THE INFLUENCE OF NOISE AND INTERFERENCE ON THE OUTPUT OF A SYNCHRONIZED OSCILLATOR

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THE INFLUENCE OF NOISE AND INTERFERENCE

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OSCILLATOR

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LIST OF SYMBOLS

Symbol	Definition
V_g	Voltage input to the grid.
V_r	Voltage returned from the feedback network.
V_1	Externally applied synchronizing voltage.
ω_1	Radian frequency of the synchronizing voltage.
ω	Radian frequency of the oscillator.
ϕ	Angle between V_g and V_r.
β	Instantaneous phase angle between the effective synchronizing voltage and the returned voltage V_r.
K_ϕ	Slope of ϕ versus ω curve.
Q	Quality factor.
ω_c	One-half the bandwidth of synchronization, $\frac{V_1}{K_\phi V_g}$.
Δ_{mn}	Difference radian frequency, $\omega_m - \omega_n$.
ω_o	Free-running radian frequency of the oscillator.
A_n	$\frac{\omega_1 I_n}{\omega_o^2 C}$.
I_1	Maximum value of the input current.
C	Capacitance.
v	Instantaneous voltage.
R	Resistance
L	Inductance.
α'	Conductance.
γ'	Coefficient of the cubic nonlinearity.

Symbol	Definition
α, γ	Constants of the differential equation of an oscillator.
a, a_o	Maximum voltages of the sinusoid in the output at ω_o.
b_{2n-1}	Maximum voltage of the sine component in the output at ω_n.
b_{2n}	Maximum voltage of the cosine component in the output at ω_n.
$P(b_1, b_2)$ $Q(b_1, b_2)$	Polynomials.
ρ_n	Normalized square amplitude in the output at $\omega_1 + \Delta_m$, $\rho_n = \dfrac{(b_{2n-1})^2 + (b_{2n})^2}{a_o^2}$.
F_n	$-\dfrac{A_n \omega_o^2}{a_o \omega_1} \doteq -\dfrac{A_n \omega_o}{a_o}$.
v_a	Instantaneous voltage output of an amplifier.
$a_1, a_2, \ldots$	Coefficients of a power series.
$(SN)_{in}$	Input signal noise voltage ratio.
$(SN)_g$	Signal noise voltage ratio at the grid.
V_{no}, V_{ni}	Output and input noise voltages.
σ_{01}	$\dfrac{\Delta_{01}}{\alpha}$.
D	Denominator polynomial.
M	Maximum radian frequency variation of the input signal.
Ω	Arbitrary phase angle.
$J_m(x)$	Bessel function of the first kind with order m and argument x.
$G(v_g)$	Gain as a function of v_g.

Symbol	Definition
V_n	Interfering voltage, $n > 1$.
v_1'	Instantaneous value of the combination of V_1 and the interfering voltages.
β_o	Fixed angle between V_1' and V_r when the oscillator is synchronized.
β_1	Variation about β_o caused by the interference.
δ_{mn}	Normalized radian frequency difference, $\frac{\Delta_{mn}}{\omega_c}$.
x,y,z	Simplifying symbols.
V_o	Maximum voltage of the free-running oscillator at the grid.
r_{ro}	Voltage ratio, $\frac{V_r}{V_o}$.
r'_{ro}	Intercept on the y-axis of a line tangent to the normalized amplitude-limiting characteristic at the operating point.
r_{mn}	Voltage ratio, $\frac{V_m}{V_n}$.
B_1, B_2	Maximum values of the cosine and sine functions of β_1, respectively.
$V_c(\omega_1+\Delta_{n1})$	Maximum value in the output of the cosine function at $\omega_1 + \Delta_{n1}$.
$V_s(\omega_1+\Delta_{n1})$	Maximum value of the sine function at $\omega_1 + \Delta_{n1}$.
b_f	Maximum value of the voltage at ω_1.
d_{2n-1}, d_{2n}	Maximum values of the sine and cosine functions at $\omega_1-\Delta_{n1}$.
ζ, ϵ	Small variations.
$G(\omega)$	Power spectral density in watts/cps.
ψ, ψ'	Functional relations.

Symbol	Definition
η	Noise spectral density in watts/cps.
W,X,Y,Z	Simplifying symbols.
k	Constant.

SUMMARY

An oscillator subjected to an externally applied signal may become synchronized by the applied signal if the frequency of the signal is near enough to the natural frequency of the oscillator, and if the amplitude of the applied signal is of sufficient magnitude. When the oscillator is synchronized, the natural or free-running frequency of the oscillator is suppressed and the output frequency of the oscillator is equal to the frequency of the externally applied synchronizing signal. However, seldom in actual application is the externally applied signal a single sinusoidal signal. In most cases the input consist of the synchronizing signal and either other sinusoidal interfering signals or noise. A synchronized oscillator will provide discrimination against the interference even though the interfering frequency is very near the synchronizing frequency.

This investigation is concerned with the influence on the output of a synchronized oscillator with either discrete signal interference or noise applied along with the synchronizing signal. The interference is considered small relative to the synchronizing signal. For the case of a single discrete signal interference, expressions have been determined for the amplitude of the synchronizing signal in the output, for the amplitude in the output at the frequency of the input interfering signal, and for the amplitude at the generated frequency resulting from intermodulation between the interfering signal and the synchronizing frequency. With multiple discrete interfering signals applied, expressions were

obtained for the output with two types of oscillator circuits. The results obtained for multiple discrete interfering signals were used to predict the output when the input was narrow-band noise.

The determination of the output effects of one discrete interfering frequency is based on Adler's equation. Adler's equation, $\frac{d\beta}{dt} = (\omega_1 - \omega_0) - \omega_c \sin \beta$, is a nonlinear differential equation involving the phase angle β between the synchronizing signal and the voltage returned to the input from the feedback network, the free-running frequency ω_o of the oscillator, the synchronizing signal frequency ω_1, and one-half the bandwidth of synchronization ω_c. The nonlinear amplitude characteristic of the active device does not explicitly appear in the differential equation, however, the imposed conditions require operation in a region of relatively severe amplitude limiting. The effect of the frequency sensitive elements is reflected in an associated phase shift but not in a variation of amplitude. The solution to Adler's equation will yield an equation for β which reaches a constant value in the limit if $\omega_c > \omega_1 - \omega_o$ or is a periodic function of time when $\omega_c < \omega_1 - \omega_o$. If an interfering signal is applied along with the synchronizing signal and the oscillator is synchronized, the phase angle β will consist of a constant magnitude β_o and a time-varying component β_1. The time-varying component results in a frequency-modulation effect which is reflected in the instantaneous frequency and an amplitude-modulation effect resulting from the dependence of the voltage returned from the feedback network on β_1. The amplitude modulation resulting is also a function of the amplitude-limiting characteristic.

The instantaneous phase of the returned voltage is the sum of the angle β_1 between the effective synchronizing voltage and the returned voltage, the angle θ between the synchronizing signal and the interfering signal and $\omega_1 t$ the instantaneous angle of the synchronizing voltage. This instantaneous phase angle yields an effective narrow-band frequency-modulation spectrum. Also the angle β_1, along with the amplitude limiting characteristic, determines the amplitude variation of the returned voltage. The combination of these two types of modulation resulting from the interfering signal yields the amplitude of the interference in the output. Since Adler's equation requires rather sharp amplitude limiting, the main contribution to the interference amplitude is from the frequency-modulation effect.

This development based on Adler's equation places in evidence the effects resulting from one discrete interfering signal externally applied along with the synchronizing signal. The conclusions to be drawn from this analysis are:

(1) The perturbation in the instantaneous phase from an interfering signal results in a combination of frequency and amplitude modulation.

(2) A synchronized oscillator provides discrimination against an interfering signal even without consideration of amplitude attenuation from the frequency sensitive elements.

(3) The discrimination against interfering signals is a minimum at the frequency of the synchronizing signal. In an analysis that included the frequency selectivity the minimum would be shifted toward the free-running frequency.

(4) The discrimination with the synchronizing frequency equal to

the free-running frequency approaches 6 db with the interfering signal frequency very near the synchronizing frequency.

(5) Under conditions of sharp amplitude limiting and without inclusion of the attenuation resulting from frequency selectivity, the generated interference term at $\omega_1-\Delta_{21}$ is approximately equal to the interference at the input interfering frequency $\omega_1+\Delta_{21}$.

(6) With the synchronizing frequency near the edge of the synchronization bandwidth and Δ_{21} small, the interference may experience gain instead of discrimination.

The development based on Adler's equation was applied to the condition of multiple discrete input signals. Also, the solution to the differential equation of an oscillator with cubic nonlinearity was found for the case of multiple discrete input interfering signals. In both approaches with $\left(\sum_{n=2}^{m} r_{n1}\right)^2 \ll V_1$, where r_{n1} is the ratio of the input interfering voltage V_n to the input synchronizing voltage V_1, it was found that the interfering amplitudes in the output of importance resulted from the intermodulation between the input interfering signals and the synchronizing signal.

An approximate method, based on the conclusions reached with multiple interfering signals, was determined to find the noise voltage spectral density in the output of a synchronized oscillator. This method is based on the approximation of a narrow-band noise spectrum by a continuous frequency band of independent discrete signals which yields the same power as the noise spectrum. The consideration of the input noise as narrow-band is justified because of the response of the oscillator which is effectively a narrow-band device. From the conclusions reached with multiple discrete interfering signals on the input, the output noise voltage spectral density can be predicted from the equations developed for one discrete interfering signal.

If the input to a synchronized oscillator consists of a continuous spectrum of discrete sinusoidal signals centered about the synchronizing frequency, then there will exist a pair of input signals equally spaced above and below the synchronizing frequency, for example one at $\omega_1-\Delta_{n1}$ and one at $\omega_1+\Delta_{n1}$. Since each input interfering signal generates an interfering signal in the output, resulting from the intermodulation with the synchronizing signal, the total voltage at any frequency $(\omega_1-\Delta_{n1})$ will be the square-root of the sum of the squares of the voltage resulting from the input interference at that frequency $(\omega_1-\Delta_{n1})$ and the voltage caused by the intermodulation from an input frequency $(\omega_1+\Delta_{n1})$ equally spaced on the opposite side of ω_1. This form of addition of the voltage results because of the noise representation by discrete signals. Therefore, the output noise voltage spectral density may be predicted from the equation derived for a discrete signal interference or by experimentally determining the input/output functional relation.

The experimental results verified with good agreement the predictions from calculations.

CHAPTER I

INTRODUCTION

Interference effects in oscillators both from discrete signals and from noise, have been considered by a number of investigators. Gartens (1), Edson (2), Mullen (3), and Golay (4) have considered noise in oscillators that were free-running, that is, not subjected to a synchronizing signal. Garstens (1) presents a method for estimating the nonlinear noise contributions in an oscillator at low levels of oscillation. His approach uses the differential equation of the oscillator with the van der Pol (8) type nonlinearity. An approximate solution is obtained by linearizing the differential equation and obtaining a particular solution with one of the noise components, represented as a sinusoid, as the forcing function. By superposition and the use of the power spectrum of a cubic nonlinearity, the nonlinear contribution is found.

The time and spectral distributions of noise effects in typical oscillators are derived by Edson (2). These spectral distributions are derived for the build-up of oscillations and for sustained oscillations.

The effects on self-excited oscillators of broad-band noise at or near the oscillator frequency is considered by Mullen (3). The results show that the noise output from noise bands around the oscillator frequency is composed of an additive noise of the shape of the oscillator resonant circuit and a very small FM broadening of the oscillator line.

Golay (4) has analyzed a two-terminal oscillator, with noise present as an RLC circuit in which a negative resistance, in parallel with R, has a slowly varying magnitude which is proportional to the mean square voltage across it. Expressions are derived for the frequency departure from the free-running value and for the bandwidth of the thermal noise generated in R.

The use of a synchronized oscillator as a filter has been considered by several authors (5,6). One of the advantages is that the unwanted signal may be very close to the synchronizing signal and still suffer considerable discrimination provided the unwanted signal is somewhat less in amplitude than the wanted signal. Secondly, the wanted or synchronizing signal may vary slightly in frequency around the free-running frequency of the oscillator and still provide discrimination against unwanted signals. It is very difficult to design a passive filter circuit to accomplish equivalent discrimination for the two cases mentioned.

Tucker and Jamieson (5) have considered the discrimination in a synchronized oscillator due to nonlinearity in the regenerative circuit by assuming the frequency selective element in the feedback path to have flat amplitude response and uniform phase shift. These restrictions, when applied to an oscillator with a frequency selective element present, limits the interfering frequencies to a narrow band near the oscillator free-running frequency or to an oscillator with amplitude characteristic practically independent of frequency. Therefore, this paper is mainly concerned with non-linear effects on discrimination.

The treatment of Spence and Boothroyd (6) to the problem of interfering signals in a synchronized oscillator overcomes the limitation due to neglect of the frequency response in Tucker and Jamieson's consideration. Spence and Boothroyd use the differential equation of an oscillator with van der Pol (8) type nonlinearity and with two forcing functions. This approach includes the frequency selective elements in the solution. However, Spence and Boothroyd treat the case of only one discrete frequency interfering signal.

Rytov (7) considers noise in oscillators that have small nonlinearity; that is, approximately conservative oscillators. Calculations of amplitude and phase fluctuations, in the steady state, are accomplished by expanding the oscillator equation in terms of a small parameter. Since the oscillator is approximately conservative, it is justifiable to use for the solution of the oscillator a method involving expansion in a small parameter. Symbolic differential equations containing random functions which describe the fluctuations are used in conjunction with methods of correlation theory. The small parameter determines slow variations of amplitude and phase both directly and indirectly. Indirectly the small parameter determines amplitude and phase variations through "slow" time, which is the product of the small parameter and real time. The explicit dependence of the function in the differential equation which contains the discrete forcing function and the nonlinear relation is assumed to be periodic with period 2π. The random force part of the differential equation is considered to consist of an in-phase component and a perpendicular component in relation to the synchronized oscillator output. A solution is assumed for a cubic nonlinearity with the third harmonic terms multiplied by the small parameter. Rytov (7) finds for the case where the continuous spectrum is relatively weak, corresponding to a large signal to noise ratio, and with the synchronization frequency at the center of the bandwidth of synchronization, the continuous noise spectrum in the output has the form of a resonance curve with respect to the normalized frequency.

The investigation in the following chapters is concerned with the interference terms in the output of a synchronized oscillator when the interfering components, discrete frequencies or noise, are introduced along with the synchronizing signal. The investigation makes use of Adler's (9) equation and of van der Pol's (8) equation. The solutions obtained to these equations, for discrete signal interference, are extended to predict the noise spectral density in the output when the input noise spectral density is known. Both mathematical and experimental results are included.

Chapter II reviews some basic theory concerning the synchronization of oscillators by sinusoidal synchronizing signals. Specifically, references (8,9) are discussed in detail as they form the basis of the later mathematical development.

Chapter III reviews papers (5,6) which deal with the problem of interference occurring in a synchronized oscillator and the use of Adler's equation with varying synchronizing signals as discussed by Jones (11).

In Chapter IV the mathematical development of the equations for an oscillator subjected to interfering signals along with the sinusoidal synchronizing signal is presented.

The results of Chapter IV to a single discrete interfering signal are extended to the consideration of multiple signals and noise in Chapter V. The solution of the differential equation, with cubic nonlinearity, to multiple signals and noise is also considered in this chapter.

Experimental results and circuits employed are presented and discussed in Chapter VI.

Chapter VII gives the summary and conclusions of the research.

CHAPTER II

REVIEW OF SYNCHRONIZATION OF AN OSCILLATOR

An oscillator subjected to an external signal, under certain conditions, may have its own free-running frequency suppressed and the frequency of operation becomes equal to the external signal. When this occurs the oscillator is said to be synchronized. Two analyses of an oscillator subjected to an external signal will be considered in this chapter. The two methods are due to Adler (9) and van der Pol (8).

Adler's Equation

Adler (9) investigated locking or synchronizing in oscillators when the synchronizing frequency was close to the free-running oscillator frequency. Another requirement on the oscillator was that the response of the oscillator not be governed by the past history of the circuit; that is, slow variations in bias were not allowed. Also the synchronizing signal must be small with respect to the free-running amplitude of oscillation. The latter condition allows neglect of the amplitude variations when the oscillator operates in a flat or near flat portion of the limiting characteristic.

A block diagram representation of an oscillator of the feedback type as analyzed by Adler is shown in Figure 1.

In Figure 1, the externally applied voltage is V_1 and in the manner applied, it can be seen that

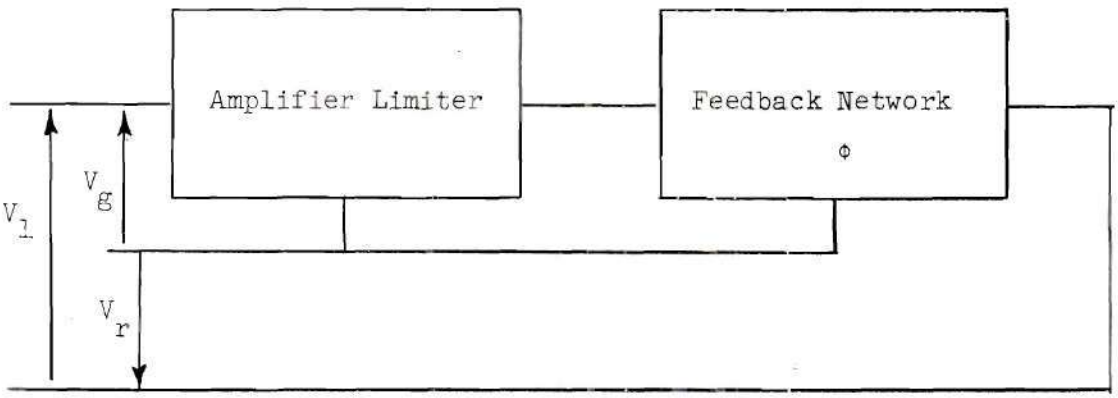


Figure 1. Block Diagram of a Feedback Oscillator with a Synchronizing Signal Applied.

$$V_g = V_r + V_1 . \tag{1}$$

If the frequency of V_r is ω and the frequency of V_1 is ω_1 then the frequency of V_g will have a frequency-modulated characteristic with its variation occurring around ω. Neglecting this variation of frequency of V_g, a phasor diagram as in Figure 2 may be drawn. In this phasor diagram the angle β is measured with respect to V_r and if V_1, the synchronizing voltage, is considered to be stationary at ω_1, $\frac{d\beta}{dt}$ will be positive if $\omega < \omega_1$.

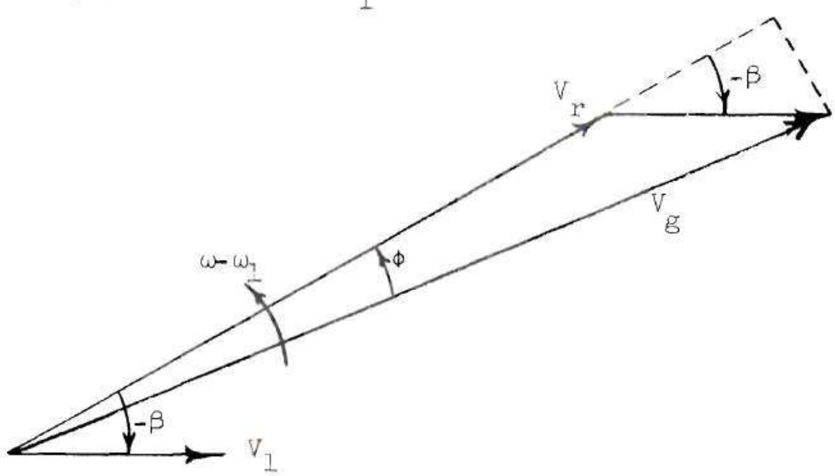


Figure 2. Phasor Diagram of an Oscillator Subjected to a Synchronizing Signal.

If the synchronizing frequency is relatively near the free-running frequency, then the phase, ϕ, may be taken as a linear function of frequency. Just how close the synchronizing frequency must be to the free-running frequency is determined by the Q of the passive elements in the feedback network. When the oscillator meets these conditions, the phase shift ϕ may be expressed as

$$\phi \doteq K_\phi (\omega - \omega_o) . \tag{2}$$

Also from Figure 2, a relation for ϕ in terms of the voltages may be obtained as

$$\sin \phi = - \frac{V_1}{V_g} \sin (-\beta) = \frac{V_1}{V_g} \sin \beta . \tag{3}$$

Since $V_1 << V_g$, equation (3) may be written

$$\phi \doteq \frac{V_1}{V_g} \sin \beta . \tag{4}$$

Introducing ω_1 into equation (2) and equating to ϕ from equation (4) gives

$$\frac{V_1}{V_g} \sin \beta = K_\phi \left[(\omega - \omega_1) + (\omega_1 - \omega_o) \right] . \tag{5}$$

The rate of change of β with time, in terms of the radian frequencies, may be found from Figure 2 as

$$\frac{d\beta}{dt} = (\omega_1 - \omega) . \tag{6}$$

Substitution of equation (6) into (5) and rearranging yields

$$\frac{d\beta}{dt} = (\omega_o - \omega_1) - \frac{V_1}{V_g K_\phi} \sin\beta \;. \tag{7}$$

When synchronization occurs, $\frac{d\beta}{dt} = 0$, and the limits of ω_1 for synchronization may be obtained from equation (7) by substituting the limits of $\sin\beta = \pm 1$. These limits for ω_1 become

$$\omega_1' = \omega_o \pm \frac{V_1}{V_g K_\phi} \;, \tag{8}$$

or in terms of the difference radian frequency ω_c,

$$\omega_c = (\omega_o - \omega_1') = \pm \frac{V_1}{V_g K_\phi} \;. \tag{9}$$

Equation (7) then becomes with the substitution of (9) and the definition of

$$\Delta_{10} = \omega_1 - \omega_o \;, \tag{10}$$

$$\frac{d\beta}{dt} = \Delta_{10} - \omega_c \sin\beta \;. \tag{11}$$

This equation (11) represents the instantaneous variation of the frequency of V_r from the synchronizing frequency, ω_1. This equation applies to variable values for Δ_{10} and ω_c as well as for constant values. This characteristic of Adler's equation will be used in later sections.

van der Pol's Equation

Van der Pol (8), in his very important contribution to oscillator theory, developed a nonlinear differential equation which applies to a wide range of oscillator types. Depending on the nonlinearity, it gives the solution for near sinusoidal as well as relaxation type oscillators. The differential equation for an oscillator with one degree of freedom and without any external forcing function applied is typically,

$$\frac{d^2v}{dt^2} + \frac{d}{dt}\left[f(v)\right] + \omega_o^2 v = 0 . \tag{12}$$

The nonlinearity is included in the term $f(v)$.

If the oscillator circuit is subjected to a time-varying forcing function, then the right side of equation (12) is altered by the addition of the differential of this time function. The amplitude of the added term is proportional to the magnitude of the forcing function, as follows:

$$\frac{d^2v}{dt^2} + \frac{d\left[f(v)\right]}{dt} + \omega_o^2 v = \omega_o^2 A_1 \sin \omega_1 t . \tag{13}$$

The equivalent circuit of one of the oscillator circuits used in the experimental work is shown in Figure 3. This circuit uses a transistor as the active element and therefore the forcing function is shown as a current source.

Summation of currents at either node yields

$$-I_1 \sin \omega_1 t = \frac{1}{L}\int v dt + C\frac{dv}{dt} + \frac{v}{R} + f(v). \tag{14}$$

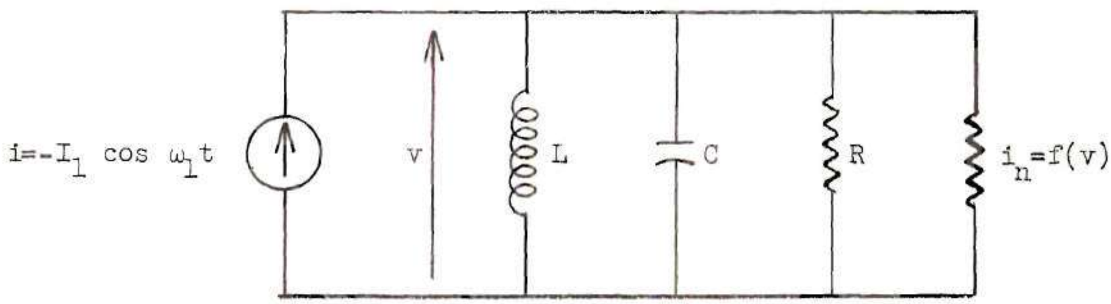


Figure 3. Two-terminal Oscillator with Constant-current Forcing Function.

If the current in the nonlinear conductance is given by

$$i_n = -\alpha' v + \gamma' v^3 , \tag{15}$$

then equation (13) becomes

$$\frac{1}{L}\int v dt + C\frac{dv}{dt} + \frac{v}{R} - \alpha' v + \gamma' v^3 = -I_1 \cos \omega_1 t . \tag{16}$$

Differentiating equation (14) with respect to time and making the coefficient of the highest order differential unity, gives

$$\frac{d^2v}{dt^2} + \frac{d}{dt}\left(\frac{v}{RC} - \frac{\alpha'}{C} v + \frac{\gamma'}{C} v^3\right) + \frac{v}{LC} = \frac{I_1\omega_1}{C} \sin \omega_1 t . \tag{17}$$

Making the following simplifying definitions;

$$\alpha = \left(\frac{\alpha'}{C} - \frac{1}{RC}\right) , \tag{18}$$

$$\gamma = \frac{\gamma'}{C} ,$$

$$\omega_o^2 = \frac{1}{LC} ,$$

$$A_1\omega_o^2 = \frac{I_1\omega_1}{C} ,$$

and incorporating these definitions in equation (17) gives an equation of the form of equation (13).

Thus,

$$\frac{d^2v}{dt^2} + \frac{d}{dt}\left(-\alpha v + \gamma v^3\right) + \omega_o^2 v = \omega_o^2 A_1 \sin \omega_1 t \ . \tag{19}$$

The solution to this equation may be obtained by assuming a solution in the form of a Fourier series. For a first order approximation the higher harmonic terms are neglected. The amplitudes of the sinusoids in the solution are assumed to be slowly varying functions of time in order to allow the stability of the solutions to be determined.

A solution is assumed as

$$v = a(t) \sin \omega_o t + b_1(t) \sin \omega_1 t + b_2(t) \cos \omega_1 t \ . \tag{20}$$

With an assumed solution in the form of equation (20), it is possible to generalize the solution for the amplitudes to find the condition for synchronization to occur; the value of the free-running amplitude; and the variation of b_1 and b_2 with frequency. Substitution of the assumed solution (20) into (19) with the condition that the following inequalities exist;

$$\dot{a} \ll \omega_o a, \ \dot{b}_1 \ll \omega_1 b_1, \ \dot{b}_2 \ll \omega_1 b_2 \ , \tag{21}$$

$$\ddot{a} \ll \omega_o \dot{a}_1, \ \ddot{b}_1 \ll \omega_1 \dot{b}_1, \ \ddot{b}_2 \ll \omega_1 \dot{b}_2 \ ,$$

yields the following differential equations:

$$2\dot{a} - \alpha + \gamma \left[\frac{3}{4} a^2 + \frac{3}{2}\left(b_1^2 + b_2^2\right)\right] = 0 \ , \tag{22}$$

$$2\dot{b}_2 - b_1(\omega_o^2 - \omega_1^2) - \alpha b_2\omega_1 + \frac{3}{4}\,\omega_1\gamma b_2(b_2^2 + 2b_1^2) = -\omega_o^2 A_1 \qquad (23)$$

$$2\dot{b}_1 + b_2(\omega_o^2 - \omega_1^2) - \alpha b_1\omega_1 + \frac{3}{4}\,\omega_1\gamma b_1(b_1^2 + 2b_2^2) = 0\ . \qquad (24)$$

Equation (22) is independent of frequency and can be used to determine free-running amplitude of oscillation with $b_1 = b_2 = 0$ and also the minimum value of $b_1^2 + b_2^2$ when synchronization of the oscillator is stable. Since with $b_1 = b_2 = 0$, $da/dt = 0$ in equation (22) because the output will have a single constant frequency ω_o, the equation becomes

$$-\alpha + \frac{3\gamma}{4}\,a_o^2 = 0\ , \qquad (25)$$

$$a_o = \left(\frac{4\alpha}{3\gamma}\right)^{\frac{1}{2}}\ .$$

This is the amplitude of oscillation with the forcing function equal to zero.

Solving equation (22) for the amplitude at frequency ω_o, yields

$$a^2 = \frac{4\alpha}{3\gamma} - 2(b_1^2 + b_2^2)\ . \qquad (26)$$

For "a" to equal zero

$$b_1^2 + b_2^2 \geqq \frac{4\alpha}{3\gamma} = \frac{a_o^2}{2}\ , \qquad (27)$$

or

$$(b_1^2 + b_2^2)^{\frac{1}{2}} \geqq \frac{a_o}{\sqrt{2}}\ .$$

Equation (27) represents one of the stability requirements of the oscillator. The output of the oscillator at the synchronizing frequency must be equal to or greater than $\frac{a_o}{\sqrt{2}}$. The other stability requirement is represented on Figure 4 and applies at frequencies very near the free-running frequency.

When the oscillator is neither synchronized nor free-running equations (22, 23, 24) may be solved for a, b_1, and b_2.

The main interest in this development is the synchronized case, $a = 0$, giving from equations (23, 24)

$$-b_1(\omega_o^2 - \omega_1^2) - \alpha b_2\omega_1 + \frac{3}{4}\,\omega_1\gamma(b_2^2 + 2b_1^2) = -\omega_o^2 A_1 \,, \qquad (28)$$

and

$$b_2(\omega_o^2 - \omega_1^2) - \alpha b_1\omega_1 + \frac{3}{4}\,\omega_1\gamma(b_1^2 + 2b_2^2) = 0 \,. \qquad (29)$$

Since equations (28, 29) represent the condition of the oscillator with only a single frequency output present, the time rate of change of amplitudes b_1 and b_2 have been set equal to zero.

The stability of the solutions to the equations (23, 24) may be determined by replacing b_1 and b_2 by $b_{10} + \delta b_1$ and $b_{20} + \delta b_2$, where δb_1 and δb_2 represent variations about constant values b_{10} and b_{20}. This substitution gives linear differential equations involving δb_1 and δb_2. The stability is indicated by the roots of these equations.

Another method described by Minorsky (10) and credited to Andronow and Witt permits a closer connection with the representation of the oscillator differential equation in the phase plane. From equations (23, 24) is found

$$\frac{db_1}{db_2} = \frac{P(b_1,b_2)}{Q(b_1,b_2)} = \frac{-\omega_o^2 A + b_1(\omega_o^2 - \omega_1^2) + \alpha b_2 \omega_1 - \frac{3}{4}\,\omega_1 b_2 \gamma (b_2^2 + 2b_1^2)}{-b_2(\omega_o^2 - \omega_1^2) + \alpha b_1 \omega_1 - \frac{3}{4}\,\omega_1 \gamma b_1 (b_1^2 + 2b_2^2)}. \tag{30}$$

Values of $\frac{db_1}{db_2}$ in the $b_1 b_2$-plane will give flow lines which will converge or diverge from singular points in the phase plane with time, depending on the nature of the stability of the point in question. The location of these singular points may be found when b_1 and b_2 are constant, which results in $P(b_1,b_2) = Q(b_1,b_2) = 0$. To establish the stability of the singular points b_1 and b_2 in equation (30) are replaced by $b_{10} + \delta b_1$ and $b_{20} + \delta b_2$ and the roots of the equations obtained for $\frac{d(\delta b_1)}{dt}$ and $\frac{d(\delta b_2)}{dt}$ are examined for stability requirements. For equation (30) the results are shown in Figure 4, where the cross-hatched region represents the unstable region.

Figure 4 shows not only the boundary of the unstable region but also shows the variation of the output in the stable region with variation of synchronizing frequency and for different values of input synchronizing signal amplitude, which is reflected in F_1. The equation for the response curves is

$$\rho_1\left(2\,\frac{\Delta_{01}}{\alpha}\right)^2 + \rho_1(1 - \rho_1)^2 = F_1^2\,, \tag{31}$$

and the region of instability includes the area $\rho_1 < 0.5$ and the area within the ellipse

$$\left(\frac{2\Delta_{01}}{\alpha}\right)^2 + (1 - \rho_1)(1 - 3\rho_1) = 0\,, \tag{32}$$

where $\rho_1 = \frac{b_1^2 + b_2^2}{a_o^2}$ and $F_1^2 = \left(-\frac{A_1\omega_o}{\alpha a_o}\right)^2$.

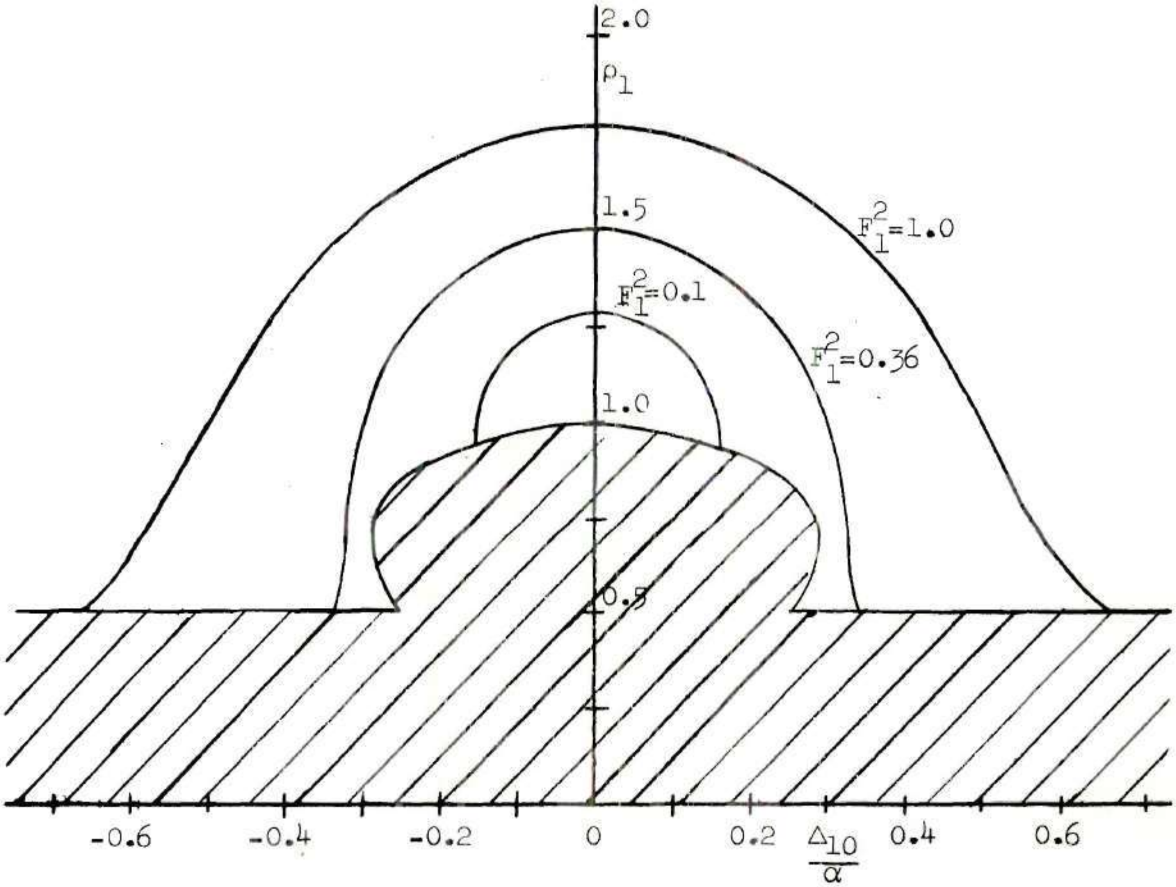


Figure 4. Variation of the Output in a Synchronized Oscillator as the Frequency of the Synchronizing Signal Varies.
$F_1^2 = \left(-\frac{A_1\omega_o}{\alpha a_o}\right)^2$.

The solutions, shown in Figure 4, are accomplished with the following approximations in equations (28, 29):

$$\frac{A_1\omega_o^2}{\omega_1} \doteq A_1\omega_o \ , \tag{33}$$

and

$$\frac{\omega_o^2 - \omega_1^2}{\omega_1} \doteq 2(\omega_o - \omega_1) = 2\Delta_{01} \ .$$

These approximations require ω_1 to be very near ω_o in order not to cause inaccuracy.

CHAPTER III

SYNCHRONIZATION WHEN THE FORCING FUNCTION IS NOT A SINGLE CONSTANT FREQUENCY SINUSOID

Effects on the Output Caused by Nonlinearity with Multiple Forcing Functions

One approach to the investigation of an oscillator subjected to a synchronizing signal that includes interfering terms is presented by Tucker and Jamieson (5). Their paper is concerned with the effect of the nonlinearity resulting from the amplifier-limiter section of the oscillator with a block diagram as shown in Figure 5.

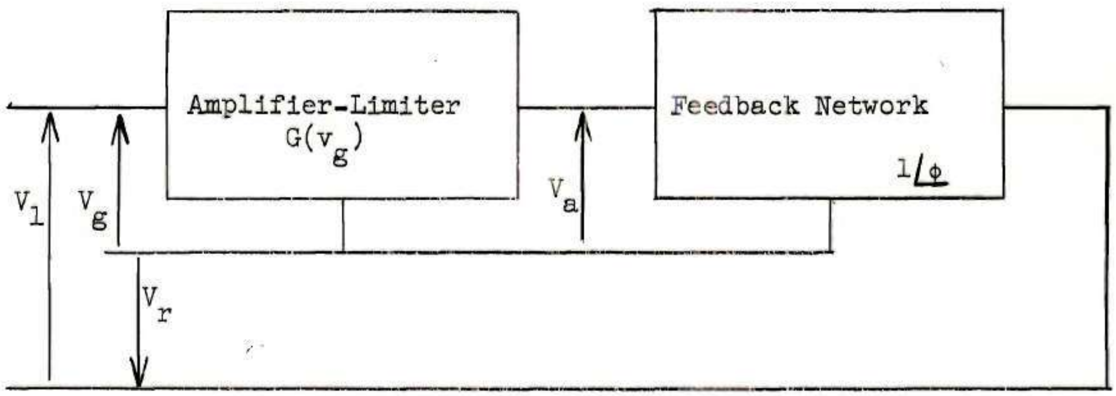


Figure 5. Block Diagram of a Feedback Oscillator with Synchronizing Signal Applied. $G(v_g) = a_1 + a_2 v_g + a_3 v_g^2$.

If the input-output characteristic of the amplifier-limiter is expressed as a power series through the cubic term, then

$$v_a = a_1 v_g + a_2 v_g^2 + a_3 v_g^3 \ . \tag{34}$$

With multiple input signals v_g is expressed as

$$v_g = V_g \left[\cos\, \omega_1 t + r_{2g} \cos(\omega_1 - \Delta_{12})t + \ldots + r_{ng} \cos(\omega_1 - \Delta_{1n})t\right], \quad (35)$$

where v_g is the instantaneous voltage applied to the grid of the amplifier-limiter, v_a is the output of the amplifier-limiter, and $r_{ng} = \dfrac{V_n}{V_g}$.

With this input signal, the output of the amplifier-limiter becomes

$$\begin{aligned}
v_a = {} & \left[a_1 V_g + a_3 V_g^3 \left(\tfrac{3}{4} + \tfrac{3}{2} \sum_{m=1}^{n} r_{mg}^2\right)\right] \cos\, \omega_1 t \qquad (36) \\
& + \sum_{q=1}^{n} \left\{ a_1 V_g r_{qg} + a_3 V_1^{\,3} \left[\tfrac{3}{4}\, r_{qg} \left(1 + \sum_{\substack{m=1 \\ m=q}}^{n} r_{mg}^2\right)\right]\right\} \cos(\omega_1 - \Delta_{1q})t \\
& + \sum_{q=1}^{n} a_3 V_g^3 \times \tfrac{3}{4}\, r_{qg} \cos\, (\omega_1 + \Delta_{1q})t \\
& + \sum_{q=1}^{n} a_3 V_g^3 \times \tfrac{3}{4}\, r_{qg}^2 \cos\, (\omega_1 - 2\Delta_{1q})t \\
& + \sum_{q=1}^{n-1} \sum_{m=q+1}^{n} a_3 V_g^3 \times \tfrac{3}{2}\, r_{qg} r_{mg} \left[\cos(\omega_1 - \Delta_{1q} + \Delta_{1m})t \right. \\
& \left. + \cos\, (\omega_1 + \Delta_{1q} - \Delta_{1m})t + \cos\, (\omega_1 - \Delta_{1q} - \Delta_{1m})t\right] \\
& + \sum_{q=1}^{n-2} \sum_{m=q+1}^{n-1} \sum_{s=m+1}^{n} a_3 V_g^3 \times \tfrac{3}{2}\, r_{qg} r_{mg} r_{sg} \left[\cos(\omega_1 + \Delta_{1q} - \Delta_{1m} - \Delta_{1s})t \right. \\
& \left. + \cos\, (\omega_1 - \Delta_{1q} + \Delta_{1m} - \Delta_{1s})t + \cos(\omega_1 - \Delta_{1q} - \Delta_{1m} + \Delta_{1s})t\right] .
\end{aligned}$$

This equation (36) gives the magnitudes in the output at the various input frequencies and at the generated frequencies because of the nonlinearity.

If only $V_g \cos \omega_1 t$ is considered to be a coherent signal in equation (35) and all the remaining terms are considered to become equal and infinite in number $(n \to \infty)$, then these terms may be taken as the noise input. The result of taking all the discrete signals at the input with the exception of v_1 as noise, yields for the output noise,

$$V_{no} = V_{ni} \left[a_1^2 + 3a_1 a_3 V_g^2 + \frac{45}{16} a_3^2 V_g^4 \left(1 + \frac{2.8}{(SN)_{in}^2} + \frac{1.2}{(SN)_{in}^4} \right) \right]^{\frac{1}{2}} \quad (37)$$

where $(SN)_{in}$ is the signal to noise voltage ratio at the input, V_{no} is the rms output noise voltage and V_{ni} is the rms input noise voltage.

This equation (37) expresses the effect of the nonlinearity of the amplifier-limiter on the output with a coherent signal and noise on the input.

To apply these results to an oscillator, Tucker and Jamieson assumed the feedback network to have flat amplitude response and zero phase shift over the band of interest. The external signal V_1 is assumed to contain only difference frequencies $\omega_1 - \Delta_{1m}$ and that the sum frequencies $\omega_1 + \Delta_{1m}$ are generated by the nonlinearity of the circuit.

The final equation relating the signal/noise ratio at the grid to the signal/noise at the input is given by

$$\frac{(SN)_g}{(SN)_{in}} \doteq \frac{1 - a_1 - \frac{3}{2} a_3 \left[V_g(\omega_1)\right]^2 \left(1 + \frac{1}{(SN)_g^2}\right)}{1 - a_1 - \frac{3}{4} a_3 \left[V_g(\omega_1)\right]^2 \left(1 + \frac{2}{(SN)_g^2}\right)} \tag{38}$$

$$\times \frac{1}{\left\{1 + \frac{9}{16} a_3^2 \left[V_g(\omega_1)\right]^4 \left(1 + \frac{6}{(SN)_g^2} + \frac{2}{(SN)_g^4}\right)\right\}^{\frac{1}{2}}} ,$$

where $V_g(\omega_1)$ is the grid voltage of frequency ω_1 with regeneration, and must be calculated from an additional relationship which is not an explicit function of V_1. The signal/noise voltage ratio on the grid is represented by $(SN)_g$ and the input signal/noise voltage ratio by $(SN)_{in}$.

This analysis by Tucker and Jamieson points out the dominant role played by the nonlinearity in suppression of interference in an oscillator. The result, however, is in a rather inconvenient form and does not include effects of phase shift or amplitude response in the feedback network.

Use of van der Pol's Equation for Discrete Signal Interference

The consideration of an oscillator with two forcing functions and the behavior of the oscillator described by a differential equation of the van der Pol type is made by Spence and Boothroyd (6). The differential equation for this case is

$$\frac{d^2v}{dt^2} + \frac{d}{dt}\left(- v + \gamma v^3\right) + \omega_o^2 v = A_1 \omega_o^2 \sin \omega_1 t + A_2 \omega_o^2 \sin \omega_2 t . \tag{39}$$

Experimental knowledge of the output from an oscillator of this type indicates the important amplitudes in the output are at frequencies ω_1, ω_2, and $2\omega_1 - \omega_2$. Therefore, the equation of the output is assumed in the form

$$\begin{aligned} v(t) = {} & b_1(t) \sin \omega_1 t + b_2(t) \cos \omega_1 t \\ & + b_3(t) \sin \omega_2 t + b_4(t) \cos \omega_2 t \\ & + b_5(t) \sin (\omega_1 + \Delta_{12}) + b_6(t) \cos (\omega_1 + \Delta_{12})t, \end{aligned} \tag{40}$$

where

$$\Delta_{12} = \omega_1 - \omega_2 .$$

Since these output terms are all related to the frequencies in the forcing function, either a phase angle must be included, or, as is done in equation (40), a sine and cosine term must be included for each frequency. The magnitudes (b's) are considered to be slowly varying functions of time.

Because of the large number of quantities that result when equation (40) is substituted into equation (39), a degree of relative magnitude of the quantities is specified in order to simplify the solution. Spence and Boothroyd (6) classify these magnitudes into three groupings "small," "very small" and "small of third order." In order to specify the order of smallness the following four assumptions are made:

(a) The oscillator output is considered sinusoidal.

(b) ω_1 and ω_2 are near ω_0.

(c) The amplitudes at ω_2 and $(\omega_1 + \Delta_{12})$ are of an order smaller than those at ω_1.

(d) The maximum values b_1 and b_2 are of the same order of magnitude as a_o.

When equation (40) is substituted into equation (39), six linear differential equations are found by retaining only terms containing frequencies ω_1, ω_2, and $\omega_1 + \Delta_{12}$, and equating the coefficients of the sine and cosine of $\omega_1 t$, $\omega_2 t$, and $(\omega_1 + \Delta_{12})t$. Only the terms that are small to the least two orders are retained in each equation. The following equations result:

$$2\dot{b}_1 + b_2(2\Delta_{01}) - \alpha b_1(1 - \rho_1) = 0 \ , \tag{41}$$

$$2\dot{b}_2 - b_1(2\Delta_{01}) - \alpha b_2(1 - \rho_1) = -\frac{A_1\omega_o^2}{\omega_1} \doteq -A_1\omega_o \ , \tag{42}$$

$$2\dot{b}_3 + b_4(2\Delta_{02}) - \alpha b_3(1 - 2\rho_1) + \alpha b_5\rho_s + \alpha b_6\rho_m = 0 \ , \tag{43}$$

$$2\dot{b}_4 - b_3(2\Delta_{02}) - \alpha b_4(1 - 2\rho_1) - \alpha b_6\rho_s + \alpha b_5\rho_m = -\frac{A_2\omega_o^2}{\omega_1} \doteq -A_2\omega_o , \tag{44}$$

$$2\dot{b}_5 + 2b_6(2\Delta_{01} - \Delta_{02}) - \alpha b_5(1 - 2\rho_1) + \alpha b_3\rho_s + \alpha b_4\rho_m = 0 \ , \tag{45}$$

$$2\dot{b}_6 - 2b_5(2\Delta_{01} - \Delta_{02}) - \alpha b_6(1 - 2\rho_1) - \alpha b_4\rho_s + \alpha b_3\rho_m = 0 \ , \tag{46}$$

where

$$\rho_s = \frac{b_1^2 - b_2^2}{a_0^2} \ , \quad \rho_m = \frac{2b_1b_2}{a_0^2} \ , \quad \text{and} \quad \Delta_{mn} = \omega_m - \omega_n \ .$$

Solutions to these equations in the steady-state gives for the normalized-squared amplitudes

$$4\rho_1^2\sigma_{01}^2 + \rho_1(1 - \rho_1)^2 = F_1^2 \ , \tag{47}$$

$$\rho_2 = F_2^2 \; \frac{(4\sigma_{01} - 2\sigma_{02})^2 + (1 - 2\rho_1)^2}{D} \; , \tag{48}$$

$$\rho_3 = F_2^2 \left[\frac{\rho_1^2}{D} \right] \; , \tag{49}$$

where

$$\sigma_{mn} = \frac{\Delta_{mn}}{\alpha} = \frac{\omega_m - \omega_n}{\alpha} \; , \; F_m^2 = \left(- \frac{A_m \omega_o}{\alpha a_o}\right)^2 \; , \; \rho_n = \frac{b_{2n-1}^2 + b_{2n}^2}{a_o^2}$$

and

$$D = \left[(2\sigma_{02})^2 + (1 - 2\rho_1)^2\right]\left[(4\sigma_{01} - 2\sigma_{02})^2 + (1 - 2\rho_1)^2\right] + \rho_1^4 - 2\rho_1^2 \left[2\sigma_{02}(4\sigma_{01} - 2\sigma_{02}) + (1 - 2\rho_1)^2\right] \; .$$

Equation (47) is the same equation as determined without the interference term at frequency ω_2; that is, with the assumptions made, the amplitude in the output at the synchronizing frequency is unchanged because of interference.

Equation (48) gives the normalized-squared amplitude in the output at frequency ω_2. It is a linear function of the input interfering signal as long as the assumptions are valid.

The component in the output generated by the nonlinearity is expressed by equation (49). Note that it is also a linear function of the input interfering signal.

Adler's Equation with Variable Frequency Synchronizing Signals

Jones (11) applied Adler's equation to an oscillator synchronized by a frequency-modulated synchronizing signal. The differential equation of the oscillator with a sinusoidal frequency-modulated signal, using Adler's approach, is

$$\frac{d\beta}{dt} = M \sin (\omega_m t + \Omega) - \omega_c \sin \beta, \tag{50}$$

where M is the maximum variation of the input signal frequency from the free-running value and Ω is an arbitrary phase angle.

A solution for β is assumed as

$$\beta = \beta_1 \sin \omega_m t + \beta_2 \sin 3\omega_m t + \beta_3 \cos 3\omega_m t \ . \tag{51}$$

Solutions for the maximum angles β_1, β_2, and β_3 are accomplished by substitution into equation (50), yielding for the resulting frequency

$$\begin{aligned} \omega = \omega_o + \omega_c \Big[2J_1(\beta_1) + \beta_3 J_2(\beta_1)\Big] \sin \omega_m t \\ - \beta_3 J_2(\beta_1) \cos \omega_m t + \Big[2J_3(\beta_1) + \beta_3 J_o(\beta_1)\Big] \sin 3\omega_m t \\ + \beta_3 \Big[1/2\ \beta_3 J_3(\beta_1) - J_o(\beta_1)\Big] \cos 3\omega_m t \ . \end{aligned} \tag{52}$$

Other solutions are accomplished by Jones (11) which include the variation of ω_c resulting from variations in amplitude of the grid voltage caused by the nonlinearity of the open-loop response curve and also resulting from amplitude variations caused by frequency sensitive elements.

CHAPTER IV

EFFECTS ON THE OUTPUT OF A SYNCHRONIZED OSCILLATOR BY INTERFERING SIGNALS

The output of an oscillator is determined by the nonlinearity of the circuit, by the feedback network characteristic, and by the external signals applied. When the oscillator is synchronized, it is possible to obtain a high degree of discrimination against interfering signals accompanying the synchronizing signal. This discrimination, that is, proportionately smaller interfering signals in the output than in the input, may be appreciable even though the frequency of the interference is very close to that of the synchronizing frequency.

In solutions for oscillators, whether isolated or synchronized, a compromise usually must be made between completeness of solution and complexity of the solution. Completely general solutions to nonlinear circuits are in most cases quite complex, if not impossible. However, with restrictions which meet practical conditions, and simplifications which result from these restrictions, useful results may be obtained. In the analyses to follow, the signal to interference ratio is considered to be large.

Mathematical Development Based on Adler's Equation

Adler's (9) equation is dimensionally in terms of radian frequency. The interest here is in amplitudes, therefore, the analysis involves a conversion of the instantaneous frequency determined from an equation based on Adler to the resulting amplitudes.

Effect of Interference on Instantaneous Frequency of an Oscillator

An oscillator subjected to two input signals is represented in Figure 6.

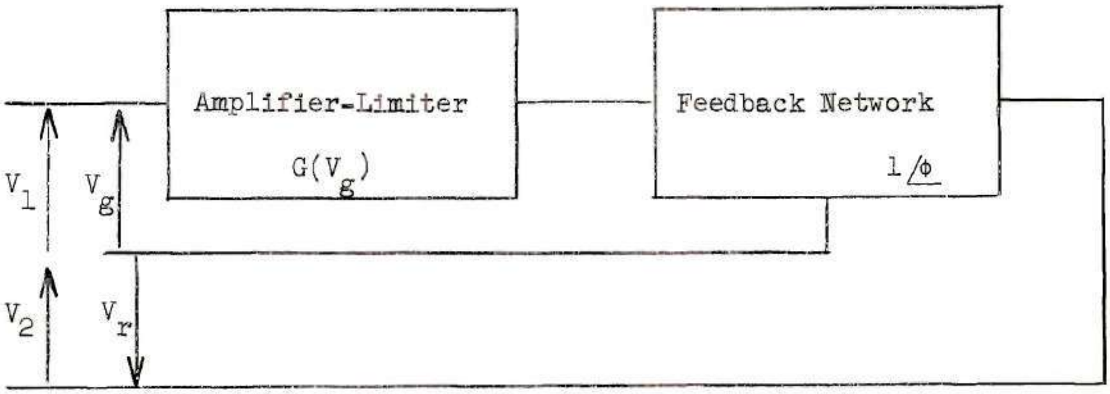


Figure 6. Block Diagram of an Oscillator With Two Externally Applied Signals.

From Figure 6 the following relations between voltages may be written:

$$V_r = V_g \; G(v_g) \; \angle\phi \; , \tag{53}$$

and

$$v_g = v_r + v_1 + v_2 \; , \tag{54}$$

where equation (54) is a relation between instantaneous voltages.

Let the voltage v_1 be the synchronizing voltage and let v_2 be the interfering voltage on the input.

The equations for v_1 and v_2 are taken as

$$v_1 = V_1 \cos \omega_1 t \; , \tag{55}$$

and

$$v_2 = V_2 \cos \omega_2 t = V_2 \cos (\omega_1 + \Delta_{21})t \ , \tag{56}$$

where Δ_{21} is the difference in radian frequency between ω_1 and ω_2,

$$\Delta_{21} = \omega_2 - \omega_1 \ . \tag{57}$$

A phasor diagram indicating magnitude and phase relations between the voltages is shown in Figure 7. If the phasor V_1 is considered to be stationary at a frequency ω_1, then the other phasors have frequencies as shown.

The frequency ω_i is the instantaneous frequency of v_1'.

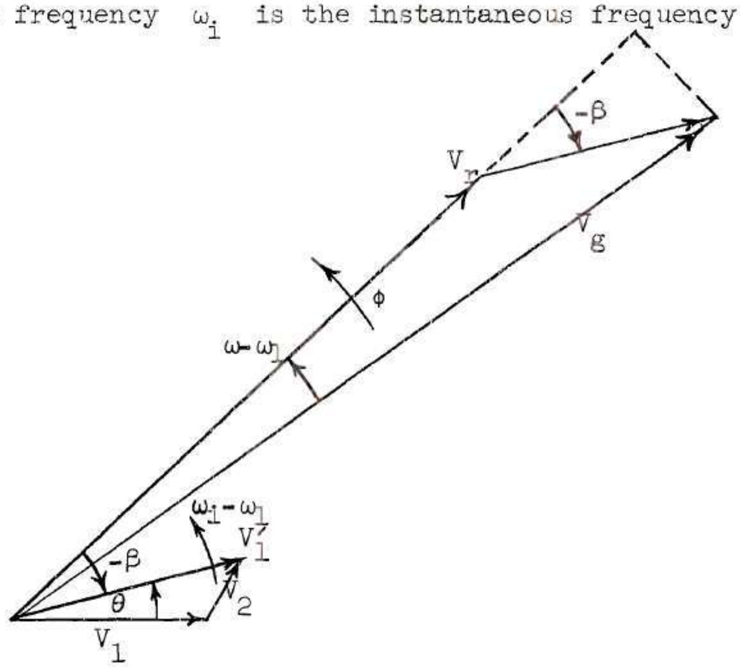


Figure 7. Phasor Diagram for an Oscillator with Synchronizing and Interfering Voltages.

The phasor V_1' is the phasor combination of V_1 and V_2. As an instantaneous value, the effective externally applied signal becomes

$$v_1' = V_1 \cos \omega_1 t + V_2 \cos (\omega_1 + \Delta_{21})t \ . \tag{58}$$

Expansion of the cosine function involving the sum, $(\omega_1 + \Delta_{21})t$, and combination of sine and cosine terms of $\omega_1 t$ yields

$$v_1' = V_1 \cos \omega_1 t \left(1 + \frac{V_2}{V_1} \cos \Delta_{21} t\right) - V_2 \sin \omega_1 t \sin \Delta_{21} t. \qquad (59)$$

This equation (59) may be written as

$$v_1' = V_1'(t) \cos (\omega_1 t + \theta(t)), \qquad (60)$$

where

$$V'_{1(t)} = \left[(V_1 + V_2 \cos \Delta_{21} t)^2 + (-V_2 \sin \Delta_{21} t)^2\right]^{\frac{1}{2}}, \qquad (61)$$

$$= \left[V_1^2 + V_2^2 + 2V_1V_2 \cos \Delta_{21} t\right]^{\frac{1}{2}},$$

and

$$\theta_{(t)} = \tan^{-1} \frac{V_2 \sin \Delta_{21} t}{V_1 + V_2 \cos \Delta_{21} t}. \qquad (62)$$

If the synchronizing voltage is much larger than the interfering voltage, $V_1^2 \gg V_2^2$, equations (61) and (62) reduce to

$$V'_{1(t)} = V_1 + V_2 \cos \Delta_{21} t, \qquad (63)$$

and

$$\theta(t) \doteq \frac{V_2}{V_1} \sin \Delta_{21} t. \qquad (64)$$

Equation (60) may now be written

$$v_1' = (V_1 + V_2 \cos \Delta_{21} t) \cos \left(\omega_1 t + \frac{V_2}{V_1} \sin \Delta_{21} t\right). \qquad (65)$$

Therefore, for interfering signals of relatively small magnitudes, the effect on the synchronizing voltage is a combination of amplitude and frequency modulation.

Referring again to the phasor diagram, Figure 7, the relations between rate of change of angles and frequencies are noted as

$$\frac{d\beta}{dt} = (\omega_i - \omega_1) - (\omega - \omega_1) \ , \tag{66}$$

and

$$\frac{d\theta}{dt} = (\omega_i - \omega_1) \ . \tag{67}$$

The instantaneous frequency ω can be found from equations (66) and (67) as

$$\omega = \omega_1 + \frac{d\theta}{dt} - \frac{d\beta}{dt} \ . \tag{68}$$

The rate of change of the angle β is

$$\frac{d\beta}{dt} = (\omega_1 - \omega) + \frac{d\theta}{dt} \ . \tag{69}$$

If $\frac{d\theta}{dt} = 0$ in equation (66), this reduces to the same relation as occurs in deriving Adler's equation.

The angle ϕ in Figure 7 between V_r and V_g represents the angle of lead of V_r with respect to V_g. If $\frac{\omega_0}{2Q} \gg |\omega_0 - \omega_1|$, that is, if one-half the bandwidth of the oscillator is much larger than the difference in the free-running and synchronizing frequency, then ϕ can be considered a linear function of frequency,

$$\phi = K_\phi(\omega - \omega_0) \ . \tag{70}$$

From Figure 7,

$$\sin\phi = -\frac{V_1'}{V_g}\sin(-\beta) \ . \tag{71}$$

For operation where the product $K_\phi(\omega - \omega_o)$ is small, then

$$\sin\phi \doteq \phi = \frac{V_1'}{V_g}\sin\beta \ . \tag{72}$$

The condition for K_ϕ to be small requires low Q of the feedback network. When the synchronizing frequency is limited to a range near the free running frequency, $|\omega_o - \omega|$ is small.

Substitution of equation (63) into (72) gives

$$\phi = \frac{(V_1 + V_2 \cos\Delta_{21}t)}{V_g}\sin\beta \ . \tag{73}$$

Equating equations (70) and (73) yields

$$K_\phi[(\omega - \omega_1) - (\omega_o - \omega_1)] = \frac{(V_1 + V_2\cos\Delta_{21}t)}{V_g}\sin\beta \ . \tag{74}$$

The combination of equations (64, 69) and (74) results in

$$\frac{d\beta}{dt} = \Delta_{10} - \frac{V_1}{K_\phi V_g}\left(1 + \frac{V_2}{V_1}\cos\Delta_{21}t\right)\sin\beta + \frac{\Delta_{21}V_2}{V_1}\cos\Delta_{21}t, \tag{75}$$

where

$$\Delta_{10} = \omega_1 - \omega_o \ . \tag{76}$$

For the condition of no interference, $V_2 = 0$, and $\frac{d\beta}{dt} = 0$, the oscillator is synchronized, with β having a constant value dependent on the circuit parameters and relative voltages that exist. Imposing the conditions $V_2 = 0$ and $\frac{d\beta}{dt} = 0$ on equation (75) reduces this equation to

$$\Delta_{10} - \frac{V_1}{K_\phi V_g} \sin \beta_o = 0 , \tag{77}$$

or

$$\sin \beta_o = \frac{\Delta_{10}}{\left(\frac{V_1}{K_\phi V_g}\right)} , \tag{78}$$

where β_o is the constant value of the angle, without interference, when synchronization occurs.

Since the limits of the sine function are ± 1, the limits of Δ_{10} to satisfy equation (77) becomes

$$\Delta'_{10} = \omega_c = \pm \frac{V_1}{K_\phi V_g} . \tag{79}$$

Therefore, the limits of the synchronization bandwidth are $\pm\omega_c$, or, ω_c is one-half the bandwidth of synchronization without interference.

Incorporating equation (79) in equation (75) gives

$$\frac{d\beta}{dt} = \Delta_{10} - \omega_c \left(1 + \frac{V_2}{V_1} \cos \Delta_{21} t\right) \sin \beta + \frac{\Delta_{21} V_2}{V_1} \cos \Delta_{21} t . \tag{80}$$

Equation (80) represents the variation with time of the angle β of an oscillator subjected to a relatively large signal plus an interfering signal of small amplitude. It is reasonable then to assume that β will consist of a constant value, corresponding to the synchronizing voltage, and a time-varying part resulting from the perturbation caused by

$V_2 \cos (\omega_1 + \Delta_{21})t$. Therefore β may be taken as the sum of β_o, the constant value, and β_1, the time varying component. Then,

$$\beta = \beta_o + \beta_1 \ . \tag{81}$$

With β defined as in equation (81), equation (80) yields

$$\frac{d\beta_o}{dt} + \frac{d\beta_1}{dt} = \Delta_{10} - \omega_c(1 + \frac{V_2}{V_1} \cos \Delta_{21}t) \sin (\beta + \beta_o) + \Delta_{21} \frac{V_2}{V_1} \cos \Delta_{21}t. \tag{82}$$

Expanding $\sin (\beta_o + \beta_1)$ by a trigonometric identity and noting that because of the assumption $V_2^2 \ll V_1^2$ and β_1 being a very small angle, there results

$$\sin (\beta_o + \beta_1) \doteq \sin \beta_o + \beta_1 \cos \beta \ . \tag{83}$$

Also, $\frac{d\beta_o}{dt} = 0$, since β_o is the constant part of the solution for β. Therefore,

$$\frac{d\beta_1}{dt} = \Delta_{10} - \omega_c(1 + \frac{V_2}{V_1} \cos \Delta_{21}t)(\sin \beta_o + \beta_1 \cos \beta_o) + \Delta_{21} \frac{V_2}{V_1} \cos \Delta_{21}t \ . \tag{84}$$

Since $\Delta_{10} - \omega_c \sin \beta_o = 0$ from equation (77),

$$\frac{d\beta_1}{dt} = -\omega_c\beta_1 \cos \beta_o(1 + \frac{V_2}{V_1} \cos \Delta_{21}t) - \omega_c \frac{V_2}{V_1} \sin \beta_o \cos \Delta_{21}t + \Delta_{21} \frac{V_2}{V_1} \cos \Delta_{21}t \ . \tag{85}$$

The resulting equation (85) for $\frac{d\beta_1}{dt}$ is a linear differential equation with varying coefficients. An exact solution for this equation may be found. However, from experimental results the voltages at the higher sum and difference terms in the output such as $\omega_1 \pm 2\Delta_{21}$, $\omega_1 \pm 3\Delta_{21}$, ... are negligible. The voltages at the sum and difference frequencies in the output, which are negligible, result from the higher harmonic terms in the solution of equation (85). Therefore to simplify the solution of equation (85), since the desired solution is the steady-state part and the higher harmonics are to be neglected, a solution is assumed as

$$\beta_1 = B_1 \cos \Delta_{21}t + B_2 \sin \Delta_{21}t \ . \tag{86}$$

The assumed solution has the same angular velocity as the forcing function of equation (85) because the steady-state solution is being sought. The constants in the assumed solution are evaluated by substitution of equation (86) into (85) and, using a method based on the principle of harmonic balance, these constants are adjusted to make the assumed solution fit the equation as well as possible. Terms of frequency other than the fundamental are neglected in this first order approximation. Both sine and cosine functions are included to account for possible phase shift.

The result of substituting equation (86) into (85) and equating like time functions of the fundamental frequency yields

$$\frac{B_1 \Delta_{21}}{\omega_c} = B_2 \cos \beta_o \ , \tag{87}$$

and

$$B_2 \frac{\Delta_{21}}{\omega_c} = -B_1 \cos \beta_o + \frac{\Delta_{21} V_2}{\omega_c V_1} - \frac{V_2}{V_1} \sin \beta_o .$$

Solutions for B_1 and B_2 are

$$B_1 = \frac{\cos \beta_o (\Delta_{21} - \omega_c \sin \beta_o)}{\Delta_{21}^2 + \omega_c^2 \cos^2 \beta_o} ,$$

and

$$B_2 = \Delta_{21} \frac{V_2}{V_1} \cdot \frac{(\Delta_{21} - \omega_c \sin \beta_o)}{\Delta_{21}^2 + \omega_c^2 \cos^2 \beta_o} , \tag{88}$$

Since

$$\sin \beta_o = \frac{\Delta_{10}}{\omega_c} , \tag{89}$$

and

$$\cos \beta_o = \frac{(\omega_c^2 - \Delta_{10}^2)^{\frac{1}{2}}}{\omega_c} ,$$

equation (88) becomes

$$B_1 = \frac{V_2}{V_1} \cdot \frac{(\Delta_{21} - \Delta_{10})(\omega_c^2 - \Delta_{10}^2)^{\frac{1}{2}}}{\Delta_{21}^2 + (\omega_c^2 - \Delta_{10}^2)} , \tag{90}$$

and

$$B_2 = \frac{V_2}{V_1} \cdot \frac{(\Delta_{21})(\Delta_{21} - \Delta_{10})}{\Delta_{21}^2 + (\omega_c^2 - \Delta_{10}^2)} .$$

Therefore,

$$\beta_1 = \frac{V_2}{V_1} \frac{(\Delta_{21} - \Delta_{10})\left[(\omega_c^2 - \Delta_{10}^2)^{\frac{1}{2}} \cos \Delta_{21}t + \Delta_{21} \sin \Delta_{21}t\right]}{\Delta_{21}^2 + \omega_c^2 - \Delta_{10}^2} . \tag{91}$$

The equation for ωt may now be found from the integration of

$$\omega = \omega_1 + \frac{d\theta}{dt} - \frac{d\beta_1}{dt} . \tag{92}$$

Thus,

$$\omega t = \omega_1 t + \theta - \beta_1 . \tag{93}$$

Substitution of equations (64) and (91) gives for ωt,

$$\omega t = \omega_1 t + \frac{V_2}{V_1} \sin \Delta_{21}t \left[1 - \frac{(\Delta_{21} - \Delta_{10})\Delta_{21}}{\Delta_{21}^2 + \omega_c^2 - \Delta_{10}^2}\right] \tag{94}$$

$$- \frac{V_2(\Delta_{21} - \Delta_{10})}{V_1(\Delta_{21}^2 + \omega_c^2 - \Delta_{10}^2)} (\omega_c^2 - \Delta_{10}^2)^{\frac{1}{2}} \cos \Delta_{12}t .$$

The instantaneous value for the voltage feedback is

$$v_r = V_r \cos \omega t . \tag{95}$$

Since ωt from equation (94) contains time-varying sinusoidal terms, the result, when substituted into equation (95), is sideband terms around ω_1 of the frequency-modulation type. The cosine function may be written in shortened form as

$$\cos \omega t = \cos (\omega_1 t + x \sin \Delta_{21}t - y \cos \Delta_{21}t) , \tag{96}$$

where

$$x = \frac{V_2}{V_1}\left[1 - \frac{(\Delta_{21} - \Delta_{10})\Delta_{21}}{\Delta_{21}^2 + \omega_c^2 - \Delta_{10}^2}\right] = r_{21}\left[1 - \frac{(\delta_{21} - \delta_{10})\delta_{21}}{1 + \delta_{21}^2 - \delta_{10}^2}\right], \quad (97)$$

and

$$y = \frac{V_2}{V_1}\left[\frac{(\Delta_{21} - \Delta_{10})(\omega_c^2 - \Delta_{10}^2)^{\frac{1}{2}}}{\Delta_{21}^2 + \omega_c^2 - \Delta_{10}^2}\right] = r_{21}\left[\frac{(\delta_{21} - \delta_{10})(1 - \delta_{10}^2)^{\frac{1}{2}}}{1 + \delta_{21}^2 - \delta_{10}^2}\right], \quad (98)$$

with

$$r_{21} = \frac{V_2}{V_1}, \quad \delta_{21} = \frac{\Delta_{21}}{\omega_c}, \quad \text{and} \quad \delta_{10} = \frac{\Delta_{10}}{\omega_c} . \quad (99)$$

The trigonometric expansion of equation (96) and the substitutions for functions that yield Bessel functions gives for $\cos\ \omega t$,

$$\begin{aligned} \cos \omega t = \cos \omega_1 t\ &[(J_o(x) + 2J_3(x) \cos \Delta_{21}t + \ldots)(J_o(y) \\ &- 2J_2(y) \cos 2\Delta_{21}t + \ldots) + (2J_1(x) \sin \Delta_{21}t \\ &+ 2J_3(x) \sin 3\Delta_{21}t + \ldots)(2J_1(y) \cos \Delta_{21}t - 2J_3(y) \\ &\cos 3\Delta_{21}t + \ldots)] - \sin \omega_1 t\ [(2J_1(x) \sin \Delta_{21}t \\ &+ 2J_3(x) \sin 3\Delta_{21}t + \ldots)(J_o(y) - 2J_2(y) \cos 2\Delta_{21}t + \ldots) \\ &- (J_o(x) + 2J_2(x) \cos 2\Delta_{21}t + \ldots)(2J_1(y) \cos \Delta_{21}t \\ &- 2J_3(y) \cos 3\Delta_{21}t + \ldots)] . \end{aligned} \quad (100)$$

Examination of the arguments, x and y, of the Bessel function indicate there are combinations of δ_{10} and δ_{21} which yield large values of the arguments. Graphs showing a family of curves for $\frac{x}{r_{21}}$ and $\frac{y}{r_{21}}$

for constant values of δ_{10}, as δ_{21} is varied, are shown in Figures 8 and 9.

Examination of the curves in Figures 8 and 9 indicate that the values of $\frac{x}{r_{21}}$ and $\frac{y}{r_{21}}$ are reasonably well behaved; that is, their magnitudes do not reach extremely large values until combinations of large values of $|\delta_{10}|$ and small values of $|\delta_{21}|$ occur at the same time. This indicates that the arguments of the Bessel functions reach large values when the synchronizing frequency is near the edge of the band of synchronization and the interfering signal is quite near the synchronizing frequency. This condition corresponds to the borderline of stability and is of little practical interest.

For values of $|\delta_{10}| \leqq 0.9$, $|x| < 1.0$ and $|y| < 2.2$, since the requirement on the development of the equations has been $r_{21}^2 \ll 1$, then the arguments of the Bessel functions will be small with the limitation on $|\delta_{10}|$. The Bessel function expansions in equation (99) may be truncated after the first term yielding,

$$\cos \omega t = \cos \omega_1 t \left[(J_o(x)\, J_o(y) + (2J_1(x) \sin \Delta_{21} t)(2J_1(y) \cos \Delta_{21} t)\right] - \sin \omega_1 t \left[(2J_1(x) \sin \Delta_{21} t(J_o(y)) - (J_o(x))(2J_1(y) \cos \Delta_{21} t)\right] . \tag{101}$$

Also the Bessel functions may be replaced by the first term in their series expansion giving for the first pair of sidebands,

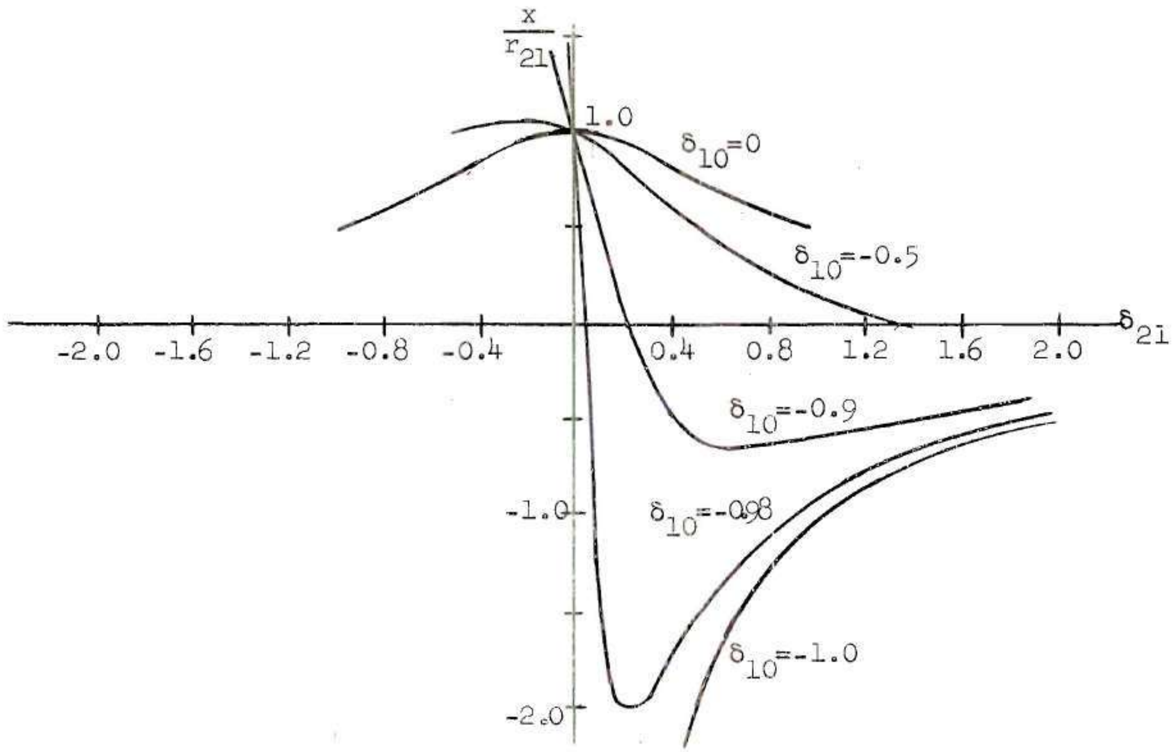


Figure 8. Variation of $\frac{x}{r_{21}}$ with δ_{21}.

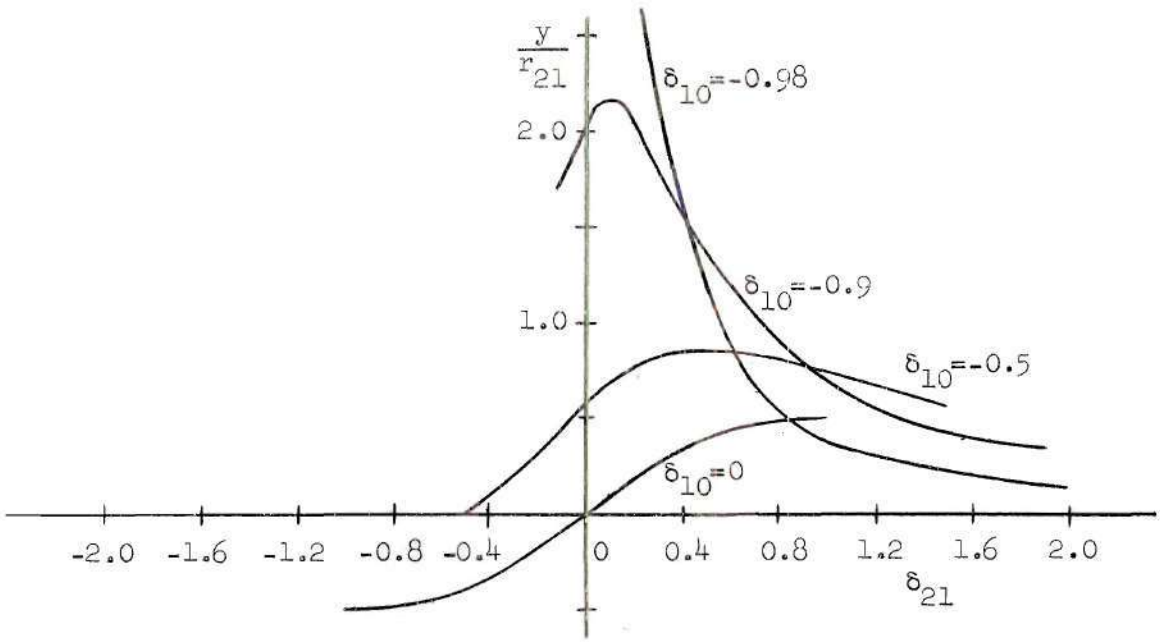


Figure 9. Variation of $\frac{y}{r_{21}}$ with δ_{21}.

$$\cos \omega t = \cos \omega_1 t - \sin \omega_1 t \left[2 \frac{x}{2!} \sin \Delta_{21} t - 2 \frac{y}{2!} \cos \Delta_{21} t\right] \quad (102)$$

$$= \cos \omega_1 t - \frac{x}{2} \left[\cos (\omega_1 - \Delta_{21})t - \cos (\omega_1 + \Delta_{21})t\right]$$

$$+ \frac{y}{2} \left[\sin (\omega_1 + \Delta_{21})t + \sin (\omega_1 - \Delta_{21})t\right] .$$

From equation (101) it can be seen that the interfering signal has caused the equivalent of narrow-band frequency modulation. The most important assumption is the restriction on the interfering to signal ratio, that is, $\left[\frac{V_2}{V_1}\right]^2 << 1$. For $0.9 \leqq |\delta_{10}| \leqq 1$, the approximations used for the Bessel functions are not valid and calculation in this region would require retention of the Bessel functions. However, cross-products between terms as in equation (100) would contribute to the lower order sidebands and an analysis in this interval of $|\delta_{10}|$ would become impractically lengthy.

Effect on the Amplitude of an Oscillator Resulting from Interference

Adler stipulated in his derivation that the amplitude variation of V_g and V_r resulting from V_1 was small in comparison to V_1. This was due to the application of a small external signal compared to V_g and to the operation of most oscillators in almost flat part of the amplitude-limiting characteristic. Neglecting the amplitude modulation entirely results in an equation for v_r as follows:

$$v_r = V_r\left[\cos \omega_1 t - \frac{x}{2} (\cos (\omega_1 - \Delta_{21})t - \cos (\omega_1 + \Delta_{21})t) \right. \quad (103)$$

$$\left. + \frac{y}{2} (\sin (\omega_1 + \Delta_{21})t + \sin (\omega_1 - \Delta_{21})t)\right] .$$

The voltage feedback V_r is taken as the free-running grid voltage

V_o, in equation (103).

In order for the analysis to be complete, even though the contribution is small, some account should be made of the amplitude variation. Since the amplitude variation of v_r is small because of the inequality $V_2 \ll V_g$, a linear approximation to the open-loop gain curve may be used.

A typical curve for an oscillator that satisfies Adler's conditions has a relatively sharp limiting characteristic and is as shown in Figure 10.

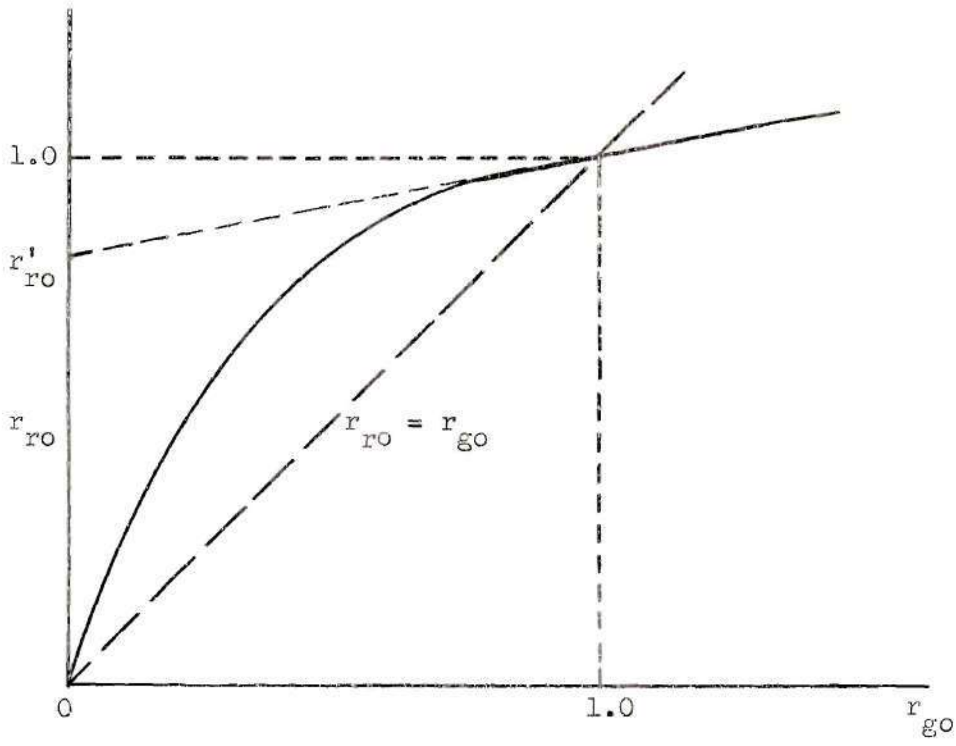


Figure 10. Open-loop Amplitude Characteristic of an Oscillator (Normalized with respect to the free-running voltage amplitude, V_o).

A tangent drawn to the curve at $r_{go} = 1.0$, will have an equation

$$r_{ro} = r'_{ro} + (1 - r'_{ro})r_{go} \tag{104}$$

$$= r'_{ro}\,(1 - r_{go}) + r_{go}\ .$$

Solving for r_{go} gives

$$r_{go} = \frac{r_{ro} - r'_{ro}}{1 - r'_{ro}} \quad . \tag{105}$$

From the phasor diagram, Figure 7,

$$V_g \doteq V_r + V'_1 \cos \beta = V_r + V'_1 \cos (\beta_o + \beta_1) \ . \tag{106}$$

Expanding the sum of angles by trigonometric identity; imposing the condition that β is small; and substituting for V'_1 from equation (63) gives for equation (106),

$$V_g = V_r + (V_1 + V_2 \cos \Delta_{21} t)(\cos \beta_o - \beta_1 \sin \beta_o) \quad , \tag{107}$$

or normalizing with respect to V_o and substituting for functions of β_o,

$$r_{go} = r_{ro} + (r_{10} + r_{20} \cos \Delta_{21} t)([1 - \delta_{10}^2]^{\frac{1}{2}} - \beta_1 \delta_{10}) \ . \tag{108}$$

From equations (105, 108)

$$r_{ro} = 1 + \frac{1 - r'_{ro}}{r'_{ro}} \ (r_{10} + r_{20} \cos \Delta_{21} t)([1 - \delta_{10}^2]^{\frac{1}{2}} - \beta_1 \delta_{10}), \tag{109}$$

where β_1 is the value stated in equation (91). Incorporating β_1 in equation (109) and neglecting higher harmonic terms yields

$$r_{ro} = 1 + \frac{1 - r'_{ro}}{r'_{ro}} \cdot (1 - \delta_{10}^2)^{\frac{1}{2}} \left[r_{10} - \frac{r_{20} \delta_{10} r_{21} (\delta_{21} - \delta_{10})}{2(1 + \delta_{21}^2 - \delta_{10}^2)} \right] \tag{110}$$

$$+ \left[r_{20} - \frac{r_{10}\delta_{10}r_{21}(\delta_{21} - \delta_{10})}{1 + \delta_{21}^2 - \delta_{10}^2} \right] \cos \Delta_{21} t$$

$$- \left[\frac{r_{10}\delta_{10}r_{21}(\delta_{21} - \delta_{10})\delta_{21}}{1 + \delta_{21}^2 - \delta_{10}^2} \right] \sin \Delta_{21} t \quad .$$

Effect on the Output Including Both Amplitude and Frequency Effects of Interference

The effects on the output due to variation of frequency is expressed by equation (103). The effects resulting from the amplitude variations caused by the amplitude limiting characteristic are expressed by equation (110). The product of these two equations (103, 110) yields the total effect of interference on the output of an oscillator that meets the restrictions set forth. The combination and solution of these two equations are included in the Appendix A. The final equations are normalized with respect to V_o and terms are retained through $\omega_1 \pm 2\Delta_{21}$. The individual terms for $\frac{V_r}{V_o}$ are as follows:

$$\frac{V_c(\omega_1)}{V_o} \cos \omega_1 t = \left\{ 1 + \left(\frac{1 - r'_{ro}}{r'_{ro}}\right) \left[(1 - \delta_{10}^2)^{\frac{1}{2}} \left(r_{10} - \frac{r_{20}\delta_{10}r_{21}[\delta_{21} - \delta_{10}]}{2D_1} \right) \right] \right\} \cos \omega_1 t, \tag{111}$$

where

$$D_1 = 1 + \delta_{21}^2 - \delta_{10}^2 , \tag{112}$$

$$\frac{V_s(\omega_1)}{V_o}\sin\omega_1 t = \left\{\frac{1}{2}\left(\frac{1-r'_{ro}}{r'_{ro}}\right)\frac{r_{21}}{D_1}\left[\delta_{21}-\delta_{10}\right]\left[r_{10}\delta_{10}\delta_{21}\left(1-\frac{\delta_{21}\left[\delta_{21}-\delta_{10}\right]}{D_1}\right)+\left(r_{20}-\frac{r_{10}\delta_{10}r_{21}\left[\delta_{21}-\delta_{10}\right]}{D_1}\right)(1-\delta_{10}^2)\right]\right\}\sin\omega_1 t\,, \tag{113}$$

$$\frac{V_c(\omega_1+\Delta_{21})}{V_o}\cos(\omega_1+\Delta_{21})t = \frac{1}{2}\left[\left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\left(r_{20}-\frac{r_{10}\delta_{10}r_{21}\left[\delta_{21}-\delta_{10}\right]}{D_1}\right)+r_{21}\left(1-\frac{\left[\delta_{21}-\delta_{10}\right]\delta_{21}}{D_1}\right)\frac{V_c(\omega_1)}{V_o}\right]\cos(\omega_1+\Delta_{21})t\,, \tag{114}$$

$$\frac{V_s(\omega_1+\Delta_{21})}{V_o}\sin(\omega_1+\Delta_{21})t = \frac{1}{2}\left[\frac{r_{21}\left[\delta_{21}-\delta_{10}\right]}{D_1}\left(-\left[\frac{1-r'_{ro}}{r'_{ro}}\right]r_{10}\delta_{10}\delta_{21}+(1-\delta_{10}^2)^{\frac{1}{2}}\frac{V_c}{V_o}(\omega_1)\right)\right]\sin(\omega_1+\Delta_{21})t\,, \tag{115}$$

$$\frac{V_c(\omega_1-\Delta_{21})}{V_o}\cos(\omega_1-\Delta_{21})t = \frac{1}{2}\left[\left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\left(r_{20}-\frac{r_{10}\,\delta_{10}r_2\left[\delta_{21}-\delta_{10}\right]}{D_1}\right)-r_{21}\left(1-\frac{\left[\delta_{21}-\delta_{10}\right]\delta_{21}}{D_1}\right)\frac{V_c(\omega_1)}{V_o}\right]\cos(\omega_1-\Delta_{21})t\,, \tag{116}$$

$$\frac{V_s(\omega_1 - \Delta_{21})}{V_o} \sin(\omega_1 - \Delta_{21})t = \frac{1}{2}\left[\frac{r_{21}[\delta_{21}-\delta_{10}]}{D_1}(r_{10}\delta_{10}\delta_{21}\right. \tag{117}$$

$$\left.\left[\frac{1-r'_{ro}}{r'_{ro}}\right] + (1-\delta_{10}^2)^{\frac{1}{2}} \frac{V_c(\omega_1)}{V_o})\right] \sin(\omega_1-\Delta_{21})t \;,$$

$$\frac{V_c(\omega_1 + 2\Delta_{21})}{V_o} \cos(\omega_1 + 2\Delta_{21})t = \frac{1}{4}\left[\frac{1-r'_{ro}}{r'_{ro}}\right][1-\delta_{10}^2)^{\frac{1}{2}} r_{21}] \tag{118}$$

$$(r_{20} - \frac{r_{10}\delta_{10}[\delta_{21}-\delta_{10}]}{D_1} r_{21})\;(1 - \frac{\delta_{21}[\delta_{21}-\delta_{10}]}{D_1}$$

$$+ \frac{r_{10}\delta_{10}r_{21}[\delta_{21}-\delta_{10}]^2}{D_1^2}) \quad \cos(\omega_1 + 2\Delta_{21})t \;,$$

$$\frac{V_s(\omega_1 + 2\Delta_{21})}{V_o} \sin(\omega_1 + 2\Delta_{21})t = \frac{1}{4}\left\{\left[\frac{(1-r'_{ro})\; r_{21}[\delta_{21}-\delta_{10}]}{r'_{ro}D_1}\right]\right. \tag{119}$$

$$\cdot\left[-r_{10}\delta_{10}\delta_{21}r_{21}\;(1 - \frac{\delta_{21}[\delta_{21}-\delta_{10}]}{D_1}) + (1-\delta_{10}^2)(r_{20} -\right.$$

$$\left.\left.\frac{r_{10}\delta_{10}r_{21}[\delta_{21}-\delta_{10}]}{D_1}\right]\right\} \sin(\omega_1 + 2\Delta_{21})t \;,$$

$$V_c(\omega_1 - 2\Delta_{21}) \cos(\omega_1-2\Delta_{21})t = -\, V_s(\omega_1+2\Delta_{21}) \cos(\omega_1-2\Delta_{21})t \;, \tag{120}$$

and

$$V_s(\omega_1-2\Delta_{21}) \sin(\omega_1-2\Delta_{21})t = V_s(\omega_1+2\Delta_{21}) \sin(\omega_1-2\Delta_{21})t \;. \tag{121}$$

These equations (111-121) reflect both the frequency-modulation effect and the amplitude-modulation effect on the output due to the interfering signal. The summation of these equations gives $\frac{v_r}{V_o}$. As would be expected the effect due to the slope of the amplitude-limiting characteristic has a minor effect when the limiting characteristic is flat or near flat, which is required for reasonable amplitude stability. The inclusion of the amplitude-modulation effect does reflect the unequal magnitudes between the interference term in the output at the frequency of the input interference and the interference image, that is, at $\omega_1 - \Delta_{21}$. Without the amplitude modulation effect these interference terms would be predicted to have equal amplitudes (equation 103). Computer calculations based on these equations indicate the $\omega_1 \pm 2\Delta_{21}$ terms are negligible compared to the $\omega_1 \pm \Delta_{21}$ terms. This is expected since their generation is entirely due to the amplitude-modulation effect which has comparatively minor overall effect. The sine function at ω_1 also proved to be negligible compared to the cosine function. Its value was never greater than 0.06 per cent of the cosine function amplitude for the calculated values.

Curves showing the discrimination to the $\omega_1 + \Delta_{21}$ term are graphed in Figure 11. The discrimination is defined as the ratio of the gain of the desired signal to the gain of the interfering signal. These curves show the discrimination as δ_{20} is varied. Each curve is for a constant value of δ_{10} and r'_{ro}. The reason that all the curves do not have a common intersection on the y-axis is due to a difference in output at ω_1 for the different conditions. The minimum discrimination occurs at the location of the synchronizing frequency in each case. In an analysis where the frequency selectivity is important these minimums are shifted toward $\delta_{10} = 0$. Also of interest for the $\delta_{10} = -0.9$ curves is the existence of a negative discrimination or gain for the interfering

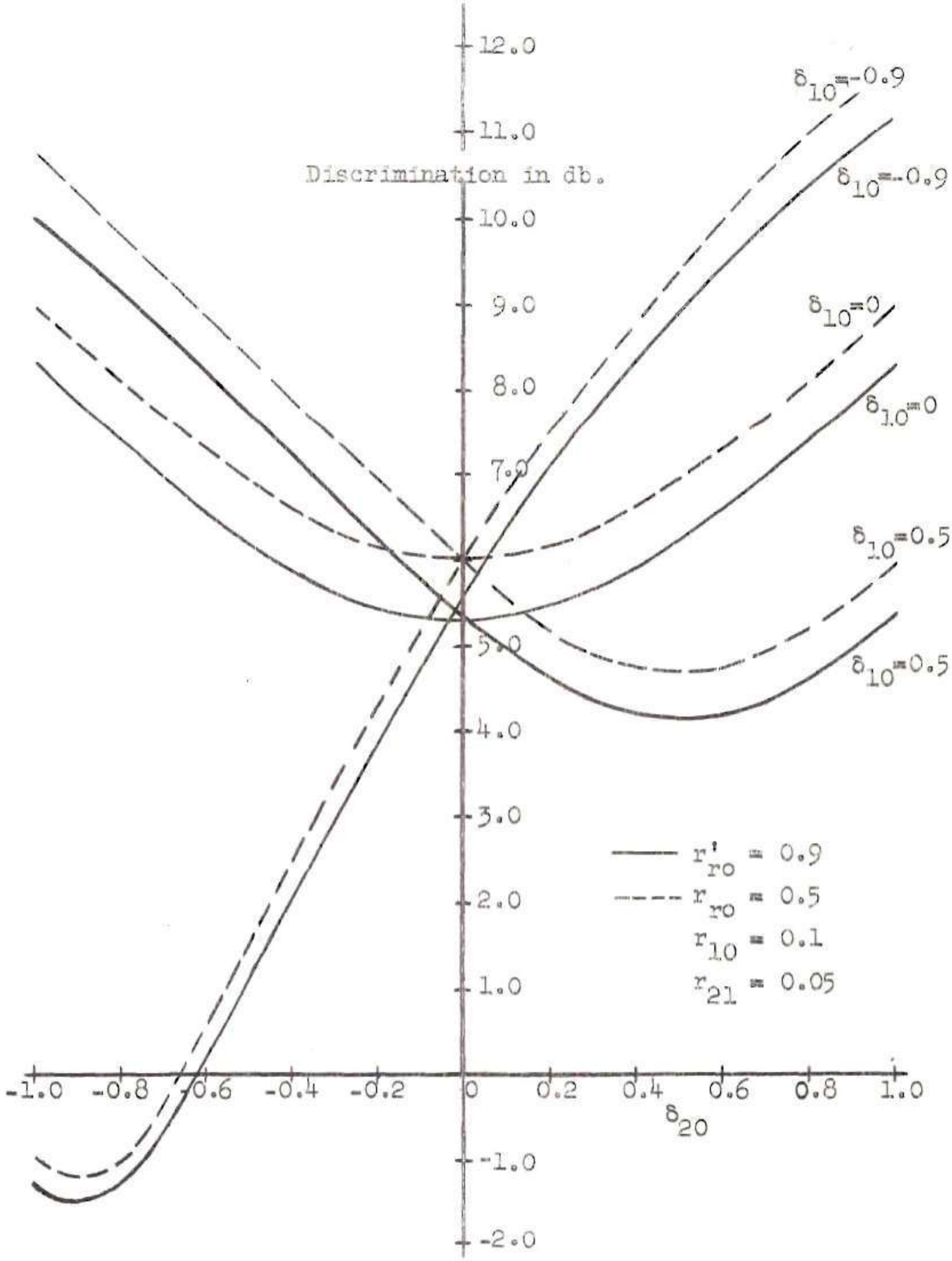


Figure 11. Discrimination to the Interfering Signal.

signal of between one and two decibels. Curves having the same magnitude for δ_{10} and r'_{ro} but of opposite sign in δ_{10} are mirror images about the y-axis. The change in r'_{ro} causes an approximately vertical shift in the discrimination curves, with the greater discrimination occurring for the larger value of r'_{ro}, reflecting less effect due to amplitude modulation.

Figure 12 shows the ratio of the interference in the output at the input interfering frequency to the interference in the output at the frequency $\omega_1 - \Delta_{21}$ as δ_{10} is varied. For the value $r'_{ro} = 0.9$ there is very little difference in the two sideband terms and the difference is practically unaffected by the location of the synchronizing frequency. The inequality is greater for the lower values of r'_{ro} and has a more pronounced change with the synchronizing frequency.

The interference at $\omega_1 + \Delta_{21}$ and at $\omega_1 - \Delta_{21}$ as δ_{20} is varied is shown in Figure 13 for constant values of δ_{10} and r'_{ro}. These curves point out the difference in amplitudes of the two interference components and also the location of each with respect to the normalized frequency. The curves for $\omega_1 - \Delta_{12}$ components are approximately the same as the curves for $\omega_1 + \Delta_{12}$ components shifted downward and rotated about the constant value of $\delta_{21} = \delta_{10}$.

The effect of the variation of the synchronizing frequency on the interference output amplitude is shown by the curves of Figure 14. If the interference is located at $\delta_{20} = 0$, there is little effect as δ_{10} is varied. However, as the interference moves away from ω_o, the magnitude increases when δ_{10} approaches δ_{20} and may reach a value where the output ratio of signal to interference is greater than the input signal to interference ratio.

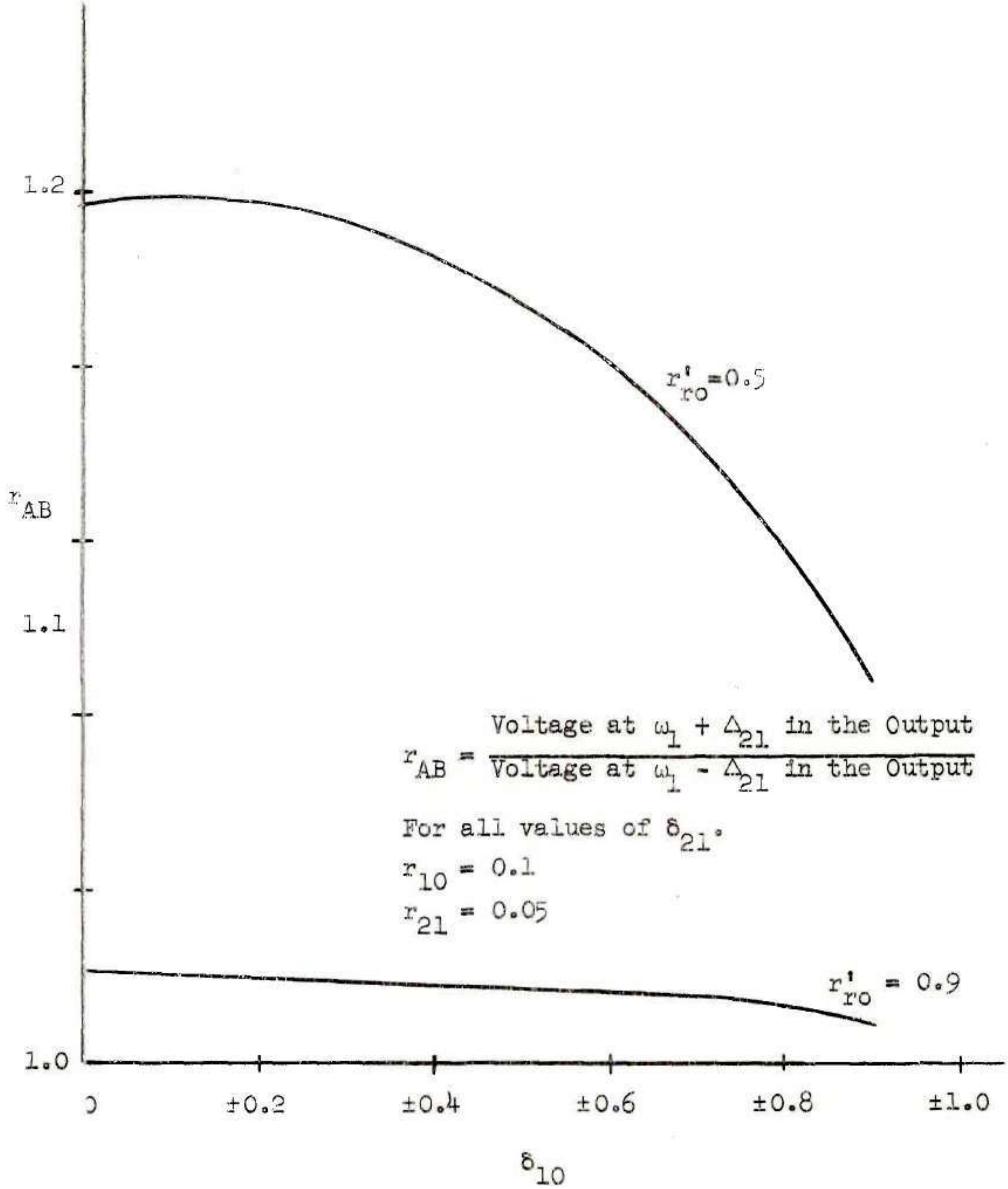


Figure 12. Ratio in the Output of the Amplitude at $\omega_1 + \Delta_{12}$ to the Amplitude at $\omega_1 - \Delta_{12}$.

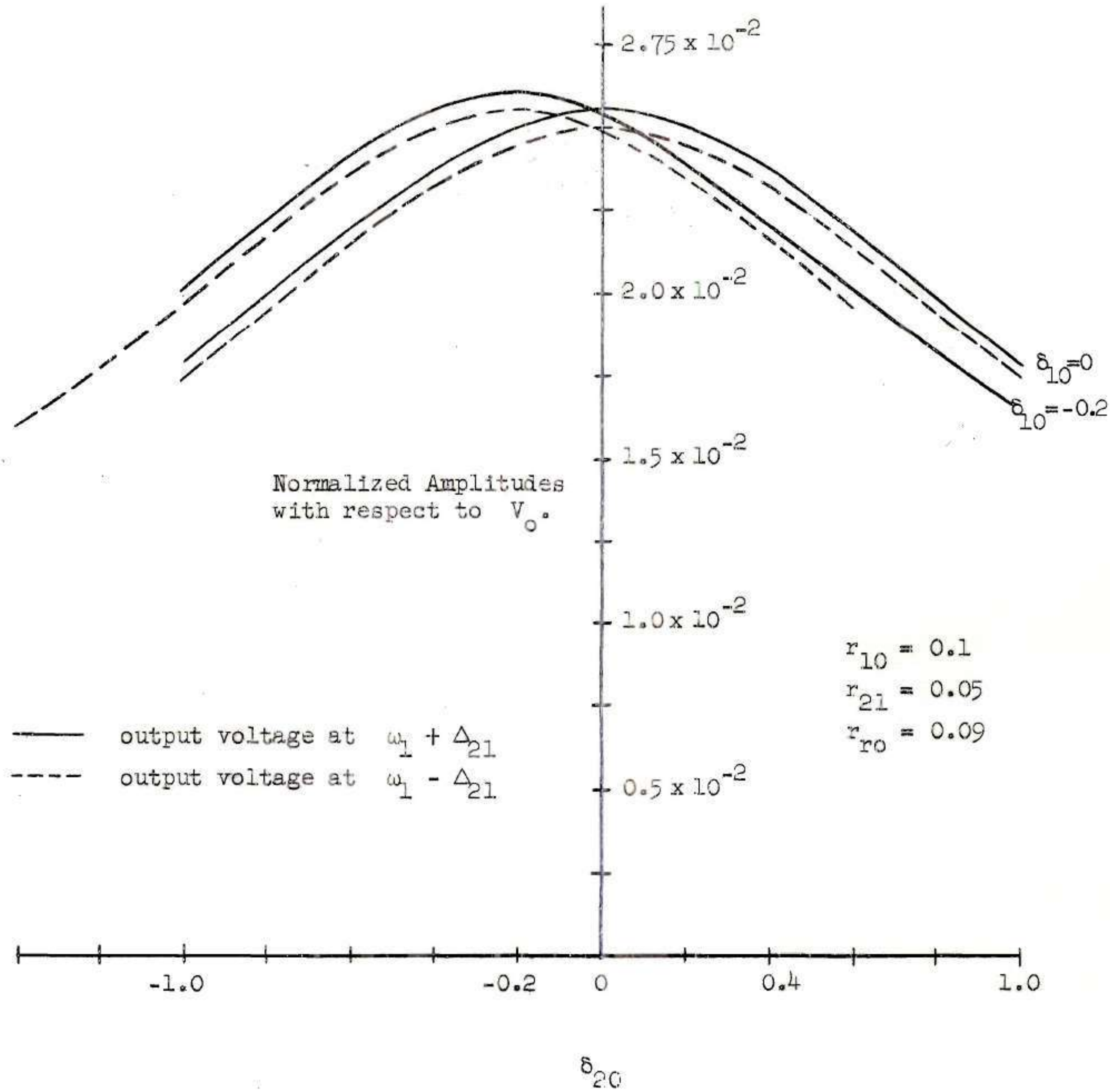


Figure 13. Normalized Amplitudes at $\omega_1 + \Delta_{21}$ and at $\omega_1 - \Delta_{21}$.

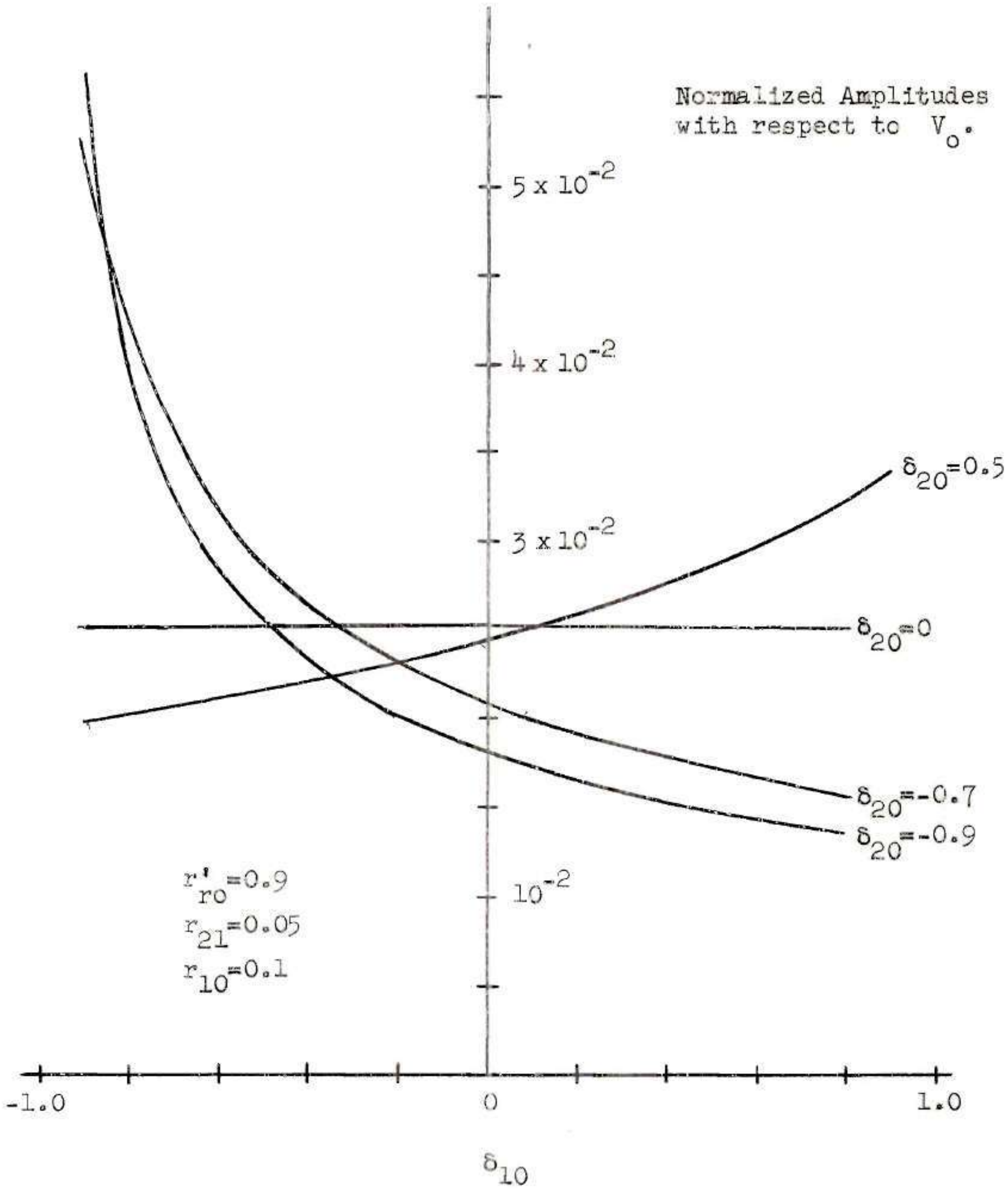


Figure 14. Variation of Interference Amplitude with Synchronizing Frequency.

CHAPTER V

EXTENSION OF EQUATIONS TO MULTIPLE SIGNALS AND NOISE

Multiple Signals Applied to Adler's Equations

In the preceding chapter equations were derived for determination of the output when the input consisted of a synchronizing signal and one interfering signal. When multiple signals are applied the procedure is similar but the restrictions placed on magnitudes must be redefined.

Let the externally applied voltage to an oscillator of the type in Figure 6 be

$$v_1' = V_1 \cos \omega_1 t + V_2 \cos (\omega_1 + \Delta_{21})t + V_3 \cos (\omega_1 + \Delta_{31})t + \ldots, \quad (122)$$

which may be written

$$v_1' = \sum_{n=1}^{m} V_n \cos \Delta_{n1} t \quad \cos \omega_1 t - \sum_{n=2}^{m} V_n \sin \Delta_{n1} t \quad \sin \omega_1 t, \quad (123)$$

in which $\Delta_{11} = 0$.

Equation (123) may be shown as a phasor diagram with dots designating the ends of the phasors representing the interference terms. Such a diagram is Figure 15.

The circle centered at the end of V_1 represents the maximum length of the phasor summation of the interfering terms. If the square of perpendicular component, $(-V_2 \sin \Delta_{21} t - V_3 \sin \Delta_{31} t - \ldots)$, is much less than the horizontal component then the resultant amplitude of V_1

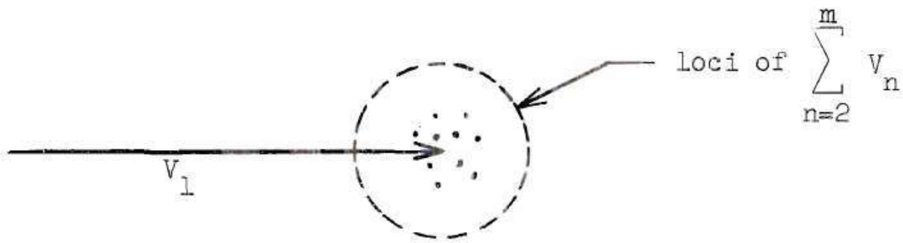


Figure 15. Phasor Diagram of Synchronizing Signal and Multiple Interfering Signals.

and the interfering terms may be taken as

$$V_1 \doteq V_1 + \sum_{n=2}^{m} V_n \cos \Delta_{n1} t \ , \tag{124}$$

and the angle $\theta(t)$ can be approximated

$$\theta(t) \doteq \sum_{n=2}^{m} \frac{V_n}{V_1} \sin \Delta_{n1} t \ . \tag{125}$$

Stated analytically the restriction discussed and now imposed is

$$\left[\frac{\sum_{n=2}^{m} V_n}{V_1}\right]^2 = \left(\sum_{n=2}^{m} r_{n1}\right)^2 \ll 1 \ . \tag{126}$$

Therefore

$$v_1 = (V_1 + \sum_{n=2}^{m} V_n \cos \Delta_{n1} t) \cos (\omega_1 t + \sum_{n=2}^{m} r_{n1} \sin \Delta_{n1} t) \ . \tag{127}$$

The angle ϕ between the phasors V_r and V_g (see Figure 7) becomes

$$\phi = \frac{V_1 + \sum_{n=2}^{m} V_n \cos \Delta_{n1} t}{V_g} \sin \beta \,. \tag{128}$$

Again, as for the single discrete interfering signal ϕ is taken as a linear relation of frequency, $\phi = K_\phi (\omega - \omega_o)$. The rate of change of β is $\frac{d\beta}{dt} = (\omega_1 - \omega) + \frac{d\theta}{dt}$.

By use of equations (125, 128),

$$\frac{d\beta}{dt} = \Delta_{10} - \omega_c (1 + \sum_{n=2}^{m} r_{n1} \cos \Delta_{n1} t) \sin \beta + \sum_{n=2}^{m} r_{n1} \Delta_{n1} \cos \Delta_{n1} t. \tag{129}$$

Since the oscillator is assumed synchronized, the perturbation caused by the interfering signals will be about a constant angle β_o. Therefore, $\beta = \beta_o + \beta_1$, where β_1 is the small angle resulting from the perturbations. Replacing β by $\beta_o + \beta_1$ in equation (129) and recalling that $\frac{d\beta_o}{dt} = 0$; that $\Delta_{10} - \omega_c \sin \beta_o = 0$; and that since β_1 is a very small angle $\sin (\beta_o + \beta_1) = \sin \beta_o + \beta_1 \cos \beta_o$, equation (129) may be reduced to

$$\frac{d\beta_1}{dt} = - \omega_c \cos \beta_o \, (1 + \sum_{n=2}^{m} r_{n1} \cos \Delta_{n1} t) \beta_1 \tag{130}$$

$$+ (\sum_{n=2}^{m} r_{n1} \cos \Delta_{n1} t)(-\omega_c \sin \beta_o + \sum_{n=2}^{m} \Delta_{n1}) \,.$$

A solution is now assumed for β_1 containing only terms of fundamental frequency as

$$\beta_1 = \sum_{n=2}^{m} \left[B_{2n-1} \cos \Delta_{n1} t + B_{2n} \sin \Delta_{n1} t \right]. \tag{131}$$

Substitution of β_1 from equation (131) into (130) and evaluation of the constants for this order of approximation from only the fundamental terms gives for the equations needed to determine the B_s' ,

$$\sum_{n=2}^{m} B_{2n}\Delta_{n1} = -\omega_c \cos \beta_o \left(\sum_{n=2}^{m} B_{2n-1} \right) + \left(\sum_{n=2}^{m} \Delta_{n1} - \omega_c \sin \beta_o \right) \left(\sum_{n=2}^{m} r_{n1} \right), \tag{132}$$

and

$$-\sum_{n=2}^{m} B_{2n-1}\Delta_{n1} = -\omega_c \cos \beta_o \left(\sum_{n=2}^{m} B_{2n} \right) . \tag{133}$$

Solution of these two equations (132, 133) yields,

$$B_{2n-1} = \frac{r_{n1}(\delta_{n1} - \delta_{10})(1 - \delta_{10}^2)^{\frac{1}{2}}}{1 + \delta_{n1}^2 - \delta_{10}^2} , \tag{134}$$

and

$$B_{2n} = \frac{r_{n1}(\delta_{n1})(\delta_{n1} - \delta_{10})}{1 + \delta_{n1}^2 - \delta_{10}^2} , \tag{135}$$

where $\delta_{n1} = \frac{\Delta_{n1}}{\omega_c}$ and n is an integer larger than 1, $n = 2, 3, \ldots$

Therefore,

$$\beta_1 = \sum_{n=2}^{m} \frac{r_{n1}(\delta_{n1}-\delta_{10})}{1+\delta_{n1}^2-\delta_{10}^2}\left[(1-\delta_{10}^2)^{\frac{1}{2}}\cos\Delta_{n1}t + \delta_{n1}\sin\Delta_{n1}t\right] . \quad (136)$$

The radian frequency of V_r is ω and $\omega = \omega_1 + \frac{d\theta}{dt} - \frac{d\beta_1}{dt}$. The integral of ω with respect to time yields $\omega t = \omega_1 t + \theta - \omega_1$.

The equation for the voltage v_r returned from the feedback network is expressed as $v_r = V_r \cos \omega t$. Examination of cos ωt with the substitutions for θ and β_1 gives,

$$\cos \omega t = \cos\left\{\omega_1 t + \sum_{n=2}^{m}\left[r_{n1}\sin\Delta_{n1}t - \frac{r_{n1}(\delta_{n1}-\delta_{10})}{1+\delta_{n1}^2-\delta_{10}^2}\left(\left[1-\delta_{10}^2\right]^{\frac{1}{2}}\cos\Delta_{n1}t + \delta_{n1}\sin\Delta_{n1}t\right)\right]\right\} . \quad (137)$$

Equation (137) may be written,

$$\cos \omega t = \cos\left(\omega_1 t + \sum_{n=2}^{m}\left[x_n \sin\Delta_{n1}t - y_n\cos\Delta_{n1}t\right]\right) , \quad (138)$$

where

$$x_n = r_{n1}\left[1 - \frac{(\delta_{n1}-\delta_{10})\delta_{n1}}{1+\delta_{n1}^2-\delta_{10}^2}\right] , \quad (139)$$

and

$$y_n = r_{n1}\frac{(\delta_{n1}-\delta_{10})(1-\delta_{10}^2)^{\frac{1}{2}}}{1+\delta_{n1}^2-\delta_{10}^2} . \quad (140)$$

For $(\sum_{n=2}^{m} r_{n1})^2 \ll 1$ and $|\delta_{10}| \leq 0.9$ this equation (138) for

$\cos\ \omega t$ may be reduced as shown in Appendix B to

$$\cos \omega t = \cos \omega_1 t + \sum_{n=2}^{m} \left\{ \frac{x_n}{2} \left[\cos(\omega_1 + \Delta_{n1})t - \cos(\omega_1 - \Delta_{n1})t \right] \right\} \tag{141}$$

$$+ \sum_{n=2}^{m} \left\{ \frac{y_n}{2} \left[\sin(\omega_1 + \Delta_{n1})t + \sin(\omega_1 - \Delta_{n1})t \right] \right\} .$$

The effect of the amplitude-limiting characteristic may be shown (Appendix B) to be

$$r_{ro} = 1 + \left[\left(\frac{1 - r'_{ro}}{r'_{ro}} \right) r_{10} (1 - \delta_{10}^2)^{\frac{1}{2}} \right] \left[1 - \sum_{n=2}^{m} \frac{\delta_{10} r_{n1} z_n}{2} \right] \tag{142}$$

$$+ \left(\frac{1 - r'_{ro}}{r'_{ro}} \right) (1 - \delta_{10}^2)^{\frac{1}{2}} \left(\sum_{n=2}^{m} \left[r_{no} - r_{10} \delta_{10} r_{n1} z_n \right] \cos \Delta_{n1} t \right)$$

$$- \left(\frac{1 - r'_{ro}}{r'_{ro}} \right) r_{10} \delta_{10} \left(\sum_{n=2}^{m} r_{n1} z_n \delta_{n1} \sin \Delta_{n1} t \right) ,$$

where

$$z_n = \frac{\delta_{n1} - \delta_{10}}{D_n} \quad \text{and} \quad D_n = 1 + \delta_{n1}^2 - \delta_{10}^2 . \tag{143}$$

The product of equation (141) and (143) to obtain $\frac{v_r}{V_o}$ is accomplished in Appendix B, giving as the individual terms of $\frac{v_r}{V_o}$,

$$\frac{V_c(\omega_1)}{V_o} \cos \omega_1 t = \left\{ 1 + \left[\left(\frac{1 - r'_{ro}}{r'_{ro}} \right) (1 - \delta_{10}^2)^{\frac{1}{2}} \right] \left[r_{10} \right. \right. \tag{144}$$

$$\left. \left. - \sum_{n=2}^{m} \frac{r_{no} \delta_{10} r_{n1} (\delta_{n1} - \delta_{10})}{D_n} \right] \right\} \cos \omega_1 t ,$$

$$\frac{V_s(\omega_1)}{V_o}\sin\omega_1 t = \frac{1}{2}\sum_{n=2}^{m}\left\{\left(\left[\frac{1-r'_{ro}}{r'_{ro}}\right]\frac{(\delta_{n1}-\delta_{10})r_{n1}}{D_n}\right)\right. \tag{145}$$

$$\left(\left[1-\delta_{10}^2\right]\left[r_{no}-\frac{r_{10}\delta_{10}r_{n1}(\delta_{n1}-\delta_{10})}{D_n}\right]\right.$$

$$\left.\left.+\,r_{10}\delta_{10}\frac{(\delta_{n1}-\delta_{10})\delta_{n1}r_{n1}}{D_n}\right)\right\}\sin\omega_1 t\ ,$$

$$\frac{V_c(\omega_1+\Delta_{n1})}{V_o}\cos(\omega_1+\Delta_{n1})t = \frac{1}{2}\left\{\left(\left(r_{n1}\left[1-\frac{(\delta_{n1}-\delta_{10})\delta_{n1}}{D_n}\right]\right)\right.\right. \tag{146}$$

$$\left(\frac{V_c(\omega_1)}{V_o}\right)+\left(\frac{1-r'_{ro}}{r'_{ro}}\right)\left(1-\delta_{10}^2\right)^{\frac{1}{2}}(r_{no}$$

$$\left.\left.-\,\frac{r_{10}\delta_{10}r_{n1}\left[\delta_{n1}-\delta_{10}\right]}{D_n}\right)\right)\right\}\cos(\omega_1+\Delta_{n1})t\ ,$$

$$\frac{V_s(\omega_1+\Delta_{n1})}{V_o}\sin(\omega_1+\Delta_{n1})t = \frac{1}{2}\left\{-\left(\frac{1-r'_{ro}}{r'_{ro}}\right)r_{10}\delta_{10}r_{n1}\left(\frac{\delta_{n1}-\delta_{10}}{D_n}\right)\delta_{n1}\right. \tag{147}$$

$$\left.+\,r_{n1}\left[\frac{(\delta_{n1}-\delta_{10})(1-\delta_{10}^2)^{\frac{1}{2}}}{D_n}\right]\left(\frac{V_c(\omega_1)}{V_o}\right)\right\}\sin(\omega_1+\Delta_{n1})t,$$

$$\frac{V_c(\omega_1-\Delta_{n1})}{V_o}\cos(\omega_1-\Delta_{n1})t = \frac{1}{2}\left\{\left(-r_{n1}\left[1-\frac{(\delta_{n1}-\delta_{10})\delta_{n1}}{D_n}\right]\right)\left(\frac{V_c(\omega_1)}{V_o}\right)\right. \tag{148}$$

$$\left.+\left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\left(r_{no}-\frac{r_{10}\delta_{10}r_{n1}(\delta_{n1}-\delta_{10})}{D_n}\right)\right\}$$

$$\cos(\omega_1-\Delta_{n1})t.$$

and

$$\frac{V_s(\omega_1-\Delta_{n1})}{V_o}\sin(\omega_1-\Delta_{n1})t \doteq \frac{1}{2}\left\{+\left(\frac{1-r'_{ro}}{r'_{ro}}\right)\frac{r_{10}\delta_{10}r_{n1}(\delta_{n1}-\delta_{10})\delta_{n1}}{D_n}\right. \tag{149}$$

$$\left. + \left[\frac{r_{n1}(\delta_{n1}-\delta_{10})(1-\delta_{10}^2)^{\frac{1}{2}}}{D_n}\right]\left(\frac{V_c(\omega_1)}{V_o}\right)\right\}\sin(\omega_1-\Delta_{n1})t \ .$$

The output amplitude at the frequency ω_1 is the square-root of the sum of the squares of the sine and cosine terms of frequency. The normalized amplitude of the cosine component is approximately one; whereas, since the square of the sine component will be much less than one, its contribution to the output at ω_1 may be neglected. The amplitude of the cosine term for the multiple signals has a form similar to that for the amplitude for one discrete interfering signal. The term from equation (144),

$$r_{10} - \sum_{n=2}^{m} \frac{r_{no}\delta_{10}r_{n1}(\delta_{n1}-\delta_{10})}{2D_n} \ ,$$

may be written as

$$r_{10}\left[1 - \sum_{n=2}^{m} \frac{r_{n1}^2\delta_{10}(\delta_{n1}-\delta_{10})}{2D_n}\right].$$

It can be shown that with $|\delta_{10}| \leq 0.9$ and $(\sum_{n=2}^{m} r_{n1})^2 \ll 1$, the second term in the brackets is much less than 1. Therefore the amplitude of the cosine function, which is also the output at frequency ω_1 (neglecting the amplitude of sine component), is approximately the same with one or with more than one interfering signal applied. The other equations for the output interference are seen to be exactly the same as for the discrete case with the amplitude of the cosine function unchanged.

Therefore, with $(\sum_{n=2}^{m} r_{n1})^2 \ll 1$ the output due to multiple discrete signals is as if each were independent of the other. The output from each interfering signal may be found by considering its interaction with only the synchronizing signal.

In the preceding analysis the dependence of the output amplitude on frequency was controlled, as far as the oscillator circuit elements were concerned, by the linear relation of the angle ϕ to frequency. This analysis did not include the amplitude variation due to frequency sensitive elements of the circuit. Even without this frequency selectivity the terms of importance are found to be the first order sideband terms. It is to be expected that in an analysis which included the frequency selectivity the terms of interest will be the first order sidebands. Therefore, in an analysis using the differential equation of the van der Pol type, which includes the frequency-sensitive elements, the preceding analysis may be used as a guide for determining the output terms.

Multiple Signals Applied to an Oscillator Described by van der Pol's Equation

The differential equation describing the behavior of an oscillator with a cubic nonlinearity and with multiple forcing functions is

$$\ddot{v} - \alpha\dot{v} + 3\gamma\dot{v}v^2 + \omega_o^2 v = \sum_{n=1}^{m} A_n \omega_o^2 \sin(\omega_1 + \Delta_{n1})t \ , \tag{150}$$

where the A_n's are related to the forcing functions, with constant current sources, as $A_n = \dfrac{I_n\ (\omega_1 + \Delta_{n1})}{\omega_o^2\ C}$, and $\Delta_{n1} = \omega_n - \omega_1$.

The restrictions on the order of magnitudes of the different components, if placed on the outputs, may require less restrictive conditions at the input because of the frequency selectivity of the circuit. It is possible for the interfering signal at the input to be larger than the synchronizing signal if it is relatively far from the free-running frequency of the oscillator. However, one of the interests in developing the equations for the output is the effect of noise accompanying the signal where the signal-to-noise ratio is relatively large. Therefore, the input amplitudes at the interfering frequencies are assumed much smaller than the amplitude of the synchronizing signal. Because of frequency selectivity the ratio of synchronizing signal to interference on the output will in most cases be larger than the corresponding ratio on the input. Since the nonlinearity is cubic the harmonic frequency generated will be of third order. Also there will be a number of frequencies generated due to intermodulation products. The third harmonic terms will suffer severe discrimination because of frequency selectivity and therefore may be neglected. Based on the preceding analysis and on experimental observations, the frequencies in the output which will be of interest are at ω_1, $\omega_1 + \Delta_{n1}$, and at $\omega_1 - \Delta_{n1}$. However, the restrictions required to allow neglect of certain terms may be derived by considering the result of applying an assumed solution to a cubic nonlinearity. Since the extension of this method is to be applied to the case of noise, in the limit the number of signals must be allowed to approach infinity and the power spectrum is of interest. Therefore, for purposes of establishing relative powers, the phase angles may be omitted.

Assume a voltage v given as

$$v = b_f \left[\cos \omega_1 t + \sum_{n=1}^{m} r_{nf} \cos (\omega_1 - \Delta_{1n}) t \right] , \tag{151}$$

with b_f the maximum value of the voltage at ω_1, applied to a cubic nonlinearity. Then, v^3 becomes

$$v^3 = b_f^3 \Big[\cos^3 \omega_1 t + 3 \cos^2 \omega_1 t \, (\sum_{n=1}^{m} r_{nf} \cos (\omega_1 - \Delta_{1n}) t) \tag{152}$$

$$+ 3 \cos \omega_1 t \, (\sum_{n=1}^{m} r_{nf} \cos (\omega_1 - \Delta_{1n}) t)^2 + (\sum_{n=1}^{m} r_{nf}$$

$$\cos (\omega_1 - \Delta_{1n}) t)^3 \Big] .$$

If terms that fall outside the pass-band are neglected and terms of the same frequency are collected,

$$v^3 = b_f^3 \left\{ \frac{3}{4} \left[1 + 2 \sum_{n=1}^{m} r_{nf}^2 \right] \cos \omega_1 t \right. \tag{153}$$

$$+ \frac{3}{4} \sum_{n=1}^{m} \left[2 r_{nf} + r_{nf}^3 + 2 \, (\sum_{\substack{q=1 \\ q \neq n}}^{m} r_{qf}^2) \, r_{nf} \right] \cos (\omega_1 - \Delta_{1n}) t$$

$$+ \frac{3}{4} \, (\sum_{n=1}^{m} r_{nf}) \cos (\omega_1 + \Delta_{1n}) t$$

$$+ \frac{3}{4} \, (\sum_{n=1}^{m} r_{nf}^2) \cos (\omega_1 - 2\Delta_{1n}) t$$

$$+ \frac{3}{2} \sum_{n=1}^{m-1} \sum_{q=n+1}^{m} r_{nf} r_{qf} \Big[\cos(\omega_1 - \Delta_{1n} + \Delta_{1q})t + \cos(\omega_1 - \Delta_{1q} + \Delta_{1n})t$$

$$+ \cos(\omega_1 - \Delta_{1q} - \Delta_{1n})t \Big]$$

$$+ \frac{3}{2} \sum_{n=1}^{m-2} \sum_{q=n+1}^{m-1} \sum_{s=q+1}^{m} r_{nf} r_{qf} r_{sf} \Big[\cos(\omega_1 - \Delta_{1n} - \Delta_{1q} + \Delta_{1s})t$$

$$+ \cos(\omega_1 - \Delta_{1q} - \Delta_{1s} + \Delta_{1n})t + \cos(\omega_1 - \Delta_{1s} - \Delta_{1n} + \Delta_{1q})t \Big].$$

An accounting of the number of terms at each frequency shows m terms for $\omega_1 + \Delta_{1n}$, $\omega_1 - \Delta_{1n}$, and $\omega_1 - 2\Delta_{1n}$; $\frac{1}{2} m(m-1)$ terms for frequencies involving two deltas, and $\frac{3}{2}$ $m(m-1)(m-2)$ terms involving three deltas.

Now, on a power basis the power for each of the frequencies is as follows:

$$\omega_1 : \frac{9}{16} b_f^6 (1 + 4mr_{nf}^2 + 4r_{nf}^4 m^2) , \tag{154}$$

$$\omega_1 - \Delta_{1n} : \frac{9}{16} m\, b_f^6\, r_{nf}^2 \left(\left[2 + r_{nf}^2\right]^2 + 4m\, r_{nf}^2 \left[2 + r_{nf}^2\right] + 4m^2 r_{nf}^2\right), \tag{155}$$

$$\omega_1 + \Delta_{1n} : \frac{9}{16} m\, b_f^6\, r_{nf}^2 , \tag{156}$$

$$\omega_1 - 2\Delta_{1n} : \frac{9}{16} m\, b_f^6\, r_{nf}^4 , \tag{157}$$

$$\omega_1 \pm \Delta_{1n} \pm \Delta_{1q} : \frac{81}{16} m\,(m-1)\, b_f^6\, r_{nf}^2 , \tag{158}$$

$$\omega_1 \pm \Delta_{1n} \pm \Delta_{1q} \pm \Delta_{1s} : \frac{9}{16} m\,(m-1)(m-2)\, b_f^6\, r_{nf}^3 \tag{159}$$

Since the output interference terms with relatively large input signal to interference ratio is proportional to the input, it is possible to allow m to approach a very large value and to allow all r_{nf}'s to become equal and very small, for noise representation, in such a way that mr_{nf}^2 is proportional to the input noise voltage ratio $(SN)_{in}^2$. Then, $(m\, r_{nf}^2)^{1/2}$ can be equated to $1/k(SN)_{in}$, with the expectation that in a synchronized oscillator $k > 1$. Substitution of $(SN)_{in}$ into equations (154 - 159) yields,

$$\omega_1 : \frac{9}{16} b_f^6 \left(1 + \frac{4}{k^2(SN)_{in}^2} + \frac{4}{k^4(SN)_{in}^4}\right) \doteq \frac{9}{16} b_f^6, \tag{160}$$

$$\omega_1 - \Delta_{1n} : \frac{9}{16} b_f^6 \; \frac{1}{k^2(SN)_{in}^2} (4 \;\; 1+\frac{1}{k^2(SN)_{in}^2}{}^{2} + 4r_{nf}^2\left[1 + \frac{1}{k^2(SN)_{in}^2}\right] \tag{161}$$

$$r_{nf}^4 \doteq \frac{9}{4} b_f^6 \frac{1}{k^2(SN)_{in}^2} ,$$

$$\omega_1 + \Delta_{1n} : \frac{9}{16} b_f^6 \frac{1}{k^2(SN)_{in}^2} , \tag{162}$$

$$\omega_1 - 2\Delta_{1n} : \frac{9}{16} b_f^6 \frac{r_{nf}^2}{k^2(SN)_{in}^2} , \tag{163}$$

$$\omega_1 \pm \Delta_{1n} \pm \Delta_{1q} : \frac{81}{16} b_f^6 \frac{1}{k^2(SN)_{in}^4} \tag{164}$$

$$\omega_1 \pm \Delta_{1n} \pm \Delta_{1q} \pm \Delta_{1s} : \frac{9}{16} b_f^6 \frac{1}{k^6(SN)_{in}^6} \tag{165}$$

The approximations are based on $(SN)_{in} > 10$.

Comparing the power at $\omega_1 - \Delta_{1n}$ plus that at $\omega_1 + \Delta_{1n}$ to the sum of powers at $\omega_1 - 2\Delta_{1n}$, $\omega_1 \pm \Delta_{1n} \pm \Delta_{1q}$, and $\omega_1 \pm \Delta_{1n} \pm \Delta_{1q} \pm \Delta_{1s}$

gives for $(SN)_{in} > 10$,

$$\frac{b_f^6}{k^2 (SN)_{in}^2} \left(\frac{9}{4} + \frac{9}{16} \right) \gg \frac{b_f^6}{k^2 (SN)_{in}^2} \left(\frac{9 r_{nf}^2}{16} + \frac{81}{16} \frac{1}{k^2 (SN)_{in}^2} + \frac{9}{16} \frac{1}{k^4 (SN)_{in}^4} \right) . \tag{166}$$

Under the restrictions imposed, terms contributing to the amplitude at the fundamental frequency that involve product terms other than b_1 and b_2 may be neglected. At the frequency $\omega_1 - \Delta_{1n}$, amplitudes that involve the square of b_1 or b_2 and one of the interfering components predominate. The assumed solution may be considered as a close approximation since terms generated that were not included in the assumed solution are negligibly small.

Then the output solution is assumed and the constants in the assumed solution evaluated by a method similar to the principle of harmonic balance. The assumed solution is

$$v = \sum_{n=1}^{m} \left\{ b_{2n-1} \sin (\omega_1 + \Delta_{n1})t + b_{2n} \cos (\omega_1 + \Delta_{n1})t + d_{2n-1} \sin (\omega_1 - \Delta_{n1})t + d_{2n} \cos (\omega_1 - \Delta_{n1})t \right\} . \tag{167}$$

In order to make all terms have the same limits on the summation indices, $d_1 = d_2 = 0$ and $\Delta_{11} = 0$. The amplitudes in equations (167) are assumed to be slowly varying which will allow testing of the stability of the equations.

The term in the differential equation due to the cubic nonlinearity causes generation of a large number of terms with an input as assumed. The process of solution is to substitute the assumed expression for v into the differential equation and to equate like functions of time. There will occur terms of different order of magnitude in this substitution and the solution will be greatly simplified if terms which may be neglected, are neglected as early in the development as possible. To help with this procedure the following assumptions are made:

(a) The interfering frequencies are not far different from the free running frequency $(\omega_1 + \Delta_{n1} \doteq \omega_1)$.

(b) The amplitudes of the individual interference terms in the output are much less than the amplitudes at the synchronizing frequency.

(c) The components of the synchronizing signal in the output is of the same order amplitude as the free-running amplitude.

(d) The variation of the amplitudes with time occurs relatively slowly.

The substitution and solution for these equations are carried out in Appendix C and they are,

$$2\dot{b}_1 + 2\Delta_{01} b_2 - \alpha b_1 (1 - \rho_1) = 0 \ , \tag{168}$$

$$2\dot{b}_2 - 2\Delta_{01} b_1 - \alpha b_2 (1 - \rho_1) = -A_1 \omega_o \ , \tag{169}$$

$$2\dot{b}_{2n-1} + 2\left[\Delta_{01} - \Delta_{n1}\right] b_{2n} - \alpha b_{2n-1} (1 - 2\rho_1) + d_{2n-1}\rho_s + d_{2n}\rho_m = 0 \tag{170}$$

$$2\dot{b}_{2n} - 2\left[\Delta_{01} - \Delta_{n1}\right] b_{2n-1} - \alpha b_{2n} (1 - 2\rho_1) + d_{2n-1}\rho_m - d_{2n}\rho_s = -A_n \omega_o, \tag{171}$$

$$2\dot{d}_{2n-1} + 2\left[\Delta_{01} + \Delta_{n1}\right] d_{2n} - \alpha d_{2n-1} (1 - 2\rho_1) + b_{2n-1}\rho_s + b_{2n}\rho_m = 0, \tag{172}$$

and

$$2\dot{d}_{2n} - 2\left[\Delta_{01} + \Delta_{n1}\right]d_{2n-1} - \alpha d_{2n}\,(1 - 2\rho_1) - b_{2n}\rho_s + b_{2n-1}\rho_m = 0 \qquad (173)$$

where,

$$\Delta_{01} = \omega_o - \omega_1 \,, \qquad (174)$$

$$\Delta_{n1} = \omega_n - \omega_1 \,,$$

$$\rho_1 = \frac{b_1^2 + b_2^2}{a_o^2} \,,$$

$$\rho_s = \frac{b_1^2 - b_2^2}{a_o^2} \,,$$

$$\rho_m = \frac{2b_1 b_2}{a_0^2} \,,$$

and

$$n = 2, 3, 4, \ldots, m \,.$$

In the manner of van der Pol, the amplitudes of the components are considered constant in the steady state, thus all the derivatives become equal to zero.

Equations (168, 169) are independent of all amplitudes except b_1 and b_2 and, therefore, may be solved for these two components,

$$b_1 = \frac{2\Delta_{01}\, A_1 \omega_o}{4\Delta_{01}^2 + \alpha^2(1 - \rho_1)^2} \,, \qquad (175)$$

and

$$b_2 = \frac{\alpha(1 - \rho_1)A_1\omega_o}{4\Delta_{01}^2 + \alpha^2(1 - \rho_1)^2} \; . \tag{176}$$

Squaring b_1 and b_2, adding and dividing by a_0^2 gives the normalized amplitude. An implicit equation for ρ_1 is as follows:

$$4\sigma_{01}^2\,\rho_1 + \rho_1\,(1 - \rho_1)^2 = F_1^2 \; , \tag{177}$$

where

$$\sigma_{01} = \frac{\Delta_{01}}{\alpha} \quad \text{and} \quad F_1^2 = \frac{A_1^2\omega_o^2}{\alpha^2 a_o^2} \; .$$

The solution of the other amplitudes are more complicated than for b_1 and b_2. However, equations (170, 171, 172, 173) gives four independent equations in b_{2n-1}, b_{2n}, d_{2n-1}, and d_{2n}. The solution of each will involve amplitudes of b_1 and b_2 and constants of the oscillator.

In determinant form the solution of b_{2n-1} becomes

$$b_{2n-1} = \frac{\begin{vmatrix} 0 & 2\left[\Delta_{01}-\Delta_{n1}\right] & \rho_m & \rho_s \\ -A_n\omega_o & -\alpha(1-2\rho_1) & -\rho_s & \rho_m \\ 0 & \rho_m & 2\left[\Delta_{01}+\Delta_{n1}\right] & -\alpha(1-2\rho_1) \\ 0 & -\rho_s & -\alpha(1-2\rho_1) & -2\left[\Delta_{01}+\Delta_{n1}\right] \end{vmatrix}}{\begin{vmatrix} -\alpha(1-2\rho_1) & 2\left[\Delta_{01}-\Delta_{n1}\right] & \rho_m & \rho_s \\ -2\left[\Delta_{01}-\Delta_{n1}\right] & -\alpha(1-2\rho_1) & -\rho_s & \rho_m \\ \rho_s & \rho_m & 2\left[\Delta_{01}+\Delta_{n1}\right] & -\alpha(1-2\rho_1) \\ \rho_m & -\rho_s & -\alpha(1-2\rho_1) & -2\left[\Delta_{01}+\Delta_{n1}\right] \end{vmatrix}} \; . \tag{178}$$

Solutions of b_{2n}, d_{2n-1} and d_{2n} proceed from a determinant such as (178). Combination of corresponding sine and cosine amplitudes to give the normalized squared amplitudes yield equations as follows:

$$\rho_{nb} = F_n^2 \frac{4(\sigma_{01} + \sigma_{n1})^2 + (1 - 2\rho_1)^2}{D_n} \tag{179}$$

$$\rho_{nd} = F_n^2 \frac{\rho_1^2}{D_n} , \tag{180}$$

where

$$\rho_{nb} = \frac{b_{2n-1}^2 + b_{2n}^2}{a_o^2} , \tag{181}$$

$$\rho_{nd} = \frac{d_{2n-1}^2 + d_{2n}^2}{a_o^2} ,$$

$$F_n = - \frac{A_n \omega_o}{a_o \alpha} ,$$

and

$$D_n = \left[4\sigma_{0n}^2 + (1 - 2\rho_1)^2\right]\left[4(\sigma_{01} + \sigma_{n1})^2 + (1 - 2\rho_1)^2\right]$$
$$+ \rho_1^4 - 2\rho_1^2\left[4\sigma_{0n}(\sigma_{01} + \sigma_{n1}) + (1 - 2\rho_1)^2\right] .$$

The stability of the solutions for b_1 and b_2, the sine and cosine components of the synchronizing voltage in the output, can be accomplished by the method of Andronow and Witt as presented in Minorsky's (10) book. The equations (168) and (169) are in the form

$$\mathring{b}_1 = P(b_1, b_2) ,$$

and

$$\mathring{b}_2 = Q(b_1, b_2) ,$$

and may be taken as variables in the phase plane. The condition for $\mathring{b}_1 = \mathring{b}_2 = 0$, corresponds to steady-state and the solution for b_1 and b_2 are as found in equations (175) and (176). To determine the stability of this point of operation the amplitudes are allowed to take on incremental variations about constant values.

$$b_1 = b_{10} + \zeta \text{ and } b_2 = b_{20} + \epsilon , \tag{182}$$

where b_{10} and b_{20} are the coordinates in the $b_1 b_2$ - plane of the singular point.

Replacement of b_1 and b_2 by these definitions in equations (152) and (153) yields,

$$2(\mathring{b}_{10} + \zeta) = -2\Delta_{01}(b_{20} + \epsilon) + \alpha(b_{10}+\zeta)\left[1 - \frac{(b_{10}+\zeta)^2 + (b_{20}+\epsilon)^2}{a_o^2}\right], \tag{183}$$

$$2(\mathring{b} + \mathring{\epsilon}) = +2\Delta_{01}(b_{10}+\zeta) + \alpha(b_{20}+\epsilon)\left[1 - \frac{(b_{10}+\zeta)^2+(b_{20}+\epsilon)^2}{a_o^2}\right]-A_1\omega_o . \tag{184}$$

Since $\mathring{b}_{10}$ and $\mathring{b}_{20}$ are zero, the equations reduce to

$$2\mathring{\zeta} = -b_{20}(2\Delta_{01}) + \alpha b_{10}\left[1 - \frac{b_{10}^2 + b_{20}^2}{a_o^2}\right] \tag{185}$$

$$+ (-2\Delta_{01} - \frac{2\alpha b_{10}b_{20}}{a_o^2})\epsilon + \alpha\zeta\ (1 - \frac{b_{10}^2+b_{20}^2+2b_{10}^2}{a_o^2}) ,$$

$$2\dot{\epsilon} = b_{10}(2\Delta_{01}) + \alpha b_{20}\left[1 - \frac{b_{10}^2 + b_{20}^2}{a_o^2}\right] - A_1\omega_o \tag{186}$$

$$+ (2\Delta_{01} - \frac{2\alpha b_{10}b_{20}}{a_o^2})\zeta + \alpha\epsilon\left[1 - \frac{b_{10}^2 + b_{20}^2 + 2b_{20}^2}{a_o^2}\right],$$

where terms involving ζ^2, ϵ^2, ... and $\zeta\epsilon$, $\zeta\epsilon^2$, ... are neglected. The terms involving only b_{10} and b_{20} are zero by equations (168) and (169) with $\dot{b}_{10}$ and $\dot{b}_{20}$ zero. The equations for $\dot{\zeta}$ and $\dot{\epsilon}$ then become

$$2\dot{\zeta} = (-2\Delta_{01} - \alpha\rho_m)\epsilon + \alpha(1 - \rho_1 - \frac{2b_{10}^2}{a_o^2})\zeta , \tag{187}$$

and

$$2\dot{\epsilon} = \alpha(1 - \rho_1 - \frac{2b_{20}^2}{a_o^2})\epsilon + (2\Delta_{01} - \alpha\rho_m)\zeta . \tag{188}$$

The nature of the singular point is determined by the roots of the characteristic equation determined from

$$\begin{vmatrix} \frac{\alpha}{2}(1 - \rho_1 - \frac{2b_{10}^2}{a_o^2}) - s & -\Delta_{01} - \frac{\alpha}{2}\rho_m \\ (\Delta_{01} - \frac{\alpha}{2}\rho_m) & \frac{\alpha}{2}(1 - \rho_1 - \frac{2b_{20}^2}{a_o^2}) - s \end{vmatrix} = 0 , \tag{189}$$

giving the equation,

$$s^2 - 2\alpha(1 - 2\rho_1) + \frac{\alpha^2}{4}(1 - \rho_1)(1 - 3\rho_1) + \Delta_{01}^2 = 0 . \tag{190}$$

Now for the system to be stable the roots of equation (190) must have negative real parts. This process may be accomplished by examining the last term in equation (190), which is the sum of the roots to determine necessary conditions that it not give a saddle point region. If the roots are real with opposite sign then a saddle point region results. A necessary condition then for the singular point not to be a saddle point is

$$(1 - \rho_1)(1 - 3\rho_1) + 4\sigma_{01}^2 \geqq 0 . \tag{191}$$

With the condition expressed by equation (191) imposed then for the roots to have negative real parts the condition

$$-2\alpha(1 - 2\rho_1) \geqq 0 , \tag{192}$$

must be met. The regions imposed by these equations are indicated in Figure 16.

Since the interfering signals are limited to small perturbations about the synchronizing voltage output, in this analysis, the solutions for these components can be expected to yield stable operation of the oscillator except when the synchronizing voltage is near the borderline between stable and unstable regions. The effect of the interference is to enlarge, slightly, the area of unstable operation.

Under the assumed conditions equations (179) and (180) are independent of all inputs except those at the synchronizing frequency and the input at frequency $\omega_1 + \Delta_{n1}$. For each input interfering term there exists an output at the frequency of the input $\omega_1 + \Delta_{n1}$ and at $\omega_1 - \Delta_{n1}$, as shown in Figure 17.

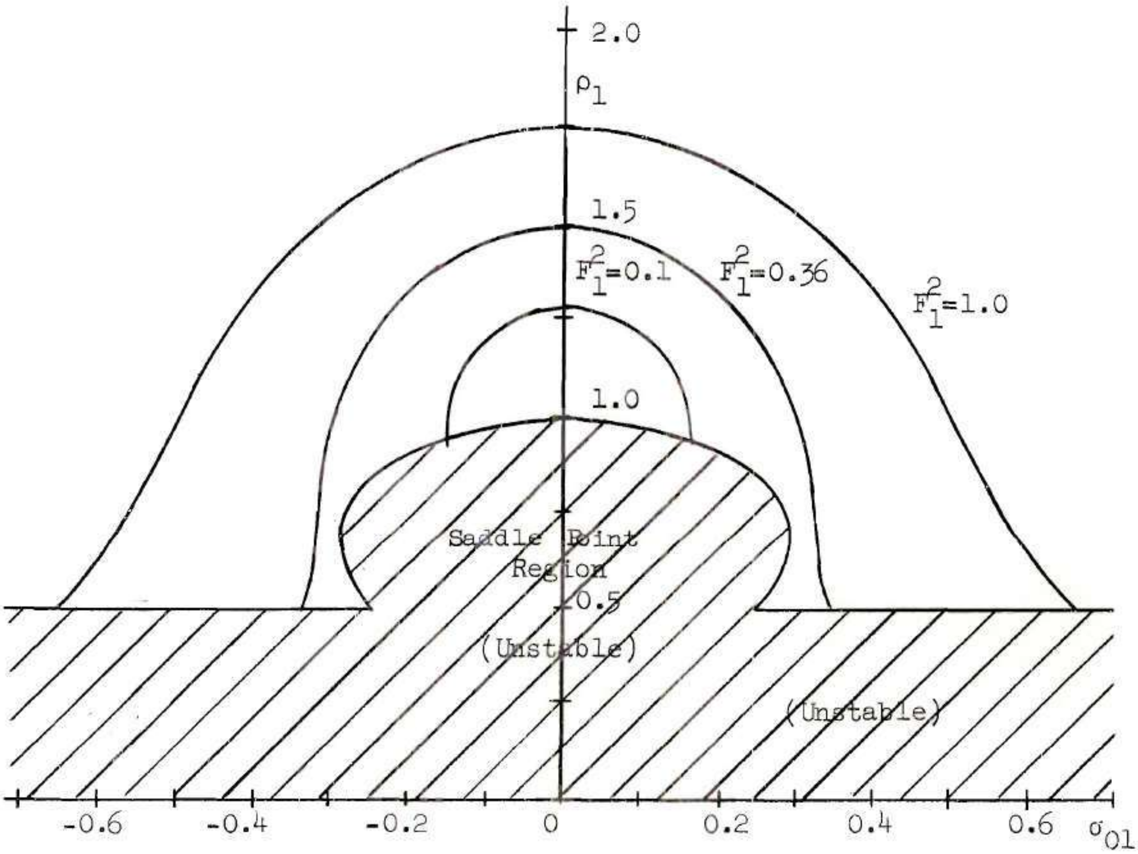


Figure 16. Response Curves of an Oscillator Showing Regions of Instability.

The ratio of the squared amplitude at $\omega_1 + \Delta_{n1}$ to the squared amplitude of its image about ω_1 is by equations (179) and (180)

$$\frac{\rho_{nb}}{\rho_{nd}} = \frac{4(\sigma_{01} + \sigma_{n1})^2 + (1 - 2\rho_1)^2}{\rho_1^2} . \tag{193}$$

The effect of σ_{01} and ρ_1 on this ratio is shown in Figure 18.

Note that for a given synchronizing signal output, the curves showing the ratio of ρ_{nb}/ρ_{nd}, are shifted on the frequency axis without change of shape when the location of the synchronizing frequency is changed.

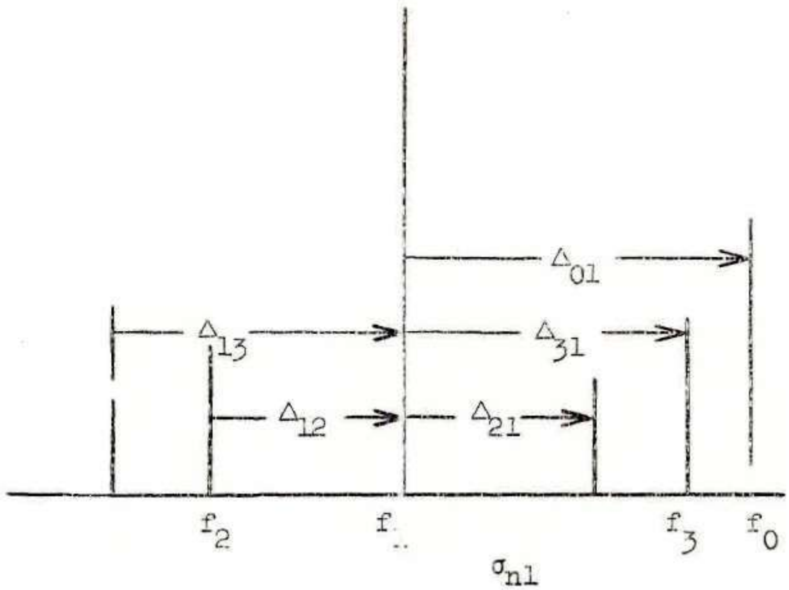


Figure 17. Spectrum in the Output with Two Interference Signals Applied to the Input.

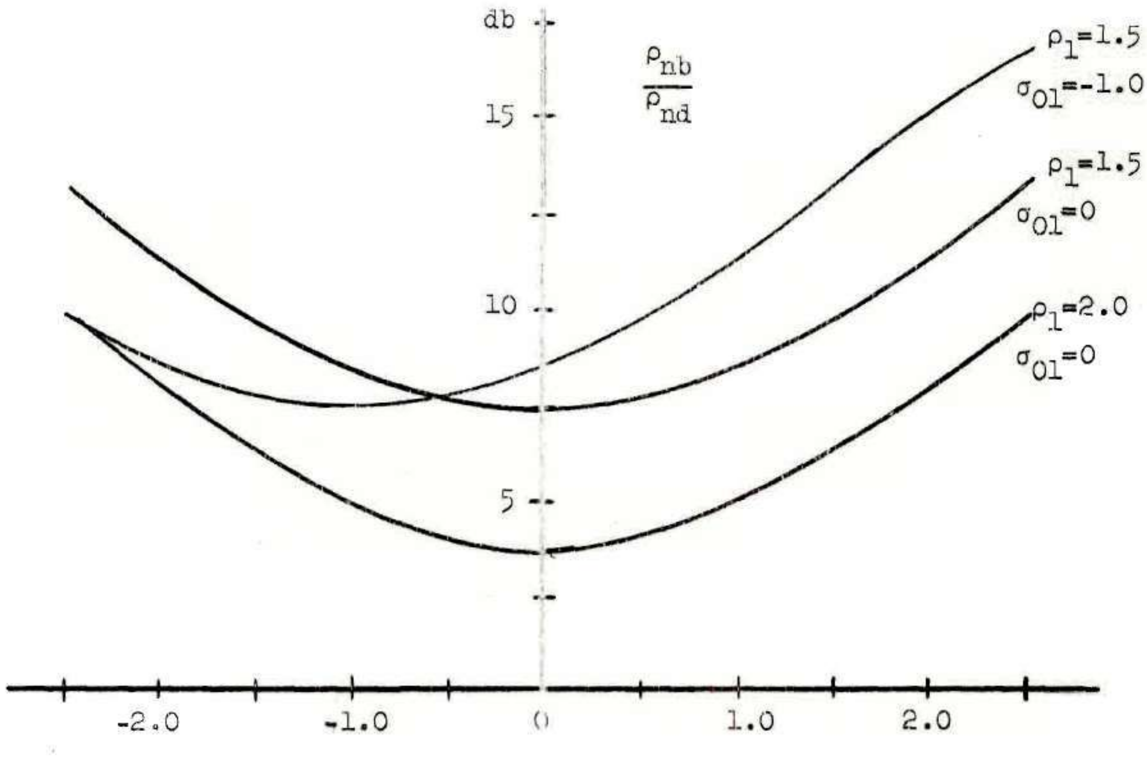


Figure 18. Ratio of Amplitude of Interfering Term in the Output at $\omega_1+\Delta_{n1}$ to the Amplitude at $\omega_1+\Delta_{n1}$.

However, the magnitudes of each component does change with synchronizing frequency. If the discrimination is defined as the ratio of the gain at the synchronizing frequency to the gain at $\omega_1 + \Delta_{n1}$, curves such as shown in Figure 19 are obtained. To get an idea of the relative magnitudes of synchronizing signal to the generated sideband term this term is treated as if it were a direct result of the interfering input. The equations used for this calculation are

$$D_b = \frac{(\rho_1)^{\frac{1}{2}}}{F_1} \cdot \frac{F_2}{(\rho_{nb})^{\frac{1}{2}}}, \qquad (194)$$

and

$$D_d = \frac{(\rho_1)^{\frac{1}{2}}}{F_1} \cdot \frac{F_2}{(\rho_{nd})^{\frac{1}{2}}}.$$

The curves in Figure 19 show the effect of the synchronizing signal output and synchronizing signal frequency on the interference and image frequency terms. As the magnitude of ρ_1 is increased the discrimination for given conditions decreased, reflecting an increased limiting effect due to the nonlinearity with increase in amplitude.

The variation with interference amplitude at the location of the synchronizing frequency is varied as shown in Figure 20. These curves are with constant input of the synchronizing signal.

Consideration of Noise Accompanying the Synchronizing Frequency

When a sinusoidal signal accompanied by noise is applied to a non-

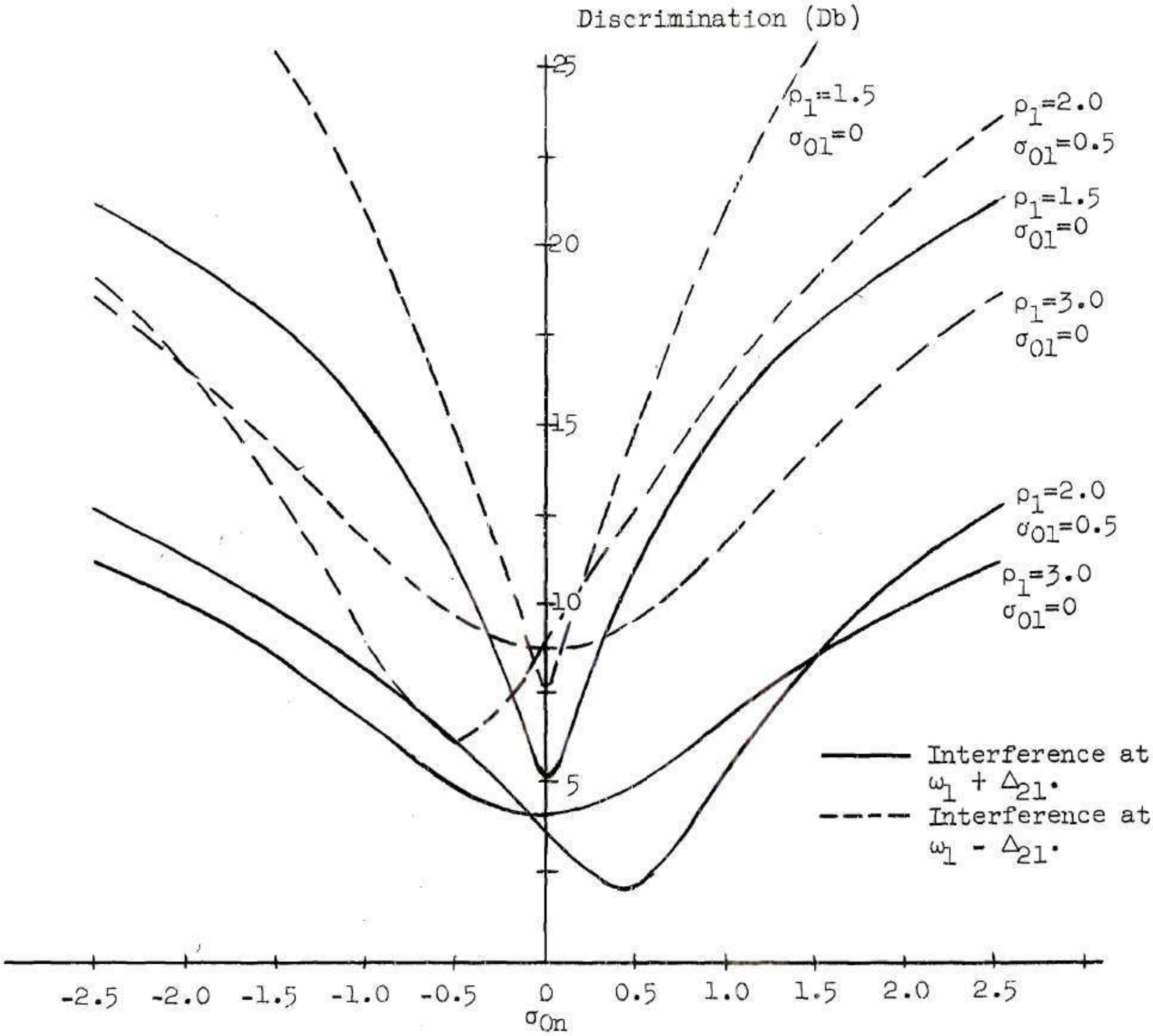


Figure 19. Discrimination for Several Values of σ_{01} and ρ_1 as σ_{on} is Varied.

linear device, the output power spectral density will contain terms due to products of the signal; due to products of the signal and noise components; and due to products of noise components. Expressed symbolically

$$G(\omega) = G_{sxs}(\omega) + G_{sxn}(\omega) + G_{nxn}(\omega) \ , \tag{195}$$

where G is the power spectral density in watts per cycle.

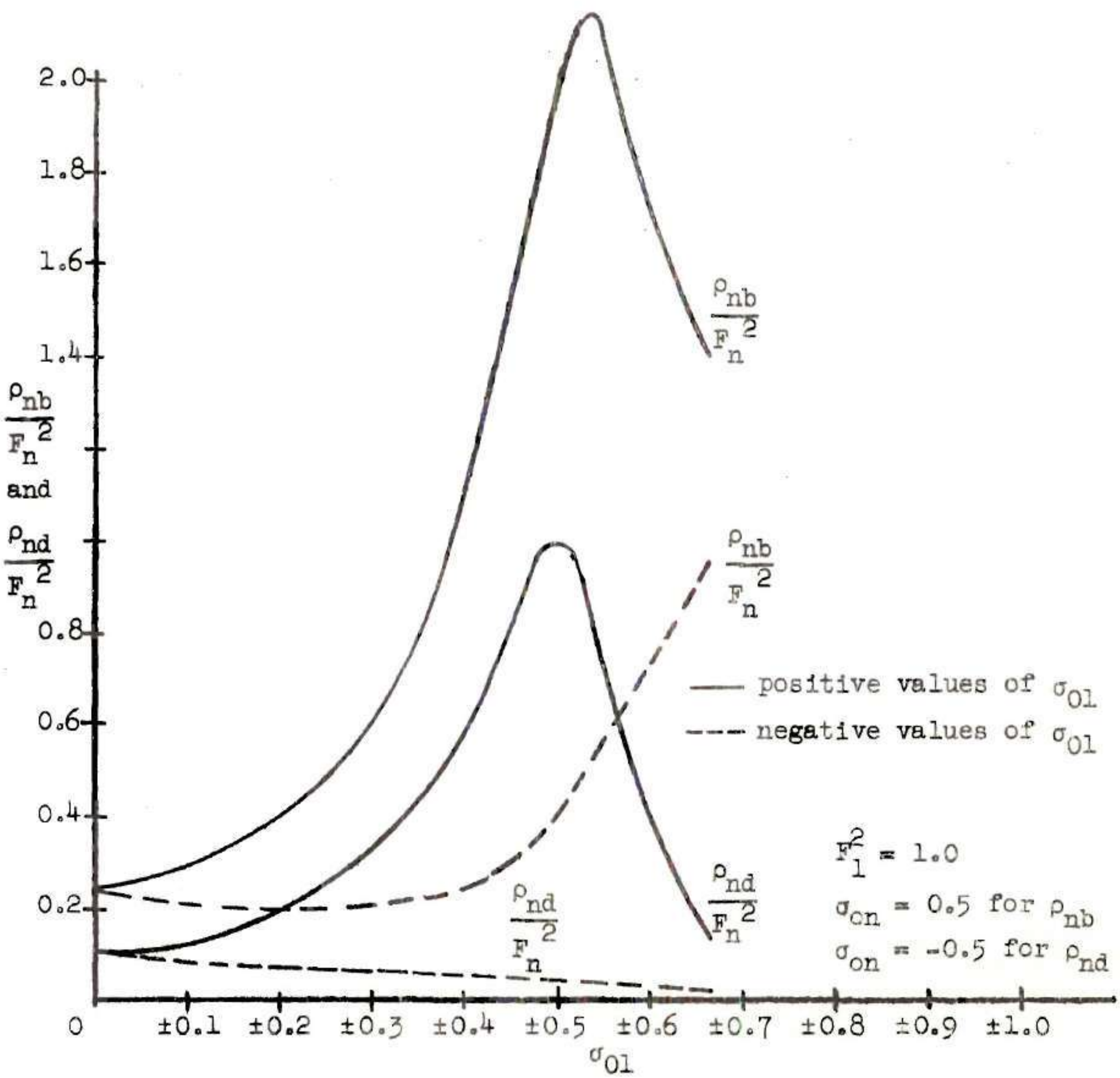


Figure 20. Variation of Interfering Amplitudes with σ_{10}. Input at f_1 Constant.

Under the action of a synchronizing signal, an oscillator acts as a narrow-band filter about the oscillator frequency. The bandwidth of this filtering action is approximately $\frac{\omega}{Q}$ where Q is the quality factor of the oscillator considering the regeneration, and ω is the frequency of oscillation. Therefore, terms of equation (195) which fall outside this narrow-band need not be considered, for example, all except one of the terms from $G_{sxs}(\omega)$ will be out of the pass-band. Therefore, with white noise, whose frequency spectrum includes the narrow-band of the oscillator, may be considered to be narrow-band noise centered about the synchronizing signal as pictured in Figure 21.

It has been shown (12) that a continuous power-density spectrum of noise may be approximated by discrete sinusoids yielding the same total power. Figure 22 a, b illustrates this approximation and indicates the power spectrum with all the power shown on the positive frequency axis.

The power contained in an incremental frequency band df is the same in the approximation as in the actual spectrum. The implicit assumption here is that the noise in any band df is independent of all noise components in other bands. Each discrete signal may then be expressed as

$$v_n = V_n \cos (\omega_1 + \Delta_{n1})t \; . \tag{196}$$

The value of V_n^2 is

$$\frac{V_n^2}{2} = \eta_n \, df \; , \tag{197}$$

where η_n is the noise power spectral density in watts/cps.

With the addition of a sinusoidal synchronizing signal the total effects of signal and noise on the input becomes,

$$v = V_1 \cos \omega_1 t + \sum_{\substack{n=2 \\ m \to \infty}}^{m} V_n \cos (\omega_1 + \Delta_{n1})t \; . \tag{198}$$

In preceding sections of this chapter it has been shown that the main contribution in the output of a synchronized oscillator, subjected to interfering signals, occurred at the frequency of the interfering signal

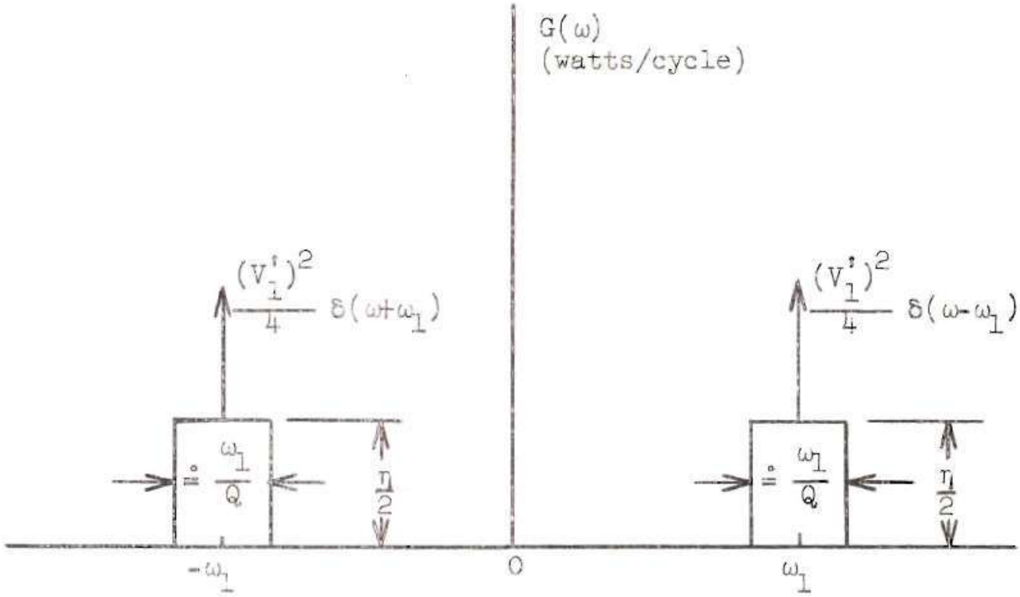


Figure 21. Representation of the Approximate Input to an Oscillator; Sine Wave Plus Noise.

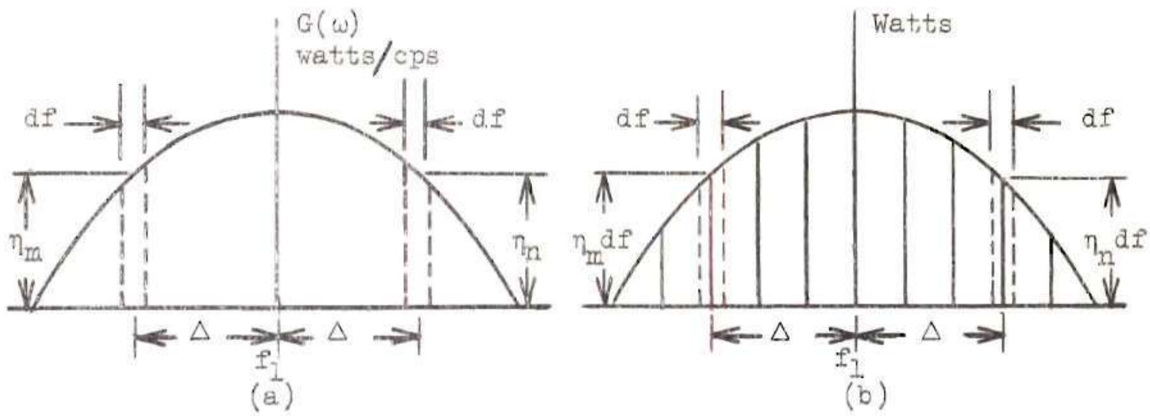


Figure 22. Noise Power Spectrum. (a) Power-density Spectrum, (b) Discrete Signal Approximation.

and at the image of this frequency about the synchronizing frequency.

With continuous spectrum centered about the synchronization signal frequency the analysis based on discrete signals predicts an overlapping of amplitudes. These amplitudes in noise consideration are taken as independent and must be combined on a mean square basis to obtain power. Figure 23 shows the variation of amplitude with Δ_{no} at $\omega_1 + \Delta_{n1}$ the input interfering frequency and at $\omega_1 - \Delta_{n1}$ the image about ω_1 of $\omega_1 + \Delta_{n1}$.

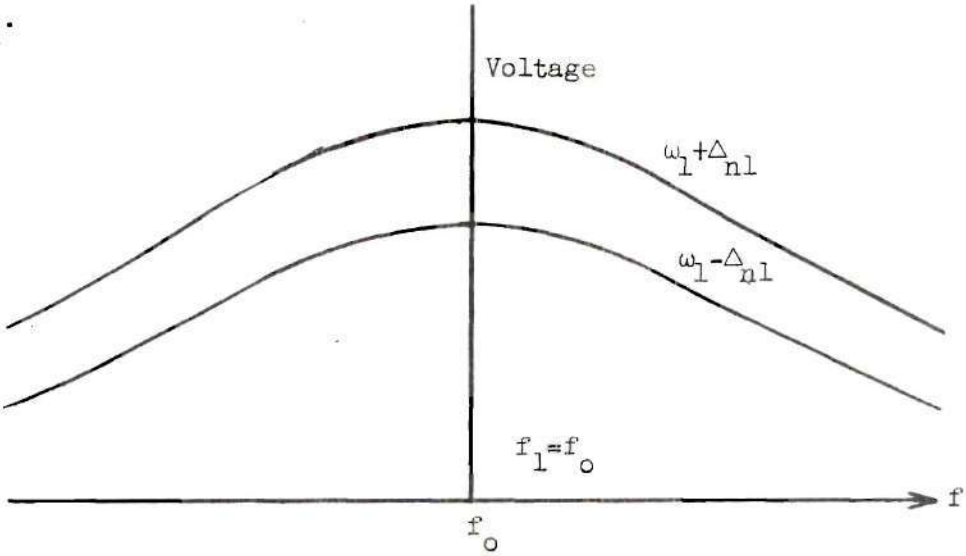


Figure 23. Variation of Amplitude at $\omega_1 + \Delta_{n1}$ and at $\omega_1 - \Delta_{n1}$ with Δ_{no}. Input Constant.

To demonstrate the overlapping that occurs in the continuous spectrum consider Figure 24. The series of diagrams show the result when two signals representing noise are spaced equally in frequency about the synchronizing frequency ω_1; one signal higher and one lower than ω_1. Figure 24 c, d show the input-output relation when the input signal is Δ_{21} above ω_1. The input-output with both the signals in 24 a, c applied is shown in 24 e, f. The addition in Figure 24 f is on a mean-square basis since the inputs are representations of narrow-band noise.

Now from Figure 23 it is possible to formalize the relation between the discrete signal approximation and the actual noise power spectrum as

$$\frac{A_n^2}{2} = \eta_n \, df \quad \text{and} \quad \frac{A_m^2}{2} = \eta_m \, df \ , \tag{199}$$

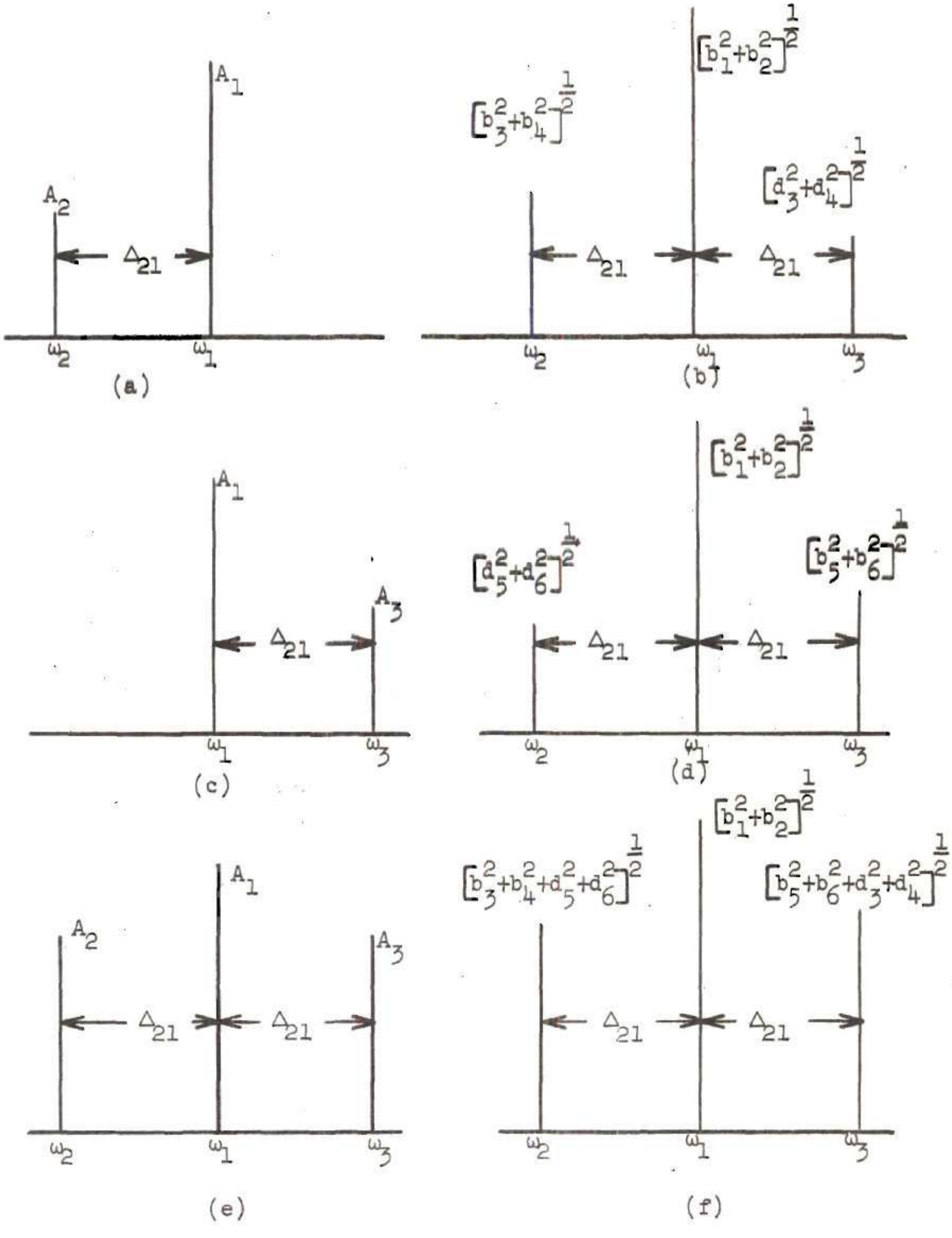


Figure 24. Effects on the Output of Interfering Signals Representing Noise Equally Spaced About the Synchronizing Frequency.

where A_n is the maximum amplitude of one of the discrete interfering terms and η_n is the noise spectral density at the same frequency as A_n.

From the previously determined relations that give curves such as graphed in Figure 23 it is possible to express

$$\left[b_{2n-1}^2 + b_{2n}^2\right]^{\frac{1}{2}} = \psi(A_1, f_1, \Delta_{n1})\, A_n \,, \tag{200}$$

and

$$\left[d_{2m-1}^2 + d_{2m}^2\right]^{\frac{1}{2}} = \psi'(A_1, f_1, \Delta_{n1})\, A_m \,, \tag{201}$$

where A_m is the input at a frequency $\omega_1 - \Delta_{n1}$, that is, equally but oppositely spaced about f_1 from the frequency with amplitude A_n. The magnitudes d_{2m-1} and d_{2m} result from A_m but are located at $\omega_1 + \Delta_{n1}$.

The mean squared amplitude in the output at $\omega_1 + \Delta_{n1}$ may be found as

$$b_{2n-1}^2 + b_{2n}^2 + d_{2m-1}^2 + d_{2n}^2 = \frac{\left[\psi(A_1, f_1, \Delta_{n1})A_n\right]^2}{2} + \frac{\left[\psi'(A_1, f_1, \Delta_{n1})A_m\right]^2}{2} \,. \tag{202}$$

Equation (202) gives the power in the region df of the output and with this representation of noise gives

$$(N_o\, df)_{f_n} = \left[\psi(A_1, f_1, \Delta_{n1})\right]^2 \frac{A_n^2}{2} + \left[\psi'(A_1, f_1, \Delta_{n1})\right]^2 \frac{A_m^2}{2} \,, \tag{203}$$

or

$$(N_o\ df)_{f_n} = \left[\psi(A_1,\ f_1,\ \Delta_{n1})\right]^2 \eta_n\ df + \left[\psi'(A_1,\ f_1,\ \Delta_{n1})\right]^2 \eta_m\ df. \quad (204)$$

Then from equation (204) it can be seen that given the input noise spectral density, the output noise spectral density may be found from a knowledge of the relation between the discrete inputs and the discrete output interfering signals. In the developments concerned with multiple signals it was shown, under the assumptions made, that the output due to any interfering input was determined by that input and the synchronizing frequency input. Therefore, to obtain the functional relations of equation (204) it is only necessary to consider one interfering signal. Experimentally this relation can be obtained by one signal source of variable frequency for the interfering term.

To obtain the total noise power in the output, integration of equation (204) is required. Graphical means may be used for this integration if the functional relation is obtained experimentally.

CHAPTER VI

EXPERIMENTAL RESULTS

The object of the experimental work is to observe the effect on the output by discrete interfering signals and by noise. Two oscillator types are used. One of the oscillators meets the requirements imposed by the analysis of Chapter IV and the other satisfies the cubic non-linearity requirement of the differential equation of the van der Pol type.

Tests on an Oscillator that Satisfies Adler's Conditions

The conditions set forth in the derivation of the equations in Chapter IV were

(a) phase-shift approximately a linear function of frequency,

(b) $V_1 \ll V_o$, synchronizing signal much less than the free-running amplitude,

(c) the response of the circuit more rapid than the difference in synchronizing and oscillator frequency,

(d) negligible change in amplitudes of the oscillator due to addition of the synchronizing voltage.

Condition (b) may also satisfy (d), however (d) implies a fairly sharp limiting characteristic of the oscillator.

The oscillator used is basically a tuned plate oscillator. To control the limiting characteristic a double Zener diode was added in the plate circuit of the 6SN7. The slope of the amplitude characteristic was controlled by the variable resistance in series with the double Zener diode. For $r'_{ro} = 0.9$ this resistance was 2000 ohms. The cathode follower is for isolation and a low impedance measuring point. In order to avoid slow bias adjustments, fixed bias was used on both the grids. The frequency of operation was approximately 10 kilocycles.

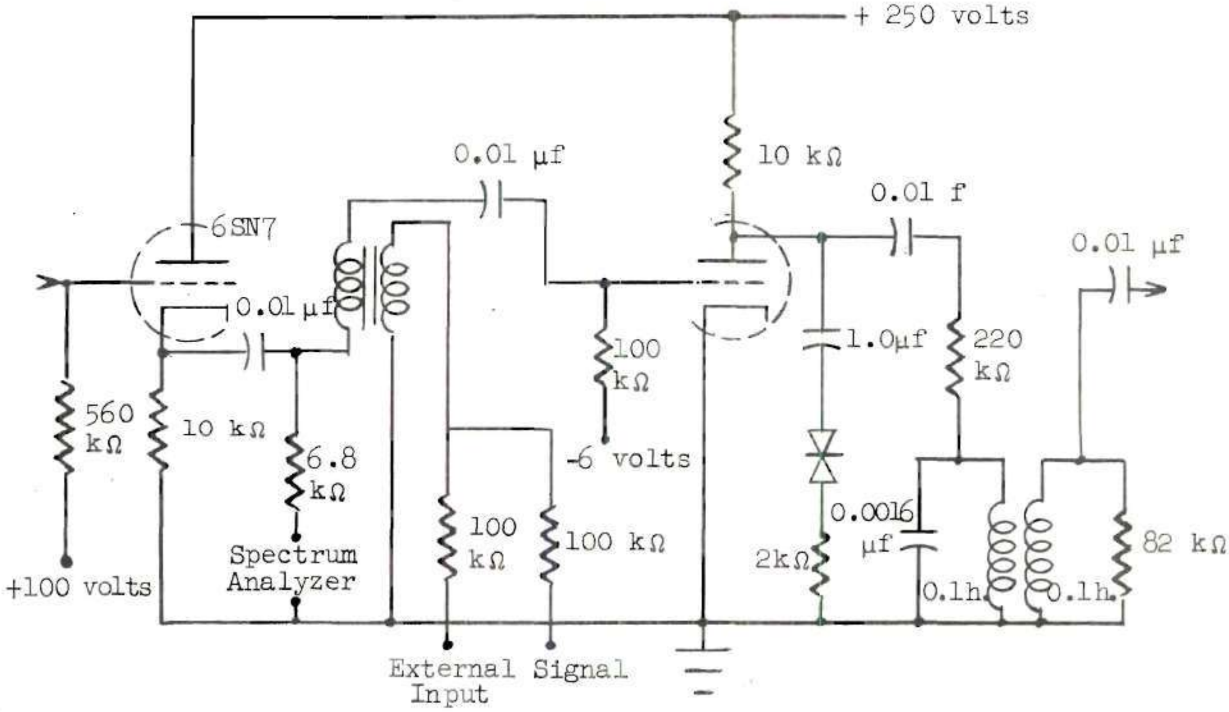


Figure 25. Schematic Diagram of a Feedback Oscillator Circuit.

From equation (9) the bandwidth of synchronization is

$$\omega_c = \omega_o - \omega_1' = \pm \frac{V_1}{K_\phi v_g} .$$

If this curve of ω_c versus V_1 yields a linear relation symmetrically placed about ω_o, the oscillator phase shift may be assumed linear in the operating range. A curve showing this relation is included as Figure 26.

The open-loop amplitude limiting characteristic was measured by breaking the circuit at the input to the amplifier-limiter triode and

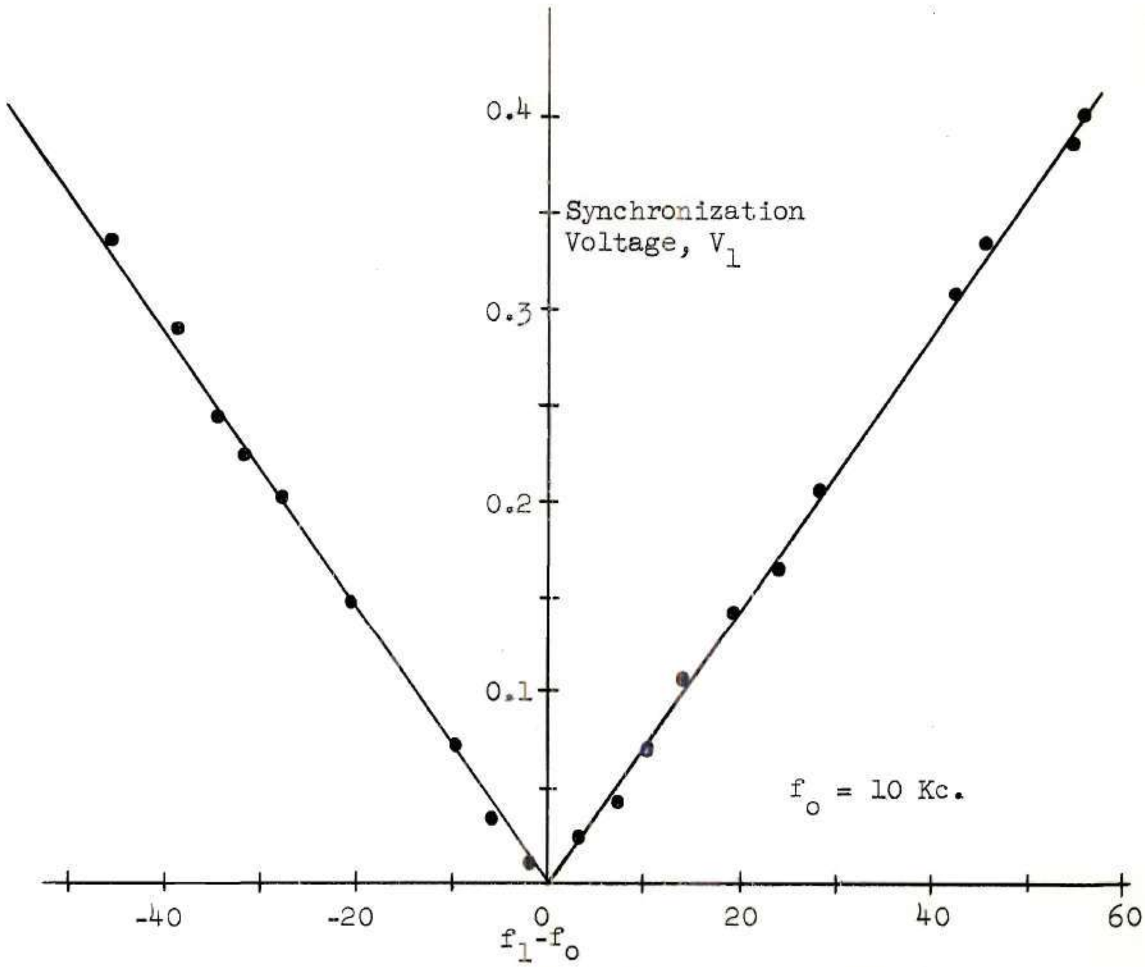


Figure 26. Synchronization Bandwidth as a Function of Synchronizing Voltage.

measuring the output at the cathode of the cathode follower. The slope of the limiting characteristic in the operating range can be controlled by the resistance in series with the Zener diode. A typical curve for the open loop characteristic is shown in Figure 27.

The synchronizing signal and the interfering signals were applied in series with the grid circuit through the use of a transformer. The 100K resistors were used between the input terminals and the primary of the transformer to prevent interaction of the signal generators. To measure frequencies a Berkley EPUT meter was used. The inputs and outputs were measured with a GR 236A Wave Analyzer. The resulting input-output data with r'_{ro} equal to 0.9 and for three locations of the synchronizing frequency relative to the free-running frequency are included in Figure 28. One curve with r'_{ro} equal to 0.5 and $\delta_{10} = 0$ is also included in Figure 28.

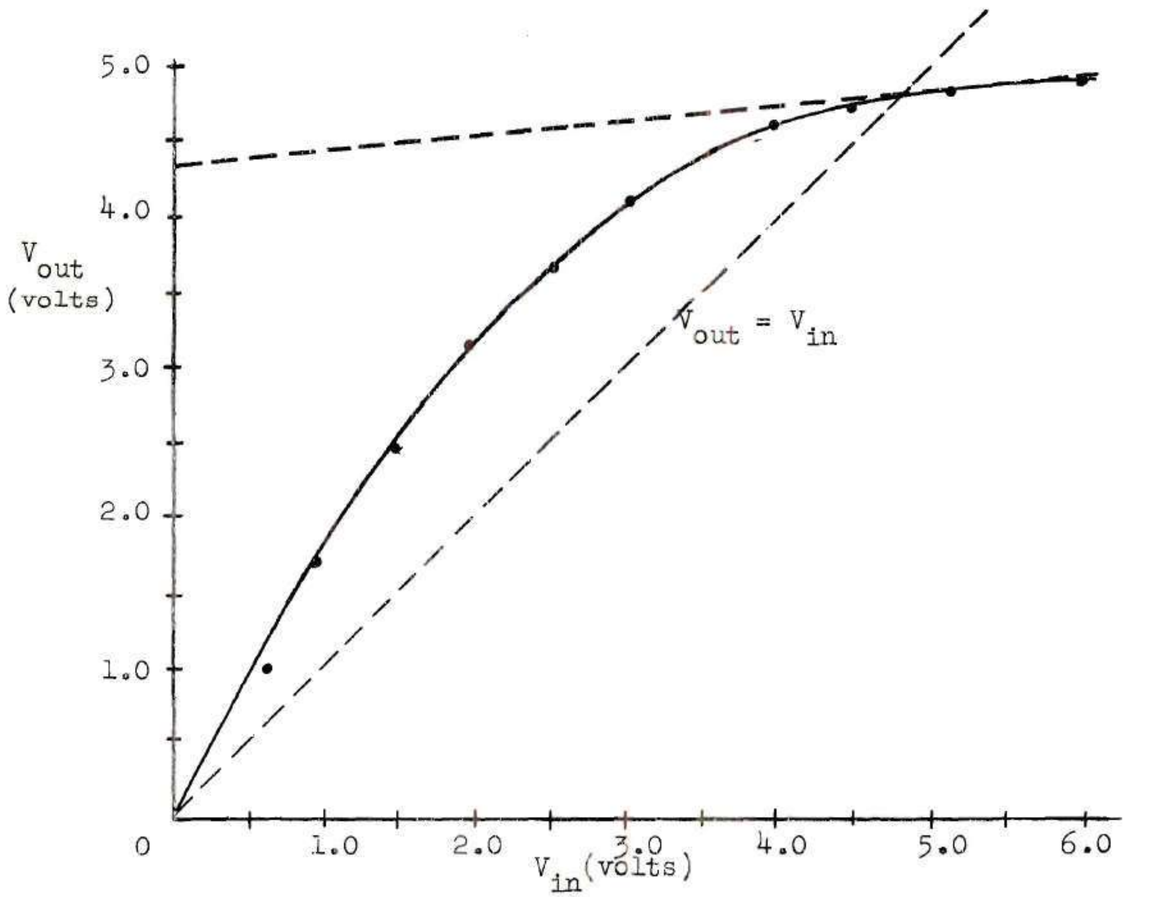


Figure 27. Open-loop Amplitude Characteristic

The data showing the effect of the slope of the open-loop amplitude characteristic on the ratio of the amplitude at $\omega_1 + \Delta_{21}$ to the amplitude at $\omega_1 - \Delta_{21}$ is included with the calculated curves in Figure 29. The sidebands, resulting from the interfering input signal, at $\omega_1 + \Delta_{21}$ and $\omega_1 - \Delta_{21}$ approach equality as the slope of the amplitude-

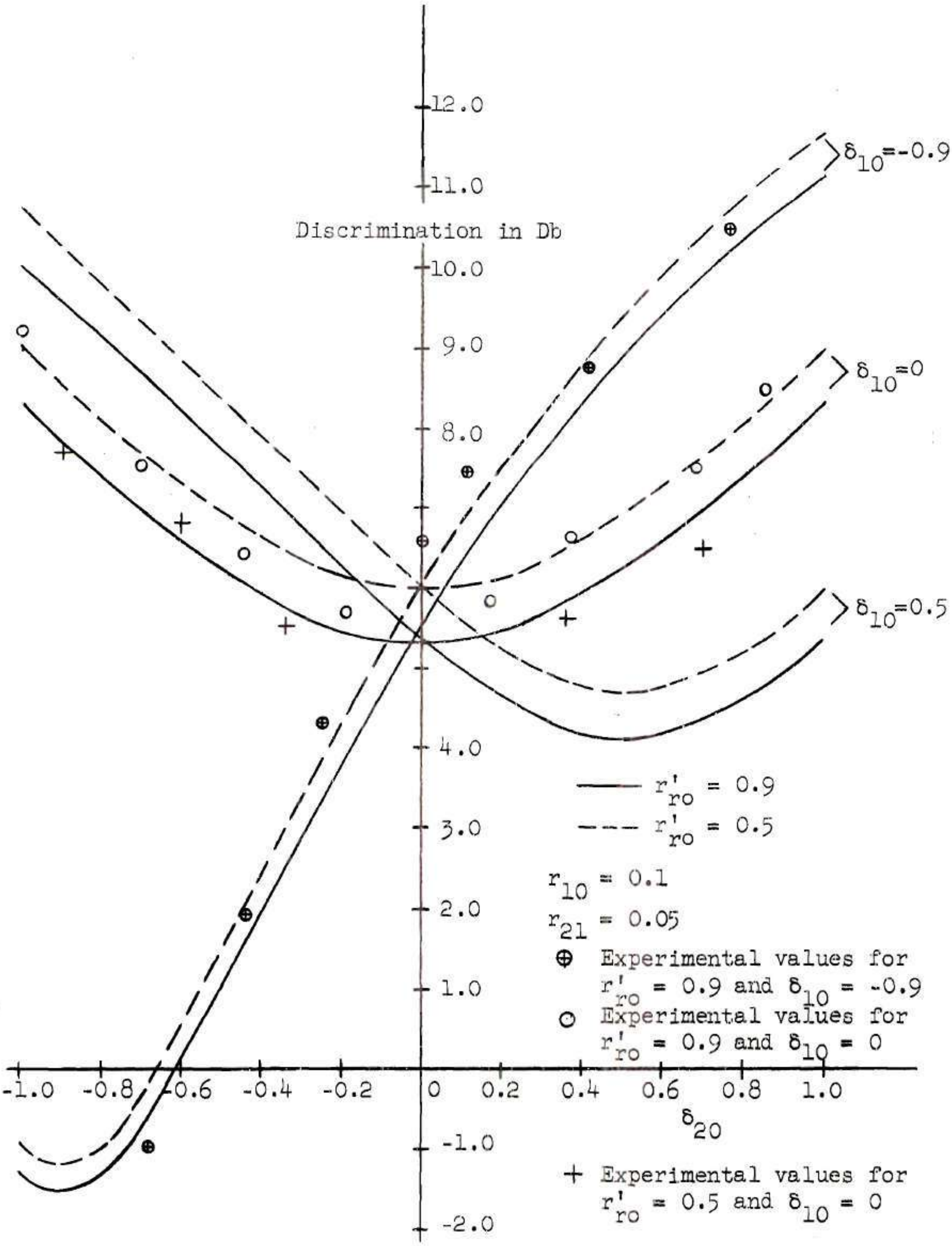


Figure 28. Variation of Discrimination with δ_{20} for Constant Values of δ_{10} and r'_{ro}.

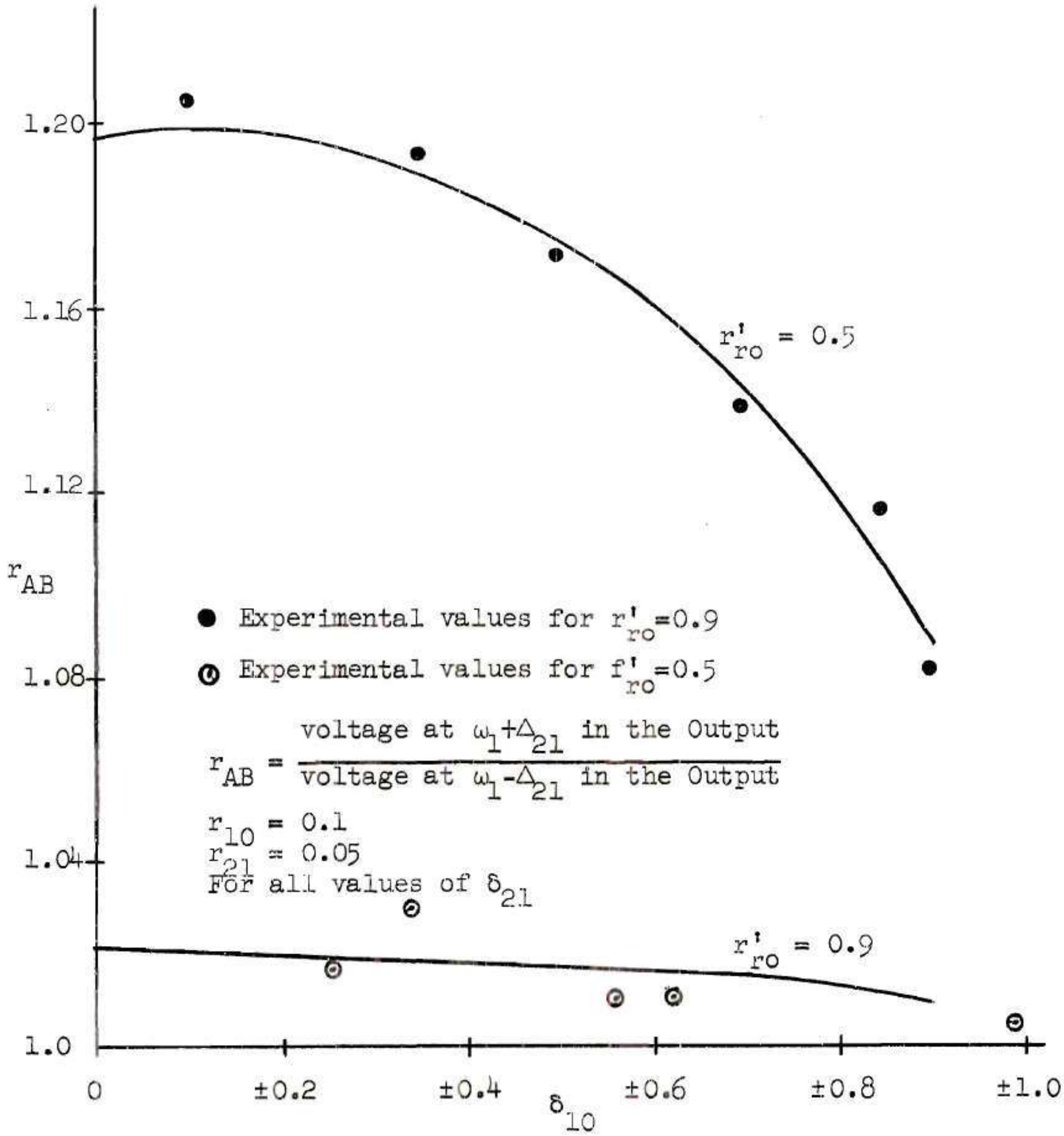


Figure 29. Ratio of the Amplitude at $\omega_1 + \Delta_{21}$ to the Amplitude at $\omega_1 - \Delta_{21}$ as δ_{10} is Varied.

limiting characteristic decreases. If r'_{ro} were equal to unity, then, the sidebands would be equal in magnitude. The variation in these curves for different values of δ_{21} is very slight, therefore, these curves are representative for all values of δ_{21}.

The variation of the amplitudes of the two interference terms in the output is included in Figure 30. These curves are normalized with

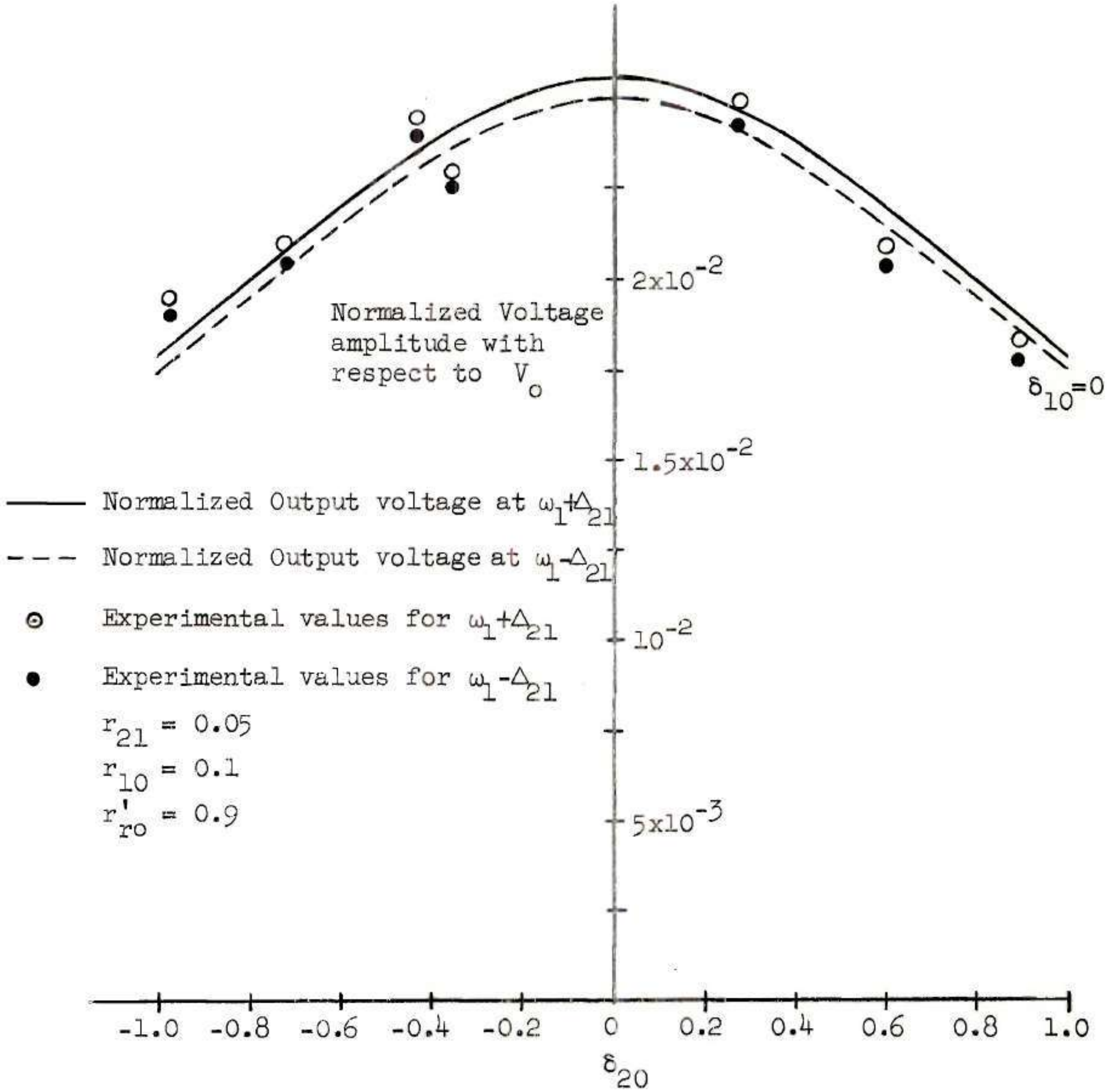


Figure 30. Normalized Voltage Amplitudes of the Interference in the Output as δ_{20} Varies.

respect to the free-running voltage amplitude of the oscillator. The value of this free-running voltage amplitude, in this circuit, was approximately 4.0 volts. The synchronizing voltage input was one-tenth the free-running voltage and the interference voltage was one-twentieth the synchronizing voltage.

To experimentally observe the input/output noise spectral density ratio, the GR Wave Analyzer was used. This wave analyzer has an effective noise bandwidth of 4.0 cps. However, the magnitude of the noise bandwidth was not important to the measurement. The fact that the bandwidth was constant in the band of frequencies used for the measurements made this instrument serve the noise measurement requirement. By tuning the wave analyzer to maximum indication on the output meter with a voltage from an audio oscillator applied, the center frequency of the 4.0 cps bandwidth of the wave analyzer could be established. The frequency of the audio oscillator was determined by use of the EPUT meter. The noise generator used in the tests was a GR 1390-B. For the tests on this oscillator the input noise in the 4.0 cps band of the wave analyzer was set to give five millivolts rms. The noise generator was used on the 20 kilocycle range. Measurements were made on the input and the output for each frequency setting. The average reading of the output meter of the wave analyzer was recorded as the rms noise voltage for a 4.0 cps bandwidth. The ratio of the input-output data for the noise is compared to the predicted ratio as calculated from the input/output data for discrete signals in Figure 31.

Results from van der Pol Type Oscillator

To obtain experimental results to verify the analysis based on van der Pol's equation an oscillator with a schematic diagram as shown in Figure 32 was used.

The transistor used in this circuit was a 2N1038 p-n-p alloy-junction germanium medium power transistor.

In order to insure linear operation of the transistor, the bias was adjusted to give linear input/output relations and the amplitude of

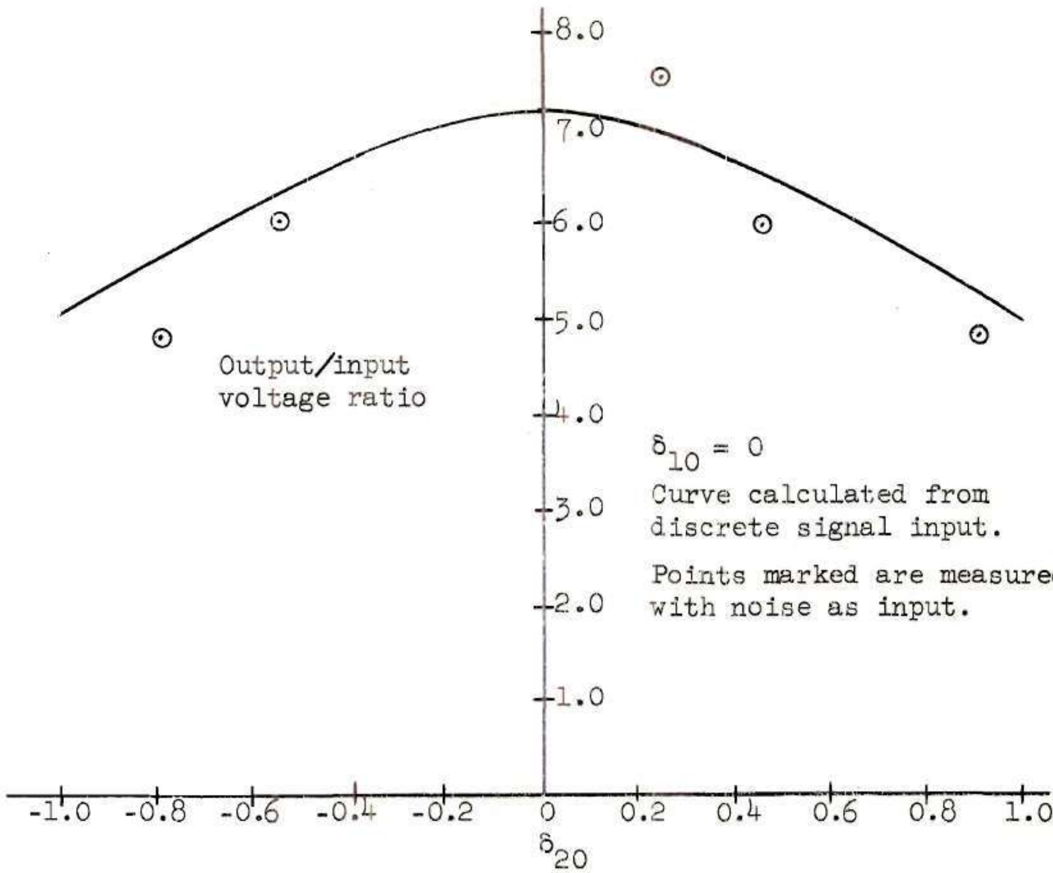


Figure 31. Comparison of the Input Output Noise Ratio to the Predicted Ratio from Discrete Signal Equations.

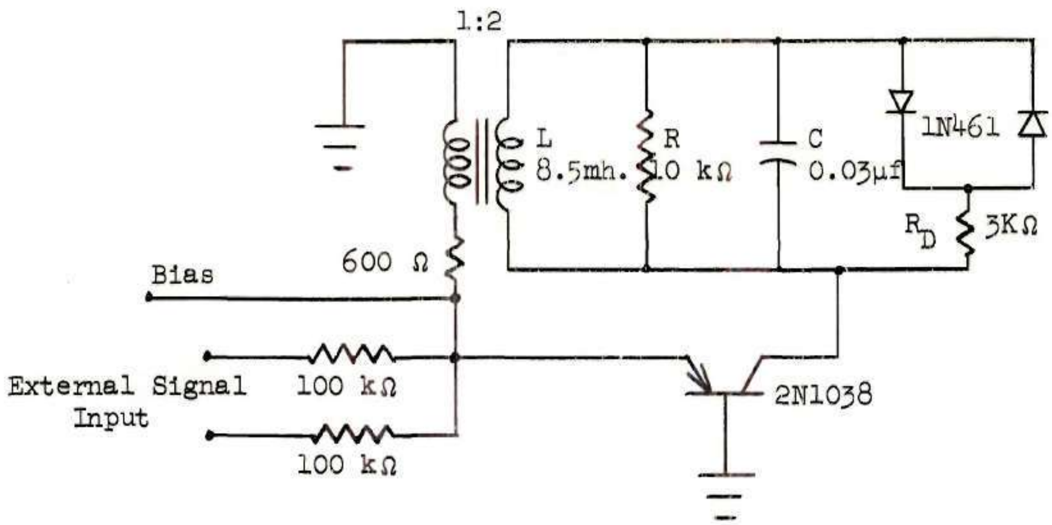


Figure 32. Schematic Diagram of Grounded-base Tuned Collector Transistor Oscillator.

oscillation was limited to a small amplitude. The free-running amplitude of oscillation was approximately 0.4 volts. The cubic nonlinearity desired was obtained by adding a diode circuit in parallel with the oscillator resonant circuit. This circuit consisted of two 1N461 semi-conductor diodes back-to-back. This arrangement eliminated the even power terms in the power series expansion of the total current to the diodes. The magnitude of the series resistance in the diode circuit could theoretically, at least, be determined to eliminate the fifth power in the series expansion of the total current to the diodes. However, it was found that experimental adjustment of this resistance to give results comparable to the calculated values of output voltage variation with synchronizing frequency did not agree with the calculated value to eliminate the fifth power term. The values of R_E and R_D were adjusted experimentally to make the operation of the oscillator agree closely with the calculated response curve. One of the calculated response curves was chosen to check the oscillator performance. This response curve specified the value of F_1. The magnitude of ρ_1, determined from F_1, was obtained by adjustment of the amplitude of the synchronizing signal. Then, R_D was varied to make the value of F_1 calculated from the circuit components agree with the value of F_1 for the response curve. The series emitter resistance was adjusted to make the frequency at the point of instability agree with the calculated value. Since the adjustments of R_E and R_D were interacting, readjustments continued until the oscillator performance agreed with the calculated performance. The equations used to check the performance of the oscillator were as follows:

$$F_1 = \frac{\omega_o^2 A_1}{\alpha a_o} = \frac{I_1 RR_o}{a_o\ (R-R_o)} = \frac{V_1\ \ R_v}{a_o\ (100k.)} \ , \tag{205}$$

and

$$\sigma_{10} = \frac{\omega_1 - \omega_o}{\alpha} = \frac{(\omega_1 - \omega_o)CRR_o}{R-R_o} = (\omega_1 - \omega_o)CR_v \ , \tag{206}$$

where R is the resistance of the oscillator resonant circuit while in a free-running condition; R_o is the value of the resistance across the resonant circuit that will stop free-running oscillations; α_T is the emitter to collector current gain of the transistor, taken as one in this circuit; and R_v is the equivalent of $RR_o/R-R_o$. The experimental procedure to determine α was to parallel the resonant circuit by a variable resistance R_v and lower its value until oscillations ceased, then, $1/R_v$ can be shown to equal αC. The implicit assumption is that α does not change with amplitude of oscillation. The experimental and calculated response curve for this oscillator is shown in Figure 33.

The discrimination against an interfering signal as the frequency of the interference is varied is shown in Figure 34. This graph includes the interference output at the frequency $\omega_1 + \Delta_{21}$ as well as the frequency $\omega_1 - \Delta_{21}$. The effect of the location of the synchronizing frequency relative to the free-running frequency is also shown.

The noise measurements were made in the same manner as for the circuit in Figure 25. The results of these measurements are displayed in Figures 35 and 36. Each of these figures show the comparison of the noise measurements to the predicted input/output noise spectral density ratio from discrete signal measured values.

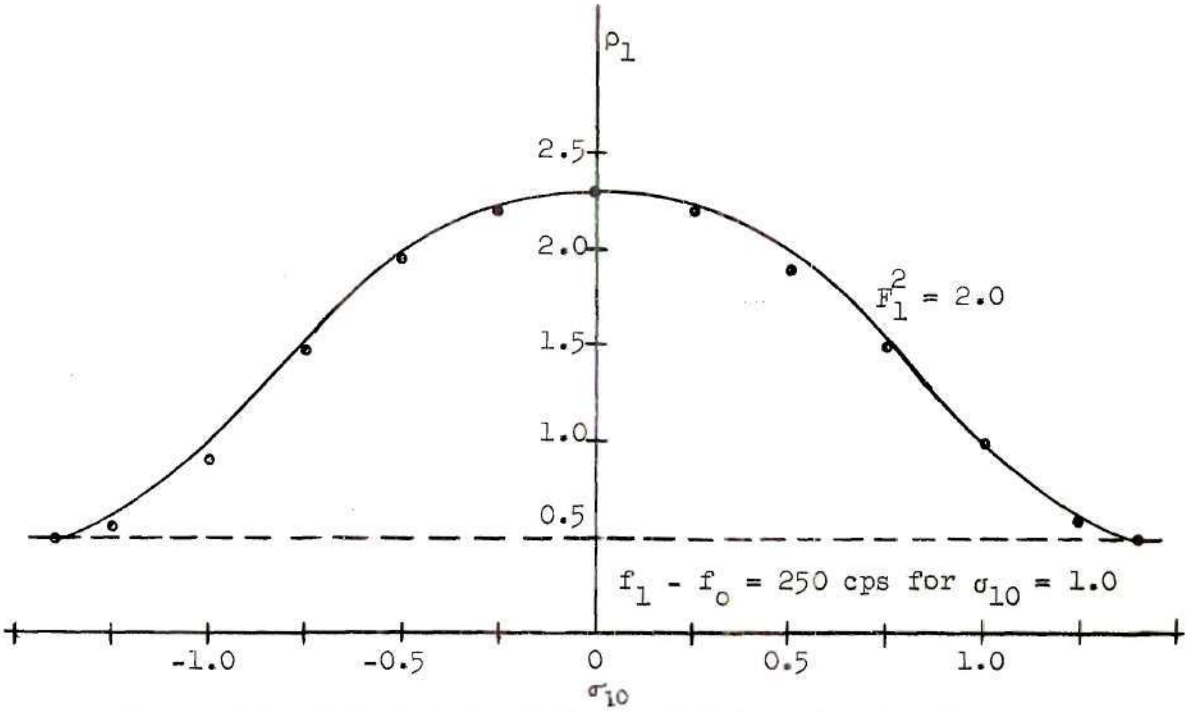


Figure 33. Calculated and Measured Values for the Response Curve of the Test Oscillator.

The bandwidth of synchronization for the transistor oscillator was wider than that of the vacuum-tube oscillator for the range of ρ_1 used. This wider bandwidth of synchronization allowed advantageous use of a Singer-Metrics Panalyzor Model SB-12b. The Panalyzor gives a visual display of the spectrum. Since this model of the Panalyzor was intended for use at higher frequencies than the 10 kc. frequency of the test oscillator, the output from the oscillator was used to amplitude modulate a signal of approximately 490 kc. from a Measurements Corp. 65-B Standard Signal Generator. One of the sidebands resulting from this modulation was examined on the Panalyzor, since it contained the output information of the test oscillator. The resulting photographs are shown in Figure 37. Figure 37 a, b are multiple exposure pictures showing the output at $\omega_1 + \Delta_{21}$

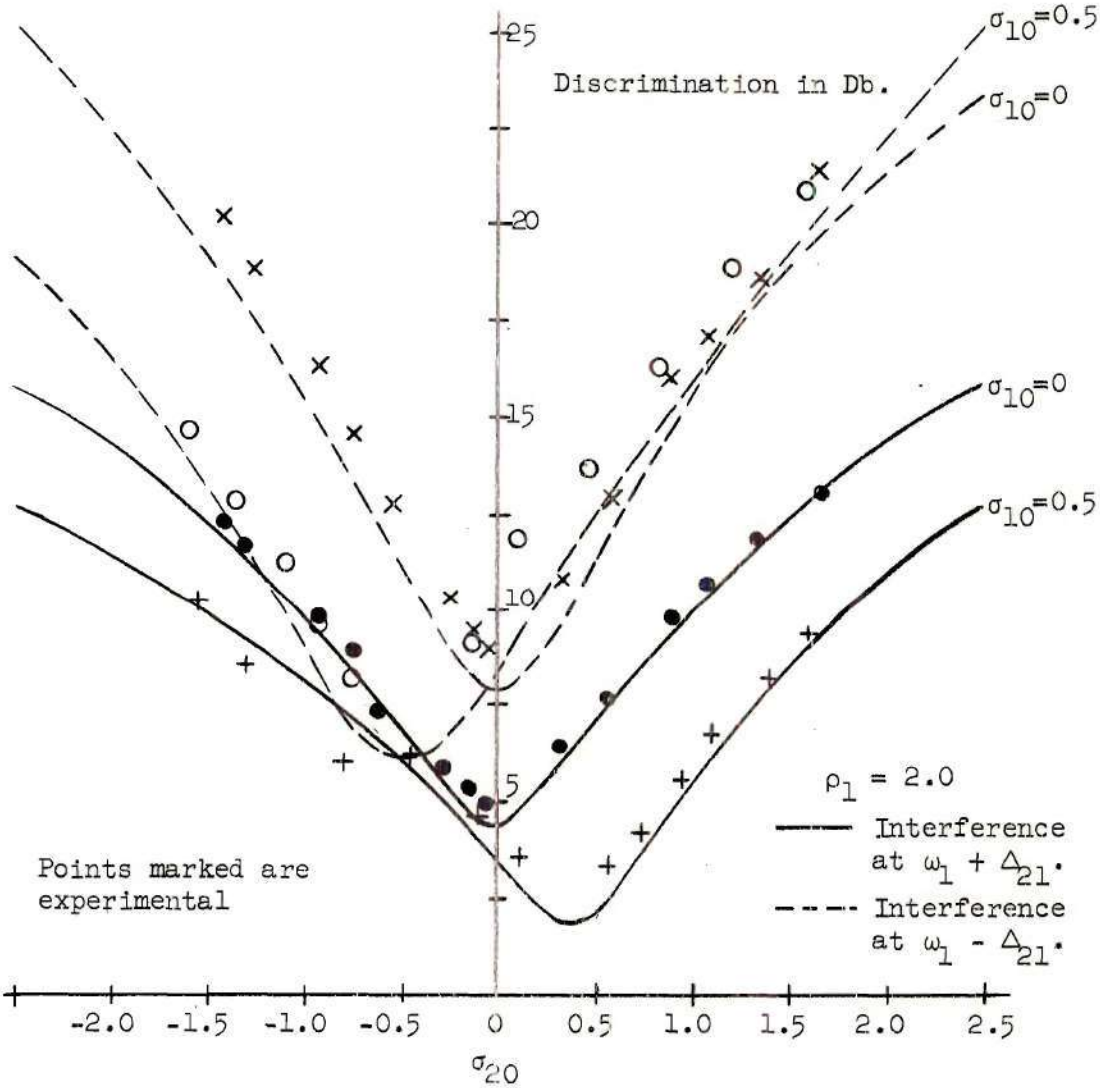


Figure 34. Discrimination Against Interference at $\omega_1 + \Delta_{21}$ and at $\omega_1 - \Delta_{21}$.

and $\omega_1 - \Delta_{21}$ as the frequency of the input interference is varied. Figure 37a is for $\sigma_{10} = -0.5$ with a linear scale on the vertical axis. The larger amplitudes are at $\omega_1 + \Delta_{21}$. Figure 37b is with $\sigma_{10} = 0$ and the vertical scale is in decibels. The interference in the output at $\omega_1 + \Delta_{21}$ is on the high-frequency side of the synchronizing signal and the output at $\omega_1 - \Delta_{21}$ on the low frequency side. Figure 37c, d shows the independence of the amplitude at one interfering signal when a second

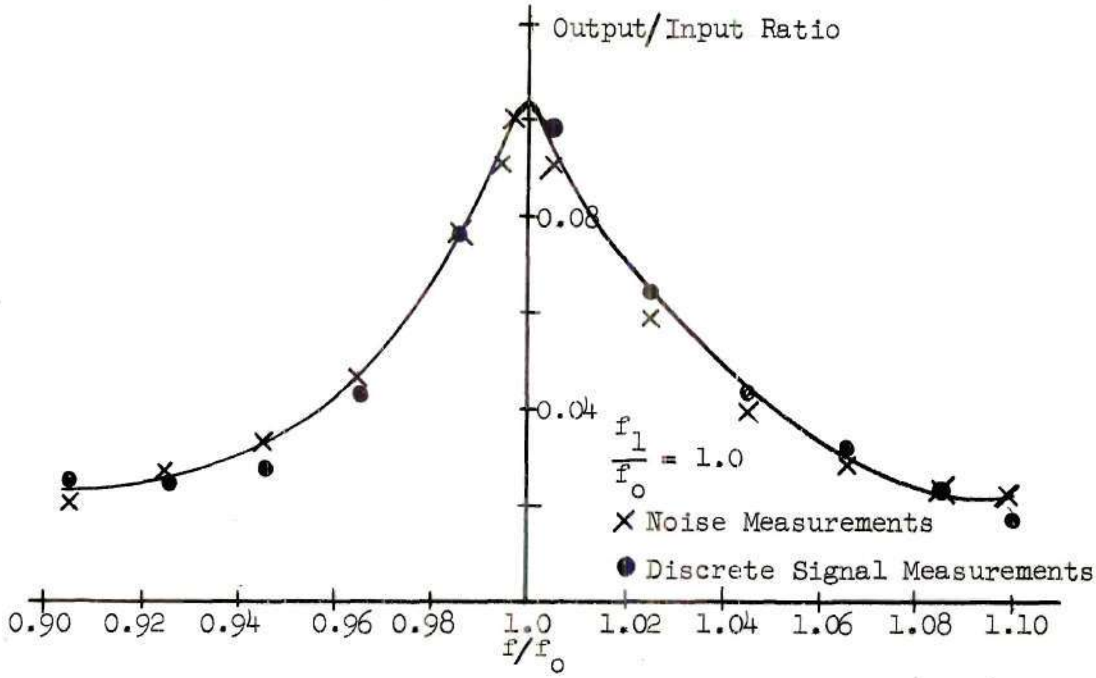


Figure 35. Input/Output Noise Spectral Density Ratio Compared to the Predicted Value from Discrete Signal Measurement. Synchronizing Frequency Equal to Free-running Frequency.

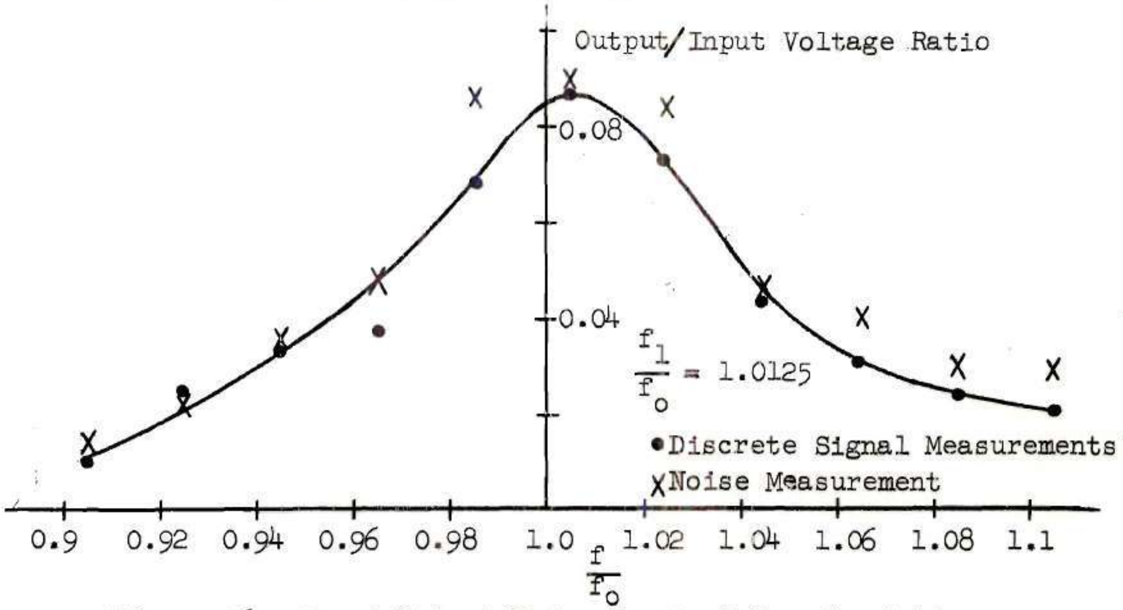
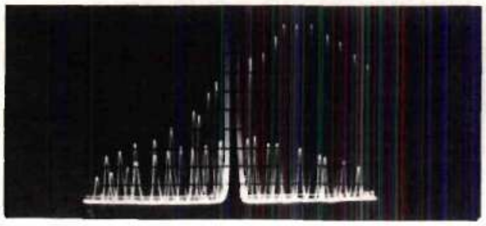


Figure 36. Input/Output Noise Spectral Density Ratio Compared to the Predicted Value from Discrete Signal Measurement.

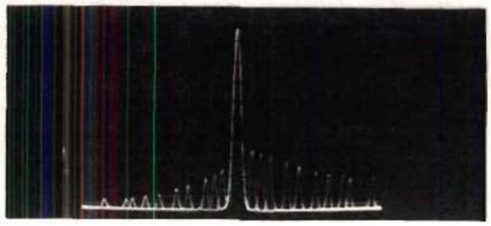
interfering signal is added. The only change from c to d was the addition of the second interfering signal on the input.

Figure 37e, f show the effect of the location of the synchronizing signal on the output noise spectrum. Figure 37e is with a log scale on the vertical axis. The markers are at ±800 cps; $\rho_1 = 2.0$; $a_o = 0.38$ volts(rms); and $\sigma_{10} = 0$. Figure 37f has a linear vertical scale; the markers are at ±1 kc. about f_1; $\rho_1 = 1.5$; $a_o = 0.38$ volts(rms); and $\sigma_{10} = -1.0$.



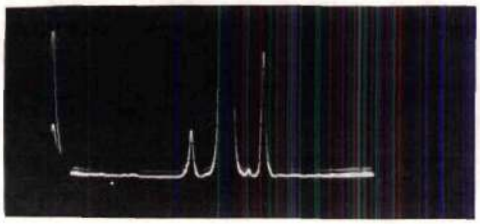
$\sigma_{01} = 1.0$, $\rho_1 = 1.5$, linear scale

(a)



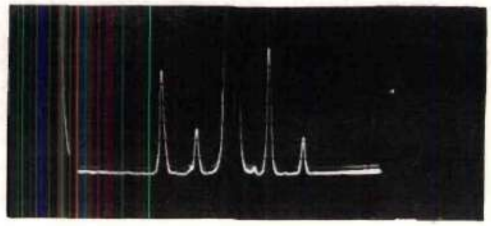
$\sigma_{01} = 0$, $\rho_1 = 2.0$, log scale

(b)



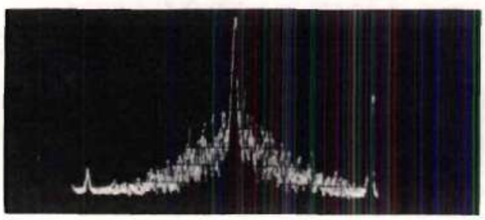
$\sigma_{01} = 1.0$, $\rho_1 = 2.0$, linear scale

(c)



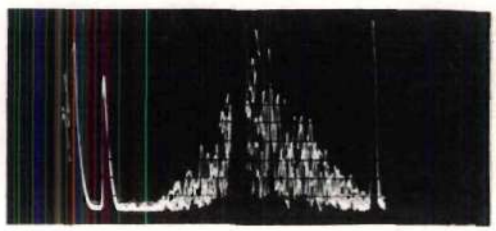
$\sigma_{01} = 0$, $\rho_1 = 2.0$, linear scale

(d)



$\sigma_{01} = 0$, $\rho_1 = 2.0$, log scale

(e)



$\sigma_{01} = 1.0$, $\rho_1 = 1.5$, linear scale

(f)

Figure 37. Photographs of Effects on the Output Produced by Interference and Noise.

CHAPTER VII

SUMMARY AND CONCLUSIONS

This investigation of the influence of noise and interference on the output of a synchronized oscillator includes a development of equations for the determination of the output interference when one discrete interfering signal is externally applied to the oscillator along with the synchronizing signal, a consideration of the effect on the output when multiple discrete interfering signals accompany the synchronizing signal, an application of the results obtained with multiple input interfering signals to the determination of the noise voltage spectral density in the output with noise applied to the input along with the synchronizing signal, and experimental verification of the theory developed.

The determination of the output effects of one discrete interfering frequency is based on Adler's equation. Adler's equation, $\frac{d\beta}{dt} = (\omega_1 - \omega_o) - \omega_c \sin \beta$, is a nonlinear differential equation involving the phase angle β between the synchronizing signal and the voltage returned to the input from the feedback network; the free-running frequency ω_o of the oscillator; the synchronizing signal frequency ω_1; and one-half the bandwidth of synchronization ω_c. The nonlinear amplitude characteristic of the active device does not explicitly appear in the differential equation, however, the imposed conditions require operation in a region of relatively severe amplitude limiting. The effect of the frequency sensitive elements is reflected in an associated phase shift but not in a variation of amplitude. The solution to Adler's equation will yield an equation for β

which approaches a constant value in the limit if $\omega_c > \omega_1 - \omega_o$ or is a periodic function of time when $\omega_c < \omega_1 - \omega_o$. If an interfering signal is applied along with the synchronizing signal and the oscillator is synchronized, the phase angle β will consist of a constant magnitude β_o and a time-varying component β_1. The time-varying component β_1 results in a frequency-modulation effect which is reflected in the instantaneous frequency and an amplitude-modulation effect resulting from the dependence of the voltage returned from the feedback network on β_1. The amplitude modulation resulting is also a function of the amplitude-limiting characteristic.

An electric circuit analogous to the equation for β_1 is shown in Figure 38.

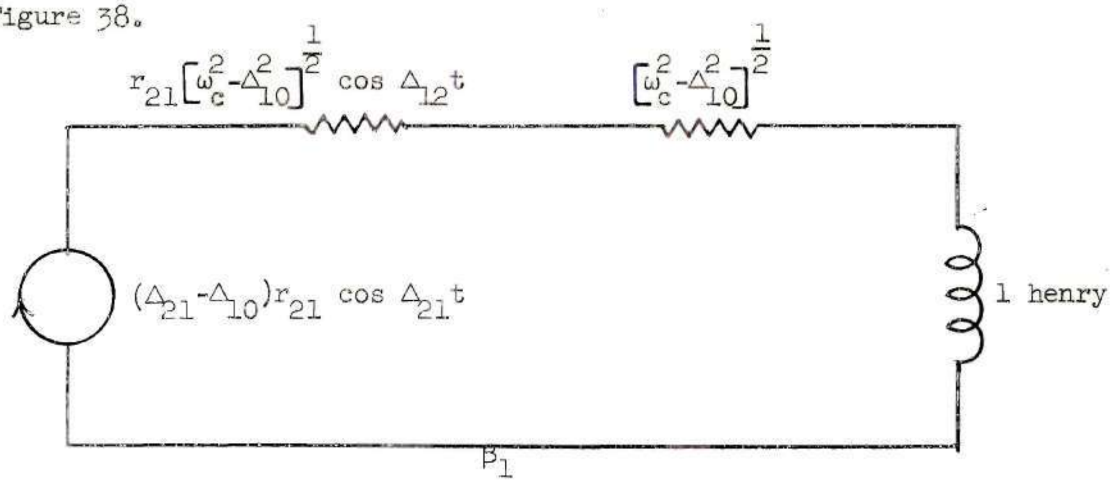


Figure 38. Electrical Circuit Analog Satisfying the Equation for β_1.

When the synchronizing voltage is equal to one-half the bandwidth of synchronization, $\Delta_{10} = \omega_c$, then the resistances in Figure 38 become zero and $\beta_1 = \dfrac{(\Delta_{21} - \Delta_{10})r_{21} \sin \Delta_{21}t}{\Delta_{21}}$. For $\Delta_{21} = \Delta_{10}$, the time-varying component β_1 reduces to zero.

The instantaneous phase of the returned voltage is the sum of the angle β_1 between the effective synchronizing voltage and the returned voltage, the angle θ between the synchronizing signal and the interfering signal, and $\omega_1 t$ the instantaneous angle of the synchronizing voltage. This instantaneous phase angle yields an effective narrow-band frequency-modulation spectrum. Also the angle β_1, along with the amplitude-limiting characteristic, determines the amplitude variation of the returned voltage. The combination of these two types of modulation resulting from the interfering signal yields the amplitude of the interference in the output. Since Adler's equation requires rather sharp amplitude limiting, the main contribution to the interference amplitude is from the frequency-modulation effect.

This development based on Adler's equation places in evidence the effects resulting from one discrete interfering signal externally applied along with the synchronizing signal. The conclusions to be drawn from this analysis are:

(1) The perturbation in the instantaneous phase from an interfering signal results in a combination of frequency and amplitude modulation.

(2) A synchronized oscillator provides discrimination against an interfering signal even without consideration of amplitude attenuation from the frequency sensitive elements.

(3) The discrimination against interfering signals is a minimum at the frequency of the synchronizing signal. In an analysis that included the frequency selectivity the minimum discrimination would be shifted toward the free-running frequency.

(4) The discrimination with the synchronizing equal to the free-

running frequency approaches 6 db with the interfering signal frequency very near the synchronizing frequency.

(5) Under conditions of sharp amplitude limiting and without inclusion of the attenuation resulting from frequency selectivity, the generated interference term at $\omega_1 - \Delta_{21}$ is approximately equal to the interference at the input interfering frequency $\omega_1 + \Delta_{21}$.

(6) With the synchronizing frequency near the edge of the synchronization bandwidth and Δ_{21} small, the interference may experience gain instead of discrimination.

The development based on Adler's equation was applied to the condition of multiple discrete input signals. Also, the solution to the differential equation of an oscillator with cubic nonlinearity was found for the case of multiple discrete input interfering signals. In both approaches with $\left(\sum_{n=2}^{m} r_{n1}\right)^2 \ll 1$, where r_{n1} is the ratio of the input interfering voltage V_n to the input synchronizing voltage V_1, it was found that the interfering amplitudes in the output of importance resulted from the intermodulation between the input interfering signals and the synchronizing signal.

An approximate method, based on the conclusions reached with multiple interfering signals, was determined to find the noise voltage spectral density in the output of a synchronized oscillator. This method is based on the approximation of a narrow-band noise spectrum by a continuous frequency band of independent discrete signals which yields the same noise power in a small frequency increment as the input noise spectrum. The consideration of the input noise as narrow-band is justified because of the response of the oscillator which is effectively a narrow-band device. From the conclusions reached with multiple discrete interfering signals on the input, the output noise voltage spectral density can be predicted from the equations developed for one discrete interfering signal.

If the input to a synchronized oscillator consists of a continuous spectrum of discrete sinusoidal signals centered about the synchronizing frequency, then there will exist a pair of input signals equally spaced above and below the synchronizing frequency, for example one at $\omega_1 - \Delta_{n1}$ and one at $\omega_1 + \Delta_{n1}$. Since each input interfering signal generates an interfering signal, resulting from the intermodulation with the synchronizing signal, the total voltage at any frequency $(\omega_1 - \Delta_{n1})$ will be the square-root of the sum of the squares of the voltage resulting from the input interference at that frequency $(\omega_1 - \Delta_{n1})$ and the voltage caused by the intermodulation from an input frequency $(\omega_1 + \Delta_{n1})$ equally spaced on the opposite side of ω_1. This form of addition of the voltage results because of the noise representation by discrete signals. Therefore, the output noise voltage spectral density may be predicted from the equations derived for a discrete signal interference or by experimentally determining the input output functional relation.

The experimental results verified with good agreement the predictions from calculations.

APPENDIX A

From equations (103, 110)

$$\frac{V_r}{V_o} = \Bigg\{ 1 + \left(\frac{1-r'_{ro}}{r_{ro}}\right)\Bigg[r_{10}(1-\delta_{10}^2)^{\frac{1}{2}} - \frac{r_{20}r_{21}\delta_{10}(1-\delta_{10}^2)^{\frac{1}{2}}(\delta_{21}-\delta_{10})}{2D_1} \tag{A1}$$

$$+ \left(r_{20}(1-\delta_{10}^2)^{\frac{1}{2}} - \frac{r_{10}\delta_{10}r_{21}(\delta_{21}-\delta_{10})(1-\delta_{10}^2)^{\frac{1}{2}}}{D_1} \right) \cos \Delta_{21} t$$

$$- \frac{r_{10}\delta_{10}r_{21}(\delta_{21}-\delta_{10})\delta_{21}}{D_1} \quad \sin \Delta_{21} t \Bigg]\Bigg\} \Bigg\{ \cos \omega_1 t$$

$$- \frac{x}{2}\left[\cos(\omega_1 - \Delta_{21})t - \cos(\omega_1 + \Delta_{21})t\right] + \frac{y}{2}\left[\sin(\omega_1 + \Delta_{21})t\right.$$

$$\left. + \sin(\omega_1 - \Delta_{21})t \right] \Bigg\}.$$

To simplify the solution of this equation (A1) make the following replacements:

$$X = \frac{1-r_{ro}}{r_{ro}} \; ; \; Y = r_{10}\,(1-\delta_{10}^2)^{\frac{1}{2}} - \frac{r_{20}r_{21}\delta_{10}(1-\delta_{10}^2)^{\frac{1}{2}}\,(\delta_{21}-\delta_{10})}{2D_1} \,, \tag{A2}$$

$$Z = r_{20}(1-\delta_{10}^2)^{\frac{1}{2}} - \frac{r_{10}\delta_{10}r_{21}(\delta_{21}-\delta_{10})(1-\delta_{10}^2)^{\frac{1}{2}}}{D_1} \,,$$

and

$$W = \frac{r_{10}\delta_{10}r_{21}(\delta_{21} - \delta_{10})\delta_{21}}{D_1}$$

Then, (A1) becomes

$$\frac{V_r}{V_o} = \left\{ 1 + X\left[Y + Z \cos \Delta_{21}t - W \sin \Delta_{21}t\right]\right\}\left\{\cos \omega_1 t \right. \tag{A3}$$

$$- \frac{x}{2}\left[\cos (\omega_1 - \Delta_{21})t - \cos(\omega_1 + \Delta_{21})t\right] + \frac{y}{2}\left[\sin(\omega_1 + \Delta_{21})t\right.$$

$$\left. + \sin (\omega_1 - \Delta_{21})t\right]\Big\}.$$

Multiplying gives

$$\frac{V_r}{V_o} = (1 + XY) \cos \omega_1 t \tag{A4}$$

$$+ XZ \cos \omega_1 t \cos \Delta_{21} t$$

$$- WX \cos \omega_1 t \sin \Delta_{21} t$$

$$- (1 + XY) \frac{x}{2}\left[\cos (\omega_1 - \Delta_{21})t - \cos (\omega_1 + \Delta_{21})t\right]$$

$$+ (1 + XY) \frac{y}{2}\left[\sin (\omega_1 + \Delta_{21})t + \sin (\omega_1 - \Delta_{21})t\right]$$

$$- XZ \frac{x}{2}\left[\cos \Delta_{21}t \cos (\omega_1 - \Delta_{21})t - \cos \Delta_{21}t \cos (\omega_1 + \Delta_{21})t\right]$$

$$+ WX \frac{x}{2}\left[\sin \Delta_{21}t \cos (\omega_1 - \Delta_{21})t - \sin \Delta_{21}t \cos (\omega_1 + \Delta_{21})t\right]$$

$$+ XZ \frac{y}{2}\left[\cos \Delta_{21}t \sin (\omega_1 + \Delta_{21})t + \cos \Delta_{21}t \sin (\omega_1 - \Delta_{21})t\right]$$

$$- WX \frac{y}{2}\left[\sin \Delta_{21}t \sin (\omega_1 + \Delta_{21})t + \sin \Delta_{21}t \sin (\omega_1 - \Delta_{21})t\right].$$

Expanding and collecting terms yields,

$$\frac{V_r}{V_o} = (1 + XY) \cos \omega_1 t + \frac{1}{2} (WXx + XZy) \sin \omega_1 t \tag{A5}$$

$$+ \frac{1}{2} (XZ + x(1 + XY)) \cos (\omega_1 + \Delta_{21})t + \frac{1}{2}(XZ - x(1+XY)) \cos(\omega_1 - \Delta_{21})t$$

$$+ \frac{1}{2} (-WX + y(1+XY)) \sin (\omega_1 + \Delta_{21})t + \frac{1}{2} (WX + y(1+XY)) \sin(\omega_1 - \Delta_{21})t$$

$$+ \frac{1}{4}(XZx + WXy)\left[\cos\,(\omega_1 + 2\Delta_{21})t - \cos(\omega_1 - 2\Delta_{21})t\right]$$

$$+ \frac{1}{4}(-WXx + XZy)\left[\sin(\omega_1 + 2\Delta_{21})t + \sin(\omega_1 - 2\Delta_{21})t\right].$$

Substitutions for the symbols X, Y, Z, W yields equations (111-121).

APPENDIX B

Equation (138) may be expanded to

$$\cos \omega t = \cos \omega_1 t + \cos\left[\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right)\right]\left[\cos\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right)\right] \quad \text{(B1)}$$

$$- \sin \omega_1 t \sin\left[\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right)\right]\left[\cos\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right)\right]$$

$$+ \sin \omega_1 t \cos\left[\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right)\right]\left[\sin\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right)\right]$$

$$+ \cos \omega_1 t \sin\left[\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right)\right]\left[\sin\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right)\right] .$$

The individual terms of $\frac{x_n}{r_{n1}}$ and $\frac{y_n}{r_{n1}}$ will have the same characteristics as shown in Figures 8 and 9.

If the limits on $|\delta_{10}|$ are again imposed, that is, $0.9 \leqq |\delta_{10}| \leqq 1$, then $\frac{x_n}{r_{n1}}$ and $\frac{y_n}{r_{n1}}$ will be limited to the values $\frac{x_n}{r_{n1}} < \frac{y_n}{r_{n1}} < 2.2$. If these limits are designated $\left(\frac{x_n}{r_{n1}}\right)$ max. and $\left(\frac{y_n}{r_{n1}}\right)$ max. then the expansion of the terms are as follows:

$$\cos\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right) \leqq \cos\left[\left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{x_n}{r_{n1}}\right) \max.\right] \tag{B2}$$

$$= 1 - \frac{1}{2!}\left(\sum_{n=2}^{m} r_{n1}\right)^2 \left[\left(\frac{x_n}{r_{n1}}\right) \max.\right]^2 + \ldots ,$$

$$\cos\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right) \leqq \left[\cos\left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{y_n}{r_{n1}}\right) \max.\right]$$

$$= 1 - \frac{1}{2!}\left(\sum_{n=2}^{m} r_{n1}\right)^2 \left[\left(\frac{y_n}{r_{n1}}\right) \max.\right]^2 + \ldots ,$$

$$\sin\left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right) \leqq \sin\left[\left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{x_n}{r_{n1}}\right) \max.\right]$$

$$= \left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{x_n}{r_{n1}}\right) \max. - \ldots ,$$

$$\sin\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right) \leqq \sin\left[\left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{y_n}{r_{n1}}\right) \max.\right]$$

$$= \left(\sum_{n=2}^{m} r_{n1}\right)\left(\frac{y_n}{r_{n1}}\right) \max. - \ldots .$$

Since $\left(\sum_{n=2}^{m} r_{n1}\right)^2 \ll 1$, the first term in the series expansion of the equations in (B2) may be used to simplify equation (B1), giving

$$\cos \omega t = \cos \omega_1 t - \sin \omega_1 t \left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right) \tag{B3}$$

$$+ \sin \omega_1 t \left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right) + \cos \omega_1 t \left(\sum_{n=2}^{m} x_n \sin \Delta_{n1} t\right)$$

$$\left(\sum_{n=2}^{m} y_n \cos \Delta_{n1} t\right) .$$

Neglecting the last term in equation (B3) because it involves the product of two small values and expanding gives

$$\cos \omega t = \cos \omega_1 t - \frac{1}{2}\sum_{n=2}^{m}\left\{x_n \left[\cos(\omega_1 - \Delta_{n1})t - \cos(\omega_1 + \Delta_{n1})t\right]\right\} \qquad \text{(B4)}$$

$$+ \frac{1}{2} \sum_{n=2}^{m}\left\{y_n \left[\sin(\omega_1 + \Delta_{n1})t + \sin(\omega_1 - \Delta_{n1})t\right]\right\} .$$

The linear approximation to the amplitude curve, equation 109 now becomes, with the substitution of β_1 from equation (136)

$$r_{ro} = 1 + \left(\frac{1-r'_{ro}}{r'_{ro}}\right)\left(r_{10} + \sum_{n=2}^{m} r_{no} \cos \Delta_{n1} t\right)\left[(1-\delta_{10}^2)^{\frac{1}{2}}\right. \qquad \text{(B5)}$$

$$\left. - \delta_{10} \left(\sum_{n=2}^{m} r_{n1} z_n\right)(1 - \delta_{10}^2)^{\frac{1}{2}} \cos \Delta_{n1} t + \Delta_{n1} \sin \delta_{n1} t\right],$$

where $z_n = \dfrac{\delta_{n1} - \delta_{10}}{D_n}$ and $D_n = 1 + \delta_{n1}^2 - \delta_{10}^2$.

Since the inclusion of the effect from the amplitude-limiting characteristic for a single discrete interfering signal produced relatively small changes in the output interference terms, it is reasonable to exclude the higher harmonic terms and cross-modulation terms that occur in equation (B5).

Then,

$$r_{ro} = 1 + \left[\left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\right]\left[r_{10} - \sum_{n=2}^{m} r_{no}\frac{\delta_{10}r_{n1}z_n}{2}\right] \qquad \text{(B7)}$$

$$+ \left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\left(\sum_{n=2}^{m}\left[r_{no}-r_{10}\delta_{10}r_{n1}z_n\right]\cos\Delta_{n1}t\right)$$

$$- \left(\frac{1-r'_{ro}}{r'_{ro}}\right) r_{10}\delta_{10}\left(\sum_{n=2}^{m} r_{n1}z_n\delta_{n1}\sin\Delta_{n1}t\right) .$$

To simplify the combination of equations (B4) and (B7) make the following substitutions in (B7):

$$1 + \sum_{q=2}^{m} W_q = 1 + \left[\left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\right]\left[r_{10} - \sum_{n=2}^{m} r_{no}\frac{\delta_{10}r_{n1}z_n}{2}\right], \qquad \text{(B8)}$$

$$\sum_{q=2}^{m} X_q\cos\Delta_{q1}t = \left(\frac{1-r'_{ro}}{r'_{ro}}\right)(1-\delta_{10}^2)^{\frac{1}{2}}\left(\sum_{n=2}^{m}\left[r_o-r_{10}\delta_{10}r_{b1}z_n\right]\cos\Delta_{n1}t\right),$$

and

$$\sum_{q=2}^{m} Y_q\sin\Delta_{q1}t = -\left(\frac{1-r'_{ro}}{r'_{ro}}\right)r_{10}\delta_{10}\left(\sum_{n=2}^{m} r_{n1}z_n\delta_{n1}\sin\Delta_{n1}t\right).$$

Therefore,

$$r_{ro} = 1 + \sum_{q=2}^{m}\left(W_q + X_q\cos\Delta_{q1}t + Y_q\sin\Delta_{q1}t\right) . \qquad \text{(B9)}$$

To find $\frac{v_r}{V_o}$,

$$\frac{v_r}{V_o} = \left\{ 1 + \sum_{q=2}^{m} (W_q + X_q \cos \Delta_{q1} t + Y_q \sin \Delta_{q1} t \right\} \left\{ \cos \omega_1 t \right. \tag{B10}$$

$$+ \frac{1}{2} \sum_{n=2}^{m} X_n \left[\cos (\omega_1 + \Delta_{n1})t - \cos (\omega_1 - \Delta_{n1})t \right]$$

$$\left. + \frac{1}{2} \sum_{n=2}^{m} Y_n \left[\sin (\omega_1 + \Delta_{n1})t + \sin (\omega_1 - \Delta_{n1})t \right] \right\}.$$

Terms of frequency ω_1 and $(\omega_1 \pm \Delta_{n1})$ which result from this product occur when $q = n$, and may be found as,

$$\frac{V_c(\omega_1)}{V_o} \cos \omega_1 t = (1 + \sum_{n=2}^{m} W_n) \cos \omega_1 t, \tag{B11}$$

$$\frac{V_s(\omega_1)}{V_o} \sin \omega_1 t = \frac{1}{2} \left(\sum_{n=2}^{m} \left[X_n y_n - Y_n x_n \right] \right) \sin \omega_1 t, \tag{B12}$$

$$\frac{V_c(\omega_1 + \Delta_{n1})}{V_o} \cos(\omega_1 + \Delta_{n1})t = \frac{1}{2} \left[(x_n)\left(\frac{V_c(\omega_1)}{V_o}\right) + X_n \right] \cos(\omega_1 + \Delta_{n1})t, \tag{B13}$$

$$\frac{V_s(\omega_1 + \Delta_{n1})}{V_o} \sin(\omega_1 + \Delta_{n1})t = \frac{1}{2} \left[Y_n + (y_n)\left(\frac{V_c(\omega_1)}{V_o}\right) \right] \sin(\omega_1 + \Delta_{n1})t, \tag{B14}$$

$$\frac{V_c(\omega_1 - \Delta_{n1})}{V_o} \cos(\omega_1 - \Delta_{n1})t = \frac{1}{2} \left[-(x_n)\left(\frac{V_c(\omega_1)}{V_o}\right) + X_n \right] \cos(\omega_1 - \Delta_{n1})t, \tag{B15}$$

$$\frac{V_s(\omega_1 - \Delta_{n1})}{V_o} \sin(\omega_1 - \Delta_{n1})t = \frac{1}{2} \left[- Y_n + (y_n)\left(\frac{V_c(\omega_1)}{V_o}\right) \right] \sin(\omega_1 - \Delta_{n1})t. \tag{B16}$$

where $n = 2, 3, \ldots m$.

Again the restriction $(\sum_{n=2}^{m} r_{n1})^2 \ll 1$, allows neglect of sideband terms of higher order than the first and also of the cross-modulation products.

APPENDIX C

The differential equation (150) for the oscillator is

$$\ddot{v} - \alpha\dot{v} + 3\gamma\dot{v}v^2 + \omega_o^2 v = \sum_{n=1}^{m} A_n \omega_o^2 \sin(\omega_1 + \Delta_{n1})t, \tag{C1}$$

and the assumed solution is

$$v = \sum_{n=1}^{m} (b_{2n-1} \sin(\omega_1 + \Delta_{n1})t + b_{2n} \cos(\omega_1 + \Delta_{n1})t \tag{C2}$$

$$+ d_{2n-1} \sin(\omega_1 - \Delta_{n1})t + d_{2n} \cos(\omega_1 - \Delta_{n1})t),$$

where

$$d_1 = d_2 = 0 \text{ and } \Delta_{11} = 0 . \tag{C3}$$

To conserve space, the following simplifying definitions are made:

$$\cos(\omega_1 + \Delta_{n1})t = C_{1+n}, \tag{C4}$$

$$\sin(\omega_1 + \Delta_{n1})t = S_{1+n},$$

$$\cos(\omega_1 - \Delta_{n1})t = C_{1-n},$$

$$\sin(\omega_1 - \Delta_{n1})t = S_{1-n}.$$

Taking the first derivative yields

$$\dot{v} = \sum_{n=1}^{m} \left[\dot{b}_{2n-1} S_{1+n} + b_{2n-1} (\omega_1+\Delta_{n1})C_{1+n}\right. \tag{C5}$$

$$+ b_{2n} C_{1+n} - b_{2n} (\omega_1 - \Delta_{n1}) S_{1+n}$$

$$+ \dot{d}_{2n-1} S_{1-n} + d_{2n-1}(\omega_1 - \Delta_{n1}) C_{1-n}$$

$$+ \dot{d}_{2n} C_{1-n} - d_{2n} (\omega_1 - \Delta_{n1}) S_{1-n} \Big] .$$

The second derivative gives

$$\ddot{v} = \sum_{n=1}^{m} \Big[\ddot{b}_{2n-1} S_{1+n} + 2\dot{b}_{2n-1} (\omega_1 + \Delta_{n1}) C_{1+n} \tag{C6}$$

$$- b_{2n-1} (\omega_1 + \Delta_{n1})^2 S_{1+n} + \ddot{b}_{2n} C_{1+n}$$

$$- 2\dot{b}_{2n} (\omega_1 + \Delta_{n1}) S_{1+n} - b_{2n} (\omega_1 + \Delta_{n1})^2 C_{1+n}$$

$$+ \ddot{d}_{2n-1} S_{1-n} + 2\dot{d}_{2n-1} (\omega_1 - \Delta_{n1}) C_{1-n}$$

$$- d_{2n-1}(\omega_1 - \Delta_{n1})^2 S_{1-n} + \ddot{d}_{2n} C_{1-n} + \dot{d}_{2n}(\omega_1 - \Delta_{n1}) S_{1-n}$$

$$- d_{2n} (\omega_1 - \Delta_{n1})^2 C_{1-n} \Big] .$$

Squaring v gives

$$v^2 = \sum_{q=1}^{m} \sum_{s=1}^{m} \Big[b_{2q-1} b_{2s-1} S_{1+q} S_{1+s} \tag{C7}$$

$$+ b_{2q-1} b_{2s} S_{1+q} C_{1+s}$$

$$+ b_{2q} b_{2s} C_{1+q} C_{1+s}$$

$$+ 2b_{2q-1} d_{2s-1} S_{1+q} S_{1-s}$$

$$+ 2b_{2q-1} d_{2s} S_{1+q} C_{1-s}$$

$$+ 2b_{2q}\, d_{2s-1}\, C_{1+q}\, S_{1-s}$$

$$+ 2b_{2q}\, d_{2s}\, C_{1+q}\, C_{1-s}$$

$$+ d_{2q-1}\, d_{2s-1}\, S_{1-q}\, S_{1-s}$$

$$+ d_{2q}\, d_{2s}\, C_{1-q}\, C_{1-s}$$

$$+ 2d_{2q-1}\, d_{2s}\, S_{1-q}\, C_{1-s} \Big] .$$

In forming the product $v^2 \dot{v}$, consideration of equation (C5) indicates the first derivatives may be neglected in $\dot{v}$. This also applies to the term $\alpha \dot{v}$. The first derivative terms result from substitution in $\ddot{v}$, $-\alpha\dot{v}$, and $3\gamma\dot{v}v^2$. If all terms in each are divided by 4α, typical terms involving $\dot{b}$ or $\dot{d}$ will be $\frac{\omega\dot{b}}{4\alpha}$; $\frac{\omega\dot{d}}{4\alpha}$, $-\dot{b}$; $-\dot{d}$, $\frac{b^2\dot{b}}{\alpha_o^2}$; $\frac{bd\dot{d}}{\alpha_o^2}$; $\frac{d^2\dot{d}}{\alpha_o^2}$, respectively. Then as far as the first derivative terms are concerned, the ones derived from $\ddot{v}$ will predominate. Therefore, it is justifiable to neglect the first derivative terms in $\dot{v}$, then $\dot{v}v^2$ gives

$$\dot{v}v^2 = \sum_{n=1}^{m} \sum_{q=1}^{m} \sum_{s=1}^{m} \Big[(\omega_1+\Delta_{n1}) b_{2n-1} C_{1+n} \Big\{ b_{2q-1} b_{2s-1} S_{1+q} S_{1+s} \qquad \text{(C8)}$$

$$+ 2b_{2q-1}\, b_{2s}\, S_{1+q}\, S_{1+s} + \dots \Big\}$$

$$- (\omega_1+\Delta_{n1}) b_{2n} S_{1+n} \Big\{ b_{2q-1} b_{2s-1} S_{1+q} S_{1+s} + 2b_{2q-1} b_{2s} S_{1+q} S_{1+s}$$

$$+ \dots \Big\} + (\omega_1 - \Delta_{n1})\, d_{2n-1} C_{1-n} \Big\{ b_{2q-1}\, b_{2s-1}\, S_{1+q}\, S_{1+s}$$

$$+ 2b_{2q-1}\, b_{2s}\, S_{1+q}\, S_{1+s} + \dots \Big\} \Big] .$$

In each of the products of equation (C8) there is involved the product of three sinusoids. These products may be expanded as follows:

$$C_{1\pm n}\, C_{1\pm q}\, C_{1\pm s} = \frac{1}{4}\left\{C_{1\pm n\pm q\pm s} + C_{1\pm q\pm s\pm n} + C_{1\pm s\pm n\pm q} + C_{3\pm n\pm q\pm s}\right\}, \quad (C9)$$

where

$$C_{3\pm n\pm q\pm s} = \cos\,(3\omega_1 t \pm \Delta_{n1} \pm \Delta_{q1} \pm \Delta_{s1}). \quad (C10)$$

$$S_{1\pm n}S_{1\pm q}C_{1\pm s} = \frac{1}{4}\left\{-\,C_{1\pm n\pm q\pm s} + C_{1\pm q\pm s\pm n} + C_{1\pm s\pm n\pm q} + C_{3\pm n\pm q\pm s}\right\}. \quad (C11)$$

$$S_{1\pm n}C_{1\pm q}C_{1\pm s} = \frac{1}{4}\left\{S_{1\pm n\pm q\pm s} - S_{1\pm q\pm s\pm n} + S_{1\pm s\pm n\pm q} + S_{3\pm n\pm q\pm s}\right\}. \quad (C12)$$

$$S_{1\pm n}S_{1\pm q}S_{1\pm s} = \frac{1}{4}\left\{S_{1\pm n\pm q\pm s} + S_{1\pm q\pm s\pm n} + S_{1\pm s\pm n\pm q} - S_{3\pm n\pm q\pm s}\right\}. \quad (C13)$$

Since the third harmonic terms are out of the pass band, the fourth terms in each expansion (C9, C11, C12, C13) may be neglected. For sinusoidal terms such as $\cos\,(\omega_1 \pm \Delta_{x1})t$ and $\sin(\omega_1 \pm \Delta_{x1})t$ to result from equations (C9, C11, C12, C13) it is necessary that one of the summation indices be equal to x and the other two equal to each other, for example, $n = x$, and $q = s$. To obtain all the resulting terms requires consideration of all combinations of n, q, s except for the special case where $n = q = s = x$. Of all the terms resulting when two of the indices are equal the largest amplitudes occur when they are equal to unity. In this case the amplitude will be $b_1^2 b_x$, $b_2^2 b_x$, $b_1^2 d_x$, or $b_2^2 d_x$. When two indices are equal but not equal unity then the amplitude product will involve all interference output amplitudes. Since by definition all "d's" are interference amplitudes, then terms involving two or more "d's" are very small. Reference to the original assumption that the square of the sum of interference components in the output is much less than the square of the synchronizing signal in the output, these terms which result when two of the indices are not equal to unity may be neglected. Actually, this approximation is less stringent than the original because it involves the sum of squares.

In summary, to obtain the output at a frequency $(\omega_1 \pm \Delta_{x1})$,

(a) set one index to x, equate the other two to unity and consider all combinations of the indices $(1,1,x)$.

(b) Neglect all products which involve more than one "d".

When two of the indices are set equal to unity, the associated difference frequency, Δ, becomes Δ_{11} which is equal to zero.

A typical amplitude term for $\cos(\omega_1 + \Delta_{x1})t$ obtained from equation (C8) becomes

$$\begin{aligned}
&\frac{1}{2}(\omega_1 + \Delta_{x1})b_{2x-1}b_1^2 + \frac{1}{2}(\omega_1 + \Delta_{x1})\, b_{2x-1}b_2^2 + \omega_1 b_1 b_2 b_{2x} \qquad \text{(C14)}\\
&+ \frac{1}{2}\,\omega_1 b_1^2 d_{2x-1} + \frac{1}{2}\,\omega_1 b_1 b_2 d_{2x} - \omega_1 b_1 b_2 b_{2x} - \frac{1}{2}\,\omega_1 b_1 b_2 d_{2x}\\
&- \frac{1}{2}\,\omega_1 b_2^2 d_{2x-1} - \frac{1}{4}(\omega_1 - \Delta_{x1})\, b_1^2 d_{2x-1} + \frac{1}{4}(\omega_1 - \Delta_{x1})\, b_2^2 d_{2x-1}\\
&- \frac{1}{2}(\omega_1 - \Delta_{x1})\, b_1 b_2 d_{2x} = (\omega_1 + \Delta_{x1})\Big(\frac{b_{2x-1}}{2}\left[b_1^2 + b_2^2\right] + \frac{d_{2x-1}}{4}\\
&(b_1^2 - b_2^2) + \frac{d_{2x}}{2}\, b_1 b_2\Big)\,.
\end{aligned}$$

In a similar manner the coefficient for the $\sin(\omega_1 + \Delta_{x1})t$ term is found as

$$(\omega_1 + \Delta_{1x})\left[-\frac{b_{2x}}{2}(b_1^2 + b_2^2) - \frac{d_{2x-1}}{2}\, b_1 b_2 + \frac{d_{2x}}{4}(b_1^2 - b_2^2)\right]. \qquad \text{(C15)}$$

The coefficient of the $\cos(\omega_1 - \Delta_{x1})t$ term is

$$(\omega_1 - \Delta_{x1})(\frac{b_{2x-1}}{4}\left[b_1^2 - b_2^2\right] + \frac{b_{2x}}{2}(b_1 b_2) + \frac{d_{2x-1}}{2}\ b_1^2 + b_2^2), \quad (C16)$$

and for the $\sin(\omega_1 - \Delta_{x1})t$ term

$$(\omega_1 - \Delta_{x1})(\frac{b_{2x}}{4}\left[b_1^2 - b_2^2\right] + \frac{b_{2x-1}}{2}(b_1 b_2) - \frac{d_{2x}}{2}(b_1^2 + b_2^2)). \quad (C17)$$

The above equations were found by taking all combinations of (1,1,x). This gives correct solution for all values of x except $x = 1$. For $x = 1$, the solutions must be divided by three.

The complete solution for $\dot{v}v^2$ then becomes

$$\dot{v}v^2 = \frac{\omega_1}{4}\left[b_1(b_1^2+b_2^2)\right]\cos\omega_1 t - \frac{\omega_1}{4}\ b_2(b_1^2 + b_2^2) \quad (C18)$$

$$+ \sum_{n=2}^{m}\left\{(\omega_1+\Delta_{n1})(\frac{b_{2n-1}}{2}\left[b_1^2+b_2^2\right] + \frac{d_{2n-1}}{4}\left[b_1^2 - b_2^2\right]\right.$$

$$\left. + \frac{d_{2n}}{2}b_1 b_2\right\}\cos(\omega_1+\Delta_{n1})t + \sum_{n=2}^{m}\left\{(\omega_1+\Delta_{n1})(\frac{-b_{2n}}{2}\left[b_1^2+b_2^2\right]\right.$$

$$\left. - \frac{d_{2n-1}}{2}b_1 b_2 + \frac{d_{2n}}{4}\left[b_1^2-b_2^2\right]\right\}\sin(\omega_1+\Delta_{n1})t + \sum_{n=2}^{m}\left\{(\omega_1-\Delta_{n1})\right.$$

$$\left.(\frac{b_{2n-1}}{4}\left[b_1^2-b_2^2\right] - \frac{b_{2n}}{2}b_1 b_2 + \frac{d_{2n-1}}{2}(b_1^2+b_2^2))\right\}\cos(\omega_1-\Delta_{n1})t$$

$$+ \sum_{n=2}^{m}\left\{(\omega_1-\Delta_{n1})(\frac{b_{2n}}{4}\left[b_1^2-b_2^2\right] - \frac{b_{2n-1}}{2}b_1 b_2 - \frac{d_{2n}}{2}(b_1^2+b_2^2)\right\}$$

$$\sin(\omega_1 - \Delta_{n1})t\ .$$

Combination of terms from $\ddot{v}$, $-\alpha\dot{v}$, $3\gamma\dot{v}v^2$, $\omega_o^2 v$, and $\sum_{n=1}^{m} A_n \omega_o^2 \sin$ $(\omega_1 + \Delta_{n1})t$ for like time functions and retaining only the larger terms yields, equating $\cos \omega_1 t$ terms

$$2\dot{b}_1\omega_1 - b_2\omega_1^2 - \alpha b_1\omega_1 + \frac{3\gamma}{4}\omega_1 b_1 (b_1^2 + b_2^2) + \omega_o^2 b_2 = 0 , \tag{C19}$$

and $\sin \omega_1 t$ terms

$$-2\dot{b}_2\omega_1 - b_1\omega_1^2 + \alpha b_2\omega_1 - \frac{3}{4}\gamma\omega_1 b_2(b_1^2 + b_2^2) + \omega_o^2 b_1 = -A_1\omega_o^2, \tag{C20}$$

which become

$$2\dot{b}_1 + 2\Delta_{01}b_2 - \alpha_{b1}(1 - \rho_1) = 0, \tag{C21}$$

$$2\dot{b}_2 - 2\Delta_{01}b_1 - \alpha b_2 (1 - \rho_1) = \frac{-A_1\omega_o^2}{\omega_1} \doteq -A_1\omega_o , \tag{C22}$$

with

$$2\Delta_{01} = \frac{2(\omega_o - \omega_1)}{\omega_1} \doteq \frac{\omega_o^2 - \omega_1^2}{\omega_1} , \tag{C23}$$

$$\rho_1 = \frac{b_1^2 + b_2^2}{a_o^2} , \tag{C24}$$

and

$$a_o^2 = \frac{4\alpha}{3\gamma} . \tag{C25}$$

Also equating $\cos(\omega_1 + \Delta_{n1})t$ terms

$$2\dot{b}_{2n-1} + 2\left[\Delta_{01} - \Delta_{n1}\right] b_{2n} - \alpha b_{2n-1}(1 - 2\rho_1) + d_{2n-1}\rho_s + d_{2n}\rho_m = 0 , \tag{C26}$$

where

$$\rho_s = \frac{b_1^2 - b_2^2}{a_o^2}\ , \ \rho_m = \frac{2b_1 b_2}{a_o^2}\ , \text{ and } n = 2, 3, \ldots, m. \tag{C27}$$

Similarly from $\sin(\omega_1 + \Delta_{n1})t$ terms

$$2\dot{b}_{2n} - 2(\Delta_{01} - \Delta_{n1})\, b_{2n-1} - \alpha b_{2n}(1 - 2\rho_1) + d_{2n-1}\,\rho_m - d_{2n}\,\rho_s \doteq - A_n\omega_o\ . \tag{C28}$$

Equating $\cos(\omega_1 - \Delta_{n1})t$ terms gives

$$2\dot{d}_{2n-1} + 2(\Delta_{01} + \Delta_{n1})d_{2n} - \alpha d_{2n-1}\,(1 - 2\rho_1) + b_{2n-1}\,\rho_s + b_{2n}\rho_m = 0\ , \tag{C29}$$

and finally from the $\sin(\omega_1 - \Delta_{n1})t$ terms,

$$2\dot{d}_{2n} - 2\left[\Delta_{01} + \Delta_{n1}\right] d_{2n-1} - \alpha d_{2n}(1 - 2\rho_1) - b_{2n}\rho_s + b_{2n-1}\,\rho_m = 0\ . \tag{C30}$$

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VITA

Jim Thomas Long was born on October 5, 1923 in Central, South Carolina. He is the son of Weston H. Long and Ollie S. Long. In 1946 he married the former Tucker Elizabeth Dickson and they are the parents of one son, Jim, Jr.

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While in the U. S. Navy he attended Pre-Radar School at Bowdoin College and Radar School at the Massachusetts Institute of Technology. From 1943 through 1963 he has been on the teaching staff of Clemson College (except for leaves of absence) where he has the rank of Associate Professor. From 1957 through 1963 he has been an Assistant Professor on the staff of Georgia Institute of Technology.

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