu^{\frac{1}{2}} |\hat{\eta}_0(k)| + |k|^{-1} \mu^{\frac{5}{2}} |\hat{v}'_{20}(k, 0)| \\
& + \mu^{\frac{3}{2} - \epsilon} |\hat{\omega}_0(k)|_{L^2_{x_2}}\big)^2 \\
\leq & C \sum_{|k| \in \frac{2\pi}{L}\mathbb{N}} |k|^{2(n_0 + n_1 + 1 - \frac{1}{q_1})} \big(|k|^{-1} |\hat{\eta}_0(k)|^2 + |k|^{-7} |\hat{v}'_{20}(k, 0)|^2 \\
& + |k|^{2\epsilon - 3} |\hat{\omega}_0(k)|^2_{L^2_{x_2}}\big) \\
\leq & C\big(|\eta_0|^2_{H^{n_0 + n_1 + \frac{1}{2} - \frac{1}{q_1}}_{x_1}} + |\partial_{x_2} v_{20}(\cdot, 0)|^2_{H^{n_0 + n_1 - \frac{5}{2} - \frac{1}{q_1}}_{x_1}} \\
& + |\omega_0|^2_{H^{n_0 + n_1 - \frac{1}{2} - \frac{1}{q_1} + \epsilon}_{x_1} L^2_{x_2}}\big).
\end{aligned}$$

The desired inequality follows from $\partial_{x_2} v_{20} = -\partial_{x_1} v_{10}$. The estimates on $\partial_t^{n_0} \eta^c$, $t\partial_t^{n_0} v_2^c$ and $t\partial_t^{n_0} \eta^c$ are obtained similarly. The inequalities on $\partial_t^{n_0}(tv_1^c)$ and $\partial_t^{n_0}(t^2 v_2^c)$ are obtained by applying Lemma 6.1.8 and Lemma 6.1.9 through a similar procedure. The estimates on Ω^c and $\Lambda_\dagger$, $\dagger = B, T$, follow directly from the estimates on their each Fourier modes given in those lemmas and

$$|\partial_{x_2}^{n_2} y_{0\pm}(k,c,\cdot)/y_{0\pm}(k,c,0)|_{L^q_{x_2}} \le C\mu^{\frac{1}{q}-n_2}, \quad n_2 = 0,1, \tag{6.2.2}$$

which is obtained using (5.0.8) and Lemma 3.3.2. The singular elliptic equations in (2.1.6) are simply from the homogeneous Rayleigh equation with $c = U(-h), U(0)$, satisfied by $y_{0\pm}$ in $(-h,0)$. THe boundary conditions of Λ_B and Λ_T are simply corollaries of their definitions and the boundary conditions (3.3.1) of $y_{0\pm}$. □

Next we consider the $(v^p(t,x), \eta^p(t,x_1))$ part of the linear solution (v,η). Let

$$\begin{aligned}
&\lambda_0 = \max\{\mathrm{Re}\,(-ic_*k) \mid k \in \tfrac{2\pi}{L}\mathbb{N},\ c_* \in R(k)\} \ge 0,\\
&N = \max\{\text{degree of root } c_* \text{ of } \mathbf{F}(k,\cdot) \mid k \in \tfrac{2\pi}{L}\mathbb{N},\ c_* \in R(k),\ \mathrm{Re}\,(-ikc_*) = \lambda_0\} \ge 1,
\end{aligned} \tag{6.2.3}$$

where the lower bounds are obtained due to the roots $c^{\pm}(k)$ for large k (Lemma 4.1.2(3)).

Proof of Theorem 2.1.2(3). On the one hand, according to Lemma 4.1.2(3), there exists $k_0 > 0$ such that $R(k) = \{c^{\pm}(k)\}$ with simple roots $c^{\pm}(k)$ for all $|k| \ge k_0$. On the other hand, (4.1.5) and Proposition 4.1.4 imply that (5.0.8) holds for all $k \in \frac{2\pi}{L}\mathbb{N}$. Along with Lemma 4.1.2(2), we obtain that, for all $k \in \frac{2\pi}{L}\mathbb{N}$ with $|k| < k_0$, the set of roots $R(k)$ is contained in a subset in the domain of analyticity of $\mathbf{F}(k,\cdot)$ uniformly in such k. Hence $R(k)$ is a discrete set and the total algebraic multiplicity of $c_* \in R(k)$ for all $k \in \frac{2\pi}{L}\mathbb{N}$ with $|k| < k_0$ is finite. This proves $\lambda_0, N < \infty$.

For any $k \in \frac{2\pi}{L}\mathbb{N}$ and $c_* \in R(k)$, let n denote the degree of c_* as a root of $\mathbf{F}(k,\cdot)$, then $\mathbf{b}$ and $\mathbf{b}_S$ are polynomials of t of degree $n-1$ (Lemma 6.1.2). Hence to prove the regularity estimates, we only need to consider $k \in \frac{2\pi}{L}\mathbb{N}$ with $|k| \ge k_0$ where all roots of $\mathbf{F}(k,\cdot)$ are simple. For such k, $R(k) = \{c^{\pm}(k)\}$ and Lemma 4.1.2(3) implies that there exists $C > 0$

such that

$$|c_*| \geq \tfrac{1}{C}|k|^{\frac{1}{2}}, \quad |\partial_c F(k, c_*)| \geq \tfrac{1}{C}|k|^{\frac{3}{2}}, \quad \forall c_* \in R(k), \; k_0 \leq |k| \in \tfrac{2\pi}{L}\mathbb{N}.$$

From the homogeneous Rayleigh equation (3.0.1), (5.0.8), and Lemma 3.3.2, it holds,

$$|\partial_{x_2}^s y_-(k, c_*, \cdot)|_{L^2_{x_2}} \leq C\mu^{\frac{3}{2}-s}, \quad \forall\, s \in [0, l_0], \; k \in \mathbb{R}, \; c_* \in R(k). \tag{6.2.4}$$

Hence Lemma 6.1.1 and Lemma 6.1.2 and the definition of v_2^p imply, for any $n_1 \in \mathbb{R}$ and $n_2 \in [0, l_0]$,

$$\begin{aligned}
&\sum_{k_0 \leq |k| \in \frac{2\pi}{L}\mathbb{N}} \mu^{-2n_1} |\hat{v}_2^p(t, k, \cdot)|^2_{H^{n_2}_{x_2}} \leq C \sum_{k_0 \leq |k| \in \frac{2\pi}{L}\mathbb{N}} \sum_{c_* = c^{\pm}(k)} \mu^{-2n_1} |\mathbf{b}(k, c_*, \cdot)|^2_{H^{n_2}_{x_2}} \\
\leq & C \sum_{k_0 \leq |k| \in \frac{2\pi}{L}\mathbb{N}} |k|^{2(n_1+n_2)-4} \big(|k|^3 |\hat{\eta}_0(k)| + |k|^{\frac{1}{2}} |\hat{v}_{20}'(k, 0)| + |k||\hat{\omega}_0(k, \cdot)|_{L^2_{x_2}}\big)^2 \\
\leq & C\big(|\eta_0|^2_{H^{n_1+n_2+1}_{x_1}} + |\partial_{x_2} v_{20}(\cdot, 0)|^2_{H^{n_1+n_2-\frac{3}{2}}_{x_1}} + |\omega_0|^2_{H^{n_1+n_2-1}_{x_1} L^2_{x_2}}\big).
\end{aligned}$$

The desired inequality follows from the divergence free condition. The expression of v_1^p involves y_-' and thus it can be differentiated in x_2 at most $l_0 - 1$ times. The procedure to obtain the estimates of v_1^p and η^p are similar and we skip the details. □

Finally we give the invariant decomposition of the phase space which proves Theorem 2.1.2(4).

Lemma 6.2.1. *Let*

$$\mathbf{X}^p = \overline{span\{range(e^{ikx_1}\mathbf{P}(k, c_*)) \mid c_* \in R(k), \, k \in \tfrac{2\pi}{L}\mathbb{Z}\}} \subset H^1(\mathbb{T}_L \times (-h, 0)) \times H^2(\mathbb{T}_L),$$

$$\mathbf{P}(v, \eta) = \oplus_{c_* \in R(k), k \in \frac{2\pi}{L}\mathbb{Z}} e^{ikx_1} \mathbf{P}(k, c_*)\big(\hat{v}(k), \hat{\eta}(k)\big),$$

$$\mathbf{X}^c = \ker \mathbf{P} \subset H^1(\mathbb{T}_L \times (-h, 0)) \times H^2(\mathbb{T}_L).$$

where $\mathbf{P}(k, c_*)$ *was defined in (6.1.8), then the following hold.*

1. $\mathbf{P}$ *is a bounded projection operator from* $H^n(\mathbb{T}_L \times (-h,0)) \times H^{n+1}(\mathbb{T}_L)$ *to* $\mathbf{X}^p \cap \big(H^n(\mathbb{T}_L \times (-h,0)) \times H^{n+1}(\mathbb{T}_L)\big)$ *for any* $n \in [1, l_0 - 1]$.

2. $\mathbf{X}^p$ *and* $\mathbf{X}^c$ *are both invariant subspaces of (2.2.1).*

3. *Moreover (2.2.1) is also well-posed on the* $L^2 \times H^1$ *completion of* $\mathbf{X}^p$ *and is a (possibly unstable) dispersive equation with the (multi-branches of) dispersion relation given by* kc_* *where* $c_* \in R(k)$.

The boundedness of $\mathbf{P}$ follows from the estimates in Theorem 2.1.2 at $t = 0$. The invariance of $\mathbf{X}^p$ and $\mathbf{X}^c$ is due to Lemma 6.1.2 and Corollary 6.1.2.1. The well-posedness of (2.2.1) on the $L^2 \times H^1$ completion of $\mathbf{X}^p$ is due to the fact that $R(k) = \{c^\pm(k)\} \subset \mathbb{R} \setminus U([-h, 0])$ except for finitely many $k \in \frac{2\pi}{L}\mathbb{Z}$. Here we did not set $\mathbf{X}^p$ and $\mathbf{X}^c$ in $L^2 \times H^1$ is due to the issue that we can not ensure $v_1(\cdot, 0) \in H_{x_1}^{-\frac{1}{2}}$ for $v \in L^2$.

6.3 Linearized capillary gravity waves in the horizontally infinite case

In this section, we consider the case where $x_1 \in \mathbb{R}$ and thus $k \in \mathbb{R}$. Throughout this section, we assume (4.1.5) for $\mathbf{K} = \mathbb{R}$. For $\dagger = c, p$, let

$$v^\dagger(t,x) = \int_{\mathbb{R}} \hat{v}^\dagger(t,k,x_2) e^{ikx_1} dk, \quad \eta^\dagger(t,x_1) = \int_{\mathbb{R}} \hat{\eta}^\dagger(t,k) e^{ikx_1} dk, \qquad v^\dagger = (v_1^\dagger, v_2^\dagger), \tag{6.3.1}$$

where $\hat{v}_1^\dagger$, $\hat{v}_2^\dagger$, and $\hat{\eta}^\dagger$ are defined in Lemma 6.1.1 and Corollary 6.1.1.1.

We first carry out the decay estimates of (v^c, η^c) based on Lemma 6.1.6–Lemma 6.1.9. Let

$$\Omega^c(x_1,x_2) = \int_{\mathbb{R}} \hat{\Omega}^c(k,x_2) e^{ikx_1} dk, \quad \Lambda_\dagger(x_1,x_2) = \int_{\mathbb{R}} \hat{\Lambda}_\dagger(k,x_2) e^{ikx_1} dk. \tag{6.3.2}$$

Proof of Theorem 2.1.3(1–3). Again the assumption of the non-existence of singular modes is given in the form of (4.1.5). According to Proposition 4.1.4, assumption (4.1.5) for $\mathbf{K} = \mathbb{R}$ implies that (5.0.8) holds and $R(k) = \{c^{\pm}(k)\}$ with all these simple roots $c^{\pm}(k)$ of $\mathbf{F}(k, \cdot)$ away from $U([-h, 0])$ for all $k \in \mathbb{R}$. Moreover, Lemma 4.1.2 yields

$$|dist(c^{\pm}(k), U([-h,0]))| \geq \tfrac{1}{C}\mu^{-\frac{1}{2}}, \quad |\partial_c F(k, c^{\pm}(k))| \geq \tfrac{1}{C}\mu^{-\frac{3}{2}}, \quad \forall k \in \mathbb{R}.$$

Like in the periodic-in-x_1 case, the proof of the decay of (v^c, η^c) is also a direct verification using Lemma 6.1.6–Lemma 6.1.9 along with (6.2.2) and the divergence free condition. We omit the details.

From Lemma 6.1.1 and Lemma 6.1.2(3), we obtain $\mathbf{b}$ and $\mathbf{b}_S$ are independent of t and satisfy, for any $n_2 \in [0, l_0]$,

$$|\partial_{x_2}^{n_2} \mathbf{b}(k, c^{\pm}(k), x_2)| \leq C\big(|k|\mu^{-\frac{1}{2}}|\hat{\eta}_0(k)| + \mu|\hat{v}_{20}'(k,0)| + |k|\mu^{\frac{3}{2}}|\hat{\omega}_0(k)|_{L^2_{x_2}}\big)\Big|\tfrac{\mu^{1-n_2}e^{\mu^{-1}(x_2+h)}}{y_-(k, c^{\pm}(k), 0)}\Big|,$$

$$|\mathbf{b}_S(k, c^{\pm}(k))| \leq C\big(|\hat{\eta}_0(k)| + |k|^{-1}\mu^{\frac{3}{2}}|\hat{v}_{20}'(k,0)| + \mu^2|\hat{\omega}_0(k)|_{L^2_{x_2}}\big).$$

The desired estimates follow from (6.2.4), $ik\hat{v}_1 = -\hat{v}_2$, and direct computations. $\square$

Similar to the periodic case, we also have the decomposition by invariant subspaces.

Lemma 6.3.1. *Let*

$$\mathbf{P}(v, \eta) = \int_{\mathbb{R}} \mathbf{P}(k, c^+(k))(v, \eta)dk + \int_{\mathbb{R}} \mathbf{P}(k, c^-(k))(v, \eta)dk,$$

$$\mathbf{X}^p = range(\mathbf{P}) \subset H^1(\mathbb{R}\times(-h,0))\times H^2(\mathbb{R}), \quad \mathbf{X}^c = \ker \mathbf{P} \subset H^1(\mathbb{R}\times(-h,0))\times H^2(\mathbb{R}),$$

where $\mathbf{P}(k, c^{\pm}(k))$ was defined in (6.1.8), then the following hold.

1. *$\mathbf{P}$ is a bounded projection operator from $H^n(\mathbb{R} \times (-h, 0)) \times H^{n+1}(\mathbb{R})$ to $\mathbf{X}^p \cap \big(H^n(\mathbb{R} \times (-h, 0)) \times H^{n+1}(\mathbb{R})\big)$ for any $n \in [1, l_0 - 1]$.*

2. *$\mathbf{X}^p$ and $\mathbf{X}^c$ are both invariant subspaces of (2.2.1).*

3. *In fact (2.2.1) is also well-posed on the $L^2 \times H^1$ completion of $\mathbf{X}^p$ and is a dispersive equation with the dispersion relation given by $kc^{\pm}(k)$.*

To end this section we show that, under assumptions (4.1.5) for $\mathbf{K} = \mathbb{R}$ and (4.1.14), due to the monotonicity of $c^{\pm}(k)$ in $k > 0$ (Lemma 4.1.7) and the asymptotics of $c^{\pm}(k)$ for $|k| \gg 1$ (Lemma 4.1.2(3)), the dynamics of the non-singular modes is conjugate to that of linear irrotational capillary gravity waves.

For $k \in \mathbb{R}$, let

$$
\begin{aligned}
e^{\pm}(k, x_2) =&(v_1, v_2, \eta) \\
=&e^{-|k|h}\Big(\mu^{-\frac{1}{2}} y_-'(k, c^{\pm}(k), x_2), -ik\mu^{-\frac{1}{2}} y_-(k, c^{\pm}(k), x_2), -\frac{y_-(k, c^{\pm}(k), 0)}{\mu^{\frac{1}{2}}(U(0) - c^{\pm}(k))}\Big),
\end{aligned}
$$

$$
e_{ir}^{\pm}(k, x_2) = (v_1, v_2, \eta) = e^{-|k|h}\Big(\mu^{-\frac{1}{2}} \cosh k(x_2 + h), -i\mu^{-\frac{1}{2}} \sinh k(x_2 + h), \frac{\sinh kh}{k\mu^{\frac{1}{2}} c_{ir}^{\pm}(k)}\Big),
$$

where $c_{ir}^{\pm}(k)$ is the wave speed of the free linear capillary gravity wave (system (1.2.2) with $U \equiv 0$ and $\nabla \times v \equiv 0$) given in (2.1.5). Here $e^{\pm}(k)$ correspond to the two non-singular modes in the k-th Fourier modes in x_1, while $e_{ir}^{\pm}$ the modes of irrotational linear capillary gravity waters waves. Define

$$
\mathcal{E}^{\pm}(f) = \int_{\mathbb{R}} f(k) e^{ikx_1} e^{\pm}(k) dk, \quad \mathcal{E}_{ir}^{\pm}(f) = \int_{\mathbb{R}} f(k) e^{ikx_1} e_{ir}^{\pm}(k) dk,
$$

$$
\mathbf{X}^{\pm} = \{\mathcal{E}^{\pm}(f) \mid f \in L^2(\mathbb{R})\}, \quad \mathbf{X}_{ir}^{\pm} = \{\mathcal{E}_{ir}^{\pm}(f) \mid f \in L^2(\mathbb{R})\}.
$$

Clearly $\mathbf{X}^+ \oplus \mathbf{X}^-$ is equal to the $L^2 \times H^1$ completion of $\mathbf{X}^p$ and $\mathcal{E}^{\pm} : L^2(\mathbb{R}) \to \mathbf{X}^{\pm}$ and $\mathcal{E}_{ir}^{\pm} : L^2(\mathbb{R}) \to \mathbf{X}_{ir}^{\pm}$ parametrize $\mathbf{X}^{\pm}$ and $\mathbf{X}_{ir}^{\pm}$ by L^2. The following proposition finishes the proof of Theorem 2.1.1(2b) and Theorem 2.1.3(4).

Proposition 6.3.2. *Assume $U \in C^3$ and (4.1.5) for $\mathbf{K} = \mathbb{R}$, then the following hold.*

1. *The mappings $\mathcal{E}^{\pm}$ and $\mathcal{E}_{ir}^{\pm}$ are isomorphisms. Moreover there exists $C > 0$ depending*

only on U such that $forallk \in \mathbb{R},\ f \in L^2(\mathbb{R})$,

$$C^{-1} \leq |e^{\pm}(k)|_{L^2}, |e^{\pm}_{ir}(k)|_{L^2} \leq C, \quad C^{-1}|f|_{L^2} \leq |\mathcal{E}^{\pm}(f)|_{L^2}, |\mathcal{E}^{\pm}_{ir}(f)| \leq C|f|_{L^2}.$$

2. *For any solution $(v(t,x), \eta(t,x_1))$ to the capillary gravity wave linearized at the shear flow $U(x_2)$, if its component (v^p, η^p) as defined in (6.3.1) belongs to $\mathbf{X}^+ \oplus \mathbf{X}^-$, then it takes the form*

$$(v^p, \eta^p) = \mathcal{E}^+(e^{-ikc^+(k)t} f_+(k)) + \mathcal{E}^-(e^{-ikc^-(k)t} f_-(k)), \tag{6.3.3}$$

for some unique $f_{\pm} \in L^2(\mathbb{R})$. Similarly, any solution $(v(t,x), \eta(t,x_1)) \in L^2$ to the free linear capillary gravity wave (system (1.2.2) with $U \equiv 0$), then it takes the form

$$(v, \eta) = \mathcal{E}^+_{ir}(e^{-ikc^+_{ir}(k)t} f_+(k)) + \mathcal{E}^-_{ir}(e^{-ikc^-_{ir}(k)t} f_-(k)), \quad f_{\pm} \in L^2. \tag{6.3.4}$$

3. *In addition, assume (4.1.14) and $0 \in U\big([-h, 0]\big)$, then there exist odd C^1 functions $\varphi_{\pm}(k)$ and $C > 0$ depending only on U such that*

$$\varphi^{\pm}(k) c^{\pm}(\varphi^{\pm}(k)) = k c^{\pm}_{ir}(k), \quad C^{-1} \leq |k|^{-1} |\varphi^{\pm}(k)|, (\varphi^{\pm})'(k) \leq C, \quad \forall k \in \mathbb{R}.$$

Define $\Phi^{\pm} : \mathbf{X}^{\pm} \to \mathbf{X}^{\pm}_{ir}$ as

$$\Phi^{\pm}\big(\mathcal{E}^{\pm}(f)\big) = \mathcal{E}^{\pm}_{ir}(f \circ \varphi^{\pm})$$

for any $\mathcal{E}^{\pm}(f) \in \mathbf{X}^{\pm}$, then $\Phi^+ + \Phi^-$ is an isomorphism from $(\mathbf{X}^+ \oplus \mathbf{X}^-) \cap (H^n \times H^{n+1})$ to $(\mathbf{X}^+_{ir} \oplus \mathbf{X}^-_{ir}) \cap (H^n \times H^{n+1})$ for any $n \in [0, l_0 - 1]$. Moreover flows (6.3.3) *and (6.3.4) are conjugate through $\Phi^+ + \Phi^-$. Namely, for any $f_\pm \in L^2$, it holds*

$$\begin{aligned}&\Phi^+\big(\mathcal{E}^+(e^{-ikc^+(k)t}f_+(k))\big) + \Phi^-\big(\mathcal{E}^-(e^{-ikc^-(k)t}f_-(k))\big)\\ =&\mathcal{E}_{ir}^+(e^{-ikc_{ir}^+(k)t}f_+(\varphi^+(k))) + \mathcal{E}_{ir}^-(e^{-ikc_{ir}^-(k)t}f_-(\varphi^-(k))).\end{aligned} \tag{6.3.5}$$

Proof. The estimates on $|e^\pm(k)|_{L^2}$ and $|e_{ir}^\pm(k)|_{L^2}$ are derived from direct computations based on Lemma 3.3.2. In particular, since $c^\pm(k) \in \mathbb{R} \setminus U([-h,0])$, formula (4.1.9) of y_- for $k=0$ and the bound on $\partial_k y_-$ are used in obtaining the lower bounds of $|e^\pm(k)|_{L^2}$ for $|k|$ close to 0. The estimates of $|\mathcal{E}^\pm(f)|_{L^2}$ and $|\mathcal{E}_{ir}^\pm(f)|_{L^2}$ follow from those of $e^\pm(f)$ and $e_{ir}^\pm(f)$ and the Parseval's identity. Statement (2) is a direct consequence of Lemma 2.2.1 and the definition of $c^\pm(k)$ and $c_{ir}^\pm(k)$.

Since $c_{ir}^\pm(0) = \sqrt{gh} \neq 0$ and $c^\pm(0) \notin U([-h,0])$, under the additional assumptions (4.1.14) and $0 \in U\big((-h,0)\big)$, Proposition 4.1.4 and Lemma 4.1.7 imply that a.) both $kc^\pm(k)$ and $kc_{ir}^\pm(k)$ are odd in k, b.) both $\pm kc^\pm(k)$ and $\pm kc_{ir}^\pm(k)$ have positive derivative for $k>0$, and c.) both are of the order $O(|k|^{\frac{3}{2}})$ for $|k| \gg 1$ and of the order $O(|k|)$ for $|k| \ll 1$. Hence $\varphi^\pm$ exist and satisfy the estimates, which implies the boundedness of Φ. The conjugacy relation (6.3.5) can be verified directly using (6.3.3), (6.3.4), and the definition of $\varphi^\pm$. $\square$

Remark 6.3.1. *Under (4.1.14), $0 \in U([-h,0])$, and $F(k, U(-h)) \neq 0$ for all $k \in \mathbb{R}$, without assuming (4.1.5), $\mathbf{X}^+ \oplus \mathbf{X}^-$ may only be a closed subspace of $\mathbf{X}^p$, but $c^\pm(k) \in \mathbb{R} \setminus U([-h,0])$ are still monotonic and isolated from the rest of the singular or non-singular modes. The exactly same argument implies that the conclusions of the above proposition still hold on $\mathbf{X}^+ \oplus \mathbf{X}^-$.*

6.4 A remark on the linearized Euler equation on a fixed 2d channel

We briefly comment on the 2-d Euler equation on a fixed channel $x_2 \in (-h, 0)$ with slip boundary condition $v_2 = 0$ at $x_2 = -h, 0$. Let $U(x_2)$ be a shear flow and we assume

$$U' > 0 \text{ and there are no singular modes.} \tag{\textbf{H}}$$

As in the literatures, singular modes mean linearized solutions in the form of $e^{ik(x_1-ct)}v(x_2)$ with $v \in H^1_{x_2}$ and $c \in U([-h, 0])$.

The approach in this paper can be easily adapted to analyze this problem. While the non-homogeneous term in the Rayleigh equation (2.2.6a) is still $-\frac{\hat{\omega}_0(k,x_2)}{U(x_2)-c}$, the main modifications are: a.) replacing $y_+(k, c, x_2)$ and $V_2(k, c, x_2)$ by $\tilde{y}_+(k, c, x_2)$ and $y_E(k, c, x_2)$ which solve the homogeneous and non-homogeneous Rayleigh equations satisfying boundary conditions

$$\tilde{y}_+(0) = y_E(0) = y_E(-h) = 0, \quad \tilde{y}'_+(0) = 1,$$

respectively, and b.) replacing $\mathbf{F}(k, c)$ by $y_-(k, c, 0)$. For the simplification of notations, we also use y_-, $\tilde{y}_+$, and y_E to denote their limits as $c_I \to 0+$. In this case of channel flow with fixed boundary, obviously the set of non-singular modes (roots of $y_-(k, c, 0)$ outside $U([-h, 0])$) for all $k \in \mathbb{R}$ is finite, actually empty if $U'' \neq 0$. Assuming (**H**), through the same procedure as in Lemma 6.1.1, the solution $v(t, x)$ to the linearized Euler equation at the shear flow $U(x_2)$ can also be split into

$$v(t, x) = v^c(t, x) + v^p(t, x)$$

associated to the continuous spectra and point spectra. Under assumption (**H**), $v^p(t, \cdot)$

belongs to the eigenspace of unstable modes which is finite dimensional if $x_1 \in \mathbb{Z}_L$. Let

$$\begin{aligned}
\hat{\Omega}^c(k, x_2) =& \hat{\omega}_0(k, x_2) + \tfrac{1}{2} U''(x_2)\big((1 + sgn(kt)) y_E(k, U(x_2), x_2) \\
&+ (1 - sgn(kt))\overline{y_E(-k, U(x_2), x_2)}\big), \\
\hat{\Lambda}_T(k, x_2) =& \frac{i\hat{\omega}_0(k, 0) y_-(k, U(0), x_2)}{kU'(0)^2 y_-(k, U(0), 0)}, \quad \hat{\Lambda}_B(k, x_2) = \frac{i\hat{\omega}_0(k, -h)\tilde{y}_+(k, U(-h), x_2)}{kU'(-h)^2 \tilde{y}_+(k, U(-h), -h)},
\end{aligned}$$

and Ω^c and $\Lambda_\dagger$, $\dagger = B, T$, be defined as in (6.2.1) for the L-periodic-in-x_1 case and in (6.3.2) for the case of $x_1 \in \mathbb{R}$.

Theorem 6.4.1. *Assume $U \in C^{l_0}$, $l_0 \geq 4$, and* (**H**) *holds for all $k \in K$ where $K = \frac{2\pi}{L}\mathbb{N}$ or $K = \mathbb{R}$, then, for any $q_1 \in [2, \infty]$, $q_2 \in (2, \infty]$, $\epsilon > 0$, $n_1 \in \mathbb{R}$, and integer $n_0 \geq 0$, there exists $C > 0$ depending only on q_1, q_2, ϵ, and U such that any solution with $\hat{v}_{10}(0, x_2) = 0$ satisfy*

$$|\partial_t^{n_0} \partial_{x_1}^{n_1} v_1^c|_{L_x^2 L_t^{q_1}(\mathbb{R})} + |\partial_t^{n_0} \partial_{x_1}^{n_1 - 1}(1 - \partial_{x_1}^2)^{\frac{1}{2}} v_2^c|_{L_x^2 L_t^{q_1}(\mathbb{R})} \leq C \big| |\partial_{x_1}|^{n_0 + n_1 - \frac{1}{q_1}} \omega_0 \big|_{H_{x_1}^{\epsilon - \frac{1}{2}} L_{x_2}^2},$$

$$\begin{aligned}
&\big| t\partial_t^{n_0} \partial_{x_1}^{n_1} (1 - \partial_{x_1}^2)^{\frac{1}{2}} v_2^c \big|_{L_x^2 L_t^{q_1}(\mathbb{R})} + \big| \partial_t^{n_0} \partial_{x_1}^{n_1} \big(\omega^c - \Omega^c(x_1 - U(x_2)t, x_2)\big) \big|_{L_x^2 L_t^{q_2}(\mathbb{R})} \\
&+ \big| \partial_t^{n_0} \partial_{x_1}^{n_1 + 1} \big(t v_1^c - U'(x_2)^{-1} \partial_{x_1}^{-1} \Omega^c(x_1 - U(x_2)t, x_2)\big) \big|_{L_x^2 L_t^{q_2}(\mathbb{R})} \\
&+ \big| \partial_t^{n_0} \partial_{x_1}^{n_1 - 1} \big(\partial_{x_2}^2 v_2^c - \partial_{x_1} \Omega^c(x_1 - U(x_2)t, x_2)\big) \big|_{L_x^2 L_t^{q_2}(\mathbb{R})} \\
\leq& C\Big(\big| |\partial_{x_1}|^{n_0 + n_1 - \frac{1}{q_1}} \omega_0 \big|_{H_{x_1}^{\epsilon + \frac{1}{2}} L_{x_2}^2} + \big| |\partial_{x_1}|^{n_0 + n_1 - \frac{1}{q_1}} \partial_{x_2} \omega_0 \big|_{H_{x_1}^{\epsilon - \frac{1}{2}} L_{x_2}^2} \Big),
\end{aligned}$$

and if $U \in C^6$,

$$\begin{aligned}
&\big|\partial_t^{n_0}\partial_{x_1}^{n_1+1}\big(t^2 v_2^c - U'(x_2)^{-2}\partial_{x_1}^{-1}\Omega^c(x_1-U(x_2)t,x_2) - \Lambda_B(x_1-U(-h)t,x_2)\\
&\quad - \Lambda_T(x_1-U(0)t,x_2)\big)\big|_{L_x^2 L_t^{q_2}(\mathbb{R})}\\
\leq &C\big(\big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\omega_0\big|_{H_{x_1}^{\epsilon+\frac{1}{2}}L_{x_2}^2} + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\partial_{x_2}\omega_0\big|_{H_{x_1}^{\epsilon-\frac{1}{2}}L_{x_2}^2}\\
&\quad + \big||\partial_{x_1}|^{n_0+n_1-\frac{1}{q_1}}\partial_{x_2}^2\omega_0\big|_{H_{x_1}^{\epsilon-\frac{3}{2}}L_{x_2}^2}\big).
\end{aligned}$$

Moreover, for any integer $n_2 = 0, 1$, and $q \in [1, \infty)$, it holds

$$\begin{aligned}
|\Omega^c - \omega_0|_{H_{x_1}^{n_1}L_{x_2}^2} &\leq C|\omega_0|_{H_{x_1}^{n_1-1+\epsilon}L_{x_2}^2},\\
|\partial_{x_2}\Omega^c - \partial_{x_2}\omega_0|_{H_{x_1}^{n_1}L_{x_2}^2} &\leq C\big(|\omega_0|_{H_{x_1}^{n_1+\epsilon}L_{x_2}^2} + |\partial_{x_2}\omega_0|_{H_{x_1}^{n_1-1+\epsilon}L_{x_2}^2}\big),\\
||k|^{n_1}\partial_{x_2}^{n_2}\hat{\Lambda}_B|_{L_k^2 L_{x_2}^q} &\leq C|\partial_{x_1}^{n_1-1}\omega_0(\cdot,-h)|_{H_{x_1}^{n_2-\frac{1}{q}}},\\
||k|^{n_1}\partial_{x_2}^{n_2}\Lambda_T|_{L_k^2 L_{x_2}^q} &\leq C|\partial_{x_1}^{n_1-1}\omega_0(\cdot,0)|_{H_{x_1}^{n_2-\frac{1}{q}}}.
\end{aligned}$$

Finally, $\Lambda_\dagger$, $\dagger = B, T$, satisfy $\hat{\Lambda}_\dagger(k=0,x_2) = 0$ and

$$\begin{cases}
-(U-U(0))\Delta\Lambda_T + U''\Lambda_T = 0, & x_2 \in (-h,0),\\
\Lambda_T(x_1,-h) = 0, \qquad \partial_{x_1}\Lambda_T(x_1,0) = -U'(0)^{-2}\omega_0(x_1,0);
\end{cases}$$

$$\begin{cases}
-(U-U(-h))\Delta\Lambda_B + U''\Lambda_B = 0, & x_2 \in (-h,0),\\
\partial_{x_1}\Lambda_B(\cdot,-h) = -U'(-h)^{-2}\omega_0(x_1,-h), \qquad \Lambda_B(x_1,0) = 0.
\end{cases}$$

Remark 6.4.1. *In the case of the Couette flow $U(x_2) = x_2$, assumption ($\mathbf{H}$) is satisfied. Obviously $\Omega^c = \omega_0$, which in fact gives the whole linearized vorticity $\omega(t,x) = \omega_0(x_1 - x_2t, x_2)$ and the leading asymptotic terms of tv_1 and $\partial_{x_2}^2 v_2$. However, t^2v_2 does also include contributions Λ_T and Λ_B from the top and bottom boundaries. These asymptotic leading order terms are essentially same as those obtained in [35] (after simplifications of (5.1) in*

Lemma 5.1 there), see also Lemma 3 in [70].

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