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THE SENSITIVITY PROBLEM IN CONTROL SYSTEM OPTIMIZATION

A THESIS

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SUMMARY

The purpose of this study is to develop a procedure for synthesizing the optimum control for a plant with a known structure but with varying parameters which will permit a priori constraints on the system's sensitivity to plant parameter changes to be imposed.

The basic problem arises from the fact that conventional optimization schemes assume the plant parameters are fixed. The controller derived from these schemes may prove to be unsatisfactory if the plant parameters do, in fact, vary from their nominal values.

The sensitivity problem exists because of the natural tendency of the plant parameters to vary. This variance could be caused by environmental changes, ageing, or the inherent stochastic nature of the parameter.

The approach of this study is to consider low sensitivity to be a figure of merit for the system similar to low system error, high speed of response, low input energy requirements, etc.

In order that the synthesis method be applicable to a wide class of problems, both linear and non-linear, time-domain variational techniques are employed. The plant equation(s) establishing the basic relations between the plant input and output are assumed to be known.

The sensitivity of a system may be described as the degree to which some state variable of the system, usually the system output, varies under changes in the values of the plant parameters. Mathematically, this sensitivity could be described by a sensitivity coefficient

defined as the partial derivative of the output with respect to a particular plant parameter of interest. In general, this sensitivity coefficient will be a function of the parameter and of time and gives an instantaneous measure of the variance of the output per a small change in the parameter value. If $\underline{x}$ is the plant output and $\underline{u}$ is the sensitivity coefficient, then $\underline{u} = \frac{\partial \underline{x}}{\partial q}$ where q is the parameter of interest.

Before an optimization scheme can be applied to the problem, a measure of sensitivity must be devised which can be included in a performance index or constrained to meet the requirements of a particular problem. Several sensitivity measures based on the sensitivity coefficient $u(t,q)$ are presented and discussed. In general, these sensitivity measures assume a form similar to the common measures of error, e.g., integral square error, maximum error magnitude in a given time range, etc.

In order to incorporate the sensitivity measure into the optimization procedure, a relation between the sensitivity coefficient and the dynamics of the plant system must be established. This is accomplished by means of the sensitivity equations. The sensitivity equations are formed by differentiating the given plant equation(s) with respect to the parameter of interest. If the plant is of order n, the result is an expanded system of order 2n consisting of the plant equations and the sensitivity equations derived from them. This 2n system can be thought of as a set of natural constraints on the sensitivity optimization problem.

An additional constraint may be applied to the problem by restricting the sensitivity measure to lie below some predetermined value.

An alternate technique for including sensitivity restrictions in the problem is to add the weighted sensitivity measure to the usual performance functional of the system. The effect of this technique is to balance sensitivity against other cost functionals of the system using some a priori weighting scheme. The more heavily the sensitivity measure is weighted the less sensitive the system resulting from the optimization becomes. This reduction in sensitivity is, of course, gained at the expense of an increase in the other cost functions of the system.

The sensitivity optimization problem has thus been put into a form which is amenable to solution by several well-known variational techniques, e.g., Pontryagin's and Bellman's optimization methods.

Application of Pontryagin's method requires the solution of a 4n order two-point boundary value problem. Methods for solution of typical boundary value problems are discussed in the study.

Application of Bellman's method requires the solution of partial differential equations which in turn leads to an n(2n+1) order initial value problem.

Assumptions required in the derivation of the synthesis method limit its application to the class of problems where:

(1) The approximation $\frac{\Delta \underline{x}}{\Delta q} \approx \frac{\partial \underline{x}}{\partial q}$ is valid.

(2) The performance index evaluated for the optimum input is continuous in all parameters throughout their range of variation.

(3) The plant parameters remain constant during a given plant but but are assumed to vary from run to run.

The overall effect of (1) and (2) above is to restrict the permissible range of parameter variation.

Various examples which demonstrate the salient features of the method are presented and discussed. A general result for the linear plant with quadratic performance index is determined.

CHAPTER I

INTRODUCTION

Definition of the Problem

The basic problem under study arises from the fact that conventional schemes for control system optimization, which assume that the parameters of the controlled plant are fixed, may yield unsatisfactory results if the plant parameters vary from their nominal values. The purpose of this study is to develop a procedure for synthesizing the optimum controller for a plant with a known structure but variable parameters. "Optimum" is defined in such a way that low sensitivity to plant parameter changes becomes a figure of merit for the system similar to low error, high speed of response, or low energy demand at the input. The result of the optimization scheme is a system which has reduced sensitivity to plant parameter changes and is "optimum" in some sense defined by the performance index and the constraints on the problem. Either the performance index includes a measure of the system's sensitivity to plant parameter changes, or a priori constraints on the system's sensitivity are imposed.

The plant itself is a physical system composed of various physical components, such as motors, amplifiers, passive elements, etc. The plant parameters are quantities such as moments of inertia, capacitance, resistance, inductance, amplifier gains, initial conditions on variables within the system, etc.

In Figure 1, a block diagram of a plant and its controller is shown. The double arrows, denoting vector quantities in general, show the flow from input to output. The output $\underline{x}$ is thus dependent upon the particular control function $\underline{y}$ selected.

To phrase the problem in concrete terms, given a plant with an input vector $\underline{y}$ and an output vector $\underline{x}$, the relations between $\underline{y}$ and $\underline{x}$ being known, find the optimum input $\hat{\underline{y}}$ such that some performance functional of $\underline{y}$ is extremized. Having determined $\hat{\underline{y}}$, a controller for the plant can be inferred.

The performance functional will contain some measure of the system's sensitivity or constraints on the system's sensitivity will be imposed. Thus, the optimization will tend either to balance the system's sensitivity against other system performance criteria or to constrain the system's sensitivity to lie below some value.

In order that the procedure be applicable to a wide class of plants, both linear and non-linear, a variational time-domain approach will be employed.

The General Optimization Problem

Before considering the sensitivity aspects of the problem, the basic nature of the optimization problem will be reviewed.

In order to optimize a controller for a plant, "optimum" must first be defined. This definition is made by the selection of a particular performance index or performance functional of $\underline{y}$. A typical form for such a performance index in control system work would be:

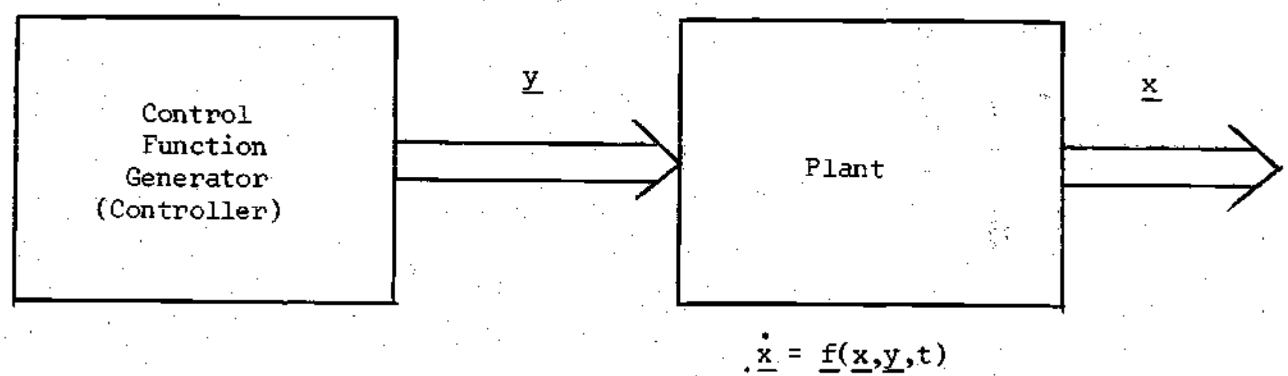

Figure 1. Block Diagram of Plant and Controller.

$$J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)\,dt + G(\underline{x}(T))$$

The system is said to be optimum when an input function $\underline{\hat{y}}$ has been determined such that the functional $J[\underline{y}]$ is minimum or maximum. The functional $J[\underline{y}]$ is often called a <u>cost functional</u> and is generally minimized. A typical performance index might include measures of system error, input energy or time of response, depending upon what was important in a particular problem. A common performance index is the so-called quadratic performance index, defined as:

$$J[\underline{y}] = \int_{t_o}^{T} (\underline{x}\cdot P\underline{x} + \underline{y}\cdot Q\underline{y})\,dt$$

where P and Q are symmetric positive definite matrices. Obviously, there must be a functional dependence of $J[\underline{y}]$ on the selection of $\underline{y}$ or there would be no problem; that is, if $J[\underline{y}]$ did not change for different values of $\underline{y}$, no minimum could be obtained and no optimum value for $\underline{y}$ would exist.

The plant equations impose natural constraints in the form of relations between $\underline{x}$ and $\underline{y}$. Additional constraints depending on the requirements of a particular problem might be imposed; for example,

$$||\,\underline{y}\,|| \le M_1 ,$$

or $\quad x_1(T_1) = M_2$

where M_1 and M_2 are known constants. Again, obviously, there must be a non-trivial relation between a constraint imposed and the inherent dynamics of the system, or else the constraint would not affect the solution of the problem. Mathematically, the additional constraints must tend to increase the performance functional to be minimized.

The optimization problem as phrased above is amenable to solution by a variety of known methods. Three major approaches to the problem are:

1. Utilization of the classical calculus of variations.
2. Utilization of the maximum principle of Pontryagin (8,10).
3. Utilization of the dynamic programming approach of Bellman (17,18).

Utilization of the classical calculus of variations presents difficulties when discontinuous functions are admitted to the class of acceptable control functions. Utilization of Pontryagin's maximum principle admits discontinuities, but requires the solution of a two-point boundary problem. The dynamic programming approach requires the solution of partial differential equations which in turn leads to a one-point boundary value problem. Existing optimization schemes (Pontryagin's or Bellman's) will be used in a modified form in the solution of the basic problem presented by this study. An outline of Pontryagin's and Bellman's method are found in Appendices I and II. The classical calculus of variations approach is well-documented in the literature.

The Sensitivity Problem

The sensitivity problem exists because of the natural tendency of the parameters of a physical system to vary from their nominal values. This variance could be caused by aging, environmental changes, or the inherent stochastic nature of the parameter. These parametric variations will cause variations in the dynamic characteristics of the plant. A system's sensitivity can be defined as the degree to which these parametric variations affect the performance of the system. A system with low sensitivity is one in which parametric variations cause only slight or insignificant changes in some desired quantity, e.g., the plant output, the plant transfer function, the performance index of the system. Thus, if consistency of the desired quantity for repeated runs of the plant is important, as in the case for most dynamic systems, a system with low sensitivity is desirable.

Most optimization schemes for the synthesis of control systems disregard the sensitivity problem; that is, these schemes assume the parameters of the plant are fixed. If the plant parameters do in fact vary, the resulting system could well prove unsatisfactory from a sensitivity point of view since the controller inferred by the optimization scheme is dependent upon the assumed values of the plant parameters. Thus, an "optimum" input which minimizes the performance functional for nominal values of the parameters may produce values of the performance functional which are no longer minimum when the parameters change slightly. Constraints applied to the design of the system assuming the plant parameters are at their nominal values may not be met if the parameters change their values.

It is the approach of this study to take these sensitivity restrictions into consideration beforehand in the system design, thus insuring an optimum system based on all constraints imposed, including sensitivity.

Before a system's sensitivity can be discussed quantitatively, some suitable measure of sensitivity must be defined. This measure could be based on various mathematical descriptions of sensitivity found in the literature. Once a valid sensitivity measure has been established, system sensitivity becomes a quantity which can be used in design, analysis or comparison of systems.

Two important mathematical descriptions of a system's sensitivity are found in the literature. They are the classical sensitivity function of Bode (1) and others and the sensitivity coefficient of Tomovic (12).

The classical sensitivity function is simply a normalized measure of the change in some desired quantity with respect to the change in some system parameter. If T is the desired quantity and q is a parameter, the classical sensitivity function is given by:

$$S_q^T = \frac{\Delta T}{\Delta q} \cdot \frac{q}{T} = \frac{q}{T} \cdot \frac{dT}{dq} \quad \text{(for small variations)}$$

Since this sensitivity measure is normalized, it yields what might be called an <u>absolute</u> measure of a system's sensitivity. It is particularly well-suited for analysis and design in the frequency domain. It has recently been employed by Dorato (3), who discusses the deviation of the performance index with respect to plant parameters in the optimum control problem and by Cruz and Perkins (4,5) who have introduced the concept of comparative sensitivity, for which the deviation of some desired quantity, e.g., a transfer function, with respect to plant parameter changes is compared for more than one implementation (open loop vs. closed loop) of the control scheme. They demonstrate the general superiority of the closed loop scheme.

Examples of design based on classical sensitivity concepts include the work of Mazer (13) and Fleischer (14). Both Mazer and Fleischer assume linear feedback systems (frequency domain) and therefore, their work would not apply directly to the approach taken in this study.

An alternate description of system sensitivity and one that will be used in this study as a basis for design is the sensitivity coefficient of Tomovic (12). The sensitivity coefficient is an unnormalized measure of the change in some desired quantity with respect to the change in some system parameters. Again, if T is the desired quantity and q is a parameter of the plant, the sensitivity coefficient is defined by

$$u(t,q) = \frac{\Delta T}{\Delta q} = \frac{dT}{dq} \quad \text{(for small changes in q)}$$

This description of sensitivity provides a simple means of defining sensitivity in the time-domain. As will be shown in Chapter II, there is a natural, simple relation between the sensitivity coefficient and the plant equations. Also, useful and relatively simple sensitivity measures based on u(t,q) can be constructed. These measures are illustrated in Chapter III.

The General Approach to the Problem

The general approach to the solution to the synthesis problem phrased above will be as follows:

First, relations between the sensitivity coefficients and the inherent dynamics of the plant will be established.

Second, sensitivity measures based on the sensitivity coefficients and satisfying the requirements of a particular synthesis problem will be constructed.

Third, the sensitivity measures will be included in a performance index or constrained to meet the requirements of a particular problem.

Fourth, conventional optimization techniques will be applied to the resulting optimization problem.

CHAPTER II

THE SENSITIVITY EQUATIONS

In this chapter, the relations between the sensitivity coefficient and the fundamental plant relations are established. This relation is described by the sensitivity equation(s) associated with a particular plant.

Definition of the Sensitivity Coefficient (12)

Let the vector $\underline{x}$ be the output of a particular plant and be the quantity of interest insofar as sensitivity is concerned. In other words we are concerned with the sensitivity of the output* to plant parameter changes. The plant parameters in question could be physical "constants" of the plant having some nominal value or the initial conditions on the plant state variables. In general, $\underline{x}$ is a function of time and the plant parameters, i.e.,

$$\underline{x} = \underline{x}(t,q)$$

where q is a particular plant parameter of interest.

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* Henceforth in this study, we will consider the output $\underline{x}$ to be the quantity of interest, although as far as the methods outlined below are concerned, the sensitivity of any quantity of the system could be handled.

Then

$$\Delta\underline{x} = \underline{x}(t,q+\Delta q) - \underline{x}(t,q) \ ; \qquad t \text{ assumed to be fixed.}$$

and

$$\underline{u}(t,q) \triangleq \lim_{\Delta q\to 0} \frac{\Delta\underline{x}}{\Delta q} = \lim_{\Delta q\to 0} \frac{\underline{x}(t,q+\Delta q) - \underline{x}(t,q)}{\Delta q}$$

$$\text{or} \qquad \underline{u}(t,q) \triangleq \frac{\partial\underline{x}(t,q)}{\partial q}$$

For small changes of the parameter or regions of the domain of q where $\underline{x}$ is linear in q

$$\underline{u}(t,q) = \frac{\partial\underline{x}}{\partial q} \simeq \frac{\Delta\underline{x}}{\Delta q}$$

The Plant and Sensitivity Equations (12)

Consider a system of plant equations written in vector form:

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},q,t), \qquad \underline{x}(t_o) = \underline{x}_o \tag{2.1}$$

where $\underline{x}$ is an n vector and $\underline{f}$ is continuous in all arguments.

If both sides of this system of equations are differentiated with respect to q, a plant parameter of interest, a new system of equations called the sensitivity equations results.

Thus:

$$\frac{d\dot{\underline{x}}}{dq} = \frac{d\underline{f}}{dq}$$

But

$$\frac{d\dot{\underline{x}}}{dq} = \frac{\partial}{\partial q}\left(\frac{\partial \underline{x}}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \underline{x}}{\partial q}\right) = \frac{\partial}{\partial t}(\underline{u}) = \dot{\underline{u}}$$

assuming the order of differentiation can be reversed.

By the chain rule,

$$\frac{d\underline{f}}{dq} = \frac{\partial \underline{f}}{\partial \underline{x}} \cdot \frac{\partial \underline{x}}{\partial q} + \frac{\partial \underline{f}}{\partial \underline{y}} \cdot \frac{\partial \underline{y}}{\partial q} + \frac{\partial \underline{f}}{\partial q}$$

Since $\underline{y}$, the input to the system, is some externally generated function and not a function of the plant parameters, $\frac{\partial \underline{y}}{\partial q} = 0$, and the second term on the right vanishes.* Thus the sensitivity equation becomes:

---

* The optimum input $\hat{y}$ is dependent upon the nominal values of the plant parameters, but once determined does not vary if the plant parameters change. Thus $\hat{y}$ is not a function of the particular values of the plant parameters, and

$$\frac{\partial \hat{\underline{y}}}{\partial q} = 0 .$$

$$\dot{\underline{u}} = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{u} + \frac{\partial \underline{f}}{\partial q} \tag{2.2}$$

where $$\frac{\partial \underline{f}}{\partial \underline{x}} \triangleq \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \quad \text{(n x n matrix)}$$

If the plant had a single input and single output, the plant equation could be written:

$$F(\overset{(n)}{x}, \ldots, \ddot{x}, \dot{x}, x, y, t, q) = 0$$

where F is continuous in all arguments.

Again differentiating with respect to q to obtain the sensitivity equation

$$\frac{\partial F}{\partial \overset{(n)}{x}} \frac{\partial \overset{(n)}{x}}{\partial q} + \ldots + \frac{\partial F}{\partial \ddot{x}} \frac{\partial \ddot{x}}{\partial q} + \frac{\partial F}{\partial \dot{x}} \frac{\partial \dot{x}}{\partial q} + \frac{\partial F}{\partial x} \frac{\partial x}{\partial q} + \frac{\partial F}{\partial q} = 0 \tag{2.3}$$

noting that $\frac{\partial \overset{(n)}{x}}{\partial q} = \frac{\partial^n}{\partial t^n}\left(\frac{\partial x}{\partial q}\right) = \overset{(n)}{u}$, equation (2.3) becomes:

$$\frac{\partial F}{\partial x^{(n)}} \overset{(n)}{u} + \ldots \frac{\partial F}{\partial \ddot{x}} \ddot{u} + \frac{\partial F}{\partial \dot{x}} \dot{u} + \frac{\partial F}{\partial x} u = - \frac{\partial F}{\partial q} \qquad (2.4)$$

Either forms (2.2) or (2.4) are acceptable forms of the sensitivity equation(s).

A general assumption made in deriving (2.2) or (2.4) was that for a particular input y or $\underline{y}$ the solutions to the plant equations are analytically dependent on the parameter in question. Insofar as the parameter q represents an initial condition on $\underline{x}$ or some other parameter of the plant system whose variance does not increase the original order of the plant system, the conditions for analytical dependence of the solutions on q are known. It is known from the theory of differential equations that the solutions of the plant equations depend continuously on the initial conditions and other parameters since F itself depends continuously on q (12). We shall assume, henceforth, that variation of the parameters does not increase the order of the plant system.

So far, we have assumed that a single varying plant parameter is of interest. The concepts outlined above are easily extended to cases where several plant parameters are of interest to the problem. Let q_i be a set of m such plant parameters, i=1,2,...m. Then a <u>set</u> of sensitivity equations can be obtained from the plant equations. If the plant equations are of the form

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{y}, q_1, q_2, \ldots q_m, t)$$

the sensitivity equations become:

$$\underline{\dot{u}}_i = \frac{\partial \underline{f}}{\partial \underline{x}} \underline{u}_i + \frac{\partial f}{\partial q_i}, \qquad i = 1,2\ldots m \tag{2.5}$$

where $\underline{\dot{u}}_i = \dfrac{\partial \underline{x}}{\partial q}$.

If the plant equation is of the form

$$F(\overset{(n)}{x}, \ldots, \ddot{x}, \dot{x}, x, y, q_1, q_2 \ldots q_m) = 0 , \tag{2.6}$$

the sensitivity equations become:

$$\frac{\partial F}{\partial x^{(n)}} \overset{(n)}{u_i} + \ldots + \frac{\partial F}{\partial \ddot{u}} \ddot{u} + \frac{\partial F}{\partial \dot{x}} \dot{u}_i + \frac{\partial F}{\partial x} u_i = - \frac{\partial F}{\partial q_i} ,$$

$$i = 1,2\ldots m$$

where $u_i = \dfrac{\partial x}{\partial q_i}$.

Initial Conditions on the Sensitivity Coefficients

In assigning values to the initial conditions on $\underline{u}_i$ or $\overset{(n)}{u_i}$, we must distinguish between two cases: The case for which the parameter of interest q_i is an initial condition on the plant and the case for which it is not. If q_i is not an initial condition on some plant state variable, it is clear that $\dfrac{\partial \underline{x}}{\partial q_i}(t_o) = \underline{u}_i(t_o) = 0$. That is, since the initial values of $\underline{x}$ are fixed, the change in $\underline{x}$ per change in the parameter evaluated at t_o must be zero. Similarly, if the plant equation is in form (2.6), the initial conditions on the sensitivity coefficients are:

$u_i^{(n)} = 0$.

If q_k is an initial condition on some state variable, say x_k, the situation is somewhat different:

From (2.1)

$$\dot{x}_k = f_k(x_1, x_2 \ldots x_n, \underline{y}, t, q_i) \tag{2.7}$$

where $q_k = x_k(t_o)$. Integrating (2.7) with respect to t yields:

$$x_k = x_k(t_o) + \int_{t_o}^{t} f_k \, dt = q_k + \int_{t_o}^{t} f_k \, dt \tag{2.8}$$

Differentiating (2.8) with respect to q_k yields:

$$u_k = \frac{\partial x_k}{dq_k} = 1 + \frac{\partial}{\partial q_k} \left(\int_{t_o}^{t} f_k \, dt \right)$$

Evaluating u_k at $t = t_o$:

$$u_k(t_o) = 1 + \frac{\partial}{\partial q_k} \left(\int_{t_o}^{t=t_o} f_k \, dt \right) = 1 + \frac{\partial}{\partial q_k} (0) = 1$$

Summarizing the above discussion:

$$u_k(t_o) = 0 \quad \text{if } q_i \neq x_k(t_o)$$

$$u_k(t_o) = 1 \quad \text{if } q_i = x_k(t_o)$$

If the plant equations are in form (2.6)

$$\overset{(n)}{u_i}(t_o) = 0 \qquad \text{if } q_i \neq \overset{(n)}{x}(t_o)$$

$$\overset{(n)}{u_i}(t_o) = 1 \qquad \text{if } q_i = \overset{(n)}{x}(t_o)$$

An Example

Consider the plant system governed by

$$\ddot{x} - a(1-x^2)\,\dot{x} + x = y \qquad x(o) = x_o \tag{2.9}$$

$$\dot{x}(o) = \dot{x}_o$$

The sensitivity equations of this system with respect to two parameters q_1 and q_2 where $q_1 = a$ and $q_2 = x_o$ will be derived.

Differentiating (2.9) first with respect to q_1 yields:

$$\ddot{u}_1 - a(1-x^2)\dot{u} + a(2x)\dot{x}u_1 - (1-x^2)\dot{x} + u = 0$$

or

$$\ddot{u}_1 - a(1-x^2)\dot{u}_1 + (1 + 2ax\dot{x})u_1 = (1-x^2)\dot{x} \tag{2.10}$$

with

$$u_1(0) = \dot{u}_1(0) = 0 \qquad (u_1 = \frac{\partial x}{\partial q_1})$$

Differentiating (2.9) with respect to q_2 yields:

$$\ddot{u}_2 - a(1-x^2)\dot{u}_2 + a(2x)\dot{x}\, u_2 + u_2 = 0$$

or

$$\ddot{u}_2 - a(1-x^2)\dot{u}_2 + (1+2ax\dot{x})u_2 = 0 \tag{2.11}$$

with

$$u_2(0) = 1 \text{ and } \dot{u}_2(0) = 0 \quad (u_2 = \frac{\partial x}{\partial q_2})$$

Equations (2.10) and (2.11) are the sensitivity equations of the system with initial conditions as shown.

CHAPTER III

SENSITIVITY MEASURES

In this chapter, the problem of developing a quantitative measure of the sensitivity of a physical system to plant parameter changes is discussed. Several reasonable measures of sensitivity applicable to particular problems are constructed and their use in performance indices or as constraints is discussed.

General Discussion of Sensitivity Measures

The problem of devising a useful measure of the sensitivity of a system is analogous to the problem of measuring other cost functions of the system; for example, the system error. There are a variety of ways error might be measured, each applicable to particular situations. These error measures would include the mean square error, the integral square error, the integral of the absolute value of the error, the instantaneous magnitude of the error at a particular time T, the maximum value of the error magnitude over a given time range and others. The choice of the particular error measure would depend upon what aspect of the error was important in a particular problem.

In a similar manner, the choice of a sensitivity measure depends upon what aspects of sensitivity are important in particular problems or situations.

Examples of Sensitivity Measures

Denoting the general sensitivity measure by U (a functional of the input y), the following are examples of various measures which would be appropriate in the given situations:

1. Assume that it is important that the output x (single output system) be consistently the same at time T for repeated runs of the plant, despite the variation of some plant parameter q. A measure of the system's sensitivity in this case might be:

$$U = |u(T,q)|$$

The sensitivity of the system at other times, $t \neq T$, is ignored. If (q_1, q_2) represents the possible range of parameter variation and $u \approx \frac{\Delta x}{\Delta q}$ where $\Delta q = q_1 - q_2$, $U = \left.\frac{|x_1 - x_2|}{|\Delta q|}\right|_{t=T}$. This measure is illustrated graphically in Figure 2.

2. Assume that it is important that the actual value of x be close to the nominal value of x over a given time range (t_o, T) despite plant parameter variation. A reasonable sensitivity measure in this case would be:

$$U = \max_{(t_o,T)} |u(t,q)|$$

Here importance is placed on the maximum deviation of x from its nominal value. This measure could be used when it was required that the maximum

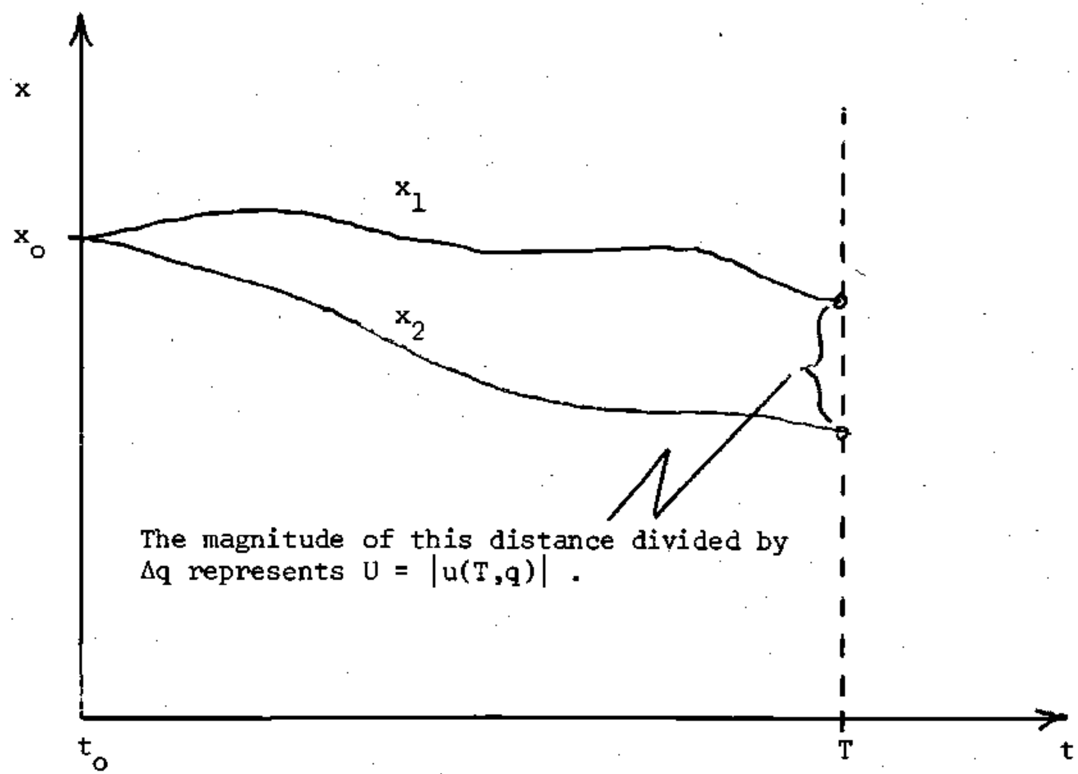

Figure 2. Graphical Representation of Sensitivity Measure $U = |u(T,q)|$.

deviation of x over (t_o, T) be bounded. If q_1 is one limit of parameter variation and q_o the nominal value of the parameter and if $u \approx \frac{\Delta x}{\Delta q}$ where $\Delta q = q_1 - q_o$, then

$$U = \max_{(t_o,T)} \frac{|x_1 - x_0|}{|\Delta q|} .$$

This measure is illustrated graphically in Figure 3.

3. Assume it is important that the integral square average value of deviations of x from its nominal value be small. This type of average weights large deviations more heavily than small deviations. In this case, a reasonable sensitivity measure would be:

$$U = \int_{t_o}^{T} u^2(t,q)\, dt$$

4. Other sensitivity measures applicable to particular situations might be:

a. $$U = \int_{t_o}^{T} |u(t,q)|\, dt$$

b. $$U = \int_{t_o}^{T} \underline{u} \cdot R \underline{u}\, dt$$

General case of 3. above; R is symmetric positive definite matrix.

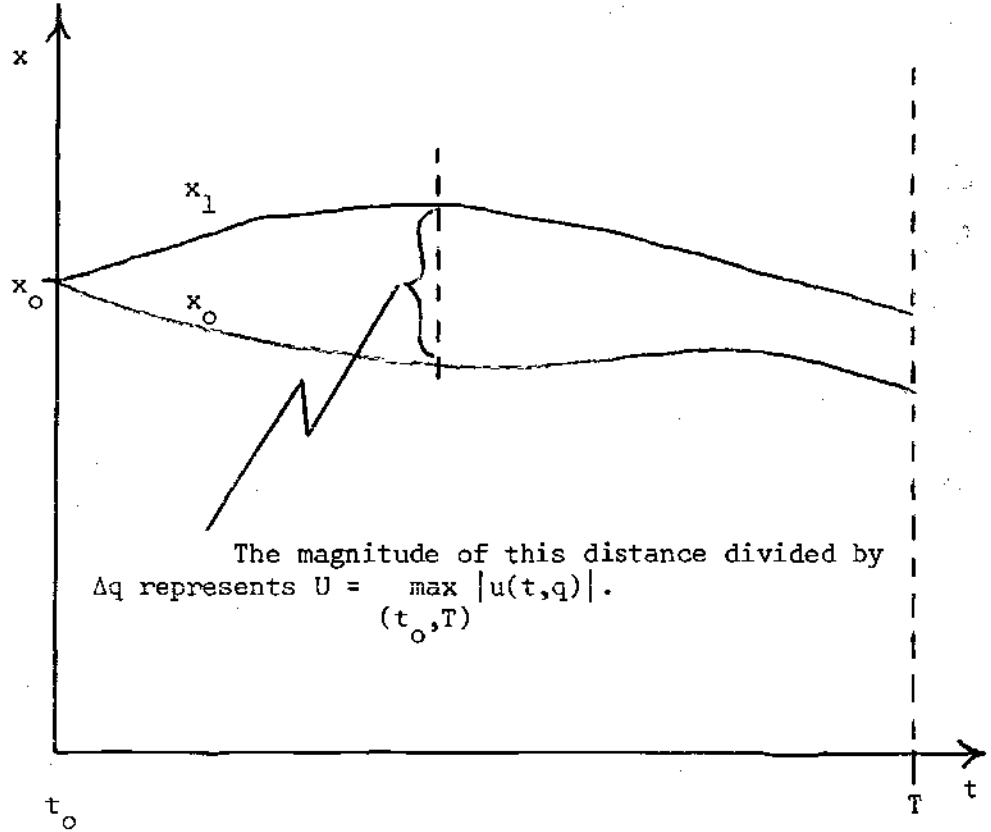

Figure 3. Graphical Representation of Sensitivity Measure $U = \max_{(t_o,T)} |u(t,q)|$.

Cases 1., 2., and 3., above are regarded as the most generally appropriate measures from the point of view of engineering usefulness and mathematical tractability.

From the discussion above, a few general remarks on sensitivity measures may be made.

First, the sensitivity measure is always zero or positive.

Second, the larger the sensitivity measure, the more sensitive a particular system is to plant parameter changes in some sense, defined by the measure.

In other words, the sensitivity measure is another cost functional that can be used to evaluate a system. It might be thought of as a type of error measure where the "error" is the deviation from the nominal value of the output caused by parameter variation instead of the deviation of the nominal value of the output from the desired value.

Applying the Sensitivity Measure to the Optimization Problem

The sensitivity measure can enter the optimization problem in one of two ways. Either the sensitivity measure is included in the performance functional of the system or the sensitivity measure is constrained to lie below some value determined by the nature of the problem.

If the sensitivity measure is included in the performance functional to be minimized, sensitivity considerations will be balanced against other typical cost functionals (error, input energy, etc.) associated with the problem.* The performance functional can be thought of

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* There is no realistic problem where the sole object of the optimization scheme is to minimize sensitivity. Obviously, the system must be required to do something other than be insensitive to parameter changes. An inert system would have a sensitivity measure of zero.

as consisting of two parts--the cost functional that measures how well the system performs its designed task for nominal values of the plant parameters and the sensitivity cost functional which measures the deviation of the output from its nominal values caused by parameter variation. The performance functional could then be the weighted sum of the two. For example:

$$J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)\, dt + \gamma\, U(T)$$

$$\text{where} \int_{t_o}^{T} F(\underline{x},\underline{y},t)\, dt$$

is the measure of the systems performance without considering sensitivity, U(T) is the sensitivity measure, and γ is a weighting factor.

If the sensitivity measure is constrained to lie below some predetermined value, the system is merely optimized with respect to its usual cost functional with this added sensitivity constraint. For example:

$$J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)\, dt, \quad U(T) \le M$$

where M is a predetermined constant.

Although these two methods of introducing the sensitivity measure into the optimization problem differ basically in intent, the actual mathematical problem of solution is quite similar. If the sensitivity measure is constrained, a new performance functional is formed by multiplying the sensitivity measure by an undetermined coefficient, say λ, and adding it to the original performance functional, i.e.,

$$J_1[\underline{y}] = J[\underline{y}] + \lambda U(T)$$

$$= \int_{t_o}^{T} F(\underline{x},\underline{y},t)\, dt + \lambda U(T)$$

The optimization scheme is then applied. The additional relation $U(T) = M^*$ permits the evaluation of λ. The only mathematical difference between the two methods is that in the first λ is known beforehand; in the second, λ is implied from the additional relation $U(T) = M$.

Engineering Estimates on Sensitivity Constraints

One of the advantages of using the sensitivity coefficient $u(t,q)$ as a basis for establishing sensitivity measures is the ease with which practical a priori sensitivity constraints can be estimated and constructed. This estimating technique will be demonstrated using a single output system and a single parameter of interest.

Let $\hat{x}$ be the system output when $q = q_o$, the nominal value of the plant parameter. Let x be the system output when $q \neq q_o$. Then

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* If $U(T)$ is a valid sensitivity measure, one which causes $J[\hat{\underline{y}}]$ to be montonically non-decreasing as M decreases, the inequality $U(T) \leq M$ can be replaced by the equality $U(T) = M$.

$$\frac{dx}{dq} = u = \frac{x-\hat{x}}{q-q_o} = \frac{x-\hat{x}}{\Delta q} \tag{3.1}$$

for small variations of the parameter near q_o or in the neighborhood of q_o where x is linear in q. Thus,

$$x - \hat{x} = \Delta q u \tag{3.2}$$

Suppose the expected range of q from its nominal value is known from the physical nature of the parameter and the conditions of the problem and that this range is such that (3.2) applies. In effect, Δq in (3.2) is thus known.

Consider now several examples of how constraints on various sensitivity measures might be applied in particular problems.

Example One

Assume the maximum deviation of x from $\hat{x}$ caused by parameter variations at a particular time T is to be kept below some value M.

$$|x(T) - \hat{x}(T)| \leq M$$

From (3.2):

$$|\Delta q \; u(T)| \leq M ,$$

yielding the sensitivity constraint

$$U(T) = |u(T)| \leq \frac{M}{|\Delta q|}$$

Example Two

Assume the maximum deviation of x from $\hat{x}$ caused by parameter variations over some time interval (0,T) is to be kept below some value M.

$$\underset{(0,T)}{\text{Max}} |x - \hat{x}| \leq M$$

From (3.2):

$$\underset{(0,T)}{\text{Max}} |\Delta q||u| \leq M$$

or,

$$|\Delta q| \underset{(0,T)}{\text{Max}} |u| \leq M$$

implying the sensitivity constraint:

$$U(T) = \underset{(0,T)}{\text{Max}} |u| \leq \frac{M}{|\Delta q|}$$

Example Three

Assume the RMS average of the deviation of x from $\hat{x}$ caused by plant parameter variations over some time interval (0,T) is to be kept below some value M.

$$\sqrt{\frac{1}{T}\int_0^T (x-\hat{x})^2\, dt} \le M$$

or,

$$\int_0^T (x-\hat{x})^2\, dt \le M^2T$$

or from (3.2):

$$(\Delta q)^2 \int_0^T u^2 dt \le M^2T$$

implying the sensitivity constraint:

$$U(T) = \int_0^T u^2 dt \le \frac{M^2T}{(\Delta q)^2} \qquad (3.3)$$

Example Four

Multiple constraints can be handled similarly.

Suppose the maximum deviation in the velocity of the output $|\dot{x}-\dot{\hat{x}}|$ caused by plant parameter changes must be kept below some value N over the time range (0,T) while at the same time the constraint of Example Three above is applied.

Differentiating (3.2) with respect to t yields:

$$\dot{x}-\dot{\hat{x}} = \Delta q\dot{u} \qquad (3.4)$$

but $$\max_{(0,T)} |\dot{x}-\hat{\dot{x}}| \leq N$$

From (3.4), $$|\Delta q| \max_{(0,T)} |\dot{u}| < N$$

implying the sensitivity constraint:

$$U_1(T) = \mathop{\mathrm{Max}}_{(0,T)} |\dot{u}| \leq \frac{N}{|\Delta q|}$$

and from (3.3) $$U_2(T) = \int_0^T u^2 dt \leq \frac{M^2 T}{(\Delta q)^2}$$

Thus there are two constraints to be applied to the optimization problem.

CHAPTER IV

THE SYNTHESIS METHOD

In this chapter, the general synthesis method for optimizing the controller for a known plant structure is outlined. Sensitivity considerations enter the problem through the inclusion of some sensitivity measure in the system performance index or through constraints applied to appropriate sensitivity measures. General stability considerations are discussed and the implications of various assumptions explored.

The Basic Approach

The basic feature of this optimization scheme, which permits the problem as originally phrased in Chapter I to be solved, is the augmentation of the plant vector differential equation with one or more vector sets of sensitivity equations derived from the plant equations.

To phrase the original problem again: Given a plant with input $\underline{y}$ and output $\underline{x}_p$ with $\underline{x}_p(t_o) = \underline{x}_{po}$ and a performance functional

$$J[\underline{y}] = \int_0^T F(\underline{x}_p, \underline{y}, t)dt + \gamma U$$

or

$$J[\underline{y}] = \int_0^T F(\underline{x}_p, \underline{y}, t)dt \text{ with } U \leq M$$

where U is some sensitivity measure based on $u_i(t, q_i)$, γ is a weighting factor and M is a predetermined constant, determine $\underline{y} = \hat{\underline{y}}$ such that $J[\underline{y}]$ is a minimum.

Then

$$\dot{\underline{x}}_p = \underline{f}_p(\underline{x}_p, \underline{y}, t),$$

an n^{th} order vector plant equation with

$$\underline{x}_p(t_o) = \underline{x}_{po}$$

and
$$\underline{u}_i = \frac{\partial \underline{f}_p}{\partial \underline{x}p}\,\underline{u}_i + \frac{\partial \underline{f}_p}{\partial q_i} = g_i(\underline{x}_p, \underline{y}, \underline{u}_i, t)$$

a set of n^{th} order sensitivity equations with $\underline{u}_i(t_o) = u_{io}$ where the set q_i, i=1,2...m, are the m plant parameters of interest in the problem and q_{io}, l=1,2...m, are their nominal values.

Now define the (m+1)n order vector $\underline{x}$:

$$\underline{x} \triangleq \begin{bmatrix} \underline{x}_p \\ \underline{u}_i \\ \underline{u}_2 \\ \cdots \\ \underline{u}_m \end{bmatrix}$$

then

$$\underline{\dot{x}} = \begin{bmatrix} \underline{f}_p(\underline{x}_p, \underline{y}, t) \\ g_1(\underline{x}_p, \underline{u}, \underline{y}, t) \\ g_2(\underline{x}_p, \underline{u}_2, \underline{y}, t) \\ \cdots \\ g_m(\underline{x}_p, \underline{u}_m, \underline{y}, t) \end{bmatrix} \triangleq \underline{f}(\underline{x}, \underline{y}, t)$$

The vector $\underline{x}$ is seen to be a system state vector whose elements include the output and the sensitivity coefficients.

The vector function $\underline{f}$ is assumed to be evaluated at $q_i = q_{io}$, the nominal values of the parameters. The effects of this assumption are discussed below.

The problem is now in the form of the general optimization problem phrased in Chapter I and is therefore amenable to solution by Pontryagin's or Bellman's methods (See Appendix I and II). These methods, applied to the problem, will yield the optimum control function $\underline{\hat{y}}$.

Stability Considerations

In this section, the stability of the optimized system is discussed. The optimum input as synthesized above is assumed to be applied to the plant input. The sense in which the optimized system is stable or unstable is defined.

For autonomous (unforced) systems, conventional system stability criteria are defined in terms of the behavior of the state variables as t becomes infinite. For non-autonomous (forced) systems, the system can be said to be stable, if for every bounded input, the output is bounded (19).

The type of systems considered here can be considered to be non-autonomous in the time range (t_o, T) and autonomous in the time range (T, ∞). In other words the driving function $\underline{y}$ is removed at T and the system operates in its unforced mode.

Since, for a realizable physical system, the output is bounded for a bounded input over a finite time range, however large, the only stability question remaining is the stability of the system after the control period (t_o, T).

This is determined by the autonomous behavior of the plant. Since we are, in general, concerned with non-linear plants, the autonomous stability of the plant will be dependent upon the point in the phase-parameter space that the state variables and the plant parameters are located when the control period terminates.

To illustrate, let the output $\underline{x}$ be an n dimensional vector and let there be m parameters q_i, i-1,2...m, of interest, each lying within a fixed neighborhood, $q_{ia} \leq q_i \leq q_{ib}$. Then there exists an n+m dimensional space Ω, consisting of all possible values of $\underline{x}(T)$ and q_i. In general, this space can be divided into two parts--the region in which the plant is stable under some definition of autonomous stability and the region in which it is unstable. If at t = T, the point defined by the state variables $\underline{x}(T)$ and the plant parameters q_i is located in the stable region of Ω, the system can be said to be stable. If it is located in the unstable region, the system can be said to be unstable.

Thus, the question of stability as far as this method is concerned is determined by the autonomous stability of the plant itself. Several methods of analyzing the stability of autonomous systems are found in the literature (19).

Assumptions and Limitations

In this section, the effects of various implicit and explicit assumptions made above will be discussed and the resulting limitations on the synthesis method explored.

An important assumption made in phrasing the optimization problem above was that the plant and sensitivity equations were evaluated at the nominal values of the parameters, $q_i = q_{io}$. This assumption was necessary in order that a definite $\hat{\underline{y}}$ result from the optimization scheme. Thus, the control function $\hat{\underline{y}}$ resulting from the optimization scheme might be said to be truly "optimum" only when the parameters are nominal. The question naturally arises, in what sense, if any, is $\hat{\underline{y}}$ an optimum when the plant parameters are not at their nominal values. This question is obviously important since the sensitivity problem would not exist if the parameters were fixed.

Let $\hat{j}$ be the value of the performance index when $\underline{y} = \hat{\underline{y}}$ as determined by the optimization scheme, $\hat{j} = J[\hat{\underline{y}}]$. Let j_1 be the value of the performance index when $\underline{y} = \underline{y}_1$, some other control function. For definiteness, assume $\underline{y}_1$ is the control function obtained by an optimization disregarding all sensitivity considerations. Both $\hat{j}$ and j_1 now can be considered to be functions of the plant parameters q_i:

$$\hat{j} = j(q_1, q_2 \ldots q_m)$$

$$j_1 = j_1(q_1, q_2 \ldots q_m)$$

Assume that both $\hat{j}$ and j_1 are continuous in all plant parameters in the region of parameter space defined by the expected range of parameter variation. In particular, $\hat{j}$ and j_1 are continuous at $q_i = q_{io}$. This assumption is easily met by typical physical systems.

If a non-trivial sensitivity problem is under consideration, then:

$$\hat{j}(q_{10},q_{20}\ldots q_{mo}) < j_1(q_{10},q_{20}\ldots q_{mo})$$

from the definition of $\underline{\hat{y}}$ and $\underline{y}_1$.

Then, arguing from the continuity of $\hat{j}$ and j_1, there exists a neighborhood of q_{io}, $N_1(q_{io})$, such that $\hat{j} < j_1$ for any value of q_i in N_1. Further, there exists a nieghborhood $N_2(q_{io})$ such that $\hat{j}$ evaluated for each value of q_i in N_2 is less than j_1 evaluated for any value of q_i in N_2. These neighborhoods are illustrated in Figure 4 for the single parameter case.

If the region of parametric space defined by the expected range of parametric variation lies within either N_1 or N_2, the control function $\underline{\hat{y}}$ may still be called optimum despite parameter variation. If the expected range of parameter variation lies within N_1, $\hat{j}(q_{i_j}) < \hat{j}_1(q_{i_j})$; i.e., $\underline{\hat{y}}$ is better than $\underline{y}_1$ for any particular set of parameter values. If the expected range of parameter variations lies within N_2, $\hat{j}(q_{i_j})$ $j(q_{i_k})$; i.e., $\underline{\hat{y}}$ is better than $\underline{y}_1$ where any two sets of parameter values have been selected for comparison. Since $\underline{y}_1$ could be the control function derived from any reasonable optimization scheme, $\underline{\hat{y}}$ could be called the optimum despite parametric variation as long as the range of parametric

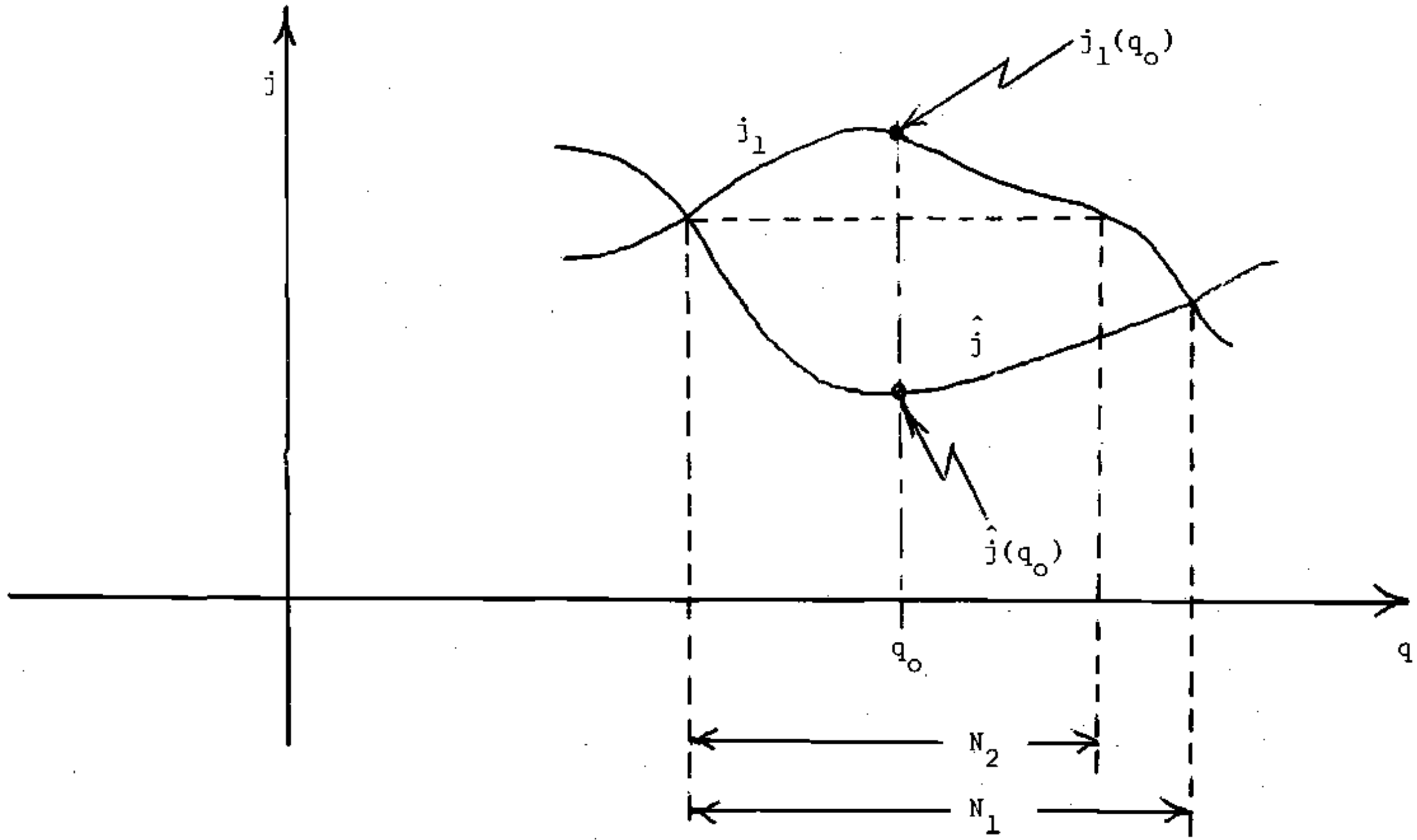

Figure 4. Variation of Performance Indices with Parameter Changes.

variation is small enough. This is assumed to be the case when applying the above described synthesis procedure.

Another assumption made in deriving the synthesis procedure and devising sensitivity measures that also serves to restrict the allowable range of parameter variation is that $\frac{\partial \underline{x}}{\partial q_i} \approx \frac{\Delta \underline{x}}{\Delta q_i}$ or that $\underline{x}$ is approximately linear in q_i over the region of the parametric range of interest. The control function $\hat{\underline{y}}$ as determined by the synthesis procedure will deviate from the true optimum to the degree that this assumption is not met.

Since the plant parameters are assumed to remain constant throughout a given control period, the synthesis procedure is applicable to the class of problems in which uniformity of the output is important for repeated runs of the plant, despite variations in the plant parameters from run to run.

CHAPTER V

THE LINEAR PLANT WITH QUADRATIC PERFORMANCE INDEX

In this chapter, an important special case of the general sensitivity problem in control system optimization is considered. This is the case where the plant is linear and the performance index is in the so-called quadratic form.

Consider the case where there is a single varying plant parameter of interest q.

The n order plant equations are of the form

$$\dot{\underline{x}}_p = A\underline{x}_p + B\underline{y} ,$$

where $$\underline{x}_p(t_o) = \underline{x}_{po} \tag{5.1}$$

A is an n x n matrix and B is a k x n matrix, where k is the order of $\underline{y}$. Any or all of the elements of both A and B could be functions of the plant parameter q. Differentiating (5.1) with respect to q yields the sensitivity equations:

$$\dot{\underline{u}} = A\underline{u} + A_q\underline{x}_p + B_q\underline{y} , \tag{5.2}$$

where $$\underline{u}(t_o) = \underline{u}_o$$

and where

$$A_q = \frac{\partial A}{\partial q} = \begin{bmatrix} \frac{\partial a_{11}}{\partial q} & \frac{\partial a_{12}}{\partial q} & \cdots & \frac{\partial a_{1n}}{\partial q} \\ \frac{\partial a_{21}}{\partial q} & \frac{\partial a_{22}}{\partial q} & \cdots & \frac{\partial a_{2n}}{\partial q} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial a_{n1}}{\partial q} & \frac{\partial a_{n2}}{\partial q} & \cdots & \frac{\partial a_{nn}}{\partial q} \end{bmatrix}$$

and

$$B_q = \frac{\partial B}{\partial q} = \begin{bmatrix} \frac{\partial b_{11}}{\partial q} & \frac{\partial b_{12}}{\partial q} & \cdots & \frac{\partial b_{1k}}{\partial q} \\ \frac{\partial b_{21}}{\partial q} & \frac{\partial b_{22}}{\partial q} & \cdots & \frac{\partial b_{2k}}{\partial q} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial b_{n1}}{\partial q} & \frac{\partial b_{n2}}{\partial q} & \cdots & \frac{\partial b_{nk}}{\partial q} \end{bmatrix}$$

The quadratic performance functional is of the form:

$$J[y] = \int_{t_o}^{T} (\underline{x}_p \cdot P\underline{x}_p + \underline{y} \cdot Q\underline{y})dt + \gamma\, U(T)$$

Let U(T), the sensitivity measure, also be in quadratic form, i.e.,

$$\gamma\, U(T) = \int_{t_o}^{T} (\underline{u} \cdot R\underline{u})\, dt$$

or $$J[\underline{y}] = \int_{t_o}^{T} (\underline{x}_p \cdot P\underline{x}_p + \underline{y} \cdot Q\underline{y} + \underline{u} \cdot R\underline{u})\, dt \qquad (5.3)$$

where P, Q, and R are positive definite symmetric matrices and determine the weighting of the cost functionals.

Form the vector $\underline{x}$:

$$\underline{x} = \begin{bmatrix} \underline{x}_p \\ \underline{u} \end{bmatrix}$$

then (5.1) and (5.2) can be written

$$\dot{\underline{x}} = C\underline{x} + D\underline{y} \qquad (5.4)$$

with

$$\underline{x}(t_o) = \begin{bmatrix} \underline{x}_p(t_o) \\ \underline{u}(t_o) \end{bmatrix} = \begin{bmatrix} \underline{x}_{po} \\ \underline{u}_o \end{bmatrix}$$

where C is the 2n matrix:

$$C = \left[\begin{array}{c|c} A & 0 \\ \hline A_q & A \end{array}\right]$$

and D is the k x 2n matrix:

$$D = \begin{bmatrix} B \\ \hline B_q \end{bmatrix}$$

The performance functional (5.3) can be written

$$J[\underline{y}] = \int_{t_o}^{T} (\underline{x}\cdot T\underline{x} + \underline{y}\cdot Q\underline{y})\, dt \tag{5.5}$$

where

$$T = \left[\begin{array}{c|c} P & 0 \\ \hline 0 & R \end{array}\right] \text{ a 2n x 2n symmetric matrix}$$

The problem now has become to find $\hat{y}$ such that

$$J[\underline{y}] = \int_{t_o}^{T} (\underline{x}\cdot T\underline{x} + \underline{y}\cdot Q\underline{y})\, dt$$

is minimum with the constraining equation

$$\dot{\underline{x}} = C\underline{x} + D\underline{y}, \quad \underline{x}(t_o) = \underline{x}_o$$

(Matrices C and D are assumed to be evaluated at $q = q_o$. The effect of this assumption is discussed in Chapter IV.)

Either Pontryagin's or Bellman's method may now be applied to the optimization problem (See Appendices I and II).

Application of Pontryagin's method yields the following set of equations to be solved:

$$D^T \underline{p} + 2Q\underline{\hat{y}} = 0 \tag{5.6}$$

$$\underline{\dot{p}} = - C^T \underline{p} + 2T\underline{x} \tag{5.7}$$

$$\underline{\dot{x}} = C\underline{x} + D\underline{\hat{y}} \tag{5.8}$$

with $\underline{x}(t_o) = \underline{x}_o$, and $\underline{p}(T) = 0$, where $\underline{p}$ is an auxiliary variable inherent in this optimization scheme. This is 4n order two-point linear boundary value problem which can be solved by a variety of known methods (See Appendix III).

If Q is non-singular matrix, (5.6) becomes

$$\underline{\hat{y}} = - 1/2\ Q^{-1} D^T \underline{p} \tag{5.9}$$

Substituting this value of $\underline{\hat{y}}$ in (5.8) yields

$$\underline{\dot{x}} = c\underline{x} - 1/2\ DQ^{-1} D^T \underline{p} \tag{5.10}$$

Equations (5.7) and (5.10) are now solved simultaneously for $\underline{p}$.

The optimum control input $\hat{\underline{y}}$ is then found from equation (5.9).

Bellman's Method

Application of Bellman's method (See Appendix II) yields the following partial differential equation to be solved, assuming Q is non-singular:

$$\frac{\partial \bar{S}}{\partial t} = T\underline{x}\cdot\underline{x} + C^T\nabla_{\underline{x}}\bar{S}\cdot\underline{x} - 1/4\ DQ^{-1}D^T\nabla_{\underline{x}}\bar{S}\cdot\nabla_{\underline{x}}\bar{S} \qquad (5.11)$$

with $\bar{S}(\underline{x},\hat{t}=0) = 0$, where $\bar{S}(\underline{x},\hat{t}) = \underset{\underline{y}\epsilon Y}{\text{minimum}} \left[\int_{t_o}^{T} \underline{x}\cdot T\underline{x} + \underline{y}\cdot Q\underline{y}\ dt\right]$, (Y is the set of admissible inputs) and $\hat{t} = T - t$.

$$\text{where} \qquad \nabla_{\underline{x}}\bar{S} \triangleq \begin{bmatrix} \frac{\partial \bar{S}}{\partial x_1} \\ \frac{\partial \bar{S}}{\partial x_2} \\ \cdots \\ \frac{\partial \bar{S}}{\partial x_{2n}} \end{bmatrix}, \text{ a 2n vector}$$

$$\text{and} \qquad \hat{\underline{y}} = -\ 1/2\ Q^{-1}\ D^T\ \nabla_{\underline{x}}\bar{S} \qquad (5.12)$$

as a result of the minimization operation.

Assuming solution of the form:

$$\bar{S}(\underline{x},\hat{t}) = S(\hat{t})\ \underline{x}\cdot\underline{x}\ ,$$

where $S(\hat{t})$ is a symmetric 2n x 2n matrix. Then

$$\frac{\partial \bar{S}}{\partial \hat{t}} = \dot{S}(\hat{t})\ \underline{x}\cdot\underline{x} \tag{5.13}$$

and

$$\nabla_{\underline{x}}\bar{S} = 2S(\hat{t})\underline{x} \tag{5.14}$$

Substituting (5.13) and (5.14) into (5.11) yields:

$$\dot{S}\ \underline{x}\cdot\underline{x} = T\underline{x}\cdot\underline{x} + 2C^{T}S\ \underline{x}\cdot\underline{x}\ - S^{T}DQ^{-1}D^{T}S\ \underline{x}\cdot\underline{x} \tag{5.15}$$

with $S(t=0)\ \underline{x}\cdot\underline{x} = 0$.

Equating like coefficients of the various products of elements of $\underline{x}$, the above separation of variables yields an initial value problem with (2n+1)n ordinary differential equations. This problem is solved for $S(\hat{t})$. The optimal control input $\hat{y}$ is then found from equation (5.12):

$$\underline{\hat{y}} = 1/2\ Q^{-1}D^{T}\nabla_{\underline{x}}\bar{S} \tag{5.12}$$

$$= Q^{-1}D^{T}S(\hat{t})\underline{x} \tag{5.16}$$

$$= K(t)\ \underline{x}$$

where $K(t) = Q^{-1}D^{T}S(\hat{t})$. Thus

$$\hat{y} = K_1(t)\underline{x}_p + K_2(t)\underline{u}$$

where

$$K = \left[\begin{array}{c|c} K_1 & K_2 \end{array}\right]$$

If $T \to \infty$, $K(t)$ becomes a constant matrix, since $\dot{S}(\hat{t}) = S(T-t) = \dot{S}(\infty)$ must equal zero and equation (5.15) becomes a set of $(2n+1)n$ algebraic equations in the elements of S.

If $\hat{\underline{y}}$ is desired as a function of time alone, $\underline{x}$ can be eliminated between (5.16) and (5.8) and the resulting differential equations solved for $\hat{\underline{y}}$.

Similarly after determining $\hat{\underline{y}}$ and the resultant output $\underline{x}$ utilizing Pontryagin's method, the parameter t can often be eliminated between $\hat{\underline{y}}$ and $\underline{x}$ yielding $\hat{\underline{y}}$ as a function of the input $\underline{x}$.

Thus, the two methods are basically equivalent.

Multi-Parameter Case

The optimization method applied to the single parameter case above can easily be extended to the case where there are several varying plant parameters of interest.

Let q_i be a set of m plant parameters, $i = 1,2,\ldots m$.

If the plant equations are given by $\dot{\underline{x}}_p = A\underline{x}_p + B\underline{y}$, a set of sensitivity equations can be determined by successive differentiation of the plant equations with respect to $q_1, q_2 \ldots q_m$.

Then

$$\dot{\underline{u}}_1 = A\underline{u}_1 + A_{q_1}\underline{x}_p + B_{q_1}\underline{y}$$

$$\dot{\underline{u}}_2 = A\underline{u}_2 + A_{q_2}\underline{x}_p + B_{q_2}\underline{y}$$

$$\cdots$$

$$\dot{\underline{u}}_m = A\underline{u}_m + A_{q_m}\underline{x}_p + B_{q_m}\underline{y}$$

where $\underline{u}_i = \dfrac{\partial \underline{x}}{\partial q_i}$

$$\text{Let } \underline{x} = \begin{bmatrix} \underline{x}_p \\ \underline{u}_1 \\ \underline{u}_2 \\ \cdots \\ \underline{u}_m \end{bmatrix}, \quad (m+1)n \text{ vector}$$

Then

$$\dot{\underline{x}} = C\underline{x} + D\underline{y} \tag{5.17}$$

where

$$C = \left[\begin{array}{c|c|c||c} A & 0 & 0 & 0 \\ \hline A_{q_1} & A & 0 & 0 \\ \hline A_{q_2} & 0 & A & 0 \\ \hline\hline A_{q_m} & 0 & 0 & A \end{array}\right] \quad \begin{array}{c} (m+1)n \\ \times \\ (m+1)n \\ \text{matrix} \end{array}$$

and

$$D = \begin{bmatrix} B \\ \hdashline B_{q_1} \\ \hdashline B_{q_2} \\ \hdashline \vdots \\ \hdashline B_{q_m} \end{bmatrix} \quad k \times (m+1)n \text{ matrix}$$

Assuming a performance index of the form:

$$J[\underline{y}] = \int_{t_o}^{T} \underline{x}_p \cdot P\underline{x}_p + \underline{y} \cdot Q\underline{y} + \underline{u}_1 \cdot R_1\underline{u}_1 + \underline{u}_2 \cdot R_2\underline{u}_2 + \ldots + \underline{u}_m \cdot R_m\underline{u}_m)dt$$

P, Q, R_i symmetric positive matrices

$i = 1,2\ldots m$

$$J[\underline{y}] = \int_{t_o}^{T} \underline{x} \cdot T\underline{x} + \underline{y} \cdot Q\underline{y}\, dt \tag{5.18}$$

$$\text{where} \quad T = \begin{bmatrix} P & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & R_2 & 0 \\ 0 & 0 & 0 & R_m \end{bmatrix} \quad \begin{array}{l} (n+1)n \\ \times \\ (m+1)n \\ \text{symmetric} \\ \text{positive matrix} \end{array}$$

Equations (5.17) and (5.18) are seen to be in the identical form as that for the single parameter case and the discussion above carries over intact.

CHAPTER VI

EXAMPLES

In this chapter, various simple examples of the application of the general synthesis method are presented. The purpose of these examples is to illustrate the problem-solving procedures and to show the effects of sensitivity restrictions on the optimization problem. Example One is an analytic solution; the rest of the examples are computer solutions. In Example One, the solution will be discussed in some detail. In the rest of the examples, the problem will be posed, explained, and the solution presented.

Example One

This is an example with a simple single-order linear plant and a quadratic performance index.

Given the plant equation:

$$\dot{x} + cx = ky \tag{6.1}$$

with $x(0) = x_o = 1$.

Let c and k be, in general, functions of a single plant parameter q. Let the measure of sensitivity be

$$U(T) = \int_0^T u^2 dt \qquad \text{where } u(t,q) = \frac{\partial x}{\partial q} \,.$$

Let the performance index be the weighted sum of an error/energy cost functional V(T) and the sensitivity measure U(T):

$$J[y] = V(T) + \gamma\, U(T)$$

$$= \int_0^T x^2 + \beta y^2 dt + \gamma \int_0^T u^2 dt$$

$$= \int_0^T x^2 + \beta y^2 + \gamma\, u^2 dt \qquad (6.2)$$

β and γ are weighting factors which assign the relative importance of error, energy and sensitivity.

The problem is to find $y = \hat{y}$ such that (6.2) is minimized.

From the manner in which the performance index is constructed and from the initial conditions on x, it may be inferred that the objective is to drive the output from its initial value, $x(0) = 1$, to zero in such a way that J[y] is minimized.

Equation (6.1) is differentiated with respect to q to yield the sensitivity equation:

$$\dot{u} + cu = k_q y - c_q x \qquad (6.3)$$

where $k_q = \frac{\partial k}{\partial q}$

$c_q = \frac{\partial c}{\partial q}$

with $u(0) = 0$.

Letting $x_1 = x$ and $x_2 = u$, evaluating c, k, c_q and k_q at $q = q_o$, (6.1) and (6.3) can be written

$$\dot{x}_1 = -c_o x_1 + k_o y \qquad = f_1 \tag{6.4}$$

$$\dot{x}_2 = -c_{qo} x_1 - c_o x_2 + k_{qo} y = f_2 \tag{6.5}$$

or

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = C\underline{x} + Dy$$

where

$$C = \begin{bmatrix} -c_o & 0 \\ -c_{qo} & -c_o \end{bmatrix} \text{ and } D = \begin{bmatrix} k_o \\ k_{qo} \end{bmatrix}$$

Applying Pontryagin's method, we form the new equation (See Appendix I):

$$\dot{x}_3 = x_1^2 + \beta y^2 + \gamma x_2^2 = f_3 \tag{6.6}$$

then:

$$\dot{p}_1 \triangleq \sum_{j=1}^{n+1=3} p_j \frac{\partial f_j}{\partial x_i} \qquad \begin{array}{l} \text{where } p_3(T) = -1 \\ \text{and } \quad p_1(T) = p_2(T) = 0 \end{array}$$

$$\dot{p}_1 = c_o p_1 + c_{q_o} p_2 + 2x_1 \tag{6.7}$$

$$\dot{p}_2 = c_o p_2 + 2\gamma x_2 \tag{6.8}$$

$$\dot{p}_3 = 0, \quad \text{implying } p_3(t) = -1$$

Form $H = \underline{p} \cdot \underline{f}$

$$= p_1(-c_o x_1 + k_o y) + p_2(-c_{q_o} x_1 - c_o x_2 + k_{q_o} y)$$

$$+ p_3(x_1^2 + \beta y^2 + \gamma x_2^2)$$

$$\frac{\partial H}{\partial y} = 0 = k_o p_1 + k_{q_o} p_2 + 2\beta p_3 y$$

$$= k_o p_1 + k_{q_o} p_2 - 2\beta y$$

$$y = \frac{k_o p_1 + k_{q_o} p_2}{2\beta} \tag{6.9}$$

Recapitulating pertinent equations above:

$$\left.\begin{aligned} \dot{x}_i &= -c_o x_1 + k_o y & x_1(0) &= 1 \\ \dot{x}_2 &= -cq_o x_1 - c_o x_2 + k_{q_o} y & x_2(0) &= 0 \\ \dot{p}_1 &= c_o p_1 + cq_o p_2 + 2x_1 & p_1(T) &= 0 \\ \dot{p}_2 &= c_o p_2 + 2\gamma x_2 & p_2(T) &= 0 \\ y &= \frac{k_o p_1 + k_{q_o} p_2}{2\beta} & & \end{aligned}\right\} \qquad (6.10)$$

Thus we are confronted with the solution of a fourth order two-point boundary value problem. This problem is solved for two special cases below.

<u>Case One</u>

Let $q = c$, k is not a function of q.

Assume $T \to \infty$.

Then $k_{q_o} = 0$, $c_{q_o} = 1$ and set (6.10) becomes:

$$\left.\begin{aligned} \dot{x}_1 &= -\,c_o x_1 + k_o y \\ \dot{x}_2 &= -\,x_1 - c_o x_2 \\ \dot{p}_1 &= c_o p_1 + p_2 + 2x_1 \\ \dot{p}_2 &= c_o p_2 + 2\gamma x_2 \\ y &= \frac{k_o p_1}{2\beta} \end{aligned}\right\} \qquad (6.11)$$

Combining set (6.11) into a single d.e. in p_1:

$$\ddot{\ddot{p}}_1 - (2c_o^2 + \frac{k_o^2}{\beta})\ddot{p}_1 + [c_o^2(c_o^2 + \frac{k_o^2}{\beta}) + \frac{\gamma k_o^2}{\beta}]p_1 = 0$$

The four roots of the characteristic equation are:

$$\pm \sqrt{c_o^2 + \frac{k_o^2}{2\beta} \pm \sqrt{\left(\frac{k_o^2}{2\beta}\right)^2 - \frac{k_o^2\gamma}{\beta}}}$$

Discarding the positive roots to meet boundary condition $p_1(T) = 0$ $(T \to \infty)$, the solution is of the form:

$$p_1 = Me^{-at} + Ne^{-bt} \tag{6.12}$$

where a, b = $\sqrt{c_o^2 + \frac{k_o^2}{2\beta} \pm \sqrt{\left(\frac{k_o^2}{2\beta}\right)^2 - \frac{k_o^2\gamma}{\beta}}}$

From set (6.11) it can be seen that x_1, x_2 and y have similar forms to (6.12). Applying the initial conditions $x_1(0) = 0$ and $x_2(0) = 0$ and solving for the undetermined constants yields the following solution:

$$x\Big|_{c=c_o} = x_1 = -\left[\frac{c_o-a}{a-b}\right]e^{-at} + \left[\frac{c_o-b}{a-b}\right]e^{-bt}$$

$$u\Big|_{c=c_o} = x_2 = \frac{1}{a-b}\,(e^{-at} - e^{-bt})$$

$$y = \hat{y} = -\frac{1}{k_o(a-b)}\,[(c_o-a)^2\, e^{-at} - (c_o - b)^2\, e^{-bt}] \qquad (6.13)$$

To demonstrate the results of this sensitivity optimization, let the following values be assigned to the plant parameters:

$$c_o = .707$$

$$k_o = 2$$

$$\beta = 1$$

Let us compare the results when sensitivity is not considered; i.e., $\gamma = 0$ and when it is ($\gamma > 0$, say $\gamma = \frac{7}{16}$).

When $\gamma = 0$, the corresponding $\hat{y}$ is applied to the plant and the parameter c permitted to vary about $c_o = .707$ in the range (.6, .8). Since $\hat{y}$ is uniquely determined by equation (6.13), x(t) and u(t) can be determined from equations (6.1) and (6.3) for each value of c and the error/energy cost functional $V = \int_0^\infty x^2 + \beta y^2 dt$ and the sensitivity measure $U = \int_0^\infty u^2 dt$ determined. Values of V and U for various values of the parameter c are found in Table 1 below:

Table 1. Values of V and U for $\gamma = 0$

	c	V	U
	.600	.375	.180
c_o =	.707	.355	.118
	.800	.341	.087

Repeating this process for $\gamma = \frac{7}{16}$, Table 2 below can be constructed:

Table 2. Values of V and U for $\gamma = \frac{7}{16}$

	c	V	U
	.600	.379	.117
c_o =	.707	.361	.0833
	.800	.350	.0652

Thus we see that the sensitivity measure U when $\gamma = \frac{7}{16}$ has been reduced by an average of about 30 per cent while the error/energy cost index V has been increased about 2 per cent. If sensitivity were an important consideration in this example, this would appear to be a good exchange.

Another method of exhibiting the results of the sensitivity optimization would be to plot the output x for various values of the parameter c when the weighting of the sensitivity measure assumes different values. In Figures 5 and 6, the output x is shown for $\gamma = 0$ and $\frac{5}{2}$ while c changes from .6 to .8. The curves for $\gamma = \frac{5}{2}$ (Figure 6) exhibit a marked "squeezing together" or improved consistency when compared with the curves for $\gamma = 0$ (Figure 5).

In Figure 7, the optimum inputs $\hat{y}$ are plotted for various values of γ. In Figure 8, the sensitivity coefficients $u(t, c_o)$ are plotted for $\gamma = 0$ and $\frac{5}{2}$.

<u>Case Two</u>

Let $q = k$; c is not a function of q.

Assume $T \to \infty$.

Then $c_{qo} = 0$ and $k_{qo} = 1$ and set (6.10) becomes:

$$
\begin{aligned}
\dot{x}_1 &= -c_o x_1 + k_o y \qquad & x_1(0) &= 1 \\
\dot{x}_2 &= -c_o x_2 + y \qquad & x_2(0) &= 0 \\
\dot{p}_1 &= c_o p_1 + 2x_1 \qquad & p_1(T) &= 0 \\
\dot{p}_2 &= c_o p_2 + 2\gamma x_2 \qquad & p_2(T) &= 0
\end{aligned}
$$

$$y = \frac{k_o p_1 + p_2}{2\beta}$$

Solving this set of differential equations in a manner similar to Case One yields solutions of the form:

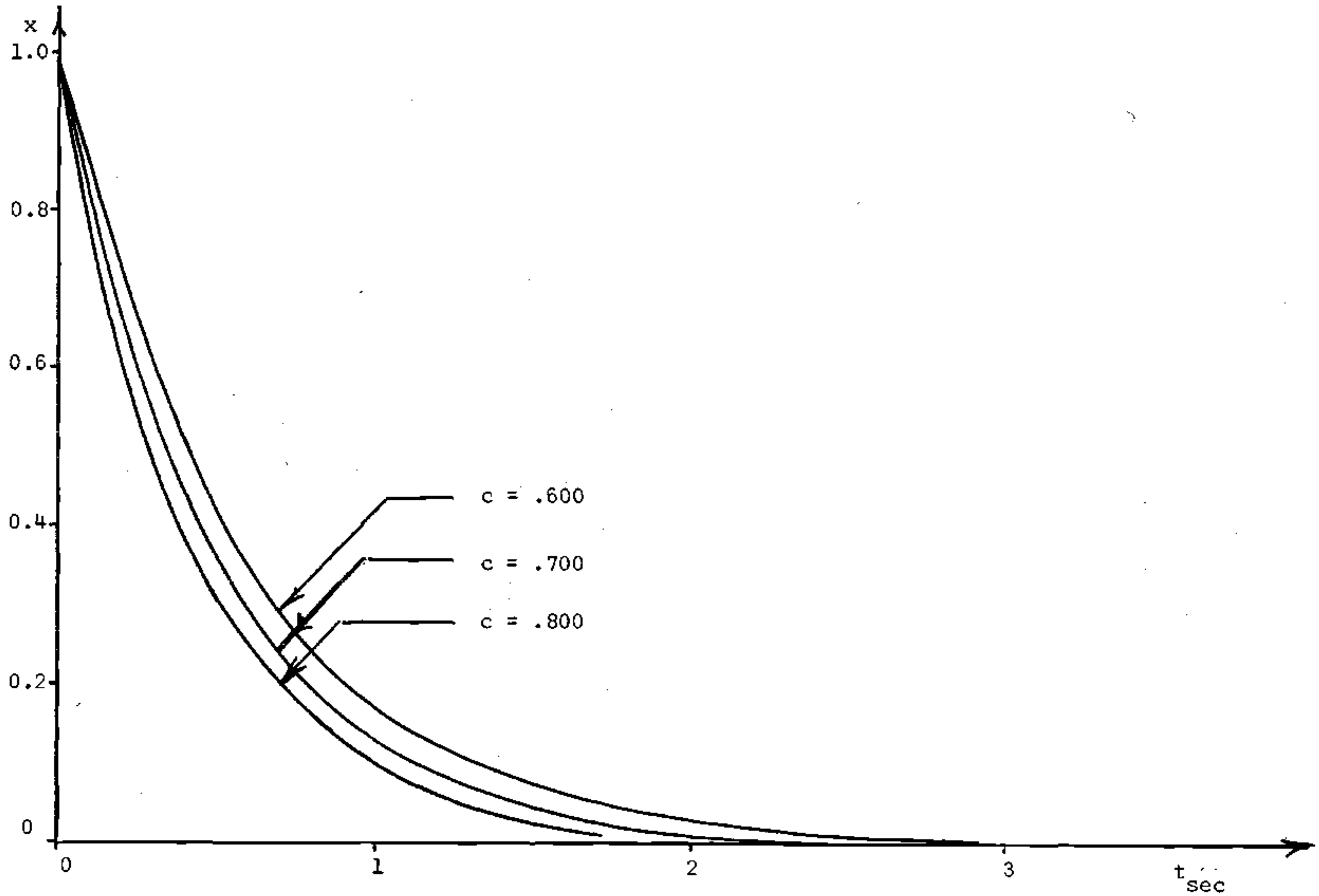

Figure 5. Plant Output for Various Parameter Values with $\gamma = 0$, Example 1, Case 1.

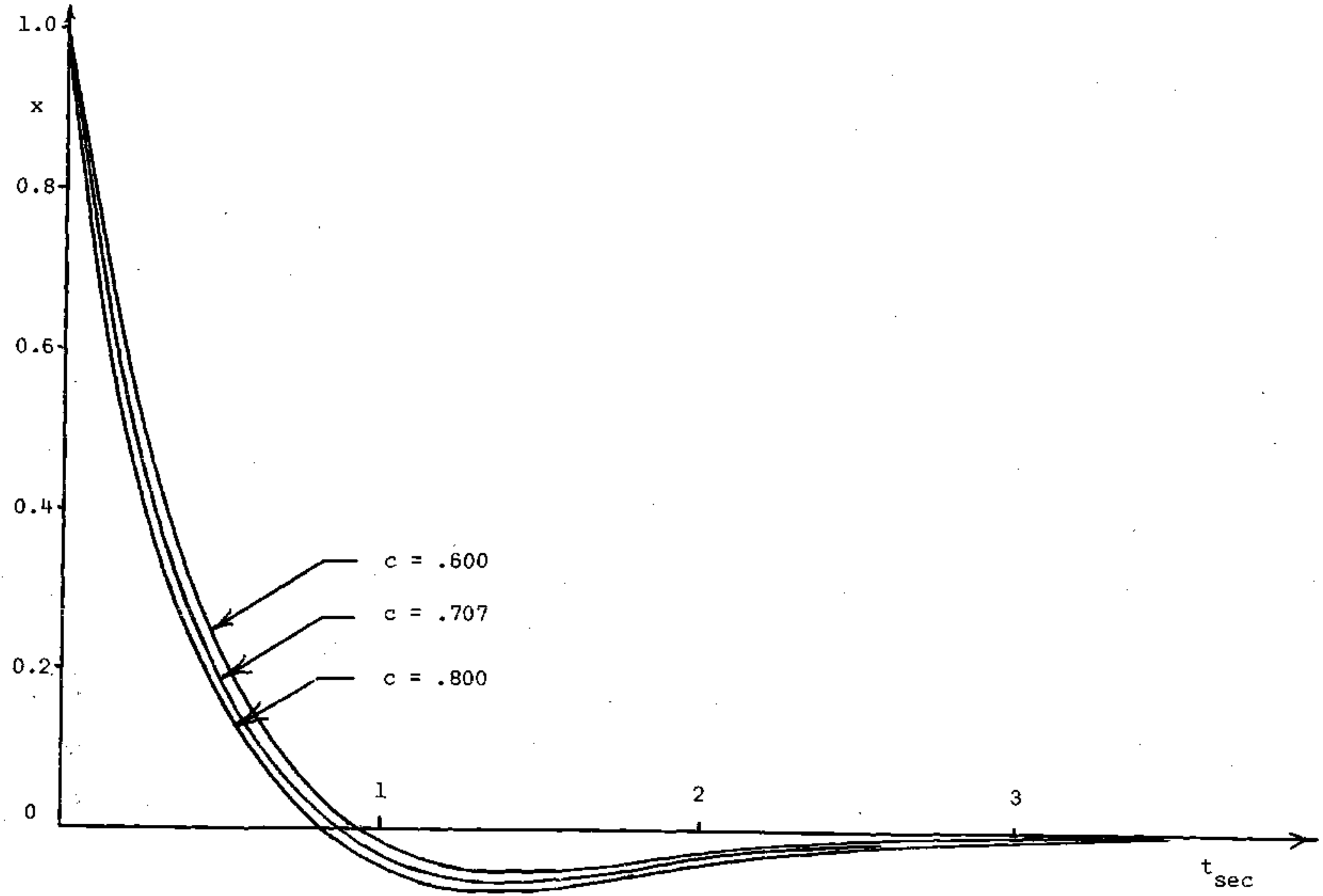

Figure 6. Plant Output for Various Parameter Values with $\gamma = \frac{5}{2}$, Example 1, Case 1.

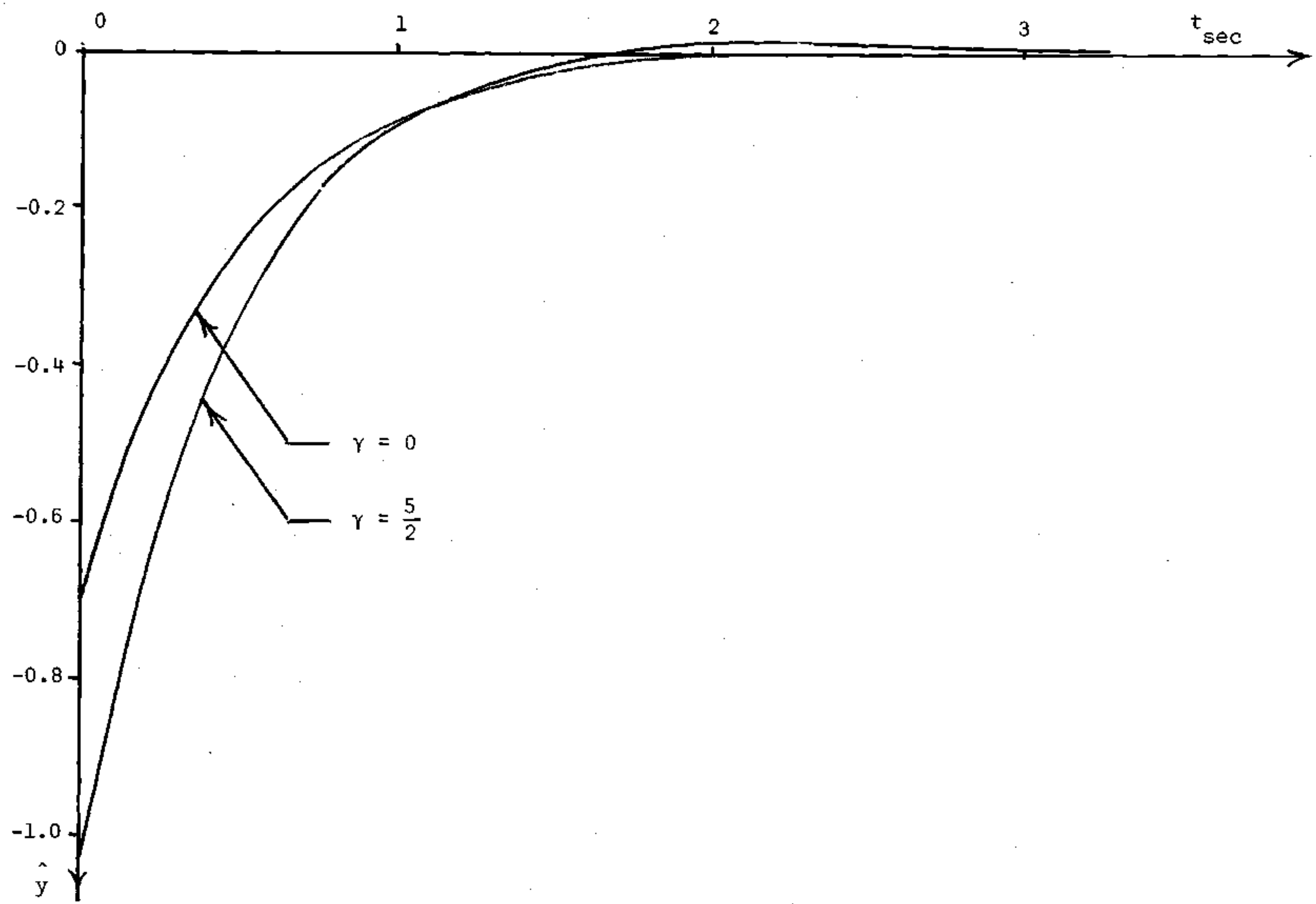

Figure 7. Optimum Input, Example 1, Case 1.

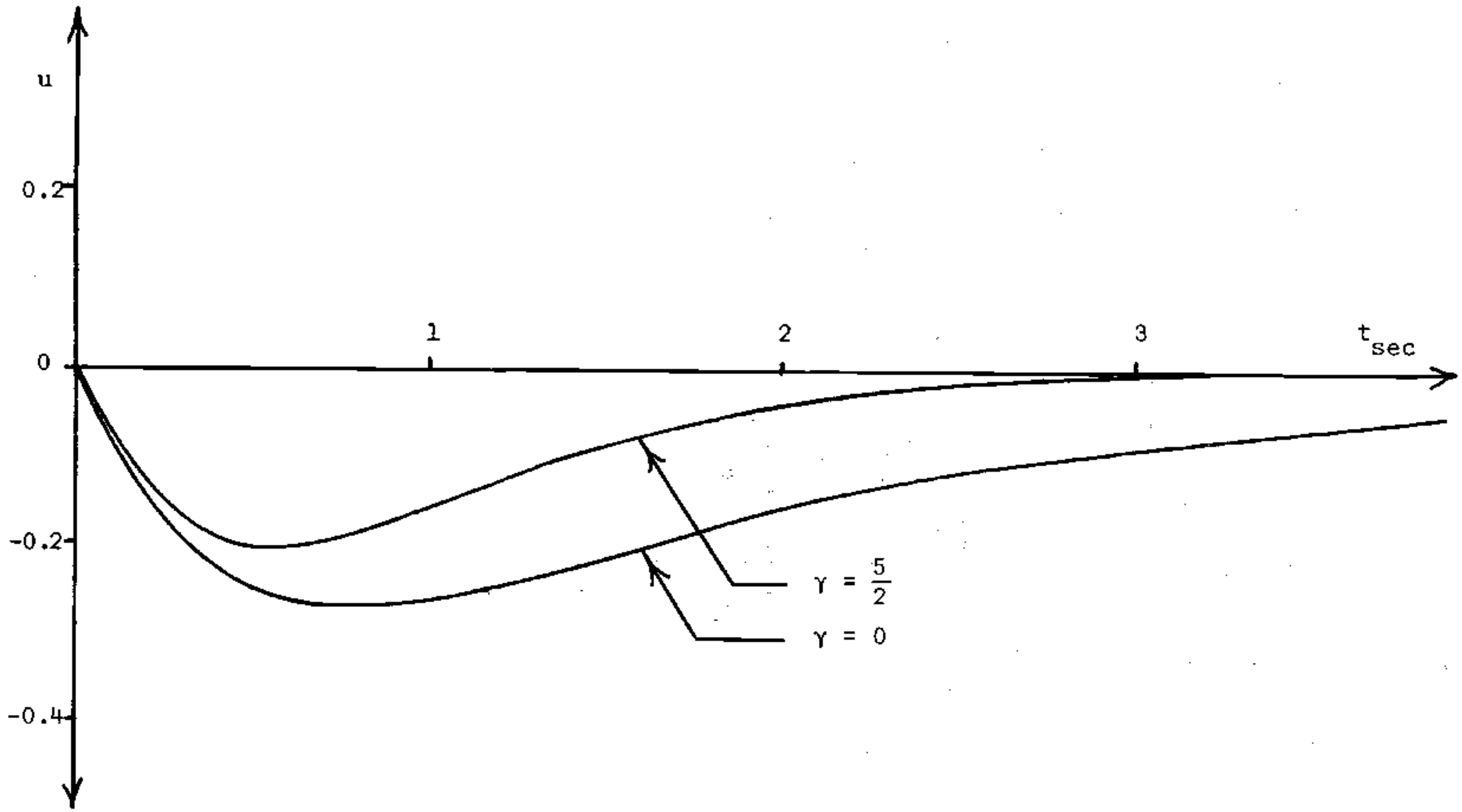

Figure 8. The Sensitivity Coefficient, $u(t,c_o)$, Example 1, Case 1.

$$Me^{-c_o t} + Ne^{-at}$$

for the variables p_1, p_2, x_1, x_2 and $\hat{y}$

where $$a = \sqrt{c_o^2 + \frac{k_o^2 + \gamma}{\beta}}$$

Applying the initial conditions $x_1(0) = 1$ and $x_2(0) = 0$ yields the solutions.

$$x\Big|_{c=c_o} = x_1 = \frac{1}{k_o^2 + \gamma} [\gamma e^{-c_o t} + k_o^2 e^{-at}]$$

$$u\Big|_{c=c_o} = x_2 = \frac{k_o}{k_o^2 + \gamma} [e^{-c_o t} - e^{-at}]$$

$$y = \hat{y} = \frac{(c_o - a)k_o}{k_o^2 + \gamma} e^{-at} \tag{6.14}$$

Again, in order to illustrate the results of the sensitivity optimization, the output x is plotted in Figures 9 and 10 for $\gamma = 0$ and $\gamma = \frac{7}{2}$ while the forward gain k is allowed to vary between 1.8 and 2.2, with $c_o = .707$, $\beta = 1$ and $k_o = 2$. The output curves when $\gamma = \frac{7}{2}$ lie closer together than when $\gamma = 0$ indicating a reduction in the system's sensitivity to plant parameter changes. Figure 11 shows plots of $\hat{y}$, the

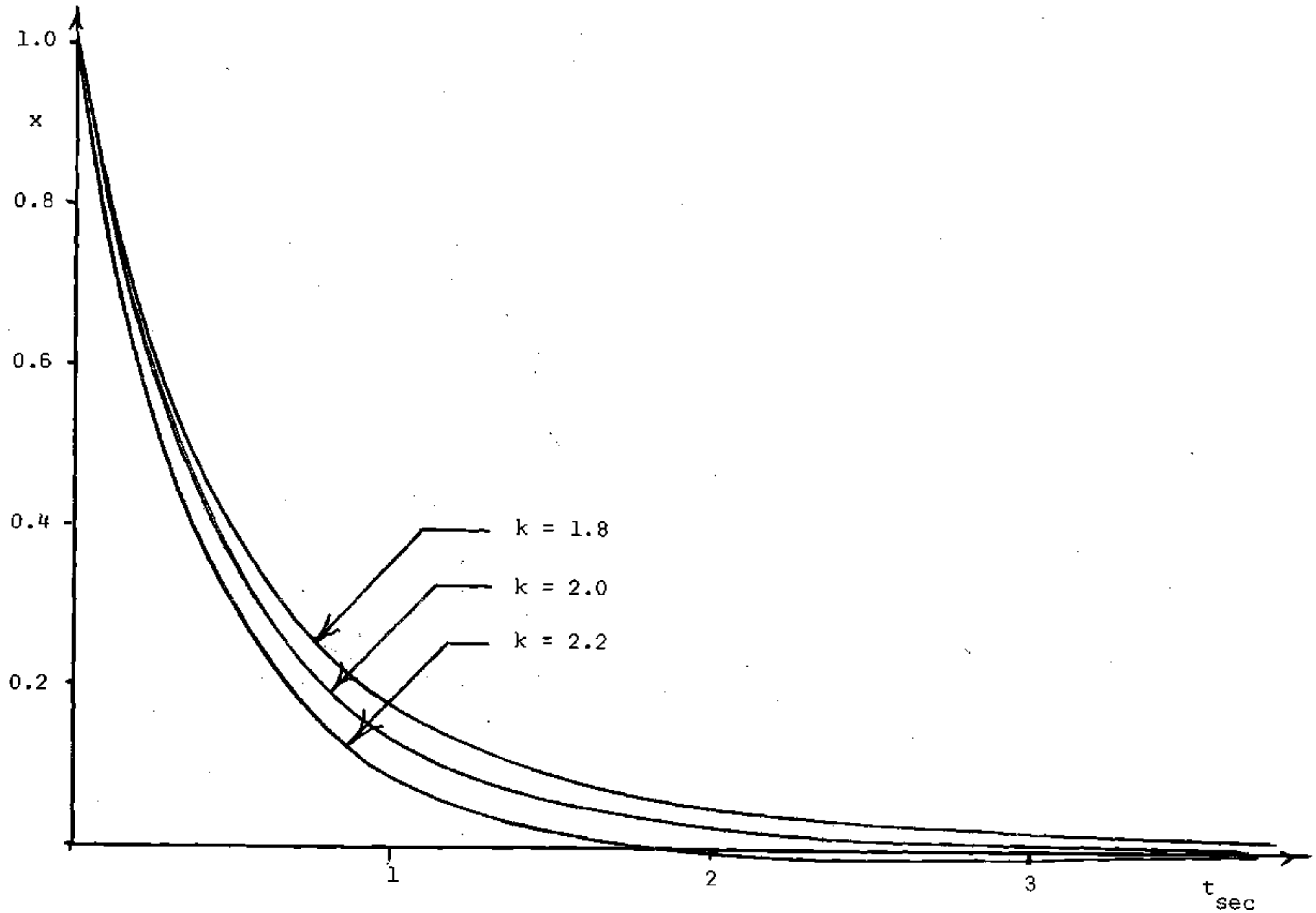

Figure 9. Plant Output for Various Parameter Values with $\gamma = 0$, Example 1, Case 1.

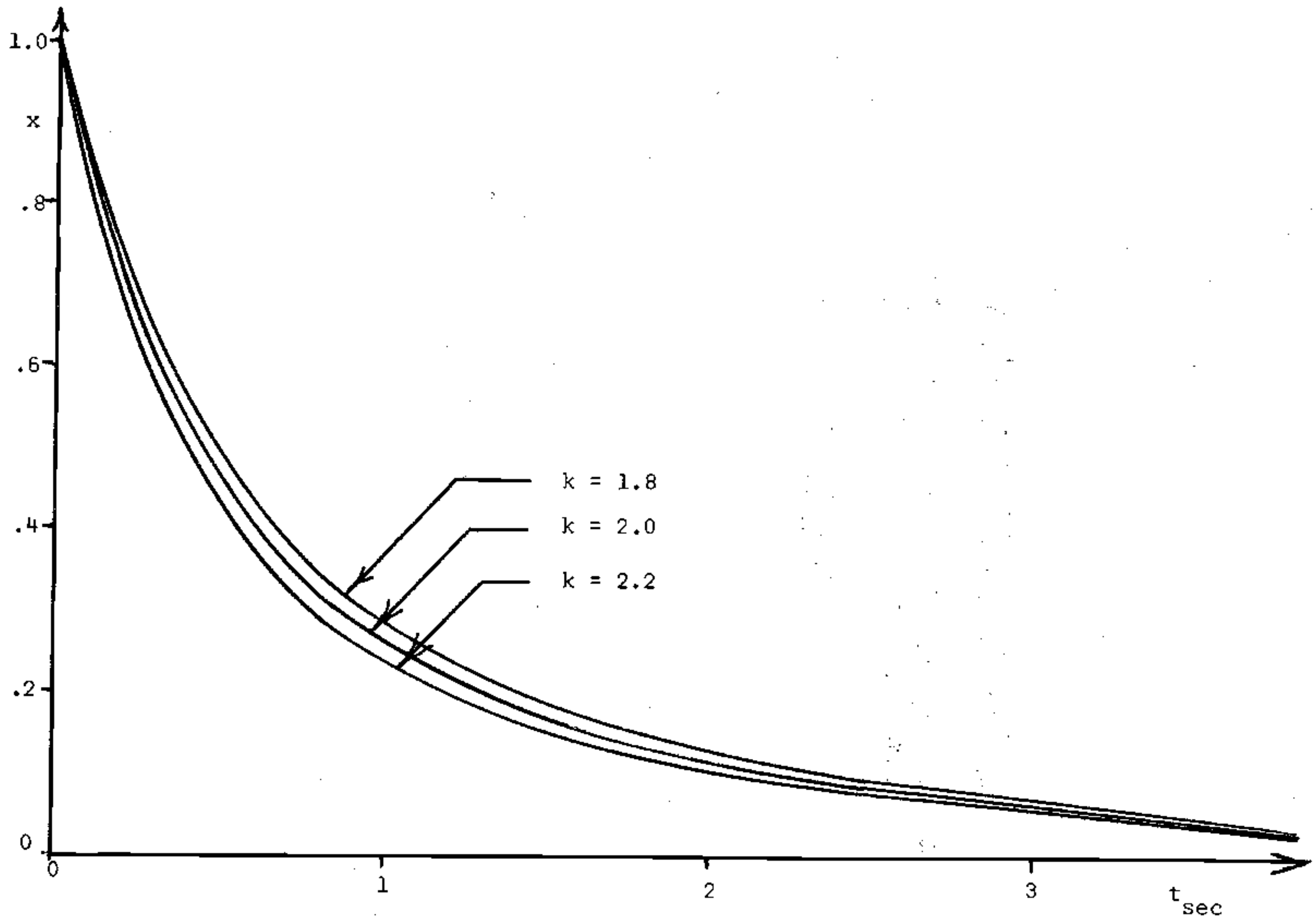

Figure 10. Plant Output for Various Parameter Values with $\gamma = \frac{7}{2}$, Example 1, Case 2.

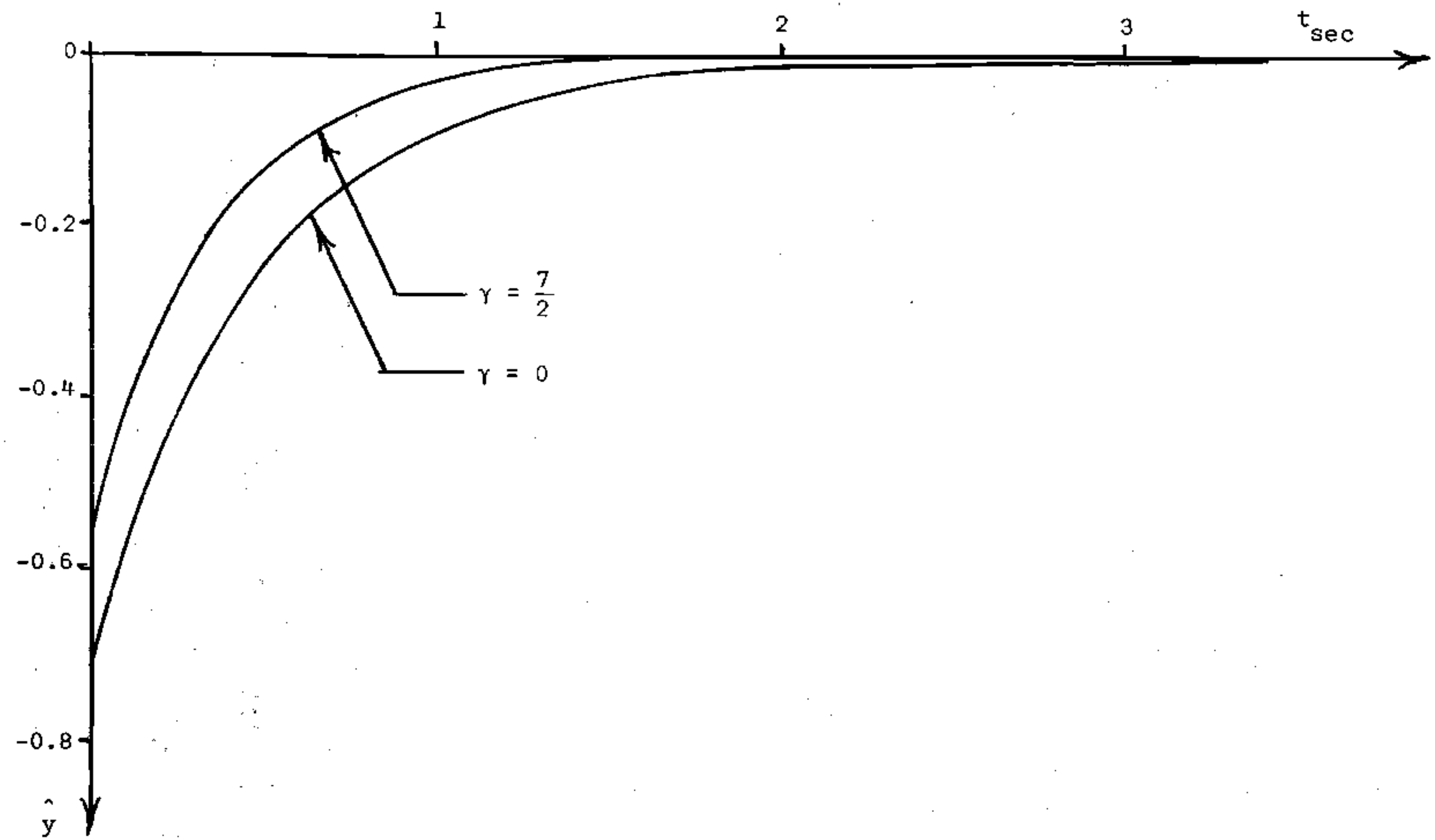

Figure 11. Optimum Input, Example 1, Case 2.

optimum control function, for $\gamma = 0$ and 7/2.

Example Two

This example is similar to Example One, except that a different sensitivity measure is used. In this problem,

$$U = \max_{(0,T)} |u(t,q)|$$

Given the plant equation:

$$\dot{x} + cx = y, \quad x(0) = 1 \tag{6.15}$$

and with c being the parameter of interest. Let

$$J[y] = \int_0^T (x^2 + \beta y^2)dt + \gamma U(T) \tag{6.16}$$

The problem is to find $\hat{y}$ so that J[y] is minimized.

Differentiating (6.15) with respect to c yields the sensitivity equation:

$$\dot{u} + cu = - x \tag{6.17}$$

Letting $x = x_1$, $u = x_2$ and evaluating at $c = c_o$, equations (6.16) and (6.17) can be written:

$$\dot{x}_1 = - c_o x_1 + y$$

$$\dot{x}_2 = - c_o x_2 - X\, x_i$$

Using Pontryagin's method:

$$\dot{x}_3 = x_1^2 + \beta_y{}^2 + \gamma\, \dot{U}(t)$$

Consider the function $U(T) = \max\limits_{(0,T)} |u(t)|$. Assume u(t) is a monotonically non-increasing negative function over the time range $(0,\tau)$ where τ is the time at which u(t) reaches its first minimum. (This assumption is motivated by the behavior of u(t) in Example One and will be shown to be true in the solution to this example.) Further assume that $|u(t)|$ is at its maximum over (0,T) at $t = \tau$. Then:

for $\quad 0 < t < \tau, \quad U(t) = - u(t)$

and for $\tau < t < T$, $\quad U(t) = |u(\tau)|$ = a constant.

Thus $\quad \dot{U}(t) = - \dot{u}(t) = - x_2(t) = cx_2 + x_1, \quad 0 < t < \tau$

$\quad = 0 \quad \tau < t < T$

and $\quad \dot{x}_3 = x_1^2 + \beta_y{}^2 + \gamma c x_2 + \gamma x_1, \quad 0 < t < \tau$

$$= x_1^2 + \beta_y{}^2 + 0, \qquad \tau < t < T$$

Applying Pontryagin's method, we obtain the following set of equations:

$$\left.\begin{aligned}
\dot{x}_1 &= -c_o x_1 + y \\
\dot{x}_2 &= -c_o x_2 - x_i \\
\dot{p}_1 &= c_o p_1 + p_2 + 2x_1 + \gamma, && 0 < t < \tau \\
&= c_o p_1 + p_2 + 2x_1, && \tau < t < T \\
\dot{p}_2 &= c_o p_2 + c_o \gamma, && 0 < t < \tau \\
&= c_o p_2, && \tau < t < T \\
y &= \frac{p_1}{2\beta}
\end{aligned}\right\} \qquad (6.18)$$

with $x_1(0) = 1 \qquad p_1(T) = 0$

$x_2(0) = 0 \qquad p_2(T) = 0$

The set of equations (6.18) can be easily mechanized on an analog computer. Solutions for $c_o = 1$, $\beta = \frac{1}{3}$ and for $\gamma = 0$ and $\gamma = 5$ with $T \to \infty$ are shown in Figures 12 through 15. In Figures 12 and 13, the output x is plotted for values of the parameter c between .9 and 1.1, with $\gamma = 0$ and $\gamma = 5$. As in Example One, the output curves are closer together for $\gamma = 5$ indicating reduced sensitivity to plant parameter changes. In Figure 14, the optimum input $\hat{y}$ is plotted for $\gamma = 0$ and 5.

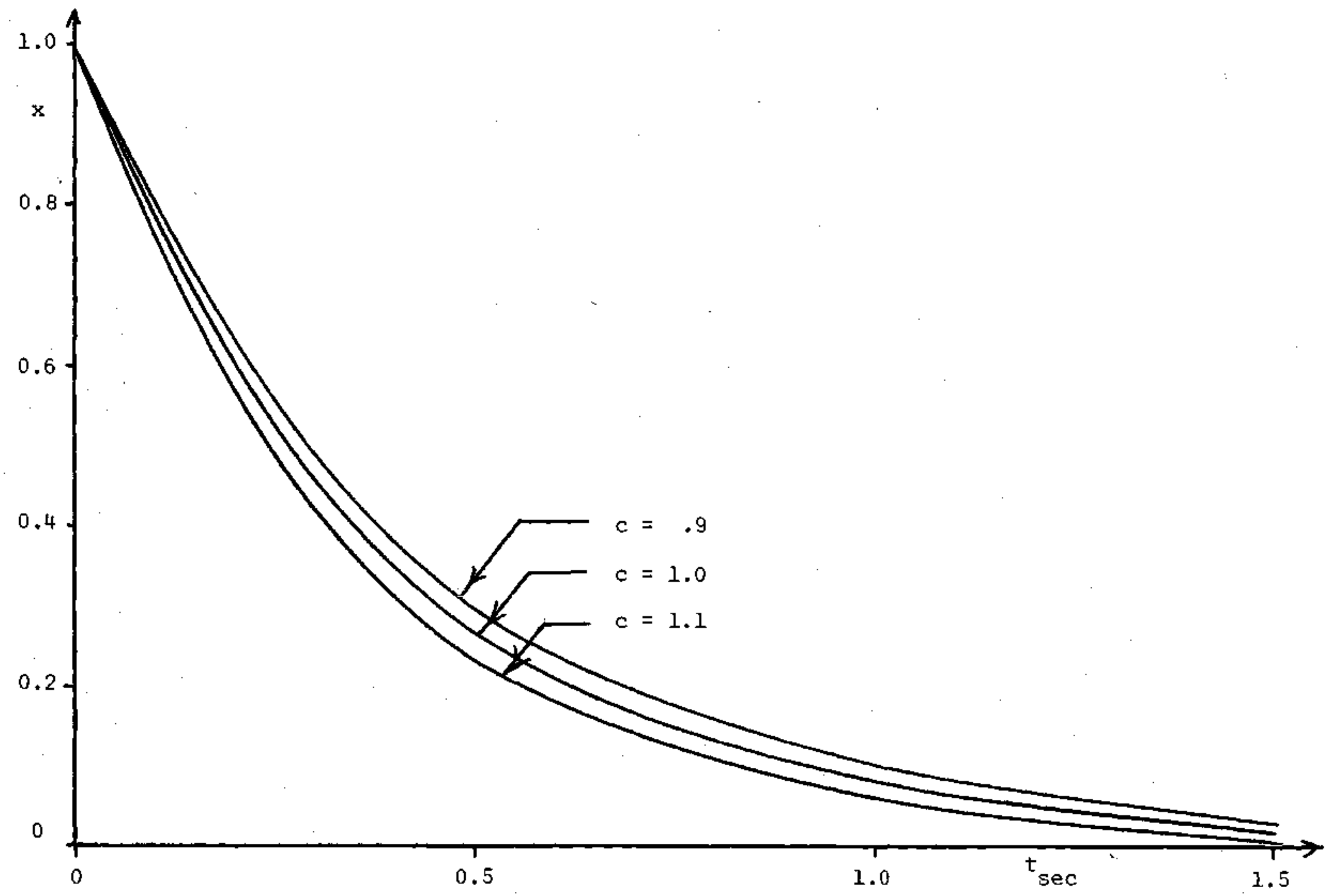

Figure 12. Plant Output for Various Parameter Values with $\gamma = 0$, Example 2.

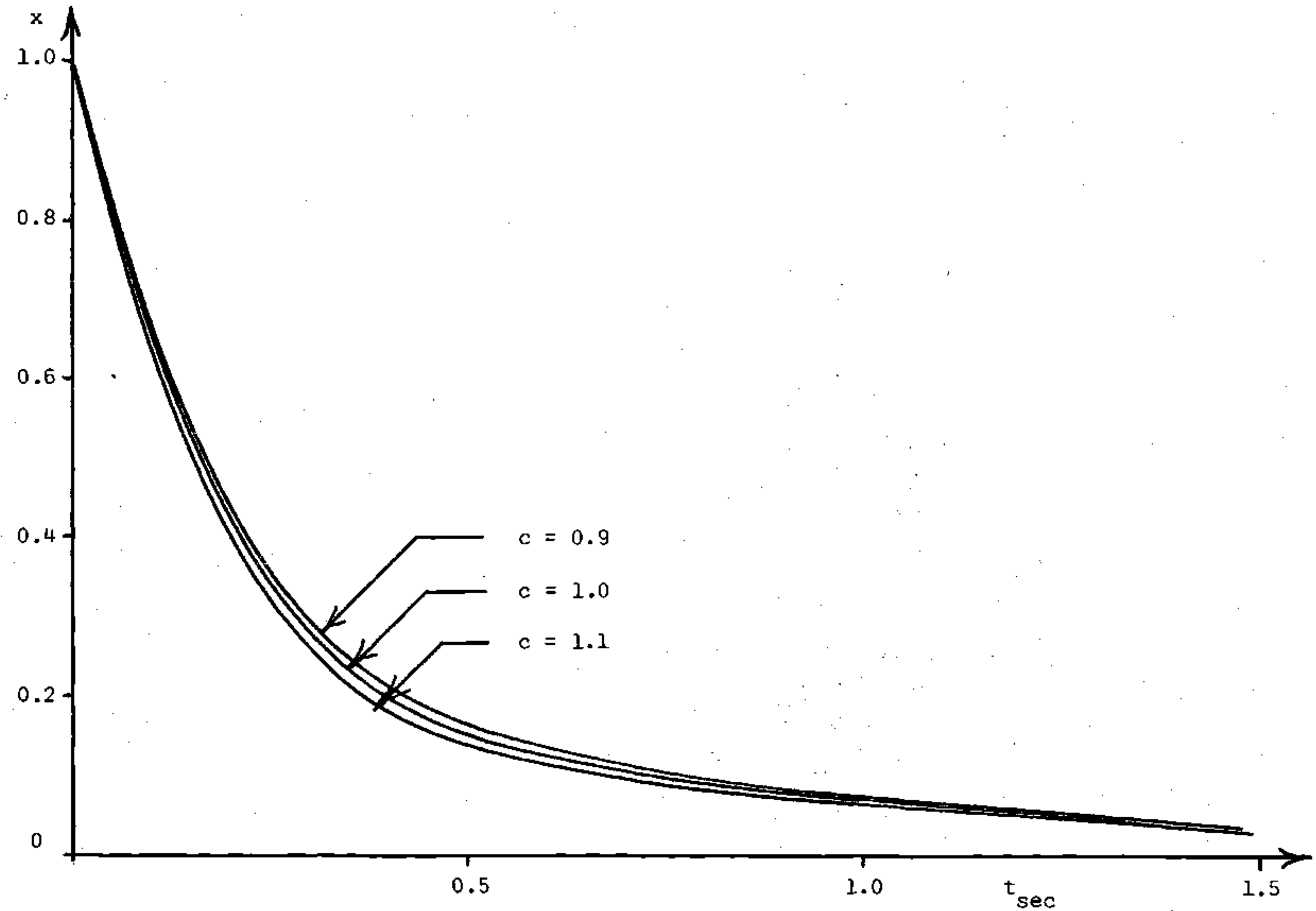

Figure 13. Plant Output for Various Parameter Values with γ = 5, Example 2.

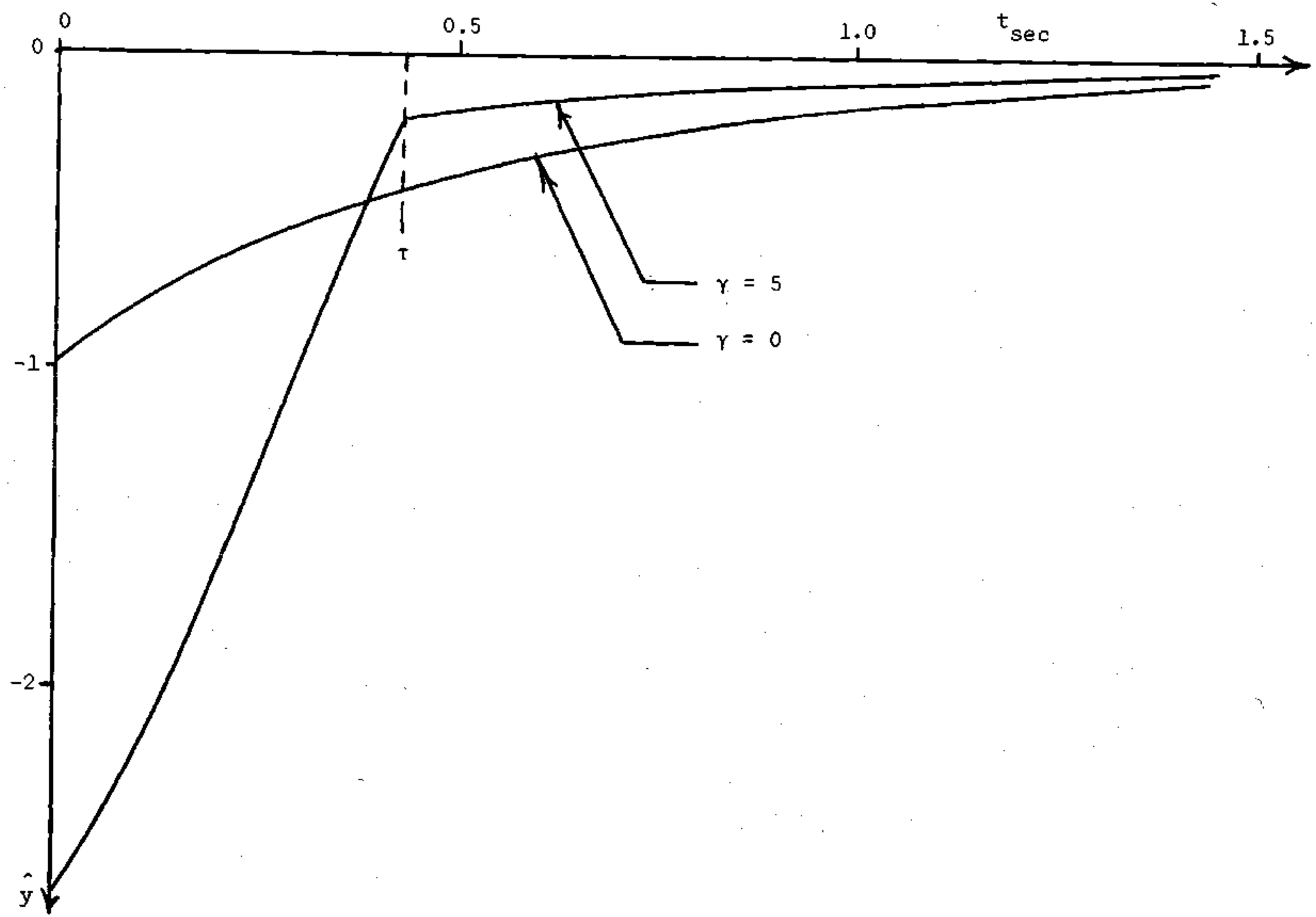

Figure 14. Optimum Input, Example 2.

In Figure 15, the sensitivity coefficient, $u(t,c_o)$ is plotted for $\gamma = 0$ and 5. The assumption above regarding the behavior of $u(t)$ is seen to be confirmed.

Example Three

This example has a second-order linear plant with a quadratic performance index. The object of the control is to drive the output to zero while minimizing the performance index.

The plant equation is:

$$\ddot{x} + b\dot{x} + cx = ky \qquad \text{with } x(0) = 1 \text{ and } \dot{x}(0) = 0 \tag{6.19}$$

The parameter of interest is the forward gain k. The performance index is:

$$J[y] = \int_0^T (x^2 + \beta y^2 + \gamma u^2)dt$$

Differentiating equation (6.19) with respect to k yields the sensitivity equation:

$$\ddot{u} + b\dot{u} + cu = y \qquad \text{with } u(0) = 0 \text{ and } \dot{u}(0) = 0 \tag{6.20}$$

Equations (6.19) and (6.20) can be written in state variable form with $x = x_1$, $\dot{x} = x_2$, $u = x_3$, $\dot{u} = x_4$ as follows:

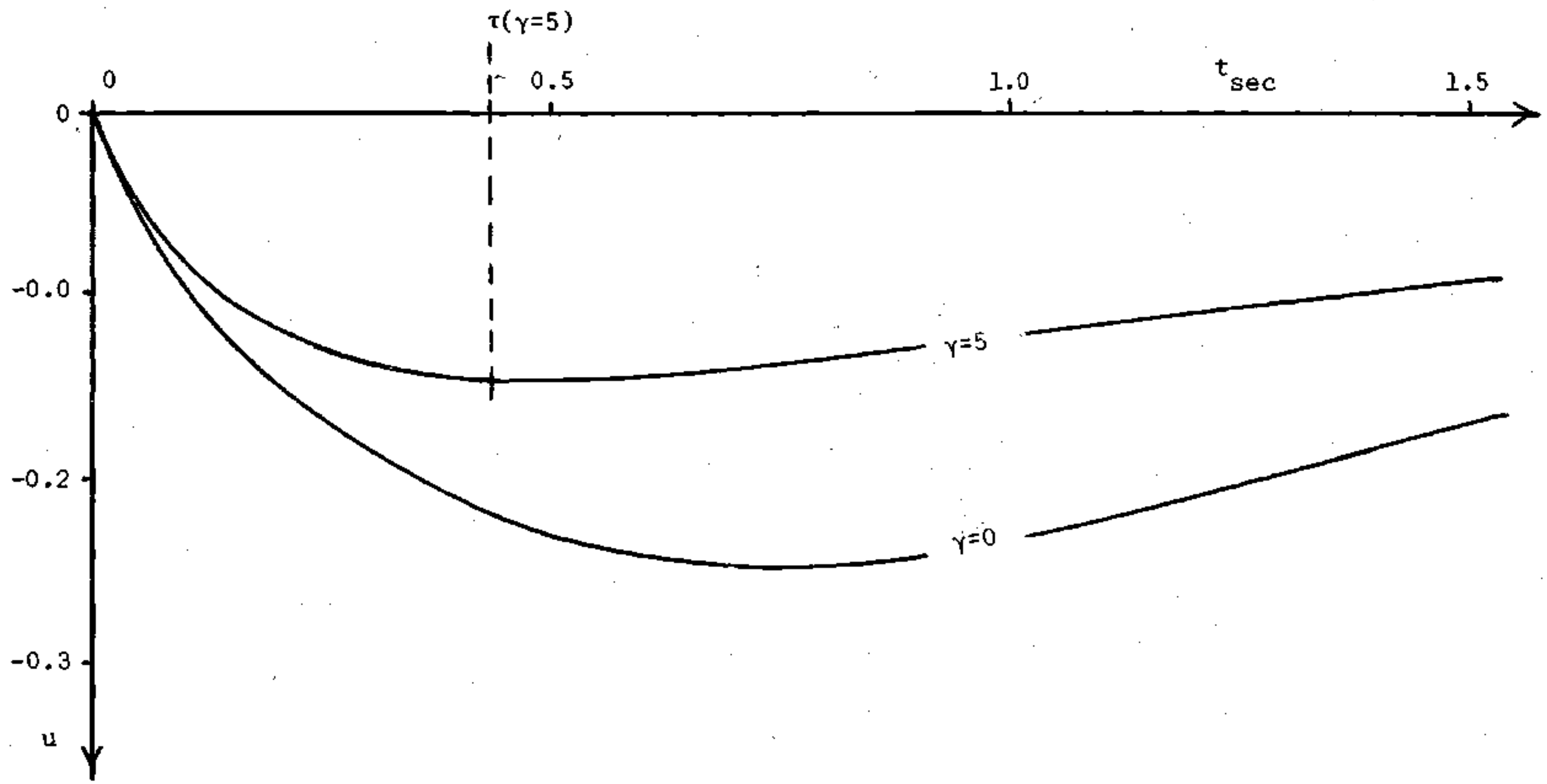

Figure 15. The Sensitivity Coefficient, $u(t,c_o)$, Example 2.

$$\left.\begin{aligned}
\dot{x}_1 &= x_2 & x_1(0) &= 1 \\
\dot{x}_2 &= -bx_2 - cx_1 + k_o y & x_2(0) &= 0 \\
\dot{x}_3 &= x_4 & x_3(0) &= 0 \\
\dot{x}_4 &= -bx_4 - cx_3 + y & x_4(0) &= 0
\end{aligned}\right\} \quad (6.21)$$

Applying Pontryagin's method (See Appendix I) yields the following additional set of equations:

$$\left.\begin{aligned}
\dot{p}_1 &= cp_2 + 2x_1 & p_1(T) &= 0 \\
\dot{p}_2 &= -p_1 + bp_2 & p_2(T) &= 0 \\
\dot{p}_3 &= cp_4 + 2\gamma x_3 & p_3(T) &= 0 \\
\dot{p}_4 &= -p_3 + bp_4 & p_4(T) &= 0 \\
y &= \frac{k_o p_2 + p_4}{2\beta} & &
\end{aligned}\right\} \quad (6.22)$$

The following numerical values are assigned to various constants to obtain typical solutions:

$$c = 1 \qquad b = 3 \qquad T \to \infty$$
$$\beta = 1 \qquad k_o = 2$$

Solutions to the two-point boundary value problem posed in (6.21) and (6.22) using the above values are shown in Figures 16 through 18.

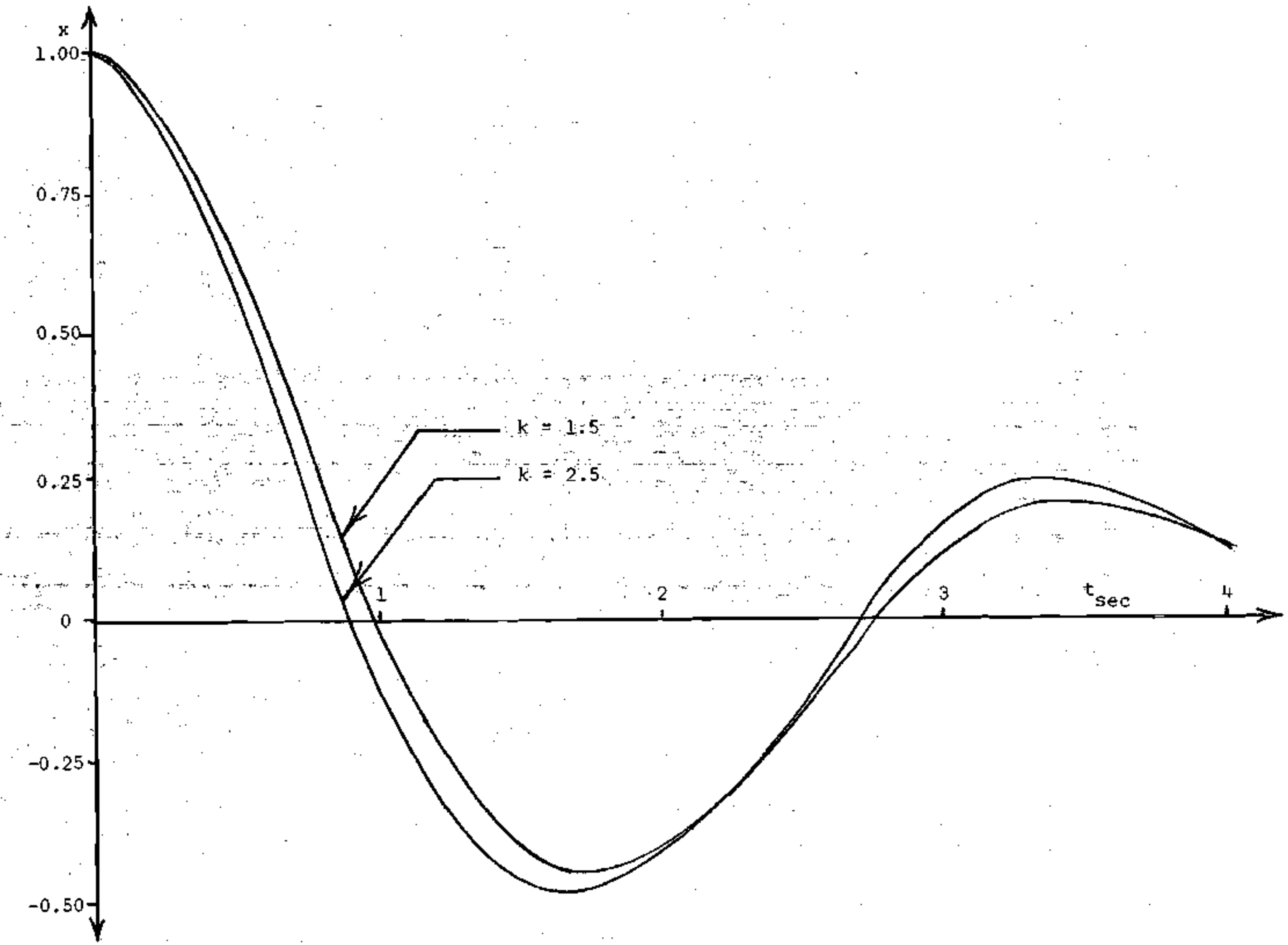

Figure 16. Plant Output for Various Parameter Values with $\gamma = 0$, Example 3.

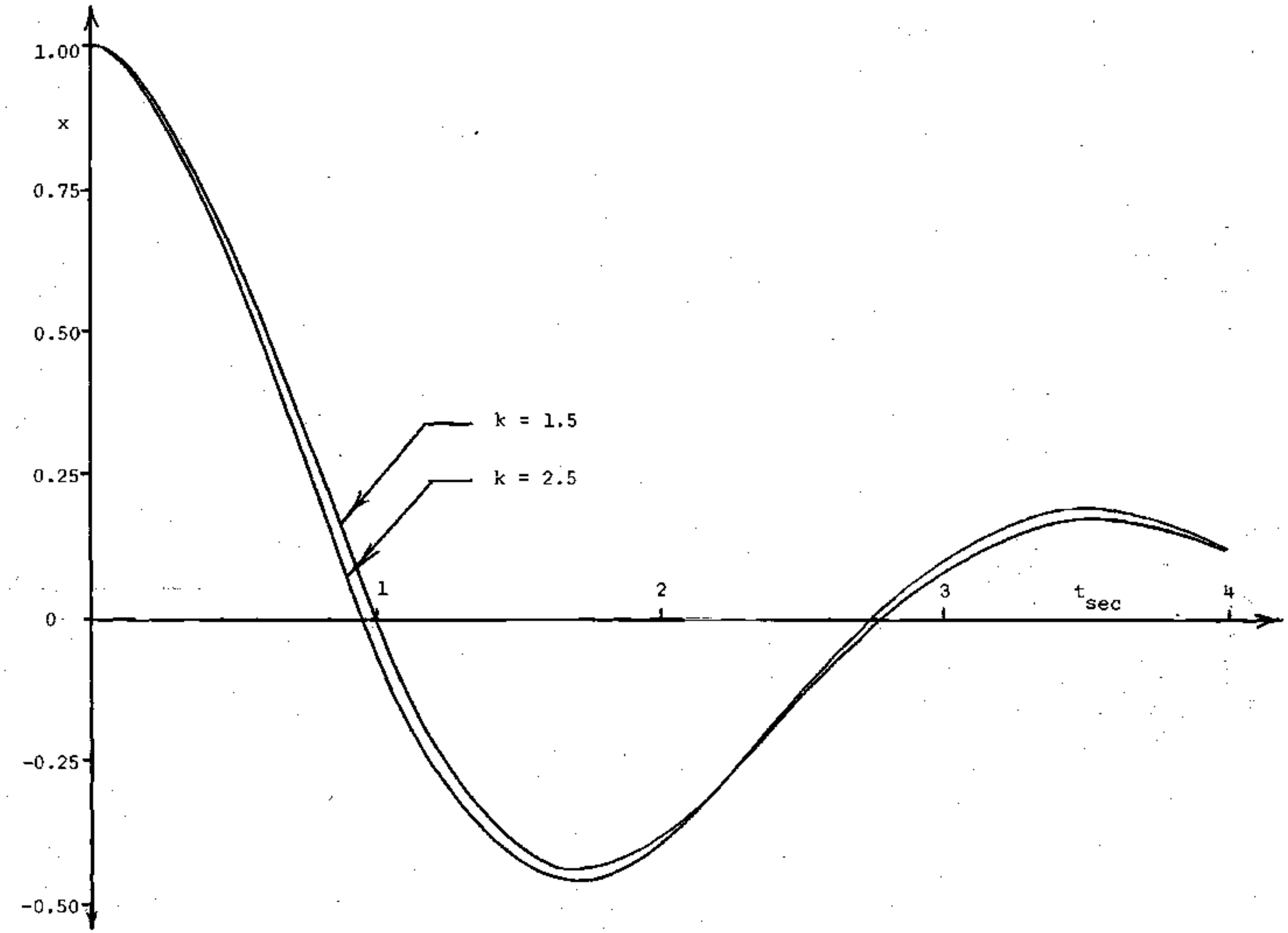

Figure 17. Plant Output for Various Parameter Values with $\gamma = 7.5$, Example 3.

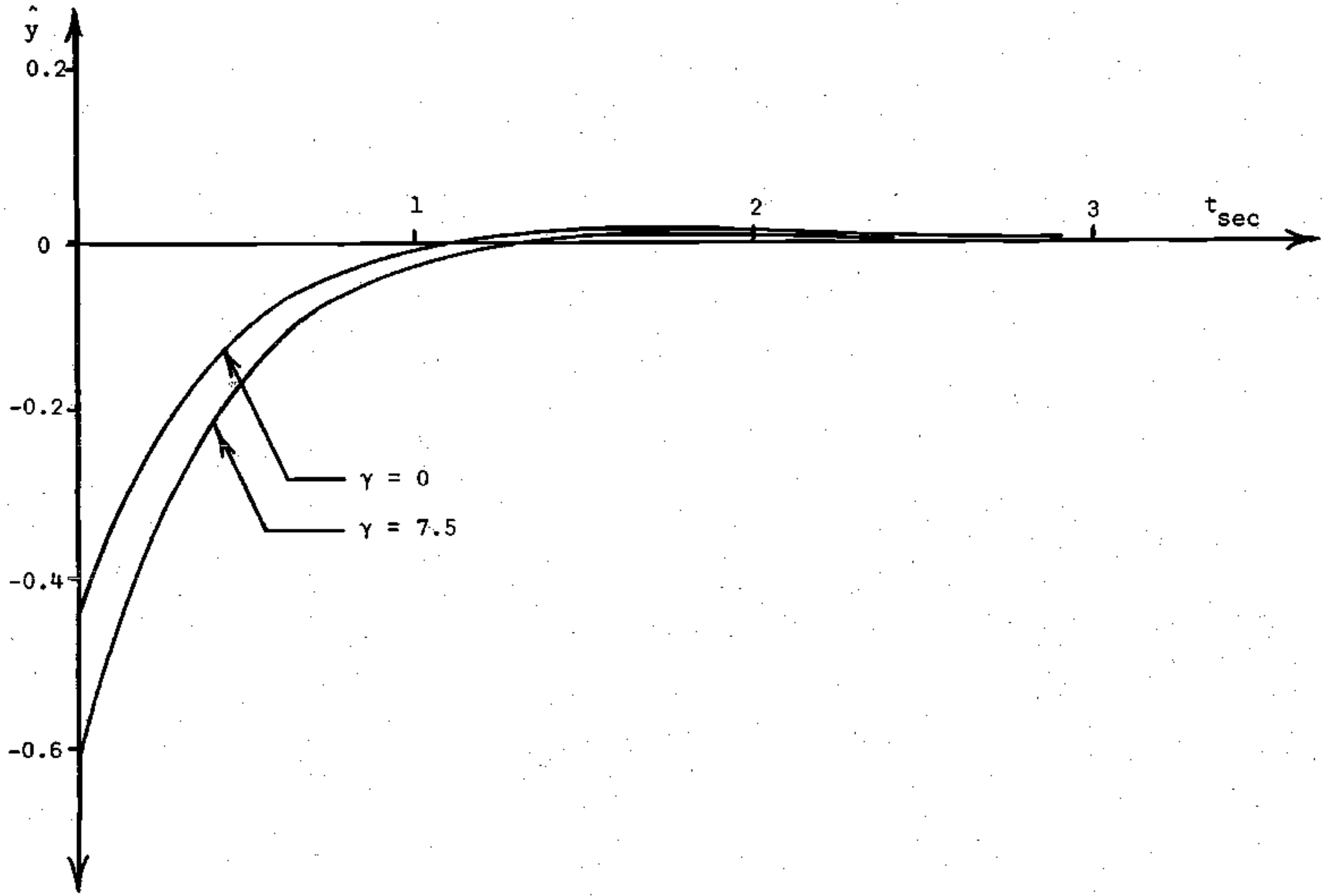

Figure 18. Optimum Input, Example 3.

The output x is plotted for various values of k^* with the sensitivity weighting factor $\gamma = 0$ in Figure 16 and with $\gamma = 7.5$ in Figure 17. Again, the output curves show the characteristic compression when the sensitivity weighting factor in increased. The optimum input $\hat{y}$ is plotted in Figure 18 for $\gamma = 0$ and 7.5.

Example Four

This is an example of a problem where the output is to be driven from its initial value of zero to some predetermined value at a particular time T with minimum energy at the input. It is desired that the output x at t = T be consistently the same despite variation of a plant parameter. A first-order linear plant is used and the sensitivity measure $|u(T)|$ constrained to be zero. For comparison purposes, the problem is re-solved with the constraint removed.

The plant equation is:

$$\dot{x} = -cx + y \qquad x(0) = 0$$

with c, the parameter of interest the sensitivity equation is:

$$\dot{u} = -cu - x$$

$$J[y] = \int_0^T y^2 dt$$

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* The curve for $k = k_o = 2$ is omitted for clarity. This curve lies between the two curves shown.

with the constraint $\qquad U = |u(T,c_o)| = 0$

Application of Pontryagin's method with a constraint, yields the following two-point boundary value problem:

$$\dot{x}_1 = -c_o x_1 + y \qquad\qquad x_1 = x,\ x_2 = u$$

$$\dot{x}_2 = -c_o x_2 - x_1 \qquad\qquad x_1(0) = 0$$

$$\dot{p}_1 = c_o p_1 + p_2 \qquad\qquad x_1(T) = x_T$$

$$\dot{p}_2 = c_o p_2 \qquad\qquad x_2(0) = 0$$

$$y = \frac{p_1}{2} \qquad\qquad x_2(T) = 0$$

Let $T = 1$ sec, $x_T = 1$ and $c_o = 1$, to permit a definite solution. With these values, the plots in Figures 19 and 21 are obtained.

Figure 19 shows the plant output for various values of the parameter c with the optimum input $\hat{y}$ applied. Despite variations in c, the value of x at t = T remains the same. This demonstrates the effect of the constraint, $|u(T,c_o)| = 0$.

The optimum input $\hat{y}$ is shown in Figure 21.

For comparison, let us re-solve the problem with the sensitivity constraint removed. Application of Pontryagin's method yields the following simple two-point boundary value problem:

$$\dot{x}_1 = -c_o x_1 + y \qquad\qquad x_1(0) = 0$$

$$\dot{p}_1 = c p_1 \qquad\qquad x_1(T) = x_T = 1$$

$$T = 1 \text{ sec},\ c_o = 1$$

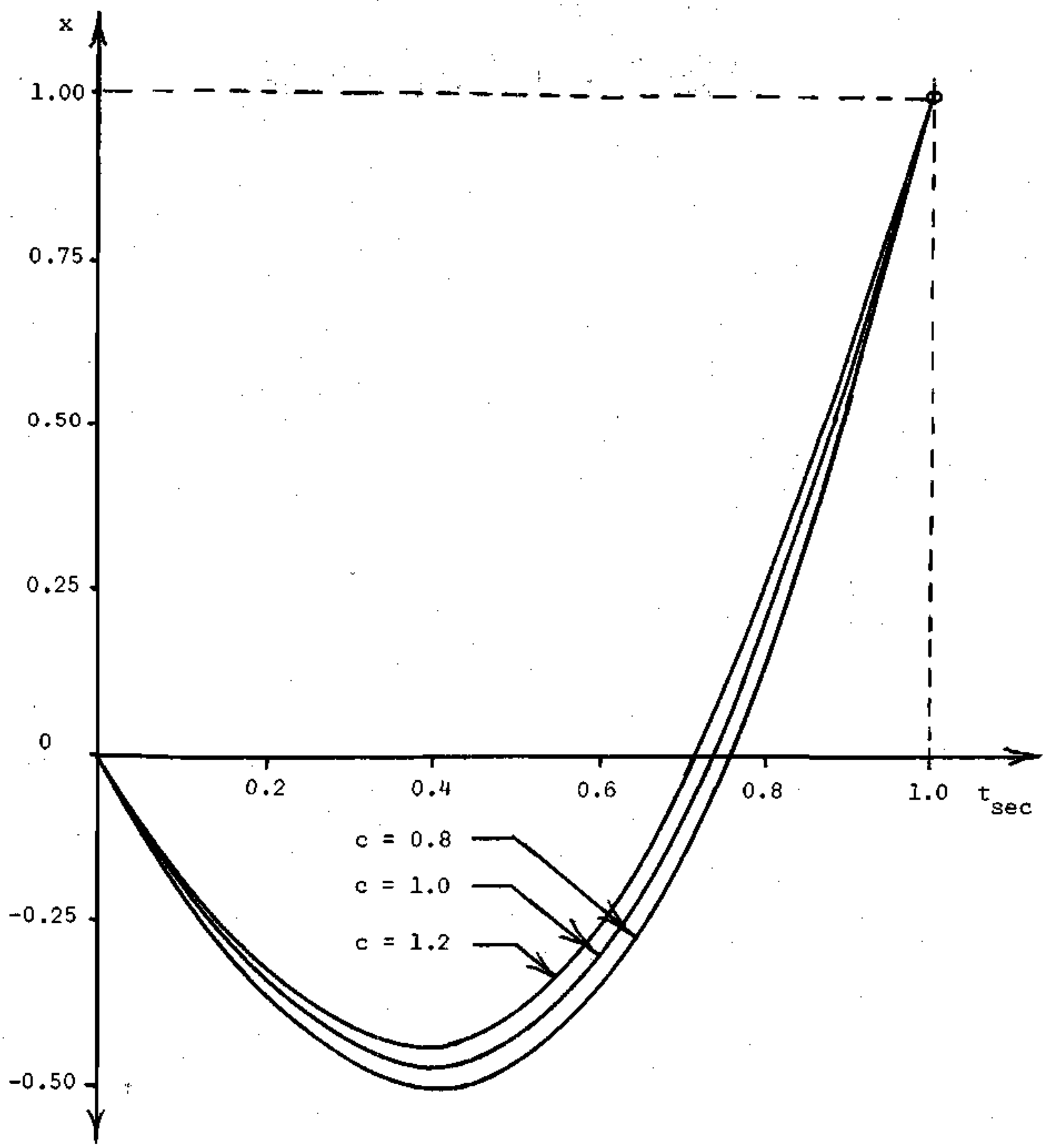

Figure 19. Plant Output for Various Parameter Values with $|u(T)| = 0$, Example 4.

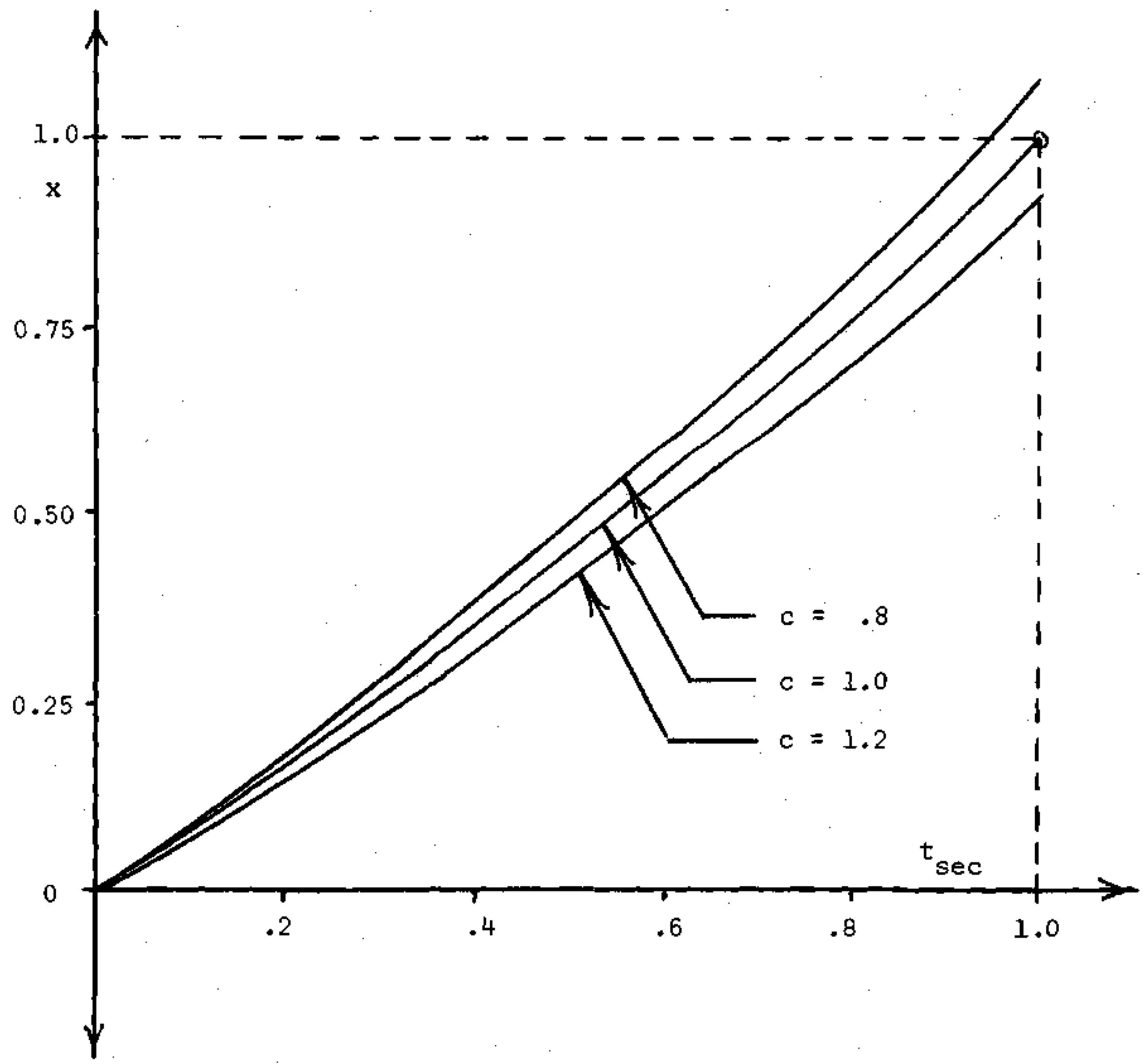

Figure 20. Plant Output for Various Parameter Values Without Sensitivity Constraint, Example 4.

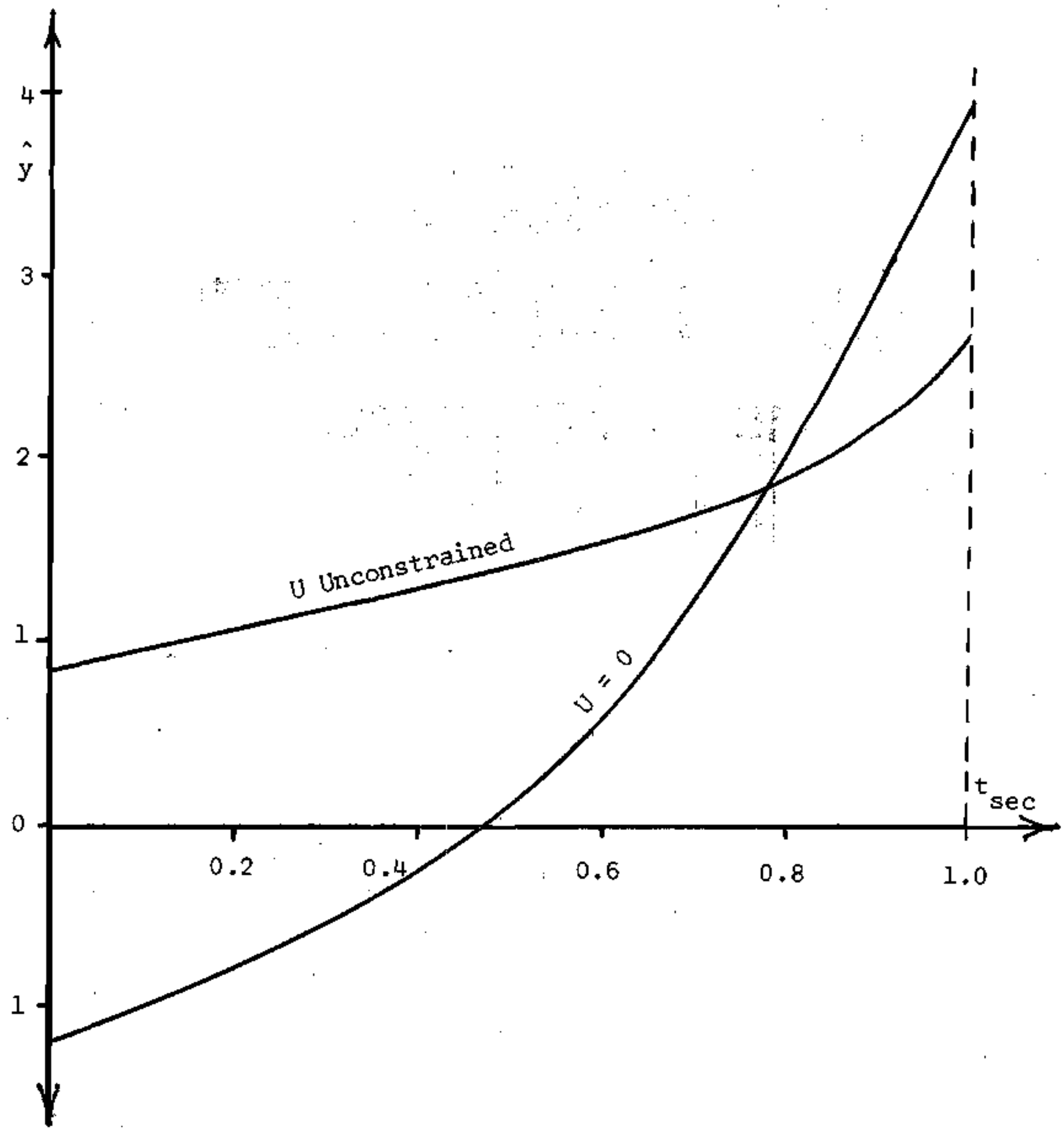

Figure 21. Optimum Input, Example 4.

Solving this problem yields the curves in Figures 20 and 21.

Figure 20 shows the plant output for various values of c with the new optimum input $\hat{y}$ applied. Variations of c in this case produce pronounced variations in the value of the output x at t = T. Thus, if it were important that x(T) be consistently the same, this system could be unsatisfactory but the system above, with the sensitivity constraint applied, would be satisfactory. Figure 21 shows the unconstrained optimum input $\hat{y}$.

Example Five

This is an example of the synthesis method applied to a simple non-linear problem.

A quadratic performance index is used.

The plant equation is:

$$\dot{x} + cx - dx^3 = ky, \qquad x(0) = x_o$$

with c the parameter of interest.

The sensitivity equation is:

$$\dot{u} + cu - 3dx^2u + x = 0, \qquad u(0) = 0$$

$$J[y] = \int_0^T x^2 + \beta y^2 + \gamma u^2 dt$$

Application of Pontryagin's method (Appendix I) yields the following two-point boundary value problem to be solved:

$$\dot{x}_1 = -c_o x_1 + d x_1^3 + ky \qquad x_1 = x,\ x_2 = u$$

$$\dot{x}_2 = -c_o x_2 + 3 d x_1^2 x_2 - x_1$$

$$\dot{p}_1 = c_o p_1 - 3 d x_1^2 p_1 + p_2 - 6 d x_1 x_2 p_2 + 2x_1$$

$$\dot{p}_2 = c_o p_2 - 3 d x_1^2 p_2 + 2\gamma x_2$$

$$y = \frac{k p_1}{2\beta}$$

$$x_1(0) = x_o$$

$$x_2(0) = 0$$

$$p_1(T) = 0$$

$$p_2(T) = 0$$

With $c_o = 0.5$, $\beta = \frac{10}{7}$, $d = 4$, $k = 1$, $T = 3$ and $x_o = 0.4$, the solutions to this problem are shown in Figures 22 through 24.

In Figure 22, the plant output $x(t)$ is shown for various values of the parameter c when $\gamma = 0$ (i.e., no sensitivity considerations) and the corresponding optimum input $\hat{y}$ is applied. In Figure 23, the output is shown when $\gamma = 4$ and the corresponding input $\hat{y}$ is applied. Comparing the curves, we note the improved consistency; i.e., reduced sensitivity when the sensitivity measure is weighted. In Figure 24, the optimum inputs $\hat{y}(t)$ are plotted for both cases, $\gamma = 0$ and $\gamma = 4$.

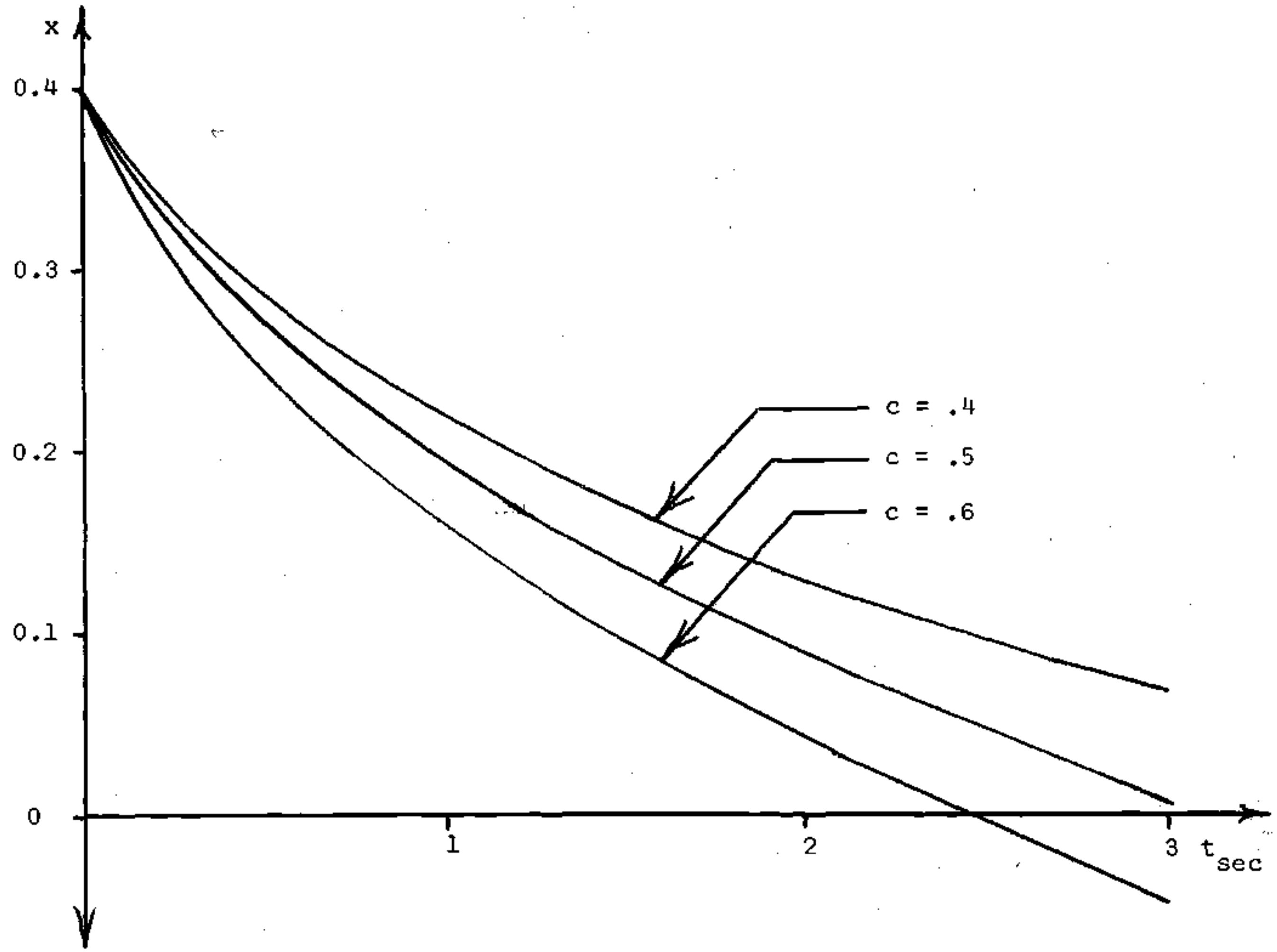

Figure 22. The Plant Output for Various Parameter Values with $\gamma = 0$, Example 5.

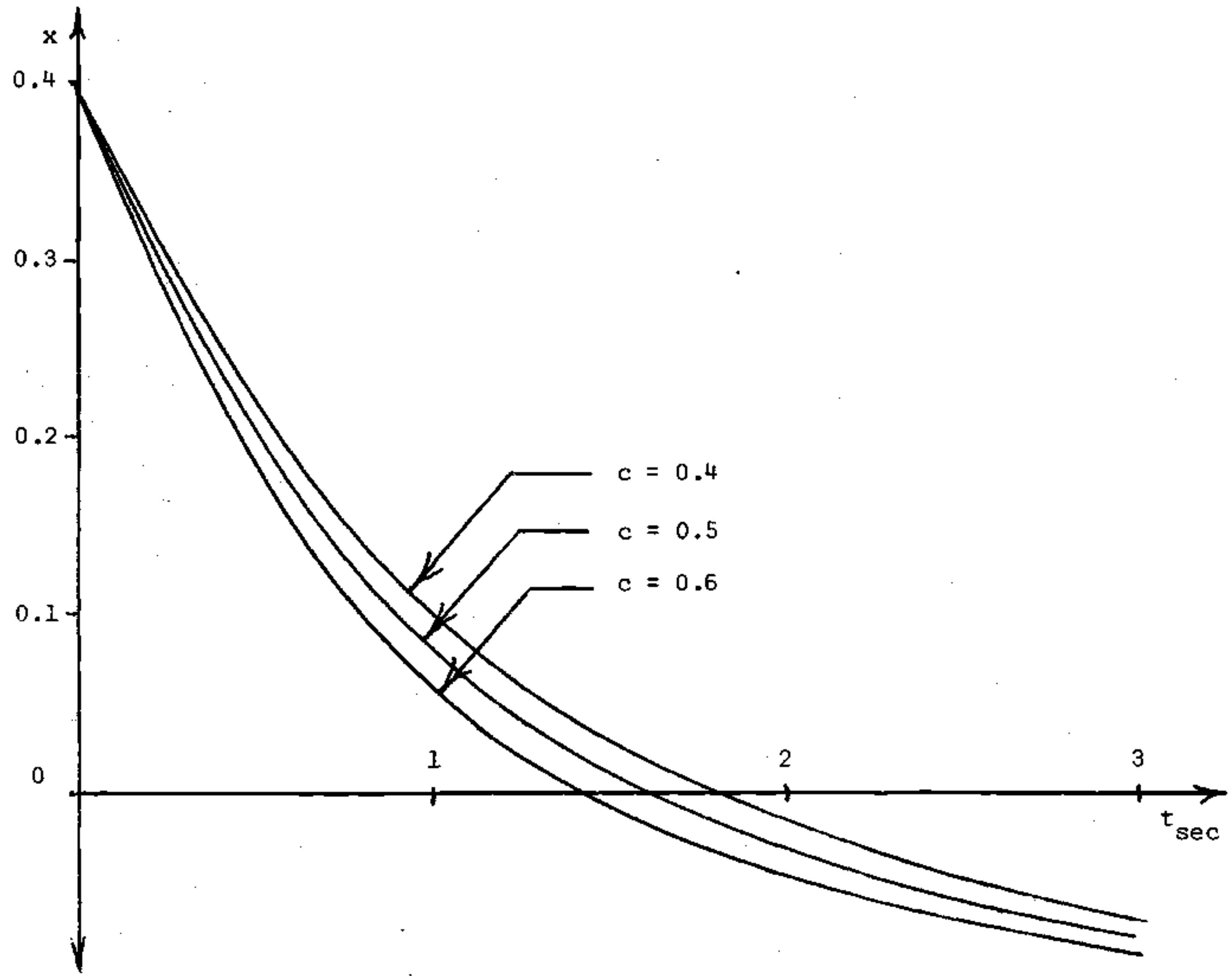

Figure 23. The Plant Output for Various Parameter Values with $\gamma = 4$, Example 5.

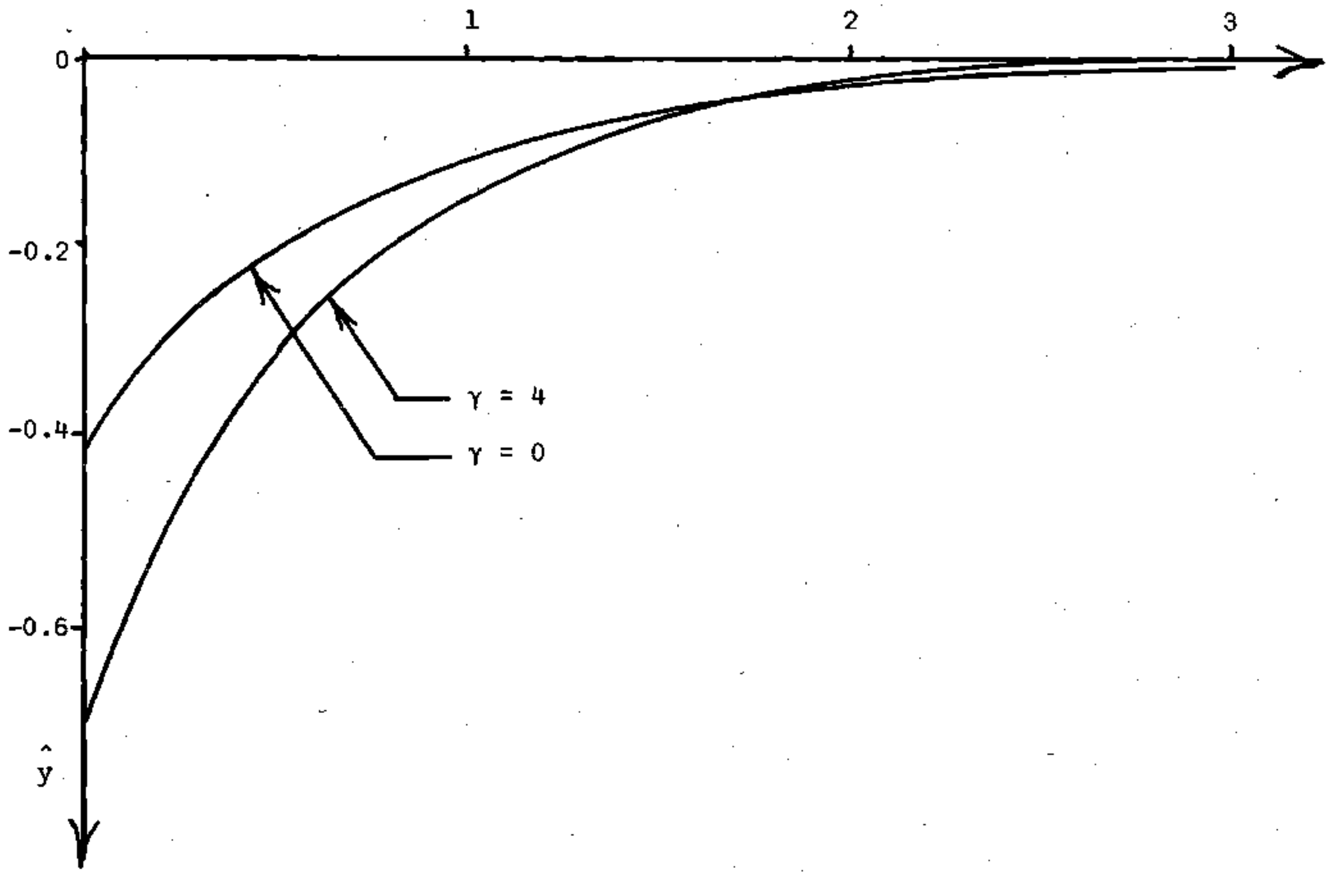

Figure 24. Optimum Input, Example 5.

Example Six

This is an example of the synthesis method applied to a second-order non-linear plant. It is desired to find the input which will drive the output from some initial value to a preselected final value with minimum energy at the input. The sensitivity measure $|u(T)|$ is to be constrained equal to zero at the final time T. In other words, we desire consistency of the output at t = T. For comparison purposes, the problem is re-solved without the sensitivity constraint. The parameter of interest will be the initial condition on the output x.

The plant equation is a form of the so-called "Duffing equation":

$$\ddot{x} + b\dot{x} + cx - dx^3 = y \tag{6.23}$$

with

$$x(0) = x_o, \quad \dot{x}(0) = \dot{x}_o$$

The parameter of interest is x_o with the nominal value x_{oo}.

Differentiating equation (6.23) with respect to x_o yields the sensitivity equation:

$$\ddot{u} + b\dot{u} + cu - 3dx^2u = 0$$

with $u(0) = 1$ and $\dot{u}(0) = 0$.

Let the desired final value of $\underline{x}(t)$ be

$$x(T) = x_T$$

$$\dot{x}(T) = \dot{x}_T$$

with the constraint $U = |u(T,x_{oo})| = 0$.

Using the performance index

$$J[y] = \int_0^T y^2 dt$$

and applying Pontryagin's method with constraints (Appendix I), we obtain the following two-point boundary problem to be solved:

Letting $\quad x_1 = x, \quad x_2 = \dot{x}, \quad x_3 = u, \quad x_4 = \dot{u},$

$$x_1 = x_2$$

$$\dot{x}_2 = - bx_2 - cx_1 + dx_1^{\,3} + y$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -bx_4 - cx_3 + 3dx_1^{\,2}\ x\ 3$$

$$\dot{p}_1 = cp_2 - 3dx_1^{\,2}p_2 - 6dx_1x_3p_4$$

$$\dot{p}_2 = - p_1 + bp_2$$

$$\dot{p}_3 = cp_4 - 3dx_1^{\,2}p_4$$

$$\dot{p}_4 = - p_3 + bp_4$$

$$y = \frac{p_2}{2}$$

with:

$$x_1(0) = x_{oo} \qquad x_2(0) = \dot{x}_o$$

$$x_1(T) = x_T \qquad x_2(T) = \dot{x}_T$$

$$x_3(0) = 1 \qquad x_3(T) = 0$$

$$x_4(0) = 0 \qquad p_4(T) = 0$$

Using the following numerical values:

$b = 0.5$	$c = 1.0$	$d = 1.0$
$x_{oo} = .2$	$\dot{x}_o = 0$	$T = 3.16$ sec.
$x_T = .5$	$\dot{x}_T = 1$	

the solutions shown in Figures 25 and 27 are obtained.

In Figure 25, with the optimum input $\hat{y}$ applied, the output x is plotted for several values of the initial condition x_o. The values of x(T) exhibit little or no variation despite the changes in initial conditions.

In Figure 27, the optimum input is shown.

For comparison, the problem is now re-solved with the sensitivity constraint removed. Applying Pontryagin's method yields the following two-point boundary value problem:

With $x_1 = x$, $x_2 = \dot{x}$.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = - bx_2 - cx_1 + dx_1^{\,3} + y$$

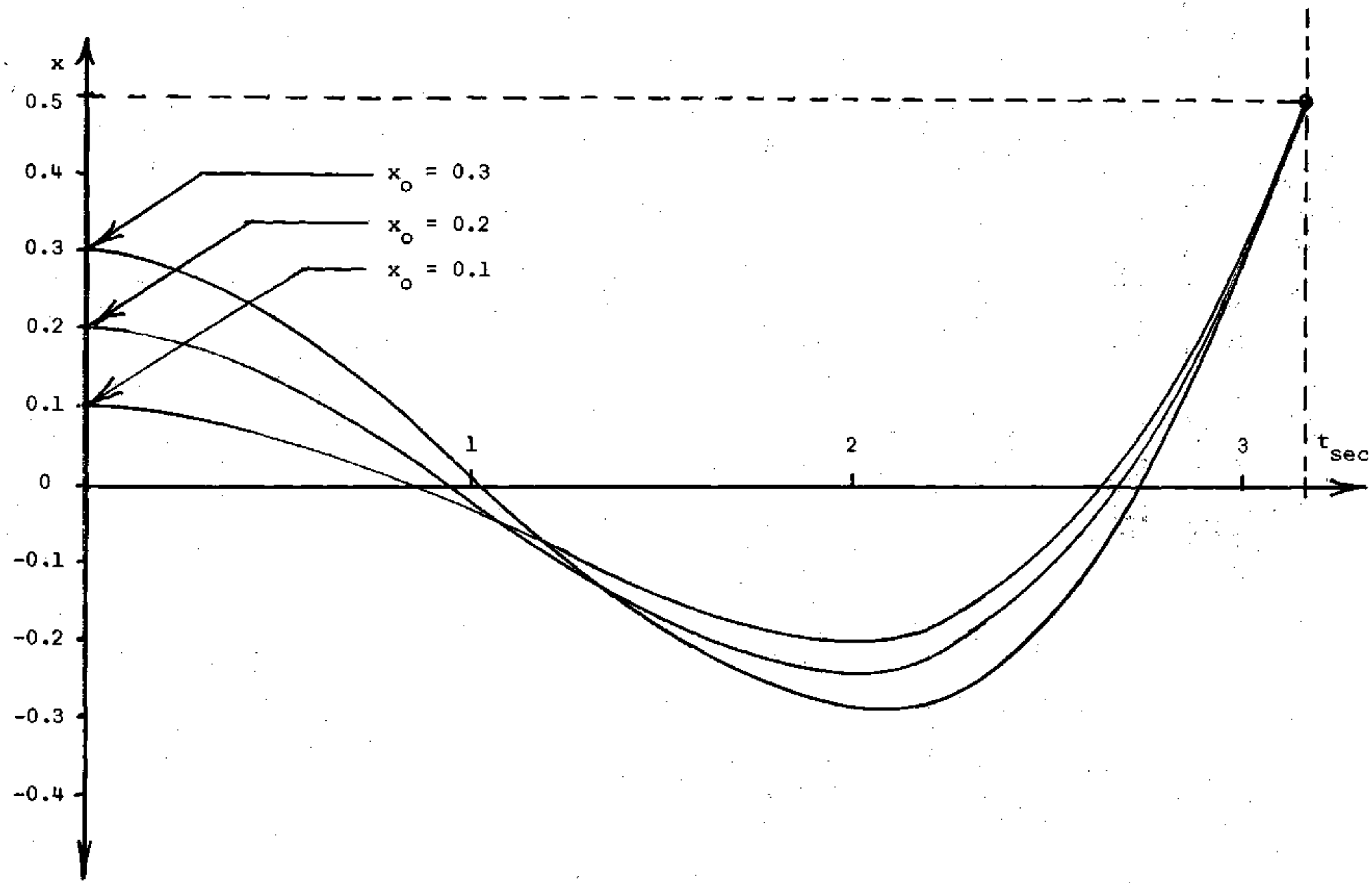

Figure 25. Plant Output for Various Initial Conditions with U = 0, Example 6.

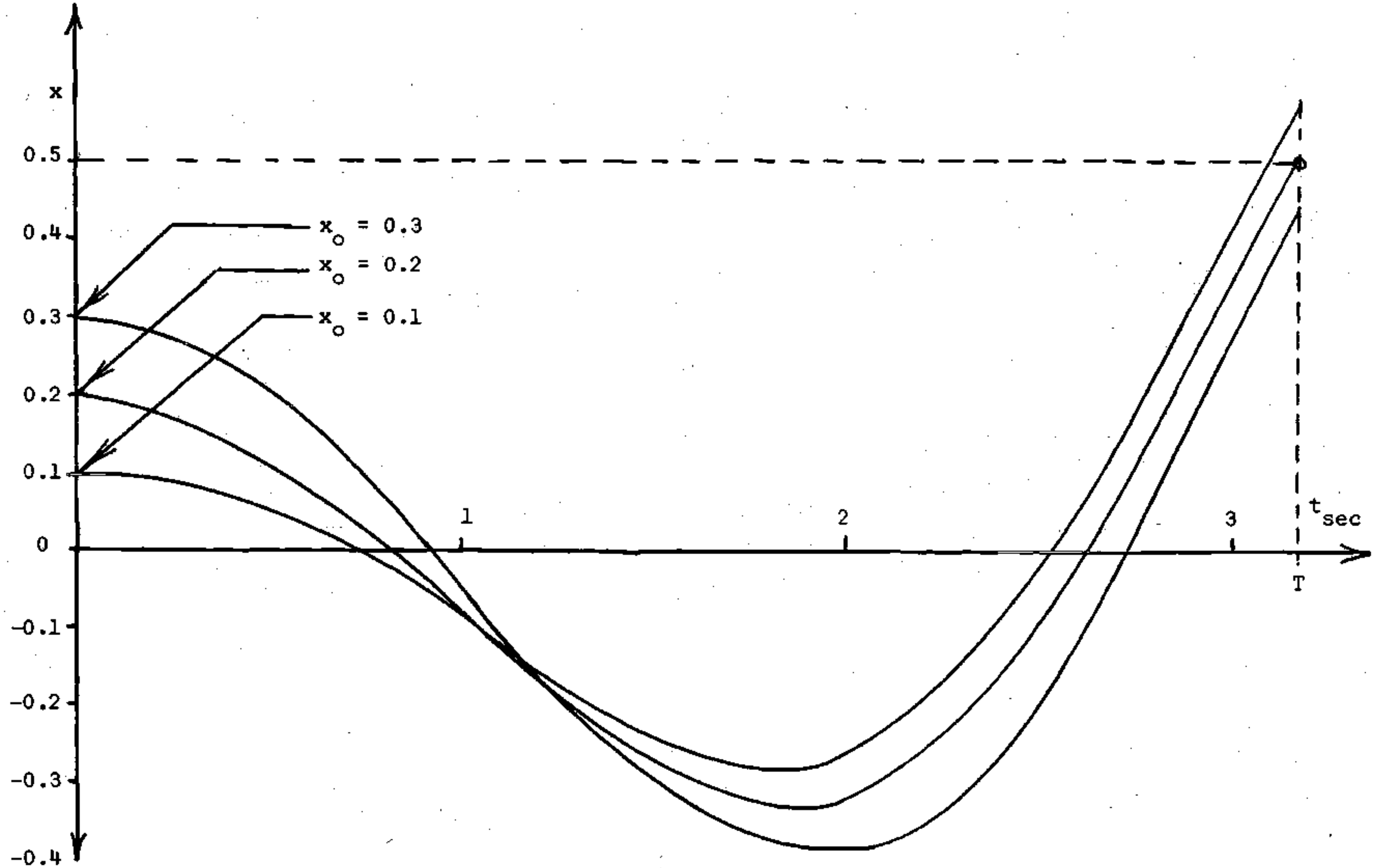

Figure 26. Plant Output for Various Initial Conditions, U Unconstrained, Example 6.

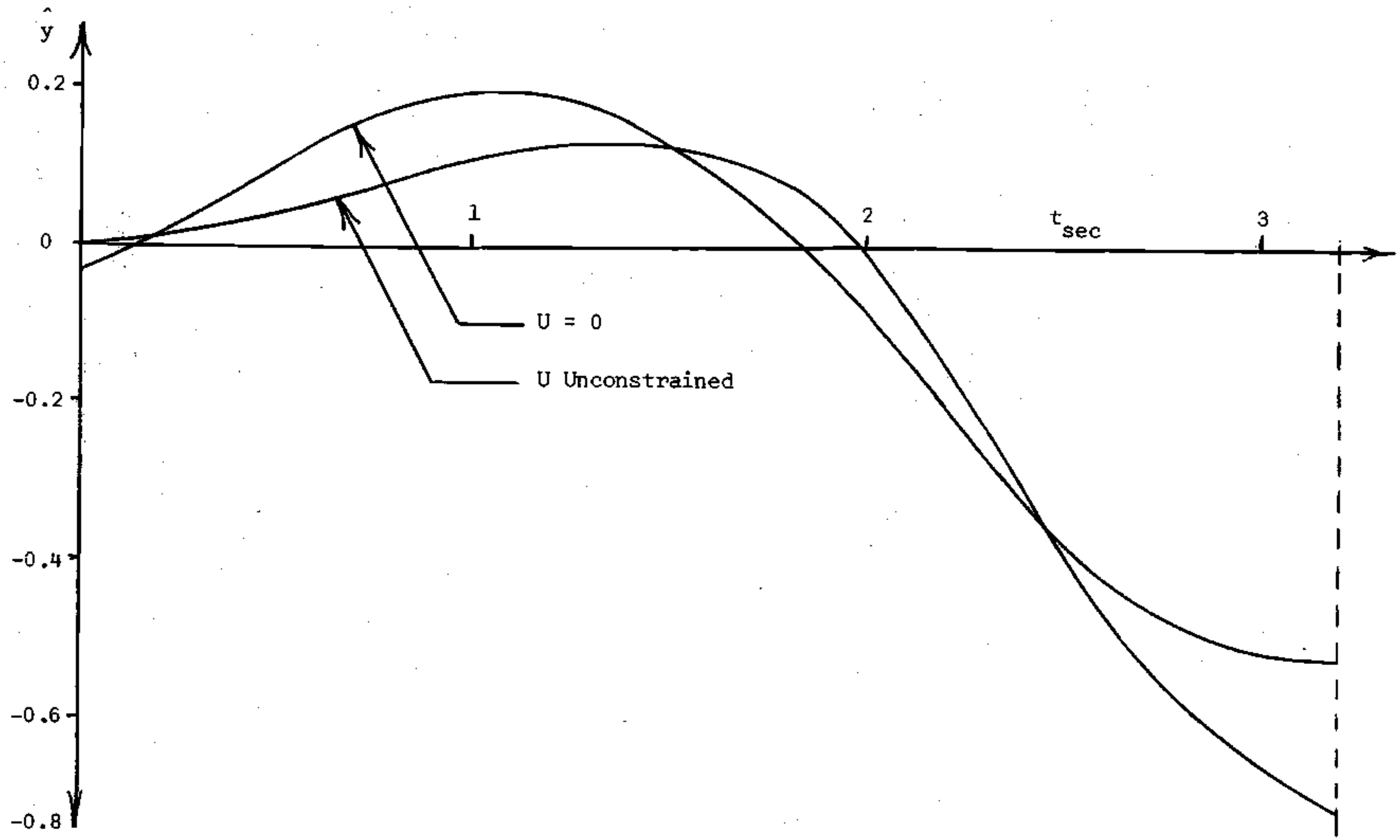

Figure 27. Optimum Input, Example 6.

$$\dot{p}_1 = cp_2 - 3dx_1^2 p_2 - 6dx_1 x_2 p_4$$

$$\dot{p}_2 = - p_1 + bp_2$$

$$y = \frac{p_2}{2}$$

where $x_1(0) = x_{oo}$ $\qquad x_2(0) = \dot{x}_o$

$x_1(T) = x_T$ $\qquad x_2(T) = \dot{x}_T$

Using the same numerical values as in the constrained example above, the solutions shown in Figures 26 and 27 are obtained.

In Figure 26, with the new optimum $\hat{y}$ applied, the output x is again plotted for several values of the initial condition x_o. A pronounced variation in x(T) is noticed when compared with the outputs x(T) in the constrained case. Thus, the system where U is constrained to equal zero, is clearly better if it were important that x(T) be insensitive to changes in the initial conditions.

In Figure 27, the optimum $\hat{y}$ for the unconstrained case is plotted.

CHAPTER VII

CONCLUSIONS

In the preceding chapters, a synthesis method for determining the optimum control vector for a class of sensitivity problems has been presented. The class of problems to which it is applicable is the one in which:

1. The plant parameters remain constant throughout a given plant run.
2. Plant parameter variation is small enough to permit the approximation:

$$\frac{\partial \underline{x}}{\partial q_i} \simeq \frac{\Delta \underline{x}}{\Delta q_i}$$

3. The performance index for the system is continuous in all parameters at the nominal value of the parameters.

For this class of problems, a control vector $\hat{\underline{y}}$ can be determined which is optimal in the sense defined in Chapter IV.

The method requires the selection of a sensitivity measure based on the sensitivity coefficients, $\underline{u}_i = \frac{\partial \underline{x}}{\partial q_i}$. This sensitivity measure is weighted and included in some performance functional or constrained in some manner appropriate to a particular problem.

The plant system of equations is augmented with the corresponding set of sensitivity equations, introducing the variables, $\underline{u}_i$ into the set of constraining equations. Conventional optimization techniques are then applied to the resulting problem.

The synthesis method has been applied above in Chapter V to the general linear plant case with quadratic performance index and to several examples in Chapter VI. These examples illustrate both the application of the method to particular problems and typical results.

The significance and limitations of the procedure have been examined in Chapter IV above.

APPENDICES

APPENDIX I

PONTRYAGIN'S METHOD (8,10)

In this appendix, an optimization scheme based on the maximum principle of Pontryagin is discussed.

Pontryagin's method is similar to classical variational optimization techniques, but has the advantage that discontinuous control functions can be handled more easily.

The General Optimization Problem

The general optimization problem can be stated as follows:

The plant relations are defined by

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},t), \tag{A-1.1}$$

where $\underline{x}$ is an n vector with $\underline{x}(t_o) = \underline{x}_o$.

The performance functional to be minimized is

$$J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)\, dt. \tag{A-1.2}$$

Now define $x_{n+1}(t)$ as follows:

$$x_{n+1}(t) \triangleq \int_{t_o}^{t} F(\underline{x},\underline{y},t)dt \qquad t_o \leq t \leq T$$

Then $$\dot{x}_{n+1}(t) = F(\underline{x},\underline{y},t) \tag{A-1.3}$$

and $$J[\underline{y}] = x_{n+1}(T)$$

Define:

$$\underline{\tilde{x}} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \\ x_{n+1} \end{bmatrix}, \quad \text{an n+1 vector}$$

$$\text{and} \quad \underline{\tilde{f}} \triangleq \begin{bmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \\ F \end{bmatrix}$$

Then from (A-1.1) and (A-1.3);

$$\dot{\underline{\tilde{x}}} = \underline{\tilde{f}}(\underline{\tilde{x}},\underline{y},t) \tag{A-1.4}$$

$$\text{with} \quad \tilde{\underline{x}}(t_o) = \begin{bmatrix} x_{10} \\ x_{20} \\ \cdots \\ x_{no} \\ 0 \end{bmatrix}$$

The problem becomes now to find $\underline{y} = \hat{\underline{y}}(t)$ such that $x_{n+1}(T)$ is minimized subject to equations (A-1.4) which act as natural constraints. The optimum control function $\hat{\underline{y}}$ is assumed to be selected from a given class Y of acceptable comparison functions.

Some generality can be added by letting the functional to be extremized be:

$$S = \underline{a} \cdot \tilde{\underline{x}}(T) = \sum_{i=1}^{n+1} a_i x_i(T)$$

which includes the above problem as the special case where $a_i = 0$, $i = 1,2\ldots n$ and $a_{n+1} = 1$.

The classical calculus of variations can be applied to the above problem to place in evidence the salient features of the maximum principle for the restricted case where Y is the class of functions with continuous first derivatives.

The extension to the discontinuous case will be stated without proof and illustrated through examples. A complete proof can be found in (10).

Rewrite A-1.5 as:

$$S = \int_{t_o}^{T} \sum_{i=1}^{n+1} a_i \dot{x}_i(t)dt + \sum_{i=1}^{n+1} a_i x_i(t_o)$$

Using the calculus of variations, we form the Lagrangian:

$$L = \sum_{i=1}^{n+1} a_i \dot{x}_i + \lambda \cdot (\underline{\tilde{\dot{x}}} - \underline{\tilde{f}})$$

$$= \sum_{i=1}^{n+1} (a_i x_i + \lambda_i (\dot{x}_i - f_i) \qquad \text{(A-1.6)}$$

The Euler-Lagrange equations for this system are:

$$L_{x_j} - \frac{d}{dt}(L_{\dot{x}_j}) = 0, \quad \text{where } L_{x_j} = \frac{\partial L}{\partial x_j}, \quad \text{and } L_{\dot{x}_j} = \frac{\partial L}{\partial \dot{x}_j}$$

and $L_{y_j} - \frac{d}{dt}(L_{\dot{y}_j}) = 0, \quad \text{where } L_{y_j} = \frac{\partial L}{\partial y_j}, \quad \text{and } L_{\dot{y}_j} = \frac{\partial L}{\partial \dot{y}_j}$

Performing the indicated operations on equation (A-1.6) the Euler-Lagrange equations become:

$$- \sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial x_j} - \dot{\lambda}_j = 0 \qquad \text{(A-1.7)}$$

and $$-\sum_{i=1}^{n} \lambda_i \frac{\partial f_i}{\partial y_j} = 0 \qquad \text{(A-1.8)}$$

$$j = 1,2 \ldots n+1$$

If no additional end constraints exist at $t = T$, T being regarded as fixed, the natural boundary conditions, determined from transversality relations, become:

$$a_i = -\lambda_i(T) \qquad \text{(A-1.9)}$$

Now let $p_j(t) = \lambda_j(t)$ and define the Hamiltonian H as

$$H(\underline{\tilde{x}},\underline{p},t) = \underline{p}\cdot\underline{\tilde{f}} = \sum_{i=1}^{n+1} p_i f_i(x,\underline{y},t) \qquad \text{(A-1.10)}$$

then the augmented plant equation (A-1.4) can be written as

$$x_i = \frac{\partial H}{\partial p_i} \qquad i = 1,2\ldots,n+1$$

and equation (A-1.7) as:

$$\dot{p}_i = \frac{\partial H}{\partial x_i} \qquad i = 1,2\ldots,n+1$$

Since $\frac{\partial H}{\partial y_i} = \sum_{j=1}^{n+1} p_j \frac{\partial f_j}{\partial y_i}$, equation (A-1.8) yields:

$$\frac{\partial H}{\partial y_i} = 0$$

Summarizing the above discussion, we can state the following theorem:

If the control vector $\underline{y} = \hat{\underline{y}}(t)$, with $\underline{y}$ belonging to the class of functions with continuous first derivatives, is optimal for $S = \sum_{i=1}^{n+1} a_i x_i(T)$ then there exists a vector $\underline{p}(t)$ such that

1. $\dot{p}_i = - \sum_{j=1}^{n+1} p_j \frac{\partial f_j}{\partial x_i}$ for $\underline{y} = \hat{\underline{y}}$. $i = 1,2\ldots,n+1$.

2. $H = \sum_{i=1}^{n+1} p_i f_i(\underline{x},\underline{y},t)$ has a stationary value for $\underline{y} = \hat{\underline{y}}$; i.e., $\frac{\partial H}{\partial y_i} = \sum_{j=1}^{n+1} p_j \frac{\partial f_j}{\partial y_i} = 0$.

3. $p_i(T) = - a_i$.

Pontryagin's Maximum Principle

The above theorem is essentially a statement of Pontryagin's maximum principle in a restricted form. The requirement that $\underline{y}$ have continuous first derivatives is unnecessary if statement 2. of the above theorem is modified to read simply:

$$H = \sum_{i=1}^{n+1} p_i f_i(\underline{x},\underline{y},t) \text{ has a maximum over } Y \text{ for } \underline{y}(t) = \hat{\underline{y}}(t).$$

Thus Pontryagin's maximum principle is simply a more general form of the theorem above and can be stated as follows:

If the control $\underline{y}(t) = \hat{\underline{y}}(t)$ is optimal for $S = \sum_{i=1}^{n+1} a_i x_i(T)$, then there exists a vector $\underline{p}(t)$ such that

$$1. \qquad \dot{p}_i = - \sum_{j=1}^{n+1} p_j \frac{\partial f_j}{\partial x_i} \text{ for } \underline{y} = \hat{\underline{y}} \tag{A-1.11}$$

$$2. \qquad H = \sum_{i=1}^{n+1} p_i f_i(\underline{x},\underline{y},t) \tag{A-1.12}$$

has a maximum over Y for $\underline{y}(t) = \hat{\underline{y}}(t)$, where Y is the class of control functions from which $\hat{\underline{y}}$ is to be selected. Ordinarily Y is the class of piecewise continuous functions.

$$3. \qquad p_i(T) = - a_i \tag{A-1.13}$$

It is to be noted that Pontryagin's maximum principle as stated deals only with necessary conditions for optimality as does the classical calculus of variation approach. The question of sufficiency is far more complex and beyond the scope of this discussion. For a discussion of sufficient conditions, see (10).

Application of Pontryagin's method yields the following two-point boundary problem defined by equation (A-1.4), (A-1.12) and (A-1.13).

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},t), \quad \text{where } \underline{x}(t_o) = \underline{x}_o \tag{A-1.4}$$

$$\dot{p}_i = -\sum_{j=1}^{n+1} p_j \frac{\partial f_j}{\partial x_i}, \quad \text{where } p_i(T) = -a_i$$

$$i = 1,2\ldots,n+1$$

or
$$\underline{\dot{p}} = \underline{h}(\underline{\tilde{x}},\underline{p},\underline{y},t), \quad \text{where } \underline{p}(T) = -\underline{a} \qquad \text{(A-1.14)}$$

An additional relation between $\underline{y}$ and $\underline{x}$, $\underline{p}$ which permits the problem to be solved can be obtained from the maximizing of $H = \underline{p}\cdot\underline{\tilde{f}}$. Substituting the value of $\underline{y}$ thus obtained in (A-1.4) and (A-1.14) yields the typical two-pound boundary value problem discussed in Appendix III:

$$\underline{\dot{z}} = \underline{r}(\underline{z},t) \quad \text{where} \quad \underline{z} = \begin{bmatrix} \underline{\tilde{x}} \\ \underline{p} \end{bmatrix}$$

and
$$z_i(t_o) = x_i(t_o) \qquad i = 1,2\ldots,n+1$$

$$z_i(T) = p_i(T) \qquad i = n+2,\ldots,2n+2$$

Examples

The following examples are presented to show the applications of Pontryagin's method to particular control problems.

Example One

$$\underline{\dot{x}} = -cx + y, \quad \text{where} \quad x(0) = x_o \quad \text{(the plant equation)}$$

$$J[y] = \int_0^T (x^2+y^2)dt$$

Y: the set of inputs, y, with continuous first derivatives

Then: $$\dot{x}_1 = - cx_1 + y$$

$$\dot{x}_2 = x_1^{\ 2} + y^2$$

$$S = a_1x_1 + a_2x_2 = x_2(T); \quad a_1 = 0, \quad a_2 = 1$$

$$H = \underline{p}\cdot\underline{\tilde{f}} = p_1(- cx_1+y) + p_2(x_1^{\ 2} + y^2)$$

$$\frac{\partial H}{\partial y} = 0 = p_1 + 2p_2y \quad \text{or} \quad y = - \frac{p_1}{2p_2}$$

$$\dot{p}_1 = - \frac{\partial H}{\partial x_1} = cp_1 - 2x_1p_2$$

$$\dot{p}_2 = - \frac{\partial H}{\partial x_2} = 0 \quad \therefore \ p_2 \text{ is a constant}$$

but $$p_2(T) = - a_2 = - 1, \quad \text{thus } p_2(t) = - 1$$

Summary: $$\dot{x}_1 = - x_1 + y \qquad\qquad x_1(0) = x_o$$

$$\dot{p}_1 = p_1 + 2x_1 \qquad\qquad p_1(T) = 0$$

$$y = \frac{p_1}{2p_2} = \frac{p_1}{2}$$

Solving these equations yields $y = \hat{y}(t)$, the optimum input.

Example Two

$$\dot{x} = -\,cx + y \qquad\qquad x(0) = x_o \quad \text{(the plant equation)}$$

$$J[y] = \int_0^T x^2 dt \;,$$

Y: the set of inputs, y, which are piecewise continuous and $|y| \leq M$.

$$\dot{x}_1 = -\,cx_1 + y$$

$$\dot{x}_2 = x_1^{\,2}$$

$$S = a_1x_1 + a_2x_2 = x_2(T); \quad a_1 = 0,\ a_2 = 1$$

$$H = p_1(\,-\,ax_1 + y) + p_2x_1^{\,2}$$

$$= ax_1p_1 + p_1y + p_2x_1^{\,2}$$

Maximizing H over Y, it is clear that H is maximum when

$y = M \text{ sign } p_1$.

$$\dot{p}_1 = cp_1 - 2x_1p_2$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 0 \quad \therefore \quad p_2 \text{ is a constant}$$

$$\text{but} \qquad p_2(T) = -a_2 = -1, \qquad \text{thus} \quad p_2(t) = -1$$

Summary:

$$\dot{x}_1 = -cx_1 + y, \quad x_1(0) = x_o$$

$$\dot{p}_1 = cp_1 - 2x_1p_2 = cp_1 + 2x_1, \quad p_1(T) = 0$$

$$y = M \text{ sign } p_1$$

Solving these equations yields $y = \hat{y}(t)$, the optimum input.

<u>Example Three</u>

General Case, Linear Plant with Quadratic Performance Index.

$\dot{\underline{x}} = C\underline{x} + D\underline{y}$, where $\underline{x}$ is an n vector,

and $\underline{y}$ is an m vector.

$\underline{x}(t_o) = \underline{x}_o$, C is an n x n matrix.

D is an m x n matrix.

$$J[\underline{y}] = \int_{t_o}^{T} (\underline{x}\cdot P\underline{x} + \underline{y}\cdot Q\underline{y})dt$$

P, Q are symmetric positive matrices

then
$$\dot{x}_{n+1} = \underline{x}\cdot p\underline{x} + \underline{y}\cdot Q\underline{y}$$

$$x_{n+1}(T) = J[\underline{y}] = S = \sum_{i=1}^{n+1} a_i x_i(T)$$

$$a_i = 0, \quad i = 1,2\ldots n, \ a_{n+1} = 1 \ .$$

$$H = \underline{p}\cdot(C\underline{x} + D\underline{y}) + p_{n+1}(\underline{x}\cdot p\underline{x} + \underline{y}\cdot Q\underline{y}); \quad \underline{p} \text{ is an n vector}$$

Let Y be the class of $\underline{y}$ with continuous first derivatives. Applying Pontryagin's principle, using vector notation:

$$\dot{\underline{p}} = - [C^T p + 2p_{n+1}P\underline{x}]$$

$$\dot{p}_{n+1} = 0 \quad \therefore \quad p_{n+1} \text{ is constant}$$

but
$$p_{n+1}(T) = - a_{n+1} = - 1, \quad \text{thus } p_{n+1}(t) = - 1$$

To maximize H:

$$\frac{\partial H}{\partial \underline{y}} = 0 \qquad \text{or} \qquad D^T\underline{p} + 2Q\underline{y} = 0 \qquad \text{(A-1.15)}$$

Summary:

$$\dot{\underline{x}} = C\underline{x} + D\underline{y} \qquad \underline{x}(t_o) = \underline{x}_o \tag{A-1.16}$$

$$\dot{\underline{p}} = C^T\underline{p} + 2P\underline{x} \qquad \underline{p}(T) = 0$$

$$D^T\underline{p} + 2Q\underline{y} = 0$$

If Q is non-singular, equation (A-1.15) can be written:

$$\underline{y} = -\frac{1}{2} Q^{-1}D^T\underline{p} \tag{A-1.17}$$

Substituting this value for $\underline{y}$ in (A-1.16) yields the following two-point boundary value problem:

$$\dot{\underline{x}} = C\underline{x} - \frac{1}{2} DQ^{-1}D^T\underline{p}; \qquad \underline{x}(t_o) = \underline{x}_o$$

$$\dot{\underline{p}} = - C^T\underline{p} + 2P\underline{x}; \qquad \underline{p}(T) = 0$$

After solution of the problem for $\underline{p}$, $\underline{y} = \hat{\underline{y}}$ is determined from (A-1.17).

End Constraints

Pontryagin's maximum principle as outlined in equations (A-1.11) through (A-1.13) can be modified slightly to permit the inclusion of additional constraints of the form:

$$\underline{g}(\underline{x}(T)) = 0$$

in the optimization problem. (Note: This is the typical form that constraints on the sensitivity measure assume in the sensitivity optimization problem.)

The modification required is that equation (A-1.13) which assigns the final values to $\underline{p}$ (i.e., $p_i(T) = - a_i$) is changed to read:

$$p_i(T) = - a_i - \sum_{j=1}^{m} \lambda_{jg}(T) \frac{\partial g_j}{\partial x_i} \tag{A-1.18}$$

where m is the order of $\underline{g}$ ($m \leq n$) and $\lambda_{jg}(T)$ are arbitrary multipliers. This relation together with the original constraint equations, $g(\underline{x}(T)) = 0$, permit evaluation of the end conditions on $\underline{p}$ and the multipliers $\lambda_{jg}(T)$.

If the elements of $\underline{g}$ are functions of a single element of $\underline{x}(T)$; i.e.,

$$g_k(\underline{x}(T)) = g_k(x_k(T))$$

where $x_k(T)$ is any element of $\underline{x}(T)$, the practical effect of this modification in the resulting boundary value problem is to replace the end condition(s) $p_k(T) = - a_k$ with the end condition $g_k(x_k(T)) = 0$.

Equation (A-1.18) is similar to the result obtained for handling end constraints in the classical calculus of variations.

Final Time T Variable

Thus far we have assumed that the problem final time T was a fixed known constant. Addition of a new relation to equations (A-1.11) through (A-1.13) permits the final time T to be variable. This relation is:

$$\sum_{i=1}^{n+1} p_i(T)f_i(\underline{x}(T),\underline{y}(T),T) = H(T) = 0 \qquad \text{for } y = \hat{y}(t)$$

Again, this result is similar to that obtained with the classical calculus of variations for this situation.

APPENDIX II

BELLMAN'S METHOD

This appendix discusses the dynamic programming approach of Bellman to the optimization problem.

Principle of Optimality

Dynamic programming is a result of the application of the principle of optimality which states in essence that any portion of an optimum trajectory is also an optimum trajectory. Thus, if $\underline{x}(t)$ is an optimum trajectory starting at $\underline{x}(t_o)$ and terminating at $\underline{x}(T)$ and passing through the intermediate points $\underline{x}(t_1) = \underline{x}_1$ and $\underline{x}(t_2) = \underline{x}_2$, then the optimal trajectory from $\underline{x}_1$ to $\underline{x}_2$ is identical to the portion of the original optimum trajectory lying between $\underline{x}_1$ and $\underline{x}_2$.

Application to the Optimization Problem

Consider the control problem:

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},t) \qquad \text{(the plant equation)}$$

with

$$\underline{x}(t_o) = \underline{x}_o$$

and

$$J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)dt$$

The objective is to find $\underline{y} = \hat{\underline{y}}(t)$ such that $J[\underline{y}]$ is a minimum,

$\hat{\underline{y}}(t)$ assumed to exist and belong to some class Y.

Let $\bar{S}(\underline{x}_o,t_o) = J[\hat{\underline{y}}]$; i.e., the value of $J[\underline{y}]$ evaluated along the optimum trajectory from $\underline{x}(t_o)$ to $\underline{x}(T)$, where the bar over S indicates evaluation along the optimum trajectory.

$$\bar{S}(\underline{x}_o,t_o) = \min_{\underline{y}\varepsilon Y} [\int_{t_o}^{T} F(\underline{x},\underline{y})dt]$$

According to the optimality principle, this is the same as

$$\bar{S}(\underline{x},t) = \min_{\underline{y}\varepsilon Y} [\int_{t}^{T} F(\underline{x},\underline{y},t)dt] \qquad \text{(A-2.1)}$$

where $t_o \leq t \leq T$ and $\underline{x} = \underline{x}(t)$.

Equation (A-2.1) can be written

$$\bar{S}(\underline{x},t) = \min_{\underline{y}\varepsilon Y} [\int_{t}^{t+\Delta t} F(\underline{x},\underline{y},t)dt + \int_{t+\Delta t}^{T} F(\underline{x},\underline{y},t)dt] \qquad \text{(A-2.2)}$$

Again, according to the optimality principle this equation (A-2.2) can be written:

$$\bar{S}(\underline{x},t) = \min_{\underline{y}\varepsilon Y} [\int_{t}^{t+\Delta t} F(\underline{x},\underline{y},t)dt + \bar{S}(\underline{x} + \Delta\underline{x},\ t + \Delta t)]$$

where $\underline{x} + \Delta\underline{x} \triangleq \underline{x}(t + \Delta t)$.

Assuming F is continuous in t and $\bar{S}$ has partial derivatives with respect to each element of $\underline{x}$ and t and applying the mean value theorem for integrals:

$$\bar{S}(\underline{x},t) = \min_{\underline{y}\varepsilon Y}[F(\underline{x},\underline{y},t)\Big|_{t=t_1}\Delta t + \bar{S}(\underline{x} + \Delta,\ t + \Delta t)] \tag{A-2.3}$$

for some time t_1 where $t \leq t_1 \leq t + \Delta t$.

Expanding $\bar{S}(\underline{x} + \Delta x,\ t + \Delta t)$ in a Taylor's series about $(\underline{x},t)$ and neglecting second and higher order terms:

$$\bar{S}(\underline{x} + \Delta x,\ t + \Delta t) = \bar{S}(\underline{x},t) + \frac{\partial \bar{S}(\underline{x},t)}{\partial t}\Delta t + \sum_{i=1}^{n} \frac{\partial \bar{S}}{\partial x_i}\Delta x_i \tag{A-2.4}$$

Noting that $\bar{S}(\underline{x},t)$ and $\frac{\partial \bar{S}}{\partial t}$ are independent of $\underline{y}$ and may thus be taken "outside" of the minimization operation, substitution of equation (A-2.4) into (A-2.3) yields:

$$-\frac{\partial \bar{S}(\underline{x},t)}{\partial t}\Delta t = \min_{\underline{y}\varepsilon Y}[F(\underline{x},\underline{y},t)\Big|_{t=t_1}\Delta t + \sum^{n} \frac{\partial \bar{S}}{\partial x_i}\Delta x_i]$$

Dividing by Δt and taking limit as $\Delta t \to 0$ yields:

$$-\frac{\partial \bar{S}(\underline{x},t)}{\partial t} = \min_{\underline{y}\varepsilon Y}[F(\underline{x},\underline{y},t) + \sum_{i=1}^{n} \frac{\partial \bar{S}}{\partial x_i}\dot{x}_i] \tag{A-2.5}$$

$$\text{Letting} \quad \nabla_{\underline{x}}\bar{S} \triangleq \begin{bmatrix} \frac{\partial \bar{S}}{\partial x_1} \\ \frac{\partial \bar{S}}{\partial x_2} \\ \cdots \\ \frac{\partial \bar{S}}{\partial x_n} \end{bmatrix}$$

and noting that $\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},t)$ from the basic plant equation, equation (A-2.5) becomes

$$-\frac{\partial \bar{S}(\underline{x},t)}{\partial t} = \min_{\underline{y}\varepsilon Y} [F(x,\underline{y},t) + \nabla_{\underline{x}}\bar{S}\cdot\underline{f}(\underline{x},\underline{y},t)] \qquad \text{(A-2.6)}$$

Performing the minimization operation indicated yields a partial differential equation in $\bar{S}$:

$$-\frac{\partial \bar{S}(\underline{x},t)}{\partial t} = F(\underline{x},\hat{\underline{y}},t) + \nabla_{\underline{x}}\bar{S}\cdot\underline{f}(\underline{x},\hat{\underline{y}},t) \qquad \text{(A-2.7)}$$

where $\hat{\underline{y}}$ is the function that minimizes $F(\underline{x},\underline{y},t) + \nabla_{\underline{x}}\bar{S}\cdot\underline{f}(\underline{x},\underline{y},t)$ with boundary conditions established by the definition of $\bar{S}$ as $\bar{S}(\underline{x},t)\big|_{t=T} = 0$.

The minimization procedure also establishes an additional relation between $\hat{\underline{y}}$ and F, $\underline{f}$, and $\nabla_{\underline{x}}\bar{S}$ which permits solution of (A-2.7).

For example, if Y is the class of functions such that y_i has continuous first derivatives, the minimization procedure yields:

$$\frac{\partial F(\underline{x},\underline{y},t)}{\partial y_i} + \sum_{j=1}^{n} \frac{\partial \bar{S}}{\partial x_j} \cdot \frac{\partial f_j(\underline{x},\underline{y},t)}{\partial y_i} = 0$$

It is sometimes convenient to invert the time scale by the substitution $\hat{t} = T - t$, putting the boundary conditions at $\hat{t} = 0$.

Summary

Summarizing the above discussion: If $\underline{y} = \hat{\underline{y}}(t) \varepsilon Y$ is optimal for $J[\underline{y}] = \int_{t_o}^{T} F(\underline{x},\underline{y},t)dt$ subject to the plant equations:

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{y},t), \quad \underline{x}(t_o) = \underline{x}_o$$

then:

1. The functional $F(\underline{x},\underline{y},t) + \nabla_{\underline{x}}\bar{S}\cdot\underline{f}$ is a minimum for $\underline{y} = \hat{\underline{y}}(t)$.

2. $$\frac{\partial \bar{S}}{\partial \hat{t}} = F(\underline{x},\hat{\underline{y}},T-\hat{t}) + \nabla_{\underline{x}}\bar{S}\cdot\underline{f}(\underline{x},\hat{\underline{y}},T-\hat{t}) \qquad \text{(A-2.8)}$$

   with $\hat{t} = T - t$, and $\bar{S}(\underline{x},\hat{t}=0) = 0$.

An Example

$$\dot{x} = - cx + y, \quad x(0) = 1$$

$$J[y] = \int_0^T (x^2 + y^2)dt$$

Y: the set of inputs, y, with continuous first derivatives. Then

$$\underline{x} = x_1 = x, \qquad \nabla_{\underline{x}}\bar{S} = \frac{\partial \bar{S}}{\partial x_1}, \qquad F = x_1^2 + y^2, \qquad \text{and } \underline{f} = f = - cx_1 + y$$

Equation (A-2.8) becomes:

$$\frac{\partial \bar{S}}{\partial \hat{t}} = \min_{y \varepsilon Y} [x_1{}^2 + y^2 + \frac{\partial \bar{S}}{\partial x_1} (- cx + y)] \tag{A-2.9}$$

The minimization operation yields

$$2\hat{y} + \frac{\partial \bar{S}}{\partial x_1} = 0$$

or

$$\hat{y} = - \frac{1}{2} \frac{\partial \bar{S}}{\partial x_1} \tag{A-2.10}$$

Substituting this value in (A-2.9) yields

$$\frac{\partial \bar{S}}{\partial \hat{t}} = x_1{}^2 + \frac{1}{4} \left[\frac{\partial S}{\partial x_1}\right]^2 - cx_1 \frac{\partial \bar{S}}{\partial x_1} - \frac{1}{2} \left[\frac{\partial \bar{S}}{\partial x_1}\right]^2$$

$$= x_1{}^2 - cx_1 \left[\frac{\partial \bar{S}}{\partial x_1}\right] - \frac{1}{4} \left[\frac{\partial \bar{S}}{\partial x_1}\right]^2 \tag{A-2.11}$$

with boundary conditions $\bar{S}(x_1, \hat{t} = 0) = 0$.

Assume solution of the form:

$$\bar{S}(\underline{x},\hat{t}) = S(\hat{t})\ \underline{x}\cdot\underline{x}$$

$$= S(\hat{t})\ x_1{}^2$$

$$\frac{\partial \bar{S}}{\partial x_1} = 2S(\hat{t})x_1 \tag{A-2.12}$$

Equation (A-2.11) becomes

$$\dot{S}(\hat{t})x_1{}^2 = x_1{}^2 - 2CS(\hat{t})\ x_1{}^2 - S^2(\hat{t})x_1{}^2$$

or
$$\dot{S}(\hat{t}) = 1 - 2CS(\hat{t}) - S^2(\hat{t}) \qquad \text{(A-2.13)}$$

with $S(0) = 0$.

This ordinary differential equation is solved for $S(t)$, and $\dot{y}$ determined from (A-2.10) and (A-2.12):

$$\hat{y}(\hat{t}) = -\frac{1}{2}\ \frac{\partial \bar{S}}{\partial x} = -\ S(\hat{t})x_1$$

or
$$\hat{y}(t) = -\ S(T-t)x_1 \qquad \text{(A-2.14)}$$

If $T \to \infty$ and system is to be stable, $\dot{S}(\hat{t}) = 0$ and equation (A-2.13) becomes an algebraic equation in S; i.e., S is some constant.

If $\hat{y}$ is desired as a function of time alone, eliminate x between equation (A-2.14) and the original plant equation.

APPENDIX III

THE TWO-POINT BOUNDARY VALUE PROBLEM

This appendix describes the manner in which the two-point boundary value (TPBV) problem resulting from conventional optimization procedures can be solved.

The TPBV problem arises frequently in optimization problems. In particular, it arises inherently from the application of Pontryagin's method to the original optimization problem posed in Chapter I. Its typical form is as follows:

$$\text{Given } \dot{\underline{z}} = \underline{f}(\underline{z},t), \quad \text{where } \underline{z} \text{ is an n vector} \tag{A-3.1}$$
$$\text{with } z_i(T) = z_{iT}, \quad i = 1,2\ldots m$$
$$\text{and } z_i(0) = z_{10}, \quad i = m+1,\ldots n$$

Find $z_i(t)$ for all t, $0 < t < T$, $i = 1,2\ldots n$.

The solution $z_i(t)$ could be immediately obtained if the values $z_i(0)$, $i = 1,2\ldots m$ were available. If $z_i(0)$, $i = 1,2\ldots m$ are known, the problem becomes an ordinary initial value problem. Thus, we will consider the problem solved once $z_i(0)$, $i = 1,2\ldots m$ are determined.

For low order systems, particularly if equation (A-3.1) is linear, analytic solutions can sometimes be obtained. Example One in Chapter VI is an example of a fourth order linear TPBV problem which can be solved analytically.

However, for higher order or non-linear systems, a computer solution is often required.

If m is small, say $m \leq 4$, solutions on the analog computer can often be obtained. The general approach is to make successive guesses at the values of $z_i(0)$, $i = 1,2...m$ until the boundary conditions $z_i(T) = z_{iT}$, $i = 1,2...m$ are met. By noting the manner in which $z_i(T)$ varies when changes in $z_i(0)$ are made, an algorithm can often be devised for successive selections of $z_i(0)$ that cause $z_i(T)$ to converge to z_{iT}. The particular algorithm is, of course, only applicable to a particular problem.

A more formal analog computer technique is based on the construction of a composite error function of the final values $z_i(T)$.

Let z_{iT} be the desired final values and $z_i(T)$ the actual final values for a particular set of guesses of $z_i(0)$, $i = 1,2...m$. Then form the composite error function E, easily mechanized on the analog computer,

$$E = \sum_{i=1}^{m} \lambda_i (z_i(T) - z_{iT})^2$$

or

$$E = \sum_{i=1}^{m} \lambda_i |z_i(T) = z_{iT}|$$

where λ_i is a set of error weighting factors.

After an original set of guesses of $z_i(0)$ $i = 1,2...m$, E is measured. Then a single initial condition on z_i is adjusted for successive computer runs until E is minimized. The process is repeated for each initial condition. If the error function is not then zero, or below some prescribed accuracy criterion, a new series of adjustments is made. The procedure is continued until the error function is within satisfactory limits, at which time the problem is said to be solved. Figure 28 shows a computer circuit for measuring E.

Examples Two through Six in Chapter VI were solved on the analog computer.

Another general approach to the problem is the application of numerical analysis techniques, with a view toward solving the problem on a digital computer. One such technique is outlined below (20,21).

Given the TPBV problem:

$$\underline{\dot{z}} = \underline{f}(\underline{z},t), \quad \text{where } \underline{z} \text{ is an n vector,} \tag{A-3.1}$$

$$\text{with } \underline{z}_i(T) = z_{iT}, \quad i = 1,2\ldots m$$

$$\text{and } \underline{z}_i(0) = z_{io}, \quad i = m+1,\ldots n$$

the problem is to select the components of $\underline{z}(0)$ which are not specified; i.e., $z_i(0)$, $i = 1,2\ldots m$, so that the conditions at $t = T$ are satisfied.

Warner's technique (21) is as follows:

Let V_i, $i = 1,2\ldots m$ be the unknown starting values for $\underline{z}(t)$ which are functions of the desired final values $z_i(T) = z_iT$, $i = 1,2\ldots m$. Thus:

$$V_1 = V_1(z_{iT},z_{2T},\ldots z_{mT}) \tag{A-3.2}$$

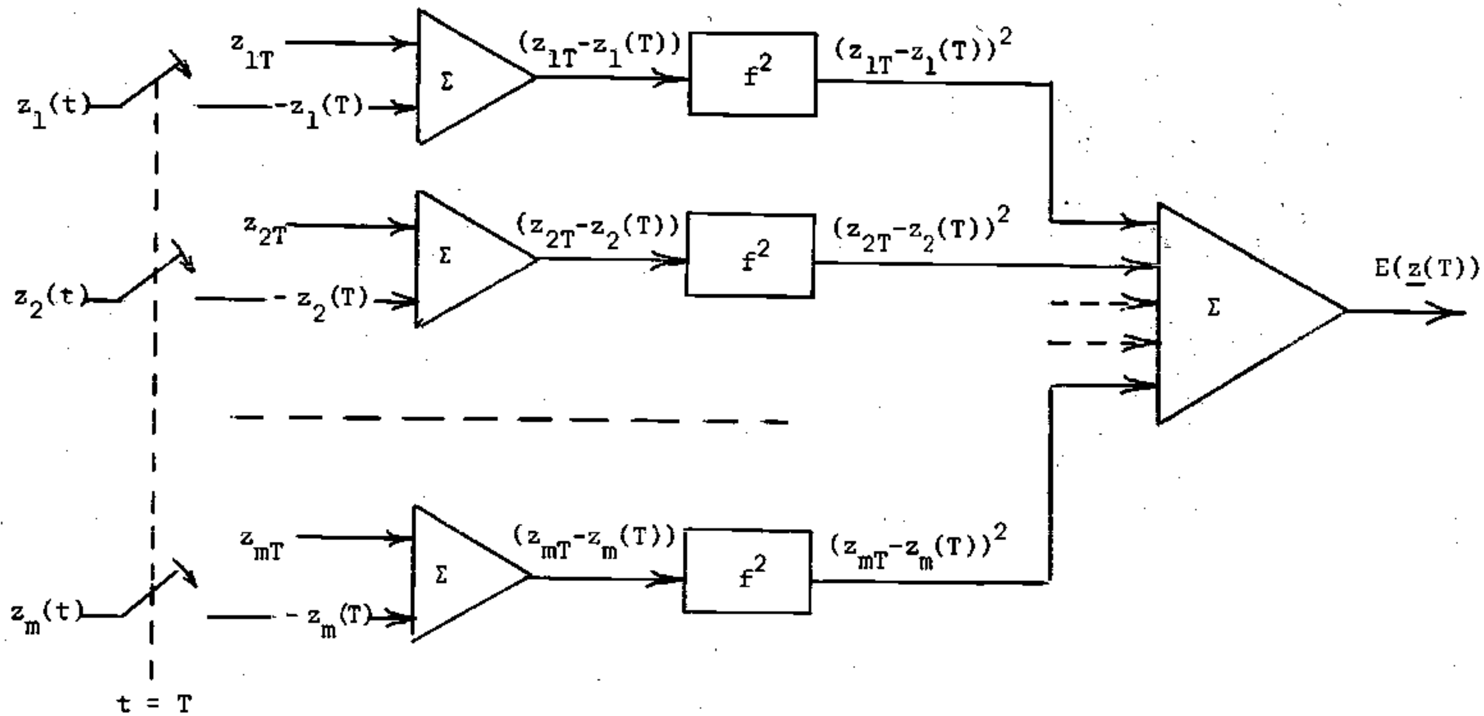

Figure 28. Analog Computer Circuit for Evaluating $E(\underline{z}(T))$.

Now, assume an initial guess for V_i. Call it v_{i1} and corresponding to this initial guess, the final value of z_i is called z_{i1}. Define $\delta z_{i1} \triangleq z_{i1} - z_{iT}$. Then expand equation (A-3.2) in a Taylor's series about the guess v_{i1}:

$$V_i = v_{i1} - \sum_{j=1}^{m} \frac{\partial V_i}{\partial z_{jT}} \delta z_{j1} + \text{(higher order terms)} \qquad \text{(A-3.3)}$$

Rewriting (A-3.3) and neglecting terms of second and higher order:

$$v_{i1} = V_i + \sum_{j=1}^{m} \frac{\partial V_i}{\partial z_{jT}} \delta z_{j1} \qquad \text{(A-3.3)}$$

We now make an additional set of m guesses for V_i. Each guess is of the form of (A-3.3):

$$v_{i2} = V_i + \sum_{j=1}^{m} \frac{\partial V_i}{\partial z_{jT}} \delta z_{j2}$$

$$v_{i3} = V_i + \sum_{j=1}^{m} \frac{\partial V_i}{\partial z_{jT}} \delta z_{j3}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$v_{i(m+1)} = V_i + \sum_{j=1}^{m} \frac{\partial V_i}{\partial z_{jT}} \delta z_{j(m+1)} \qquad \text{(A-3.4)}$$

The above sets of equations (A-3.3 and A-3.4) may be conveniently placed in matrix form as shown below:

$$\begin{bmatrix} 1 & \delta z_{11} & \delta z_{21} & \cdots & \delta z_{m1} \\ 1 & \delta z_{12} & \delta z_{22} & \cdots & \delta z_{m2} \\ \cdot & \cdots & \cdots & \cdots & \cdots \\ 1 & \delta z_{1(m+1)} & \delta z_{2(m+1)} & \cdots & \delta z_{m(m+1)} \end{bmatrix} \begin{bmatrix} V_1 & V_2 & \cdots & V_m \\ \frac{\partial V_1}{\partial z_{1T}} & \frac{\partial V_2}{\partial z_{1T}} & \cdots & \frac{\partial V_m}{\partial z_{1T}} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial V_1}{\partial z_{mT}} & \cdots & \cdots & \frac{\partial V_m}{\partial z_{mT}} \end{bmatrix} =$$

$$\begin{bmatrix} v_{11} & v_{21} & \cdots & v_{m1} \\ v_{12} & v_{22} & \cdots & v_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ v_{1(m+1)} & \cdots & \cdots & v_{m(m+1)} \end{bmatrix}$$

The first matrix on the left is evaluated by m+1 integrations of equation (A-3.1) using the guesses in the right hand matrix as initial conditions.

Since V_i are the best starting values (at least to first order) of z_i, $i = 1,2\ldots m$, we solve the matrix equation (A-3.5) for V_i, $i = 1,2\ldots m$. This requires an inversion of the first matrix on the left and a partial matrix multiplication. Note that the partials $\frac{\partial V_i}{\partial z_{jT}}$ do not have to be evaluated. The V_i thus determined are the "best" set of guesses available.

The general procedure now is to replace the "worst" set of guesses by the "best" set V_i and repeat the process until some predetermined accuracy level has been reached.

The "worst" set of guesses will be defined as the set with the greatest error where the error may be calculated in one of several ways; e.g., letting E_k be the error associated with the set of guesses v_{ik}:

$$E_k = \sum_{i=1}^{m} (\delta z_{ik})^2$$

or

$$E_k = \sum_{i=1}^{m} |\delta z_{ik}|$$

After determining which error is the largest, say E_k, we replace the row v_{ik} in the right hand matrix with the values V_i, $i = 1,2...m$, and determine a new set of δz_{ik}, $i = 1,2...m$, by a single integration of equation (A-3.5).

Another matrix inversion and partial matrix multiplication determines a better estimate of the starting values V_i, $i = 1,2...m$ and the process is repeated until $E_k < E_{min}$, an a priori exit criterion.

The advantage of this method is that integration is traded for matrix inversion, a good trade in most computers. Convergence considerations are briefly discussed elsewhere (21) but no serious problems are evident.

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