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THE ANALYTICAL THEORY OF COPLANAR MOTION
APPLIED TO APPROXIMATE FOUR-BAR STRAIGHT LINE MECHANISMS

A THESIS

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APPLIED TO APPROXIMATE FOUR-BAR STRAIGHT LINE MECHANISMS

Approved:

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SUMMARY

The principal objective of this work is to provide the analytical foundation for the analysis of approximate straight line four-bar linkages. Graphical procedures developed by previous investigators have been largely ignored except for those cases which can easily be interpreted analytically. Nonetheless, no significant area of linkage design has been omitted. In addition to a thorough discussion of the advantages of pinned linkages for producing approximate straight line motion, there are five major areas of study:

1. The development of the theory and terminology of curvature theory. Areas included deal with order of contact, the inflection circle, the cubic of stationary curvature, the Burmester points, alternate linkages, Ball-double Burmester points, and a sequence of design procedures using alternate four-bar linkages of specified slider-crank mechanisms.

2. The development of the analytical theory of four and five finitely separated positions of the moving plane in coplanar motion. Original derivations are given for the cubic circle point curve, the quartic equation for the four Burmester points of the moving plane, the quadratic equation for the unknown Burmester points of a given four-bar linkage, the cubic equation for the three unknown Burmester points where the fourth Burmester point is prescribed in five finitely separated positions on a straight line, and finally the quadratic equation for the two unknown Burmester points of a given slider-crank mechanism.

3. The development of the analytical theory for combinations of contacts of the coupler curve with a straight line. Generally, a set of contacts having six points of the sixth degree coupler curve in common with a straight line requires an iteration of the design parameters. The input-output displacement function is derived in closed form for the slider-crank mechanism. A technique is given so that the alternate four-bar linkages of given slider-crank mechanisms generate a symmetrical coupler curve. Multiple position design techniques are given for the cases of two finitely separated inflections on a symmetrical coupler curve, one prescribed intersection with a Ball point, two prescribed intersections with a Ball point, one prescribed tangent with a Ball point, and one prescribed intersection with a Ball-Burmester point.

4. Except for the symmetrical linkages, classical mechanisms have previously been designed graphically. Analytical interpretations are now given in their most general form for the Watt, Evans, and conchoidal straight line mechanisms.

5. The evaluation of linkages from areas one and four above is undertaken by means of the digital computer. The primary information obtained is the length of the approximate straight line output for a specified accuracy. Additional information is also obtained, including transmission angles, rotation angles of the cranks, type of mechanism, linkage dimensions, deviation values, and the second derivative of the deviation from a straight line. A detailed explanation of the principal segments of the computer program is also given. The linkages for which computer data were obtained are compared on the basis of length of approximate straight line output, transmission angles, and higher derivatives

of the deviation curve. The data for the set of linkages having a Ball-double Burmester point are developed in the form of nomographs to assist the designer. Data for the other groups are also available for this type of presentation.

CHAPTER I

INTRODUCTION

The production of approximate straight line motion by means of the simplest of planar linkages, the four-bar linkage, has been the subject of investigation of many eminent kinematicians. Analytical and graphical design procedures for general solutions of approximate straight line motion by four-bar linkages have been developed by Chebychev (1), Burmester (2), Mueller (3), Allievi (4), and other more recent investigators (5, 6, 7, 8). The earlier investigations were very thorough, and much significant theoretical information was produced.

Roberts (9) and Cayley (10) developed some basic but complex mathematical theory related to the properties of the coupler curve of a four-bar linkage. Freudenstein and Primrose (11) have recently explained much of this theory to make it more useful to the researcher in kinematics. These publications are of interest since they contain the derivation of the equation of the coupler curve in rectangular coordinates.

Finding a general method for the synthesis of straight line mechanisms was the problem which motivated Burmester (2) to develop that important part of kinematics which is called Burmester theory. The graphical method which he developed for synthesizing a four-bar linkage which guides a point through up to five finitely separated positions is explained in recent publications (12, 13, 14, 15). A digital computer program, published by Freudenstein and Sandor (16), eliminates many of the limitations of this graphical approach. A graphical modification

of Burmester's concept is the arc match method given by Kearney and Wright which they have used in the design of radio station indicators (15). Freudenstein and Sandor have developed a digital computer method for synthesis of path generating mechanisms, including approximate straight line mechanisms (17).

R. Mueller (3) considered the same problem as Burmester but directed his study towards the concept of single position design. His investigations were purely geometrical and many important concepts of coplanar motion first appeared in his papers. Application of his results to the generation of approximate straight line motion led to some important but complex graphical procedures. During the same period, L. Allievi (4) made a study of coplanar motion on a purely analytical basis. His comprehensive book also contains single position design procedures for designing four-bar straight line mechanisms. In addition, he set forth much of the basic terminology that applies to kinematics in general.

Other recent works include the important and comprehensive studies on straight line mechanism design by W. Meyer zur Capellen (6, 18) and A. E. R. De Jonge (5). Isolated design procedures appear in books by K. Hain (19), R. Beyer (20), A. S. Hall, Jr. (14), and A. Cowie (21).

Surprisingly, little useful quantitative information has been published by the previous investigators concerning the variation from exact straight line motion. Most investigators have been content to state general qualitative information. Chebychev (1) derived statements for the error and length of the approximating straight line portion of the coupler curve. These statements depend on certain parameters which must be arrived at through his method of analysis. Furthermore, they are

valid only for symmetrical linkages. Mueller (3) also derived an expression to compare one of his linkages to those of Chebychev. Wunderlich (22) gives a nomograph for symmetrical linkages which allows one to determine the accuracy of the straight line output for an undetermined length of travel. Volmer (23) thoroughly analyzes a group of slider-crank mechanisms which are closely associated to a special form of the Evans type of straight line motion linkage. It is indeed remarkable that more usable data are not presently available to designers and that none now exist in the English literature.

Application

General Comments

It is the author's contention that pinned linkages have definite, desirable properties which make them particularly attractive for the production of straight line motion. These are listed below:

(a) With the use of properly designed bearings, friction at the working elements can be reduced to a very low level. Lubrication of the bearings is generally a simple matter. This is not the case for slides on ways where a constant thickness of the lubricant film must be maintained for high accuracy in the motion. The application of the driving forces on the slider which normally vary in magnitude and direction during the cycle also makes it more difficult to maintain the desired constant film thickness. For higher speeds, the sliding pair is designed to produce a hydrodynamic wedge action in the lubricant. Such designs allow backlash (wobble) between the pairing elements during the return stroke and create an undesirable condition when the sense of the motion is changed at each end of the stroke.

(b) Since the four-bar linkage has at least three independent design parameters, this linkage has inherently a great flexibility for application to a wide number of design situations.

(c) The four-bar linkage is light, simple, and easily manufactured with high accuracy using standard machine tools. Its application, therefore, would not require an expensive outlay of new tooling. This is in marked contrast to the manufacture of sliding pairs which are often made as matched pairs and require highly skilled craftsmanship.

(d) Linkages can be used (in contrast to the sliding pair) to efficiently amplify motion. The output element can be a low inertia producing mass, removed from the working elements. In oscillating motions, such as that of the slider-crank mechanism, undesirably high peaks in the acceleration occur at the end of the stroke. Furthermore, these peak accelerations act on an element having, by necessity, a large mass.

(e) Because of the cylindrical working elements of the bearings, linkages can be expected to give longer wear when compared with mechanisms that have reciprocating surface contact between mating surfaces. Furthermore, the bearings may be easily replaced at relatively small cost. Bearings used in precision equipment are often designed to allow for small adjustments in the location of the center of rotation.

(f) Of primary importance is the ability of four-bar linkages to be largely unaffected by errors of manufacture. This characteristic was first enunciated by Mueller (24) in reference to the use of four-bar linkages to produce accurate straight line motion. The author has investigated this concept with a digital computer in order to indicate its

precise meaning. For a particular four-bar straight line mechanism, an error of 0.025 in was introduced in one of its 5.0 in cranks. This reduced the accuracy from $\pm 1.0 \times 10^{-6}$ in to $\pm 5.0 \times 10^{-6}$ in over the same range of motion. Instead of reducing the accuracy to $\pm 5.0 \times 10^{-3}$ in, the linkage compensated for the introduced error and produced a straight line output which was 1,000 times better than might generally be expected. Much of the merit of this dissertation depends on this fact, since a truly accurate straight line motion can be obtained from a four-bar linkage in practice.

Uses of Straight Line Motion

Past and Present Uses. The following list is an indication of the actual applications made of approximate straight line motion mechanisms:

Guidance of piston rods by Watt

Steam indicators

Radio station indicators

Granite gang saws

Pull rod carriers

Transport mechanisms

Film projectors

Unloading mechanisms

(The center of gravity of the load travels on a horizontal line)

Minimum energy jib crane (Wippkrane)

(Important application made by the Germans)

Oil pump rod guide

Auger hole driller

Quick return straight line mechanisms

Automatic machines such as:

Wrapping

Packaging

Printing

Weaving

Vending

New and Proposed Uses. Some new uses, proposed by the author, require extremely accurate straight line motion, the design of which is now made possible.

The proper combination of two straight line linkages in a dwell mechanism can give a dwell period, for half the cycle, that has an angular rotation during the dwell of $\pm 1/2$ minute or less in the output link (25). Such a mechanism has been built in model form and the error measured is considered to be very small since the length of any of the component links could have had an error as large as 0.005 in. Such dwell linkages can substitute adequately for periodic mechanisms such as cams and ratchet, and indexing gears to give higher possible speeds.

Another proposed use is in the area of milling machines. Special milling machines cutting curves defined by a controlling four-bar linkage are in use in Germany (26). It seems quite possible that reasonably accurate and inexpensive milling machines could be designed based on very accurate straight line motion produced by the simple four-bar linkage.

A seismograph designed with the basic supporting mechanisms being four-bar linkages having very accurate straight line motion over a short range is the third proposed use. Since the straight line motion, based

on the author's design equations, is extremely accurate near the design position, the small amplitudes usually recorded by a seismograph would be unaffected by the control linkage. Four-bar linkages that are properly manufactured can practically eliminate the friction limitations of other designs.

Scope of this Study

Many publications concerned with approximate straight line four-bar linkages appear in the kinematics literature. Previous results have been primarily concerned with giving the designer a method for finding the necessary link dimensions. Little work has been done to provide the designer with data on the length and accuracy of the approximate straight line output. There are, in general, a minimum of four independent design parameters in a four-bar linkage system which results in a totality of ∞^4 possible solutions to the designer's problem. It is the primary purpose of this study to reduce the "trial and error" effort now required of the designer by providing a deviation analysis and by adding restrictions based on curvature theory to interrelate some of the independent parameters. The result should reduce the designer's problem by providing a systematic set of choices to optimize the resulting design.

Mechanisms to be Investigated

The groups of mechanisms analyzed in this investigation are those having the best possibility of producing sufficiently accurate approximate straight line motion. Each group is based on a distinct design procedure. The groups are:

(1) The classic mechanisms of Evans, Watt, Roberts, and Chebychev as well as other linkages found in the literature.

(2) Those mechanisms having fourth order contact at Ball's point; including:

- (a) general Ball-Burmester point
- (b) double Burmester point coincident with Ball's point
- (c) Ball-Burmester point at the inflection pole.

(3) Those mechanisms having combinations of finitely separated coupler curve contacts with a straight line; including:

- (a) Ball point and two intersections
- (b) Ball point and one tangent
- (c) Ball-Burmester point and one intersection
- (d) two finitely separated inflections on a symmetrical coupler curve.

Linkages of the first group depend on well known concepts which are developed analytically by the author. Design techniques for the second group are original with the author although graphical and analytical design procedures were given by Mueller and Allievi for the general case of a Ball-Burmester point. To the author's knowledge, the third group of linkages has not been attempted before by other investigators. The first two groups have been analyzed by computer oriented programs for such properties as:

1. dimensions of the links
2. deviation and length of the approximate straight line output
3. type of mechanism
4. dynamic characteristics of the deviation curve

5. transmission angles

6. portion of cycle spent producing the straight line output.

The third group has been analyzed primarily to develop the theory required to determine link dimensions. Much of the computer programming for the first two groups will also apply for the third group.

Analytical Theory of Finitely Separated Coupler Plane Positions

An analytical theory of the motion of the coupler plane through finitely separated positions has also been developed in this dissertation. The work of Burmester on this problem was completely geometrical and resulted in graphical design procedures. The basic concepts proposed by Bottema (27) are used to derive the cubic circle point curve equation, the quartic equation for the Burmester points in the coupler plane when it moves through five finitely separated positions, as well as the design equations to find the unknown Burmester points for a given four-bar linkage (or a given slider-crank mechanism) which is displaced in five finitely separated positions.

CHAPTER II

CURVATURE THEORY

Basic Concepts and TerminologyOrder of Contact

General. Much of the terminology of curvature theory is based on the concept of the order of contact between algebraic curves. This order of contact indicates how intimately the curves are associated in the region of their contact. The particular case where one curve is a straight line and the other the sixth degree coupler curve of the four-bar linkage is of special interest to this dissertation. Any line has six real or imaginary intersections with the sixth degree coupler curve. The generality of approximate straight line motion as produced by a four-bar linkage depends on the arrangement and spacing of these intersections.

Zero Order. If two curves cross each other at an oblique angle (which does not approach zero in the sense of a contingence angle), the curves will have one point in common. In this case, the curves show no interrelation at the contact.

First Order Contact. If the two intersections of a secant to a curve (Fig. 1) approach each other such that $\Delta S \rightarrow 0$, the secant approaches a tangent condition and has two infinitesimally separated points in common with the curve. This is first order contact. Note that the tangent is a fair approximation to the curve in the region of the contact. The general case of first order contact between two general algebraic curves occurs, obviously, when the two curves have a unique common

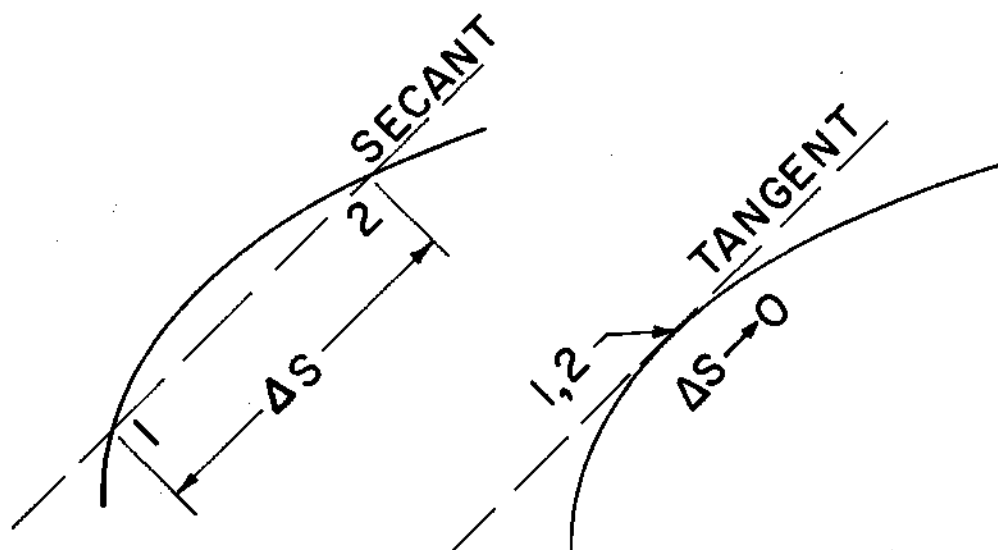


Figure 1. First Order Contact

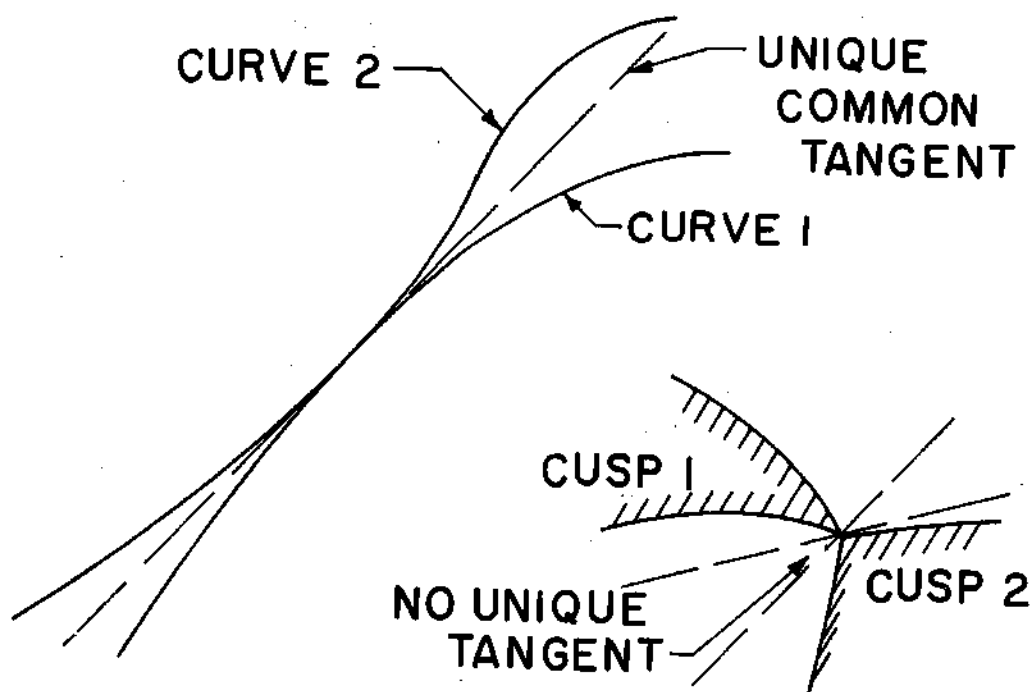


Figure 2. Two Curves in First Order Contact.

tangent at the contact. This insures that the curves have two common infinitesimally separated points. Cusps in contact do not exhibit first order contact since there is no unique common tangent (Fig. 2).

Second Order Contact. Second order contact occurs when there are three infinitesimally separated points in common between the two curves. This case can be best introduced (Fig. 3) by first considering three finitely separated intersections. The path element is defined to be the length of the path between the two external intersections. If $\Delta S_{12} = \Delta S_{23} = \Delta S$ represents one-half of the path element and if $\Delta\psi$ represents the contingence angle between the finitely separated tangents defined by points 1 and 2 and 2 and 3, then

$$\frac{\Delta S}{2} = \rho \sin \frac{\Delta\psi}{2}$$

where ρ is the average length of the perpendicular bisectors of the chords between points 1 and 2 and 2 and 3. As the value $\Delta S \rightarrow 0$ in the limit (Fig. 4), it follows that

$$\lim_{\Delta S \rightarrow 0} \frac{\sin(\Delta\psi)}{\Delta S} = \frac{d\psi}{dS} = \frac{1}{\rho}$$

This demonstration graphically displays the fact that three infinitesimally separated points are necessary to define the curvature of an algebraic curve. In general two curves have second order contact (Fig. 5) if there exists:

1. three common infinitesimally separated points.
2. two common infinitesimally separated tangents.
3. one common radius of curvature.

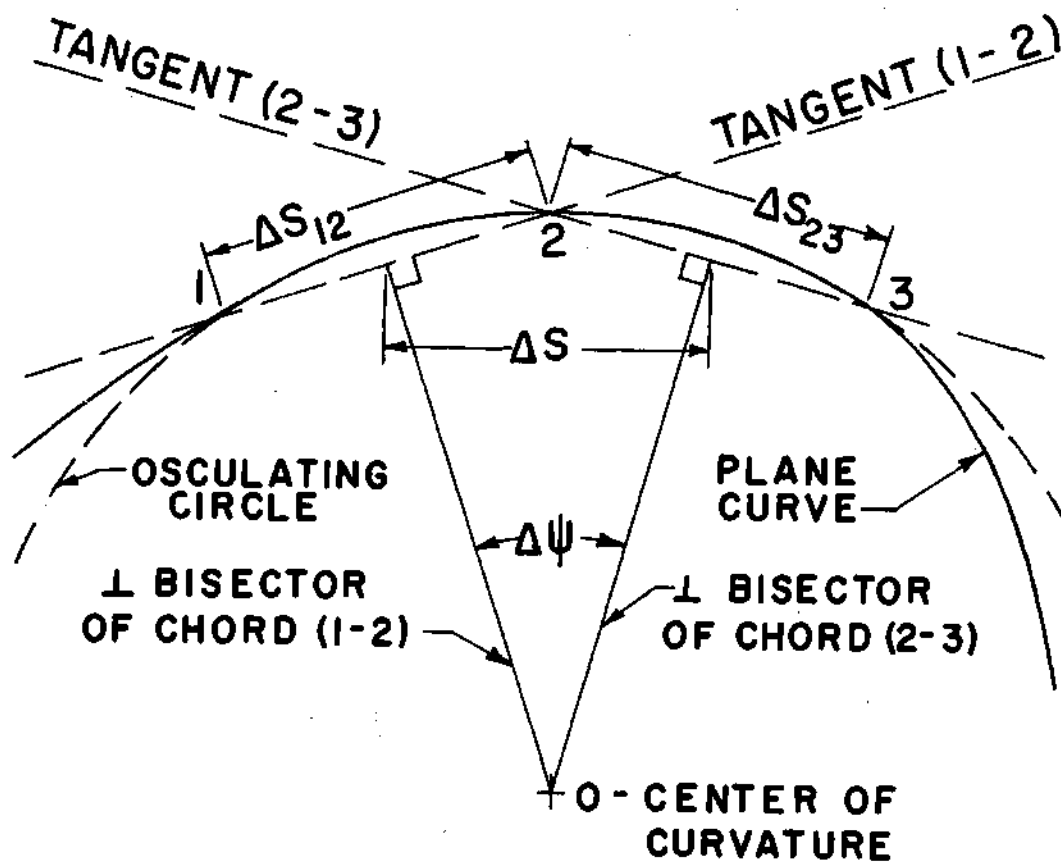


Figure 3. Three Common Infinitesimally Separated Points.

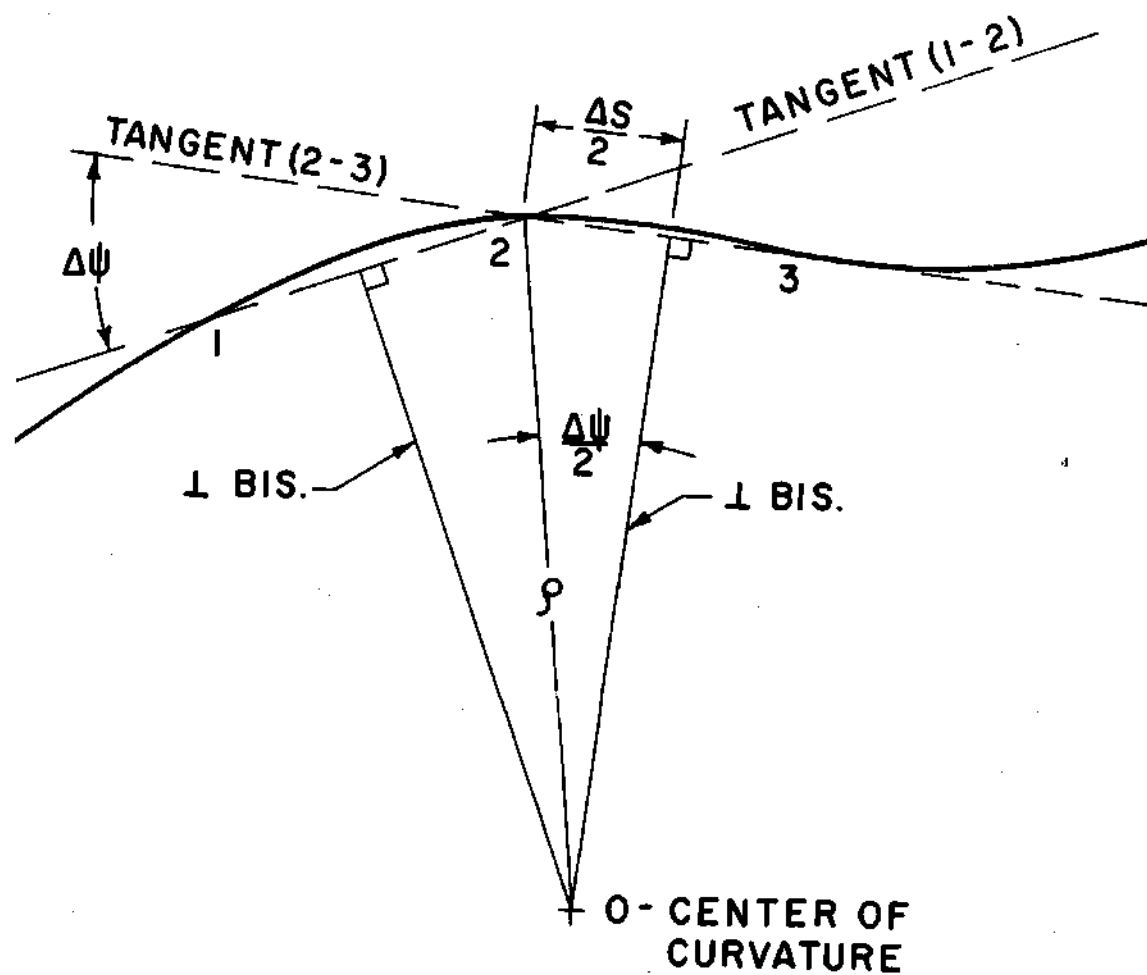


Figure 4. Two Common Infinitesimally Separated Tangents.

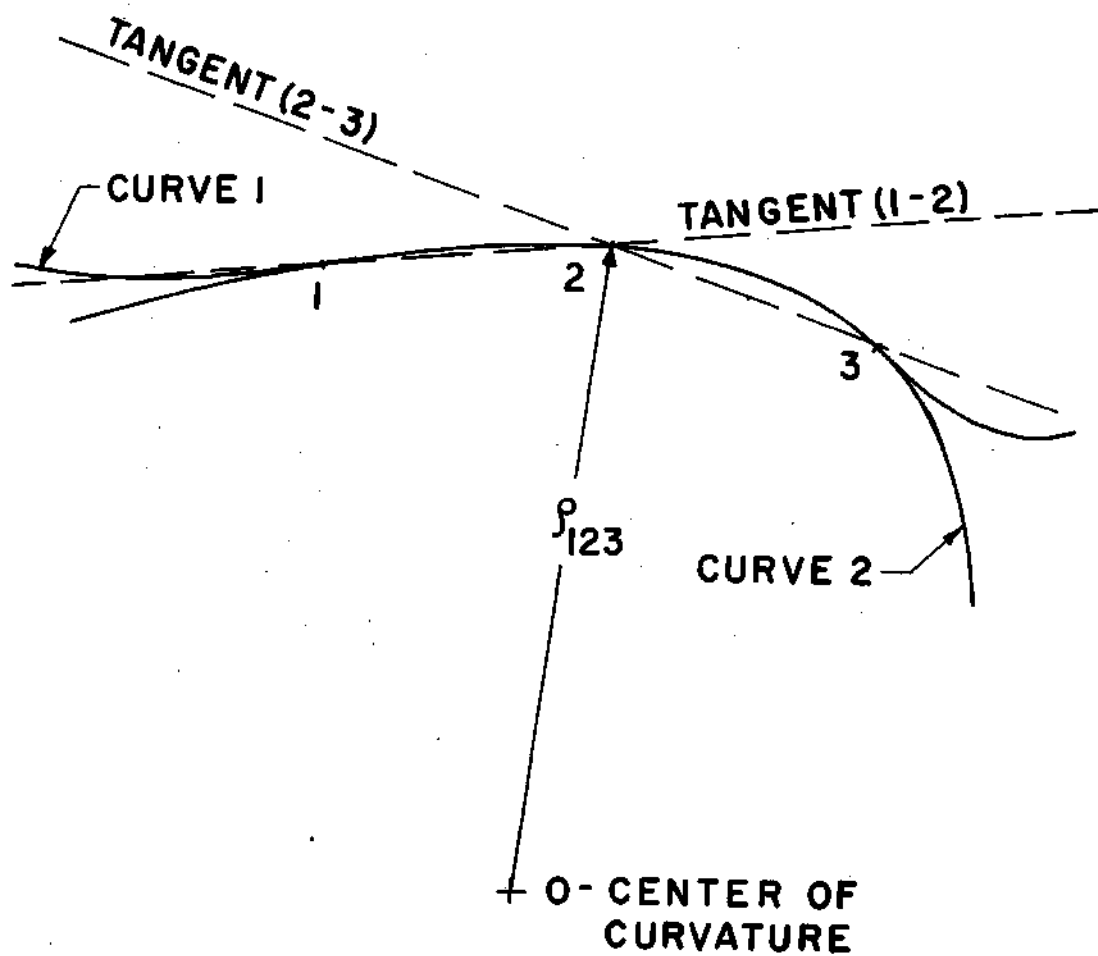


Figure 5. Second Order Contact (Curvature).

Nth Order Contact. Extension of the preceding sections, results in a general statement of n^{th} order contact in the form:

$n + 1$ --- Common Infinitesimally Separated Points
 n --- Common Infinitesimally Separated Tangents
 $n - 1$ --- Common Infinitesimally Separated Radii

where:

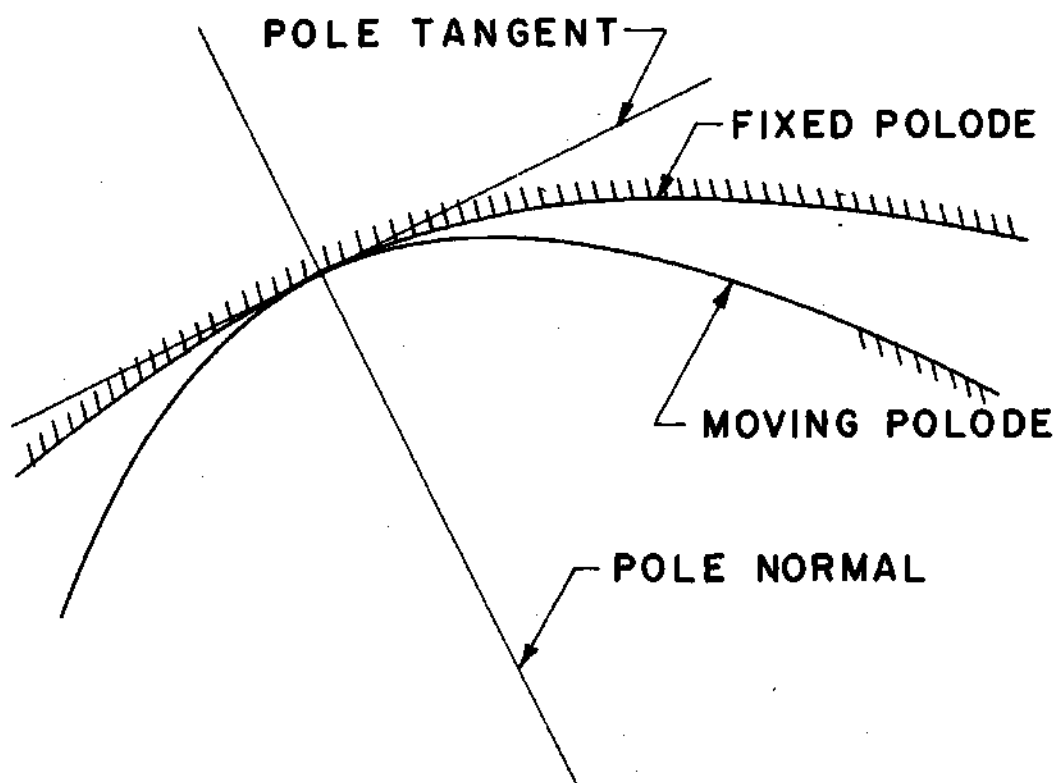
$n = 0$: Single Point of Intersection
 $n = 1$: Single Tangent
 Two Points
 $n = 2$: Single Radius of Curvature
 Two Tangents
 Three Points

Etc.

Single Position Design Kinematics

General. In the consideration of general coplanar motion,¹ the instant center (pole) of the motion is well defined. It is that point in the moving plane that has zero velocity with respect to the fixed plane. During the process of the motion the pole will trace out a locus (fixed polode) in the fixed plane as well as a locus (moving polode) in the moving plane. These polodes have first order contact and define, therefore, a unique common tangent called the pole tangent (Fig. 6). With the pole normal (a line perpendicular to the pole tangent through the pole), the pole tangent defines a coordinate system with which the basic formulas can be derived. This system is applicable for only one position of the moving plane; i.e., the formulas based on this system are single position design formulas.

¹The general case requires that the inflection circle be neither infinite nor zero in diameter and that the pole be located in a "finite" location in the moving plane.



**COPLANAR MOTION --- RESULTS FROM PURE ROLLING OF
THE POLODES AT THE POLE
(THE POLODES HAVE FIRST ORDER
CONTACT AT THE POLE, i.e.,
A COMMON TANGENT)**

Figure 6. Coplanar Motion (Pole and Polodes).

Inflection Circle. The inflection circle (Fig. 7) is the locus of those points in the moving plane which are passing through path elements of their point paths that:

1. are inflections of their point paths.
2. have an infinite radius of curvature.
3. have 2nd order contact with the tangent to the path element.
4. have three infinitesimally separated points in common with the inflection tangent.
5. have a stationary tangent.

Using the polar coordinate system formed by the pole tangent and the pole normal, the equation of the inflection circle is

$$r = PJ \sin \alpha$$

The fundamental equation of single position design kinematics is the quadratic relationship known as the Euler-Savary equation

$$\rho = \frac{r^2}{r - (PJ) \sin \alpha}$$

The equation is an analytical statement allowing the computation of the radius of curvature of all the path elements momentarily being described by points in the moving plane. (It does not include the point coincident with the pole.)

Cubic of Stationary Curvature. The cubic of stationary curvature is the locus of points in the moving plane that are momentarily passing through path elements (Fig. 8) which have:

- a. the same osculatory circle (i.e., the same radius of curvature) for two infinitesimally separated positions.

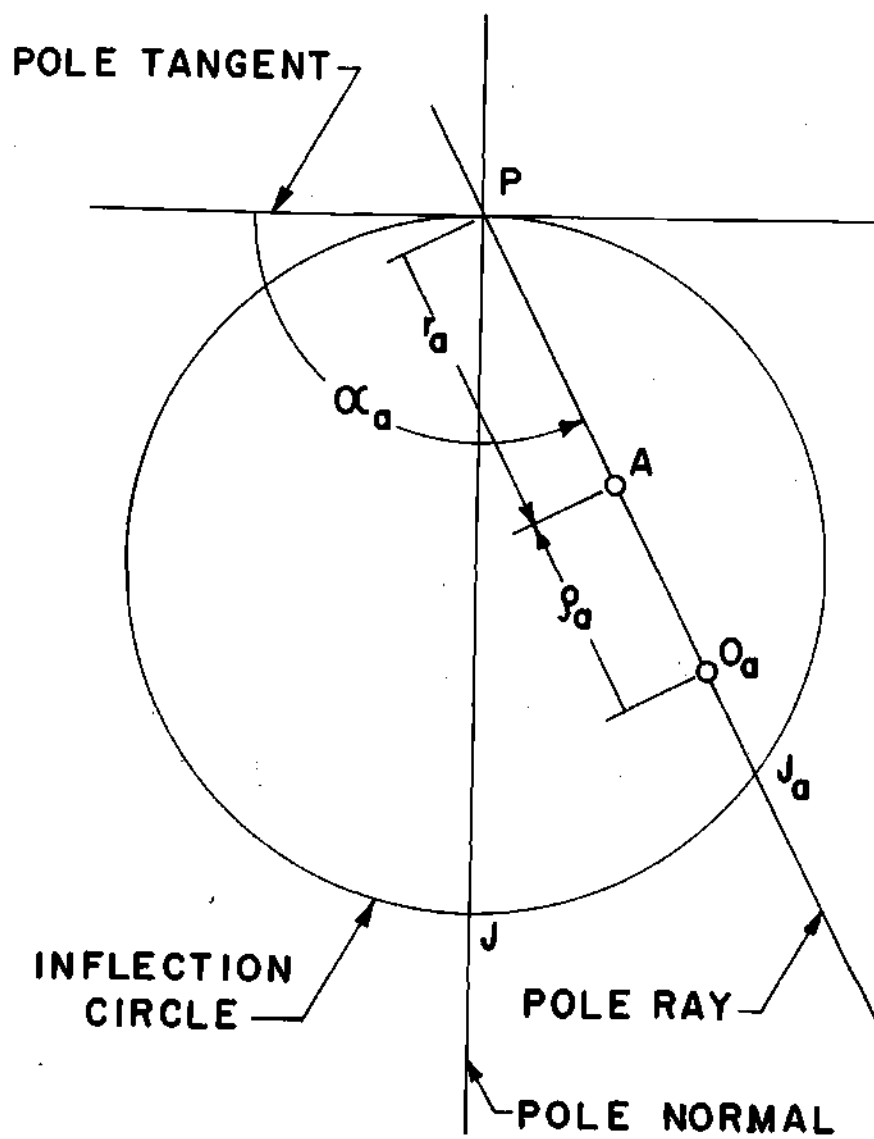
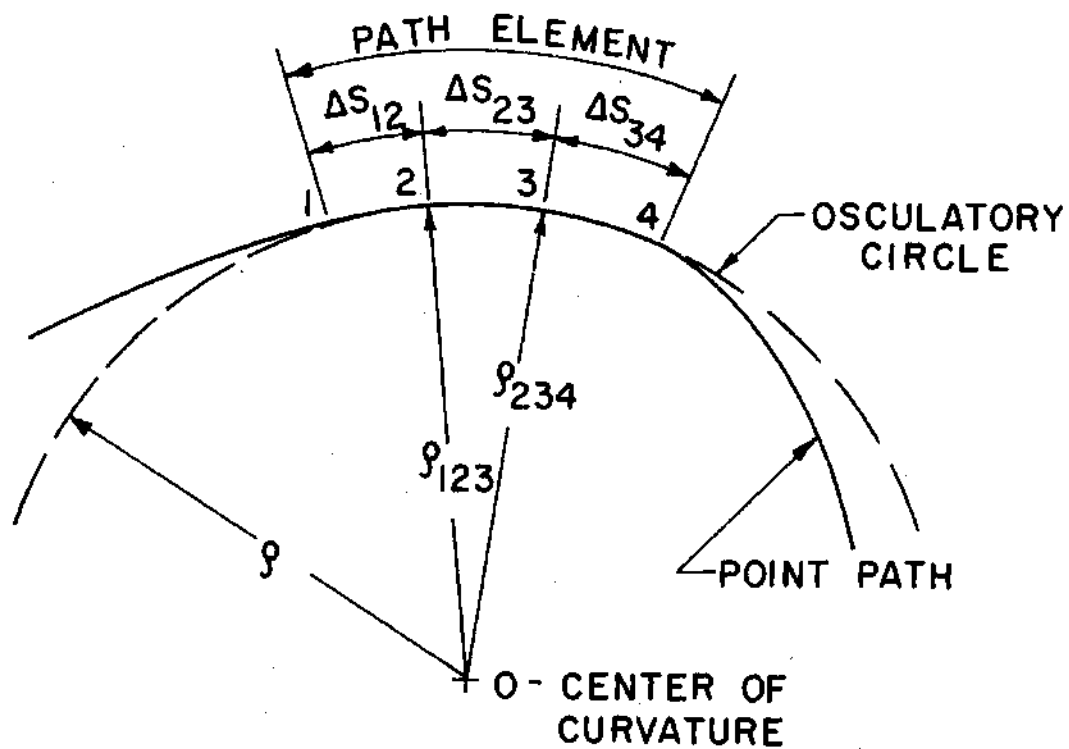


Figure 7. Inflection Circle.



IF

$$\Delta S = \Delta S_{12} = \Delta S_{23} = \Delta S_{34}$$

AND

$$\frac{d\rho}{ds} = \lim_{\Delta S \rightarrow 0} \left[\frac{\rho_{123} - \rho_{234}}{\Delta S} \right] = 0$$

THEN

$$\rho_{123} = \rho_{234}$$

Figure 8. Third Order Contact (Stationary Curvature).

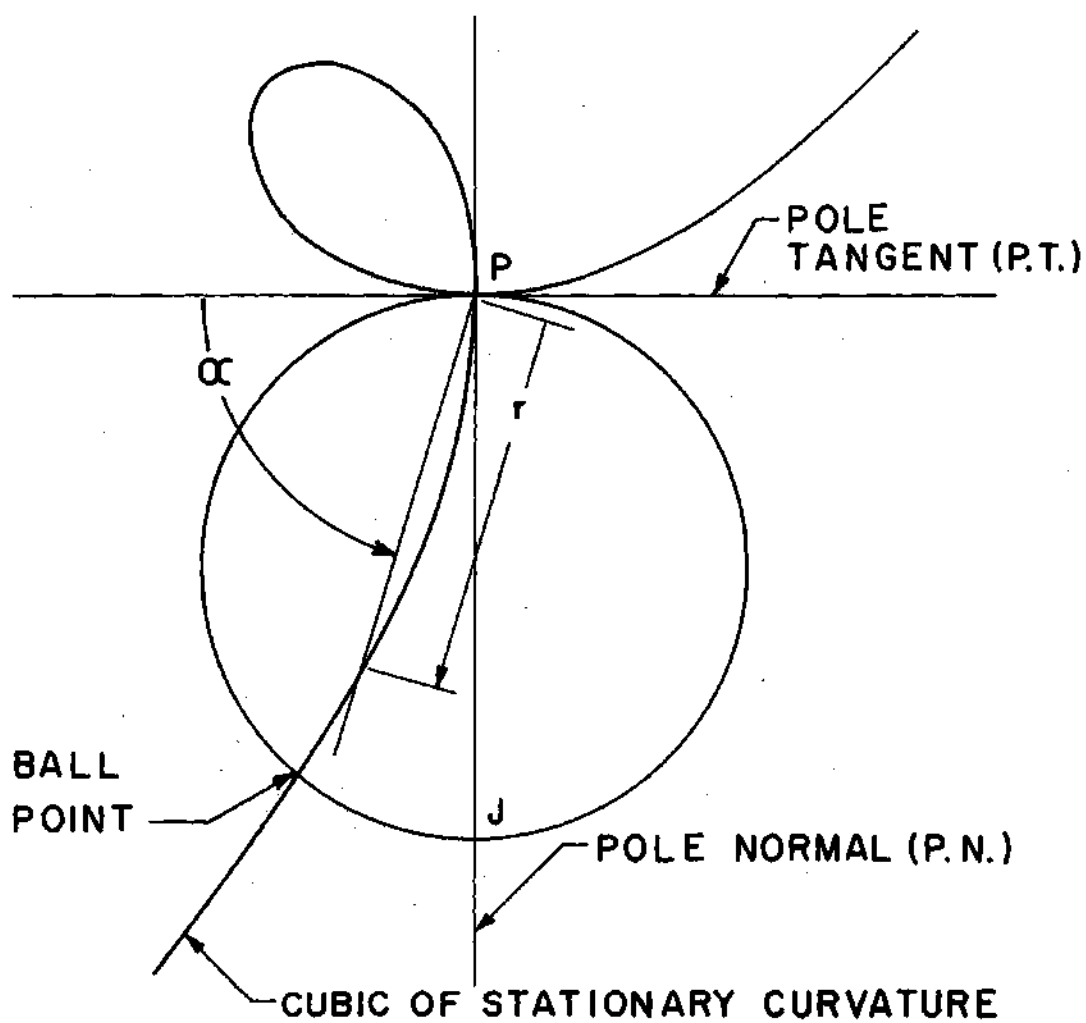
- b. four points in common with the stationary osculatory circle.
- c. third order contact with a circular arc.
- d. a zero first derivative of the radius of curvature with respect to distance along the path ($dp/ds = 0$).

The polar equation of the cubic curve (Fig. 9) is

$$\frac{1}{r} = \frac{1}{M \sin \alpha} + \frac{1}{N \cos \alpha}$$

where M and N are constants which can be found if two points on the curve are known from the constraints on the motion (Ex: the pin joints of a four-bar linkage). The derivations of the inflection circle equation, the Euler-Savary equation, and the cubic of stationary curvature equation are not given here because they are widely known. A classical derivation is given by Hall (14) and a derivation using the method of instantaneous invariants is given by Bottema (28) and Veldkamp (29).

The intersection of the inflection circle and the cubic of stationary curvature, known as Ball's point, is a point on the body which traces a path element having an infinite radius of curvature and the rate of change of the radius of curvature is momentarily zero. Four infinitesimally separated positions of the point lie on a straight line. That is, the point path has third-order contact with the contacting tangent. Ball's point, in general, can be determined for any position of the coupler plane of any given four-bar linkage and this point will trace an approximate straight line path over a limited range of motion. In a sense, all points on the inflection circle have such path elements but Ball's point is that point on the inflection circle which should best approximate a straight line.



EQUATION OF CUBIC

$$\frac{1}{r} = \frac{1}{M \sin \alpha} + \frac{1}{N \cos \alpha}$$

Figure 9. Cubic of Stationary Curvature.

Burmester Points. There are some points of a body moving with general coplanar motion which describe, momentarily, path elements having both the first and second derivatives of their radii of curvature with respect to distance along their point paths equal to zero for any given position of the moving body. These points are called the Burmester points¹ and have fourth order contact with their osculatory circles. As the body moves, five infinitesimally separated positions of a Burmester point lie on a circular arc (Fig. 10). A four-bar linkage synthesized such that one of the Burmester points of the plane of the coupler link lies on the inflection circle coincident with Ball's point should produce very satisfactory results as an approximate straight line mechanism. Five infinitesimally separated positions of the Ball-Burmester point of the coupler link lie on a straight line (Fig. 11).

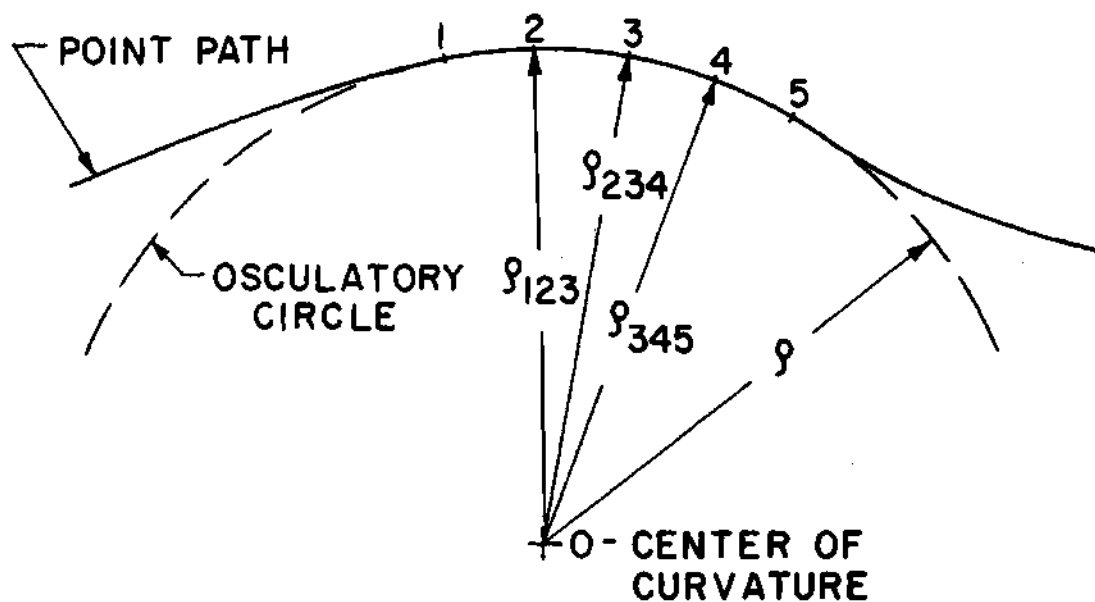
Locating Ball's Point for the Coupler Link of
a Given Four-Bar Mechanism

The equation of the cubic of stationary curvature in polar coordinates is

$$r = \frac{MN \sin \alpha \cos \alpha}{N \cos \alpha + M \sin \alpha} \quad (2.1)$$

The point of zero relative velocity, or pole, is the origin of the coordinate system. The distance from the origin to a point on the curve is r , and the ray from the pole to the point forms an angle α with the pole tangent. Quantities M and N are constants for any

¹As will be shown later, similar points exist for five finitely separated positions of the moving plane.

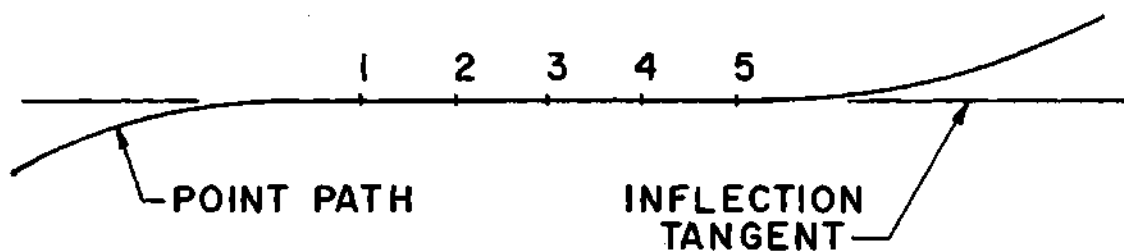


$$\rho = \rho_{123} = \rho_{234} = \rho_{345}$$

$$\text{WHEN } \frac{d\rho}{ds} = \frac{d^2\rho}{ds^2} = 0$$

$$\text{ASSUMING } \Delta S_{12} = \Delta S_{23} = \Delta S_{34} = \Delta S_{45}$$

Figure 10. Fourth Order Contact (Burmester Points).



$$\rho = \rho_{123} = \rho_{345} = \rho_{234} = \infty$$

Figure 11. Ball-Burmester Point.

particular positions of the moving plane. In applying Eq. (2.1) to the coupler plane of a four-bar linkage, two sets of values for r and α are known since the pin joints at the ends of the cranks must lie on the cubic. Designating the coordinates of the pin joints with r_a , α_a and r_b , α_b , substituting them into Eq. (2.1); and solving the two resulting equations simultaneously for M and N gives

$$M = \frac{\cot \alpha_a - \cot \alpha_b}{(1/r_a) \cos \alpha_a - (1/r_b) \cos \alpha_b} \quad (2.2)$$

$$N = \frac{\tan \alpha_a - \tan \alpha_b}{(1/r_a) \sin \alpha_a - (1/r_b) \sin \alpha_b} \quad (2.3)$$

The equation of the inflection circle in the same coordinate system is

$$r = D \sin \alpha \quad (2.4)$$

where D is the diameter of the inflection circle. This can be determined from the equation

$$D = \frac{-(PO_a)(PA)}{(O_aA) \sin \alpha_a} \quad (2.5a)$$

or

$$D = \frac{-(PO_b)(PB)}{(O_bB) \sin \alpha_b} \quad (2.5b)$$

The directions P to A and P to B are taken as positive. Note that D is always positive if α is measured from the pole tangent in a direction such that line PJ coincides with the line $\alpha = 90^\circ$. The

angle α from the pole tangent to the pole ray through Ball's point is determined by solving Eq. (2.1) and Eq. (2.4) simultaneously, giving

$$\alpha = \tan^{-1} \left\{ \frac{MN - DN}{DM} \right\} \quad (2.6)$$

Substituting the value of α obtained from Eq. (2.6) into Eq. (2.1) or Eq. (2.4) gives the distance from the origin to Ball's point.

Synthesis of Four-Bar Mechanisms

With a Burmester Point on the Inflection Circle

When a Burmester point is on the inflection circle, five infinitesimally separated positions of the point are on a straight line. In general, a better straight line mechanism results than for the case where Ball's point is not also a Burmester point. A familiar example of a mechanism with a Burmester point on the inflection circle is Watt's straight-line mechanism. For this particular mechanism, the inflection circle is a straight line (infinitely large circle).

Assume the inflection circle for the coupler link of a four-bar mechanism is given and that one of the coupler link's Burmester points, D, lies on the inflection circle. Points A and B are the pin joints at the ends of the cranks. The angles formed by the pole rays PD, PA, and PB with the pole tangent are α_d , α_a , and α_b . Suppose these angles are known but distances PA and PB are unknown. If distances PA and PB can be determined, then the fixed pivots can be located using the Euler-Savary equation and the linkage is completely determined.

Points A, B, and D must satisfy Eq. (2.1) since they trace paths having stationary curvature:

$$\frac{1}{PA} = \frac{1}{M \sin \alpha_a} + \frac{1}{N \cos \alpha_a} \quad (2.7a)$$

$$\frac{1}{PB} = \frac{1}{M \sin \alpha_b} + \frac{1}{N \cos \alpha_b} \quad (2.7b)$$

$$\frac{1}{PD} = \frac{1}{M \sin \alpha_d} + \frac{1}{N \cos \alpha_d} \quad (2.7c)$$

These three equations contain the four unknowns PA, PB, M, and N.

It has been shown (4, 30) that angle α_d , locating the pole ray on which Burmester point D is found, must satisfy the Allievi-Wolford equation

$$\begin{aligned} \tan^2 \alpha + \left[\tan \alpha_a + \tan \alpha_b + \frac{N(R - M)}{RM} \right] \tan \alpha \\ + \frac{N^2(M - 2R)/RM^2}{(\tan \alpha_a)(\tan \alpha_b)} = 0 \end{aligned} \quad (2.8)$$

with $\alpha = \alpha_d$.

The radius of curvature of the moving polode R is related to the diameter of the inflection circle D and the constant M by the equation

$$\frac{1}{M} = \frac{1}{3} \left(\frac{1}{D} + \frac{1}{R} \right)$$

or $R = M/(3 - M)$ where $D = 1$.

Eq. (2.7a), (2.7b), (2.7c) and (2.8) are solved simultaneously for the four unknowns PA, PB, M, and N resulting in the following expressions for PA and PB:

$$PA = \frac{[(3W + 1) \tan \alpha_d + VW] \sin \alpha_a}{(W + 1) \tan \alpha_a + W(2 \tan \alpha_d + V)} \quad (2.9)$$

$$PB = \frac{[(3W + 1) \tan \alpha_d + VW] \sin \alpha_b}{(W + 1) \tan \alpha_b + W(2 \tan \alpha_d + V)} \quad (2.10)$$

where $V = \tan \alpha_a + \tan \alpha_b$ and $W = (\tan \alpha_a) \cdot (\tan \alpha_b)$. The Euler-Savary equation is then used to locate the fixed pivots and the mechanism is determined.

Points A, B, and D are three of the four Burmester points for coupler link AB. The fourth Burmester point C can be found by determining the second root of Eq. (2.8), α_c , and then substituting this value of α into Eq. (2.1) or (2.4) and solving for the corresponding value of r (or PC). As a check on the correctness of the synthesized mechanism, the fourth Burmester point should lie on line AB. Mueller pointed out from geometric considerations that when one of the Burmester points lies on the inflection circle, the other three lie on a straight line. This fact has also been proved analytically by the author (31).

An ∞^3 number of approximate straight-line mechanisms can be synthesized using the various combinations of α_a , α_b , and α_d .

Alternate Linkages

A Burmester pair is any particular Burmester point and its corresponding Burmester center. For the case of general coplanar motion exhibited by the four-bar linkage, the Burmester pairs represented by the pin joints and the corresponding fixed pivots are separated by a fixed distance provided by the cranks of the linkage. A new link, however,

could be added to rigidly separate either of the other two Burmester pairs. This new link would allow the primary cranks to give the original motion to the coupler link for five infinitesimally separated positions. Beyond this infinitesimal range of movement, in general, only two of the three cranks can be used to control the coupler link (Fig. 12). If all four Burmester pairs are real, it is possible to determine six different mechanisms giving the same motion for five infinitesimally separated positions to all the points in the coupler plane. Linkages derived in this manner shall be termed -- alternate linkages.

Of particular interest is the case where a Burmester point coincides with Ball's point. The Burmester point D traces a coupler curve with five points in common with its tangent in the design position. Thus, the linkage produces a "five point exact" straight line motion and was first designated as such by Mueller (32). The remaining Burmester point C must lie on the coupler line AB and is located, in general, by

$$\tan \alpha_c = - \frac{2 \tan \alpha_d + V}{W + 1} \quad (2.11)$$

and

$$r_c = \frac{[(3W + 1) \tan \alpha_d + VW] \sin \alpha_c}{(W - 1)(2 \tan \alpha_d + V)} \quad (2.12)$$

where the diameter of the inflection circle is unity.

Ball-Double Burmester Point

If, for general coplanar motion, one of the Burmester points coincides with the Ball point, the remaining three Burmester points

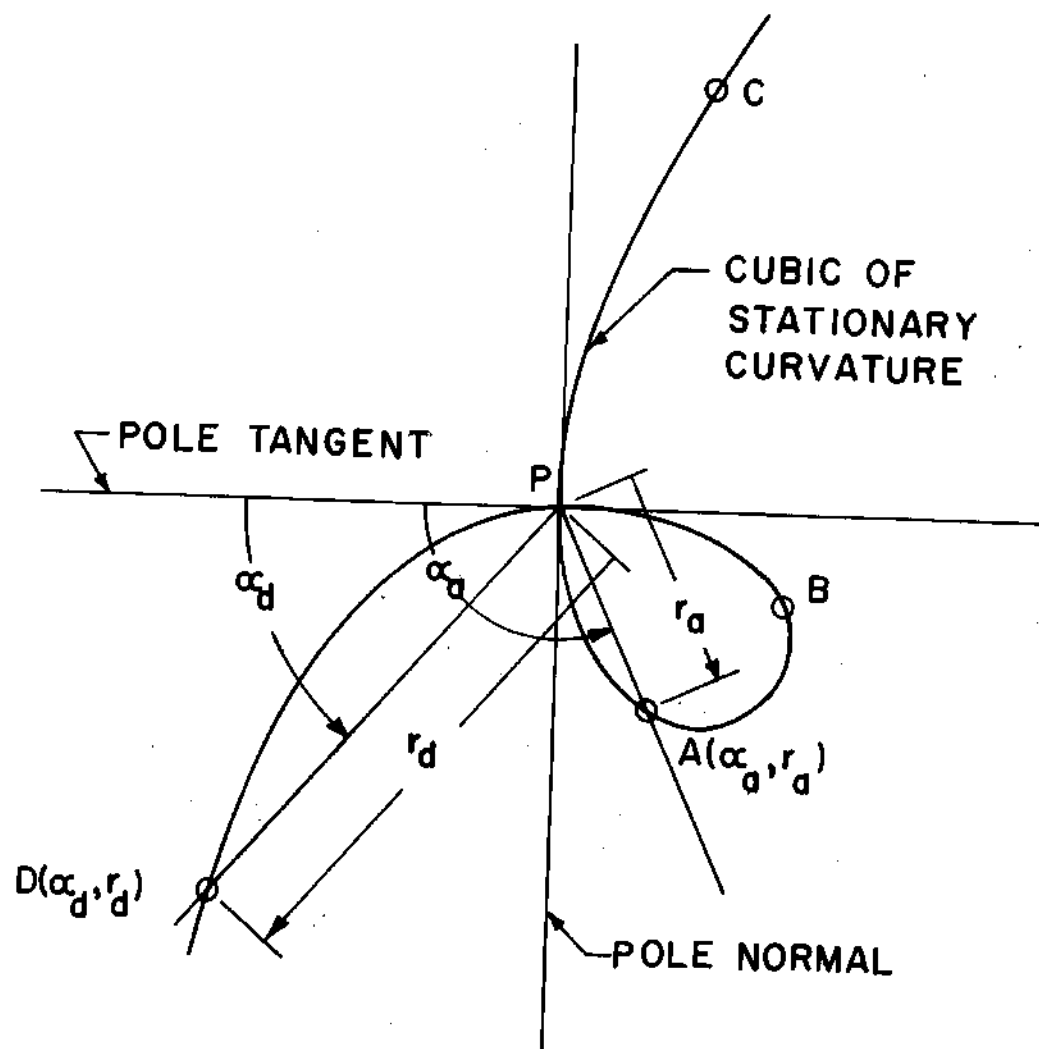


Figure 12. Alternate Linkages.

are collinear. If, in addition, a second Burmester point coincides with the Ball point, it would mean that the Ball-double Burmester point would lie on the coupler center line of a four-bar linkage formed by using the remaining Burmester pairs as rigid cranks.

By introducing the requirement that $\alpha_c = \alpha_d$ in Eq. (2.11), the result

$$\tan \alpha_d = - \frac{V}{(W + 3)} \quad (2.11a)$$

is obtained.

Substitution of this information into Eq. (2.9) and (2.10) results in

$$PA = \frac{V(W - 1) \sin \alpha_a}{(W + 3) \tan \alpha_a + VW} \quad (2.9a)$$

$$PB = \frac{V(W - 1) \sin \alpha_b}{(W + 3) \tan \alpha_b + VW} \quad (2.10a)$$

where again $V = \tan \alpha_a + \tan \alpha_b$ and $W = (\tan \alpha_a)(\tan \alpha_b)$. The remaining necessary design relations obtained by use of the Euler-Savary equation are:

$$O_a A = \frac{PA}{PA - \sin \alpha_a}, \quad PO_a = \frac{PA \sin \alpha_a}{PA - \sin \alpha_a}$$

$$O_b B = \frac{PB}{PB - \sin \alpha_b}, \quad PO_b = \frac{PB \sin \alpha_b}{PB - \sin \alpha_b}$$

Note that the added restriction of a double Burmester point has reduced the number of independent design parameters to two.

Alternate Linkages of Slider-Crank Mechanisms

In many design situations, an analytical single position design procedure may be preferred to a graphical approach. In the case of a general slider-crank mechanism (Fig. 13), A and D are known to be Burmester points. The other two Burmester points (if they are real) will generate coupler curves which should result in a particularly good match to circular arcs. In fact, for single position design, the portion of a point path described by a Burmester point near the initial position should be the best approximation to an arc available.

A method is developed next to find the alternate linkages of a given slider-crank mechanism. To accomplish this, Eq. (2.8) will be written in terms of the parameters of a general slider-crank mechanism. As before, consider the diameter PJ of the inflection circle to be equal to unity. Since the pin joint D lies on the inflection circle, the polar coordinate $r = PD$ can be determined from Eq. (2.4) as

$$PD = \sin \alpha_d \quad (2.13)$$

Also, the coordinates of the pin joints, A and D, satisfy the cubic of stationary curvature equation and result in

$$\frac{1}{PA} = \frac{1}{M \sin \alpha_a} + \frac{1}{N \cos \alpha_a} \quad (2.14)$$

and

$$\frac{1}{PD} = \frac{1}{M \sin \alpha_d} + \frac{1}{N \cos \alpha_d} \quad (2.15)$$

A relationship between PA and PO_a in the form

$$PA = \frac{PO_a \sin \alpha_a}{PO_a + \sin \alpha_a} \quad (2.16)$$

can be obtained from the Euler-Savary equation. Solving Eqs. (2.13), (2.14), (2.15), and (2.16) simultaneously for the constants, M and N gives

$$M = \frac{N}{N - \tan \alpha_d} \quad (2.17)$$

with

$$N = \frac{PO_a}{\sin \alpha_a} (\tan \alpha_a - \tan \alpha_d) \quad (2.18)$$

Now, remembering that, if $PJ = 1$, the expression for the instantaneous radius of curvature of the moving polode can be written in terms of the constant M as

$$R = \frac{M}{3 - M} \quad (2.19)$$

the constants of Eq. (2.8) are then defined by

$$\frac{N(R - M)}{RM} = 2 \tan \alpha_d - N \quad (2.20)$$

and

$$\frac{N^2(M - 2R)/RM^2}{(\tan \alpha_a)(\tan \alpha_d)} = \frac{\tan \alpha_d - N}{\tan \alpha_a} \quad (2.21)$$

Thus, the quadratic equation for locating the two unknown Burmester points of a general slider-crank mechanism may be written as:

$$\tan^2 \alpha + [\tan \alpha_a + 3 \tan \alpha_d - N] \tan \alpha + \frac{\tan \alpha_d - N}{\tan \alpha_a} = 0 \quad (2.22)$$

Solving this modified Allievi-Wolford equation produces two roots, $\tan \alpha_b$ and $\tan \alpha_c$. The polar coordinates, α_b and α_c , for the Burmester points, B and C, respectively, must also satisfy the cubic of stationary curvature equation giving the other coordinates:

$$PB = \frac{N \sin \alpha_b}{N + K_b}, \quad PC = \frac{N \sin \alpha_c}{N + K_c} \quad (2.23)$$

where $K_b = \tan \alpha_b - \tan \alpha_d$ and $K_c = \tan \alpha_c - \tan \alpha_d$. Also, the Euler-Savary equation can be used to give the necessary expressions for locating the center of curvature of the point paths traced by the Burmester points. These expressions are:

$$PO_b = \frac{N \sin \alpha_b}{K_b}, \quad PO_c = \frac{N \sin \alpha_c}{K_c} \quad (2.24)$$

In order to simplify the calculations, let

$$k = \frac{PO_a}{\sin \alpha_a}$$

or

$$PO_a = k \sin \alpha_a \quad (2.25)$$

then, from Eq. (2.16) it follows that

$$PA = \frac{PO_a}{k + 1} \quad (2.16a)$$

and from Eq. (2.18) that

$$N = k (\tan \alpha_a - \tan \alpha_d) \quad (2.18a)$$

Example: Alternate Linkages to a General Slider-Crank Mechanism

The design parameters are k , α_a , and α_d . By assuming values for these parameters, the slider-crank mechanism is completely specified.

Assume:

$$k = 0.90$$

$$\alpha_a = 72.5^\circ$$

$$\alpha_d = 45^\circ$$

for the mechanism shown in Fig. (2-14). From Eqs. (2.13), (2.25) and (2.16a), we obtain

$$PD = \sin \alpha_d = 0.7071 \text{ units}$$

$$PO_a = k \sin \alpha_a = 0.8765 \text{ units}$$

$$PA = \frac{PO_a}{k + 1} = 0.4613 \text{ units}$$

Substituting the values for α_a , α_d , and N into Eq. (2.22), we obtain

$$\alpha_b = 103.13^\circ$$

$$\alpha_c = 4.015^\circ$$

Calculating the values for K_b and K_c and using these values in Eq. (2.23), we obtain

$$PB = \frac{N \sin \alpha_b}{N + K_b} = -0.5711 \text{ units}$$

and

$$PC = \frac{N \sin \alpha_c}{N + K_c} = 0.1316 \text{ units}$$

At this point, the pin joints A, B, and C should be laid out graphically. If the points lie on a straight line, the solution is probably correct. The author has found that this is a valuable check on the calculations. Now, by using Eq. (2.24), the following values are obtained

$$PO_b = \frac{N \sin \alpha_b}{K_b} = -0.3600 \text{ units}$$

$$PO_c = \frac{N \sin \alpha_c}{K_c} = -0.1451 \text{ units}$$

In Fig. (14), the point paths of the pin joints B and C, fixed to the connecting rod AD, are drawn and match their corresponding circular arcs in the vicinity of the design position.

In Fig. (15) and (16), the alternate linkages $O_c O_b BC$ and $O_b O_a AB$, respectively, are represented with their straight line output. In Fig. (17), the third alternate linkage $O_a O_c CA$ is drawn with its two cognates (determined graphically according to Roberts' law), and all three linkages (alternate and two cognates) produce the same straight line output.

GIVEN:

$$k = 0.90$$

$$\alpha_a = 72.5^\circ$$

$$\alpha_d = 45^\circ$$

CALCULATED:

$$PA = 0.4613$$

$$PO_a = 0.8765$$

$$\alpha_b = 103.13^\circ$$

$$\alpha_c = 4.02^\circ$$

$$PB = -0.5711$$

$$PO_b = -0.3600$$

$$PC = 0.1316$$

$$PO = -0.1451$$

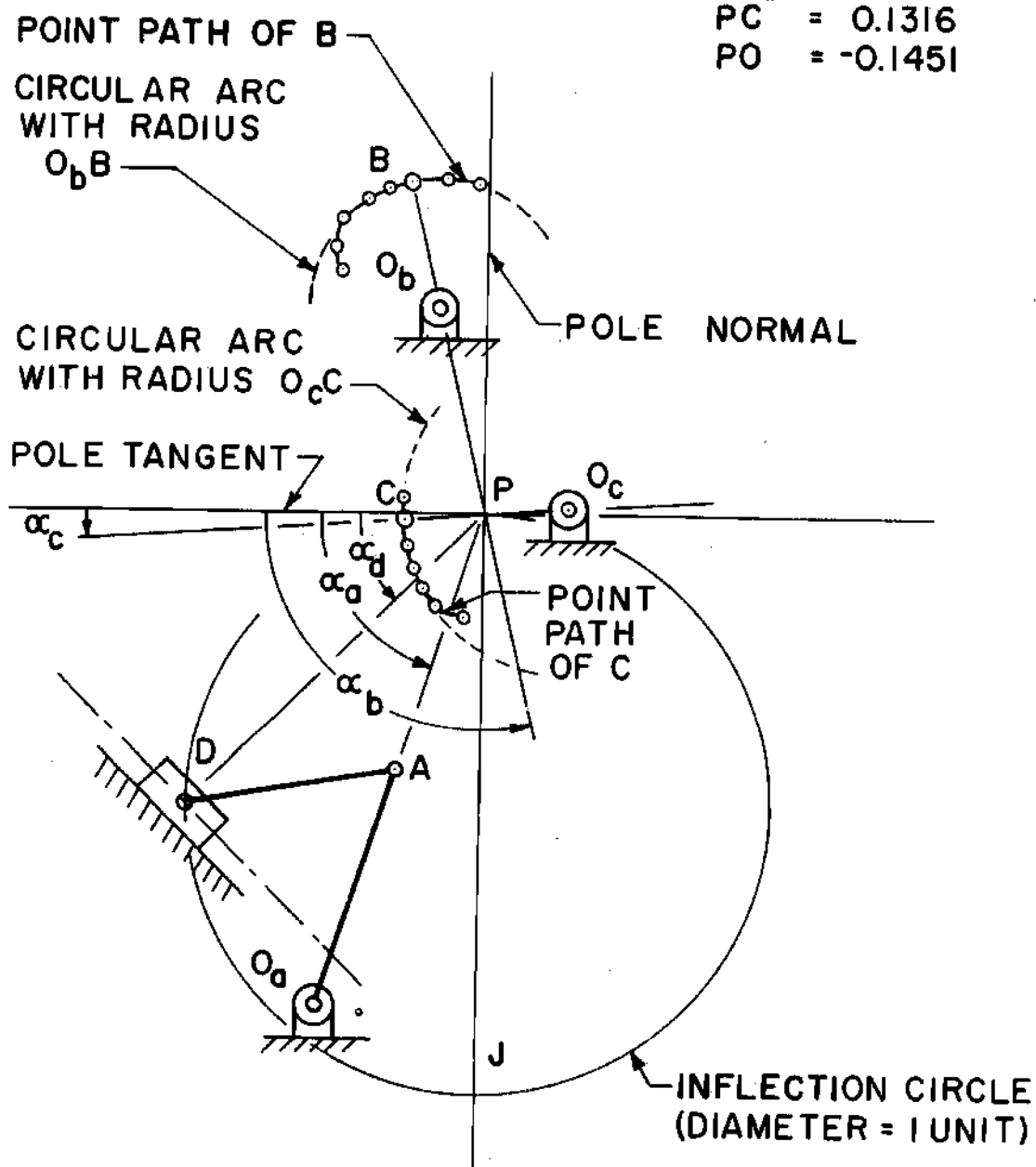


Figure 14. Burmester Points Located in Coupler Plane of Slider-Crank

GIVEN:

$$\begin{aligned}
 k &= 0.90 \\
 \alpha_a &= 72.5^\circ \\
 \alpha_d &= 45^\circ
 \end{aligned}$$

CALCULATED:

$$\begin{aligned}
 PB &= -0.5711 \\
 PO_b &= -0.3600 \\
 \alpha_b &= 103.13^\circ \\
 \alpha_c &= 4.02^\circ \\
 PO_c &= -0.1451 \\
 PC &= 0.1316
 \end{aligned}$$

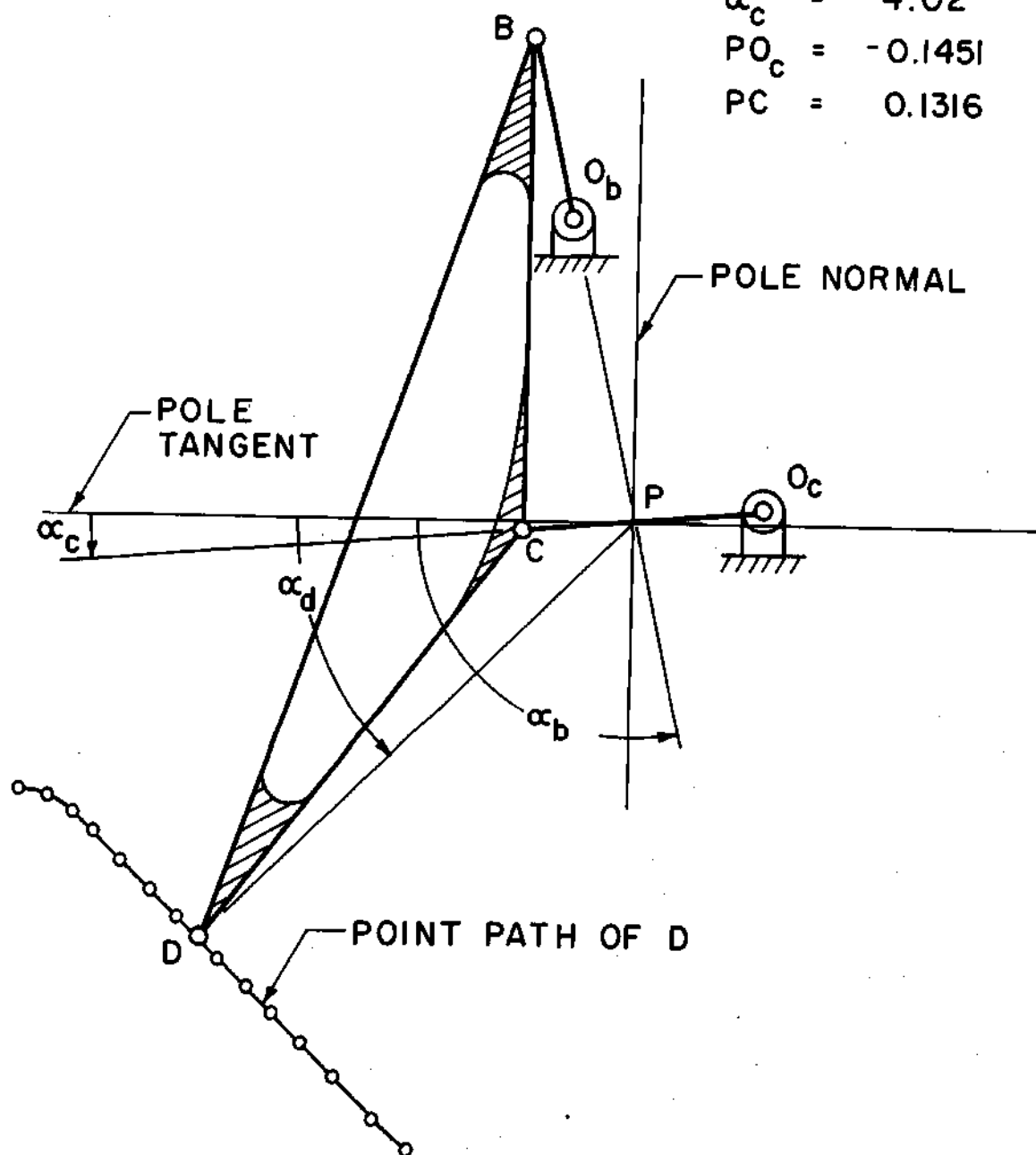


Figure 15. Alternate Using Burmester Pairs B and C.

GIVEN:

$$\begin{aligned} k &= 0.90 \\ \alpha_d &= 72.5^\circ \\ \alpha_d &= 45^\circ \end{aligned}$$

CALCULATED:

$$\begin{aligned} PA &= 0.4613 \\ PO_d &= 0.8765 \\ \alpha_b &= 103.13^\circ \\ PO_b &= -0.3600 \\ PB &= -0.5711 \end{aligned}$$

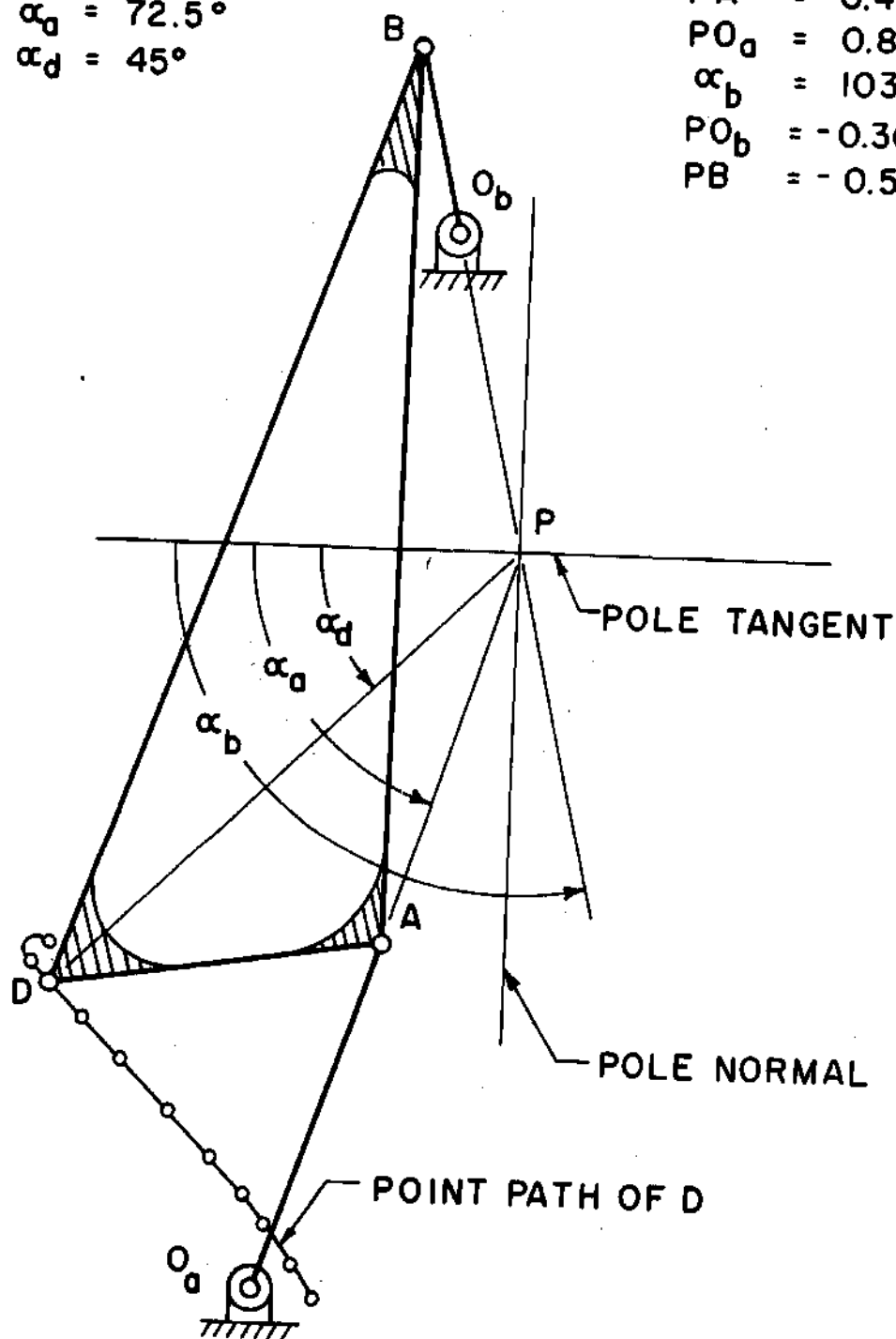


Figure 16. Alternate Using Burmester Pairs A and B.

GIVEN:

$$\begin{aligned} k &= 0.90 \\ \alpha_d &= 72.5^\circ \\ \alpha_d &= 45^\circ \end{aligned}$$

CALCULATED:

$$\begin{aligned} PA &= 0.4613 \\ PO_d &= 0.8765 \\ \alpha_c &= 4.02^\circ \\ PO_c &= -0.1451 \\ PC &= 0.1316 \end{aligned}$$

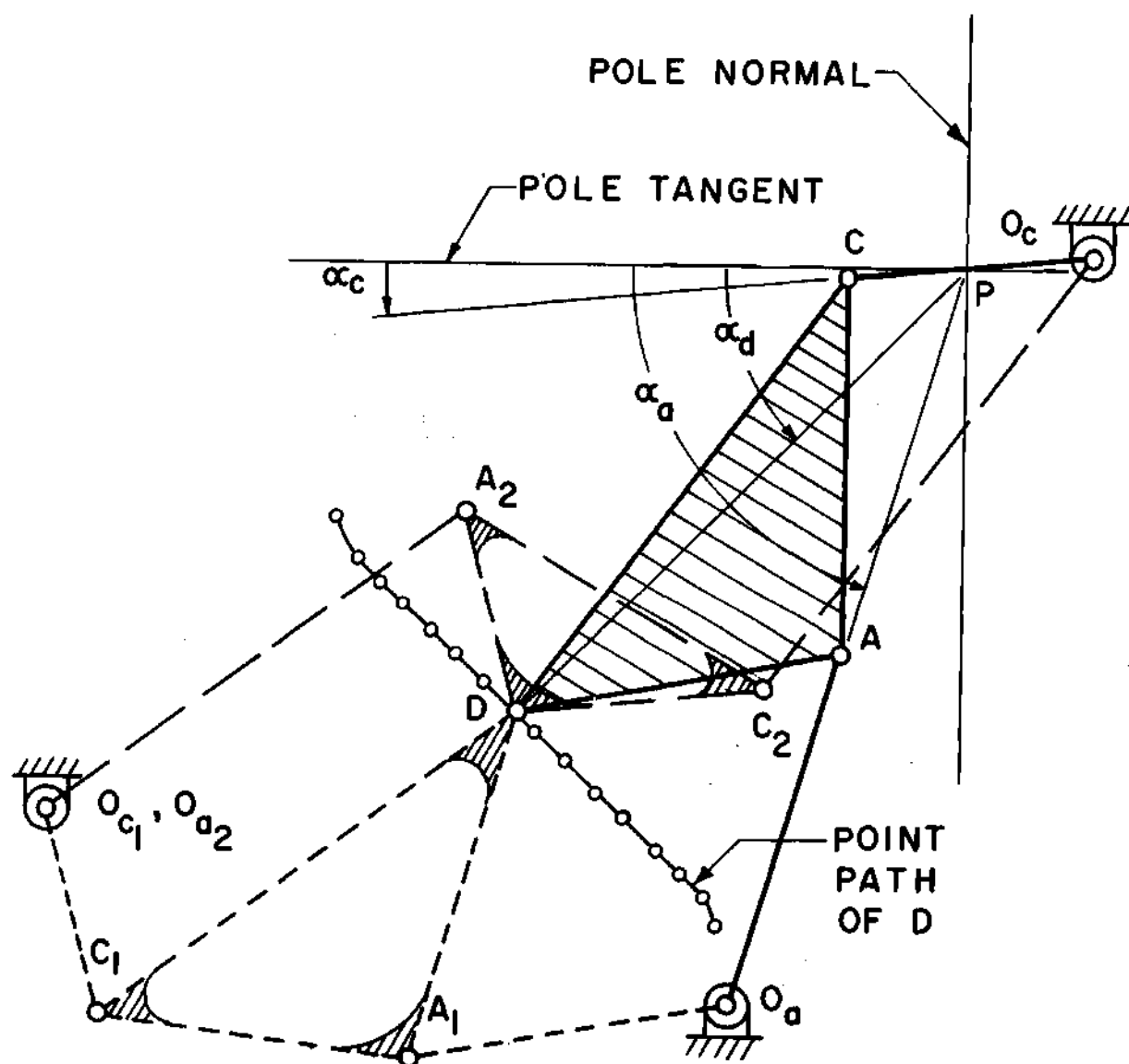


Figure 17. Alternate Using Burmester Pairs A and C Showing Two Cognates.

Thus, it is possible to determine nine different four-bar linkages derived from a given slider-crank mechanism that produce a "five point exact" straight line motion at the coupler point D.

Pin Joint D of the Slider at the Inflection Pole¹

General Case. The constants in Eq. (2.22) for finding the Burmester points of a general slider-crank mechanism become infinitely large when $\alpha_d = 90^\circ$. To develop the necessary design equations, the numerator and denominator of the right term of Eq. (2.9) are divided by $\tan \alpha_d$, and then the value $\alpha_d = 90^\circ$ is substituted into the equation for α_d . If

$$PA = \frac{k \sin \alpha_a}{k + 1}$$

this results in

$$\frac{k}{k + 1} = \frac{3W + 1}{2W}$$

where $W = (\tan \alpha_a)(\tan \alpha_b)$. Thus this is an expression from which $\tan \alpha_b$ can be determined. Developing this expression gives

$$\tan \alpha_b = -\frac{k + 1}{k + 3} \cdot \frac{1}{\tan \alpha_a} \quad (2.26)$$

Using the same procedure as was used with Eq. (2.9), Eq. (2.23) and (2.24) become

$$PO_b = k \sin \alpha_b \quad (2.23a)$$

$$PB = \frac{PO_b}{k + 1} \quad (2.24a)$$

¹These design procedures give results similar to those obtained by W. Meyer zur Capellen in Ref. (6).

Eq. (2.16a) and (2.25) remain unchanged. Therefore, if the Ball-Burmester point (the pin joint of the slider) coincides with the inflection pole such that $J \equiv D$, the pin joints are found on a circle whose diameter on the pole normal is $k/(k+1)$, and the pivot joints are found on a circle whose diameter on the pole normal is k (Fig. 18). Furthermore, the coupler link is parallel to the fixed link.

From Eq. (2.11), it is obvious that if $\alpha_d = 90^\circ$, then $\tan \alpha_c = \infty$, i.e., $\alpha_c = 90^\circ$. Using the same technique as before, Eq. (2.12) yields

$$PC = \frac{k}{2k + 4} \quad (2.28)$$

and writing the Euler-Savary equation in the form

$$\frac{1}{PC} - \frac{1}{PO_c} = \frac{1}{PJ_c}$$

gives

$$PO_c = \frac{k}{k + 4} \quad (2.29)$$

D Coincides with the Inflection Pole - Equal Cranks. If the pole normal bisects the angle between the cranks of the alternate linkage of a slider-crank mechanism that has its pin joint D coincident with the inflection pole, the cranks will be equal in length. This requires, then, that $\alpha_a + \alpha_b = 180^\circ$ and gives

$$\tan \alpha_a = -\tan \alpha_b$$

or from Eq. (2.26)

$$\tan^2 \alpha_a = \frac{k+1}{k+3}$$

i.e., choose any k which will give a real solution to

$$\tan \alpha_a = \pm \left(\frac{k+1}{k+3} \right)^{\frac{1}{2}} \quad (2.30)$$

For this case the slider-crank is completely specified by choosing k .

D Coincides with the Inflection Pole - A Double Burmester Point.

If it is desired to have a double Burmester point coincident with Ball's point at the inflection pole J , i.e., if the four Burmester points are to lie on a straight line, then

$$PJ = PC = \frac{k}{2k+4} = 1$$

or

$$k = -4$$

Thus, in this case, the only parameter is α_a and Eq. (2.26) becomes

$$\tan \alpha_b = \frac{-3}{\tan \alpha_a} \quad (2.27a)$$

CHAPTER III

THE ANALYTICAL THEORY OF FINITELY SEPARATED
POSITIONS OF THE MOVING PLANE IN COPLANAR MOTION

Recently, Bottema (27) and Veldkamp (29) have made a fundamental analysis of coplanar motion based on the well known transformation existing between two rectangular coordinate systems. The problem of finitely separated positions was investigated on a purely geometrical basis by Burmester (2). For the synthesis of kinematical problems, Burmester supplied involved graphical procedures based on his results. Bottema (27) has outlined the theory for the problem of having four finitely separated positions on a straight line and five finitely separated positions on a circle (a Burmester point). Based on these fundamental concepts, this chapter includes the original derivation of:

(a) The equation of the cubic circle point curve which is the locus of all points in the moving plane that pass through four finitely separated positions on a circular arc.

(b) The quartic equation for the general case of five finitely separated positions of a point of the moving plane on a circular arc.

(c) The quadratic equation for the unknown Burmester points of a given four-bar linkage which is specified in five finitely separated positions.

(d) The cubic equation for the three unknown Burmester points where the fourth Burmester point is prescribed in five finitely separated positions on a straight line.

(e) The quadratic equation for the unknown Burmester points of a given slider-crank mechanism which is specified in five finitely separated positions.

Four Finitely Separated Positions

Let OXY and oxy be cartesian coordinate systems arbitrarily located in (but rigidly attached to) the fixed and moving planes, respectively. The two sets of coordinates, x, y and X, Y of a point in the moving system are related by the transformation

$$X = x \cos \phi - y \sin \phi + a \quad (3.1)$$

$$Y = x \sin \phi + y \cos \phi + b$$

A displacement of the moving plane is determined by the three parameters a, b, ϕ . The motion of the plane is known when these parameters are specified as functions of time t . Since our interest is purely geometrical, ϕ may be considered as the independent variable ($\phi = t$) with $a = a(\phi)$ and $b = b(\phi)$.

It is well-known from Burmester theory that there is a locus of points (x, y) in the moving plane that have four finitely separated positions on a circular arc when the moving plane assumes the four prescribed positions defined by a_j, b_j, ϕ_j . Taking the general form of the equation of these circular arcs in the fixed plane to be

$$Q_0(x^2 + y^2) + 2Q_1x + 2Q_2y + Q_3 = 0 \quad (3.2)$$

the following set of four equations is obtained:

$$\begin{aligned}
& Q_0 \left[(x \cos \phi_j - y \sin \phi_j + a_j)^2 + (x \sin \phi_j + y \cos \phi_j + b_j)^2 \right] \\
& + 2Q_1 \left[x \cos \phi_j - y \sin \phi_j + a_j \right] + 2Q_2 \left[x \sin \phi_j + \right. \\
& \left. y \cos \phi_j + b_j \right] + Q_3 = 0 \quad \text{for } j = 0, 1, 2, 3 \quad (3.3)
\end{aligned}$$

by using the transformation of Eq. (3.1). The equation representing the initial position ($j = 0$) is identically satisfied if the oxy and OXY coordinate systems are coincident ($a_0 = b_0 = \phi_0 = 0$) in that position. Expanding Eq. (3.3) and noting that

$$Q_0(x^2 + y^2) + 2Q_1x + 2Q_2y + Q_3 \equiv 0$$

for all prescribed positions of the plane, the system

$$\begin{aligned}
& Q_0 \left[a_j^2 + b_j^2 + 2x(a_j \cos \phi_j + b_j \sin \phi_j) + 2y(b_j \cos \phi_j - a_j \sin \phi_j) \right] \\
& + 2Q_1 \left[a_j - x \cos \phi_j - y \sin \phi_j + x \right] + 2Q_2 \left[b_j + y \cos \phi_j + \right. \\
& \left. x \sin \phi_j - y \right] = 0 \quad \text{for } j = 1, 2, 3 \quad (3.4)
\end{aligned}$$

results. Temporarily considering x and y to have known values, Eq. (3.4) is a set of three linear homogeneous equations in terms of the unknowns Q_0 , Q_1 , and Q_2 . It has a solution only if the Δ determinant of the coefficients of Q_0 , Q_1 , Q_2 is zero. That is, if Eq. (3.4) is written in the form

$$Q_0 D_j + Q_1 E_j + Q_2 F_j = 0, \quad j = 1, 2, 3 \quad (3.5)$$

then the following equality

$$\begin{vmatrix} D_1 & E_1 & F_1 \\ D_2 & E_2 & F_2 \\ D_3 & E_3 & F_3 \end{vmatrix} = 0 \quad (3.6)$$

must hold. Eq. (3.6) is a cubic equation in terms of the coordinates x, y which is the equation of the circle point curve in the moving plane corresponding to the prescribed positions a_j, b_j, ϕ_j .

When $Q_0 = 0$, Bottema has shown that, in general, there is only one point in the moving plane which passes through four positions on a straight line. If this line in the fixed plane is taken to be the X axis (such that $b_j = 0$), then the x, y coordinate system is uniquely established in the fixed plane and the origin of the moving system has four positions on the X axis. This choice of location for the fixed system does not markedly limit the generality of the results to be obtained, but it does considerably reduce the magnitude of the resulting equations.

Even so, the expansion of the cubic equation represented by Eq. (3.6) is quite lengthy and will not be given in detail here. Using the symbols

$$R_k = g_{k+1} n_{k+2} - n_{k+1} g_{k+2}$$

$$S_k = m_{k+1} h_{k+2} - h_{k+1} m_{k+2}$$

$$T_k = n_{k+1} h_{k+2} - h_{k+1} n_{k+2} + g_{k+1} m_{k+2} - m_{k+1} g_{k+2}$$

$$U_k = a_{k+1}^2 m_{k+2} - m_{k+1} a_{k+2}^2$$

$$V_k = a_{k+1}^2 n_{k+2} - n_{k+1} a_{k+2}^2$$

where the additional symbols

$$g_k = 2a_k \cos \phi_k, \quad h_k = 2a_k \sin \phi_k$$

$$m_k = 1 - \cos \phi_k, \quad n_k = -\sin \phi_k$$

are used for brevity; the cubic equation is

$$\begin{aligned} \sum_{k=1}^3 & \left[y^3(n_k s_k) + x^3(m_k R_k) + xy^2(T_k n_k + m_k s_k) \right. \\ & + yx^2(T_k m_k + n_k R_k) + y^2(a_k s_k + n_k U_k) + x^2(a_k R_k + m_k V_k) \\ & \left. + xy(T_k a_k + m_k U_k + n_k V_k) + x(a_k V_k) + y(a_k U_k) \right] = 0 \quad (3.7) \end{aligned}$$

The subscript k in Eq. (3.6) is cyclic and corresponds to the values for $j = 1, 2, 3$.

The centers X, Y of the circular arcs, upon which four homologous positions of the points of the circle point curve are located, form the locus known as the center point curve. In general, the centers are located at

$$X = -\frac{Q_1}{Q_0}, \quad Y = -\frac{Q_2}{Q_0}$$

when $Q_0 \neq 0$. Considering x, y to be values for points on the circle point curve, Eq. (3.5) for $j = 1, 2$ is solved to obtain

$$\begin{aligned} X &= \frac{E_1 F_2 - E_2 F_1}{D_1 F_2 - D_2 F_1} \\ Y &= \frac{D_1 E_2 - D_2 E_1}{F_1 E_2 - F_2 E_1} \end{aligned} \quad (3.8)$$

the locus of the desired center point curve corresponding to the x, y values of the circle point curve.

Five Finitely Separated Positions

Bottema's results¹ concerning the Burmester points for five distinct positions of the moving plane may be particularized to provide a method of determining the unknown Burmester points of the coupler plane of a given mechanism. Of primary interest are the four-bar and slider-crank mechanisms where the two pin joints on the coupler link are known to be Burmester points. The alternate four-bar linkages of a slider-crank mechanism would then have a coupler curve with five precision points on a straight line. The alternate four-bar linkages of a given four-bar would have five precision points on a circular arc thus providing an approximation to a circular arc which could be used as the basis for a dwell mechanism.

The Burmester point (x_0, y_0) of the moving plane will have five distinct homologous positions on the Burmester circle

$$Q_0(x^2 + y^2) + 2Q_1x + 2Q_2y + Q_3 = 0 \quad (3.9)$$

in the fixed plane and the corresponding Burmester center will be at

$$x_c = -\frac{Q_1}{Q_0}, \quad y_c = -\frac{Q_2}{Q_0} \quad (3.10)$$

If $Q_0 = 0$, the Burmester center is located at infinity; for the present, however, we will assume that $Q_0 \neq 0$. For the distinct positions of the moving plane, the relation between the coordinates of the Burmester

¹Eq. (3.9) through (3.21) of this article are due to Bottema (27).

points in the moving and fixed systems are:

$$\begin{aligned} X_j &= x_o \cos \phi_j - y_o \sin \phi_j + a_j \\ Y_j &= x_o \sin \phi_j + y_o \cos \phi_j + b_j \end{aligned} \quad (3.11)$$

The Burmester point must lie on the Burmester circle in the specified positions; or X_j, Y_j must satisfy Eq. (3.2)

$$\begin{aligned} Q_0 &[(x_o \cos \phi_j - y_o \sin \phi_j + a_j)^2 + (x_o \sin \phi_j + \\ &y_o \cos \phi_j + b_j)^2] + 2Q_1(x_o \cos \phi_j - y_o \sin \phi_j + a_j) \\ &+ 2Q_2(x_o \sin \phi_j + y_o \cos \phi_j + b_j) + Q_3 = 0 \end{aligned} \quad (3.12)$$

There are 15 specified values, a_j, b_j, ϕ_j for the prescribed positions of the moving plane. There are five unknowns

$$x_o, y_o, Q_1/Q_0, Q_2/Q_0, Q_3/Q_0$$

and five independent equations. In the present form, however, the equations are non-linear. Rearranging Eq. (3.12) so that all the known factors play the role of coefficients and using an expanded set of unknowns

$$\begin{aligned} Z_0 &= Q_0, \quad Z_1 = -Q_0 x_o, \quad Z_2 = -Q_0 y_o, \quad Z_5 = Q_1, \quad Z_6 = Q_2 \\ Z_3 &= -Q_1 x_o - Q_2 y_o, \quad Z_4 = -Q_1 y_o + Q_2 x_o \\ Z_7 &= Q_0(x_o^2 + y_o^2) + 2Q_1 x_o + 2Q_2 y_o + Q_3, \end{aligned} \quad (3.13)$$

Eq. (3.12) is linearized in the form

$$\begin{aligned} & \frac{1}{2}(a_j^2 + b_j^2)Z_0 + \underline{a}_j Z_1 + \underline{b}_j Z_2 + a_j Z_5 + b_j Z_6 \\ & + (1 - \cos \phi_j) Z_3 + (\sin \phi_j) Z_4 + \frac{1}{2} Z_7 = 0 \end{aligned} \quad (3.14)$$

$$\text{for } j = 0, 1, 2, 3, 4$$

Note that $Z_7 \equiv 0$, for the prescribed positions of the moving plane and that

$$\begin{aligned} \underline{a}_j &= -a_j \cos \phi_j - b_j \sin \phi_j \\ \underline{b}_j &= a_j \sin \phi_j - b_j \cos \phi_j \end{aligned} \quad (3.15)$$

are the a_j, b_j values for the inverse motion. If, in the zero position, the coordinate systems coincide ($a_0 = b_0 = \phi_0 = 0$) then the first of Eq. (3.14) is identically satisfied. Hence

$$\begin{aligned} & \frac{1}{2}(a_j^2 + b_j^2)Z_0 + \underline{a}_j Z_1 + \underline{b}_j Z_2 + a_j Z_5 + b_j Z_6 \\ & + (1 - \cos \phi_j) Z_3 + \sin \phi_j Z_4 = 0 \end{aligned} \quad (3.16)$$

$$\text{for } j = 1, 2, 3, 4$$

is a set of four independent equations in terms of seven homogeneous unknowns (disregarding Z_7). Since Eq. (3.12) is a determinate system, Eq. (3.16) must also be determinate. This requires that there be two additional relations among the new unknowns. There are the quadratic equations

$$\begin{aligned}
 z_0 z_3 &= z_1 z_5 + z_2 z_6 \\
 z_0 z_4 &= z_2 z_5 - z_1 z_6
 \end{aligned}
 \tag{3.17}$$

Eq. (3.16) and (3.17) form a determinate set of six equations in the six unknowns $(z_i/z_0, i = 1, 2, 3, 4, 5, 6)$ which has, in general, four solutions. Consequently, there are four distinct Burmester pairs for the general case of coplanar motion. If the solution takes the form

$$\begin{aligned}
 (z_i/z_0)^k &= C_i^k \quad k = 1, 2, 3, 4 \\
 i &= 1, 2, 3, 4, 5, 6
 \end{aligned}$$

then from Eq. (3.13) and (3.10), we obtain

$$\left. \begin{aligned}
 x_o^k &= -C_1^k, \quad y_o^k = -C_2^k \\
 x_c^k &= -C_5^k, \quad y_c^k = -C_6^k
 \end{aligned} \right\} \begin{array}{l} \text{Burmester} \\ \text{Point} \\ \\ \text{Burmester} \\ \text{Center} \end{array} \left. \vphantom{\begin{aligned} x_o^k \\ x_c^k \end{aligned}} \right\} \begin{array}{l} \text{Burmester} \\ \text{Pair} \end{array} \tag{3.18}$$

If one of the Burmester centers lies at infinity, then $Q_0 = 0$ and from Eq. (3.13) we note that

$$z_0 = z_1 = z_2 = 0$$

such that the system, Eq. (3.16), reduces to

$$a_j z_5 + b_j z_6 + (1 - \cos \phi_j) z_3 + \sin \phi_j z_4 = 0 \tag{3.19}$$

for that particular Burmester pair. For Eq. (3.19) to have a solution, the Δ determinant

$$\Delta = \begin{vmatrix} a_1 & b_1 & 1 - \cos \phi_1 & \sin \phi_1 \\ a_2 & b_2 & 1 - \cos \phi_2 & \sin \phi_2 \\ a_3 & b_3 & 1 - \cos \phi_3 & \sin \phi_3 \\ a_4 & b_4 & 1 - \cos \phi_4 & \sin \phi_4 \end{vmatrix} \quad (3.20)$$

must be zero. Bottema (27) has shown that when $\Delta = 0$ for one of the Burmester points (five homologous positions on a straight line), then the three remaining Burmester points in the moving plane must be collinear.

In considering the solution for the general case of five distinct positions of the moving plane, let Z_0, Z_1, Z_2 temporarily assume known values. Rearranging Eq. (3.16)

$$\begin{aligned} (1 - \cos \phi_j) Z_3 + (\sin \phi_j) Z_4 + a_j Z_5 + b_j Z_6 \\ = -\frac{1}{2} (a_j^2 + b_j^2) Z_0 - a_j Z_1 - b_j Z_2 \end{aligned} \quad (3.21)$$

and using Kramer's rule for linear systems, where Δ_{ji} represents the cofactor of the coefficients of the unknowns of Eq. (3.21) for the m, n positions (based on standard matrix notation) in the determinant, the solution takes the form

$$\begin{aligned} (Z_{i+2})_{\Delta} = Z_0 \left[-\frac{1}{2} \sum_{j=1}^4 (-1)^{j+i} (a_j^2 + b_j^2) \Delta_{ji} \right] \\ + Z_1 \left[-\sum_{j=1}^4 (-1)^{j+i} a_j \Delta_{ji} \right] \\ + Z_2 \left[-\sum_{j=1}^4 (-1)^{j+i} b_j \Delta_{ji} \right] \end{aligned} \quad (3.22)$$

where Δ is the determinant of the coefficients. A reduced form of Eq. (3.22) is obtained by using the symbols

$$A_i = -\frac{1}{2} \sum_{j=1}^4 (-1)^{j+i} (a_j^2 + b_j^2) \Delta_{ji}$$

$$B_i = -\sum_{j=1}^4 (-1)^{j+i} a_j \Delta_{ji}$$

$$C_i = -\sum_{j=1}^4 (-1)^{j+i} b_j \Delta_{ji}$$

such that

$$Z_{i+2} = \frac{A_i}{\Delta} Z_0 + \frac{B_i}{\Delta} Z_1 + \frac{C_i}{\Delta} Z_2 \quad (3.23)$$

where Δ , A_i , B_i , C_i are completely determined by the parameters a_j , b_j , ϕ_j . Substituting Eq. (3.23) in Eq. (3.17) gives

$$C_4 \left(\frac{Z_2}{Z_0} \right)^2 + \frac{Z_2}{Z_0} \left[(C_3 + B_4) \frac{Z_1}{Z_0} + (A_4 - C_1) \right]$$

$$+ B_3 \left(\frac{Z_1}{Z_0} \right)^2 + (A_3 - B_1) \frac{Z_1}{Z_0} - A_1 = 0$$

$$C_3 \left(\frac{Z_2}{Z_0} \right)^2 + \frac{Z_2}{Z_0} \left[(B_3 - C_4) \frac{Z_1}{Z_0} + (A_3 - C_2) \right]$$

$$- B_4 \left(\frac{Z_1}{Z_0} \right)^2 - (A_4 + B_2) \frac{Z_1}{Z_0} - A_2 = 0 \quad (3.24)$$

and removing the unknown Z_2/Z_0 gives the quartic

$$\begin{aligned}
 & \left(\frac{Z_1}{Z_0}\right)^4 \left[(c-h)^2 - f(c-h)(a-f) + h(a-f)^2 \right] \\
 & + \left(\frac{Z_1}{Z_0}\right)^3 \left[(h-c)[2(k-d) + f(b-g)] + (a-f)[f(k-d) \right. \\
 & + g(h-c) + k(a-f) + 2h(b-g)] + \left(\frac{Z_1}{Z_0}\right)^2 \left[(k-d)^2 \right. \\
 & + 2(h-c)(l-e) + (b-g)[f(k-d) + g(h-c) + h(b-g)] \\
 & + (a-f)[f(l-e) + g(k-d) + l(a-f) + 2k(b-g)] \left. \right] \\
 & + \left(\frac{Z_1}{Z_0}\right) \left[(2+g)(l-e)(k-d) + (b-g)[f(l-e) + g(k-d) \right. \\
 & + 2l(a-f) + k(b-g)] + \left[(l-e)^2 + g(l-e)(b-g) \right. \\
 & + l(b-g)^2 \left. \right] = 0 \tag{3.25}
 \end{aligned}$$

where:

$$\begin{aligned}
 a &= \frac{C_3 + B_4}{C_4} & f &= \frac{B_3 - C_4}{C_3} \\
 b &= \frac{A_4 - C_1}{C_4} & g &= \frac{A_3 - C_2}{C_3} \\
 c &= \frac{B_3}{C_4} & h &= -\frac{B_4}{C_3} \\
 d &= \frac{A_3 - B_1}{C_4} & k &= -\frac{A_4 + B_2}{C_3} \\
 e &= -\frac{A_1}{C_4} & l &= -\frac{A_2}{C_3}
 \end{aligned}$$

Alternate Linkages of a Specified Four-Bar Linkage

For a known mechanism (such as a four-bar linkage or a slider-crank mechanism), two of the Burmester points are known to be the pin joints A, B of the coupler link. Hence, two solutions to the above quartic are known

$$x_o^a = -\left(\frac{z_1}{z_o}\right)^a, \quad x_o^b = -\left(\frac{z_1}{z_o}\right)^b$$

where the superscript denotes the Burmester pairs of the similarly labeled pin joints. The quartic can be reduced to a quadratic equation by using the relations that exist between the roots and coefficients of these equations; therefore

$$\begin{aligned} &\left(\frac{z_1}{z_o}\right)^2 + \left\{x_o^a + x_o^b + (h-c)[2(k-d) + f(b-g)] \right. \\ &\quad \left. + (a-f)[f(k-d) + g(h-c) + k(a-f) + 2h(b-g)]\right\} \left(\frac{z_1}{z_o}\right) \\ &\quad + \frac{(l-e)^2 + g(l-e)(b-g) + l(b-g)^2}{(x_o^a)(x_o^b)} = 0 \end{aligned} \quad (3.26)$$

Considering Eq. (3.26) in the form

$$\left(\frac{z_1}{z_o}\right)^2 + E\left(\frac{z_1}{z_o}\right) + F = 0$$

it is apparent that the two remaining Burmester pairs are imaginary if

$$E^2 - 4F < 0$$

If the solutions are real, the corresponding values for Z_2/Z_0 can be obtained from

$$\frac{Z_2}{Z_0} = \frac{(Z_1/Z_0)^2 (c-h) + (Z_1/Z_0)(k-d) + l-e}{g(a-f) + (b-g)} \quad (3.27)$$

which is arrived at by eliminating the factor $(Z_2/Z_0)^2$ from Eq. (3.24). Then the values for Z_5/Z_0 and Z_6/Z_0 may be calculated by using Eq. (3.23). Finally, the coordinates of the Burmester pairs are found by using the formulas outlined in Eq. (3.18). Any two Burmester pairs will provide the necessary constraints to take the moving plane through the five prescribed positions (a_j, b_j, ϕ_j) . Hence, there are six linkages which are alternates of each other.

Five Finitely Separated Positions on a Line

The application of the preceding results depends on the determination of the position parameters a_j, b_j, ϕ_j for a given linkage. In the initial position, the coordinate systems must coincide, but there is no restriction upon the location and orientation of the systems. It is apparent, then, that given a slider-crank mechanism in five successive positions, the values $b_j = 0, a_j, \phi_j$ would result if the X axis were coincident with the line of motion of the pin joint of the slider D. Letting a_j be the location of the pin joint D along the X axis, the values for ϕ_j would correspond to positions of the x axis rigidly fixed to the coupler link. If a_j were chosen to give Chebychev spacing for the pin joint of the slider, then the deviation curve of the straight line motion that would result by using the alternate four-bar linkages to the slider-crank would have nearly equal maximums between the

precision points. Note that the Δ determinant, Eq. (3.20), is zero when $b_j = 0$.

With $b_j = 0$, Eq. (3.16) becomes

$$\begin{aligned} \frac{1}{2} a_j^2 Z_0 - a_j \cos \phi_j Z_1 + a_j \sin \phi_j Z_2 + (1 - \cos \phi_j) Z_3 \\ + \sin \phi_j Z_4 + a_j Z_5 = 0 \quad \text{for } j = 1, 2, 3, 4 \end{aligned} \quad (3.28)$$

Note that Z_6 does not appear in this set of linear independent equations. Now, letting Z_5 and Z_0 temporarily be assumed as known, Eq. (3.28) may be rearranged

$$\begin{aligned} -a_j \cos \phi_j Z_1 + a_j \sin \phi_j Z_2 + (1 - \cos \phi_j) Z_3 + \sin \phi_j Z_4 \\ = -\frac{1}{2} a_j^2 Z_0 - a_j Z_5 \end{aligned} \quad (3.29)$$

which has the set of solutions

$$\begin{aligned} (Z_i)_{\Delta} &= Z_0 \left[-\frac{1}{2} \sum_{j=1}^4 (-1)^{j+i} (a_j)^2 \Delta_{ji} \right] \\ &+ Z_5 \left[-\sum_{j=1}^4 (-1)^{j+i} a_j \Delta_{ji} \right] \end{aligned} \quad (3.30)$$

and using the symbols

$$M_i = -\frac{1}{2} \sum_{j=1}^4 (-1)^{j+i} (a_j)^2 \Delta_{ji}$$

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a result similar to Eq. (3.23) is obtained

$$(Z_i) = \frac{Z_0}{\Delta} M_i + \frac{Z_5}{\Delta} N_i \quad i = 1, 2, 3, 4 \quad (3.31)$$

Substituting this relationship into Eq. (3.17), the result is a cubic equation

$$\begin{aligned} & \left(\frac{Z_5}{Z_0}\right)^3 (N_1^2 + N_2^2) + \left(\frac{Z_5}{Z_0}\right)^2 (2M_1N_1 - N_1N_3 + 2N_2M_2 - N_2N_4) \\ & + \left(\frac{Z_5}{Z_0}\right) (M_1^2 + M_2^2 - M_1N_3 - N_1M_3 - N_2M_4 - N_4M_2) \\ & - (M_3M_1 + M_2M_4) = 0 \end{aligned} \quad (3.32)$$

in terms of the unknown Z_5/Z_0 . The three roots to the above cubic equation correspond to the three desired Burmester pairs. In this form (with $b_j = 0$), a_j and ϕ_j are arbitrary and do not depend on a given slider-crank mechanism. The values for Z_i/Z_0 , $i = 1, 2, 3, 4$ can be calculated by using Eq. (3.31) and then the value for Z_6/Z_0 becomes

$$Z_6/Z_0 = \frac{(Z_3/Z_0) - (Z_1/Z_0)(Z_5/Z_0)}{(Z_2/Z_0)} \quad (3.33)$$

which is arrived at from Eq. (3.17).

Alternate Linkages of a Specified Slider-Crank Mechanism

If two Burmester pairs are specified in advance, for example as the pin joints A, D of a general slider-crank mechanism, then the cubic equation can be reduced further to a quadratic equation. The quadratic equation would have the remaining Burmester pairs as solutions.

This would require that $\phi_j = f(a_j)$ as determined by the constraint of the slider-crank mechanism on the moving system.

By comparing the properties between the roots and coefficients of cubic and quadratic polynomials, we obtain

$$\left(\frac{z_5}{z_0}\right)^2 + \left[X_c^a + \frac{2M_1N_1 - N_1N_3 + 2N_2M_2 - N_2N_4}{N_1^2 + N_2^2} \right] \left(\frac{z_5}{z_0}\right) - \frac{M_3M_1 + M_2M_4}{(N_1^2 + N_2^2) X_c^a} = 0$$

where X_c^a corresponds to the known center of rotation of the crank.

The solution is completed by using Eq. (3.33), (3.31) and (3.17).

CHAPTER IV

MULTIPLE POSITION DESIGN

The design procedures of the preceding chapters indicate that, in general, there are at least three independent design parameters. At best, the designer can satisfactorily master the duality resulting from two independent parameters. This capacity probably disappears when the designer is confronted with three parameters, and the outcome is a trial and error procedure which may or may not give satisfactory results. It is unlikely that the design could be optimized. It may be argued, then, that the designer should restrict himself to procedures involving only two parameters. At present the accuracy and length of the approximate straight line point path does not depend directly on the number of independent parameters. Curvature theory and finitely separated position theory can be combined to reduce the number of design parameters and at the same time, increase the length and accuracy of the approximate straight line segment of the coupler curve.

Except for the special case of symmetry and a closed form solution obtained by Mueller (33), no solutions for the case where the coupler curve has six points in common with a straight line are available. In general, an iterative procedure will be necessary to extend the previous solutions so that six points, some infinitesimally separated and some finitely, lie on a straight line. The most direct approach is to design a slider-crank mechanism to satisfy the curvature theory of infinitesimally separated points (2nd, 3rd, and 4th order) and to displace the slider

to the desired finitely separated positions. There will, generally, be some points in the coupler plane of the slider-crank mechanism that will produce coupler curves which can be approximated by an appropriate circular arc (hence a rigid crank) which satisfies the geometry of both the finitely and infinitesimally separated positions. The analysis of this approach to the design of approximate straight line mechanisms is the topic of this chapter.

Input-Output Displacement Function for the Slider-Crank Mechanism

The slider-crank mechanism designed in the initial position to satisfy curvature theory is to be displaced to the finitely separated positions in the anticipated design procedures. Consequently, it becomes necessary to derive a closed form displacement function (the travel s of the slider being the position parameter) similar to that obtained for the four-bar linkage. In Fig. (19) is represented a general slider-crank mechanism with an offset of d . The coordinates of the pin joints A, D of the coupler in the x, y coordinate system are:

$$\begin{aligned} x_a &= R \cos \omega & y_a &= R \sin \omega \\ x_d &= s & y_d &= d \end{aligned} \quad (4.1)$$

Requiring that the coupler AD be inextensible leads to

$$M^2 = (s - R \cos \omega)^2 + (d - R \sin \omega)^2 \quad (4.2)$$

Using the trigonometric formulas

$$\sin \omega = \frac{2 \tan \omega/2}{1 + \tan^2 \omega/2}, \quad \cos \omega = \frac{1 - \tan^2 \omega/2}{1 + \tan^2 \omega/2}$$

in the expanded form of Eq. (4.2) gives

$$2Rs \left[\frac{1 - \tan^2 \omega/2}{1 + \tan^2 \omega/2} \right] + 2Rd \left[\frac{2 \tan \omega/2}{1 + \tan^2 \omega/2} \right] = s^2 + R^2 + d^2 - M^2 \quad (4.3)$$

which reduces to the quadratic in $\tan \omega/2$

$$\begin{aligned} \left[(s+R)^2 + d^2 - M^2 \right] \tan^2 \omega/2 + \left[4Rd \right] \tan \omega/2 \\ + \left[(s-R)^2 + d^2 - M^2 \right] = 0 \end{aligned} \quad (4.4)$$

The general solution of this quadratic is

$$\omega = 2 \tan^{-1} \left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right] \quad (4.5)$$

where

$$A = (s + R)^2 + d^2 - M^2$$

$$B = 4Rd$$

$$C = (s - R)^2 + d^2 - M^2$$

The output angle has two possible values depending on which of the signs on the radical is used. Graphically, the negative sign is used when the coupler link is below the line O_2D .

Coupler Point Triangulation for a Slider-Crank Mechanism

In the following articles it will be necessary to determine the

U,V coordinates of location of the coupler point B of a slider-crank mechanism. Assume that a_b and PB are known. Then

$$\begin{aligned}
 M &= \left[(PA)^2 + (PD)^2 - 2(PA)(PD) \cos (\alpha_d - \alpha_a) \right]^{\frac{1}{2}} \\
 N &= \left[(PB)^2 + (PD)^2 - 2(PB)(PD) \cos (\alpha_d - \alpha_b) \right]^{\frac{1}{2}} \\
 S &= \left[(PA)^2 + (PB)^2 - 2(PA)(PB) \cos (\alpha_b - \alpha_a) \right]^{\frac{1}{2}} \\
 \epsilon &= \cos^{-1} \left[\frac{S^2 + M^2 - N^2}{2MS} \right]
 \end{aligned} \tag{4.6}$$

are the dimensions of the coupler triangle obtained by using the cosine law (Fig. 20).

In general, the angle β of the coupler center line relative to the x axis (Fig. 19) is given by

$$\beta = \tan^{-1} \left[\frac{d - R \sin \omega}{s - R \cos \omega} \right] \tag{4.7}$$

The x,y coordinates of point B are

$$\begin{aligned}
 x_b &= R \cos \omega + S \cos (\beta - \epsilon) \\
 y_b &= R \sin \omega + S \sin (\beta - \epsilon)
 \end{aligned} \tag{4.8}$$

and the U,V coordinates are calculated by

$$\begin{aligned}
 U_b &= -x_b \sin \alpha_d - y_b \cos \alpha_d + PO_a \cos \alpha_a \\
 V_b &= x_b \cos \alpha_d - y_b \sin \alpha_d + PO_a \sin \alpha_a
 \end{aligned} \tag{4.9}$$

Symmetrical Coupler Curves Produced by Alternate
Four-Bar Linkages of Slider-Crank Mechanisms

Generally, symmetrical coupler curves can easily be obtained from the general theories for the production of approximate straight line motion by properly adjusting the available parameters of the four-bar linkage system. This is not the case where the linkage is developed as an alternate of a given slider-crank mechanism. A general development is given here to relate the parameters k , α_a , and α_d of a slider-crank mechanism so that any resulting alternate four-bar linkage will generate a symmetrical coupler curve at the coupler point D.

The crank of the mechanism (Fig. 20) is completely specified by the parameters k and α_a in the system defined by the pole tangent and pole normal

$$PA = \frac{k \sin \alpha_a}{k + 1} \quad (4.10)$$

$$PO_a = k \sin \alpha_a \quad (4.11)$$

and the location of the slider on the inflection circle is

$$PD = \sin \alpha_d \quad (4.12)$$

where the diameter of the inflection circle is unity.

If the length of the proposed coupler link AB of the alternate four-bar linkage is S , it will produce a symmetrical coupler curve at the coupler point D when

$$R = M = S \quad (4.13)$$

The parameters k , α_a , α_d are related by means of the first of these equalities.

The length of the crank of the slider-crank mechanism is given by

$$R = |PO_a - PA| = \left| \frac{k^2 \sin \alpha_a}{k + 1} \right| \quad (4.14)$$

The length of the coupler link $AD = M$ is obtained by using the cosine law in the form

$$M^2 = (PA)^2 + (PD)^2 - 2(PA)(PD) \cos (\alpha_d - \alpha_a) \quad (4.15)$$

But $M = R = |PO_a - PA|$, such that Eq. (4.15) becomes

$$(PO_a)^2 - 2(PA)(PO_a) = (PD)^2 - 2(PA)(PD) \cos (\alpha_d - \alpha_a) \quad (4.16)$$

and, using the previous design equations, this may be reduced to

$$\begin{aligned} (k^3 - k^2 + k) \cos 2\alpha_a + k^3 + k^2 + 1 \\ = (\cos 2\alpha_d)(1 + k \cos 2\alpha_a) + (k \sin 2\alpha_a) \sin 2\alpha_d \end{aligned} \quad (4.17)$$

In this form, Eq. (4.17) is a transcendental equation and cannot be solved easily. However, the trigonometric formulas

$$\sin 2\alpha_d = \frac{2 \tan \alpha_d}{1 + \tan^2 \alpha_d}, \quad \cos 2\alpha_d = \frac{1 - \tan^2 \alpha_d}{1 + \tan^2 \alpha_d}$$

may be used to form a quadratic equation in terms of $\tan \alpha_d$

$$A \tan^2 \alpha_d + B \tan \alpha_d + C = 0 \quad (4.18)$$

where

$$A = (k^3 - k^2 + k) \cos 2\alpha_a + k \sin 2\alpha_a + (k^3 + k^2 + 1)$$

$$B = 2 + 2k \cos 2\alpha_a$$

$$C = (k^3 - k^2 + k) \cos 2\alpha_a - k \sin 2\alpha_a + (k^3 + k^2 + 1)$$

Hence, given the parameters k and α_a , the value of α_d can be obtained from Eq. (4.18) by using the roots

$$\alpha_d = \tan^{-1} \left[\frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \right] \quad (4.19)$$

Coupler point D of the coupler plane of a four-bar linkage having pin joint B on a circle of radius $S = R = M$ and centered at pin joint A will then describe a symmetrical coupler curve (Fig. 21). Furthermore, the axis of symmetry of the resulting coupler curve is located by a line through O_a and oriented by the angle $\epsilon/2$ from the fixed link in the same sense that ϵ is measured in the coupler triangle. For the proof of these symmetry properties see Ref. (34).

Two Inflections on a Symmetrical

Coupler Curve

The results of the preceding article will be useful for the production of approximate straight line motion if the two inflections of the symmetrical coupler curve have the same tangent. This special case occurs when the axis of symmetry of the coupler curve is parallel to pole ray PD . There is, as expected, no apparent closed form solution to this problem, and an iterative procedure will be the result.

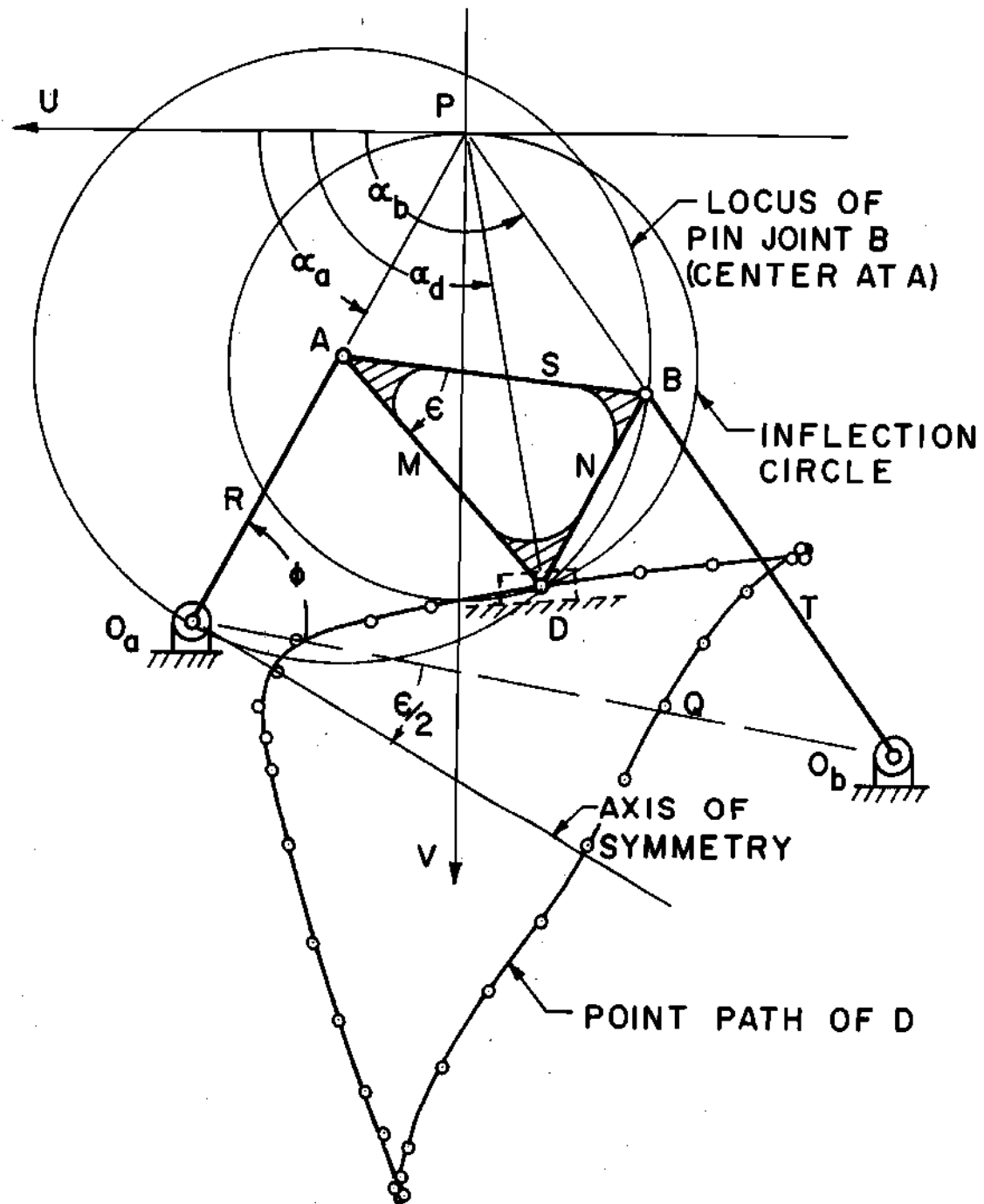


Figure 21. Slider-Crank Having Alternate Four-Bar Linkages with Symmetrical Coupler Curves.

The angle Φ between the crank R and the side M of the coupler triangle is

$$\Phi = \cos^{-1} \left[\frac{1}{2k} + \frac{k}{2} - \frac{\sin^2 \alpha_d}{2k^3 \sin \alpha_a} \right] \quad (4.20)$$

by using the cosine law with triangle PAD. Choosing a tentative value of ϵ to locate point B on the circular locus, the value for PB is

$$PB = \frac{k \sin \alpha_a}{k + 1} \left[1 + k^2 - 2k \cos (\Phi + \epsilon) \right] \quad (4.21)$$

by again using the cosine law with triangle PAB. Note that the second side of the coupler triangle N has the value

$$N = \frac{2k^2 \sin \alpha_a}{k + 1} \sin \epsilon/2 \quad (4.22)$$

The value for α_b can be obtained by using the cosine law with triangle PDN

$$\alpha_b = \alpha_d + \cos^{-1} \left[\frac{(PB)^2 + (PD)^2 - N^2}{2(PB)(PD)} \right] \quad (4.23)$$

and the dimensions of crank B become

$$PO_b = \frac{PB \sin \alpha_b}{\sin \alpha_b - PB}$$

$$O_b B = \frac{(PB)^2}{\sin \alpha_b - PB} \quad (4.24)$$

The alternate four-bar linkage is now completely determined. It may not, however, satisfy the requirement that the axis of symmetry of the coupler curve be parallel to pole ray PD. The length of the coupler link $Q = O_a O_b$ is

$$Q = [(PO_a)^2 + (PO_b)^2 - 2(PO_a)(PO_b) \cos(\alpha_b - \alpha_a)]^{\frac{1}{2}} \quad (4.25)$$

and the initial position of crank R relative to the fixed link is

$$\phi = \cos^{-1} \left[\frac{Q^2 + (PO_a)^2 - (PO_b)^2}{2(Q)(PO_a)} \right] \quad (4.26)$$

If the following equality

$$\alpha_d - \alpha_a + \frac{\epsilon}{2} + \phi = 180^\circ \quad (4.27)$$

holds, the alternate four-bar due to the choice of ϵ satisfies the requirement that two inflections on a symmetrical coupler curve have the same tangent. Generally, it will be necessary to iterate the value of ϵ until Eq. (4.27) is satisfied for the given parameter values k and α_a .

Ball Point with One Prescribed Intersection or One Prescribed Tangent

As given in the chapter on curvature theory, a slider-crank mechanism (Fig. 22) is completely specified by the parameters k , α_a , α_d and the formulas

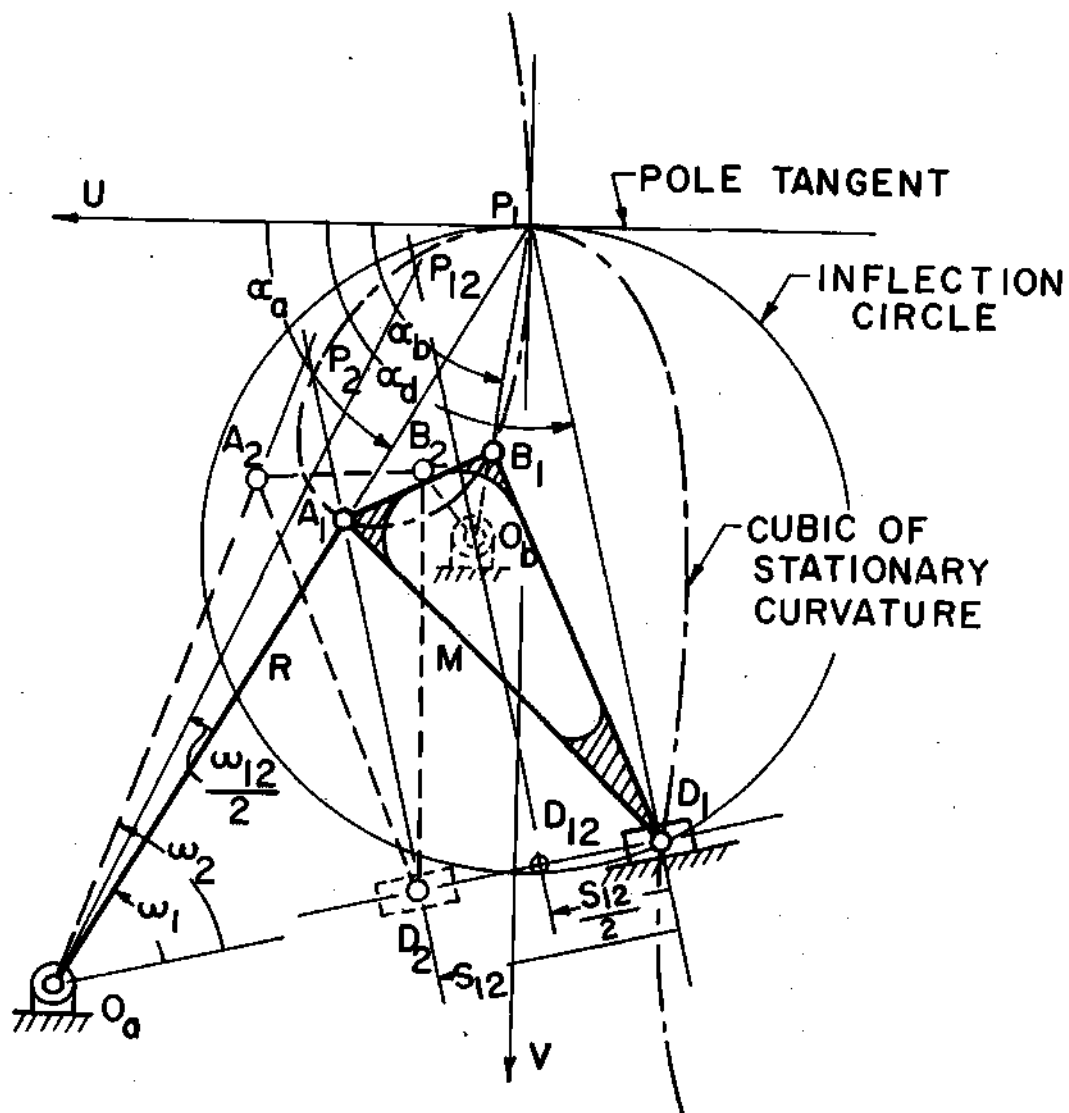


Figure 22. General Slider-Crank with Slider Displacement.

$$\begin{aligned}
 PO_a &= k \sin \alpha_a \\
 PA &= \frac{k \sin \alpha_a}{k + 1} \\
 PD &= \sin \alpha_d
 \end{aligned}
 \tag{4.28}$$

The constants M, N of the cubic of stationary curvature equation

$$\frac{1}{r} = \frac{1}{M \sin \alpha} + \frac{1}{N \cos \alpha}
 \tag{4.29}$$

are defined by

$$\begin{aligned}
 N &= k(\tan \alpha_a - \tan \alpha_d) \\
 M &= \frac{N}{N - \tan \alpha_d}
 \end{aligned}
 \tag{4.30}$$

for the given system position. Any other point B on the cubic of stationary curvature and its corresponding radius of curvature, when replaced by a rigid crank $O_b B$, will provide the necessary constraint so that point D generates a path element having third order contact with its tangent.

The above mentioned procedure entails the specification of four parameters (k, α_a, α_d , and α_b). Generally, four design parameters would not allow the designer to optimize his solution. It is possible, however, to reduce the number of parameters and also assure an extended approximate straight line output. This can be accomplished by specifying an additional contact or contacts between the coupler curve and the line being approximated. These contacts (intersections or a tangent) are finitely displaced by prescribed distances from the Ball point contact.

Suppose the slider is displaced by s_{12} along its path of travel as shown in Fig. (22). Using the displacement function, Eq. (4.5), the position of the crank is determined and the change in its position is $\Delta\omega = \omega_2 - \omega_1$. The instant center (pole) for the initial position is P_1 and for the final position it is P_2 . Since the coupler curve to be produced by the desired alternate four-bar linkage is required to be tangent at D_2 , the location of P_2 will be the instant center of that linkage in that position also. The intermediate pole for the finite motion of the coupler plane is P_{12} . Any point B (located by specifying a_b and calculating PB by using Eq. (4.29)), which is to serve as the second crank of the alternate four-bar must have its center of rotation O_b on the line $P_{12}D_{12}$. The equation for the locus of centers corresponding to the points on the cubic of stationary curvature is

$$\frac{1}{p} = -\frac{\tan \alpha_d}{N \sin \alpha} + \frac{1}{N \cos \alpha} \quad (4.31)$$

obtained by substituting the following form of the Euler-Savary equation

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{\sin \alpha} \quad (4.32)$$

into Eq. (4.29). The polar equation of line $P_{12}D_{12}$ is

$$\frac{1}{m} = \frac{s_{12}}{2 \sin (\theta - \alpha_d)} \quad (4.33)$$

and for point O_b on that line

$$\frac{1}{PO_b} = \frac{s_{12}}{2 \sin (a_b - \alpha_d)} \quad (4.34)$$

where $m = PO_b$. Solving the statement of Eq. (4.29) in terms of point B and Eq. (4.34) for s_{12} gives

$$s_{12} = \frac{2}{N} \left\{ \frac{1}{\cos \alpha_b} - \frac{\tan \alpha_d}{\sin \alpha_b} \right\} \sin (\alpha_b - \alpha_d) \quad (4.35)$$

or solving for $\tan \alpha_b$ gives the quadratic

$$\begin{aligned} \tan^2 \alpha_b \cos \alpha_d + \tan \alpha_b \left(-2 \sin \alpha_d - \frac{s_{12} N}{2} \right) \\ + \tan \alpha_d \sin \alpha_d = 0 \end{aligned} \quad (4.36)$$

which has the solutions

$$\tan \alpha_b = \frac{2 \sin \alpha_d + \frac{s_{12} N}{2} \pm \sqrt{\left(\frac{N s_{12}}{2} \right)^2 - (2 \sin \alpha_d) s_{12} N}}{2 \cos \alpha_d} \quad (4.37)$$

Eq. (4.37) does not in itself provide sufficient information to insure that a tangent exists at D_2 as desired. If only an intersection is needed at D_2 , then Eq. (4.37) is the appropriate design equation.

If, however, the instant center P_2 , the fixed pivot O_b , and the final position B_2 of pin joint B are collinear; then the coupler curve of the alternate four-bar will indeed have a tangent at D_2 . The (U_b, V_b) coordinates of O_b are

$$(PO_b \cos \alpha_b, PO_b \sin \alpha_b)$$

The coordinates (U_{P_2}, V_{P_2}) of the pole P_2 are given by

$$U_{P_2} = -s_{12} \sin \alpha_d - s_{12} \tan \omega \cos \alpha_d + PO_a \cos \alpha_a \quad (4.38)$$

$$V_{P_2} = s_{12} \cos \alpha_d - s_{12} \tan \omega \sin \alpha_d + PO_a \sin \alpha_a$$

and the coordinates (U_{b_2}, V_{b_2}) of B_2 can be obtained by using Eq. (4.8) and (4.9). These points are collinear if the determinant

$$\begin{vmatrix} 1 & U_{O_b} & V_{O_b} \\ 1 & U_{P_2} & V_{P_2} \\ 1 & U_{b_2} & V_{b_2} \end{vmatrix} = 0$$

If the value of s_{12} is of primary importance, then an iteration of the preceding design equations may be accomplished by choosing a sequence of values for α_d . When the above determinant is zero, the coupler curve will have a Ball point in the initial position D_1 and a tangent in the final position D_2 .

Ball Point with Two Prescribed Intersections

If two displacements s_{12}, s_{13} of the slider pin joint D are specified, the resulting coupler curve of the alternate four-bar will have two intersections in addition to the Ball point contact in common with a straight line. Because of the availability of Eq. (4.37) to locate a possible second crank O_bB for the desired four-bar, this design procedure does not entail a special analysis. The displacement function Eq. (4.5) is used to determine the location of the coupler plane in the displaced positions. Then by appropriate triangulation, the locations of B_1, B_2 , and B_3 can be calculated for these positions. Use

of Eq. (4.37) requires that two positions of point B (say B_1 and B_2) be on a circular arc about the center O_b which has the coordinates $(PO_b \cos \alpha_b, PO_b \sin \alpha_b)$. The location of $B_3 (U_{b3}, V_{b3})$ when substituted into the equation for the circle described by pin joint B

$$(U_3 - PO_b \cos \alpha_b)^2 + (V_3 - PO_b \sin \alpha_b)^2 - (O_b B)^2 = \delta \quad (4.39)$$

will not, in general, make δ zero. An iteration can be accomplished by taking a sequence of values for α_d until $\delta = 0$.

Ball-Burmester Point With One Prescribed Intersection

From curvature theory, the two unknown Burmester points of the coupler plane of a general slider-crank mechanism (specified by the parameters k, α_a, α_d) are obtained by solving the Allievi-Wolford equation

$$\tan^2 \alpha + [\tan \alpha_a + 3 \tan \alpha_d - N] \tan \alpha + \frac{\tan \alpha_d - N}{\tan \alpha_a} = 0 \quad (4.40)$$

for the roots $\tan \alpha_b$ and $\tan \alpha_c$. The constant N is determined by

$$N = k (\tan \alpha_a - \tan \alpha_d)$$

The dimensions related to points B and C are calculated by using

$$PB = \frac{N \sin \alpha_b}{N + K_b} \quad PC = \frac{N \sin \alpha_c}{N + K_c} \quad (4.41)$$

and

$$PO_b = \frac{N \sin \alpha_b}{K_b} \quad PO_c = \frac{N \sin \alpha_c}{K_c} \quad (4.42)$$

where:

$$K_b = \tan \alpha_b - \tan \alpha_d, \quad K_c = \tan \alpha_c - \tan \alpha_d$$

Either crank $O_b B$ or $O_c C$ can be used to replace the constraint provided by the slider on the motion of the original coupler link to form an alternate four-bar linkage. If the slider is given the displacement s_{12} to D_2 , the equations from the preceding articles can be used to find the coordinates (U_{b2}, V_{b2}) and (U_{c2}, V_{c2}) of the displaced positions of points B and C , respectively. The equation of a circle centered at either O_b or O_c may be used to evaluate how closely the coupler curve approaches the desired intersection D_2 on the approximating straight line. That is

$$\begin{aligned} (U_{b2} - U_{O_b})^2 + (V_{b2} - V_{O_b})^2 - (O_b B)^2 &= \delta_b \\ (U_{c2} - U_{O_c})^2 + (V_{c2} - V_{O_c})^2 - (O_c C)^2 &= \delta_c \end{aligned} \quad (4.44)$$

give values for δ_b (or δ_c) which approximate the "error" which results when B_2 (or C_2) does not lie on the desired circle. A sequence of values for α_d can be used until δ_b (or δ_c) approaches zero. When $\delta = 0$, the coupler curve of the alternate four-bar linkage has a Ball-Burmester point in the initial position and an intersection at D_2 in the final position.

CHAPTER V

CLASSICAL MECHANISMS

The linkages investigated in this chapter represent a useful and perhaps complete (5) assessment of those mechanisms designed by methods not dependent on curvature theory. Other design techniques are available, but they are in reality based on one of the groups to be considered or depend on curvature theory (25). The mechanisms to be considered are the Watt, Evans, conchoidal, symmetrical (including both the Roberts and Chebychev types), and those mechanisms based on Burmester theory having five finitely spaced precision points of the coupler curve on a straight line. Design procedures for these groups are given in their most general form.

Watt Mechanisms

The lemniscoid or Watt motion was perhaps the first mechanism actually used to produce approximate straight line motion by means of a pinned linkage. In the design position, the control cranks are parallel so that their intersection (the instant center for the coupler link) is at infinity. Consequently, the coupler link has in this position pure translatory motion and all points in the coupler plane momentarily describe path elements that have zero curvature. In general, the cranks need not be of equal length. If T is the larger crank, the output point C (Fig. 23 and 24) on the coupler link center line AB is located (35) from pin joint A by the distance

$$r = \frac{T}{T + nR} S \quad (5.1)$$

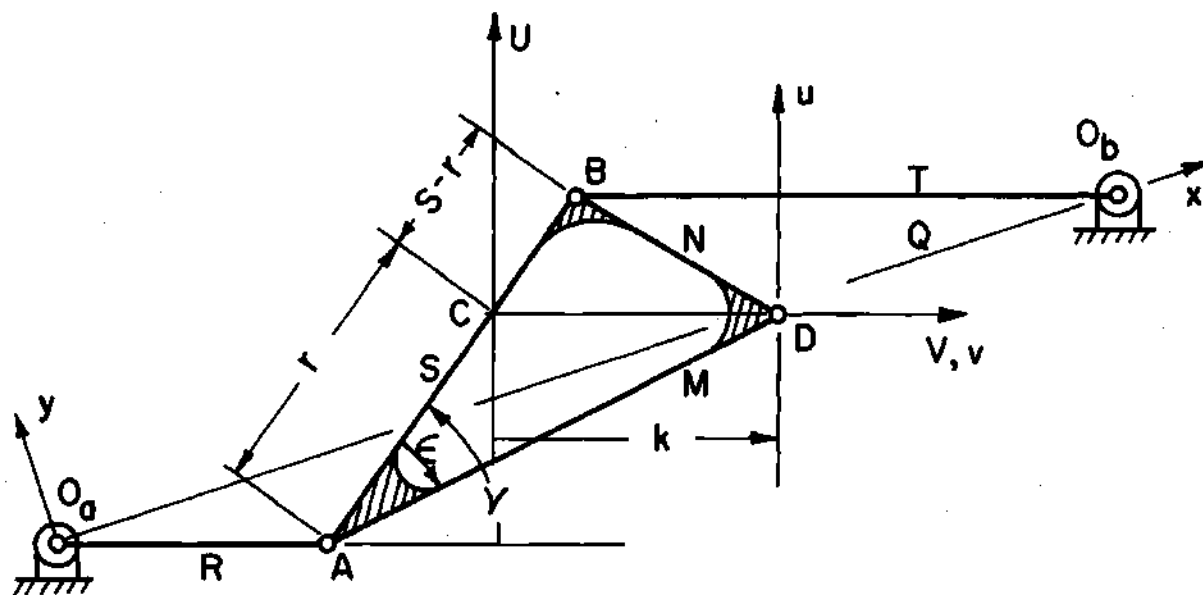


Figure 23. Watt Mechanism (Crossed Type).

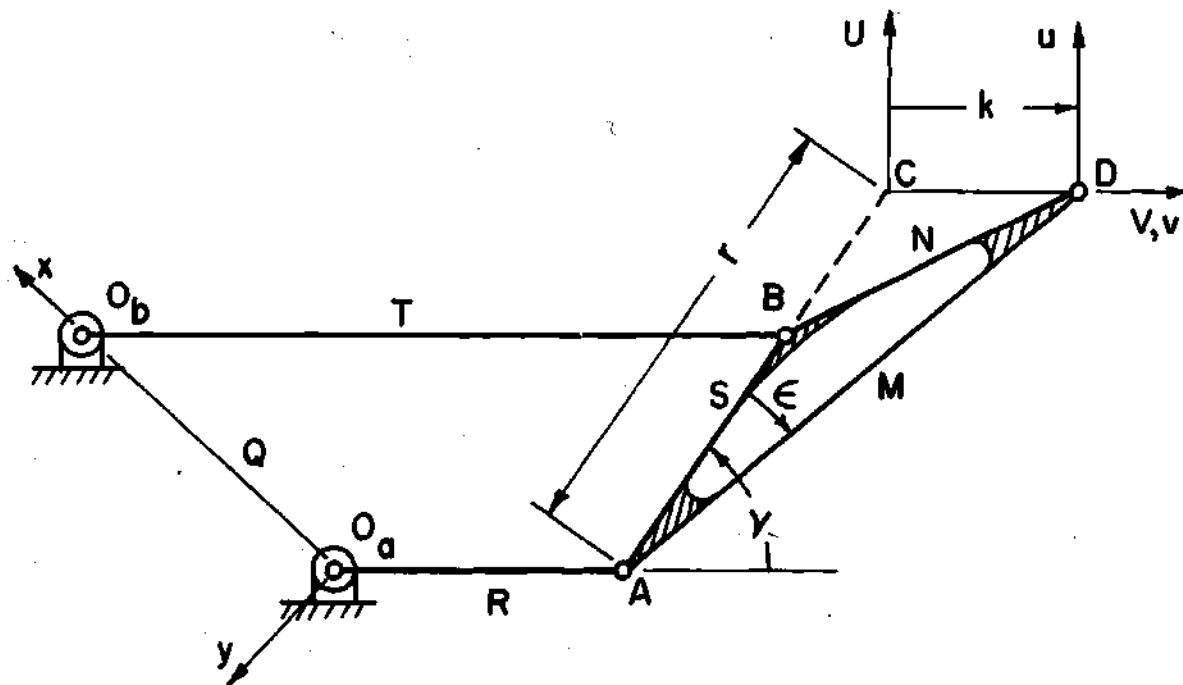


Figure 24. Watt Mechanism (Open Type).

where $n = +1$ represents the crossed type and $n = -1$, the open type. A more general form of the linkage is possible. Some points D, not lying on the coupler center line, may also produce satisfactory approximate straight line motion. These points are taken to be those lying on a line from the instant center through point C and specified by the parameter $k = CD$.

The location of the fixed pivot O_a in the U, V system is given by

$$\begin{aligned} U_1 &= -r \sin \gamma \\ V_1 &= -(R + r \cos \gamma) \end{aligned} \quad (5.2)$$

where $S = 1.0$ and γ represents the angle formed by the coupler link AB with the cranks in the design position. The pin joint A is located by

$$\begin{aligned} UP_1 &= U_1 \\ VP_1 &= V_1 + R \end{aligned} \quad (5.3)$$

The coordinates of the fixed pivot O_b are

$$\begin{aligned} V_2 &= V_1 + R + \cos \gamma + nT \\ U_2 &= U_1 + \sin \gamma \end{aligned} \quad (5.4)$$

and the coordinates of the pin joint B are

$$\begin{aligned} VP_2 &= V_2 - nT \\ UP_2 &= U_2 \end{aligned} \quad (5.5)$$

Finally, the location of the output point D is given by

$$\begin{aligned}V_d &= k \\U_d &= 0\end{aligned}\tag{5.6}$$

such that

$$\begin{aligned}M &= [(UP_1)^2 + (VP_1 - k)^2]^{\frac{1}{2}} \\N &= [(UP_2)^2 + (VP_2 - k)^2]^{\frac{1}{2}}\end{aligned}\tag{5.7}$$

The length of the fixed link is

$$Q = [(U_2 - U_1)^2 + (V_2 - V_1)^2]^{\frac{1}{2}}\tag{5.8}$$

and the orientation of the x axis is given by

$$\theta = \tan^{-1} \left[\frac{V_2 - V_1}{U_2 - U_1} \right]\tag{5.9}$$

To obtain the value of the coupler angle ϵ , the cosine law is used in the form

$$\epsilon = \pm \cos^{-1} \left[\frac{M^2 + 1 - N^2}{2M} \right]\tag{5.10}$$

where the negative sign is used when the product $(k)(n)$ is greater than zero.

Evans Mechanisms

The general form of the Evans mechanism is based on the Cardanic

circles which produce a special form of cycloidal motion. This cycloidal coplanar motion is generated by a circle rolling on the inner surface of a fixed circle of twice the diameter of the other. All points rigidly attached to the smaller circle (such as A) describe elliptical point paths. Those points on the surface of the smaller circle (such as D and B) trace exact straight line point paths along diameters of the larger fixed circle (Fig. 25).

If any two points (Ex.: points B and D in Fig. (26)) of a moving plane are momentarily tracing exact straight line point paths, the remaining points (Ex.: point A) of the moving plane trace elliptical coupler curves, segments of which may be approximated by a circular arc (Ex.: the arc described by the radius $O_a A$). If point B is guided by a slider and A is controlled by a rigid crank $O_a A$, the resulting slider-crank mechanism will have an approximate straight line output at all points such as D. If the straight line point path of B is satisfactorily approximated by a circular arc of sufficiently large radius ($O_b B$) the resulting four-bar linkage is known as an Evans mechanism. The accuracy of the straight line output at D will depend on the quality of the approximations at A and B.

The initial position of the coupler plane (denoted by the subscript o) and the central position (denoted by the subscript c) have well defined instant centers P_o and P'_c . Any crank which is to be added as a constraint to the motion such as $O_a A$ must be on pole rays from these instant centers. The point P'_c , rigidly attached to the outer surface of the smaller Cardan circle, moves along the diameter $2(OP_c)$, (Fig. 26), of the larger circle so that P'_c coincides with P_c in the

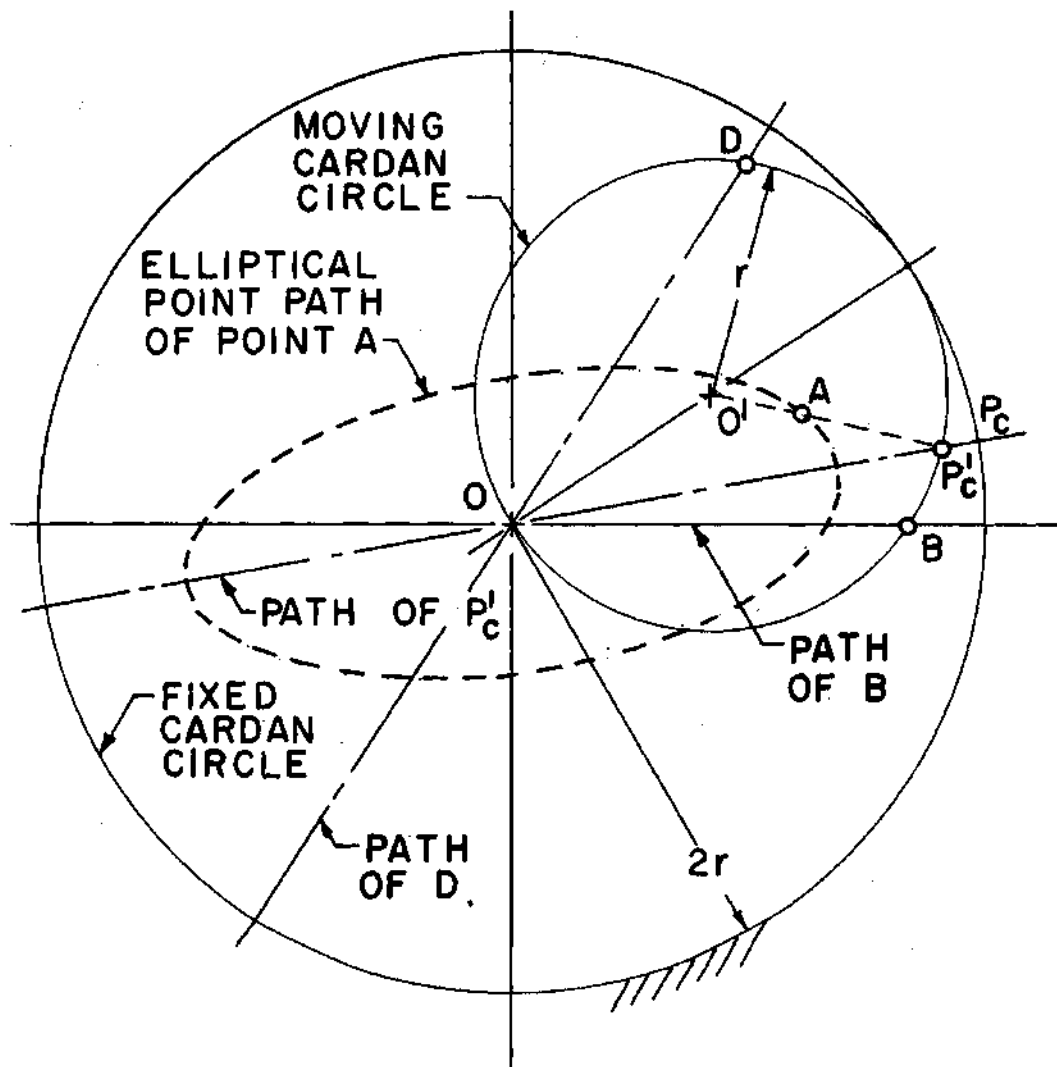


Figure 25. Cardan Motion.

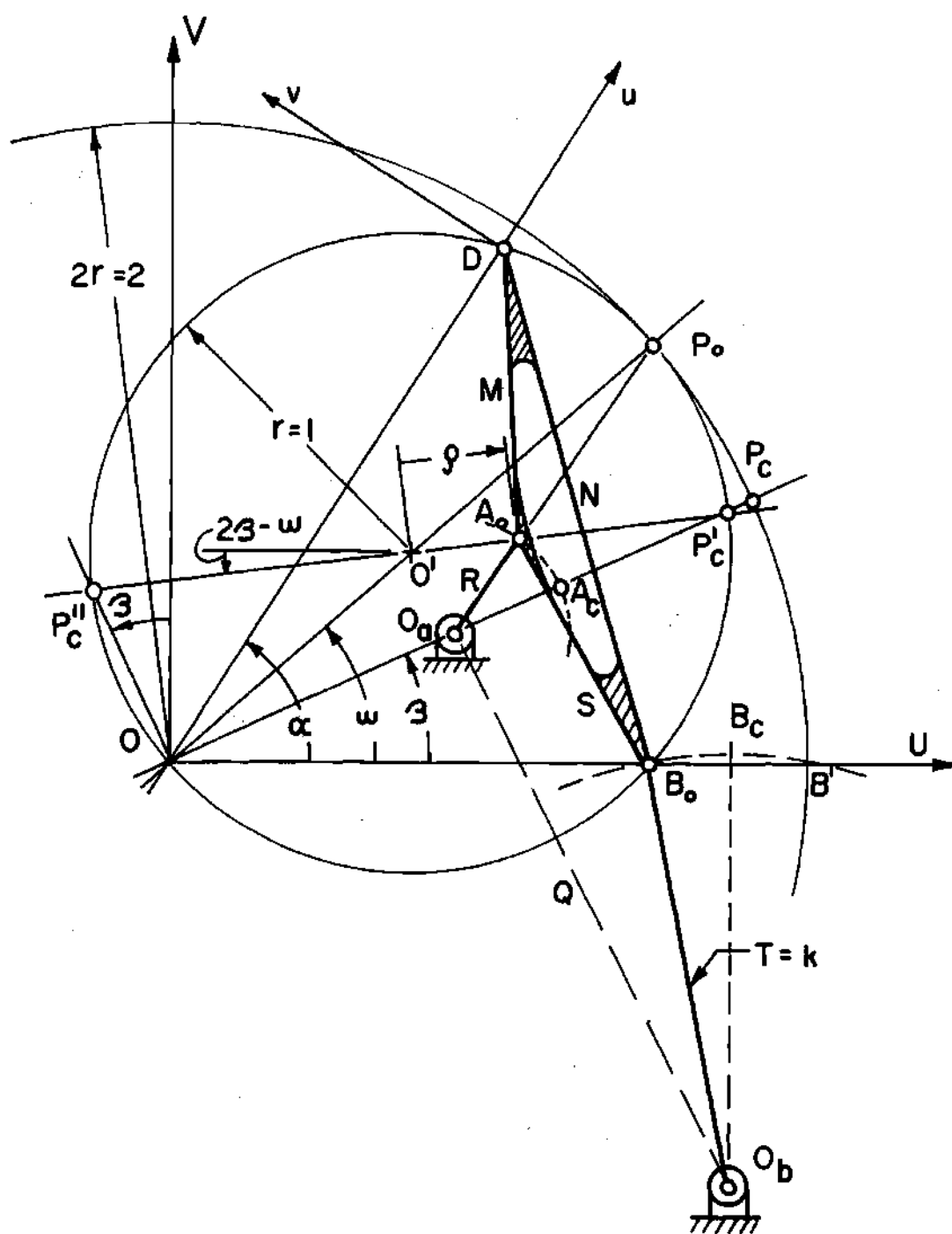


Figure 26. General Form of the Evans Linkage.

central position. Hence $P_c' O P_c''$ is the diameter of the smaller circle which coincides with the diameter $2(OP_c)$ of the larger circle in the central position. The result is that the ellipse being traced by point A has $2(OP_c)$ as one of its axes of symmetry. If the crank $O_a A$ is to approximate the elliptical path of A, its center of rotation must lie on OP_c .

The smaller Cardan circle has its center O' on a ray (defined by ω) from the origin of the U, V coordinate system (the center of the fixed Cardan circle) and its radius is $r = 1$. This circle must pass through the origin of the U, V system. The radius of the outer circle is $OP_c = 2$. As shown in the figure, U_2 is the average of the U coordinates for B_o and B' so that arc $B_o B_c$ best approximates the straight line which point B on the smaller Cardan circle describes. This gives the coordinates of O_b to be

$$U_2 = 1 + \cos \omega$$

$$V_2 = k \quad (5.11)$$

where k will take on sufficiently large positive or negative values to insure that arc $B_o B_c$ satisfactorily approximates a straight line. In the initial position, the instant center P_o is located at

$$(2 \cos \omega, 2 \sin \omega)$$

Note that the slope of the diameter $P_c'' P_c'$ is governed by the angle $(2\beta - \omega)$. Letting the directed distance of the pin joint A from the center O' of the smaller circle be ρ , the coordinates of A_o are

$$\begin{aligned}
 VP_1 &= \sin \omega + \rho \sin (2\beta - \omega) \\
 UP_1 &= \cos \omega + \rho \cos (2\beta - \omega)
 \end{aligned}
 \tag{5.12}$$

The slope of the line $A_o P_o$ is

$$m = \frac{\sin \omega - \rho \sin (2\beta - \omega)}{\cos \omega - \rho \cos (2\beta - \omega)} \tag{5.13}$$

such that the equation of this line in the U, V system is

$$V = 2 \sin \omega + m(U - 2 \cos \omega) \tag{5.14}$$

The equation of OP_c is

$$U = V \tan \beta \tag{5.15}$$

Lines OP_c and $A_o P_o$ intersect at the desired fixed pivot O_a which has the coordinates

$$\begin{aligned}
 V_1 &= \frac{2 \sin \omega - 2m \cos \omega}{1 - m \tan \beta} \\
 U_1 &= V_1 \tan \beta
 \end{aligned}
 \tag{5.16}$$

The location of D on the outer surface of the smaller Cardan circle is defined by the parameter α , such that

$$\begin{aligned}
 U_d &= 2 \cos \alpha \cos (\alpha - \omega) \\
 V_d &= 2 \sin \alpha \cos (\alpha - \omega)
 \end{aligned}
 \tag{5.17}$$

The orientation of the u, v system which has its origin at D is given by the parameter α . Since the coordinates of all of the pertinent

points (A, B, O_a, O_b, D) are known, the linkage is completely determined. The determination of the usual link dimensions $(Q, R, S, T, M, \epsilon)$ is not reviewed here since it is quite straight-forward.

Note that the angle ω is not a parameter for the size of the linkage in that it specifies the initial position of the linkage. The difference angle $(\omega - \beta)$ is significant in predicting the accuracy of the resulting approximate straight line motion. If $\omega - \beta$ is small, the approximation will be quite accurate over a small range. If, however, $\omega - \beta$ is large, then the approximation will be poorer but the range will have been increased.

Special cases result when the linkage parameters α , β and ρ take on particular values. These are:

$\beta = 0$ This is the centric oblique case where O_a lies on the U axis but $\alpha \neq 90^\circ$.

$\alpha = 90^\circ$ This is the rectangular case where the straight line paths of B and D are perpendicular.

$\alpha = 90^\circ; \beta = 0$ This is the centric rectangular case which has the points D, A, and B lying on a straight line.

$\alpha = 90^\circ$ This is the well-known Scott-Russell Mechanism.
 $\beta = \rho = 0$

Conchoidal Straight-Line Motion

The equation of a conchoid (Fig. 27) in polar coordinates is

$$\rho = a + \frac{b}{\cos \phi} \quad (5.18)$$

The conchoid is generated by all points of a line which has point D constrained to move along the U axis and which slides through a fixed

point O_b on the V axis.

The portion of the conchoid between points A and A' may be approximated by a circular arc such that A and A' lie on a circle with its center at O_a . The general equation of this circle is:

$$U^2 + (V - V_1)^2 = R^2 \quad (5.19)$$

There are two unknowns in this equation. These may be determined since two points are known to lie on the circle. For point A'

$$(r - V_1)^2 = R^2 \quad (5.20)$$

and for point A

$$(\tan \alpha - r \sin \alpha)^2 + (r \cos \alpha - V_1)^2 = R^2 \quad (5.21)$$

from which the equality

$$(r - V_1)^2 = (\tan \alpha - r \sin \alpha)^2 + (r \cos \alpha - V_1)^2 \quad (5.22)$$

is obtained. The only unknown in Eq. (5.22) is V_1 so that

$$V_1 = \frac{\tan^2 \alpha}{2r} \left[\frac{1 - 2r \cos \alpha}{1 - \cos \alpha} \right] \quad (5.23)$$

It follows from Eq. (5.20) that

$$R = |r - V_1| \quad (5.24)$$

Note that the coordinates for A are

$$\begin{aligned} VP_1 &= r \cos \alpha \\ UP_1 &= \tan \alpha - r \sin \alpha \end{aligned} \quad (5.25)$$

The coordinates for point O_b are

$$V_2 = 1.0 \quad U_2 = 0$$

and for the pin joint B

$$VP_2 = 1 + kn \sin \alpha$$

$$UP_2 = kn \cos \alpha \quad (5.26)$$

where $k = T$ and $n = -1$ when point B lies in the lower half of the plane. The crank O_bB is made sufficiently long in order to approximate the sliding motion of the generating line DA through O_b . The length of the fixed link is

$$Q = |V_1 - V_2| \quad (5.27)$$

The length of the coupler link is

$$S = [(UP_2 - UP_1)^2 + (VP_2 - VP_1)^2]^{\frac{1}{2}} \quad (5.28)$$

and the sides of the coupler triangle are given by

$$M = r$$

$$N = [(\tan \alpha - UP_2)^2 + (VP_2)^2]^{\frac{1}{2}} \quad (5.29)$$

where the coordinates of the output point D are

$$U_d = \tan \alpha$$

$$V_d = 0$$

The enclosed coupler angle ε is obtained by using the cosine law in the

$$\varepsilon = n \cos^{-1} \left[\frac{M^2 + S^2 - N^2}{2MS} \right] \quad (5.30)$$

with the added restriction that $\varepsilon = -|\varepsilon|$ if $r > 1$. The orientation of the x axis is $\theta = -90^\circ$ for all cases where $r < 1$. If $r > 1$ then $\theta = 90^\circ$.

Symmetrical Linkages

In a recent paper (22), W. Wunderlich investigates the accuracy of the straight line output of symmetrical linkages for an unspecified length of the approximate straight line output. Special cases of symmetrical linkages are the Roberts, Chebyshev, and those resulting from curvature theory. To consider the totality of symmetrical linkages, no special design procedure need be used. The linkages may be specified (Fig. 28) by the following parameters:

- a This parameter orients the fixed pivots about the center line of the linkage. If a is negative, the cranks are crossed.
- d This parameter gives the spacing between the moving and fixed links in the central position.
- c This parameter is the altitude of the coupler triangle from the coupler link AB.

In the following formulas, all dimensions of the links are related to the coupler link AB which is taken to have a fixed magnitude $S = 2.0$. The following formulas can be used to determine the link dimensions:

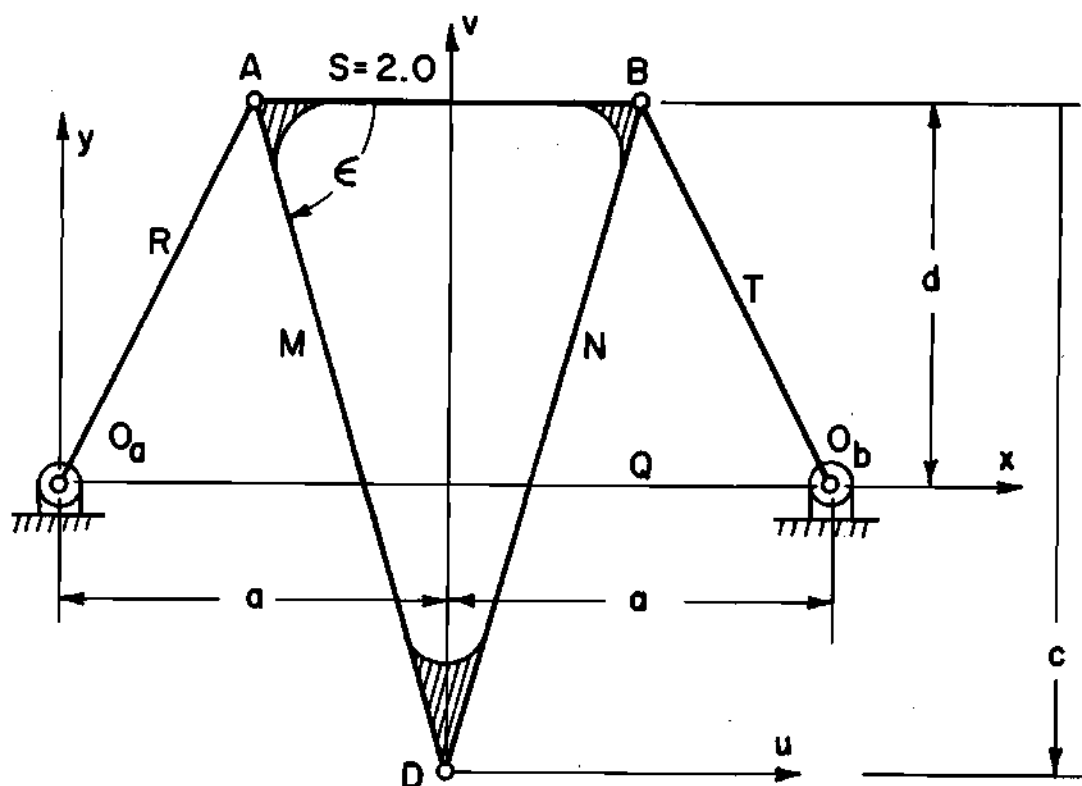


Figure 28. Symmetrical Linkage

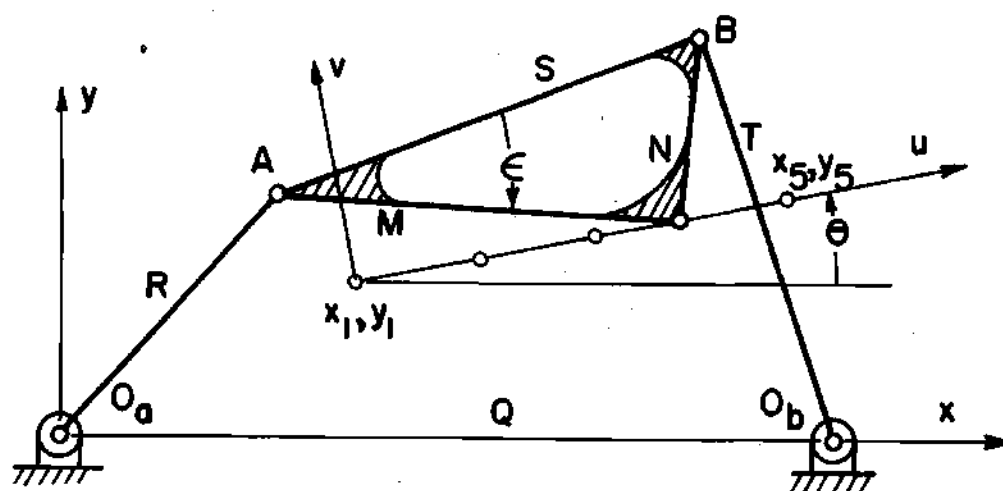


Figure 29. Five Finitely Separated Precision Points.

$$\begin{aligned}
 M &= N = [1 + c^2]^{\frac{1}{2}} \\
 \epsilon &= \tan^{-1} (-c) \\
 Q &= |2a| \\
 R &= T = [d^2 + (1 - a)^2]^{\frac{1}{2}} \quad (5.31)
 \end{aligned}$$

and the u, v system is given in terms of the x, y system as

$$\begin{aligned}
 u &= x - |a| \\
 V &= y - d \pm c \quad (5.32)
 \end{aligned}$$

where the minus sign corresponds to the minus sign of the parameter a .

Mechanisms Based on Burmester Theory

The investigation of those linkages which generate a coupler curve passing through five precision points lying on a straight line should provide a wide range of mechanisms for use by the designer. The design procedure is based on Burmester theory recently explained by Hall (14) and expressed in terms of coordinate transformations by Bottema (27). Using the fundamental concepts given by Bottema, the theory for four and five finitely separated positions of the moving plane is developed in Chapter III. The problem of four and five finitely separated positions on a straight line is also considered. Neither of these procedures will be reviewed here because of their complexity.

It is possible, however, to analyze those linkages already existing in the literature (2). The dimensions of the links may be scaled directly from the existing figures. The x, y coordinates of at least

two of the precision points are necessary to define the u,v system which has the orientation

$$\theta = \tan^{-1} \left[\frac{y_5 - y_1}{x_5 - x_1} \right] \quad (5.34)$$

relative to the x axis (Fig. 29).

CHAPTER VI

COMPUTER PROGRAMS

The computer programming associated with what appears to be a very simple mechanism, the four-bar linkage, is in reality very complex. The average length of each of seven programs written for the Burroughs 220 in the algol language was 750 statements. Each program contained approximately 50 switch statements which made "debugging" very difficult. A representative program is included for reference in the last section of this chapter. The primary information desired is the length of the approximate straight line output for a specified accuracy. Much more information was obtained, however, to assist the designer to optimize the use of the primary information. The additional information included data for the transmission angles, the amount of rotation of the cranks, the type of mechanism (whether crank and lever, double lever, or double crank), the linkage dimensions, as well as information on the higher derivatives of the deviation curve and the deviation curve itself.

Coordinate Systems

The coordinates of the pin joints, the fixed pivots, and the output point are calculated in the initial position by using the supporting theory of each group of linkages studied. The location of the coordinate system is dictated by the requirements of the design equations. For the linkages designed by using curvature theory, the coordinate system is attached to the pole tangent and the pole normal. For the other linkages,

the coordinate system is attached to the fixed link with the origin at one of the pivots and the x-axis directed through the other.

The set of coordinate systems needed to analyze the linkages based on curvature theory is represented in Fig. (30). The supporting theory allows the calculation of the coordinates in the U,V system of the following essential points necessary to define the linkage:

$$\left. \begin{array}{l} O_a (U_i, V_i) \\ A (UP_i, VP_i) \end{array} \right\} \text{the coordinates of the end-} \\ \text{points of the } i^{\text{th}} \text{ crank}$$

$$\left. \begin{array}{l} O_b (U_{i+1}, V_{i+1}) \\ B (UP_{i+1}, VP_{i+1}) \end{array} \right\} \text{the coordinates of the end-} \\ \text{points of the } (i+1)^{\text{th}} \text{ crank}$$

$$D (UP_1, VP_1) \quad \text{the coordinates of the output} \\ \text{point in the coupler plane.}$$

The subscript i is used to allow an orderly iteration between the three linkages that result from the theory of alternate linkages. For all other cases i has the value of one. In the computer program at the end of this chapter, these coordinates are calculated by lines 35 through 50.

The x_i, y_i coordinate system has its origin at (U_i, V_i) and is oriented with its x_i axis rotated by θ_i from the U axis. The angle θ_i is obtained by the following

$$\theta_i = \tan^{-1} \left[\frac{V_{i+1} - V_i}{U_{i+1} - U_i} \right] \quad (6.1)$$

This computation is accomplished by lines 60 through 62 in the computer program.

The u,v coordinate system is used to represent the travel u of the output point D along the path tangent and the deviation v of the coupler curve from that tangent. Its origin is located at the Ball-Burmester point with the coordinates $(UP1, VP1)$. The orientation of the u axis with respect to the U axis is given by the angle $\alpha_d + 90$.

Coordinate Transformations

One of the important, and sometimes quite confusing, concepts used in the computer programs is the transformation of the coordinates from one system to another. The coordinate systems of Fig. (30) are shown removed from the linkage in Fig. (31). If the (x_i, y_i) coordinates of a point are known, the (U,V) coordinates are found by using

$$\begin{aligned} U_i &= x_i \cos \theta_i - y_i \sin \theta_i + U_i \\ V_i &= x_i \sin \theta_i + y_i \cos \theta_i + V_i \end{aligned} \quad (6.2)$$

or if the (U,V) coordinates of the point are known, the (x_i, y_i) coordinates are found by using

$$\begin{aligned} x_i &= (U - U_i) \cos \theta_i + (V - V_i) \sin \theta_i \\ y_i &= - (U - U_i) \sin \theta_i + (V - V_i) \cos \theta_i \end{aligned} \quad (6.3)$$

Eq. (6.3) is obtained by inverting the transformation represented by Eq. (6.2). Another transformation that is found to be useful occurs between the (u,v) and (U,V) systems

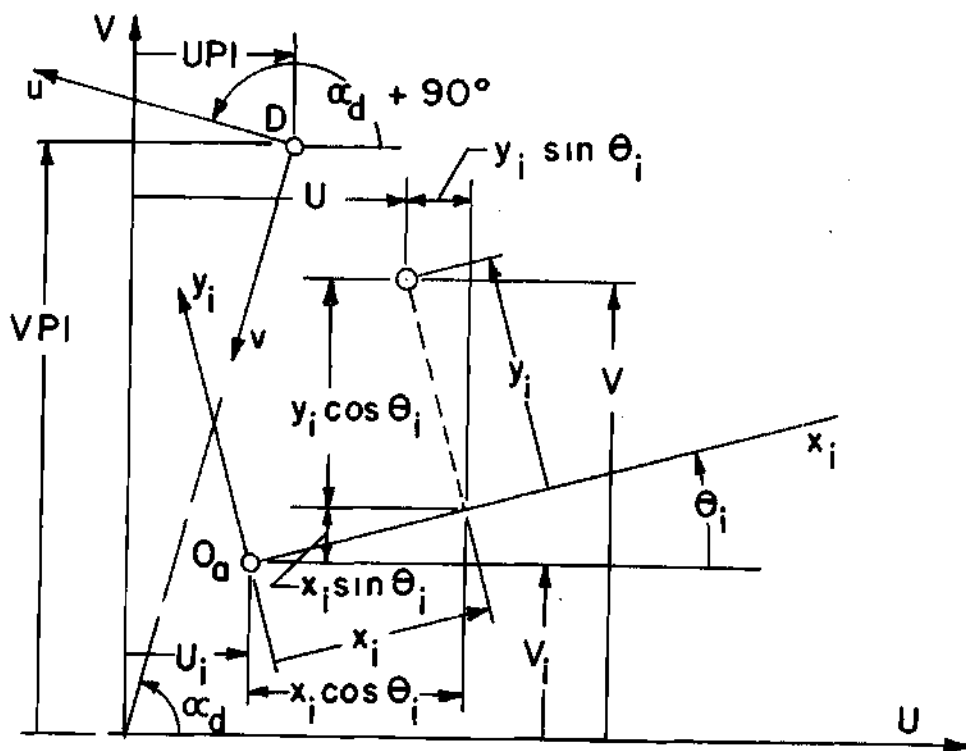


Figure 31. Transformation of Coordinates

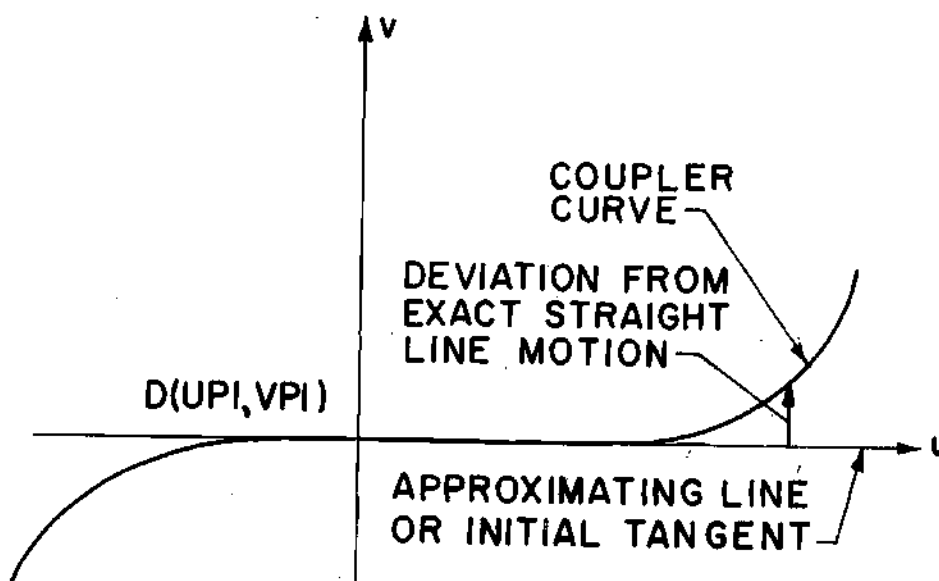


Figure 32. Deviation Curve.

$$\begin{aligned}
 u &= (UP_1 - U) \sin \alpha_d - (VP_1 - V) \cos \alpha_d \\
 v &= (UP_1 - U) \cos \alpha_d + (VP_1 - V) \sin \alpha_d
 \end{aligned}
 \tag{6.4}$$

where the 90° rotation angle is taken into account by using the trigonometric addition formulas.

As shown in Fig. (32), the u axis coincides with the coupler curve tangent and represents a close approximation to the distance traveled along the curve, while v represents the deviation of the coupler curve from its tangent. A direct transformation between the (x_i, y_i) and (u, v) systems is found to be too cumbersome to use except in special cases. The transformation is best accomplished by first using Eq. (6.2) and then Eq. (6.4)

Computation of Linkage Dimensions

Since the coordinates of the points A, B, O_a, O_b are known from the theory, the link dimensions $Q = O_a O_b$, $R = O_a A$, $S = AB$, and $T = O_b B$ can be best calculated by using the following formulas:

$$\begin{aligned}
 Q &= [(U_{i+1} - U_i)^2 + (V_{i+1} - V_i)^2]^{\frac{1}{2}} \\
 R &= [(UP_i - U_i)^2 + (VP_i - V_i)^2]^{\frac{1}{2}} \\
 S &= [(UP_{i+1} - UP_i)^2 + (VP_{i+1} - VP_i)^2]^{\frac{1}{2}} \\
 T &= [(UP_{i+1} - U_{i+1})^2 + (VP_{i+1} - V_{i+1})^2]^{\frac{1}{2}}
 \end{aligned}
 \tag{6.5}$$

The computation of these dimensions is carried out by lines 51 through 59 of the program.

Although the dimensions M , N and ϵ of the coupler triangle are calculated in a different manner, their computation by using the above technique is included here for completeness. These supplementary formulas are:

$$\begin{aligned} M &= [(UP1 - UP_i)^2 + (VP1 - VP_i)^2]^{\frac{1}{2}} \\ N &= [(UP1 - UP_{i+1})^2 + (VP1 - VP_{i+1})^2]^{\frac{1}{2}} \end{aligned} \quad (6.6)$$

and

$$\epsilon = \cos^{-1} \left[\frac{S^2 + M^2 - N^2}{2MS} \right] \quad (6.7)$$

The result from Eq. (6.7) does not specify whether the angle is measured in a clockwise or counterclockwise direction. This uncertainty can be removed by requiring that the apex of the coupler triangle actually coincide with the specified Ball-Burmester point in the initial position. Generally the technique used in the computer programs is to find the coordinates of the Ball-Burmester point (xJ, yJ) by using Eq. (6.3) where $U = UP1$ and $V = VP1$ and the components

$$\begin{aligned} Mx &= xJ - R \cos \phi_0 \\ My &= yJ - R \sin \phi_0 \end{aligned} \quad (6.8)$$

in the formula

$$M = [(Mx)^2 + (My)^2]^{\frac{1}{2}} \quad (6.9)$$

where ϕ_0 is the initial position of the crank R relative to the fixed link as shown in Fig. (33). Then the value for ϵ can be calculated by using the expression

$$\epsilon = \tan^{-1} (My/Mx) - \chi_0 \quad (6.10)$$

where χ_0 is the initial position of the coupler center line relative to the fixed link.

In the program at the end of this chapter, still a different approach was used for the special case where $\epsilon = 0.0^\circ$ or $\epsilon = 180.0^\circ$. This computation was accomplished by lines 83 through 89.

Initial Position of the Linkage

The initial positions ϕ_0 , χ_0 , and ψ_0 of the links R, S, and T, respectively, must be known later in the program. Using a transformation similar to Eq. (6.3), the values

$$\begin{aligned} xxP &= (UP_i - U_i) \cos \theta_i + (VP_i - V_i) \sin \theta_i \\ yyP &= - (UP_i - U_i) \sin \theta_i + (VP_i - V_i) \cos \theta_i \\ xPx &= (UP_{i+1} - U_{i+1}) \cos \theta_i + (VP_{i+1} - V_{i+1}) \sin \theta_i \\ yPy &= - (UP_{i+1} - U_{i+1}) \sin \theta_i + (VP_{i+1} - V_{i+1}) \cos \theta_i \end{aligned} \quad (6.11)$$

are computed. The following formulas

$$\begin{aligned} \phi_0 &= \tan^{-1} (yyP/xxP) \\ \psi_0 &= \tan^{-1} (yPy/xPx) \\ \chi_0 &= \tan^{-1} \left[\frac{yPy - yyP}{xPx + Q - xxP} \right] \end{aligned} \quad (6.12)$$

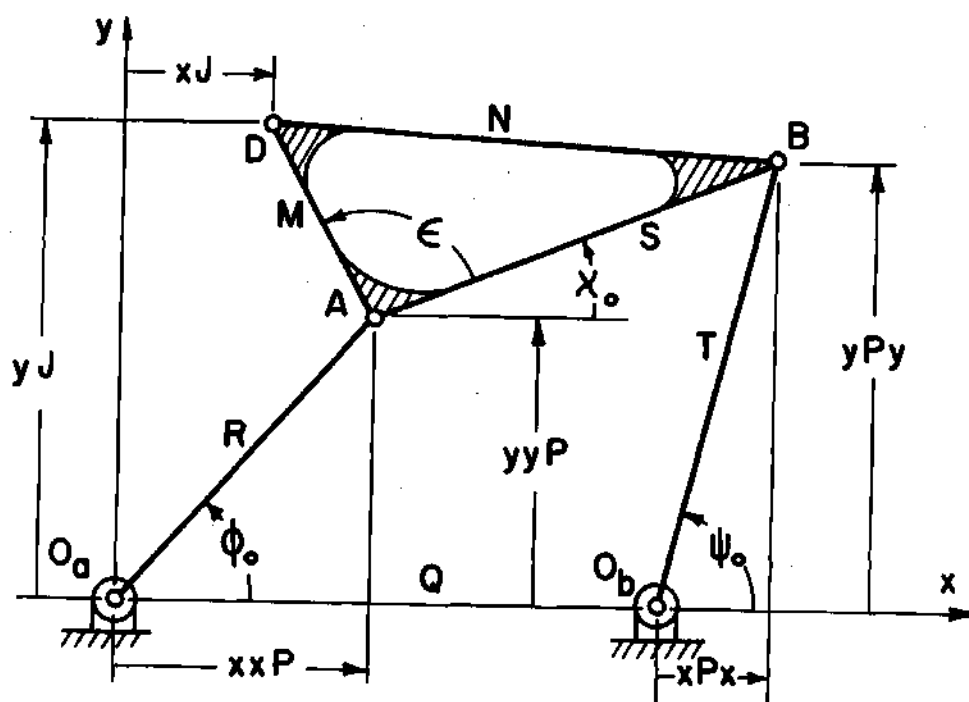


Figure 33. Initial Position of the Linkage.

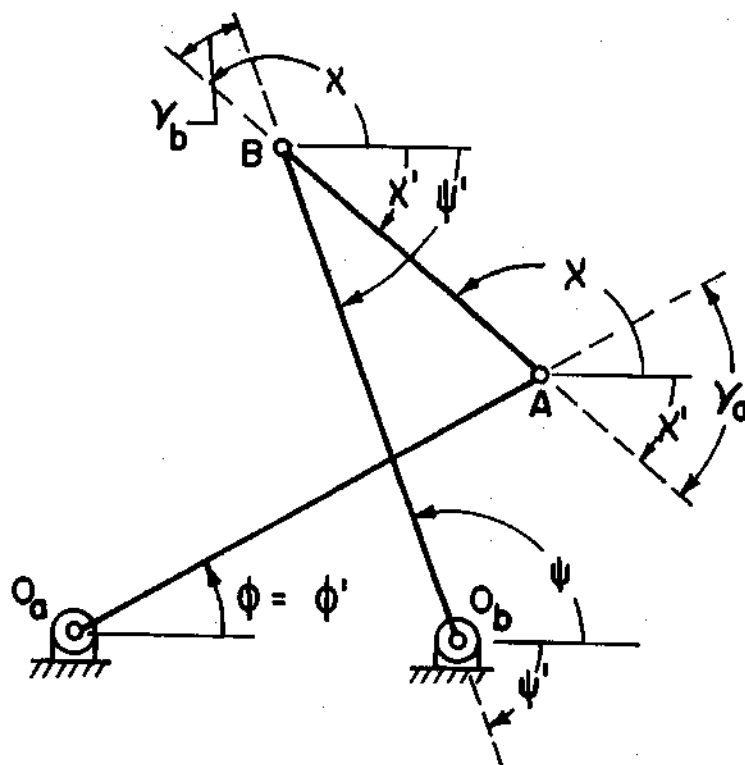


Figure 34. Transmission Angles.

may be used to calculate the initial position of the linkage. These calculations are executed by lines 63 through 80 in the computer program. Note that the computer can evaluate only the principal value of the arc-tangent function, therefore necessitating an appropriate corrective action.

Type of Mechanism

Grashof's rules are applied in the computer program (lines 238 through 251) to determine whether the linkages are crank and lever, double lever, or double crank mechanisms. In the present statement of the problem, the links are:

Q the fixed link

S the coupler link

R,T the control cranks.

It is first necessary to order the sums of the opposing links in magnitude and label these links according to the inequalities

$$a + b > c + d \quad (6.13)$$

and

$$a > b, \quad c > d \quad (6.14)$$

such that each pair itself is also ordered in magnitude. Now if

$$a - b < c - d \quad (6.15)$$

then the shortest link m of this group can make a full revolution with respect to each of the others. If this shortest link is Q , then a double crank results. If m is R or T , then a crank and lever

results with that particular link being a crank. If, however, relation (6.15) is reversed, then a double lever results and no link can rotate completely relative to any of the three remaining links.

Note the use of the Max and Min functions which are available in the algol language. If

$$a > b > c > d$$

are ordered magnitudes as indicated above, then

$$a = \text{Max } (a,b,c,d)$$

and

$$d = \text{Min } (a,b,c,d)$$

Transmission Angles

An important consideration for the designer of mechanisms is the value of the transmission angle at pin joint A or B. The closer the transmission angle approaches 90° , the better the linkage is capable of transmitting a usable force. Since either of the links R or T might be used as the follower link, both are studied for the value of the transmission angle γ during the production of the approximate straight line portion of the coupler curve.

In Fig. (34) a four-bar linkage is represented with transmission angles γ_a and γ_b located at pin joints A and B. In the evaluation of the transmission angle, the direction of the link center line (without regard to the sense of the line) is significant. Consequently, the principal value (represented by primes in the figure) of the arctangent function is sufficient for the present purpose. (The principal value of the arctangent function has the range $-\pi/2$ to $\pi/2$).

Consider the evaluation of γ_a for the position shown in Fig. (6.5). If both ϕ' and χ' are positive, then

$$\gamma_a = |\phi' - \chi'| \quad (6.16)$$

Otherwise, if $|\phi'| + |\chi'|$ is larger than 90° , then

$$\gamma_a = 180^\circ - \{|\phi'| + |\chi'|\} \quad (6.17)$$

or if $|\phi'| + |\chi'|$ is less than 90° , then

$$\gamma_a = |\phi'| + |\chi'| \quad (6.18)$$

For the calculation of γ_b , the values of ψ' and χ' are used in the above relations. This sequence of computations is represented by what is known as a procedure statement labeled "angle" (lines 13 through 22). It can be used at any time in the program as illustrated by lines 81 and 157.

Computation of the Deviation Curve

Most of the preceding discussion of this chapter refers to the initial position of the linkage. To calculate the deviation of the coupler curve from an exact straight line, it is necessary to consider the linkage in a range of positions on either side of the initial position. This requires a special technique of computation due to Denavit and Hartenberg (34). It represents a closed form expression for the output angle ψ if the value for the input angle ϕ is specified. Because of the high accuracy required of the computations, a closed form solution is virtually essential. It has the limitation, however, of not being able to pass through certain dead center positions of the cranks.

Derivation of the Input-Output Displacement Function

Because of the importance of the input-output displacement function of the four-bar linkage to this investigation, it is reproduced here in a form similar to that given by Denavit and Hartenberg. Referring to Fig. (35), it is apparent that

$$\begin{aligned}x_a &= R \cos \phi & x_b &= T \cos \psi + Q \\y_a &= R \sin \phi & y_b &= T \sin \psi\end{aligned}\quad (6.19)$$

and requiring that the coupler link is inextensible, it follows that

$$(x_b - x_a)^2 + (y_b - y_a)^2 = S^2 \quad (6.20)$$

Substitution of the values for the coordinates of the pin joints A and B from Eq. (6.19) into Eq. (6.20) results in

$$\begin{aligned}T^2 + Q^2 - S^2 + R^2 - 2RQ \cos \phi = \\[2RT \cos \phi - 2QT] \cos \psi + [2RT \sin \phi] \sin \psi\end{aligned}$$

and rearranging

$$A \sin \psi + B \cos \psi = C \quad (6.21)$$

where

$$\begin{aligned}A &= \sin \phi \\B &= \cos \phi - Q/R \\C &= \frac{Q^2 + R^2 + T^2 - S^2}{2RT} - \frac{Q}{T} \cos \phi\end{aligned}$$

Eq. (6.21) is not explicit in the desired variable ψ . To obtain an

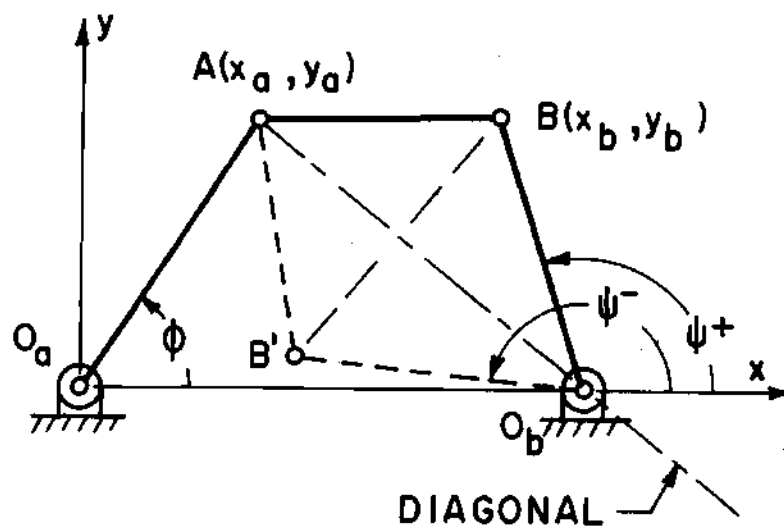


Figure 35. Symbols for the Input-Output Displacement Function.

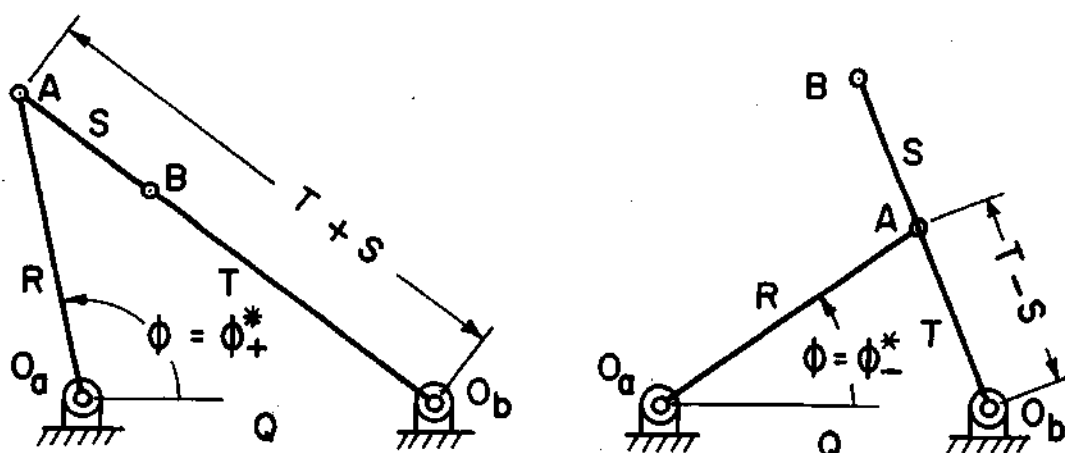


Figure 36. Dead Center Positions Along Follower.

explicit expression, use the substitutions

$$\sin \psi = \frac{2 \tan \psi/2}{1 + \tan^2 \psi/2}, \quad \cos \psi = \frac{1 - \tan^2 \psi/2}{1 + \tan^2 \psi/2}$$

which results in the quadratic in $\tan \psi/2$

$$(B + C) \tan^2 \psi/2 - 2A \tan \psi/2 + (C - B) = 0 \quad (6.22)$$

from which

$$\psi = 2 \tan^{-1} \left\{ \frac{A \pm \sqrt{A^2 + B^2 - C^2}}{B + C} \right\} \quad (6.23)$$

This is the required explicit form for ψ . As mentioned earlier, it has certain limitations. The square root term can not have a negative argument and there exists the uncertainty associated with its sign. As shown in Fig. (35), there are two positions of the follower which satisfy Eq. (6.23). Graphically, the positive sign is to be used when the coupler link center line does not pass between the fixed pivots. It must be pointed out that the explicit form, Eq. (6.23), has a distinct advantage over the transcendental form, Eq. (6.21). The explicit form does not require an iteration as does the transcendental form. This means that the explicit form, where it is applicable, is inherently much more accurate and consumes much less machine time. Both of these criteria are of primary importance to this investigation.

In general, the use of the explicit form was found to be applicable for a wide range of motion. That is, normally the entire approximate straight line segment of the coupler curve was successfully calculated by this approach. Given a faster machine such as the Burroughs

B 5000, the implicit form should be investigated for its usefulness by evaluating the amount of machine time required and the resulting accuracy.

The following approach to eliminate the limitations of the explicit form has not been used in the computer programs of this investigation (primarily because the memory space of the computer was already limited). The difficulty results when a dead center position at the follower pin-joint is reached. This difficulty does not occur for the drag link mechanism. If the input variable is always applied to the crank of those mechanisms that are crank and lever mechanisms, dead center positions at the follower do not occur.

During the motion of those double lever mechanisms which allow a full 360° rotation of the coupler link, the square root term of the explicit form of the input-output function approaches zero twice. In those positions the pin joint B lies on the diagonal O_bA . The sign of the square root term must be reversed for these positions; and, in addition, the motion of the input lever must be reversed (Fig. 36). The values for the input variable ϕ corresponding to these positions are represented as ϕ^* values calculated by

$$\begin{aligned}\phi_+^* &= \cos^{-1} \left[\frac{Q^2 + R^2 - (S + T)^2}{2RQ} \right] \\ \phi_-^* &= \cos^{-1} \left[\frac{Q^2 + R^2 - (T - S)^2}{2RQ} \right]\end{aligned}\tag{6.24}$$

The double lever mechanisms which do not allow a full rotation of the coupler link have two symmetrically placed positions about the fixed axis. The dead center positions are

$$\begin{aligned}\phi_+^{**} &= \cos^{-1} \left[\frac{Q^2 + R^2 - (S + T)^2}{2RQ} \right] \\ \phi_-^{**} &= 360 - \phi_+^{**}\end{aligned}\tag{6.25}$$

Application of the Displacement Function

The sign of the square root term is chosen by lines 93 through 103 of the computer program. Once this choice has been made the sign is not reversed during the rest of the computation. The limitations of this procedure have been discussed in the preceding article. The computation of the angle ψ for increasing ϕ is effected by lines 114 through 129 and for decreasing ϕ by lines 176 through 192. The position of the coupler link is determined by evaluating the angle χ as shown in lines 133 through 138. The x and y coordinates of the location of the output point are then obtained from

$$\begin{aligned}x &= R \cos \phi + M \cos (\epsilon + \chi) \\ y &= R \sin \phi + M \sin (\epsilon + \chi)\end{aligned}\tag{6.26}$$

The u and v coordinates are calculated by using the transformations of Eq. (6.2) and (6.4) in succession. These computations are carried out in lines 139 through 145 where D is the sought after deviation.

Lines 159 through 173 are used to determine the length of the approximate straight line segment of the coupler curve for three different specified accuracies. The values of $\Delta\phi$, $\Delta\psi$, γ_a , and γ_b corresponding to the linkage positions at the extremes of the approximating segment of the curve are also recorded.

Lines 254 through 258 represent the calculation of dv/du and d^2v/du^2 for the deviation curve by finite difference approximations. These derivatives are indicative of the dynamic character of the deviation curve.

Discussion of the Computer Output

In order to make the data resulting from the computer oriented problem more useful, a value for the unit length of the mechanism is required. Since the diameter of the inflection circle is largely independent of the size of the mechanism, it does not fulfill this requirement. The unit length that has been used is the average of the four links Q, R, S, T and the average of M and N, such that

$$\text{Unit Length} = \frac{Q + R + S + T + \frac{M + N}{2}}{5} \quad (6.27)$$

This unit value gives very satisfactory results. All data resulting from the computations of the computer program are adjusted to this unit length. In addition, no link is allowed to be five times larger than the smallest link. This requirement removes from consideration those linkages that may be considered to have poor proportions. This is accomplished by lines 90 through 92 in the computer program. At the end of this article is a representative output for a linkage having a Ball-double Burmester point (Fig. 37). The manner in which the data are interpreted depends somewhat on the type of mechanism. Three sets of data for the length of travel, the range of rotation of a crank, and the transmission angles, are given corresponding to the three specified accuracies of the output curve.

In all cases, the range of the rotation of the cranks $\Delta\theta$ or $\Delta\psi$ will be important. For example if $\Delta\theta$ for the crank of a crank and lever mechanism is 180° , this tells the designer that half the cycle is spent producing the segment of the coupler curve approximating a straight line. If it is greater than 180° , the linkage might be considered to be a quick return mechanism. Normally, the transmission angle at the driven pin joint is of importance. For the drag link mechanism, the driven crank will most likely be the one having the minimum rotation during the production of the approximate straight line portion of the curve. Hence, the largest range of rotation of the cranks and the transmission angle of the opposite crank are quoted in the graphical presentation of the linkage (Fig. 37). For the crank and lever mechanism, the range of rotation of the crank and the transmission angle at the follower are listed. Since double lever linkages will most likely be used as guiding mechanisms with the input at the coupler point, the larger range of rotation, $\Delta\theta$ or $\Delta\psi$, of the levers and the smaller value of the transmission angles, γ_a and γ_b , would be significant.

In Figure (37), the dimensions of the levers R and T are given with the other linkage dimensions. If R or T or both are cranks, they will be starred. No star is interpreted as a double lever mechanism, one star as a crank and lever mechanism, and two stars as a drag link mechanism. Fig. (38) is the graphical plot of the deviation v and d^2v/du^2 versus the distance u traveled along the tangent to the coupler curve. Note the smoothness of the curves and the high degree of "flatness" exhibited by the curve near the design position ($u = 0$). These desirable properties are characteristic of those coupler curves whose mechanisms are designed on the basis of curvature theory.

PARAMETERS

$$\alpha_a = 65.0 \quad \alpha_b = 165.0$$

DIMENSIONS ($\alpha_d = 142.270$)

$$Q = 1.13015 \quad S = 1.03814$$

$$R^* = 0.39118 \quad M = 1.42187$$

$$T = 1.53773 \quad \epsilon = 0.0$$

RESULTS ($\phi_0 = 231.600$)

| | | | |
|--------------|--------|-------|-------|
| D | 0.0001 | 0.01 | 0.05 |
| L | 0.398 | 1.020 | 1.320 |
| $\Delta\phi$ | 62 | 163 | 234 |
| γ_b | 53.8 | 38.4 | 35.2 |

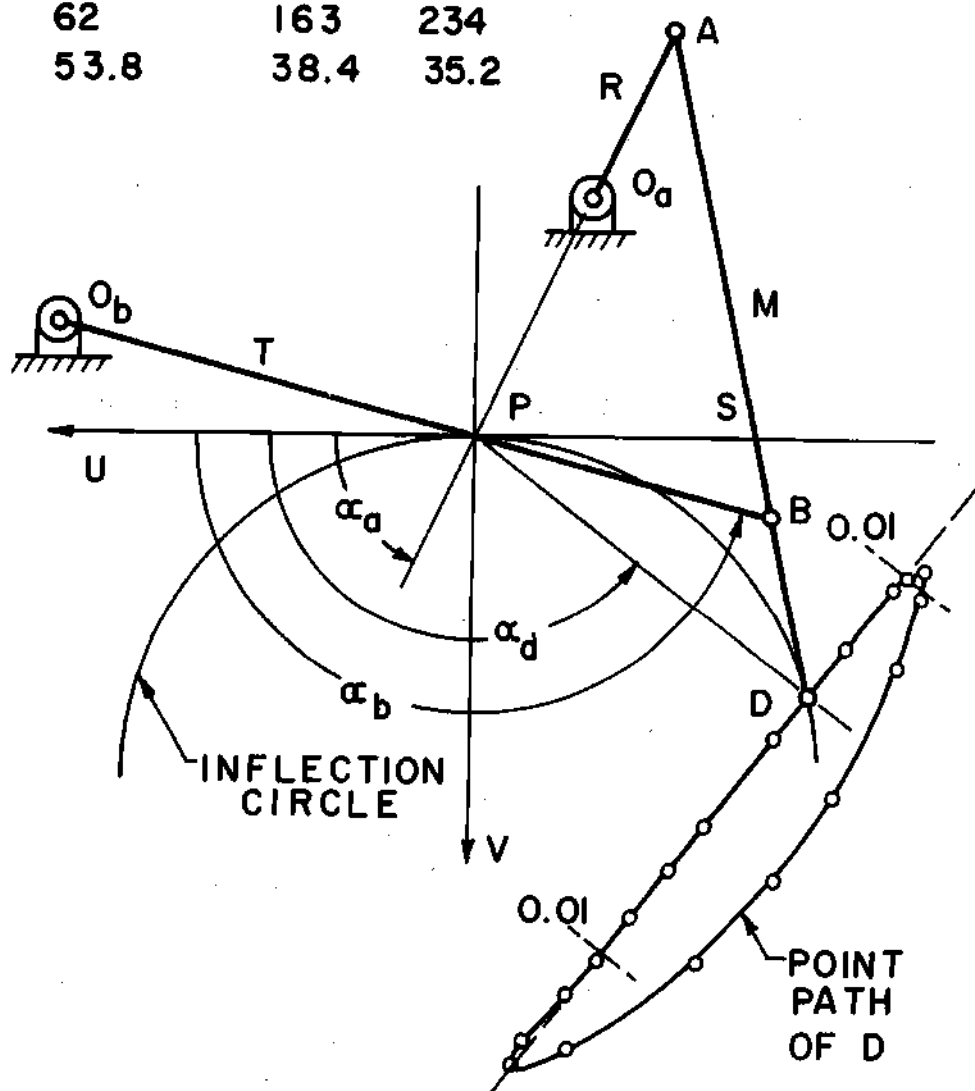


Figure 37. Representative Linkage.

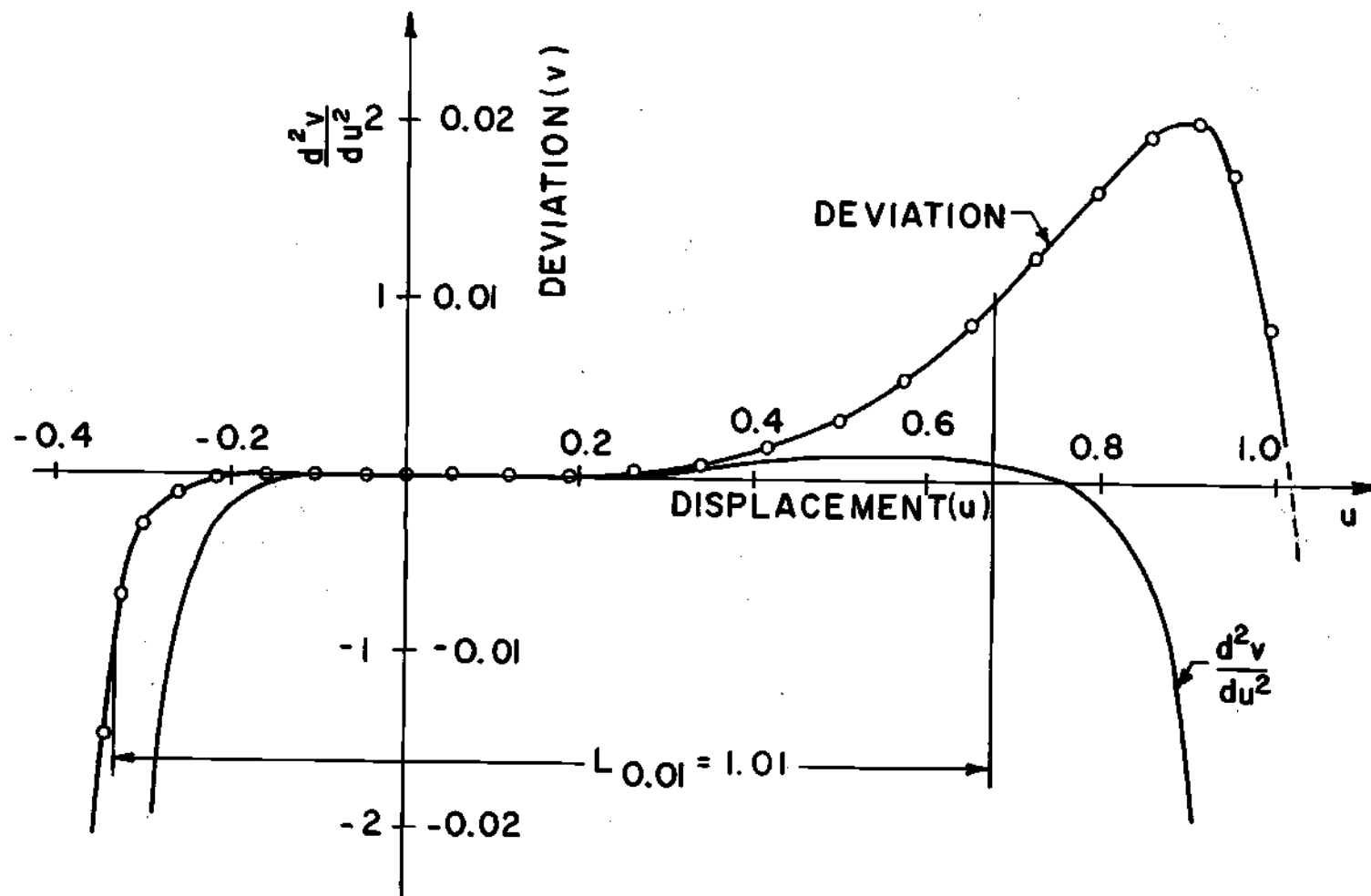


Figure 38. Representative Deviation Curve.

Representative Output

FIVE POINT EXACT STRAIGHT LINE MECHANISMS

AA= 65.00 AB= 165.00 AD= .14227029, 03
1 IS CRANK OF 1,2 CRANK AND LEVER MECHANISM
MECHANISM 1,2 UNIT LENGTH = .69321822, 00

LINK LENGTHS AND COUPLER DIMENSIONS

Q= .11301506, 01 R= .39117811, 00 S= .10381410, 01 T= .15377302, 01
EPSILON = .00000 SIDE 1 = .14218706, 01 SIDE 2 = .38372948, 00
INITIAL ANGLES
PHI= 231.600 PSI= 151.600 GAMMA A= 37.270 GAMMA B= 62.729
LENGTH = .39753037, 00 DL = .11262542, -03 DR = .10999133, -03

CORRESPONDING LIMITS FOR INPUT OUTPUT, AND PRESSURE ANGLES

PHI= 30.000 PSI= 4.279 GAMMA A= 20.418 GAMMA B= 53.860
PHI= +32.000 PSI= -6.483 GAMMA A= 57.189 GAMMA B= 68.326
LENGTH = .10096978, 01 DL = .10418407, -01 DR = .10076307, -01

CORRESPONDING LIMITS FOR INPUT OUTPUT, AND PRESSURE ANGLES

PHI= 72.000 PSI= 6.337 GAMMA A= 4.084 GAMMA B= 38.421
PHI= -91.000 PSI= -21.923 GAMMA A= 76.327 GAMMA B= 65.404
LENGTH = .13200716, 01 DL = .20448077, -01 DR = .20140843, -03

CORRESPONDING LIMITS FOR INPUT OUTPUT, AND PRESSURE ANGLES

PHI= 81.000 PSI= 5.777 GAMMA A= 10.289 GAMMA B= 35.149
PHI= -153.000 PSI= -34.114 GAMMA A= 7.840 GAMMA B= 46.769

FIVE POINT EXACT STRAIGHT LINE MECHANISMS

MECHANISM 1,2

UNIT LENGTH = .69321822, 00

| LOCATION | DEVIATION | VELOCITY | ACCELERATION |
|-----------|-----------|----------------|----------------|
| .97570900 | .00183950 | -.34803430,-02 | -.14204014,.02 |
| .97225119 | .00882146 | .95703827, 00 | -.18084719, 03 |
| .96334456 | .01393513 | .37244756, 00 | -.26391150, 02 |
| .94960080 | .01740326 | .17063867, 00 | -.84074768, 01 |
| .93158280 | .01946277 | .71932804,-01 | -.36385718, 01 |
| .90980668 | .02035311 | .16410163,-01 | -.17928213, 01 |
| .88474688 | .02030559 | -.16659259,-01 | -.96581596, 00 |
| .85683841 | .01953475 | -.36468443,-01 | -.51393934, 00 |
| .82648077 | .01823561 | -.47735711,-01 | -.25285230, 00 |
| .79404086 | .01658060 | -.53371987,-01 | -.10394546, 00 |
| .75985562 | .01471675 | -.55057770,-01 | -.68380164,-02 |
| .72423382 | .01276698 | -.53989427,-01 | .49571176,-01 |
| .68745797 | .01082992 | -.51049621,-01 | .95904700,-01 |
| .64978541 | .00898209 | -.46866213,-01 | .12084603, 00 |
| .61144926 | .00727868 | -.41899168,-01 | .13381694, 00 |
| .57265915 | .00575613 | -.36564222,-01 | .14072584, 00 |
| .53360258 | .00443490 | -.31123072,-01 | .13402258, 00 |
| .49444502 | .00332077 | -.25824623,-01 | .13426227, 00 |
| .45533076 | .00240914 | -.20851296,-01 | .12604396, 00 |
| .41638486 | .00168662 | -.16340321,-01 | .10873422, 00 |
| .37771317 | .00113319 | -.12371264,-01 | .97153264,-01 |
| .33940468 | .00072571 | -.90024789,-02 | .82010364,-01 |
| .30153226 | .00043864 | -.62494740,-02 | .65698634,-01 |
| .26415527 | .00024697 | -.41112160,-02 | .54027916,-01 |
| .22732123 | .00012720 | -.24954218,-02 | .37519886,-01 |
| .19106758 | .00005830 | -.13794555,-02 | .24515674,-01 |
| .15542446 | .00002278 | -.67239598,-03 | .15231174,-01 |
| .12041689 | .00000717 | -.26848533,-03 | .39509112,-03 |
| .08606704 | .00000168 | -.71462755,-04 | .12051201,-02 |
| .05239677 | .00000031 | .73595873,-05 | .46168998,-02 |

| | | | |
|-------------|-------------|----------------|----------------|
| .01943042 | .00000022 | -.27651575,-05 | -.15352068,-02 |
| - .01280350 | .00000027 | .51735844,-05 | -.39514020,-02 |
| - .04426964 | .00000019 | -.14626064,-04 | -.45705798,-02 |
| - .07492291 | -.00000089 | -.75399906,-04 | -.57549880,-02 |
| - .10470595 | -.000000596 | -.31740957,-03 | -.15485328,-01 |
| - .13354783 | -.00002293 | -.94897389,-03 | -.30759618,-01 |
| - .16135922 | -.00006553 | -.22950742,-02 | -.65285206,-01 |
| - .18803117 | -.00015665 | -.48363480,-02 | -.13236529, 00 |
| - .21343110 | -.00033064 | -.93333825,-02 | -.22886516, 00 |
| - .23739808 | -.00063552 | -.16852344,-01 | -.41444884, 00 |
| - .25973891 | -.00113661 | -.29184300,-01 | -.72352354, 00 |
| - .28022252 | -.00191883 | -.49198090,-01 | -.13101116, 01 |
| - .29857351 | -.00309044 | -.82061554,-01 | -.24600122, 01 |
| - .31446264 | -.00478514 | -.13824123, 00 | -.50871186, 01 |
| - .32749828 | -.00716417 | -.24298295, 00 | -.12799208, 02 |
| - .33721499 | -.01041840 | -.47783821, 00 | -.47411232, 02 |
| - .34306067 | -.01476598 | -.13457974, 01 | -.55081088, 03 |

Representative Computer Program

```

COMMENT STUDY OF THE DEVIATION FROM EXACT STRAIGHT LINE MOTION FOR THE
LINKAGES WHICH HAVE A DOUBLE BURMESTER-BALL POINT SOMEWHERE ON THE
INFLECTION CIRCLE. THE OUTPUT WILL GIVE THE PRESSURE ANGLE, TYPE OF
MECHANISM, THE LENGTH OF THE STRAIGHT LINE OUTPUT FOR THREE DIFFERENT
ACCURACY SPECIFICATIONS --- DELBERT TESAR, M.E., G.I.T., OCT 1, 1963
DUMP
MONITOR Z,X,Y,US,VS
INTEGER I,J,L,MEN,LADY,Z,F,FAT,NAT
ARRAY A(3),AR(3),U(2),V(2),UP(2),VP(2), UU(2),
VV(2),UUP(2),VVP(2),UPU(2),VPV(2),NS(330),DL(330)
G=1.7453293**-2 $ Z=0 $ W=(180.0,G)
PROCEDURE ANGLE(X,Y$V)
BEGIN S=ABS(X/0.017453293)$ T=ABS(Y/0.017453293)
EITHER IF (X.Y) GTR 0.0
BEGIN V=ABS(S-T) $ RETURN END
OTHERWISE
EITHER IF (S+T) GTR 90.0
BEGIN V=ABS(180.0-(S+T)) $ RETURN END
OTHERWISE
BEGIN V=S+T $ RETURN END
END ANGLE ( )
I=1 $ J=2
FOR A(1)=75.0,80.0,85.0
FOR A(2)=145.0,150.0,155.0,160.0,165.0
BEGIN AR(1)=(G)(A(1)) $ AR(2)=(G)(A(2))
VEW=TAN(AR(1))+TAN(AR(2))
WEV=(TAN(AR(1)))(TAN(AR(2)))
ARM=ARCTAN(-VEW/(WEV+3.0))
EITHER IF ARM LSS 0.0 $ AR(3)=ARM+W
OTHERWISE $ AR(3)=ARM $ A(3)=AR(3)/G
AIR=AR(3)
IF ABS(A(1)-A(2)) LSS 0.001 $ GO TO JOB

```

| | | | | |
|--------|--|----|---------------------------|-------|
| | IF ABS(A(1)-A(3)) LSS 0.001 | \$ | GO TO JOB | \$ 34 |
| | PA=(VEW)(WEV-1.0)(SIN(AR(1)))/((WEV+3.0)(TAN(AR(1))) | | | \$ 35 |
| | | | VEW(WEV) | \$ 36 |
| | PB=(VEW)(WEV-1.0)(SIN(AR(2)))/((WEV+3.0)(TAN(AR(2))) | | | \$ 37 |
| | | | VEW(WEV) | \$ 38 |
| | POA=(PA)(SIN(AR(1)))/(SIN(AR(1))-PA) | | | \$ 39 |
| | POB=(PB)(SIN(AR(2)))/(SIN(AR(2))-PB) | | | \$ 40 |
| | V(1)=(POA)(SIN(AR(1))) | | | \$ 41 |
| | U(1)=(POA)(COS(AR(1))) | | | \$ 42 |
| | VP(1)=(PA)(SIN(AR(1))) | | | \$ 43 |
| | UP(1)=(PA)(COS(AR(1))) | | | \$ 44 |
| | V(2)=(POB)(SIN(AR(2))) | | | \$ 45 |
| | U(2)=(POB)(COS(AR(2))) | | | \$ 46 |
| | VP(2)=(PB)(SIN(AR(2))) | | | \$ 47 |
| | UP(2)=(PB)(COS(AR(2))) | | | \$ 48 |
| | VP1=(SIN(AR(3)))(SIN(AR(3))) | | | \$ 49 |
| | UP1=(COS(AR(3)))(SIN(AR(3))) | | | \$ 50 |
| | UU(I)=U(J)-U(I) | | | \$ 51 |
| | VV(I)=V(J)-V(I) | | | \$ 52 |
| | UUP(I)=UP(J)-UP(I) | | | \$ 53 |
| | VVP(I)=VP(J)-VP(I) | | | \$ 54 |
| | UPU(I)=UP(I)-U(I) | \$ | UPU(J)=UP(J)-U(J) | \$ 55 |
| | VPV(I)=VP(I)-V(I) | \$ | VPV(J)=VP(J)-V(J) | \$ 56 |
| | Q = SQRT((UU(I))(UU(I))+(VV(I))(VV(I))) | | | \$ 57 |
| | S = SQRT((UUP(I))(UUP(I))+(VVP(I))(VVP(I))) | | | \$ 58 |
| | R=ABS(POA-PA) | \$ | T=ABS(POB-PB) | \$ 59 |
| EITHER | IF UU(I) LSS 0.0 | \$ | THR=ARCTAN(VV(I)/UU(I))+W | \$ 60 |
| | OTHERWISE | \$ | THR=ARCTAN(VV(I)/UU(I)) | \$ 61 |
| | IF THR LSS 0.0 | \$ | THR=THR+2.0*W | \$ 62 |
| | XXP = UPU(I)COS(THR) + VPV(I)SIN(THR) | | | \$ 63 |
| | YYP = -UPU(I)SIN(THR) + VPV(I)COS(THR) | | | \$ 64 |
| | XPX = UPU(J)COS(THR) + VPV(J)SIN(THR) | | | \$ 65 |
| | YPY = -UPU(J)SIN(THR) + VPV(J)COS(THR) | | | \$ 66 |
| | PHHI=ARCTAN(YPY/XXP) | | | \$ 67 |
| EITHER | IF XXP LSS 0.0 | \$ | PPHIR=PHHI+W | \$ 68 |
| | OTHERWISE | \$ | PPHIR=PHHI | \$ 69 |

```

IF PPHIR LSS 0.0 $ PPHIR=PPHIR+2.0.W $ 70
PPHI=PPHIR/G $ 71
PPSI=ARCTAN(YPY/XPX) $ 72
EITHER IF (XPX) LSS 0.0 $ SSIR=PPSI+W $ 73
OTHERWISE $ SSIR=PPSI $ 74
IF SSIR LSS 0.0 $ SSIR=SSIR+2.0.W $ 75
SSI=SSIR/G $ 76
BEGIN IKKIR=ARCTAN((YPY-YPY)/(XPX +Q-XXP)) $ 77
EITHER IF (XPX-XXP+Q) LSS 0.0$ KKIR=IKKIR+W $ 78
OTHERWISE $ KKIR=IKKIR $ 79
IF KKIR LSS 0.0 $ KKIR=KKIR+2.0.W END $ 80
ANGLE(PHHI,IKKIR$GGA) $ 81
ANGLE(PPSI,IKKIR$GGB) $ 82
PD=SIN(AR(3)) $ 83
N=SQRT(ABS(PB.PB+PD.PD-(2.0)(PB.PD)COS(AR(2)-AR(3)))) $ 84
M=SQRT(ABS(PA.PA+PD.PD-(2.0)(PA.PD)COS(AR(1)-AR(3)))) $ 85
EITHER IF ABS(S-(M+N)) LSS 0.00001 $ E=0.0 $ 86
OTHERWISE $ 87
BEGIN EITHER IF (M-N) LSS 0.0$ E=180.0 $ 88
OTHERWISE E=0.0 END $ ER=(E)(G) $ 89
MM = MAX(Q,R,S,T,((M+N)/2.0)) $ 90
NN = MIN(Q,R,S,T,(M+N)) $ 91
IF MM/NN GTR 5.0 $ GO TO JOB $ 92
AA = SIN(PPHIR) $ 93
BB = COS(PPHIR)-(Q/R) $ 94
CC = ((Q.Q+R.R+T.T-S.S)/(2.0.R.T))-(Q/T)COS(PPHIR) $ 95
IF (AA.AA+BB.BB-CC.CC) LSS 0.0$ GO TO JOB $ 96
DD = SQRT(AA.AA+BB.BB-CC.CC) $ 97
EITHER IF ((AA+DD)/(BB+CC)) GEQ 0.0 $ BEGIN $ 98
EITHER IF ABS(SSIR-(2.0)ARCTAN((AA+DD)/(BB+CC))) LSS 0.001 $ 99
CA = 1.0 $ OTHERWISE $ CA = -1.0 END $ 100
OTHERWISE $ BEGIN $ 101
EITHER IF ABS(SSIR-(2.0)(W+ARCTAN((AA+DD)/(BB+CC)))) LSS 0.001 $ 102
CA = 1.0 $ OTHERWISE $ CA = -1.0 END $ 103
WRITE($$TL1) $ 104
WRITE($$AN1,TL2) $ 105

```

| | | | | |
|-----------|---|---------------------------|-----------------------|--------|
| | UN= | (Q+R+S+T+((M+N)/2.0))/5.0 | | \$ 106 |
| | MEN=0 | \$ | SV=0.0 | \$ 107 |
| | SAVE=Z=SU=GIN=SIR=SOT=0.0 | \$ | D=0.0 | \$ 108 |
| | UMAX1=UMAX2=UMAX3=UMIN1=UMIN2=UMIN3=0.0 | | \$ L=1 | \$ 109 |
| | GA1=GA2=GA3=AG1=AG2=AG3=0.0 | | | \$ 110 |
| | GB1=GB2=GB3=BG1=BG2=BG3=0.0 | | | \$ 111 |
| | IP1=IP2=IP3=IIP1=IIP2=IIP3=0.0 | | | \$ 112 |
| | IS1=IS2=IS3=IIS1=IIS2=IIS3=0.0 | | | \$ 113 |
| | FOR Z=(0,1,180) | | | \$ 114 |
| BEGIN | PHIR = PPHIR + (G.Z)\$ | | PHI = PPHI + Z | \$ 115 |
| | LA = SIN(PHIR) | \$ | F=Z+180 | \$ 116 |
| | LB = COS(PHIR)-(Q/R) | | | \$ 117 |
| | LC = ((Q.Q+R.R+T.T-S.S)/(2.0.R.T)) - (Q/T)COS(PHIR) | | | \$ 118 |
| | IF (LA.LA+LB.LB-LC.LC) LSS 0.0 | | | \$ 119 |
| EITHER | IF MEN EQL 0 | | | \$ 120 |
| BEGIN | Z=Z-L | \$ | L=L+1 | \$ 121 |
| OR | IF MEN EQL 1 | | | \$ 122 |
| BEGIN | Z=Z-L | \$ | L=L+1 | \$ 123 |
| OTHERWISE | | | | \$ 124 |
| BEGIN | Z=Z-L | \$ | | \$ 125 |
| | LD = SORT(LA.LA+LB.LB-LC.LC) | | | \$ 126 |
| | SIR = (2.0)ARCTAN((LA+CA.LD)/(LB+LC)) | | | \$ 127 |
| EITHER | IF SIR GTR 0.0 | \$ | SI=(SIR-SSIR)/G | \$ 128 |
| | OTHERWISE | \$ | SI=(SIR-SSIR+2.0.W)/G | \$ 129 |
| EITHER | IF (SIR GTR W/2.0) AND (SIR LEQ W) | \$ | SIRI=SIR-W | \$ 130 |
| OR | IF (SIR LSS -W/2.0) AND (SIR GEQ -W) | \$ | SIRI=SIR+W | \$ 131 |
| | OTHERWISE | \$ | SIRI=SIR | \$ 132 |
| | AX = R.COS(PHIR) | \$ | AY = R.SIN(PHIR) | \$ 133 |
| | BX = T.COS(SIR) + Q | \$ | BY = T.SIN(SIR) | \$ 134 |
| BEGIN | IK=ARCTAN((BY-AY)/(BX-AX)) | | | \$ 135 |
| EITHER | IF (BX-AX) LSS 0.0 | \$ | KIR=IK+W | \$ 136 |
| | OTHERWISE | \$ | KIR=IK | \$ 137 |
| | IF KIR LSS 0.0 | \$ | KIR=KIR+2.0.W | \$ 138 |
| | | | END | \$ 139 |
| | X=((R)(COS(PHIR)))+(M)(COS(ER+KIR)) | | | \$ 140 |
| | Y=((R)(SIN(PHIR)))+(M)(SIN(ER+KIR)) | | | \$ 141 |
| | US=((X)(COS(THR)))-(Y)(SIN(THR))+U(I) | | | \$ 141 |

| | | | | |
|-----------|---|---------------|--------------|--------|
| | VS=((X)(SIN(THR)))+(Y)(COS(THR))+(V(1)) | | | \$ 142 |
| | SU=((UP1-US)SIN(A1R)+(VS-VP1)COS(A1R))/UN | | | \$ 143 |
| | SV=((UP1-US)COS(A1R)-(VS-VP1)SIN(A1R)) | | | \$ 144 |
| | D=ABS(SV/UN) | | | \$ 145 |
| | IF Z EQL 1.0 \$ GIN=SIGN(SU-SOT) | | | \$ 146 |
| | IF Z EQL 0.0 \$ SOT=SU | | | \$ 147 |
| | IF (GIN)(SU-SAVE) LSS 0.0 | | | \$ 148 |
| EITHER | IF MEN EQL 0 | | | \$ 149 |
| BEGIN | Z=Z-L \$ L=L+1 \$ | GO TO DUT END | | \$ 150 |
| OR | IF MEN EQL 1 | | | \$ 151 |
| BEGIN | Z=Z-L \$ L=L+1 \$ | GO TO GUT END | | \$ 152 |
| OTHERWISE | | | | \$ 153 |
| BEGIN | Z=Z-L \$ | GO TO MUT END | | \$ 154 |
| | DL(F)=SV/UN | | | \$ 155 |
| | NS(F)=SU-SOT \$ | SAVE=SU | | \$ 156 |
| | ANGLE((PHHI+Z.G),IK\$GA) | | | \$ 157 |
| | ANGLE(SIRI,IK\$GB) | | | \$ 158 |
| | SWITCH MEN,(BAD,GOOD) | | | \$ 159 |
| | IF D GTR 0.0001 | | | \$ 160 |
| DUT.. | BEGIN UMAX1=SU-SOT \$ | MEN = 1 | | \$ 162 |
| | GA1 = GA \$ | GB1 = GB \$ | DL1=D | \$ 163 |
| | IP1 = Z \$ | IS1 = SI | END | \$ 164 |
| BAD.. | IF D GTR 0.01 | | | \$ 165 |
| GUT.. | BEGIN UMAX2=SU-SOT \$ | MEN = 2 | | \$ 166 |
| | GA2 = GA \$ | GB2 = GB \$ | DL2=D | \$ 167 |
| | IP2 = Z \$ | IS2 = SI | END | \$ 168 |
| GOOD.. | IF D GTR 0.05 | | | \$ 169 |
| MUT.. | BEGIN UMAX3=SU-SOT \$ | FAT=F | | \$ 170 |
| | GA3 = GA \$ | GB3 = GB \$ | DL3=D | \$ 171 |
| | IP3 = Z \$ | IS3 = SI | | \$ 172 |
| | GO TO PUT | | END END | \$ 173 |
| PUT.. | LADY=0 \$ | SV=0.0 | | \$ 174 |
| | SAVE=Z=SU=GIN=SIR=0.0 | \$ | D=0.0 \$ L=1 | \$ 175 |
| | FOR Z = (-1,-1,-180) | | | \$ 176 |
| BEGIN | PHIR = PPHIR + (G.Z)\$ | F=Z+180 | | \$ 177 |
| | PHI = PPHI + Z | | | \$ 178 |

| | | | | |
|-----------|--|----|-----------------------|-------------------------|
| | LA = SIN(PHIR) | | | \$ 179 |
| | LB = COS(PHIR)-(Q/R) | | | \$ 180 |
| | LC = ((Q.Q+R.R+T.T-S.S)/(2.0.R.T)) -(Q/T)COS(PHIR) | | | \$ 181 |
| | IF (LA.LA+LB.LB-LC.LC) LSS 0.0 | | | \$ 182 |
| EITHER | IF LADY EQL 0 | | | \$ 183 |
| BEGIN | Z=Z+L | \$ | L=L+1 | \$ GO TO WUT END \$ 184 |
| OR | IF LADY EQL 1 | | | \$ 185 |
| BEGIN | Z=Z+L | \$ | L=L+1 | \$ GO TO YUT END \$ 186 |
| OTHERWISE | | | | \$ 187 |
| BEGIN | Z=Z+L | \$ | | \$ GO TO CUT END \$ 188 |
| | LD = SQRT(LA.LA+LB.LB-LC.LC) | | | \$ 189 |
| | SIR = (2.0)ARCTAN((LA+CA.LD)/(LB+LC)) | | | \$ 190 |
| EITHER | IF SIR GTR 0.0 | \$ | SI=(SIR-SSIR)/G | \$ 191 |
| | OTHERWISE | \$ | SI=(SIR-SSIR+2.0.W)/G | \$ 192 |
| EITHER | IF (SIR GTR W/2.0) AND (SIR LEQ W) | \$ | SIRI=SIR-W | \$ 193 |
| OR | IF (SIR LSS -W/2.0) AND (SIR GEQ -W) | \$ | SIRI=SIR+W | \$ 194 |
| | OTHERWISE | \$ | SIRI=SIR | \$ 195 |
| | AX = R.COS(PHIR) | \$ | AY = R.SIN(PHIR) | \$ 196 |
| | BX = T.COS(SIR) + Q | \$ | BY = T.SIN(SIR) | \$ 197 |
| BEGIN | IK=ARCTAN((BY-AY)/(BX-AX)) | | | \$ 198 |
| EITHER | IF (BX-AX) LSS 0.0 | \$ | KIR=IK+W | \$ 199 |
| | OTHERWISE | \$ | KIR=IK | \$ 200 |
| | IF KIR LSS 0.0 | \$ | KIR=KIR+2.0.W | \$ 201 |
| | | | END | \$ 202 |
| | X=((R)(COS(PHIR)))+(M)(COS(ER+KIR)) | | | \$ 203 |
| | Y=((R)(SIN(PHIR)))+(M)(SIN(ER+KIR)) | | | \$ 204 |
| | US=((X)(COS(THR)))-((Y)(SIN(THR)))+(U(I)) | | | \$ 205 |
| | VS=((X)(SIN(THR)))+(Y)(COS(THR)))+(V(I)) | | | \$ 206 |
| | SU=((UP1-US)SIN(AIR)+(VS-VP1)COS(AIR))/UN | | | \$ 207 |
| | SV= (UP1-US)COS(AIR)-(VS-VP1)SIN(AIR) | | | \$ 208 |
| | D=ABS(SV/UN) | | | \$ 209 |
| | IF Z EQL -1.0 | \$ | GIN=SIGN(SU-SOT) | \$ 210 |
| | IF (GIN)(SU-SAVE) LSS 0.0 | | | \$ 211 |
| EITHER | IF LADY EQL 0 | | | \$ 212 |
| BEGIN | Z=Z+L | \$ | L=L+1 | \$ GO TO WUT END \$ 213 |
| OR | IF LADY EQL 1 | | | \$ 214 |
| BEGIN | Z=Z+L | \$ | L=L+1 | \$ GO TO YUT END \$ 214 |

| | | | | | |
|-----------|--------------------------|----|---------------|---------|---------------|
| OTHERWISE | | | | | \$ 215 |
| BEGIN | Z=Z+L | \$ | GO TO CUT END | | \$ 216 |
| | DL(F)=SV/UN | | | | \$ 217 |
| | NS(F)=SU-SOT | \$ | SAVE=SU | | \$ 218 |
| | ANGLE((PHHI+Z.G),IK\$GA) | | | | \$ 219 |
| | ANGLE(\$IRI,IK\$GB) | | | | \$ 220 |
| | SWITCH LADY,(GIRL,BOY) | | | | \$ 221 |
| | IF D GTR 0.0001 | | | | \$ 222 |
| WUT.. | BEGIN UMIN1=SU-SOT | \$ | LADY = 1 | | \$ 223 |
| | AG1 = GA | \$ | BG1 = GB | \$ | \$ 224 |
| | IIP1 = Z | \$ | IIS1 = SI | DR1=D | \$ 225 |
| GIRL.. | IF D GTR 0.01 | | | END | \$ 226 |
| YUT.. | BEGIN UMIN2=SU-SOT | \$ | LADY = 2 | | \$ 227 |
| | AG2 = GA | \$ | BG2 = GB | \$ | \$ 228 |
| | IIP2 = Z | \$ | IIS2 = SI | DR2=D | \$ 229 |
| BOY.. | IF D GTR 0.05 | | | END | \$ 230 |
| CUT.. | BEGIN UMIN3=SU-SOT | \$ | NAT=F | | \$ 231 |
| | AG3 = GA | \$ | BG3 = GB | \$ | \$ 232 |
| | IIP3 = Z | \$ | IIS3 = SI | DR3=D | \$ 233 |
| | GO TO RUT | | | END END | \$ 234 |
| RUT.. | X =ABS(UMAX1)+ABS(UMIN1) | | | | \$ 235 |
| | Y =ABS(UMAX2)+ABS(UMIN2) | | | | \$ 236 |
| | P =ABS(UMAX3)+ABS(UMIN3) | | | | \$ 237 |
| EITHER | IF (Q+S) GTR (T+R) | | | | \$ 238 |
| BEGIN | SA = MAX(Q,S) | \$ | SB=Q+S-SA | \$ | \$ 239 |
| | OTHERWISE | | SC=MAX(T,R) | \$ | SD=T+R-SC END |
| BEGIN | SA=MAX(T,R) | \$ | SB=T+R-SA | \$ | \$ 240 |
| EITHER | IF (SA-SB) LSS (SC-SD) | | SC=MAX(Q,S) | \$ | SD=Q+S-SC END |
| BEGIN | SM = MIN(SA,SB,SC,SD) | | | | \$ 241 |
| | IF ABS(SM-Q) LSS 0.00001 | | | | \$ 242 |
| | WRITE(\$\$AN2,TL3) | | | | \$ 243 |
| | IF ABS(SM-R) LSS 0.00001 | | | | \$ 244 |
| | WRITE(\$\$AN3,TL4) | | | | \$ 245 |
| | IF ABS(SM-T) LSS 0.00001 | | | | \$ 246 |
| | WRITE(\$\$AN4,TL4) | | | | \$ 247 |
| | OTHERWISE | | | END | \$ 248 |
| | | | | | \$ 249 |
| | | | | | \$ 250 |

| | | |
|--------|--|---------|
| | WRITE(\$\$AN2,TL5) | \$ 251 |
| | Q=Q/UN \$ R=R/UN \$ S=S/UN \$ T=T/UN \$ M=M/UN \$ N=N/UN | \$ 252 |
| | WRITE(\$\$AN10,TL10) | \$ 253 |
| | FOR F=(NAT+1,1,FAT-1) | \$ 254 |
| BEGIN | AC = (2.0) (((DL(F+1)-DL(F))/(NS(F+1)-NS(F))) - | 255 |
| | ((DL(F)-DL(F-1))/(NS(F)-NS(F-1)))/(NS(F+1)-NS(F-1))) | \$ 256 |
| | VE = (DL(F+1)-DL(F-1))/(NS(F+1)-NS(F-1)) | \$ 257 |
| | WRITE(\$\$AN17,TL17) | \$ 258 |
| | END | \$ 259 |
| JOB.. | END | \$ 260 |
| OUTPUT | AN1(A(1),A(2),A(3)) | \$ 261 |
| OUTPUT | AN2(I,J) | \$ 262 |
| OUTPUT | AN3(I,I,J) | \$ 263 |
| OUTPUT | AN4(J,I,J) | \$ 264 |
| OUTPUT | AN10(I,J,UN,Q,R,S,T,E,M,N,PPHI,SSI,GGA,GGB, | 265 |
| | X,DL1,DR1,IP1,ISI,GA1,GB1,IIP1,IISI,AG1,BG1, | 266 |
| | Y,DL2,DR2,IP2,IS2,GA2,GB2,IIP2,IISI,AG2,BG2, | \$ 267 |
| | P,DL3,DR3,IP3,IS3,GA3,GB3,IIP3,IISI,AG3,BG3,I,J,UN) | \$ 268 |
| OUTPUT | AN17(NS(F),DL(F),VE ,AC) | \$ 269 |
| FORMAT | TL1(B20,*FIVE POINT EXACT STRAIGHT LINE MECHANISMS*,W3,W4,W6) | \$ 270 |
| FORMAT | TL2(B5, *AA=*,X7.2,B9, *AB=*,X7.2,B9,*AD=*, | \$ 271 |
| | F14.8,W2) | \$ 272 |
| FORMAT | TL3(B5,I1,*,*,I1,* IS A DRAG LINK MECHANISM*,W0) | \$ 273 |
| FORMAT | TL4(B5,I1,* IS CRANK OF *,I1,*,*,I1,* CRANK AND LEVER *, | \$ 274 |
| | *MECHANISM*,W0) | \$ 275 |
| FORMAT | TL5(B5,I1,*,*,I1,* IS A DOUBLE LEVER MECHANISM*,W0) | \$ 275A |
| FORMAT | TL17(B5,X11.8,X17.8,F20.8,F17.8,W0) | 276 |
| FORMAT | TL10(B5,*MECHANISM *,I1,*,*,I1,B6,*UNIT LENGTH =*,F14.8,W4,W2, | 277 |
| | B5,*LINK LENGTHS AND COUPLER DIMENSIONS*,W2, | 278 |
| | B5,*Q=*,F14.8,B2,*R=*,F14.8,B2,*S=*,F14.8,B2,*T=*,F14.8, | 279 |
| | W2, | 280 |
| | B5,*EPSILON =*,X10.5,B3,*SIDE 1 =*,F14.8,B3,*SIDE 2 =*, | 281 |
| | F14.8,W2, | 282 |
| | B5,*INITIAL ANGLES*,W2, | 283 |
| | B5,*PHI=*,X8.3,B3,*PSI=*,X8.3,B3,*GAMMA A=*,X8.3,B3, | 284 |
| | *GAMMA B=*,X8.3,W0, | 285 |
| | B5,*LENGTH =*,F14.8,B3,*DL =*,F14.8,B3,*DR =*,F14.8,W4,W0, | |

| | |
|---|--------|
| B5,*CORRESPONDING LIMITS FOR INPUT OUTPUT, AND *, | 286 |
| *PRESSURE ANGLES*,W0, | 287 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 288 |
| *GAMMA B=* ,X8.3,W0, | 289 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 290 |
| *GAMMA B=* ,X8.3,W0, | 291 |
| B5,*LENGTH =*,F14.8,B3,*DL =*,F14.8,B3,*DR =*,F14.8,W4,W0, | 292 |
| B5,*CORRESPONDING LIMITS FOR INPUT OUTPUT, AND *, | 293 |
| *PRESSURE ANGLES*,W0, | 294 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 295 |
| *GAMMA B=* ,X8.3,W0, | 296 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 297 |
| *GAMMA B=* ,X8.3,W0, | 298 |
| B5,*LENGTH =*,F14.8,B3,*DL =*,F14.8,B3,*DR =*,F14.8,W4,W0, | 299 |
| B5,*CORRESPONDING LIMITS FOR INPUT OUTPUT, AND *, | 300 |
| *PRESSURE ANGLES*,W0, | 301 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 302 |
| *GAMMA B=* ,X8.3,W0, | 303 |
| B5,*PHI=* ,X8.3,B3,*PSI=* ,X8.3,B3,*GAMMA A=* ,X8.3,B3, | 304 |
| *GAMMA B=* ,X8.3,W0, | 305 |
| B20,*FIVE POINT EXACT STRAIGHT LINE MECHANISMS*,W3,W4,W6, | 306 |
| B5 ,*MECHANISM *,I1,* ,*,I1,B6,*UNIT LENGTH =*,F14.8,W4,W2, | 307 |
| B8,*LOCATION*,B9 ,*DEVIATION*,B8 ,*VELOCITY*, | 308 |
| B7,*ACCELERATION*,W2) | \$ 309 |
| FINISH | \$ 310 |

CHAPTER VII

EVALUATION AND COMPARISON OF THE LINKAGES STUDIED

The material included in this chapter is intended to survey the results of the analysis derived from the theory of the preceding chapters. It is by no means to be considered a complete interpretation of the full scope of the data that might be obtained. It should, however, provide the fundamental methods of reducing the data to a meaningful form. The linkages considered are based on the design equations of Chapters II and V. Similar studies can be made on the material given in Chapters III and IV.

The available data consist of approximately 3000 linkages. Of these, sixteen are used as representative linkages for the purpose of displaying the primary information (the length of the approximate straight line output for a specified accuracy). The analysis for the primary information of those linkages having a Ball-double Burmester point is presented in nomograph form. Although additional data are available, it can not be generalized here because of the extent of the data.

Comparison of Linkages

The linkages and the respective deviation and second derivative curves are presented in Fig. (39-70, 72, and 73). Fig. (39-44) are the three four-bar linkages which are alternates to the original slider-crank mechanism O_aAD . These linkages have a Burmester point coincident with

PARAMETERS (1,2)

$$\alpha_a = 50.0 \quad \alpha_b = 20.0 \quad \alpha_d = 80.0$$

DIMENSIONS ($\alpha_c = 96.343$)

$$Q = 0.78412 \quad S = 0.59214$$

$$R = 2.10674 \quad M = 0.52451$$

$$T = 0.94794 \quad \epsilon = -66.343$$

RESULTS ($\phi_o = 344.373$)

$$D \quad 0.0001 \quad 0.01 \quad 0.05$$

$$L \quad 0.432 \quad 1.164 \quad 1.392$$

$$\Delta\phi \quad 17 \quad 43 \quad 56$$

$$Y_b \quad 46.2 \quad 20.3 \quad 5.0$$

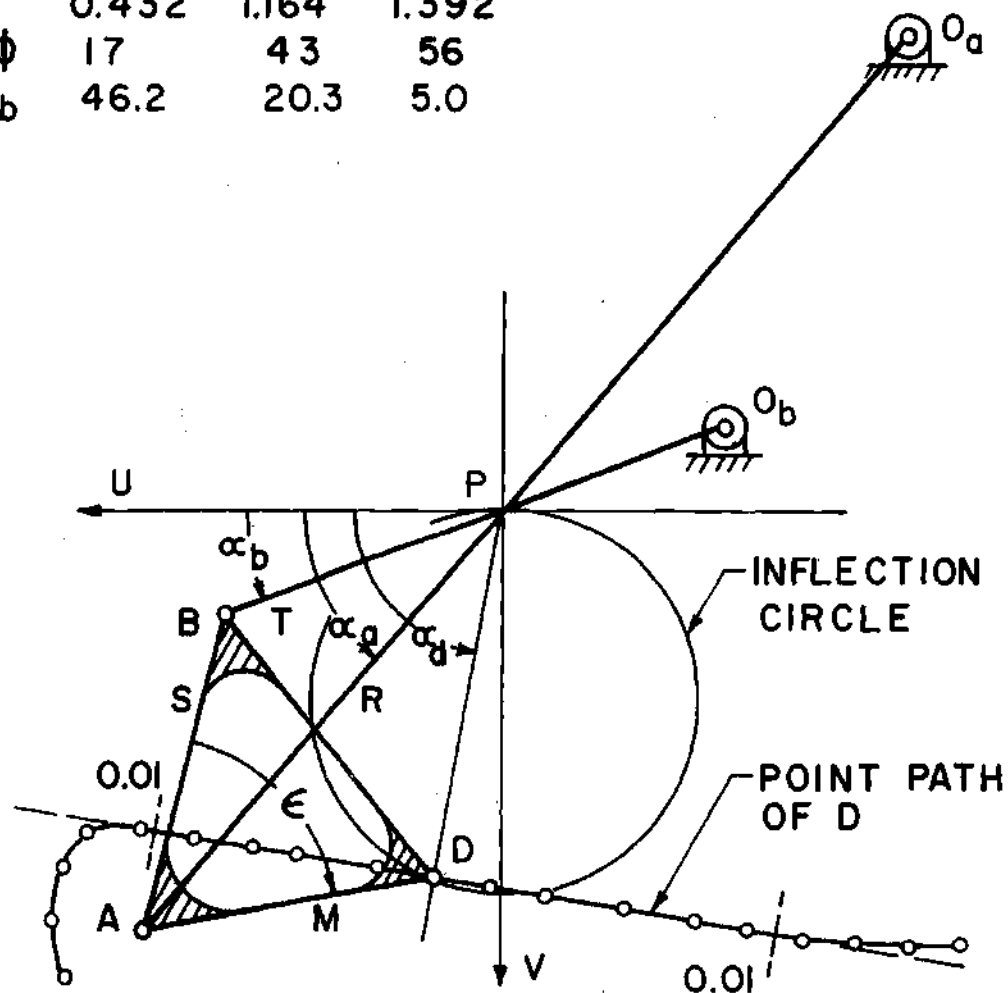


Figure 39. Alternate 1,2 for General Ball-Burmester Point.

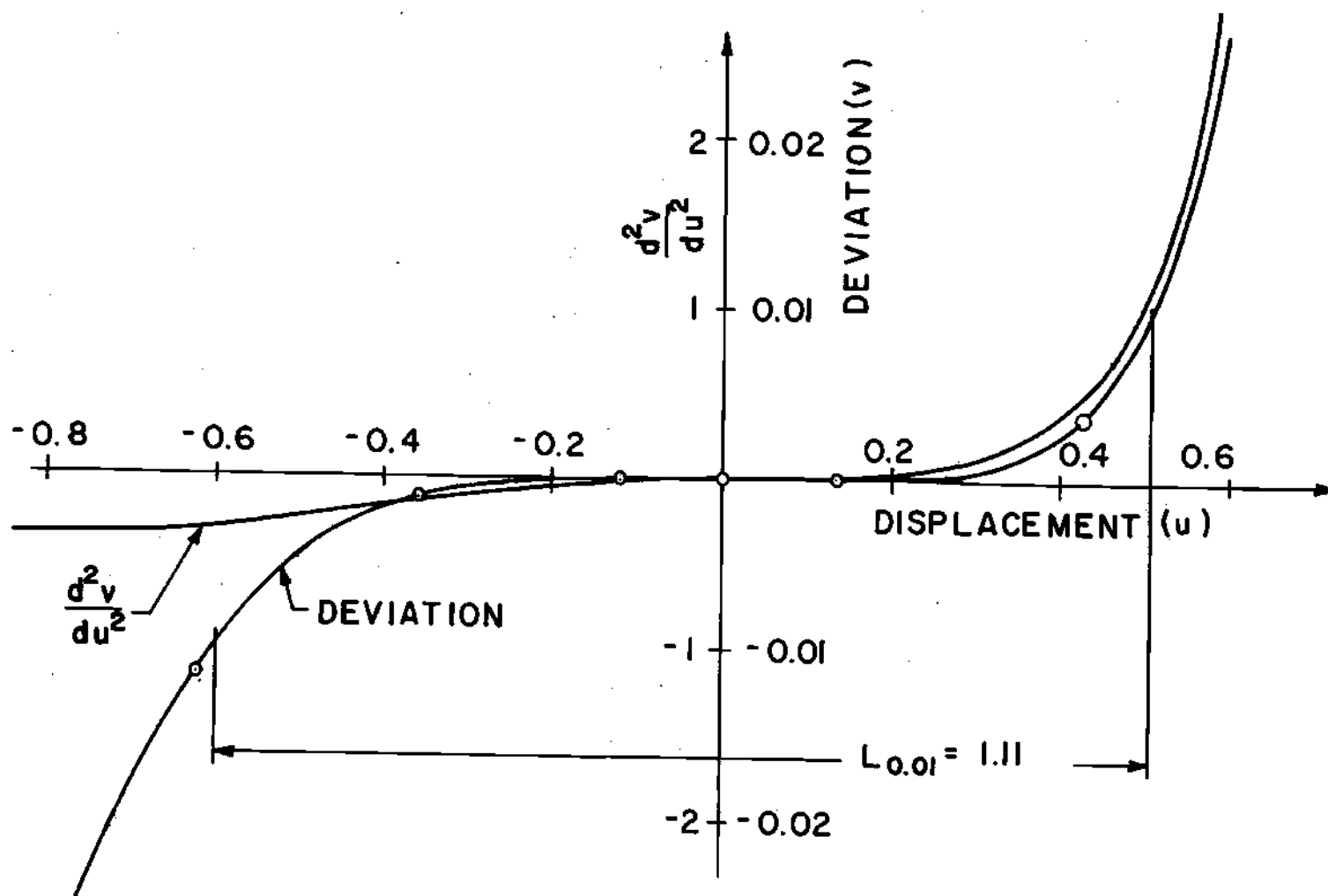


Figure 40. Deviation-Second Derivative Curves for Alternate 1,2 of General Ball-Burmester Point.

PARAMETERS (2,3)

$$\alpha_a = 50.0 \quad \alpha_b = 20.0 \quad \alpha_d = 80.0$$

DIMENSIONS ($\alpha_c = 96.343$)

$$Q = 0.52678 \quad S = 1.46780$$

$$R = 0.93190 \quad M = 0.60321$$

$$T = 0.82065 \quad \epsilon = -128.465$$

RESULTS ($\phi_0 = 53.657$)

$$D \quad 0.0001 \quad 0.01 \quad 0.05$$

$$L \quad 0.355 \quad 0.810 \quad 1.061$$

$$\Delta\psi \quad 47 \quad 103 \quad 117$$

$$Y_b \quad 12.1 \quad 3.7 \quad 3.7$$

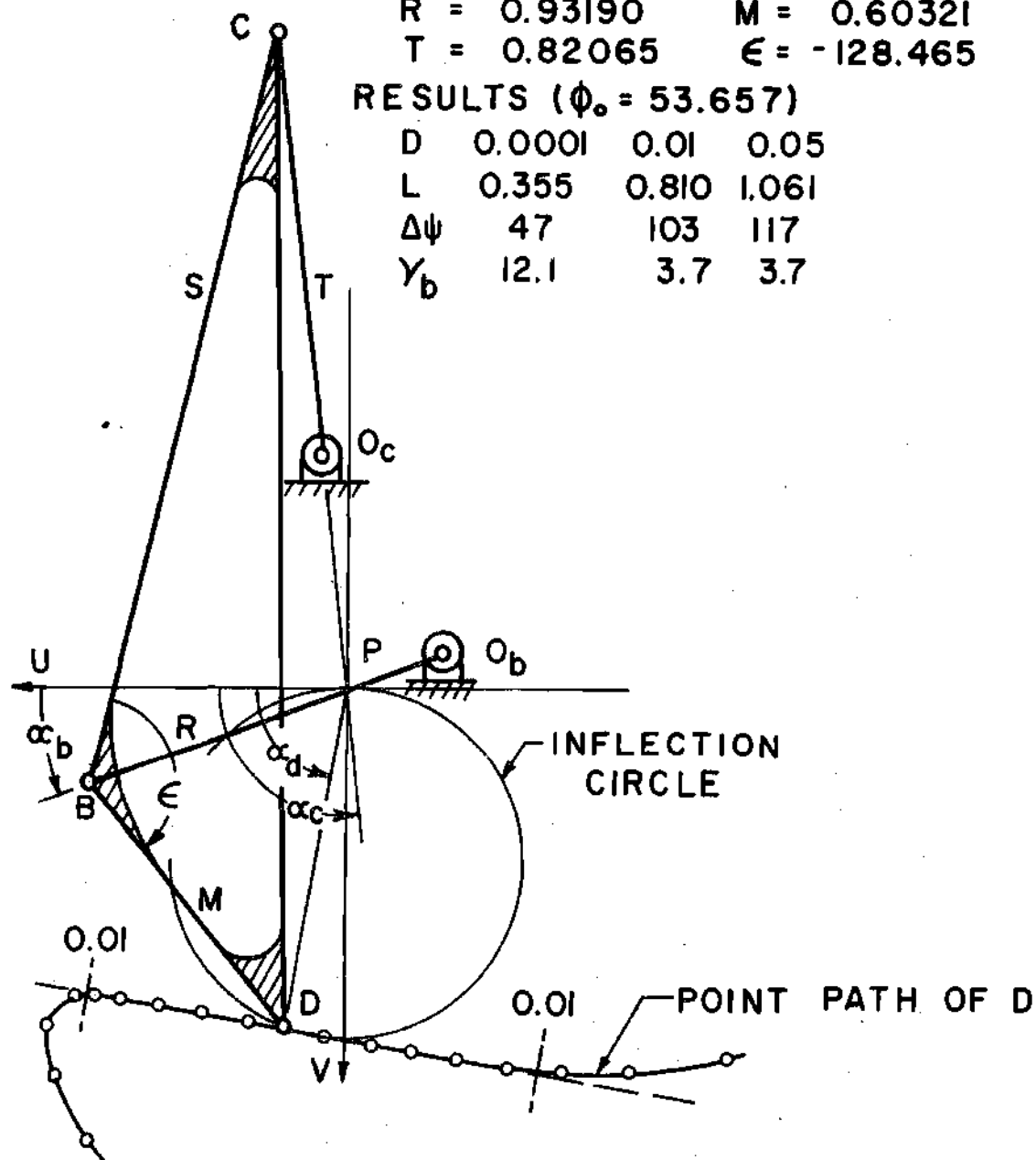


Figure 41: Alternate 2,3 for General Ball-Burmester Point.

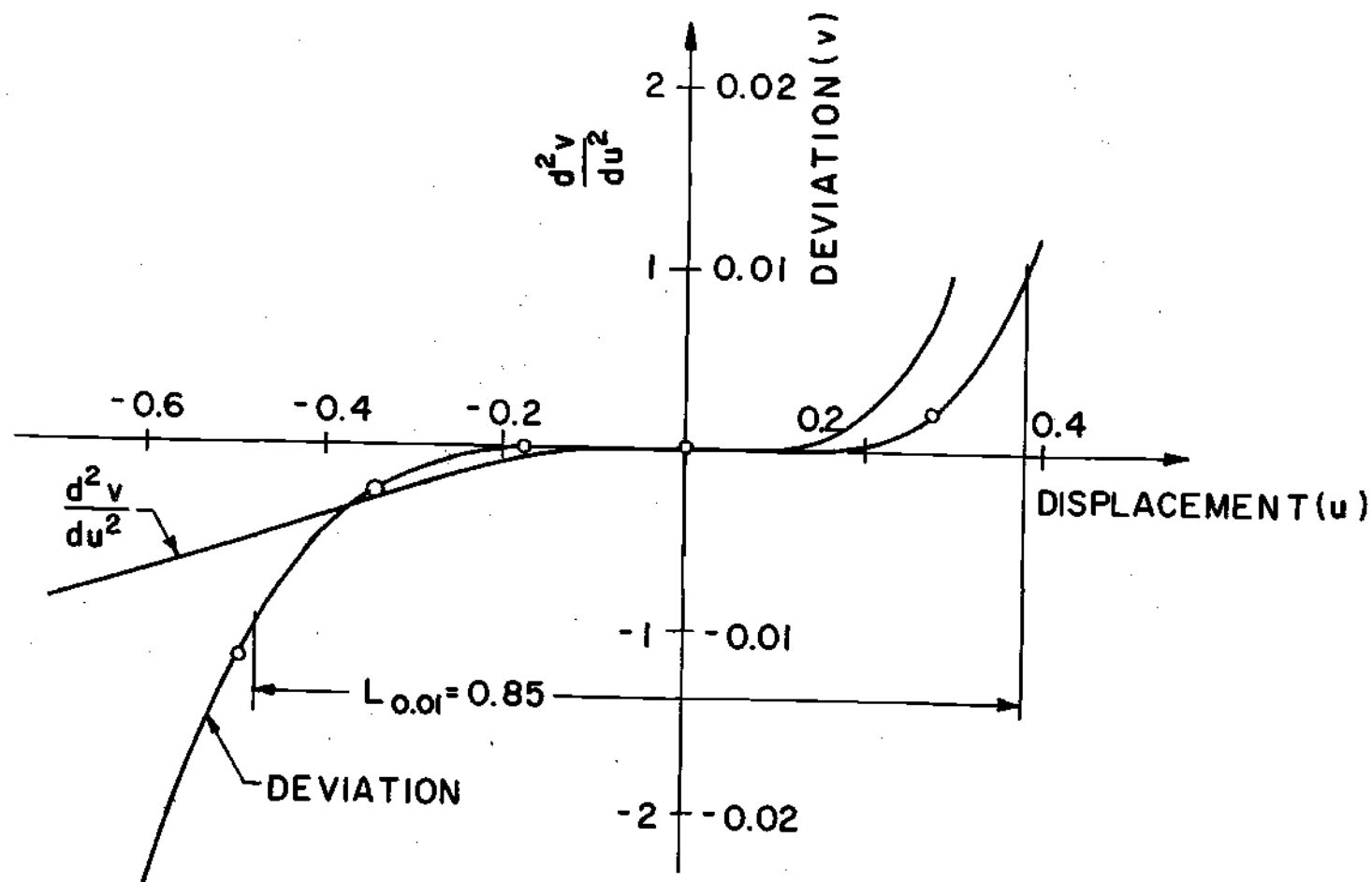


Figure 42. Deviation-Second Derivative Curves for Alternate 2,3 of General Ball-Burmester Point.

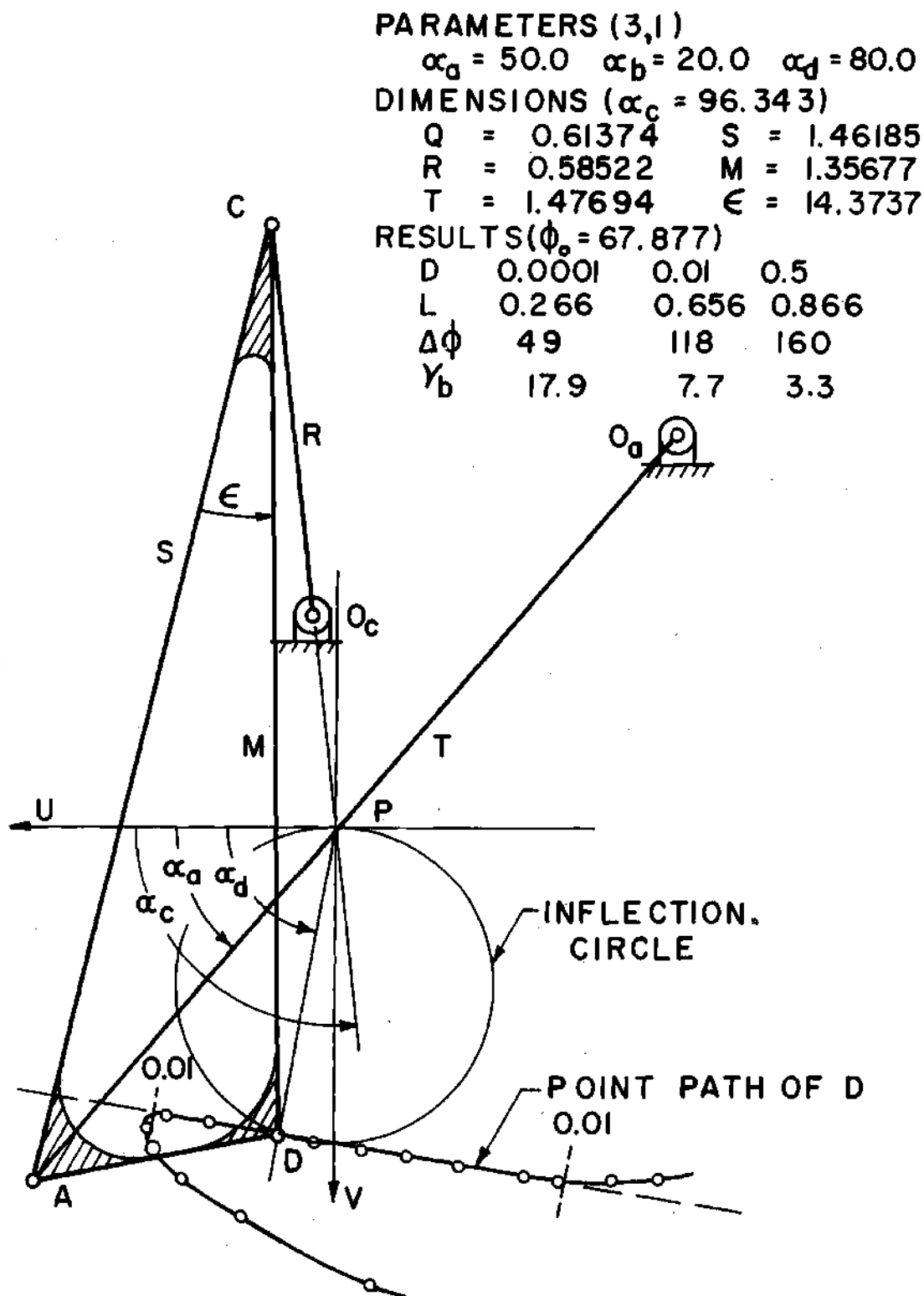


Figure 43. Alternate 3,1 for General Ball-Burmester Point.

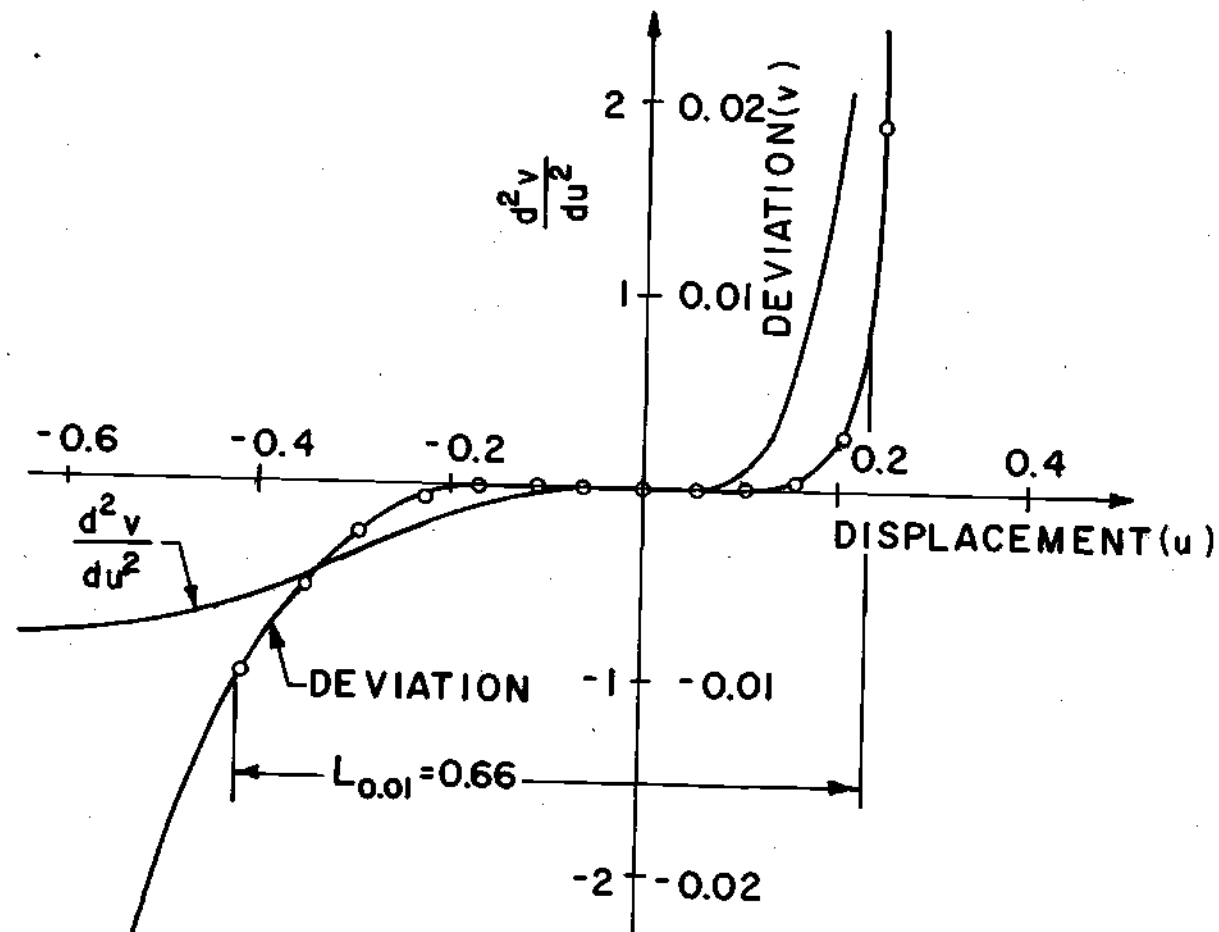


Figure 44. Deviation-Second Derivative Curves for Alternate 3,1 of General Ball-Burmester Point.

the Ball point D on the inflection circle (Eq. 2.22-2.25 of Chapter II). The design parameters are a_a , a_b , and a_d . The dimensions of each linkage are given in the figure based on the value of unity determined by Eq. (6.27). The results of the digital computation that may be used by the designer are also listed. The approximating line of the motion of point D is drawn with the limiting positions where the coupler curve has a deviation of 0.01. In Figure (40), the deviation¹ and second derivative of the deviation of the coupler curve from exact straight line motion is illustrated for alternate 1,2. As has been indicated earlier, the deviation is extremely small near the design position. The fact that the second derivative curve is also very close to zero in the design position indicates a very intimate contact between the coupler curve and its tangent. These comments apply to all designs based on curvature theory and generally hold more fully for the higher orders of contact. For fourth order contact, $d^4v/du^4 = 0$, thus indicating that the second derivative curve d^2v/du^2 has an inflection in the design position. Since all derivatives up to the fourth are zero, the inflection of the second derivative curve is parallel to the u axis. Similar diagrams for alternates 2,3 and 3,1 are represented in Fig. (41-44). Note the remarkable similarity of the second derivative curves for these alternates. All three deviation curves have similar shapes and similar second derivative curves, which is indicative of the equivalence of these alternate linkages.

¹The circles along the deviation curve represent 10° increments of rotation of the input crank O_aA .

Fig. (45-50) are the corresponding representations of three alternate four-bar linkages of a slider-crank mechanism having a Ball-Burmester point coincident with the inflection pole (Eq. 2.26-2.29 of Chapter II). Remarks similar to those for the case of the Ball-Burmester point can also be made.

Fig. (51-56) represent three linkages designed by Burmester.¹ These linkages are alternates of a slider-crank mechanism in the sense that five finitely separated positions of the coupler plane (hence, also output point D) of these linkages are the same as those of the original mechanism. The alternate linkages produce remarkably high quality results. Note that there is no restriction on the deviation curve between the five specified positions. (The curves are not completely accurate in this case since the dimensions were obtained by direct measurement from the figures given by Burmester.) Nonetheless, the deviation curves are quite long and accurate and the second derivative curves are also close to zero over an extended range of the motion. Burmester obtained these linkages by means of an involved graphical procedure with which accuracy is very difficult to maintain. Generally, the deviation between the specified "precision" points can be expected to be considerably poorer.²

An important fact can be derived from Burmester's results. Finite position theory provides no control on the motion between the specified positions. This theory is presented in analytical form in Chapter III

¹Burmester, Fig. (665-670).

²The author wishes to express the admiration he holds for these results. Burmester must have indeed worked long and hard to find such high quality linkages out of the multitude that is possible.

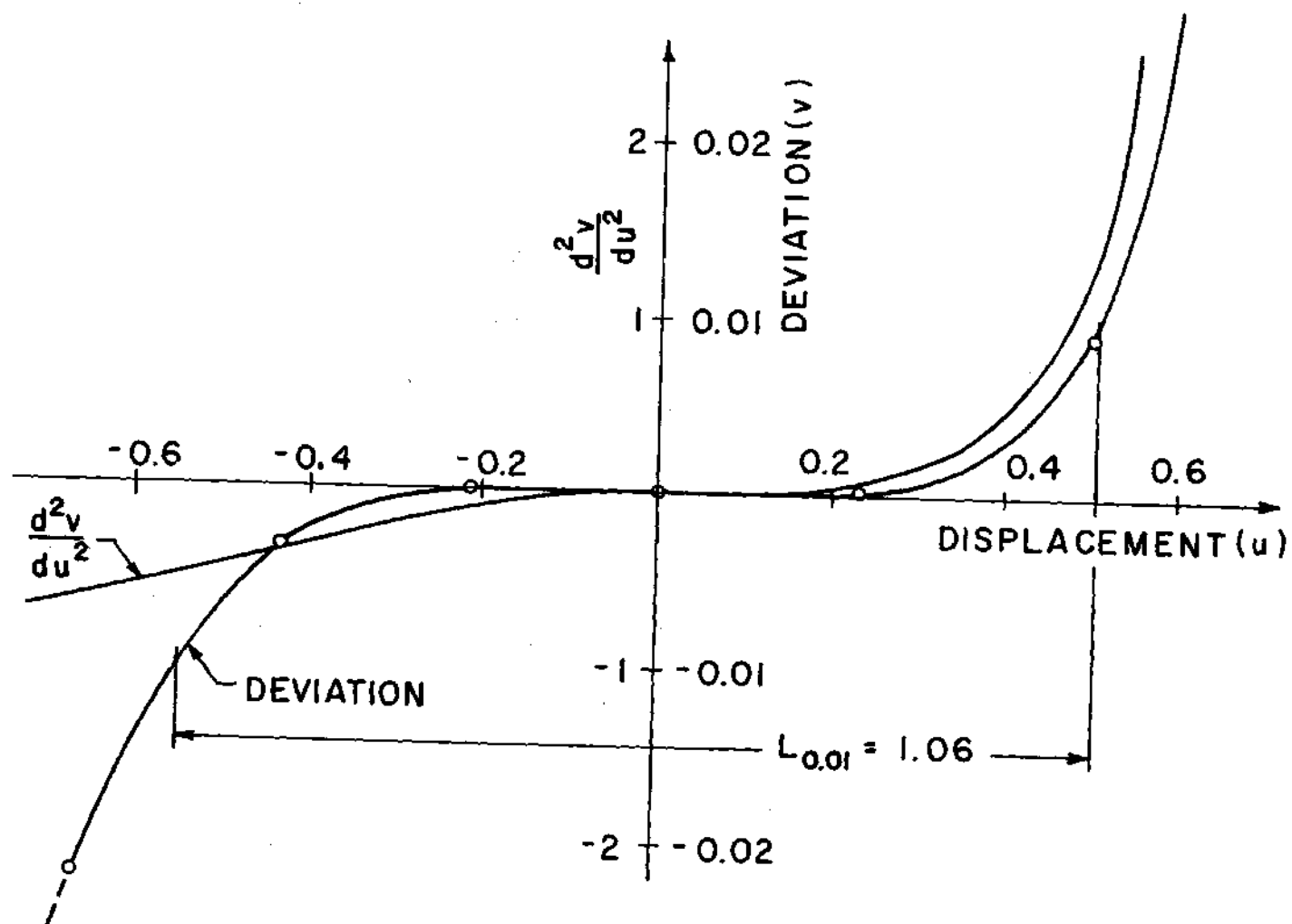


Figure 46. Deviation-Second Derivative Curves for Alternate 1,2 for Ball-Burmester Point at Inflection Pole.

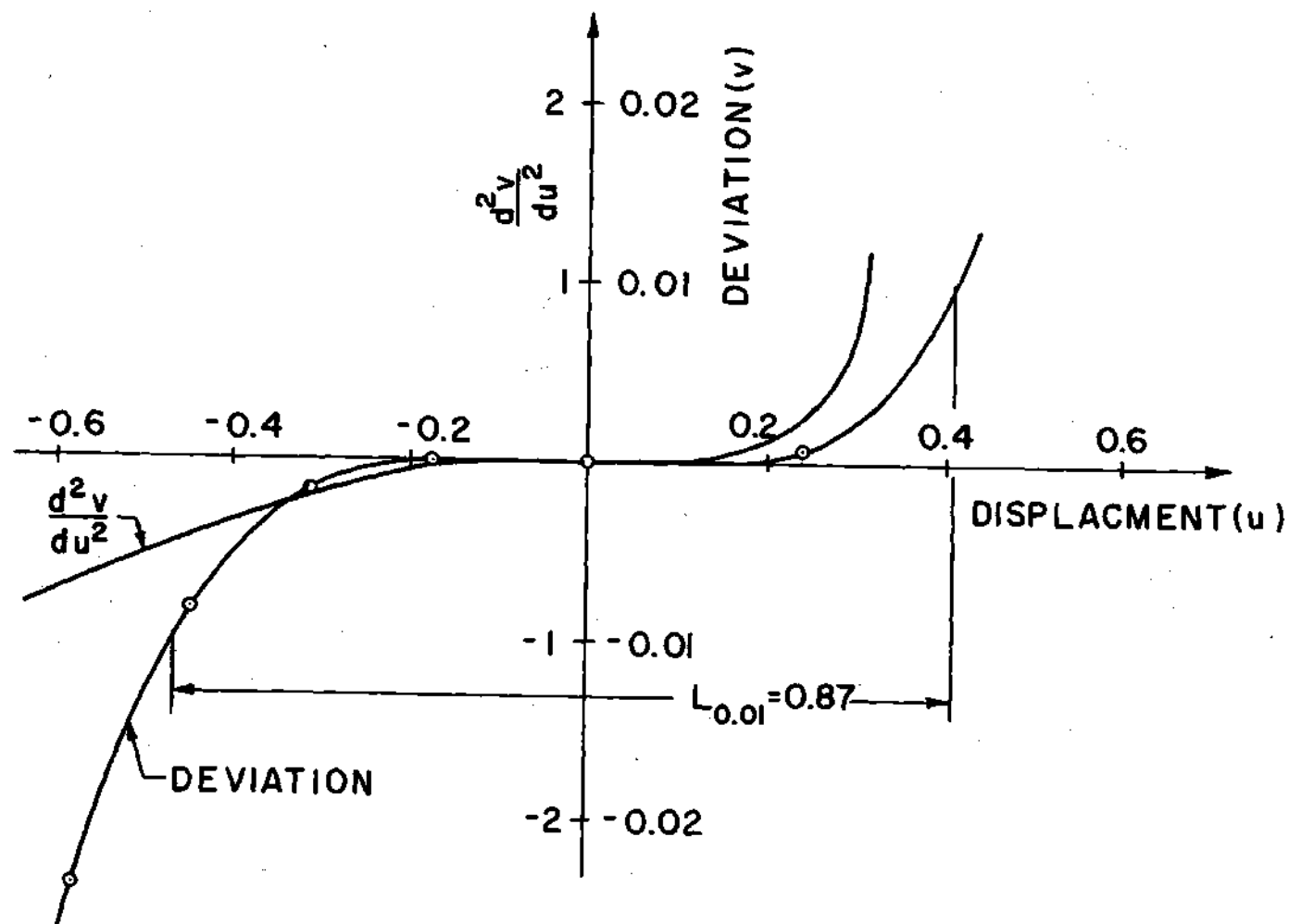


Figure 48. Deviation-Second Derivative Curves for Alternate 2,3 for Ball-Burmester Point at Inflection Pole.

PARAMETERS (3,1)

$$k = -2.50 \quad \alpha_a = 75.0$$

DIMENSIONS ($\alpha_b = 38.794$)

$$Q = 0.40645 \quad S = 0.45964$$

$$R = 1.85443 \quad M = 0.66759$$

$$T = 1.79124 \quad \epsilon = -23.794$$

RESULTS ($\phi_o = 223.186$)

$$D \quad 0.0001 \quad 0.01 \quad 0.05$$

$$L \quad 0.353 \quad 0.860 \quad 1.174$$

$$\Delta\phi \quad 27 \quad 63 \quad 82$$

$$Y_a \quad 15.0 \quad 4.4 \quad 1.0$$

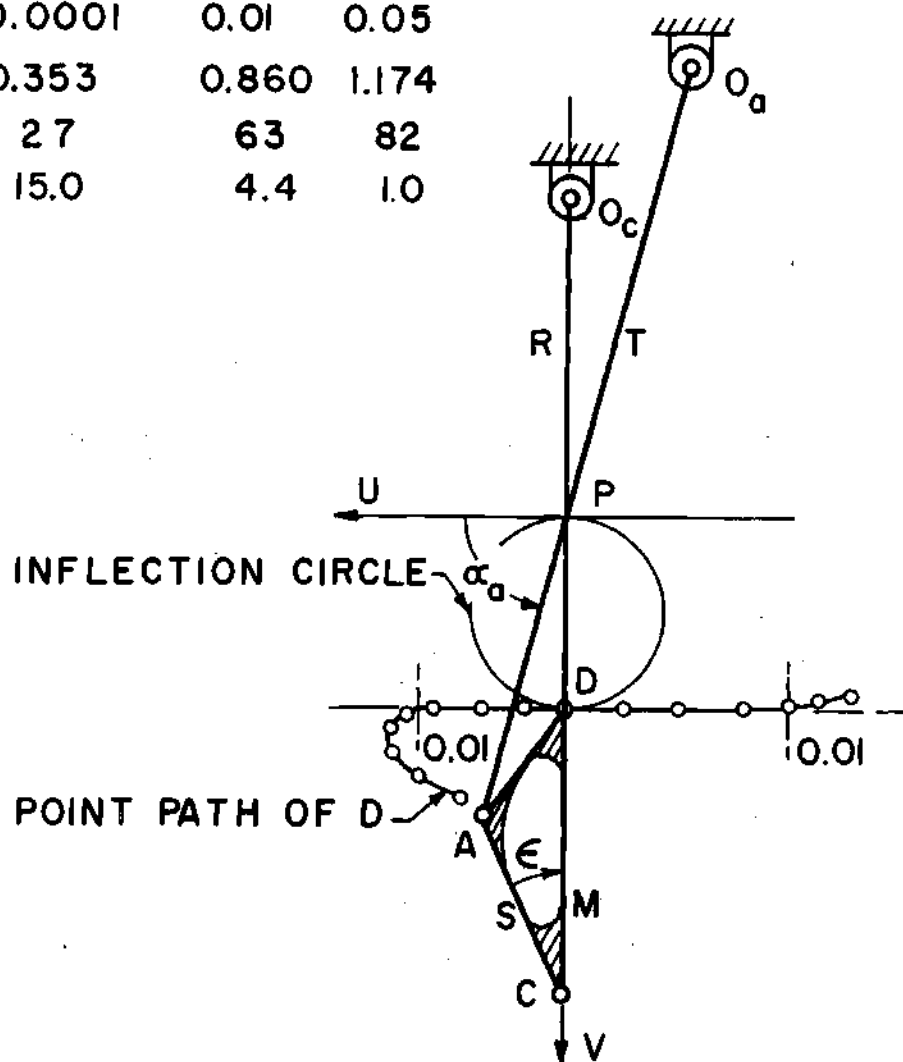


Figure 49. Alternate 3,1 for Ball-Burmester Point at Inflection Pole.

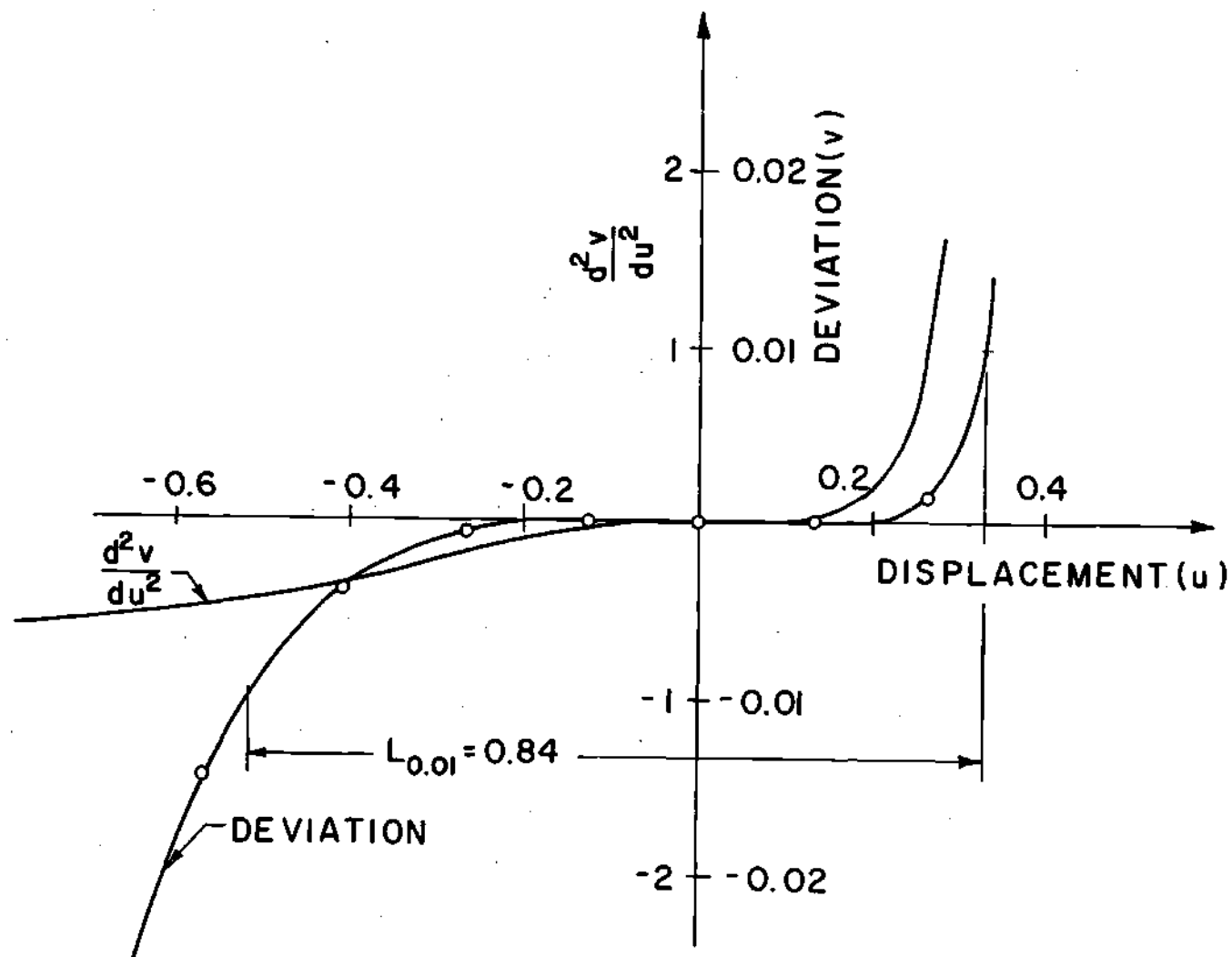


Figure 50. Deviation-Second Derivative Curves for Alternate 3,1 for Ball-Burmester Point at Inflection Pole.

DIMENSIONS (A,B)

$$Q = 1.815 \quad S = 0.764$$

$$R = 0.963 \quad M = 0.778$$

$$T = 0.528 \quad \epsilon = -89.65$$

RESULTS ($\phi_0 = 42.75$)

| | | | |
|--------------|------|------|------|
| D | 0.01 | 0.05 | 0.1 |
| L | 1.31 | 1.36 | 1.38 |
| $\Delta\phi$ | 50 | 57 | 62 |
| γ_b | 9.6 | 9.6 | 9.6 |

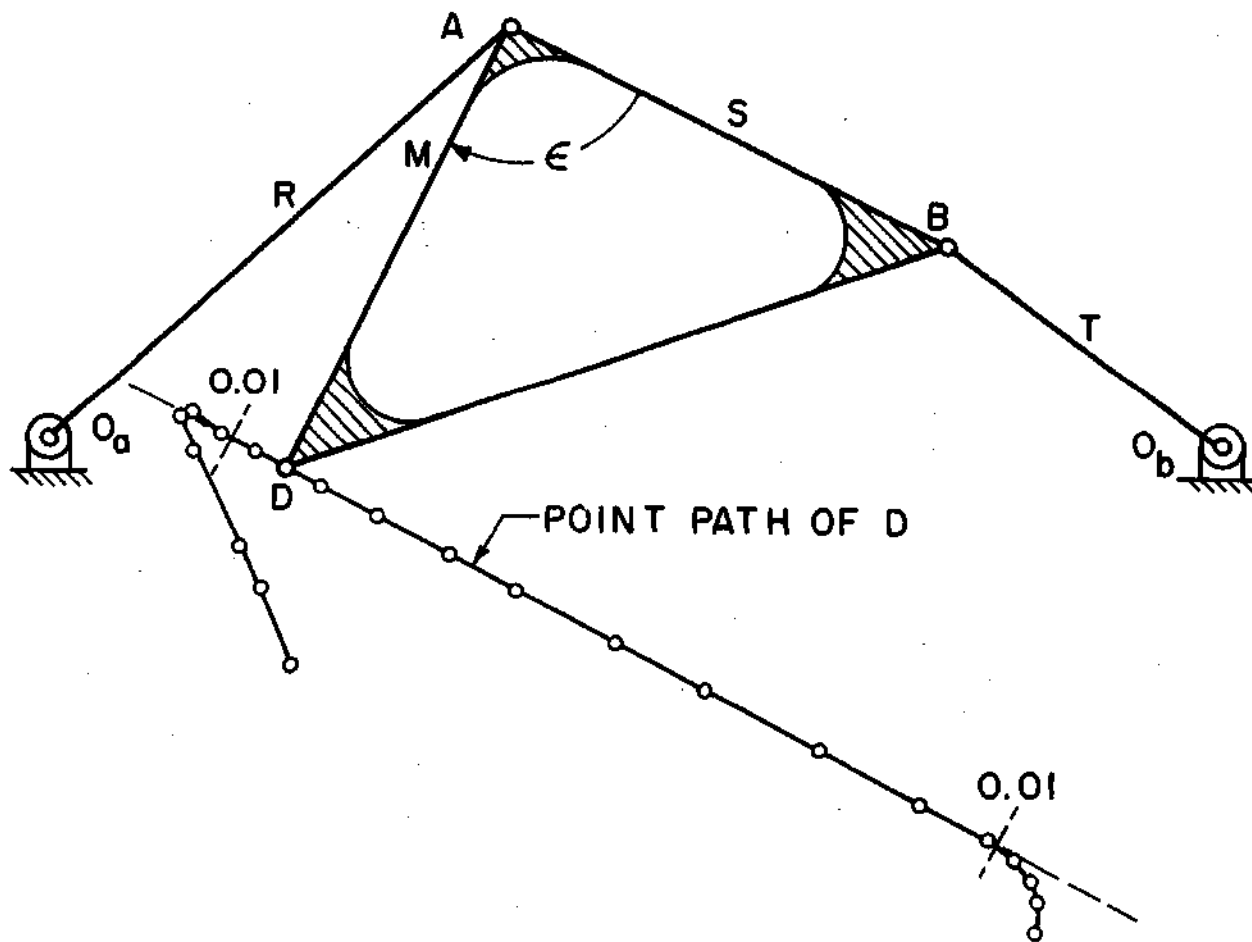


Figure 51. Linkage A,B by Burmester.

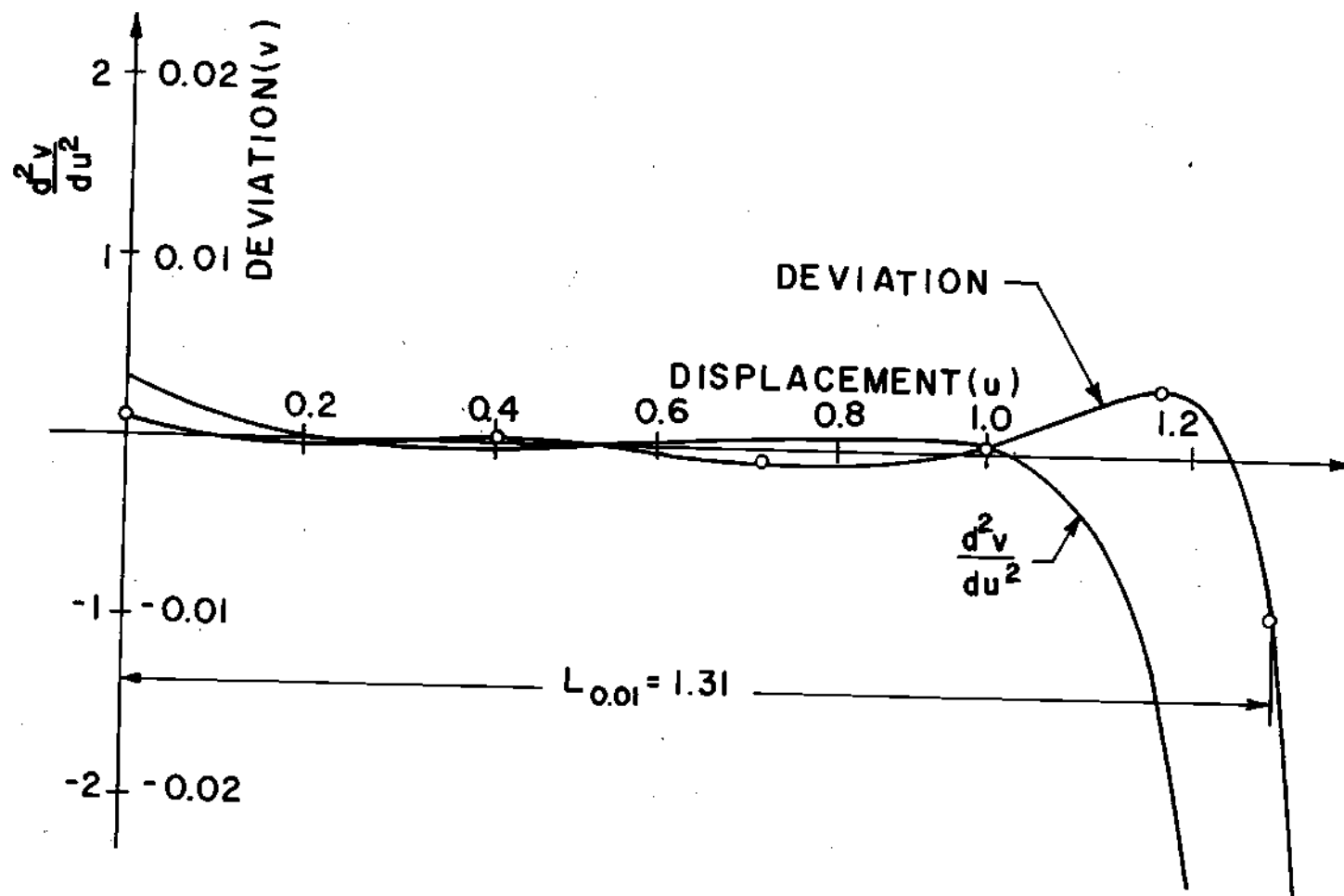


Figure 52. Deviation-Second Derivative Curves for Linkage A,B by Burmester.

DIMENSIONS (B,C)

$$Q = 1.316 \quad S = 0.624$$

$$R = 0.290 \quad M = 1.627$$

$$T = 0.963 \quad \epsilon = -116.90$$

RESULTS ($\phi_0 = 80.20$)

$$D \quad 0.01 \quad 0.05 \quad 0.1$$

$$L \quad 1.70 \quad 2.28 \quad 2.42$$

$$\Delta\phi \quad 60 \quad 146 \quad 175$$

$$\gamma_b \quad 58.1 \quad 4.5 \quad 4.5$$

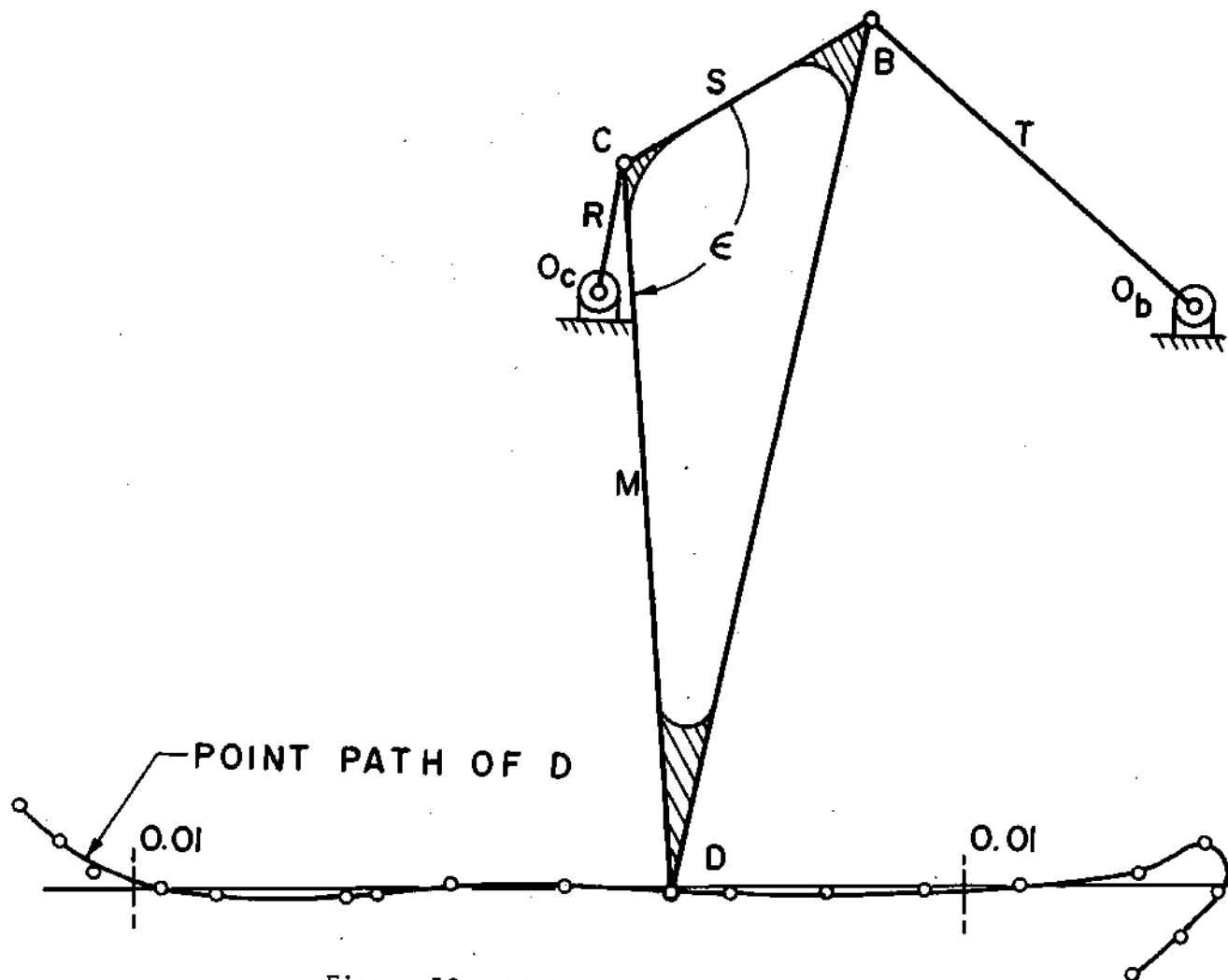


Figure 53. Linkage B,C by Burmester.

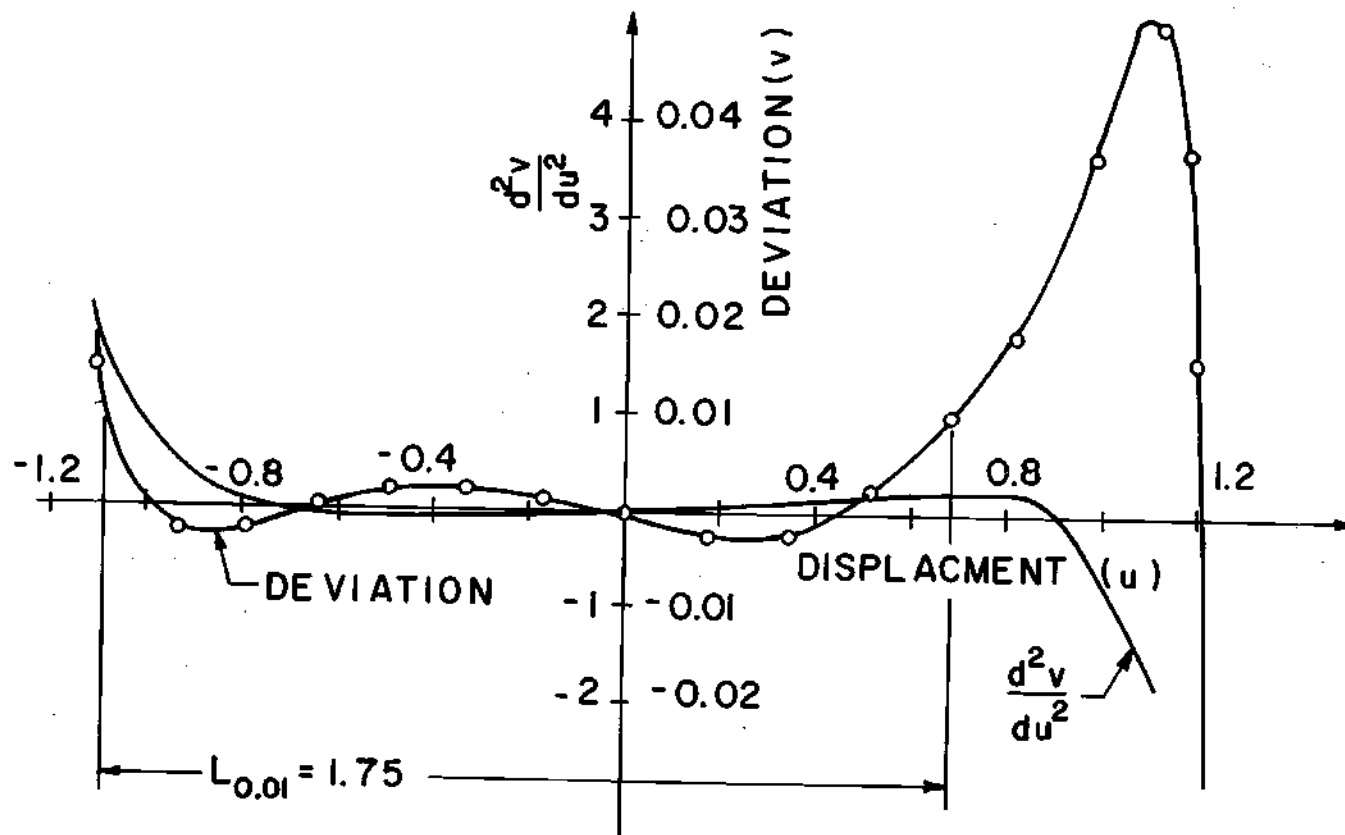


Figure 54. Deviation-Second Derivative Curves for Linkage B,C by Burmester.

DIMENSIONS (A,C)

| | |
|-------------|---------------------|
| $Q = 1.710$ | $S = 0.587$ |
| $R = 0.216$ | $M = 1.225$ |
| $T = 1.339$ | $\epsilon = -60.90$ |

RESULTS ($\phi_0 = 67.70$)

| | | | |
|--------------|------|------|------|
| D | 0.01 | 0.05 | 0.1 |
| L | 1.38 | 1.52 | 1.53 |
| $\Delta\phi$ | 127 | 169 | 170 |
| γ_a | 1.5 | 1.5 | 1.5 |

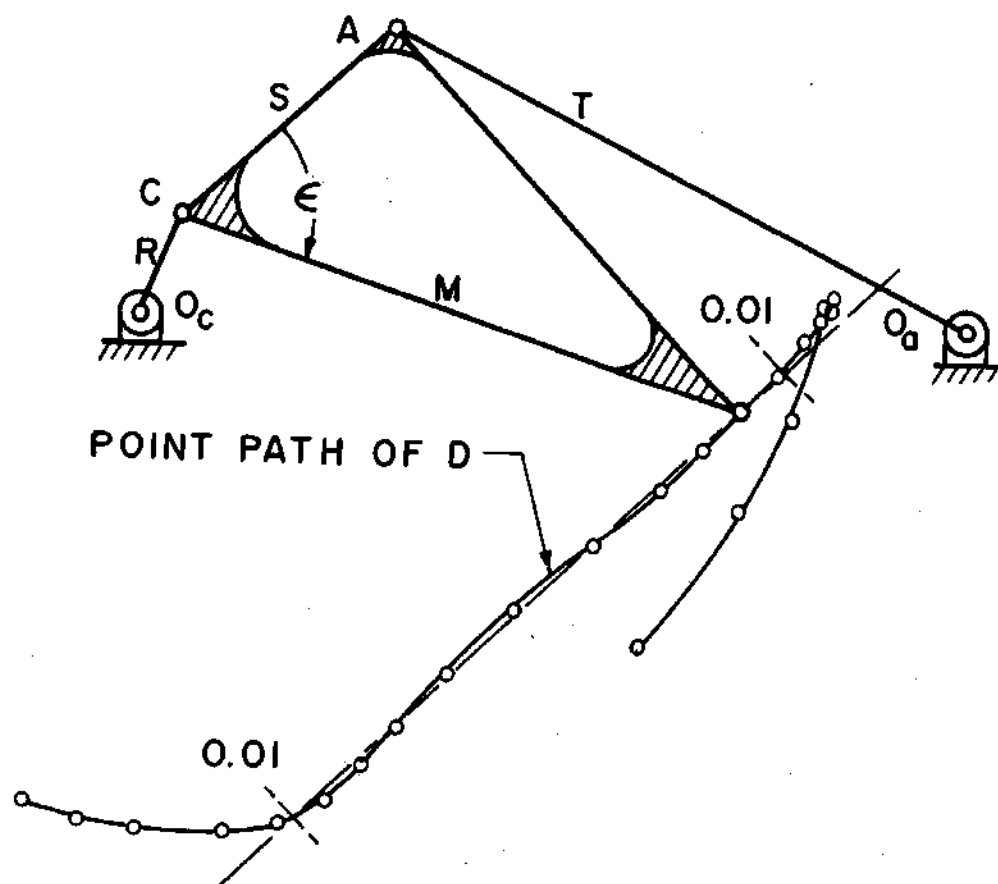


Figure 55. Linkage A,C by Burmester.

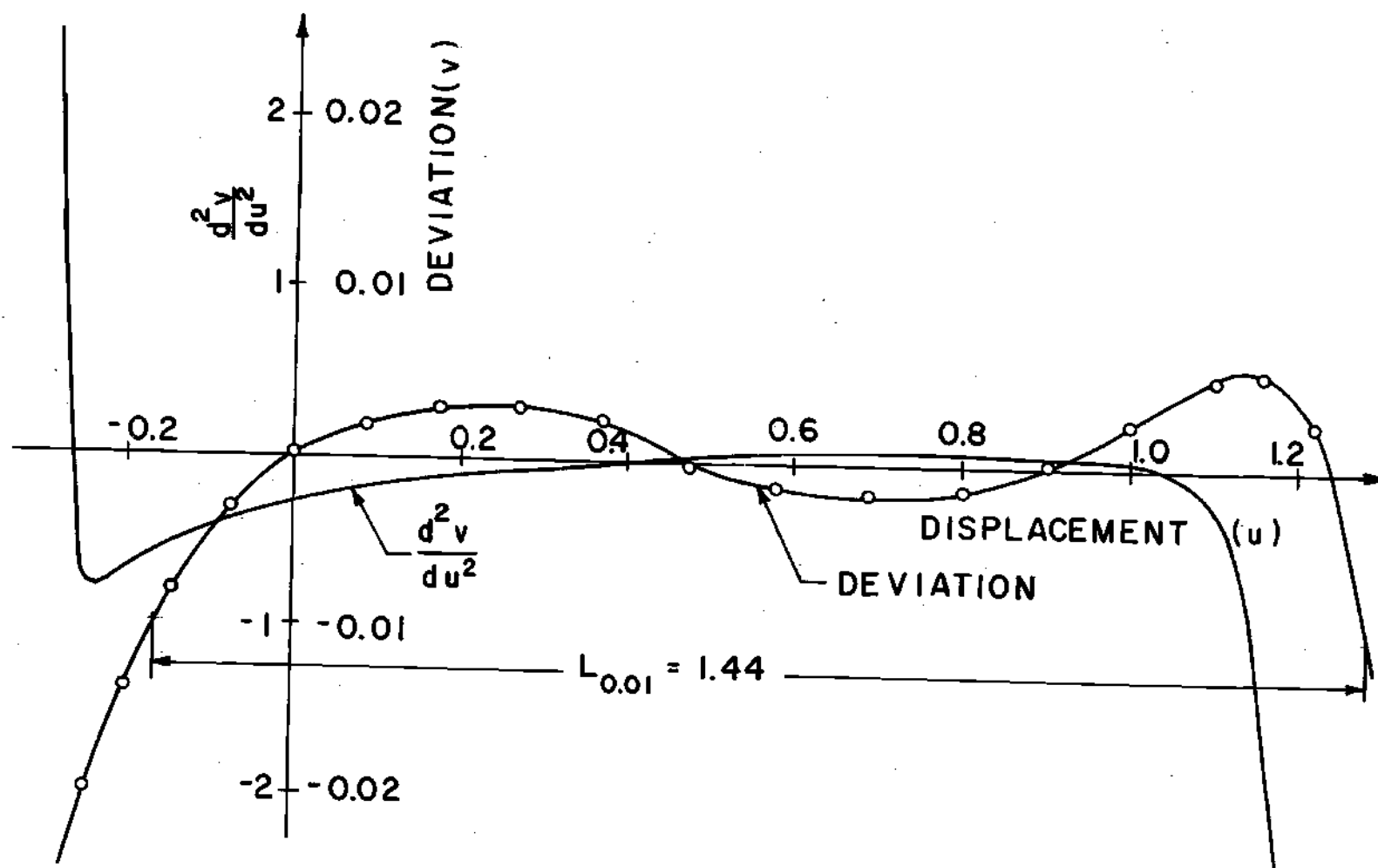


Figure 56. Deviation-Second Derivative Curves for Linkage C,A by Burmester.

of this dissertation. Since it is likely that far too many linkages (∞^{10}) are possible, further restrictions on the higher derivatives (suppose that the first derivative of the deviation curve be controlled in the Chebychev sense) might be considered. In other words, it will be necessary to interrelate the a_j, ϕ_j parameters of the motion. The magnitude of the problem can be reduced to ∞^5 by choosing a slider-crank mechanism in five finitely specified positions (ϕ_j is the a function of a_j). During the process of the motion, the two unknown Burmester points will trace point paths which approximate circular arcs. The quality of this approximation will depend on the variation of the deviation and its higher derivatives. By placing restrictions on this approximation, the digital computer can be used to select those alternate linkages giving good results.

Two Watt linkages are illustrated in Fig. (57-60). Since the output point D is actually a Burmester point (five points infinitesimally separated), the deviation and second derivative curves have a character similar to those of linkages based on curvature theory.

A symmetrical linkage of the Chebychev type is represented in Fig. (61,62). A Roberts type linkage is represented in Fig. (63,64). These linkages have inferior but extended straight line outputs. Linkages of this group that are based on Chebychev minimum deviation theory and those based on curvature theory will probably provide the best results.

The next six figures (Fig. 65-70) represent the three distinct configurations of the centric-rectangular Evans linkage. The linkage of Fig. (65) is closely associated with the Scott-Russell exact straight

DIMENSIONS

$$\begin{array}{ll}
 Q = 1.94913 & S = 0.87168 \\
 R = 0.87168 & M = 0.43584 \\
 T = 0.87168 & \epsilon = 0.0
 \end{array}$$

RESULTS

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.900 | 1.146 | 1.192 |
| $\Delta\phi$ | 62 | 83 | 92 |
| y_b | 39.8 | 14.1 | 14.1 |

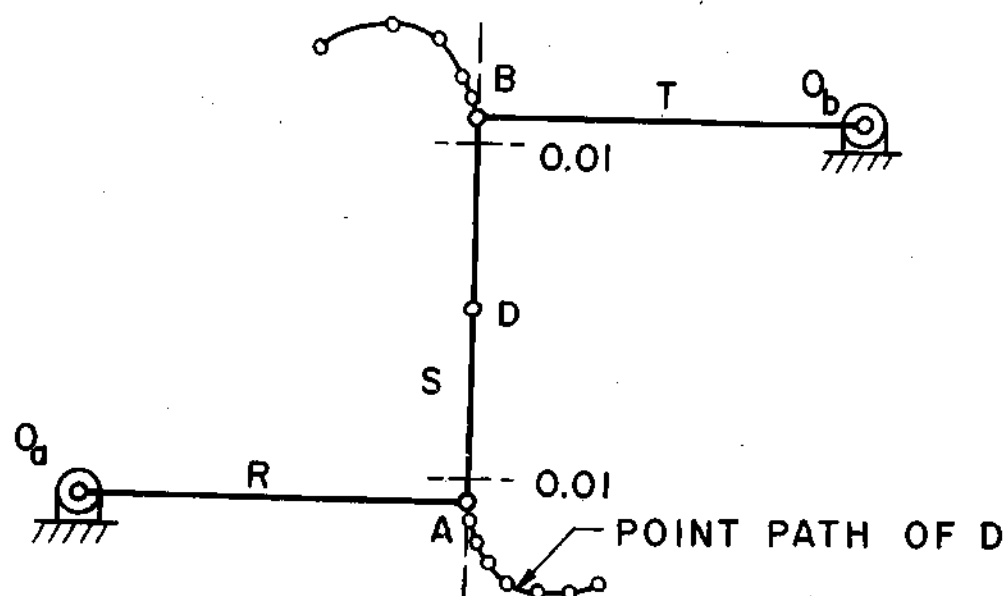


Figure 57. Watt Linkage with Equal Cranks.

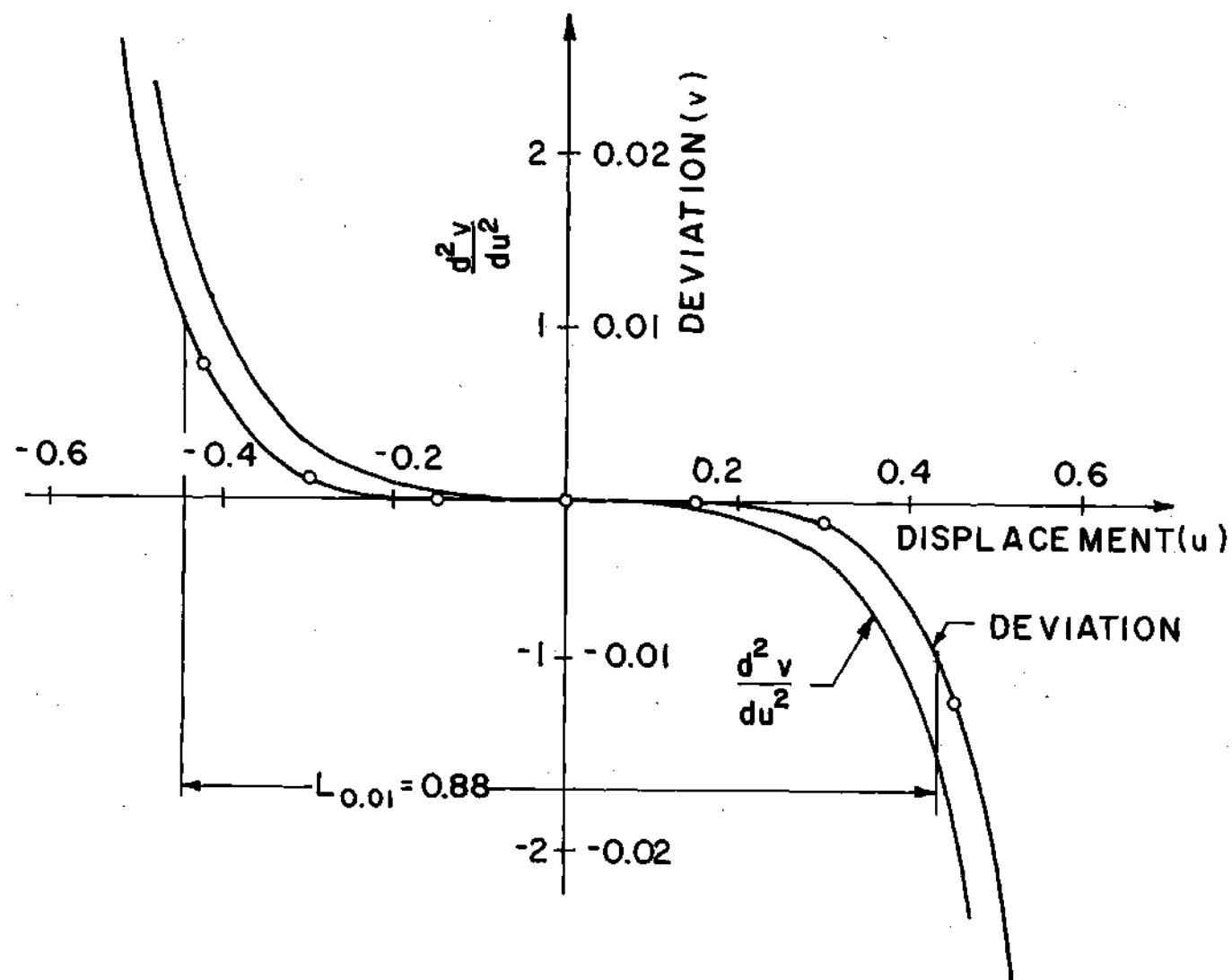


Figure 58. Deviation-Second Derivative Curves for Watt Linkage With Equal Cranks.

DIMENSIONS

| | |
|---------------|------------------|
| $Q = 1.87681$ | $S = 1.04106$ |
| $R = 0.52053$ | $M = 0.69404$ |
| $T = 1.04106$ | $\epsilon = 0.0$ |

RESULTS

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.854 | 1.054 | 1.074 |
| $\Delta\phi$ | 104 | 138 | 154 |
| γ_b | 36.1 | 11.9 | 6.4 |

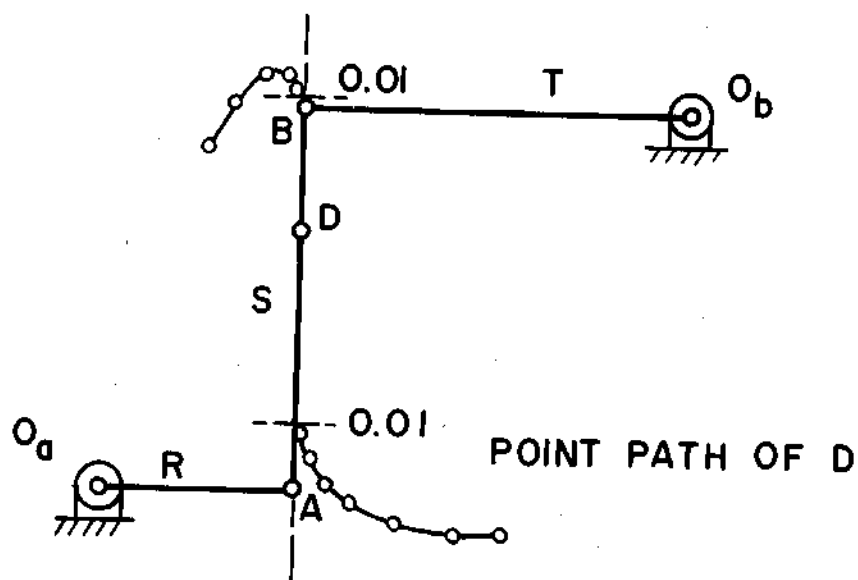


Figure 59. Watt Linkage with Unequal Cranks.

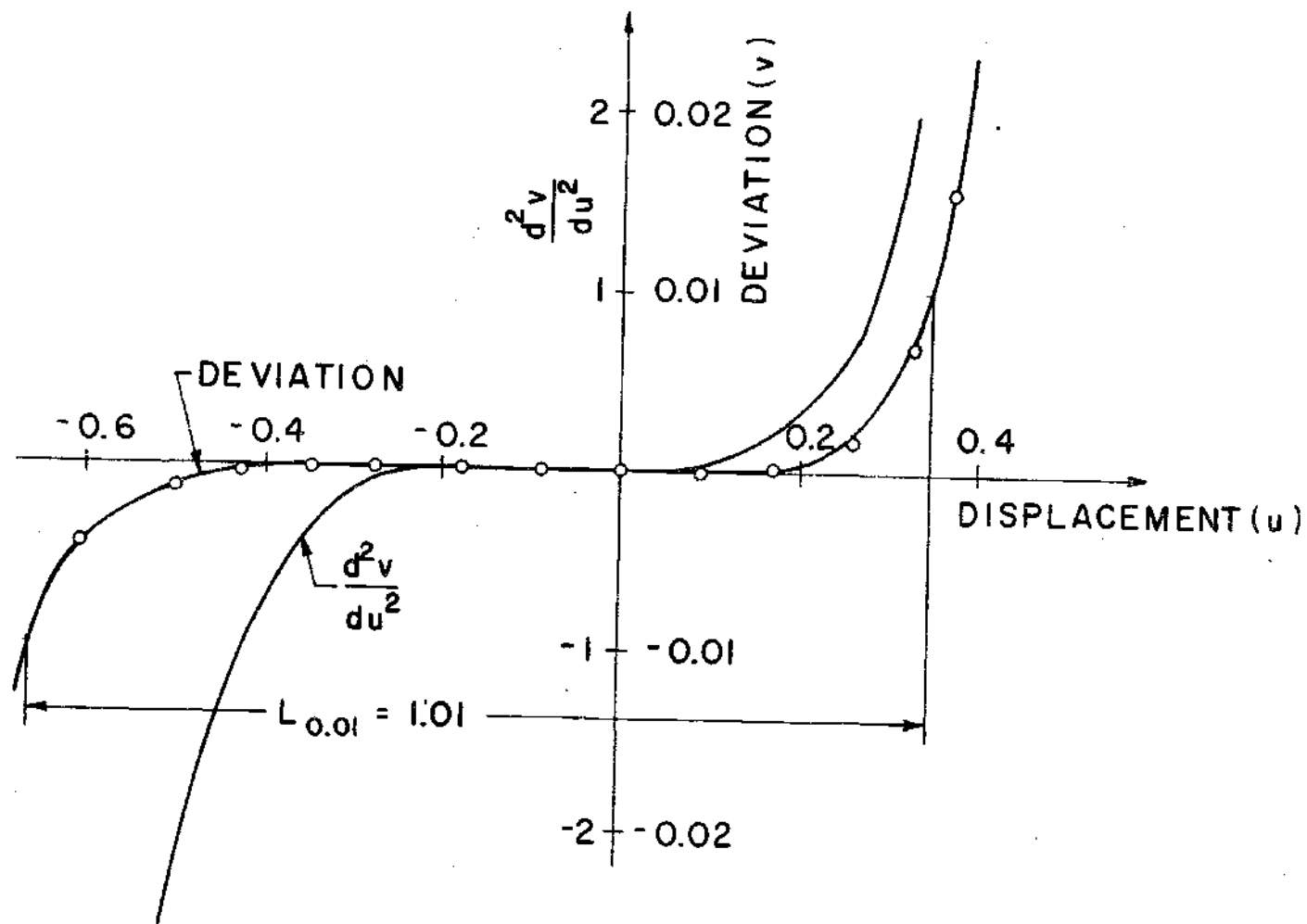


Figure 60. Deviation-Second Derivative Curves for Watt Linkage with Unequal Cranks.

DIMENSIONS

| | |
|---------------|---------------------|
| $Q = 0.91387$ | $S = 0.60925$ |
| $R = 1.56815$ | $M = 0.34058$ |
| $T = 1.56815$ | $\epsilon = 26.565$ |

RESULTS ($\phi_0 = 60.945$)

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.360 | 1.778 | 1.805 |
| $\Delta\phi$ | 21 | 81 | 83 |
| γ_b | 44.0 | 5.0 | 0.0 |

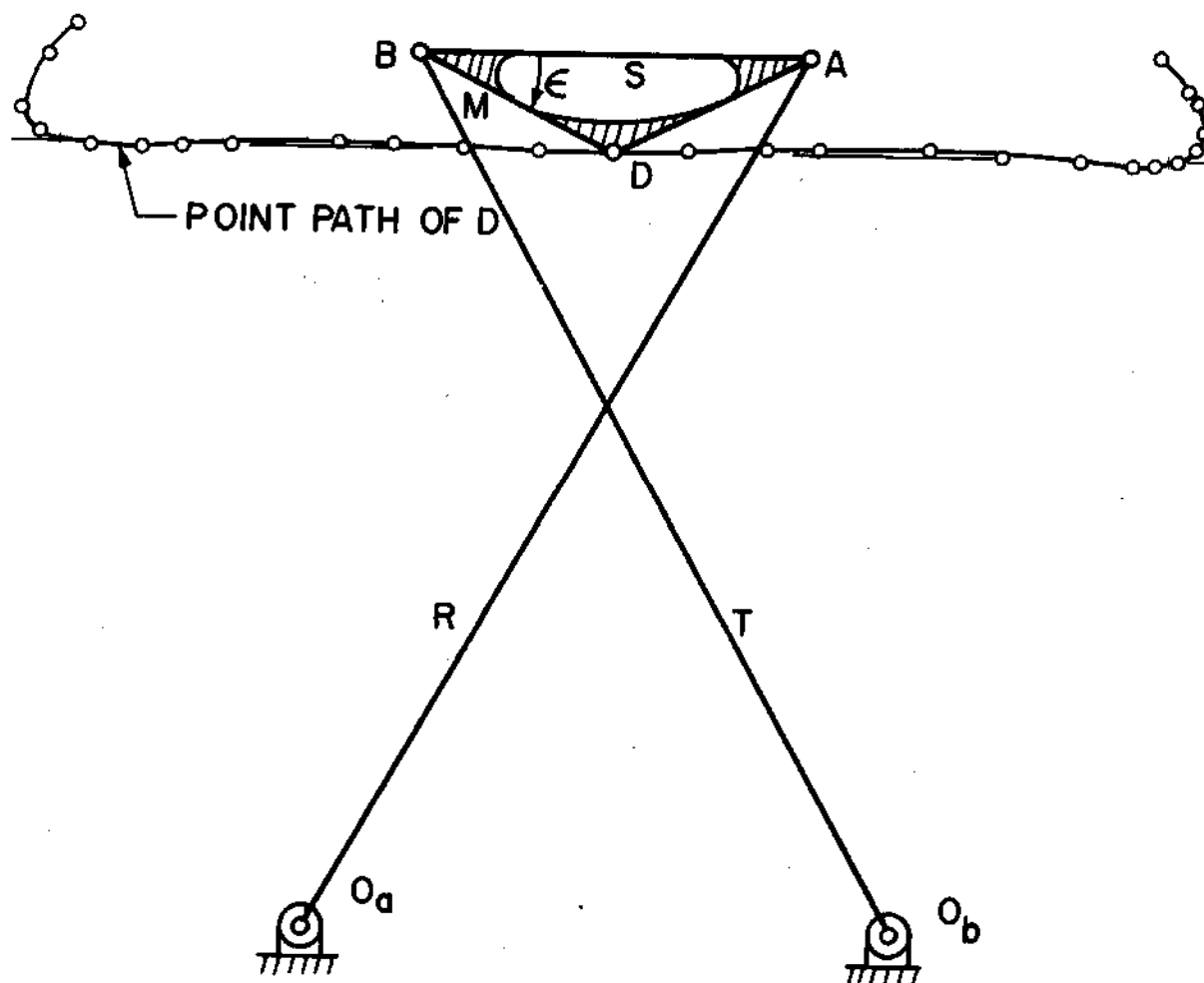


Figure 61. Symmetrical Linkage (Chebychev Type).

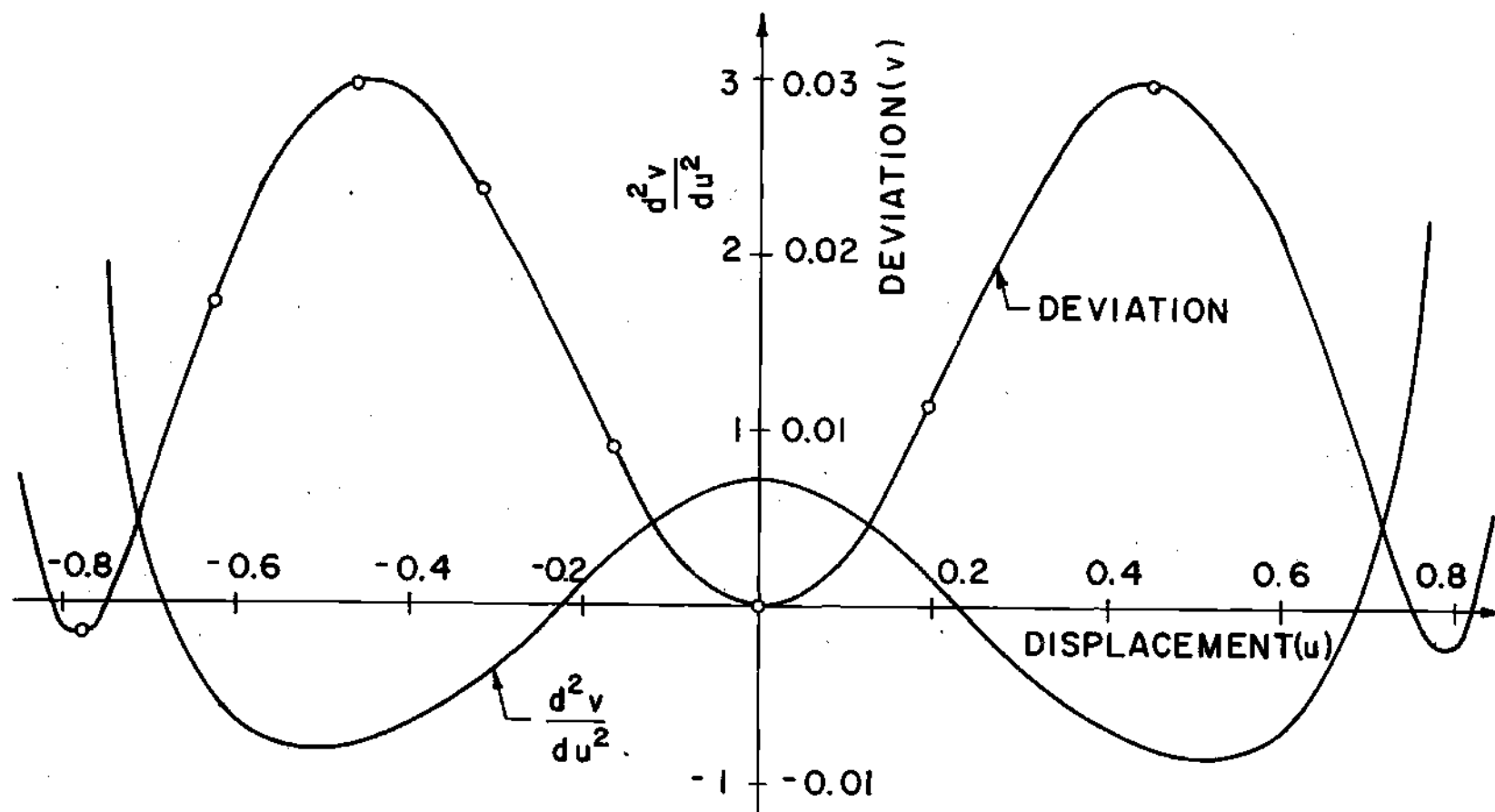


Figure 62. Deviation-Second Derivative Curves for Chebychev Type Symmetrical Linkage.

DIMENSIONS

| | |
|---------------|----------------------|
| $Q = 1.13111$ | $S = 0.75407$ |
| $R = 0.59615$ | $M = 1.92252$ |
| $T = 0.59615$ | $\epsilon = -78.690$ |

RESULTS ($\phi_0 = 71.565$)

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.607 | 1.968 | 2.123 |
| $\Delta\phi$ | 23 | 72 | 78 |
| Y_b | 57.0 | 0.0 | 12.0 |

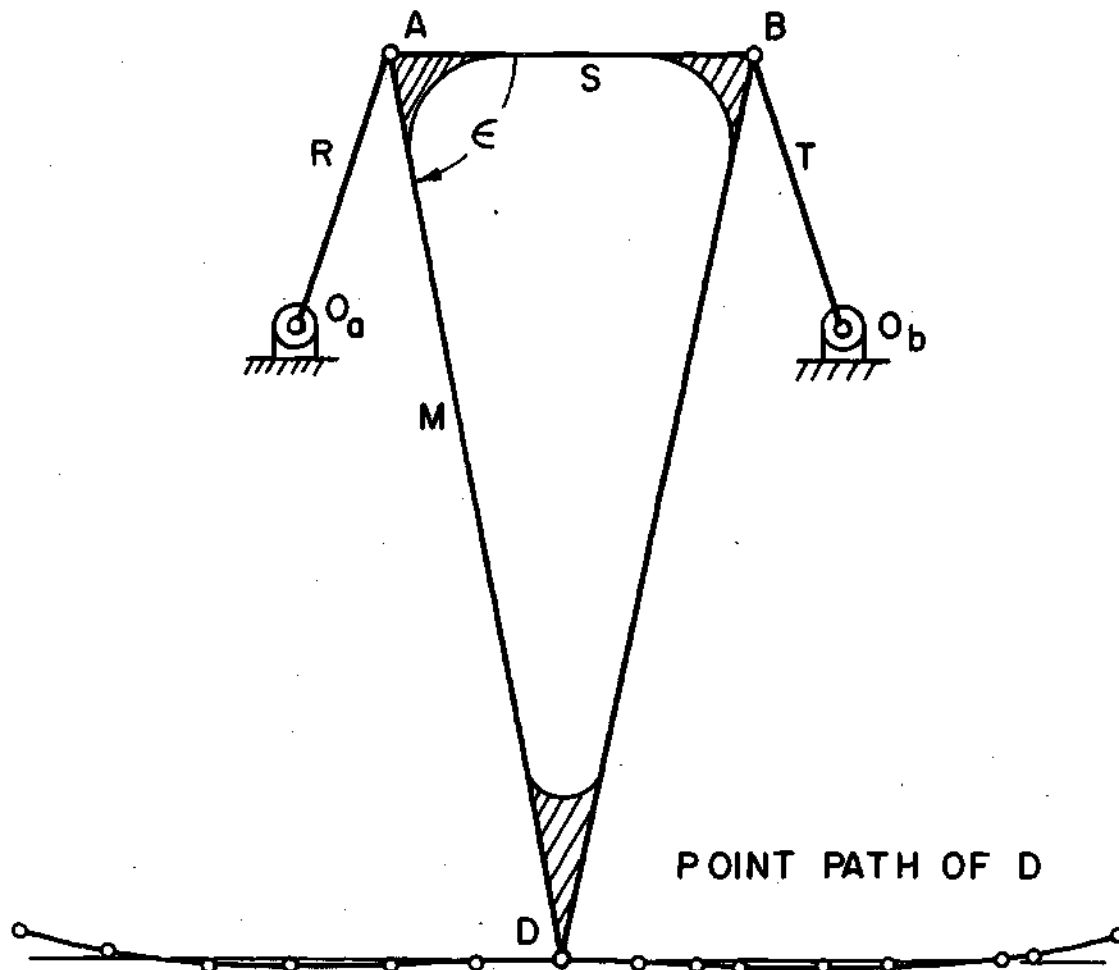


Figure 63. Symmetrical Linkage (Roberts Type).

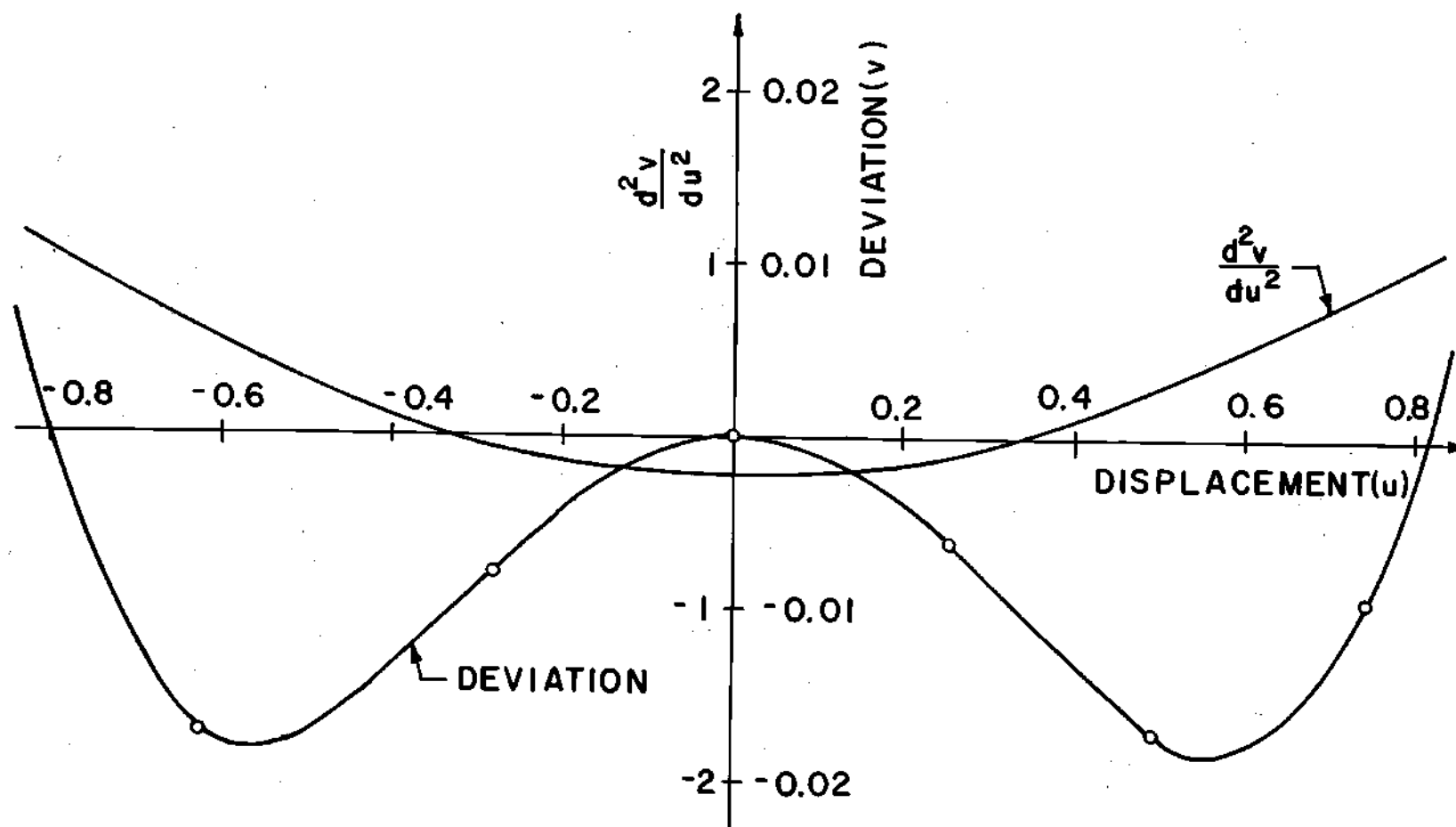


Figure 64. Deviation-Second Derivative Curves for Roberts type Symmetrical Linkage.

DIMENSIONS ($\alpha = 25.0$)

$Q = 1.53993$ $S = 0.61279$
 $R = 0.45739$ $M = 0.85791$
 $T = 1.22558$ $E = 180.0$

RESULTS ($\phi_0 = 272.777$)

| D | 0.01 | 0.05 | 0.1 |
|--------------|-------|-------|-------|
| L | 1.800 | 2.042 | 2.164 |
| $\Delta\phi$ | 108 | 129 | 142 |
| γ_b | 33.1 | 17.4 | 5.4 |

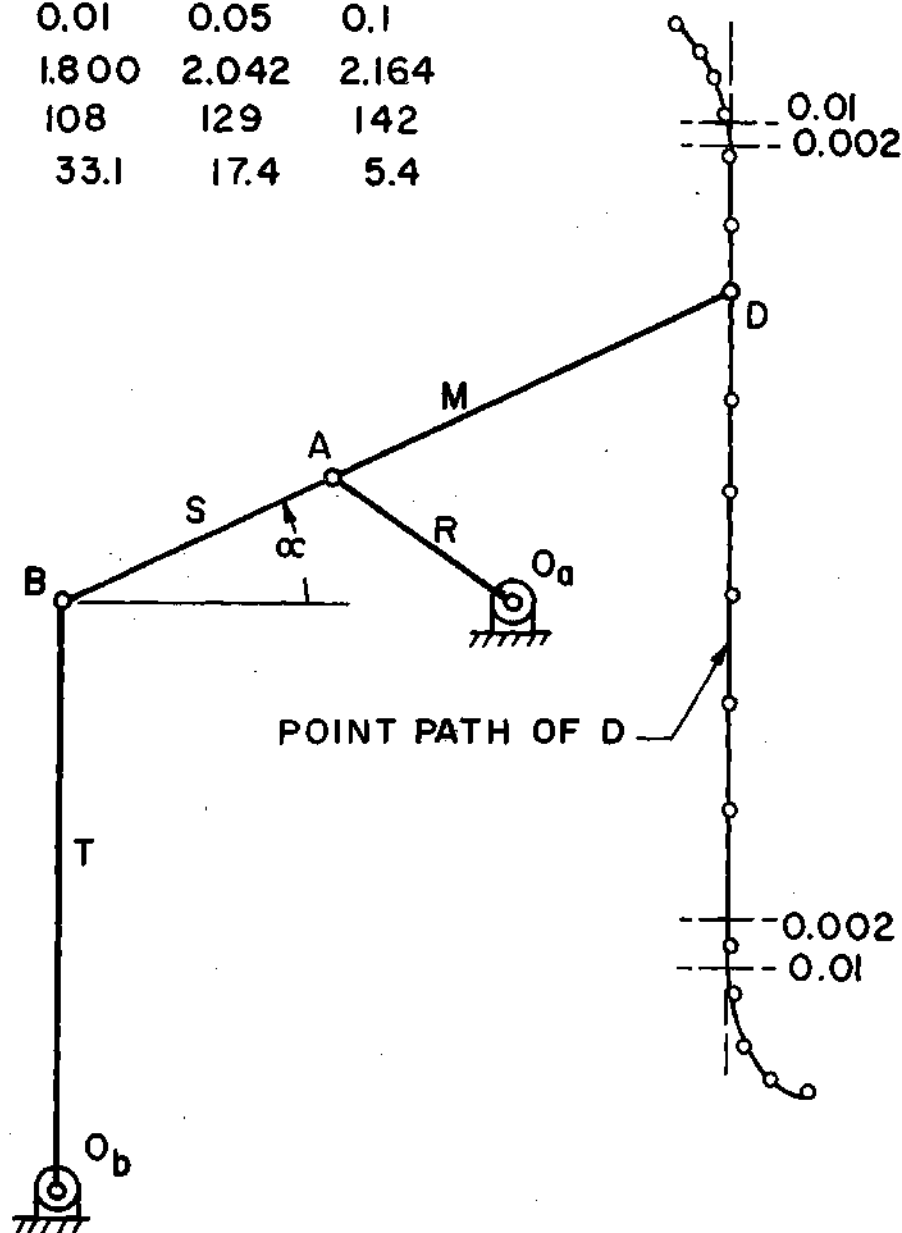


Figure 65. Centric-Rectangular Evans Linkage (Configuration 1).

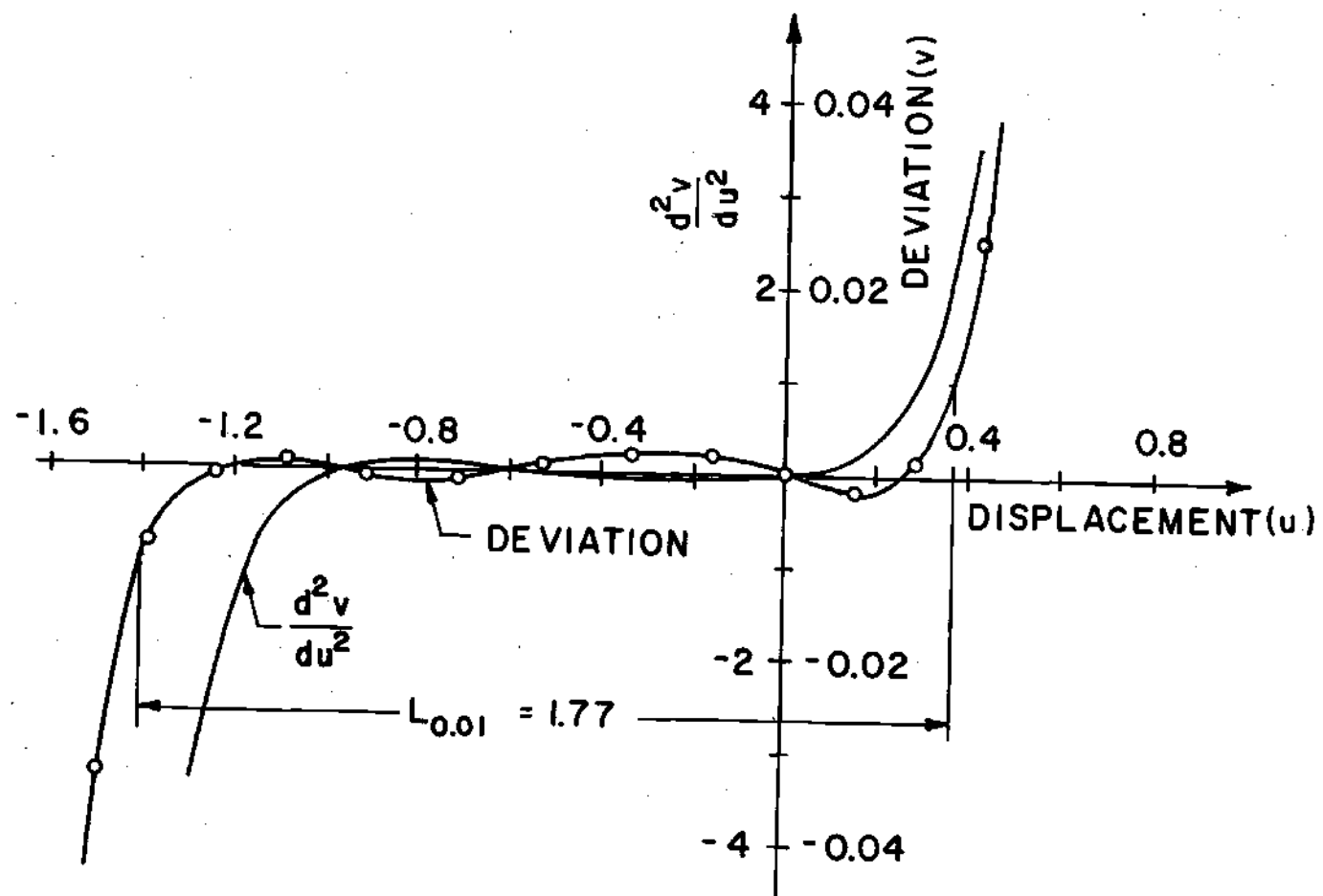


Figure 66. Deviation-Second Derivative Curves for Evans Linkage (Configuration 1).

DIMENSIONS ($\alpha = 20.0$)

$$Q = 1.55247 \quad S = 0.51136$$

$$R = 1.65776 \quad M = 0.15341$$

$$T = 1.02272 \quad \epsilon = 0.0$$

RESULTS ($\phi_0 = 47.262$)

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.602 | 0.766 | 0.808 |
| $\Delta\phi$ | 17 | 38 | 47 |
| γ_b | 28.6 | 15.4 | 7.5 |

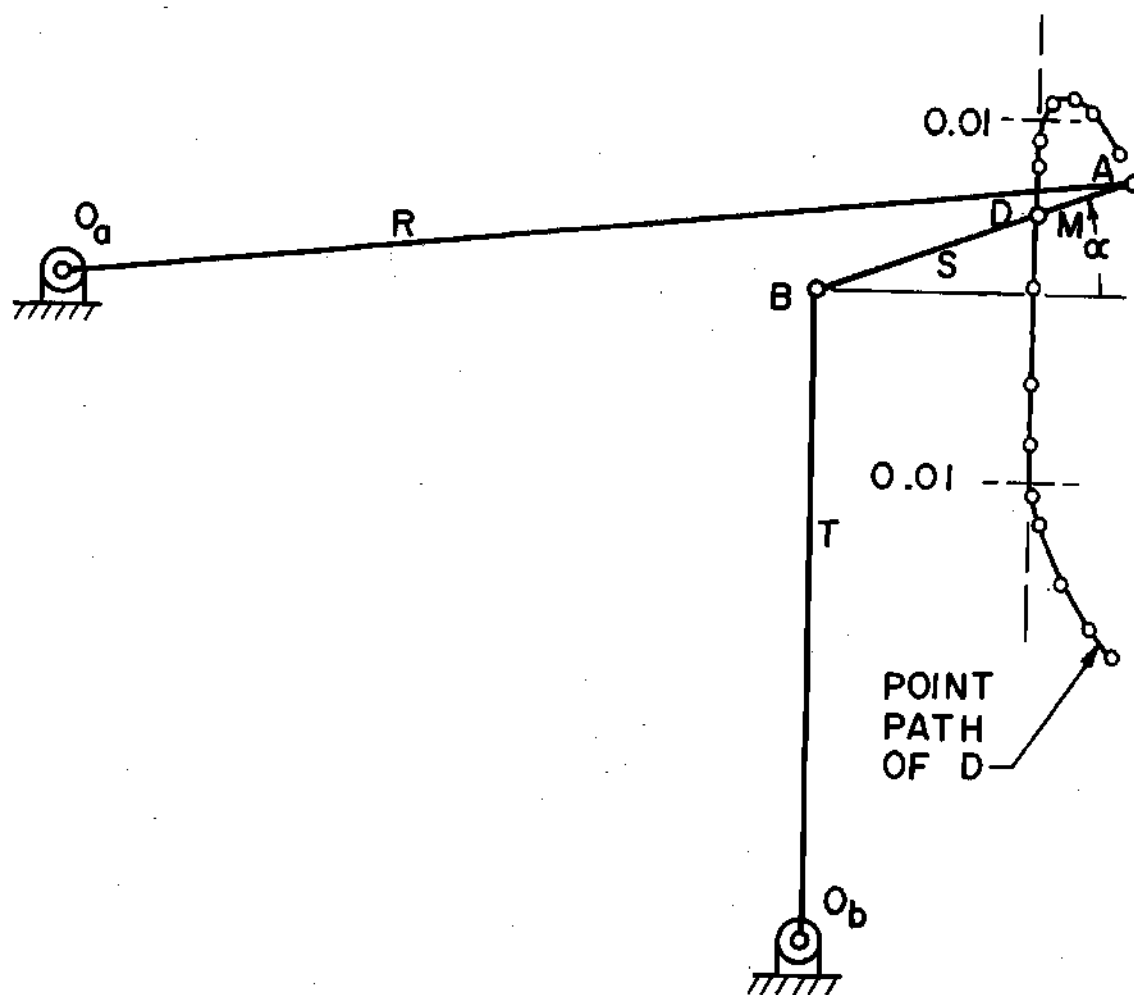


Figure 67. Centric-Rectangular Evans Linkage (Configuration 2).

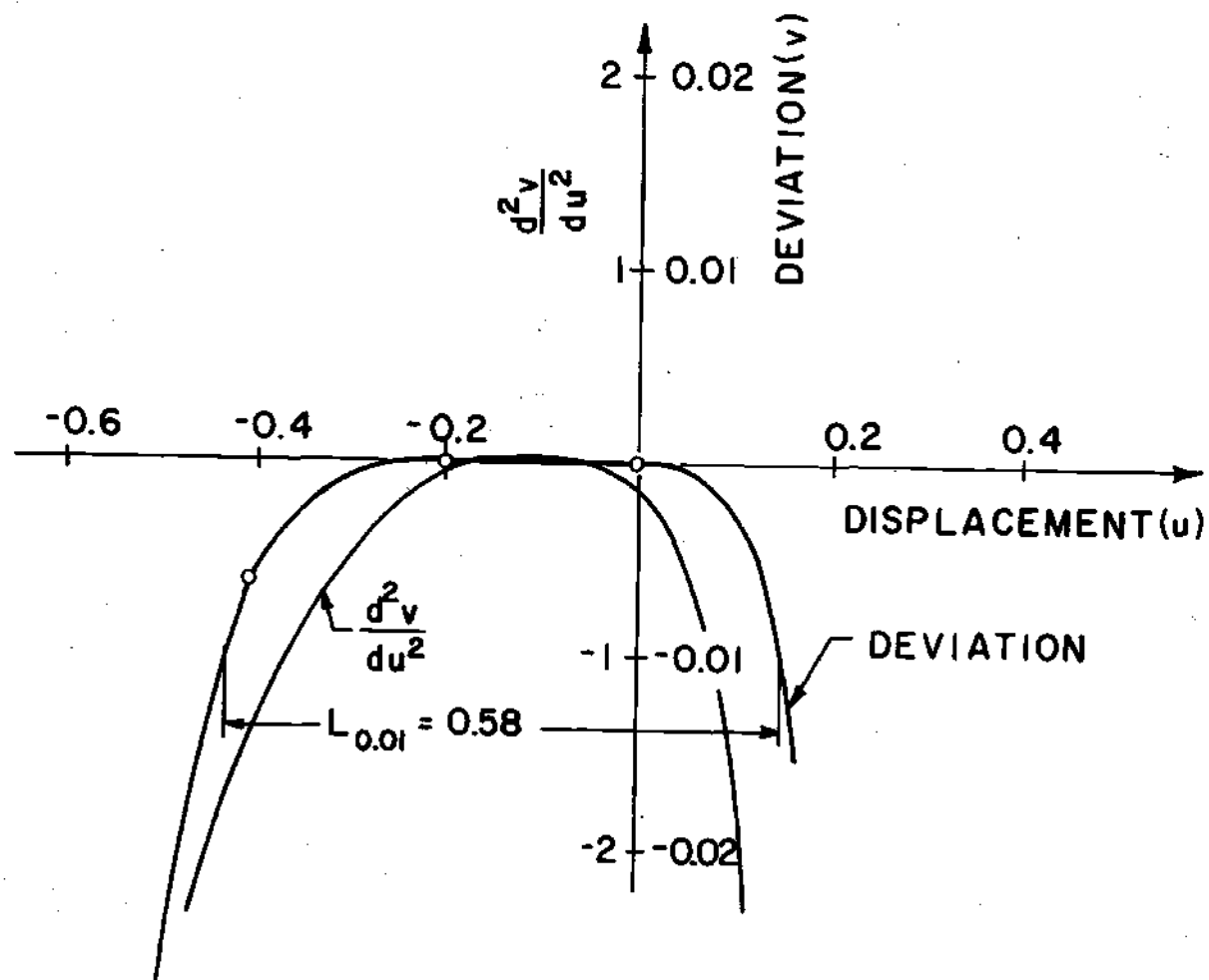


Figure 68. Deviation-Second Derivative Curves for Evans Linkage (Configuration 2).

DIMENSIONS ($\alpha = 20.0$)

$Q = 1.42634$ $S = 0.68629$
 $R^* = 0.34807$ $M = 1.50985$
 $T = 1.37259$ $\epsilon = 0.0$

RESULTS ($\phi_0 = 296.624$)

| | | | |
|--------------|-------|-------|-------|
| D | 0.01 | 0.05 | 0.1 |
| L | 0.676 | 0.793 | 0.835 |
| $\Delta\phi$ | 108 | 143 | 166 |
| γ_b | 64.7 | 57.6 | 54.7 |

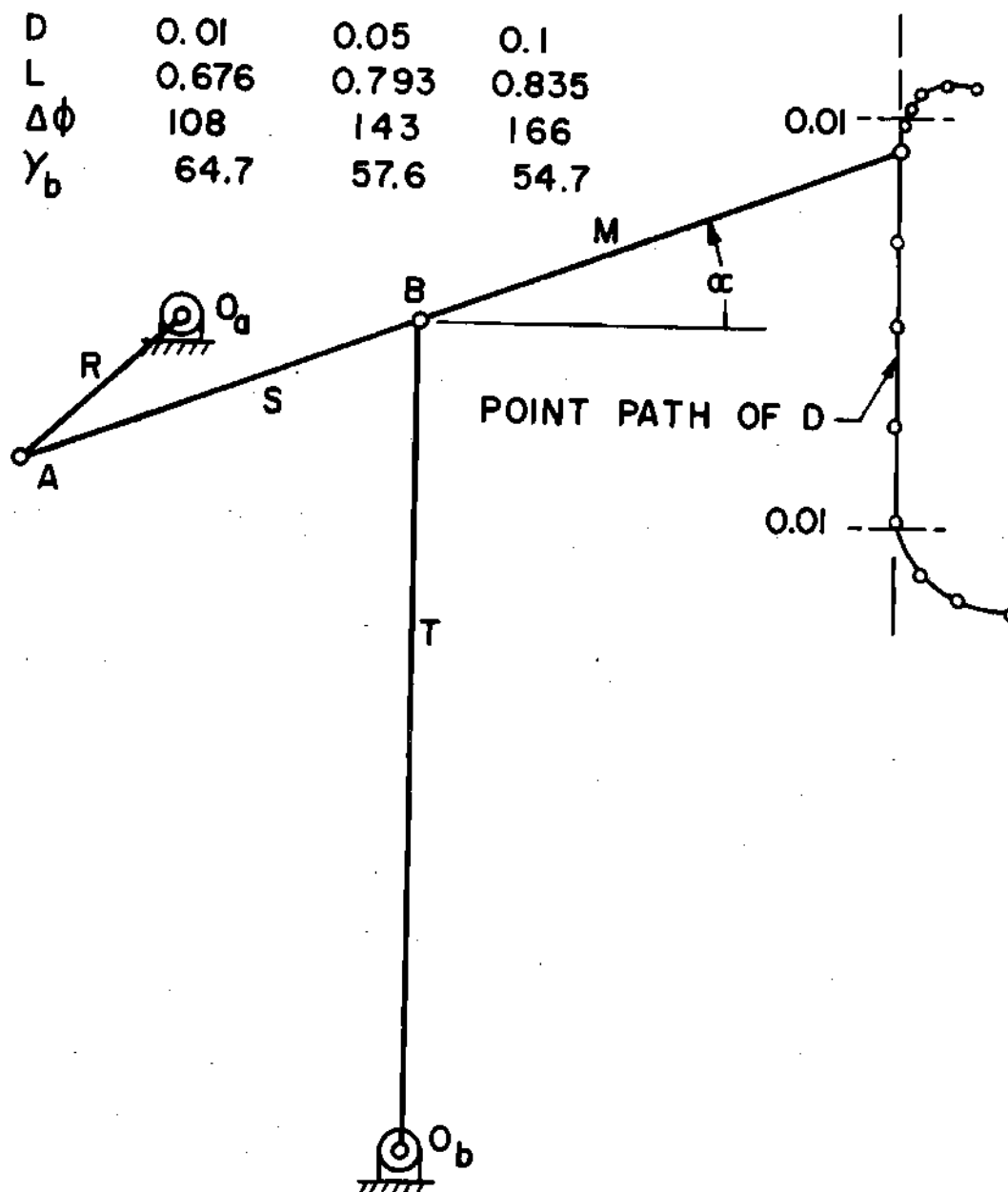


Figure 69. Centric-Rectangular Evans Linkage (Configuration 3).

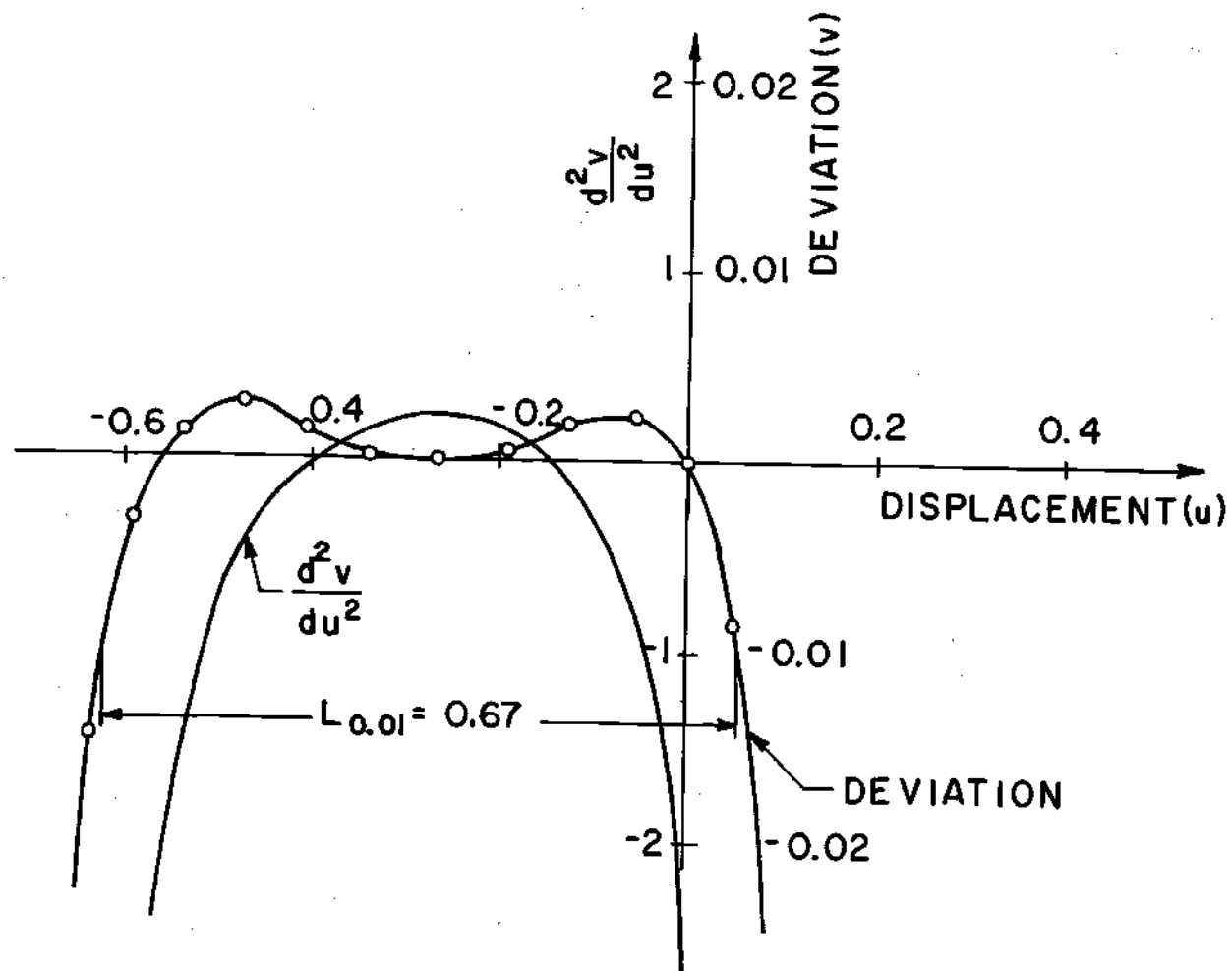


Figure 70. Deviation-Second Derivative Curves for Evans Linkage (Configuration 3).

line, slider-crank mechanism and therefore gives very good results. Based on the results of the digital computations the other configurations do not seem to give very satisfactory results. In general, the theory provides four-finitely separated intersections of the coupler curve with a straight line. The first configuration, however, apparently provides five intersections.

Design Procedure for One Set of Linkages Based On Computer Output

Up to this point, the data has not been reduced to a form which might be useful to the designer. In this section the set of linkages having a Ball-double Burmester point will be analyzed to provide the designer with a technique enabling him to choose the design parameters α_a , α_b for a specified quality and length of the straight line output. Previously, it was necessary for him to judiciously choose the design parameters in a trial and error approach to get a satisfactory straight line output. Furthermore, it was difficult to assess the quality of the resulting approximation to a straight line.

Fig. (71) is a nomograph of the primary information plotted on the basis of the parameters α_a , α_b . The length of the approximate straight line output for an accuracy of 0.01 was calculated by the digital computer for the present set of linkages. A grid of this data was then formed and curves having a constant value for the primary information were interpolated. Linkages having the ratio of the longest and shortest links greater than five are not represented on the nomograph. This stipulation reduced the number of linkages that were considered by a factor of about five. These blank regions should be considered with a

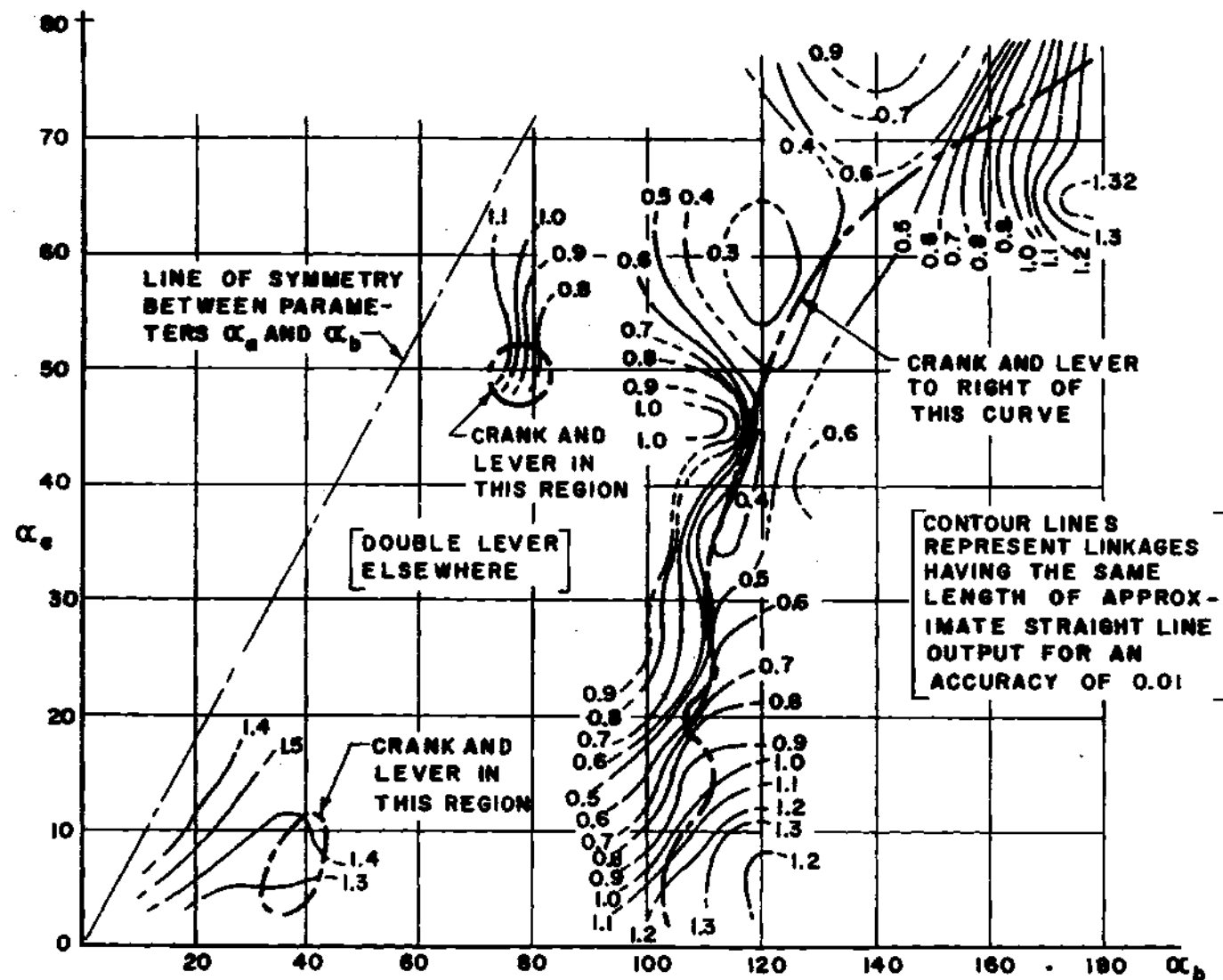


Figure 71. Ball-Double Burmester Point Nomograph.

larger grid to supply information where it is presently ambiguous. Note that there is a symmetry in the design equations (Eq. 2.9a-2.11a) in that α_a and α_b are interchangeable. Hence, α_b is allowed to vary from 0 to 180° and α_a from 0 to 90° with no values of α_a being considered that are simultaneously greater than those for α_b .

Through the center region of the nomograph there is a curve dividing the linkages that are crank and lever and those which are double lever. Local areas that are not well defined occur when $\alpha_b < 90^\circ$. This information enables the designer to predict in advance whether the linkage will be a crank and lever mechanism. It is also possible for him to choose linkages that will likely have good proportions since only those linkages are represented. Other nomographs on transmission angles, range of motion of the cranks, and dynamic characteristics of the deviation curve could also assist the designer to isolate that group of linkages having properties best fulfilling his requirements.

Suppose the designer wants a fairly long approximate straight line output to be produced by a crank and lever mechanism. Choosing $\alpha_a = 5^\circ$ and $\alpha_b = 40^\circ$ satisfies this requirement and the expected length of the output is given to be 1.30. This linkage and the corresponding curves are represented in Fig. (72, 73). Note that the transmission angle γ_a remains within very desirable limits during the production of the straight line portion of the coupler curve. But since the choice for α_a and α_b is very close to the dividing curve between crank and lever and double lever mechanisms, γ_a will most likely approach zero during the complete cycle of the crank.

PARAMETERS

$$\alpha_a = 5.0 \quad \alpha_b = 40.0$$

DIMENSIONS (163.223)

$$Q = 0.88328 \quad S = 1.02568$$

$$R = 1.25723 \quad M = 0.91848$$

$$T^* = 0.40249 \quad \epsilon = 180.0$$

RESULTS ($\phi_0 = 306.314$)

| | | | |
|--------------|--------|-------|-------|
| D | 0.0001 | 0.01 | 0.05 |
| L | 0.545 | 1.322 | 1.749 |
| $\Delta\phi$ | 57 | 143 | 229 |
| γ_b | 62.1 | 51.4 | 43.0 |

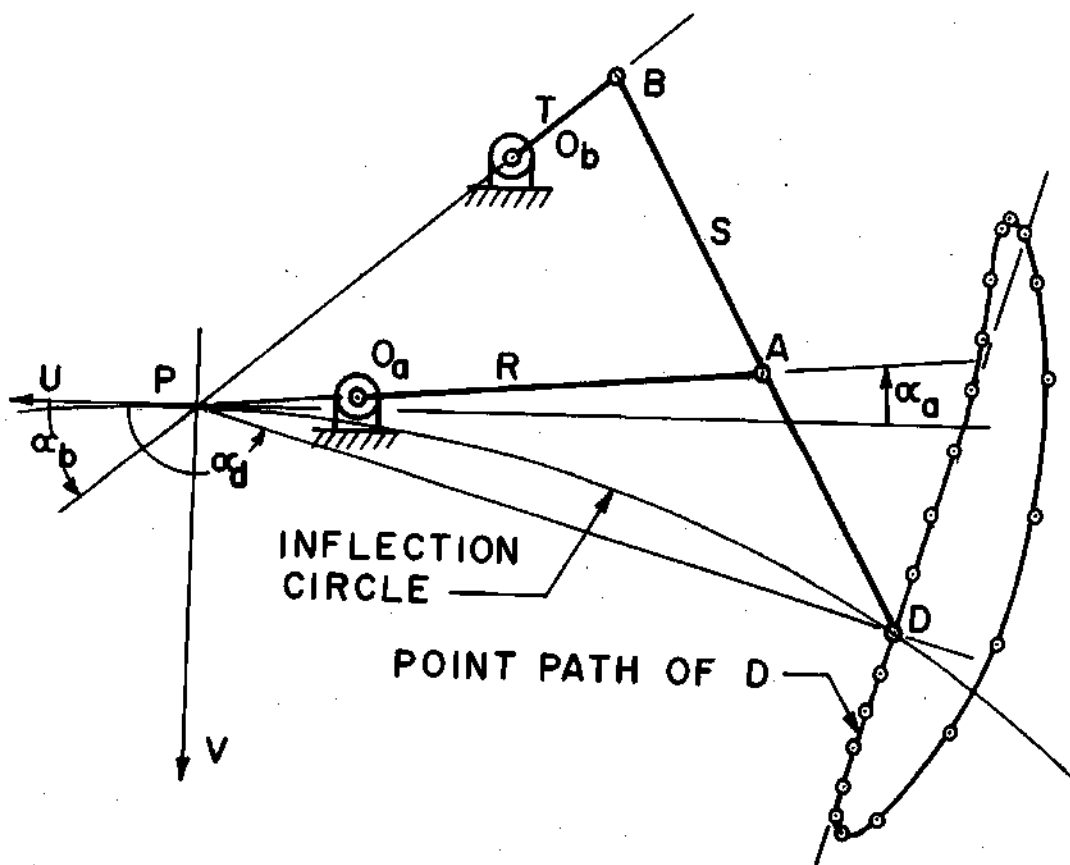


Figure 72. Ball-Double Burmester Point Linkage.

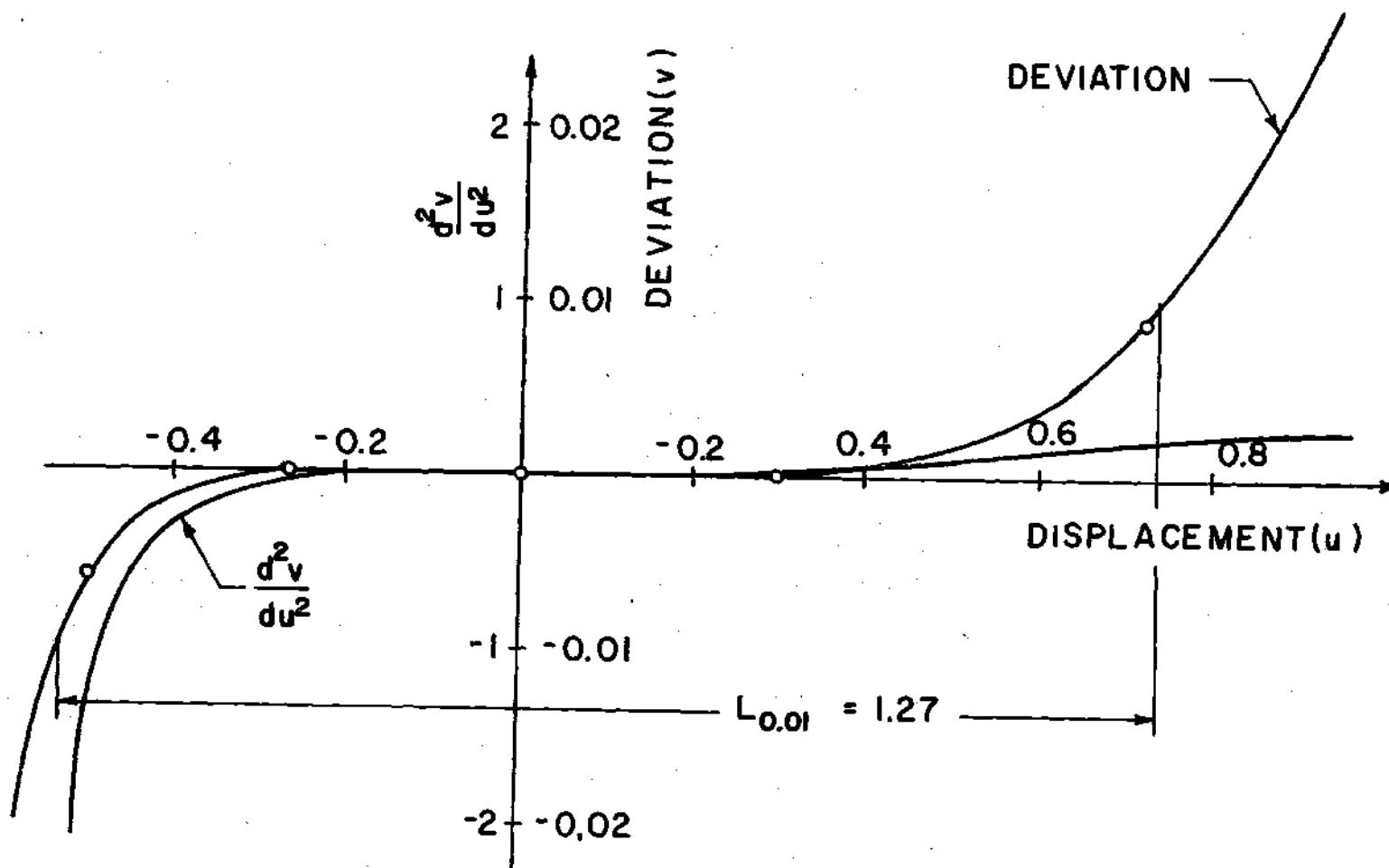


Figure 73. Deviation-Second Derivative Curves for Ball-Double Burmester Point Linkage.

The computer data given with the linkages should enable the designer to determine whether they would satisfy his design requirements. The quoted parameters are those used in the design equations to determine the dimensions of the links. Again note that a star represents those control cranks which can make a full rotation. The dimensions are all based on a unit of length closely associated with the average lengths of the links. The initial position of the input crank ϕ_0 relative to the fixed link gives the designer a reference position of the linkage to use in applying the remainder of the data. In each figure, data for the length of the approximate straight line L , the range of rotation of one of the control cranks $\Delta\phi$ (or $\Delta\psi$), and the appropriate transmission angle γ_b (or γ_a) are given for three different accuracy requirements (usually $D = 0.001, 0.01, 0.05$). These values are reached when the coupler curve approaches the corresponding values of the deviation D . The transmission angles γ of the probable driven crank correspond to the lowest values that occur during the production of each segment of the approximating coupler curve.

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Delbert Tesar was born on September 2, 1935 in Beaver Crossing, Nebraska, a small town near Lincoln, Nebraska. There he attended elementary and secondary schools, graduating from high school in 1953. He entered the University of Nebraska in the fall of 1953 to pursue a degree in Mechanical Engineering (receiving the B.Sc. M.E. in January, 1958). He entered the graduate division at the University of Nebraska in 1958 and obtained a M.Sc. degree in Engineering Mechanics in January, 1960 under the supervision of Dr. J. C. Wolford.

He married Rogene E. Kresak in February, 1957 and presently has two daughters, Vim Lee and Aleta Anne.

In January, 1960, he took a position as instructor in the Applied Mechanics department of Kansas State University where he remained until January, 1962. He then entered the Graduate Division at the Georgia Institute of Technology to pursue the Doctor of Philosophy in Mechanical Engineering.

His primary goal is to become a teacher in Mechanical Engineering with specific interest in teaching machine design and allied courses. Research interest is centered in theoretical and applied kinematics.