NON-LINEAR BENDING VIBRATIONS OF BEAMS

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NOTATION

A (in sq)	Beam cross-sectional area
A_n, A_r	Coefficient in a cosine series
B_n, B_r	Coefficient in a sine series
C_1, C_2, C_3, C_4	Constants
D_1, D_2, D_3, D_4	Integration constants
E (lbs/sq in)	Modulus of elasticity
E^* (lbs/sq in)	Constant = γE
F(x,t) (lbs)	External force, function of x and t
I (in^4)	Second moment of inertia of the beam cross-section = $\int Z^2 \, dA$
J (in^6)	Constant = $\int Z^4 \, dA$
K (in^2)	Ratio of (E^* J) to (E I), Constant
L (y)	Differential equation for which y is the dependent variable
M (lb in)	Bending moment
R (in)	Radius of curvature of the beam elastic line
T	A function in (t) only
V (lbs)	Shearing force
X	A function in (x) only
a_n	Coefficient in a cosine series, also undetermined parameters
b_n	Coefficient in a sine series
a_1	Arbitrary amplitude
c_n	Undetermined parameters, constants

d	Distance from the origin
$f(x)$	A given displacement at the start of the motion
$f_1(x)$	A given velocity at the start of the motion
g	Gravitational acceleration, 32.2 ft/sec^2
$i = \sqrt{-1}$	Imaginary unit
j	An index
k	Radius of gyration of the beam element about an axis through its center of gravity
ℓ	Total length of the beam
m	Constant $= \frac{\pi}{\ell}$
n	An index. Subscript meaning mode number or approximation number
p	Natural frequency of the linearized equation of motion
q	Natural frequency of the non-linear differential equation of motion
r	A positive integer, an index
t	Time
v	Initial velocity
w	Weight of beam per unit length
x	A distance on the x-axis
y	Beam deflection
y', y''	First, second, ..., etc. derivatives of y with respect to x
$\bar{y}$	Assumed approximate solution
$\dot{y}$	Differentiation of y with respect to t, velocity
Ξ	Euler's integral of the variational problem
β	An index

γ	Dimensionless constant
δ	A small portion of the beam
ε (in/in)	Strain referred to the median line of the undeformed beam
η	Ratio of q to p
λ^4	Constant = $\frac{\rho}{E\ I}$
ρ (lb sec^2/in^2)	Mass of beam per unit length
σ (lb/sq in)	Stress
τ	Period of oscillation
ϕ	Slope of the deflection curve
ψ	A function in x and t

SUMMARY

In this study, the Bernoulli-Euler equation of bending of beams is first reformulated to include the effect of the non-linear elastic properties of a material. The properties of the material are expressed by a stress-strain relation of the form

$$\sigma = E(\varepsilon-\gamma\varepsilon^3)$$

of which γ is constant.

Next the complete formula of the radius of curvature is used and a non-linear differential equation of motion is obtained. This equation is expanded to fifth order terms.

Galerkin's variational method is applied to the non-linear differential equation of motion and the natural frequency is determined. Application of the method to the simply supported beam as well as the cantilever beam is discussed. The four possible combinations of linearized and non-linear curvature and stress-strain relation are presented.

For each case, a frequency equation is obtained showing the natural frequency of the non-linear system as related to the natural frequency of the linearized system for an arbitrarily chosen amplitude. A numerical example is given and the frequency curves are discussed.

One major result of this study lies in the verification of the fact that added non-linear terms to the equation of motion influence the shape of the amplitude versus frequency curve. These terms limit the infinite increase of amplitudes predicted by the linear theory.

A second result lies in the comparison of the effect of non-linear curvature and non-linear stress-strain relation on the natural frequency.

A third result consists of the comparison of the choice of two assumptions for the approximate solution.

The fourth result reveals the insignificant effect of the added fifth order terms, thus indicating that third order terms are sufficient for the purpose of this investigation.

CHAPTER I

INTRODUCTION

1.1 Definition and Scope of the Problem

The subject of the present study is the determination of fundamental natural frequency of beams with the inclusion of the non-linear effects. This problem is of considerable importance in the design of a great number of engineering components. It includes the analysis of aircraft wings, high speed aircrafts, space boosters, etc. The trend of this research is linked with the solution of one of the basic problems, that of resonance determination (i.e., to move by proper choice of the operating conditions away from a position at which detrimental vibrational effects would occur).

The mathematical solution of straight uniform beams in simple bending which contains trigonometric and hyperbolic functions is well known [1]*. The use of a linear differential equation to describe the beam vibratory motion, amounts to certain restrictive assumptions of linearity. It is adequate only if the occurring amplitudes can be considered small. With increasing amplitudes the influence of the non-linear effects becomes more and more apparent.

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* Figures within brackets refer to items in the Bibliography. See page 93.

1.2 Types of Non-Linearities

The elementary theory of bending of beams does not account for the beam non-linear effects. Beam non-linearity takes place in the formulation of the equation of motion due to one or more of the following:

1. Non-Linear Curvature

According to the Bernoulli-Euler equation [2], the bending moment M of any portion of the beam is proportional to the change in the curvature caused by the action of the load. In other words

$$\frac{M}{EI} = \frac{1}{R} \tag{1.1}$$

in which $\frac{1}{R}$ is the curvature of the beam center line, E is the modulus of elasticity, and I is the moment of inertia of the beam cross-section.

If the curvature is expressed in rectangular coordinates, then

$$\frac{1}{R} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{1.2}$$

A combination of Equations (1.1) and (1.2) results in a second order non-linear differential equation

$$\frac{M}{EI} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} \tag{1.3}$$

In the classic theory of bending of beams the term $(\frac{dy}{dx})^2$ is neglected as being very small compared to unity, provided the deflection is small compared to the length of the beam. This assumption can lead to large errors when applied to conditions involving large deflections. Euler, who is credited with the beam equation, has discussed several problems in which he retained the term $(\frac{dy}{dx})^2$ [3]. Euler investigated the deflection of a cantilever beam due to vertical load at the free end. His results exhibit a hardening characteristic.

2. Non-Linear Stress-Strain Relation

The classic theory of pure bending in which the bending stress is calculated by the formula

$$\sigma = \frac{M \cdot Z}{I} \tag{1.4}$$

was developed on the basis of two assumptions; first, that plane cross-sections of the beam before bending remain plane after bending, and second, that Hooke's law is valid. These two assumptions reveal that the theory is good only for linear stress-strain characteristics of a material. A typical indealized stress-strain curve for an elastic material, however, shows some deviations as exhibited in Figure 1.

For small values of σ, the relation is a straight line indicating direct proportionality in accordance with Hooke's law. Reduction of the stress to zero within the segment 1 - 2 causes the strain to revert to zero also. Beyond point 2, there is a portion of the curve 2 - 3 over which the variation of σ with ε is non-linear, but reduction of σ to zero still causes ε to vanish. It follows that within the elastic deformation of a material, the non-linear elastic properties can also be considered for an elastic material with linear properties. Beyond point 3, the stress strain variation is non-linear, but reduction of

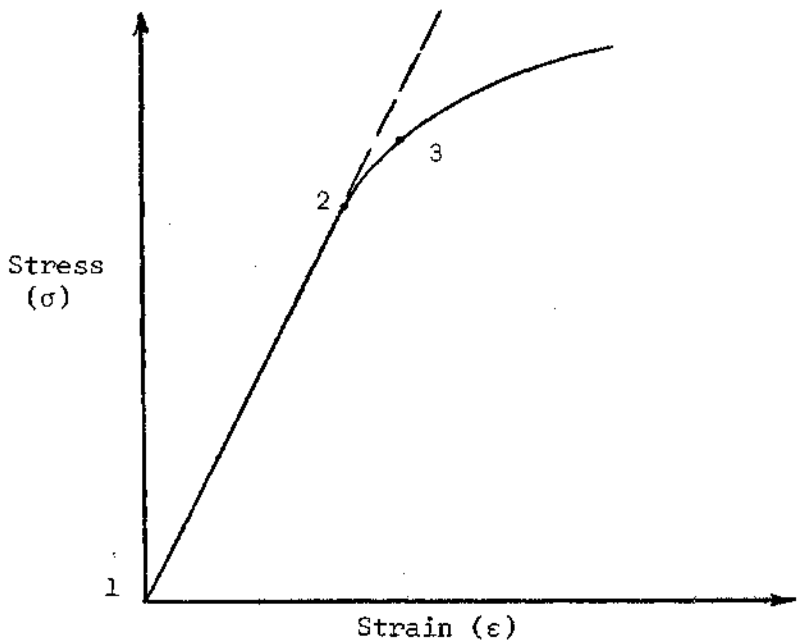


Figure 1. A Typical Stress-Strain Curve for an Elastic Material.

σ to zero results in a non-zero residual strain.

Several authors [4] have expressed the stress-strain relationship mathematically by considering stress as a function of strain or vice versa.

Ramberg and Osgood [5] showed that the formula

$$\varepsilon = \sigma + K\,\sigma^{n} \tag{1.5}$$

in which K is constant, is applicable to a wide variety of materials. Wang [6] proposed a stress-strain relation depending on the form of the cross-section of beams and the relative dimensions of the cross-section.

Bolotin [7] assumed a relation in the form

$$\sigma = E(\varepsilon - \gamma\varepsilon^3) \tag{1.6}$$

in which γ is constant. He reasoned that for the majority of known materials the stress σ and the strain ε satisfy the inequality

$$\frac{d^2\sigma}{d\varepsilon^2} \leq 0 \tag{1.7}$$

3. Non-Linear Damping

This is encountered due to the condition which has been experimentally verified [8], that energy loss during vibrations depends not only on velocity, but on material internal friction as well. The latter is usually called structural damping.

4. Non-Linear Inertia

This is due to the assumption that in vibrations, every section of the beam goes through some longitudinal displacement of higher order in comparison to the beam displacement. The resulting distributed loading due to the inertia forces acting on the beam would be equal to the mass times the acceleration. This load would slightly influence the shape of the vibration.

In this study we shall limit ourselves to the first two cases of beam non-linearity. These are referred to as non-linearities of static origin, whereas the last two cases are known as non-linearities of dynamic origin.

1.3 Methods of Investigating a Solution

The determination of the natural frequency often requires an elaborate analysis. Several methods are now widely used. The majority of these methods start with an assumed shape of the vibrating beam. The determination of the lowest natural frequency permits some freedom in the choice of an assumed deflection curve as long as it satisfies the beam end conditions.

Lord Rayleigh [9] in his text on sound suggested an approximate method by which the fundamental natural frequency of a beam is determined with reasonable accuracy. The method is based on the equivalence of the maximum potential energy and the maximum kinetic energy.

A second method originated by Stodola [10] allows the determination of all modes of vibrations. In this method, the differential equation is solved by iterative integration. It becomes quite time-consuming for higher modes of vibrations.

A third method devised by Myklestad [11] is a sequence of calculations whereby the beam is divided into a number of segments and the distributed mass of each segment is replaced by a discrete mass at the center of gravity of the segment (lumped mass method).

The above methods have been widely used in solving vibration problems described by linear differential equations. Several analytical methods are also available for the evaluation of the non-linear differential equations. For a complete survey of the subject the reader is referred to references [12] and [13].

For the present problem Galerkin's variational method is employed. This method depends on minimizing the error averaged by integration. Duncan [14] has shown that the use of Galerkin's method for solving beam vibration problems is equivalent to the application of Rayleigh's principle.

1.4 Approach to the Problem

The non-linear bending equation of motion of beams, with the effect of an assumed non-linear stress-strain relation, is derived in Chapter II. The exact formula of curvature is used to evaluate the Bernoulli-Euler equation of bending. Four possible combinations of curvature and stress-strain relation are discussed.

In Chapter III, the linearized equation of motion is solved by Bernoulli's separation method. The application of the general solution in calculating the fundamental frequencies is discussed for the simply supported beam and the cantilever beam.

Chapter IV gives the solutions of the non-linear differential equation of motion. Galerkin's method is explained and applied to the equation of motion. The non-linear natural frequency is determined as a function of the natural frequency of the linearized system and an arbitrary amplitude. Chapter V outlines the results obtained.

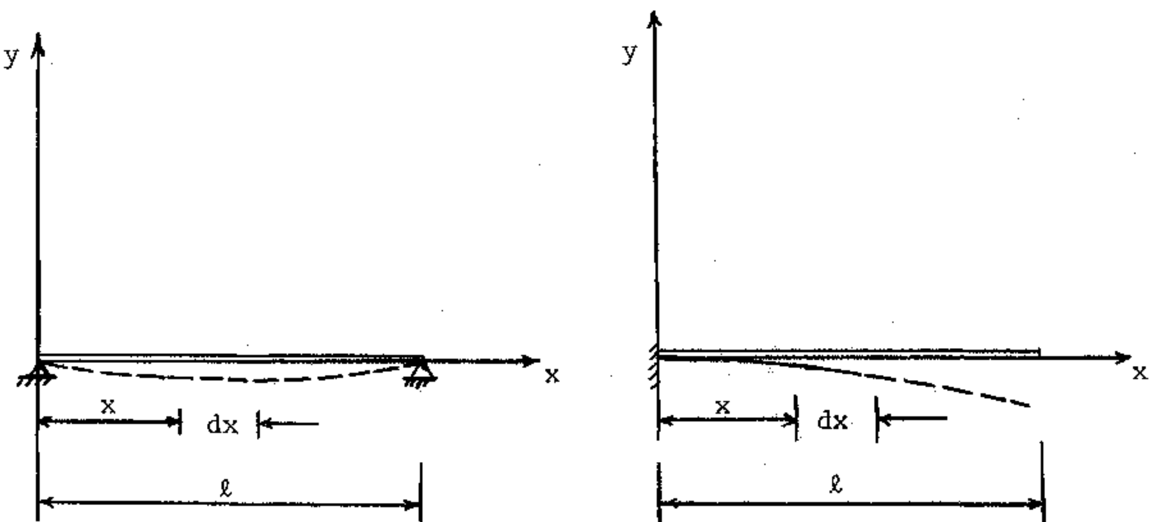
In the Appendix, a numerical example is given, and frequency curves are presented. These curves exhibit soft restoring characteristics.

CHAPTER II

EQUATION OF MOTION

2.1 Derivation of the Basic Equation of Motion

Let us assume that a uniform beam, fixed horizontally in any manner, is oscillating in one plane containing one of the principal axes of inertia of the beam cross-section. Let the x-axis be taken along the center line of the beam in its unstrained position and the y-axis perpendicular to the x-axis in the plane of motion, Figure 2.



(a) Simply Supported Beam (b) Cantilever Beam

Figure 2. The Vibrating System.

Let Q and Q' be two points on the central line of the beam. The length QQ' represents an element dx. At any instant of motion, let the inclination of the beam center line at these points with respect to the x-axis be ϕ and $(\phi + d\phi)$, Figure 3. Since the position of point Q varies with time, the vertical displacement y is a function of t at a given x. But y is also a function of x at any particular instant; hence, y is a function of x and t, and the acceleration is $\frac{\partial^2 y}{\partial t^2}$. The

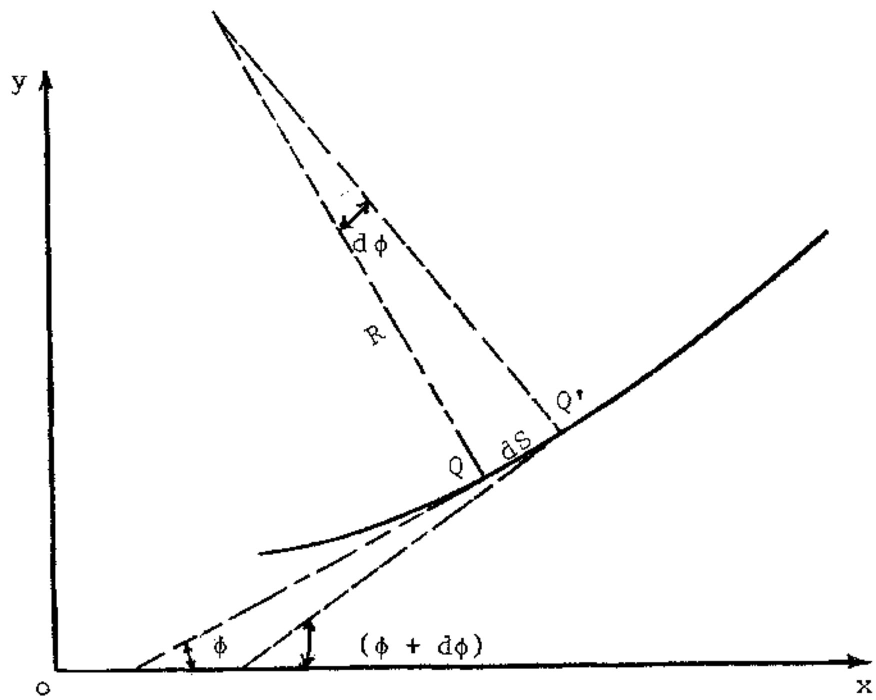


Figure 3. Beam Center-Line.

component of the acceleration in the x-direction will be ignored as being of small effect in comparison with $(\frac{\partial^2 y}{\partial t^2})$.

The slope of the beam center curve is $\tan \phi = \frac{\partial y}{\partial x}$ and if ϕ is small, it is approximately

$$\tan \phi = \frac{\partial y}{\partial x} = \phi \qquad (2.1)$$

Let V and M denote the shearing force and the bending moment at point Q, (V + dV) and (M + dM) denote the shearing force and the bending moment at point Q'. Let ω be the weight of the beam per unit length.

Resolving in the direction of the y-axis for the motion of the

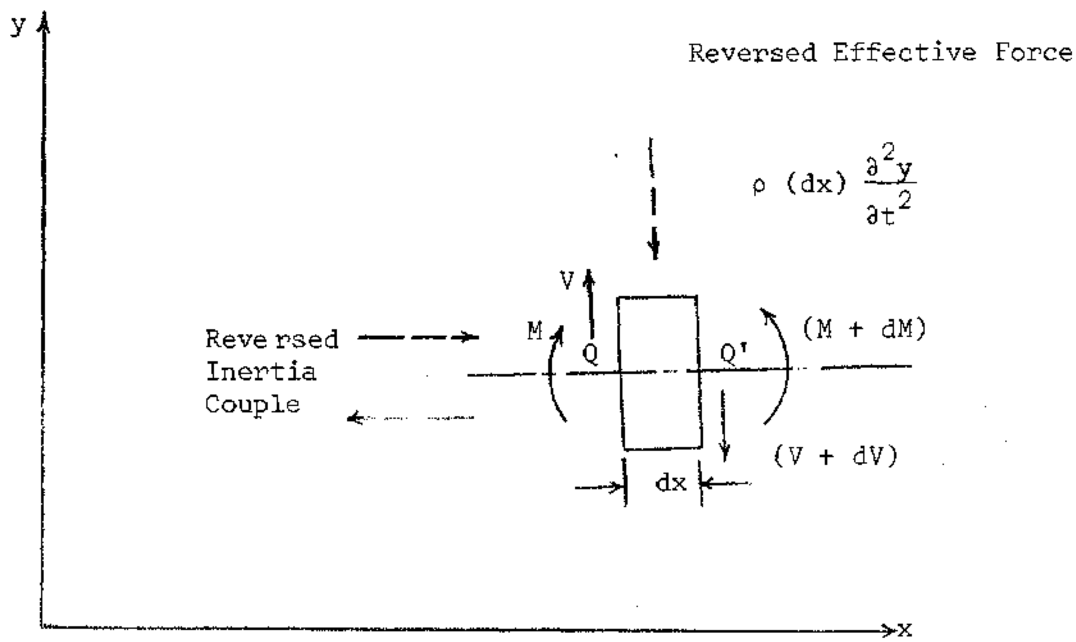


Figure 4. Free Body of a Beam Element

element Q Q', and assuming that there are no forces acting on the element dx except the shearing force and the bending moment at its ends, we get

$$dV + \rho\ \left(\frac{\partial^2 y}{\partial t^2}\right)\ (dx) = 0$$

that is

$$\frac{dV}{dx} = - \rho \frac{\partial^2 y}{\partial t^2} \tag{2.2}$$

To get the relation between V and M, we consider the moments about the center of gravity of the element dx. To do this, we must pay attention to the fact that during bending vibrations, the cross sections of the beam are not only displaced normally to its length (displacement y), but they also rotate by an angle ϕ given by the slope of the deflection at the corresponding point; i.e., $\frac{\partial y}{\partial x}$.

The angular velocity and acceleration follow from Equation (2.1)

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2 y}{(\partial x)(\partial t)} \text{ , and}$$

$$\frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^3 y}{(\partial x)(\partial t^2)} \tag{2.3}$$

Summing moments about the center of gravity of the element Q Q', we get, for the motion of this element

$$I \frac{\partial^2 \phi}{\partial t^2} + [(V+dV) + V] \cos \phi \, [\frac{dx}{2}] - dM = 0 \tag{2.4}$$

where I is the moment of inertia of the beam element about an axis through its center of gravity perpendicular to the beam center-line. If k denotes the radius of gyration of a section of the beam about the axis through its center of gravity perpendicular to the plane of motion, then

$$I = \rho \; (dx) \; [k^2]$$

Neglecting small quantities of higher order in the Equation (2.4), we arrive at

$$[\rho \; (dx) \; k^2] \; \frac{\partial^2 \phi}{\partial t^2} + V(dx) - dM = 0$$

Dividing the last equation all through by dx, the result becomes

$$[\rho \; k^2] \; \frac{\partial^2 \phi}{\partial t^2} + V - \frac{\partial M}{\partial x} = 0$$

which, when in the limit becomes

$$\frac{\partial M}{\partial x} = V + \rho k^2 \; \frac{\partial^2 \phi}{\partial t^2} \tag{2.5}$$

Differentiating Equation (2.5) with respect to x, we get

$$\frac{\partial^2 M}{\partial x^2} = \frac{\partial V}{\partial x} + \rho \; k^2 \; \frac{\partial}{\partial x} \left[\frac{\partial^2 \phi}{\partial t^2}\right] \tag{2.6}$$

Now making use of Equations (2.2) and (2.3), the Equation (2.6) becomes

$$\frac{\partial^2 M}{\partial x^2} = - \rho \frac{\partial^2 y}{\partial t^2} + \rho k^2 \frac{\partial^4 y}{\partial x^2 \cdot \partial t^2}$$

that is,

$$\frac{\partial^2 M}{\partial x^2} = - \rho \left[\frac{\partial^2 y}{\partial t^2} - k^2 \frac{\partial^4 y}{\partial x^2 \partial t^2}\right] \tag{2.7}$$

The last term in Equation (2.7) is due to the rotary inertia of the beam. If the beam is thin k is small, then the term due to the rotary inertia in the above equation is small compared with the other two terms in the equation of motion for ordinary beam. It has been shown by Rayleigh [15] that added terms in the equation of motion due to rotary inertia influence the vibrations significantly at higher modes. At lower frequencies they are insignificant and may be omitted. Then the final equation of motion for free oscillation of thin uniform beams is

$$\frac{\partial^2 M}{\partial x^2} = - \rho \frac{\partial^2 y}{\partial t^2} \tag{2.8}$$

where ρ is the mass per unit length of the beam,

M is the bending moment, and

$y(x,t)$ is the deflection.

When beams are short, further changes must be made to Equation (2.8). The shearing force will also contribute to the deflection of the beam. The relative contribution being greater the shorter the beam [16].

2.2 Evaluation of the Bending Moment

Let us assume the properties of the beam material to be expressed by the relation (1.6), which states that

$$\sigma = E(\varepsilon - \gamma\varepsilon^3)$$

where σ is the stress, ε is the strain, and γ is a dimensionless constant.

Assuming plane sections to remain plane, it is

$$M = \int_A \sigma Z \, dA \qquad (2.9)$$

where the integration is taken over the entire normal cross-section. With

$$\sigma = E\varepsilon - E^*\varepsilon^3 \quad \text{where } E^* = \gamma E, \quad \text{and} \quad \varepsilon = \frac{Z}{R}$$

we obtain

$$\sigma = E\left(\frac{Z}{R}\right) - E^*\left(\frac{Z}{R}\right)^3 \qquad (2.10)$$

Equations (2.9) and (2.10) yield

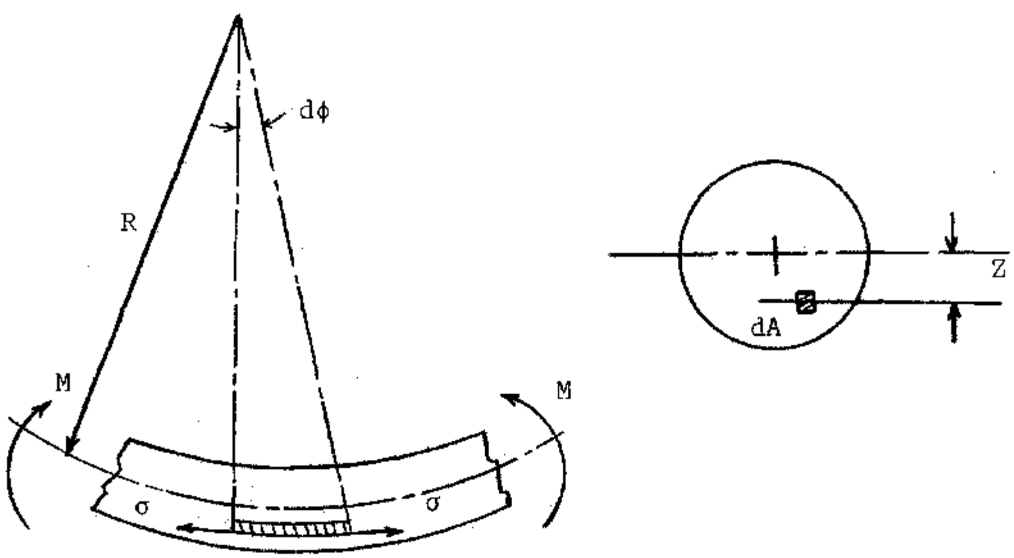


Figure 5. Strain of Beam Element.

$$M = \int_A \left[E\left(\frac{Z}{R}\right) - E^*\left(\frac{Z}{R}\right)^3 \right] Z \, dA \;, \quad \text{or}$$

$$M = \frac{EI}{R} - \frac{E^* J}{R^3} \qquad (2.11)$$

where $I = \int_A Z^2 \, dA$

$J = \int_A Z^4 \, dA$

Replacing $\frac{1}{R}$ by its value from Equation (1.2), we obtain

$$M = E\ I\left[y''(1+y'^2)^{-\frac{3}{2}}\right] - E^*J\left[y''^3(1+y'^2)^{-\frac{9}{2}}\right] \tag{2.12}$$

where the prime indicates differentiation with respect to the variable x.

Differentiating Equation (2.12) with respect to x, we get

$$\frac{\partial M}{\partial x} = E\ I\left[y'''(1+y'^2)^{-\frac{3}{2}} - 3\ y'y''^2(1+y'^2)^{-\frac{9}{2}}\right]$$

$$- E^*J\left[3\ y''^3\ y'''\ (1+y'^2)^{-\frac{9}{2}} - 9\ y'\ y''^4(1+y'^2)^{-\frac{11}{2}}\right]$$

The second derivative yields

$$\frac{\partial^2 M}{\partial x^2} = E\ I\left[y^{iv}(1+y'^2)^{-\frac{3}{2}} - 9\ y'\ y''\ y'''\ (1+y'^2)^{-\frac{5}{2}} - 3\ y''^3(1+y'^2)^{-\frac{5}{2}}\right.$$

$$\left. + 15\ y'^2\ y''^3\ (1+y'^2)^{-\frac{7}{2}}\right]$$

$$- E^*J\left[6\ y''\ y'''^2(1+y'^2)^{-\frac{9}{2}} + 3\ y''^2\ y^{iv}\ (1+y'^2)^{-\frac{9}{2}}\right.$$

$$- 63\ y'\ y''^3\ y'''\ (1+y'^2)^{-\frac{11}{2}} - 9\ y''^5(1+y'^2)^{-\frac{11}{2}}$$

$$\left. + 99\ y'^2\ y''^5(1+y'^2)^{-\frac{13}{2}}\right]$$

Making use of the Binomial Theorem, we expand the quantities

$$(1+y'^2)^{-\left(\frac{2\beta+1}{2}\right)} \qquad \beta = 1,\ 2,\ 3,\ldots$$

This yields

$$\frac{\partial^2 M}{\partial x^2} = E\ I \Big[y^{iv} [1 - \frac{3}{2} y'^2 + \frac{15}{8} y'^4 \ \ldots]$$

$$- 9\ y'\ y''\ y''' [1 - \frac{5}{2} y'^2 + \frac{35}{8} y'^4 \ \ldots]$$

$$- 3\ y''^3\ [1 - \frac{5}{2} y'^2 + \frac{35}{8} y'^4 \ \ldots]$$

$$+ 15\ y'^2\ y''^3\ [1 - \frac{7}{2} y'^2 + \frac{63}{8} y'^4 \ \ldots] \Big]$$

$$- E^* J \Big[6\ y''\ y'''^2 [1 - \frac{9}{2} y'^2 + \frac{99}{8} y'^4 \ \ldots]$$

$$+ 3\ y''^2\ y^{iv} [1 - \frac{9}{2} y'^2 + \frac{99}{8} y'^4 \ \ldots]$$

$$- 63\ y'\ y''^3\ y''' [1 - \frac{11}{2} y'^2 + \frac{143}{8} y'^4 \ \ldots]$$

$$- 9\ y''^5 [1 - \frac{11}{2} y'^2 + \frac{143}{8} y'^4 \ \ldots]$$

$$+ 99\, y'^2\, y''^5 \,[1 - \frac{13}{2}\, y'^2 + \frac{195}{8}\, y'^4 \;\ldots] \Big]$$

We obtain the first non-linear approximation by retaining terms of the third order in the expansion. The second non-linear approximation is accomplished by retaining third and fifth order terms in the above equation. Neglecting higher order terms, we arrive at

$$\frac{\partial^2 M}{\partial x^2} = E\, I\, y^{IV} - E\, I \Big[[\frac{3}{2}\, y^{IV}\, y'^2 - 9\, y'\, y''\, y''' + 3y''^3]$$

$$- [\frac{15}{8}\, y^{IV}\, y'^4 + \frac{45}{2}\, y'^3\, y''\, y'''$$

$$+ \frac{45}{2}\, y'^2\, y''^3] \Big]$$

$$- E^* J \Big[[6\, y''\, y'''^2 + 3\, y''^2\, y^{IV}]$$

$$- [27\, y'^2\, y''\, y'''^2 + \frac{27}{2}\, y'^2\, y''^2\, y^{IV}$$

$$+ 63\, y'\, y''^3\, y''' + 9\, y''^5] \Big]$$

The first term in Equation (2.13) corresponds to the usual linear approximation in the theory of bending.

A substitution of Equation (2.13) into the equation of motion (2.8), yields

$$- \rho \frac{\partial^2 y}{\partial t^2} = E\ I\ y^{IV} - E\ I\ [\frac{3}{2}\ y^{IV} y'^2 + 9\ y'\ y''\ y''' + 3\ y''^3]$$

$$+ E\ I\ [\frac{15}{8}\ y^{IV} y'^4 + \frac{45}{2}\ y'^3\ y''\ y''' + \frac{45}{2}\ y'^2\ y''^3]$$

$$- E^* J\ [6\ y''\ y'''^2 + 3\ y^{IV} y''^2]$$

$$+ E^* J\ [27\ y'^2\ y''\ y'''^2 + \frac{27}{2}\ y'^2\ y''^2\ y^{IV}$$

$$+ 63\ y'\ y''^3\ y''' + 9\ y''^5]$$

Rearranging the terms, we get the non-linear bending equation of motion

$$y^{IV} + \lambda^4 \ddot{y} = [\frac{3}{2}\ y^{IV} y'^2 - 9\ y'\ y''\ y''' + 3\ y''^3] \qquad (2.14)$$

$$- [\frac{15}{8}\ y^{IV} y'^4 + \frac{45}{2}\ y'^3\ y''\ y''' + \frac{45}{2}\ y'^2\ y''^3]$$

$$+ K\ [6\ y''\ y'''^2 + 3\ y^{IV}\ y''^2]$$

$$- K\ [27\ y'^2\ y''\ y'''^2 + \frac{27}{2}\ y'^2\ y''^2\ y^{IV}$$

$$+ 63\ y'\ y''^3\ y''' + 9\ y''^5]$$

where $K = \frac{E^* J}{E \ I}$, and

$$\lambda^4 = \frac{\rho}{E \ I}$$

On the right hand side non-linear terms of third and fifth order are presented. Setting the right hand side equal to zero yields the linearized problem.

2.3 Forms Derivable from the Differential Equation of Motion

If the equation of motion (2.14) is to be linearized by neglecting the non-linear stress-strain effect and the non-linear terms in the curvature expression, the equation of motion is

$$y^{IV} + \lambda^4 \ddot{y} = 0 \qquad (2.15)$$

Linearizing the curvature expression by neglecting the term y'^2 in Equation (2.12) as being very small compared to unity, and retaining the non-linear stress-strain relationship, the equation of motion yields

$$\frac{\partial^2}{\partial x^2} [E \ I \ y'' - E^* J y''^3] = - \rho \ \frac{\partial^2 y}{\partial t^2}$$

or

$$y^{IV} + \lambda^4 \ddot{y} = K \ [6 \ y'' \ y'''^2 + 3y''^2 \ y^{IV} \ ] \qquad (2.16)$$

Retaining the non-linear terms in the curvature expression and linearizing the stress-strain relationship; i.e., $\gamma = 0$, the equation of motion will be

$$y^{iv} + \lambda^4 \ddot{y} = [\frac{3}{2} y^{iv} y'^2 + 9 y' y'' y''' + 3 y''^3] \tag{2.17}$$

$$- [\frac{15}{8} y^{iv} y'^4 + \frac{45}{2} y'^3 y'' y''' + \frac{45}{2} y'^2 y''^3]$$

where higher order terms, above fifth, are neglected. Equation (2.17) can be attained by equating K to zero in Equation (2.14).

Finally we have Equation (2.14) which includes the non-linear stress-strain effect as well as the non-linear curvature.

Since the assumed stress-strain relation is of the third power; i.e., ε^3, the right hand side of Equation (2.16) will always yield third order terms. Fifth order terms appear in the right hand side of Equations (2.14) and (2.17).

CHAPTER III

LINEAR SOLUTIONS

3.1 General Solutions

Equations (2.14), (2.15), (2.16), and (2.17), previously derived, possess periodic solutions $y(x,t)$. The theory of differential equations requires that such periodic solutions be unique once the initial conditions of displacement and velocity are specified. Non-periodic solutions can also be verified, but for the present problem, only periodic solutions are investigated.

Let us first consider the linearized differential equation of motion

$$y'''' + \lambda^4 \ddot{y} = 0$$

where $\lambda^4 = \frac{\rho}{EI}$. With Bernoulli's separation method, the solution is assumed in the form

$$y = X(x) \cdot T(t) \tag{3.1}$$

where X and T are functions of x and t, respectively. Substituting Equation (3.1) into Equation (2.15), we obtain

$$T \frac{d^4X}{dx^4} + \lambda^4 X \frac{d^2T}{dt^2} = 0$$

Dividing the above equation by XT and re-arranging the terms, we get

$$\frac{1}{\lambda^4} \cdot \frac{1}{X} \cdot \frac{d^4X}{dx^4} = -\frac{1}{T}\frac{d^2T}{dt^2} \tag{3.2}$$

The independent variables are now separable, and since the right hand side is a function of x and the left hand side a function of t, each side must be equal to a constant, say (p^2).

$$\frac{1}{\lambda^4} \cdot \frac{1}{X}\frac{d^4X}{dx^4} = -\frac{1}{T}\frac{d^2T}{dt^2} = P^2 \tag{3.3}$$

Equation (3.3) may be separated into two independent equations in X and T.

$$\frac{d^2T}{dt^2} + P^2T = 0 \ , \quad \text{and} \tag{3.4}$$

$$\frac{d^4X}{dx^4} - \lambda^4P^2X = 0 \quad \text{or}$$

$$\frac{d^4X}{dx^4} - m^4X = 0 \tag{3.5}$$

where $m^4 = \lambda^4P^2$.

The general solution of Equation (3.4) is

$$T_n(t) = a_n \cos P_n t + b_n \sin P_n t \tag{3.6}$$

where a_n and b_n are constants depending on the initial conditions of displacement and velocity at the start of the motion. This solution is restricted to positive values of P^2. If $P^2 \leq 0$ the solutions for T(t) cannot be periodic.

Equation (3.5) is a fourth order ordinary homogeneous differential equation. The solution can be assumed in the form

$$X = e^{rx} \tag{3.7}$$

Substituting Equation (3.7) into Equation (3.5), we get the characteristic equation

$$r^4 - m^4 = 0 \tag{3.8}$$

which has the roots

$$r = m, -m, im, -im \tag{3.9}$$

where $i = \sqrt{-1}$, and the solution of Equation (3.5) becomes

$$X(x) = D_1 e^{mx} + D_2 e^{-mx} + D_3 e^{imx} + D_4 e^{-imx} \tag{3.10}$$

in which D_1, D_2, D_3, and D_4 are integration constants.

Equation (3.10) can be written in another form in terms of trigonometric and hyperbolic functions as:

$$X(x) = C_1 \sin mx + C_2 \cos mx + C_3 \sinh mx + C_4 \cosh mx \qquad (3.11)$$

where C_1, C_2, C_3, and C_4 are constants to be determined in each particular case from the boundary conditions of the beam.

By superimposing all possible normal vibrations, the general solution of the linearized equation of motion becomes

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x) \, [a_n \cos P_n t + b_n \sin P_n t] \qquad (3.12)$$

The application of the above equation in calculating the fundamental frequencies will be discussed for the simply supported beam and the cantilever beam.

3.2 Simply Supported Beam

Let us first determine the constants for the simply supported beam. The end conditions in this case are:

1. $y(0,t) = 0$ — at the end $x = 0$
2. $y(\ell,t) = 0$ — at the end $x = \ell$
3. at $x = 0$ — $M_o = 0$; i.e., $y''_o = 0$
4. at $x = \ell$ — $M_\ell = 0$; i.e., $y''_\ell = 0$

where M is the bending moment at the respective ends of the beam.

Applying the end conditions to Equation (3.11), we get:

from condition (1)

$$C_2 + C_4 = 0$$

applying condition (3), we get

$$C_4 - C_2 = 0$$

hence

$$C_2 = C_4 = 0$$

and Equation (3.11) reduces to

$$X(x) = C_1 \sin mx + C_3 \sinh mx$$

from conditions (2) and (4), we obtain

$$C_3 \sinh m\ell + C_1 \sin m\ell = 0 \text{ , and}$$

$$C_3 \sinh m\ell - C_1 \sin m\ell = 0$$

for the non-trivial solution, the above equations are satisfied for

$$\sinh (m\ell)\cdot\sin (m\ell) = 0$$

and since sinh $(m\ell)$ cannot be zero, then

$$\sin (m\ell) = 0 \tag{3.13}$$

which is the frequency equation for the simply supported beam. The roots of this equation are

$$m\ell = n\pi \qquad n = 1, 2, \ldots \tag{3.14}$$

Substituting for m its value from Equation (3.5), we obtain

$$P_n^2 = \frac{n^4\pi^4}{\ell^4\lambda^4} = \frac{n^4\pi^4}{\ell^4} \cdot \left(\frac{EI}{\rho}\right)$$

of which the natural frequency is

$$P_n = \frac{n^2\pi^2}{\ell^2} \sqrt{\frac{EI}{\rho}} \qquad n = 1,2,\ldots \tag{3.15}$$

The corresponding period of oscillation will be

$$\tau_n = \frac{2\pi}{P_n} = \frac{2\ell^2}{\pi n^2} \sqrt{\frac{\rho}{EI}}$$

The shape of the deflection curve for the various modes of vibrations is determined by the displacement function, Equation (3.11). For the simply supported beam, the displacement function takes the form

$$X = C_n \sin mx$$

where C_n are constants.

If m is replaced by its value $\frac{n\pi}{\ell}$, we obtain an infinite number of possible vibrations of sinusoidal shape;

$$X_n = C_n \sin \frac{n\pi x}{\ell} \tag{3.17}$$

Figure 6 presents the first four characteristic functions for the simply supported beam.

The general solution of the equation of motion for the simply supported beam is now obtained from Equations (3.17) and (3.12).

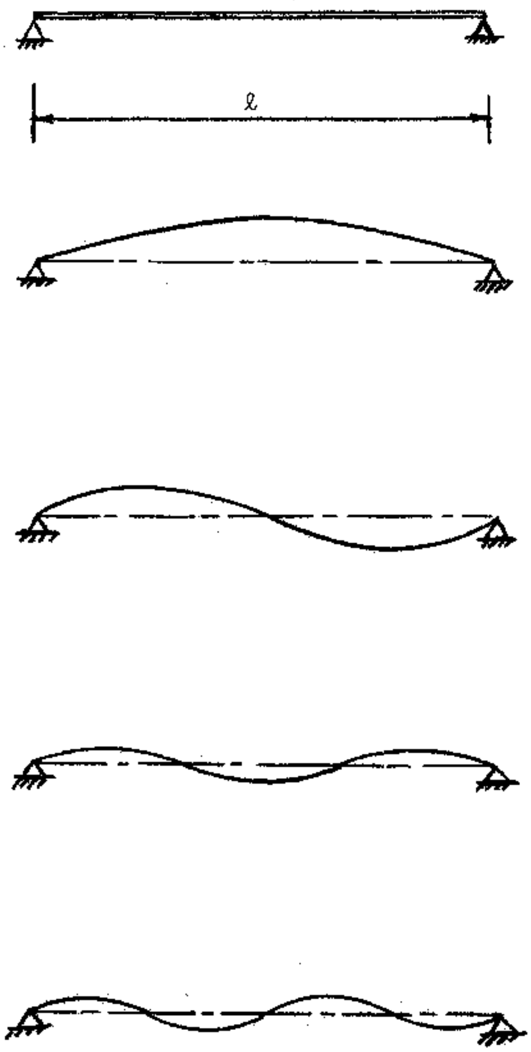
$$y = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell} [A_n \cos P_n t + B_n \sin P_n t] \tag{3.18}$$

in which A_n and B_n are constants depending on the initial conditions at time $t = 0$.

Assuming that the initial conditions are given by

$$y = f(x) \qquad \text{at } t = 0, \text{ and} \tag{3.19}$$

$$\frac{dy}{dt} = f_1(x) \qquad \text{at } t = 0$$



Fundamental

$$P_1 = \left(\frac{\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}}$$

Second Mode

$$P_2 = \left(\frac{2\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}}$$

Third Mode

$$P_3 = \left(\frac{3\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}}$$

Fourth Mode

$$P_4 = \left(\frac{4\pi}{\ell}\right)^2 \sqrt{\frac{EI}{\rho}}$$

Figure 6. Shapes of the First Four Characteristic Functions for a Simply Supported Beam.

Substituting Equations (3.19) into Equation (3.18), we obtain

$$y = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{\ell} = f(x) \tag{3.20}$$

$$\frac{dy}{dt} = \sum_{n=1}^{\infty} [P_n B_n] \sin \frac{n\pi x}{\ell} = f_1(x)$$

Multiplying Equations (3.20), both sides, by $\sin \frac{n\pi x}{\ell}$ and making use of the orthogonality conditions to evaluate the coefficients, we obtain

$$A_n = \frac{2}{\ell} \int_o^{\ell} f(x) \sin \frac{n\pi x}{\ell} dx \ , \quad \text{and}$$

$$B_n = \frac{2}{\ell} \cdot \frac{1}{P_n} \int_o^{\ell} f_1(x) \sin \frac{n\pi x}{\ell} dx$$

Assuming that in the initial moment $t = 0$ the axis of the beam is straight and that due to impact an initial velocity, say v, is given to a short portion δ of the beam at the distance d from the origin. Then

$$y = f(x) = 0 \ , \quad \text{and} \tag{3.22}$$

$f_1(x)$ also is equal to zero at all points except the point $x = d$ for which $f_1(x) = v$.

Substituting this into Equations (3.21), we obtain

$$A_n = 0 \tag{3.23}$$

and,
$$B_n = \frac{2}{\ell} \cdot \frac{1}{P_n} (v \cdot \delta) \sin \frac{n\pi d}{\ell}$$

Substituting equations (3.23) into the general solution of the simply supported beam, Equation (3.18), we get

$$y = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell} \left[\frac{2}{\ell} \frac{(v \cdot \delta)}{P_n} \sin \frac{n\pi d}{\ell} \sin P_n t\right] \tag{3.24}$$

If $d = \ell/2$; i.e., the impact is produced at the middle point of the beam span, we obtain

$$y = \frac{2}{\ell} (v \cdot \delta) \left[\frac{1}{P_1} \sin \left(\frac{\pi x}{\ell}\right) \sin P_1 t - \frac{1}{P_3} \sin \frac{3\pi x}{\ell} \sin P_3 t + \frac{1}{P_5} \sin \frac{5\pi x}{\ell} \sin P_5 t - \ldots\right] \tag{3.25}$$

3.3 Cantilever Beam

If the origin is taken at the clamped end of the beam, the end conditions for the cantilever beam will be:

1. at $x = 0$ $\quad y(0,t) = 0$
2. at $x = 0$ $\quad y'(0,t) = 0$
3. at $x = \ell$ $\quad y''(\ell,t) = 0$
4. at $x = \ell$ $\quad y'''(\ell,t) = 0$

Applying the above boundary conditions to the displacement function, Equation (3.11), we obtain:

from condition (1)

$$C_2 + C_4 = 0 \ , \quad \text{or} \tag{3.26}$$

$$C_2 = -C_4$$

Applying condition (2), we get

$$C_1 = -C_3 \tag{3.27}$$

Eliminating C_3 and C_4 from Equation (3.11), and applying conditions (3) and (4), we obtain

$$C_1 \, [\sin m\ell + \sinh m\ell] + C_2 \, [\cos m\ell + \cosh m\ell] = 0 \ , \quad \text{and}$$

$$C_1 \, [\cos m\ell + \cosh m\ell] + C_2 \, [\sinh m\ell - \sin m\ell] = 0 \tag{3.28}$$

Eliminating C_1 and C_2 in Equations (3.28), we obtain the characteristic equation of $m\ell$

$$1 + \cosh (m\ell) \cdot \cos (m\ell) = 0 \tag{3.29}$$

Equation (3.29) determines $m\ell$, and therefore determines the frequency p, since all other quantities involved in m are known. This equation can be solved graphically. Let

$$y_1 = \cos m\ell \text{ , and} \tag{3.30}$$

$$y_2 = -1/(\cosh m\ell) = -\text{sech } m\ell$$

The roots of Equation (3.29) will be abscissas of the points of intersection of the two curves y_1 and y_2, Figure 7. There is an

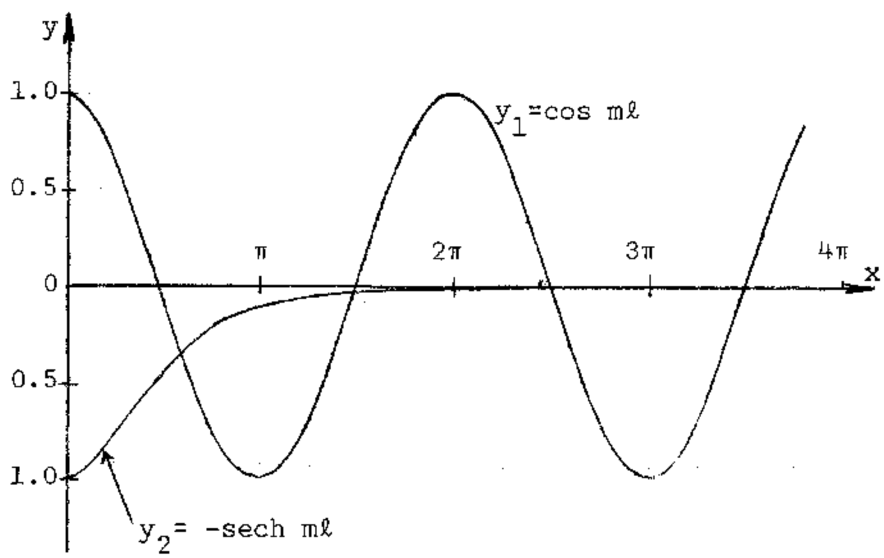


Figure 7. Graphical Solution of the Equation.
($1 + \cos m\ell \cosh m\ell = 0$)

infinite number of roots of the frequency equation. The first three roots are nearly

$$m\ell = 0.6\pi \ , \quad 1.494\pi \ , \quad 2.50\pi \tag{3.31}$$

Since sech $m\ell$ becomes quite small for large values of $m\ell$, the higher modes are given with satisfactory accuracy from the equation

$$\cos m\ell = 0 \ , \quad \text{or} \tag{3.32}$$

$$m\ell = (n - 1/2)\pi$$

The second root, for example would thus be 1.50π which differs slightly from the more correct value of the second root 1.494π.

The natural frequency of the cantilever beam is given by

$$P_n = \frac{(m\ell)^2}{\ell^2} \sqrt{\frac{EI}{\rho}} \tag{3.33}$$

The period of oscillation corresponding to each value of P is given by

$$\tau = \frac{2\pi}{P} \tag{3.34}$$

Substituting the roots of the frequency equation into Equation (3.28), the ratio (C_1/C_2) for the corresponding modes of vibration can be calculated, and the shape of the deflection curve is obtained from the displacement equation

$$X(x) = C_1[\sin mx - \sinh mx] + C_2\ [\cos mx - \cosh mx] \qquad (3.35)$$

Figure 8 gives the shapes of the first four characteristic functions for the cantilever beam.

The general solution of the linearized equation of motion for the cantilever beam is now obtained by substituting Equation (3.35) into Equation (3.12). The result is

$$y = \sum_{n=1}^{\infty} X_n(x)\ [A_n \cos P_n t + B_n \sin P_n t] \qquad (3.36)$$

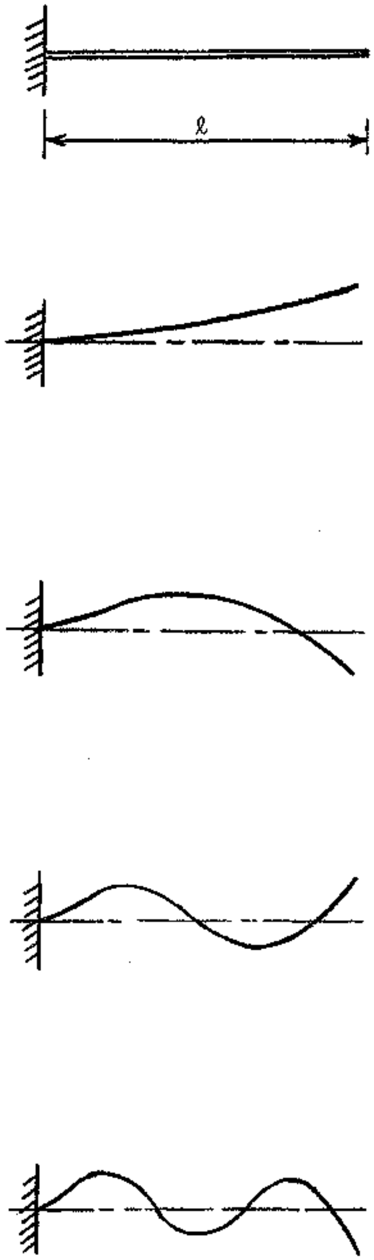
where A_n and B_n are constants.

Assuming the initial conditions at the moment $t = 0$ are given by Equation (3.19), to satisfy the initial displacement conditions, we must have

$$y = f(x) = \sum_{n=1}^{\infty} A_n\ X_n(x) \qquad (3.37)$$

and to satisfy the initial velocity conditions, we must have

$$\frac{dy}{dt} = f_1(x) = \sum_{n=1}^{\infty} (B_n\ P_n)\ X_n(x) \qquad (3.38)$$



Fundamental $\quad P_1 = (0.60)^2 \dfrac{\pi^2}{\ell^2} \sqrt{\dfrac{EI}{\rho}}$

Second Mode $\quad P_2 = (1.49)^2 \dfrac{\pi^2}{\ell^2} \sqrt{\dfrac{EI}{\rho}}$

Third Mode $\quad P_3 = (2.50)^2 \dfrac{\pi^2}{\ell^2} \sqrt{\dfrac{EI}{\rho}}$

Fourth Mode $\quad P_4 = (3.50)^2 \dfrac{\pi^2}{\ell^2} \sqrt{\dfrac{EI}{\rho}}$

Figure 8. Shapes of the First Four Characteristic Functions for a Cantilever Beam.

CHAPTER IV

NON-LINEAR SOLUTIONS

4.1 Method of Solution

In calculating the fundamental frequency of a vibrating system governed by a non-linear differential equation, approximate methods are employed. Two methods are widely used. The first method is associated with Galerkin, the other one which is closely related to Galerkin's method is that of Ritz. This method [17], generally referred to as the minimizing method, is a further development of Raleigh's principle. Galerkin's method is usually known as the averaging method. They may be formulated in general terms as follows [18], [19].

Let

$$L\,(y) = 0 \tag{4.1}$$

be the differential equation to be solved, where y is a function of (x,t) and L is some differential operator of which the solution satisfies the given boundary conditions.

$L\,(y) = 0$ can be considered to be the Euler equation of the corresponding variational problem

$$\Xi = \int_0^{\ell} \int_0^{2\pi} L\,(y, y', y'', \ldots, t)\; dx\; dt = \text{minimum} \tag{4.2}$$

Assuming an approximate solution in the form

$$\bar{y}_n(x,t) = \sum_{r=1}^{n} c_r \, \psi_r(x,t) \tag{4.3}$$

where $\psi_r(x,t)$ for $r = 1,2,\ldots n$ are chosen functions each of which satisfies the given boundary conditions, and c_r are undetermined parameters.

Introducing Equation (4.3) into Equation (4.2), the integral Ξ becomes a finite set of equations in the unknown parameters c_r. These parameters are determined by the condition

$$\frac{\partial \Xi}{\partial c_r} = 0 \qquad r = 1,2,3,\ldots n \tag{4.4}$$

(Minimizing Method)

In order that $\bar{y}_n(x,t)$ be the exact solution of the given Equation (4.1), it is necessary that L (y) be identically equal to zero. This requirement is equivalent to the requirement of the orthogonality of the expression L (y) to all the chosen functions

$$\psi_r(x,t) \quad \text{for} \quad r = 1,2,3,\ldots n$$

and since we have n constants c_1, c_2, c_3,...c_n, then we can satisfy n conditions of orthogonality expressed by

$$\iint L\,[\bar{y}(x,t)]\,\psi_r(x,t)\,dx\,dt \tag{4.5}$$

$$= \iint L\,[\sum_{j=1}^{n} c_j\,\psi_j(x,t)]\,[\psi_r(x,t)]\,dx\,dt = 0$$

for $r = 1,2,3,\ldots n$

(Averaging Method)

Equations (4.5) serve to determine the coefficients c_r.

In the case L (y) is non-linear, the n-equations of amplitude will also be non-linear. Once c_r is determined, then the approximate solution $\bar{y}_n(x,t)$ is obtained.

It can be shown [20] that the set of Equations (4.4) is equivalent to the set of Equations (4.5). Although the two methods lead to the same approximate solution, Galerkin's method makes possible the simpler and the more direct setting of the equations. In fact for applying it, one needs to know only the differential equation, irrespective of whether or not a variational expression exists [21]. For the present study we are going to follow Galerkin's procedure.

The accuracy of the method depends on the choice of the functions ψ_r and the number of terms used in the displacement function. Duncan [22], [23] has introduced sets of displacement functions of the polynomial form which satisfy the beam geometrical end conditions. Rauscher [24] introduced displacement functions that depend on the elastic properties of the structure, whereas Duncan's functions do not depend on a particular structure. Some advance knowledge of the expected solution will facilitate the choice of the functions and the quality of the solution.

Klotter [25] has shown that a good choice is often possible by restricting the assumptions to one single term approximation. The complexity of the problem increases with the number of terms in the solution.

Since ψ_r is a function of x and t, it is customary to assume $\psi_r(x,t)$ in the form

$$\psi_r(x,t) = X_r(x)\,[A_r \cos P_r t + B_r \sin P_r t] \qquad (4.6)$$

If only the natural frequencies P_r and the associated amplitudes $X_r(x)$ are required, then the function $\psi_r(x,t)$ may be simplified to

$$\psi_r(x,t) = X_r(x)\,[\cos P_r t] \quad \text{or to}$$

$$\psi_r(x,t) = X_r(x)\,[\sin P_r t]$$

This procedure is correct if no damping is present.

For the present study, the function $\psi_r(x,t)$ will be considered as

$$\psi_r(x,t) = X_r(x)\,[\cos P_r t] \quad \text{for the linear system} \qquad (4.7)$$

$$\psi_r(x,t) = X_r(x)\,[\cos q_r t] \quad \text{for the non-linear system}$$

where P_r is the natural frequency of the linearized equation, and q_r is the natural frequency of the non-linear equation. The ratio of q_r to P_r will be furnished for both the simply supported beam and the cantilever beam.

4.2 Simply Supported Beam

Let us assume a function

$$\psi_r(x,t) = X_r(x) \cdot \cos P_r t \qquad r = 1,2,3,\ldots$$

satisfying the simply supported beam end conditions

$$\psi_r(o,t) = \psi_r(\ell,t) = 0$$

$$\psi_r''(o,t) = \psi_r''(\ell,t) = 0$$

Such solutions can be adapted in the form

$$X_r = \sin rmx \qquad \text{such that}$$

$$\psi_r = (\sin rmx)(\cos p_r t) \qquad r = 1,2,3,\ldots \qquad (4.8)$$

where P_r is the natural frequency of the linearized equation of motion, and $m = \frac{\pi}{\ell}$ or, in the form [26]

$$X_r(x) = (\frac{x}{\ell}) - 2\,(\frac{x}{\ell})^3 + (\frac{x}{\ell})^4 \qquad \text{for } r = 1$$

and,

$$X_r(x) = (\frac{x}{\ell})^{r+1}\,[1 - (\frac{x}{\ell})]^{r+1} \qquad \text{for } r \geq 2$$

such that

$$\psi_r = [(\frac{x}{\ell}) - 2\,(\frac{x}{\ell})^3 + (\frac{x}{\ell})^4]\cos P_r t \qquad r = 1 \quad (4.9)$$

$$\psi_r = \left(\frac{x}{\ell}\right)^{r+1} \left[1 - \left(\frac{x}{\ell}\right)\right]^{r+1} \cos P_r t \qquad r \geq 2$$

The approximate solution then becomes

$$\bar{y}_n(x,t) = \sum_{r=1}^{n} a_r \, \psi_r \tag{4.10}$$

A substitution of Equation (4.10) into the differential equation of motion, results in the following Galerkin condition

$$\int_0^{\ell} \int_0^{\frac{2\pi}{P}} L \left[\sum_{a=1}^{n} a_j \, \psi_j \, (x,t)\right] [\psi_r(x,t)] \, dx \, dt = 0 \tag{4.11}$$

which yields the set of equations for the unknown constants a_j.

4.2.1 Linearized Curvature and Linearized Stress-Strain Relation

The differential equation of motion to be considered here is Equation (2.15) or

$$L\,(y) = y^{1V} + \lambda^4 \, \ddot{y} = 0$$

where $$\lambda^4 = \frac{\rho}{EI}$$

Let us first consider the function ψ_r described by Equation (4.8), with one term approximation, we assume a solution

$$\bar{y}_1 = a_1 \sin mx \cos pt \tag{4.12}$$

Substituting $\bar{y}_1$ into Equation (2.15), we obtain Galerkin's condition

$$\int_0^{\ell} \int_0^{\frac{2\pi}{P}} [\bar{y}^{iv} - \lambda^4 \, P^2 \, \bar{y}_1] \, [\psi_1] \, dx \, dt = 0 \ , \quad \text{or}$$

$$a_1 [m^4 - \lambda^4 \, P^2] \, [\tfrac{\ell}{2}] \, [\tfrac{\pi}{P}] = 0$$

from which we have

$$m^4 - \lambda^4 \, P^2 = 0 \ , \quad \text{or}$$

$$P^2 = \frac{m^4}{\lambda^4} = \frac{\pi^4}{\ell^4} \cdot \frac{EI}{\rho}$$

The natural frequency of the system is therefore

$$P = \frac{\pi^2}{\ell^2} \sqrt{\frac{EI}{\rho}} \tag{4.13}$$

which is identical with Equation (3.15). One can conclude that the choice of Equation (4.8) yields the correct answer [27].

Let us now consider the set of Equations (4.9). For n = 1, we have

$$X_1(x) = (\tfrac{x}{\ell}) - 2 \, (\tfrac{x}{\ell})^3 + (\tfrac{x}{\ell})^4 \ , \quad \text{and}$$

$$\psi_1(x,t) = [(\tfrac{x}{\ell}) - 2 \, (\tfrac{x}{\ell})^3 + (\tfrac{x}{\ell})^4] \cos pt \tag{4.14}$$

The corresponding solution will be

$$\bar{y}_1 = a_1 \left[\left(\frac{x}{\ell}\right) - 2 \left(\frac{x}{\ell}\right)^3 + \left(\frac{x}{\ell}\right)^4 \right] \cos pt \qquad (4.15)$$

Substituting $\bar{y}_1$ into Equation (4.11), we obtain

$$\int_0^{\ell} \int_0^{\frac{2\pi}{P}} [\bar{y}_1^{iv} - \lambda^4 P^2 \bar{y}_1] [\psi_1] \, dx \, dt = 0$$

The above equation, when evaluated results in

$$4.80 - \lambda^4 \ell^4 P^2 (0.0492) = 0$$

from which we have

$$\lambda^4 P^2 \ell^4 = 97.5610$$

and the natural frequency will be

$$P = \frac{9.8775}{\ell^2} \sqrt{\frac{EI}{\rho}} \qquad (4.16)$$

whereas the exact value is

$$P = \frac{9.8696}{\ell^2} \sqrt{\frac{EI}{\rho}}$$

A plot of the two displacement functions, Equations (4.8) and

(4.9), with one term approximation, is shown in Figure 9.

Let us now consider a two-term approximation; i.e., $n = 2$, we have

$$\psi_1 = [(\frac{x}{\ell}) - 2(\frac{x}{\ell})^3 + (\frac{x}{\ell})^4] \cos Pt \tag{4.17}$$

$$\psi_1 = (\frac{x}{\ell})^3 [1 - (\frac{x}{\ell})]^3 \cos Pt$$

and the corresponding approximate solution is

$$\bar{y}_2 = a_1 \psi_1 + a_2 \psi_2 \tag{4.18}$$

Substituting $\bar{y}_2$ into the linearized equation of motion (2.15), we arrive at Galerkin's variational equations

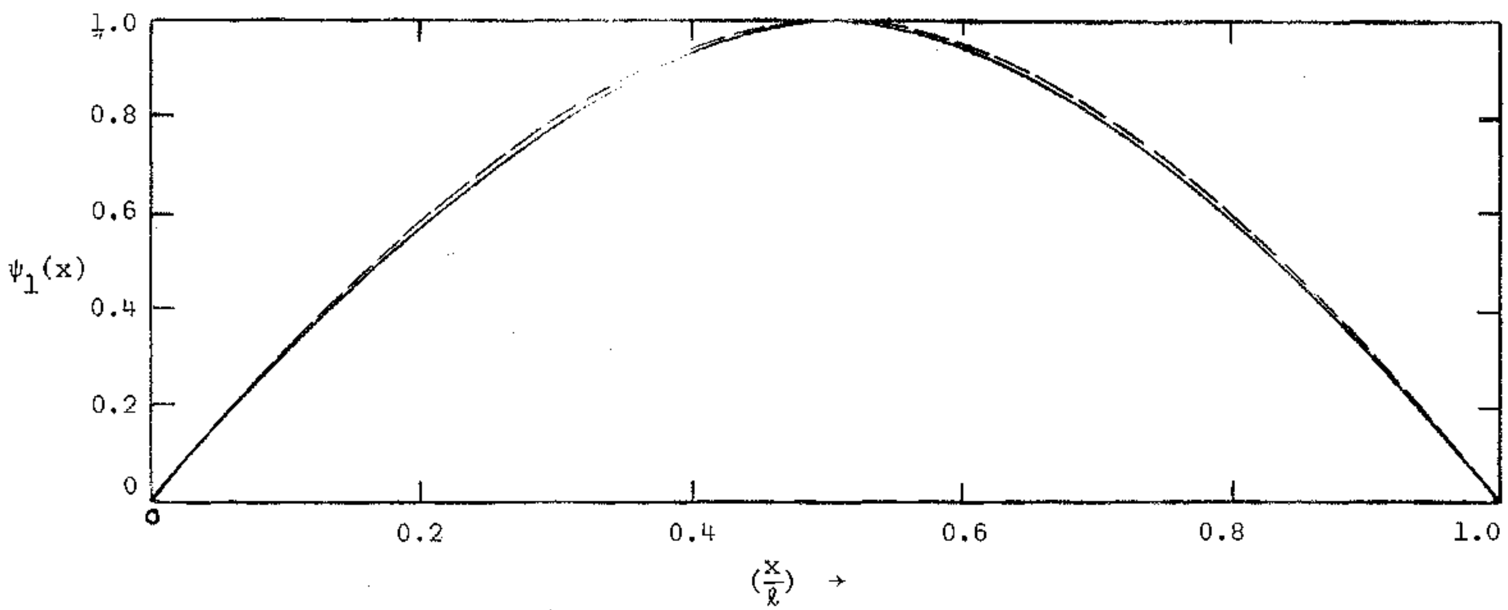
$$\int_0^\ell \int_0^{\frac{2\pi}{P}} L(\bar{y}_2)\, \psi_1 \, dx\, dt = 0 \text{ , and}$$

$$\int_0^\ell \int_0^{\frac{2\pi}{P}} L(\bar{y}_2)\, \psi_2 \, dx\, dt = 0$$

An evaluation of these two equations results in

$$\frac{a_1}{a_2} \int_0^\ell [\psi_1^{iv} \cdot \psi_1 - \lambda^4 P^2 \psi_1 \cdot \psi_1] \, dx \tag{4.19}$$

$$+ \int_0^\ell [\psi_2^{iv} \cdot \psi_1 - \lambda^4 P^2 \psi_2 \cdot \psi_1] \, dx = 0$$



——— $\psi_1(x) = \sin m x$

------- $\psi_1(x) = (\frac{x}{\ell}) - 2(\frac{x}{\ell})^3 + (\frac{x}{\ell})^4$ [Approximate Solution]

Figure 9. Graphical Presentation of Two Displacement Functions for a Simply Supported Beam.

and,

$$\frac{a_1}{a_2} \int_o^{\ell} [\psi_1^{iv} \cdot \psi_1 - \lambda^4 P^2 \psi_1 \cdot \psi_2] \, dx$$

$$+ \int_o^{\ell} [\psi_2^{iv} \cdot \psi_2 - \lambda^4 P^2 \psi_2 \cdot \psi_2] \, dx = 0$$

substituting ψ_1 and ψ_2 in Equation (4.19) yields

$$\frac{a_1}{a_2} [\frac{24}{5} - \lambda^4 P^2 \ell^4 \frac{31}{630}] + [\frac{6}{35} - \lambda^4 P^2 \ell^4 \frac{3}{1540}] = 0$$

and

$$\frac{a_1}{a_2} [\frac{6}{35} - \lambda^4 P^2 \ell^4 \frac{3}{1540}] + [\frac{2}{35} - \lambda^4 P^2 \ell^4 \frac{1}{12012}] = 0$$

The characteristic equation will then be

$$[\frac{6}{35} - \lambda^4 P^2 \ell^4 \frac{3}{1540}]^2 - [\frac{24}{5} - \lambda^4 P^2 \ell^4 \frac{31}{630}] [\frac{2}{35} - \lambda^4 P^2 \ell^4 \frac{1}{12012}] = 0$$

or, when simplified, becomes

$$[\lambda^4 P^2 \ell^4]^2 - 8479.3 [\lambda^4 P^2 \ell^4] + 816326.5 = 0$$

and the roots of this equation are

$$\frac{1}{2}\,[8479.3 \pm \sqrt{68633222.49}\,]$$

The lowest root is

$$\frac{1}{2}\,(8479.30 - 8284.517034) = \frac{194.782966}{2} = 97.3915 \qquad (4.20)$$

and the natural frequency of the system will be

$$P = \frac{9.8687}{\ell^2}\,\sqrt{\frac{EI}{\rho}} \qquad (4.21)$$

This may be compared to the exact answer

$$P = \frac{9.8696}{\ell^2}\,\sqrt{\frac{EI}{\rho}}$$

Comparing the results of P obtained for one and two-term approximation, we arrive at

Exact Answer of $P^2\ \lambda^4\ \ell^4$ = 97.4090

Equation (4.8), One-Term Approximation = 97.4090

Equation (4.9), One-Term Approximation = 97.5610

Equation (4.9), Two-Term Approximation = 97.3915

4.2.2 Linearized Curvature and Non-Linear Stress-Strain Relation

Equation (2.16) is the differential equation of motion to be solved in this section, which for convenience is given here again

$$L(y) = y^{iv} + \lambda^4 \ddot{y} - K [6 y'' y'''^2 + 3 y''^2 y^{iv}] = 0$$

For $$\bar{y}_1 = a_1 \sin mx \cos qt$$

Galerkin's condition yields

$$\int_0^{\ell} \int_0^{\frac{2\pi}{q}} \Big[(a_1 m^4 - a_1 \lambda^4 q^2) (\sin mx \cos qt)$$

$$- K [6 (-a_1^3 m^8 \cos^2 mx \sin mx + 3a_1^3 m^8 \sin^3 mx)] \cdot$$

$$[(\cos^3 qt)] \Big] \{\sin mx \cos qt\} \, dx \, dt = 0$$

where q is the natural frequency of the non-linear system.

Making use of the definite integrals:

$$\int_0^{\ell} \sin^2 mx \, dx = \frac{\ell}{2}$$

$$\int_0^{\ell} \sin^4 mx \, dx = \frac{3}{8} \ell$$

$$\int_0^{\ell} \sin^2 mx \cos^2 mx \, dx = \frac{1}{8} \ell$$

$$\int_0^{\frac{2\pi}{q}} \cos^2 qt \, dt = \frac{\pi}{q}$$

$$\int_0^{\frac{2\pi}{q}} \cos^4 qt \, dt = \frac{3}{4} \frac{\pi}{q}$$

this results in the frequency equation

$$a_1 \, [m^4 - \lambda^4 \, q^2] \, \frac{\ell}{2} \cdot \frac{\pi}{q} - K \, [6(-a_1^3 \, m^8 \cdot \frac{\ell}{8}) + 3(a_1^3 \, m^8 \cdot \frac{3}{8} \, \ell)] \; .$$

$$[\frac{3}{4} \, \frac{\pi}{q}] = 0$$

Dividing the above equation by $[\frac{\pi}{q} \, m^4 \, a_1 \, \frac{\ell}{2}]$ the result is

$$1 - \frac{\lambda^4}{m^4} \, q^2 - K[- \frac{6}{8} \, a_1^2 \, m^4 + \frac{9}{8} \, a_1^2 \, m^4] \frac{3}{2} = 0$$

but $\frac{\lambda^4}{m^4} = \frac{1}{P^2}$. Hence,

$$1 - \frac{q^2}{P^2} - \frac{9}{16} \, K \, a_1^2 \, m^4 = 0$$

or

$$\frac{q^2}{P^2} = 1 - \frac{9}{16} \, K \, a_1^2 \, m^4 \; , \quad \text{and}$$

$$q^2 = P^2 \, [1 - 54.78 \, \frac{a^2{}_1}{\ell^4} \, K] \tag{4.22}$$

Equation (4.22) exhibits the behavior of a softening characteristic where the frequency decreases as the amplitude increases.

4.2.3 Non-Linear Curvature and Linearized Stress-Strain Relation

Retaining the non-linear terms in the curvature formula and linearizing the stress-strain relation, the equation of motion yields

$$L(y) = y^{iv} + \lambda^4 \ddot{y} - [\frac{3}{2} \, y^{iv} \, y'^2 + 9 \, y' \, y'' \, y''' + 3y''^3] \tag{2.17}$$

$$+ [\frac{15}{8} y^{iv} y'^4 + \frac{45}{2} y'^3 y'' y''' + \frac{45}{2} y'^2 y''^3] = 0$$

The assumed approximate solution is

$$\bar{y}_1 = a_1 \sin mx \cos qt$$

Substituting $\bar{y}_1$ into Equation (2.17) and requiring the resulting function to be orthogonal to the selected function ψ_1, we obtain

$$\int_0^{\ell}\int_0^{\frac{2\pi}{q}} \Big[(a_1 m^4 - a_1 \lambda^4 q^2) \sin mx \cos qt$$

$$- [\frac{3}{2} a_1^3 m^6 \cos^2 mx \sin mx + 9 a_1^3 m^6 \cos^2 mx \sin mx$$

$$- 3 a_1^3 m^6 \sin^3 mx] \cos^3 qt$$

$$+ [\frac{15}{8} a_1^5 m^8 \cos^4 mx \sin mx + \frac{45}{2} a_1^3 m^8 \cos^4 mx \cdot$$

$$\sin mx - \frac{45}{2} a_1^5 m^8 \cos^2 mx \sin^3 mx] \cdot$$

$$\cos^5 qt\Big] \{\sin mx \cos qt\} \, dx \, dt = 0$$

Evaluating this yields:

$$\int_0^{\frac{2\pi}{q}} \Big[\frac{\ell}{2} \cdot a_1 m^4 (1 - \frac{\lambda^4}{m^4} q^2) \cos^2 qt - a_1^3 m^6 (\frac{3}{2} \cdot \frac{\ell}{8} + 9 \cdot \frac{\ell}{8}$$

$$- 3 \cdot \frac{3\ell}{8}) \cos^4 qt + a_1^5 m^8 (\frac{15}{8} \cdot \frac{\ell}{16} + \frac{45}{2} \cdot \frac{\ell}{16}$$

$$\left. - \frac{45}{2} \cdot \frac{\ell}{16}) \cos^6 qt \right] dt = 0$$

with

$$\int_o^{\frac{2\pi}{q}} \cos^2 qt \; dt = \frac{\pi}{q}$$

$$\int_o^{\frac{2\pi}{q}} \cos^4 qt \; dt = \frac{3}{4} \frac{\pi}{q}$$

$$\int_o^{\frac{2\pi}{q}} \cos^6 qt \; dt = \frac{1}{2} \frac{\pi}{q}$$

we obtain the expression

$$[1 - \frac{\lambda^4}{m^4} q^2] [a_1 m^4 \cdot \frac{\ell}{2} \cdot \frac{\pi}{q}] - [\frac{3}{16} \cdot \ell \cdot a_1^3 m^6 \cdot \frac{3}{4} \frac{\pi}{q}]$$

$$+ [\frac{15}{128} \ell \cdot a_1^5 m^8 \cdot \frac{\pi}{2q}] = 0$$

Dividing the above equation by $[a_1 m^4 \cdot \frac{\ell}{2} \cdot \frac{\pi}{q}]$, we obtain

$$1 - \frac{\lambda^4}{m^4} q^2 - \frac{9}{32} a_1^2 m^2 + \frac{15}{128} a_1^4 m^4 = 0$$

or

$$1 - \frac{q^2}{p^2} - \frac{9}{32} a_1^2 m^2 + \frac{15}{128} a_1^4 m^4 = 0$$

where

$$\frac{m^4}{\lambda^4} = P^2$$

from which we have

$$\frac{q^2}{P^2} = 1 - \frac{9}{32} a_1^2 m^2 + \frac{15}{128} a_1^4 m^4$$

and the natural frequency of the non-linear system will be

$$q^2 = P^2[1 - 2.77 \frac{a_1^2}{\ell^2} + 11.40 \frac{a_1^4}{\ell^4}] \tag{4.23}$$

where the first term exhibits the fundamental frequency of the linearized system. For increasing values of the amplitude, the ratio q/p will always decrease.

4.2.4 Non-Linear Curvature and Non-Linear Stress-Strain Relation

The equation of motion under consideration is

$$L\ (y) = [y^{1v} + \lambda^4 \ddot{y}] - [\frac{3}{2} y^{1v} y'^2 + 9\ y'\ y''\ y''' + 3\ y''^3]$$

$$+ [\frac{15}{8} y^{1v} y'^4 + \frac{45}{2} y'^3\ y''\ y''' + \frac{45}{2} y'^2\ y''^3]$$

$$- K[6\ y''\ y'''^2 + 3^{1v}\ y'^2]$$

$$+ K[27\ y'^2\ y''\ y'''^2 + \frac{27}{2} y'^2\ y''^2 + 63\ y'\ y''^3\ y''' + 9\ y''^5] = 0$$

where

$$K = \frac{E^* J}{EI}, \quad \text{and}$$

$$\lambda^4 = \frac{\rho}{EI}$$

The first four brackets in the above equation have been already evaluated. To evaluate the last bracket, we use the same approximate solution

$$\bar{y}_1 = a_1 \sin mx \cos qt$$

Following Galerkin's procedure, we get

$$\int_0^{\ell} \int_0^{\frac{2\pi}{q}} [(\text{Terms Already Evaluated})]$$

$$+ K[27\, y'^2\, y''\, y'''^2 + \frac{27}{2}\, y'^2\, y^{iv}\, y''^2 + 63\, y'\, y''^3\, y''' + 9\, y''^5] \cdot$$

$$[\sin mx \cos qt]\, dx\, dt = 0$$

or
$$[\qquad\qquad] + \int_0^{\ell}\int_0^{\frac{2\pi}{q}} \Big[K[(-)\ 27(a_1^5\, m^{10} \cos^4 mx \sin mx$$

$$+ \frac{27}{2} a_1^5\, m^{10} \cos^2 mx \sin^3 mx + 63\, a_1^5\, m^{10} \cos^2 mx \sin^3 mx$$

$$- a_1^5\, m^{10} \sin^5 mx] \cos^5 qt \Big] \{\sin mx \cos qt\}\, dx\, dt = 0$$

After computing the integrals, we obtain

$$[\qquad\qquad] + K\, a_1^5\, m^{10} \left[-27 \frac{\ell}{16} + \frac{27}{2} \cdot \frac{\ell}{16} + 63 \cdot \frac{\ell}{16} - \frac{5}{16}\, \ell\right] \frac{\pi}{2q} = 0$$

Dividing the above equation by $[a_1\, m^4\, \ell\, \frac{\pi}{2q}]$, we obtain an additional term, $K\, a_1^4\, m^6\, (\frac{89}{32})$, and the complete frequency equation is

$$1 - \frac{q^2}{P^2} - \frac{9}{32}\, a_1^2\, m^2 + \frac{15}{128}\, a_1^4\, m^4 - \frac{9}{16}\, K\, a_1^2\, m^4 + \frac{89}{32}\, K\, a_1^4\, m^6 = 0$$

The natural frequency of the non-linear system is therefore

$$q^2 = P^2\left[1 - 2.77\, \frac{a_1^2}{\ell^2} + 11.40\, \frac{a_1^4}{\ell^4} - 54.78\, K\, \frac{a_1^2}{\ell^4} + 2672.16\, \frac{a_1^4}{\ell^6}\right] \qquad (4.24)$$

The above frequency equation is but the summation of Equations (4.21), (4.22), and (4.23) with one additional term. The added quantity $[\frac{89}{32}\, K\, a_1^4\, m^6]$ is very small and will not effect the softening behavior of the system.

A numerical example to be given in the Appendix will illustrate the amplitude-frequency relation.

4.3 Cantilever Beam

Let us assume the function

$$\psi_r(x,t) \qquad \text{for} \quad r = 1,2,\ldots$$

satisfying the cantilever beam end conditions; i.e.,

$$\psi_r(o,t) = \psi_r'(o,t) = 0$$

$$\psi_r''(\ell,t) = \psi_r'''(\ell,t) = 0$$

where the fixed end of the cantilever beam is taken at the origin.

The general solution of the linearized equation of motion is

$$\psi_r = [C_1(\sin mx - \sinh mx) + C_2(\cos mx - \cosh mx)]\,[\cos pt]$$

The application of these functions to the non-linear equation requires an elaborate analysis and the problem becomes complicated even with one term approximation. Without losing the general shape of the deflection curve and fulfilling the given boundary conditions, the functions ψ_r can be selected in polynomial form. The choice of ψ_r in polynomial form has the advantage that the process consists in the manipulation of small functions.

A proper set of functions is

$$\psi_r = \frac{(r+2)(r+3)(r+4)}{3(3r+8)}\left(\frac{x}{\ell}\right)^{r+1} - \frac{r(r+3)(r+4)}{2(3r+8)}\left(\frac{x}{\ell}\right)^{r+2} + \frac{r(r+1)(r+2)}{6(3r+8)}\left(\frac{x}{\ell}\right)^{r+4}$$

Another set of functions derived by Duncan [28] is

$$\psi_r = \frac{1}{6}(r+1)(r+3)\ (\frac{x}{\ell})^{r+1} - \frac{1}{3}\ r(r+3)\ (\frac{x}{\ell})^{r+2} + \frac{1}{6}\ r(r+1)\ (\frac{x}{\ell})^{r+3} \qquad (4.26)$$

Similar sets of functions can easily be derived, once the terms in the linearized solution are expanded into power series. Some advance knowledge of the expected answer of applying the above sets of functions would favor the selection of sets (4.25). In fact, if we restricted ourselves to ψ_1, then Equation (4.25) makes a good choice. To obtain the same results from Equation (4.26) both ψ_1, and ψ_2 are required.

4.3.1 Linearized Curvature and Linearized Stress-Strain Relation

Let us first consider r = 1, [29]; then Equation (4.25) becomes

$$\psi_1(x) = \frac{20}{11}\ (\frac{x}{\ell})^2 - \frac{10}{11}\ (\frac{x}{\ell})^3 + \frac{1}{11}\ (\frac{x}{\ell})^5 \qquad (4.27)$$

whereas Equation (4.26) yields

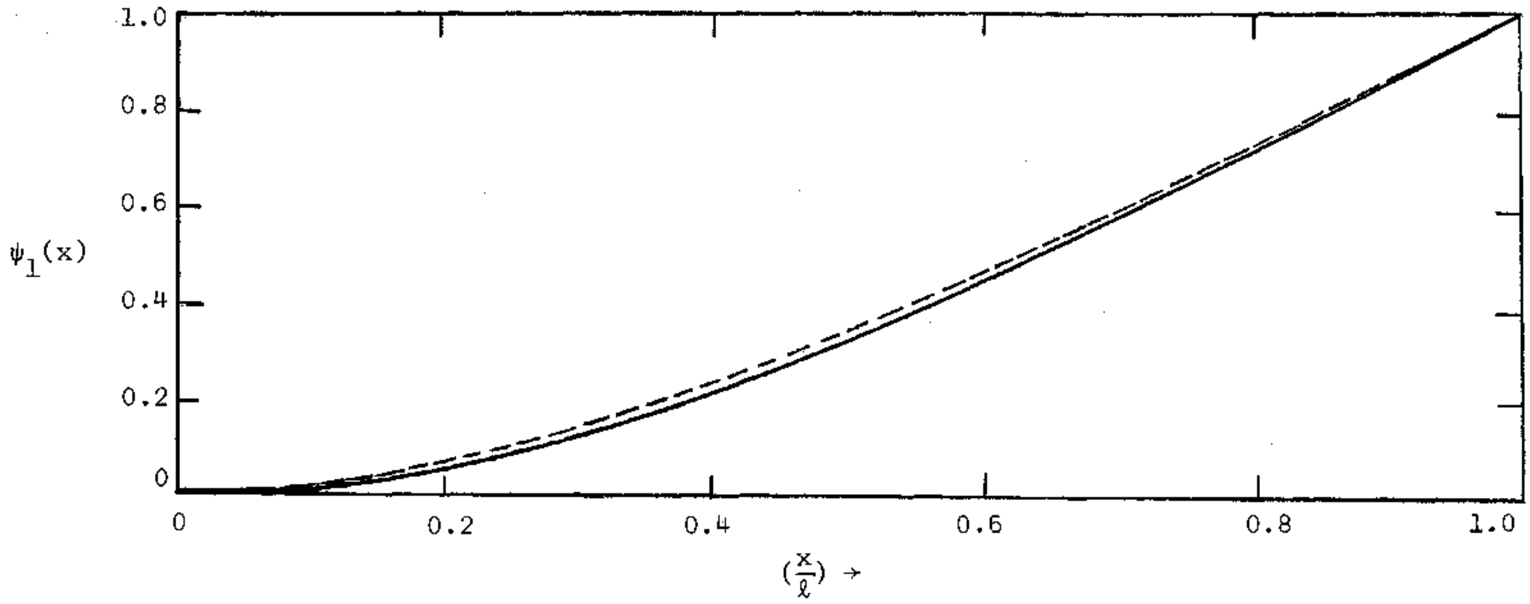
$$\psi_1(x) = 2(\frac{x}{\ell})^2 - \frac{4}{3}\ (\frac{x}{\ell})^3 + \frac{1}{3}\ (\frac{x}{\ell})^4 \qquad (4.28)$$

A plot of these two displacement functions is shown in Figure 10, where it can be seen that they differ only slightly.

Let us now apply Equation (4.27) to the linearized differential equation of motion. The approximate solution is now:

$$\bar{y}_1 = a_1 \cos pt\ [\frac{20}{11}\ (\frac{x}{\ell})^2 - \frac{10}{11}\ (\frac{x}{\ell})^3 + \frac{1}{11}\ (\frac{x}{\ell})^5\ ]$$

Substituting $\bar{y}_1$ into Equation (2.15), Galerkin's condition yields



——— $\psi_1(x) = (\frac{20}{11})\ (\frac{x}{\ell})^2 - (\frac{10}{11})\ (\frac{x}{\ell})^3 + (\frac{1}{11})\ (\frac{x}{\ell})^5$

- - - - $\psi_1(x) = 2\ (\frac{x}{\ell})^2 - \frac{4}{3}\ (\frac{x}{\ell})^3 + \frac{1}{3}\ (\frac{x}{\ell})^4$

Figure 10. Graphical Presentation of Two Displacement Functions for a Cantilever Beam.

$$\int_0^{\ell}\int_0^{\frac{2\pi}{P}} \left[\frac{120}{11}\,\frac{x}{\ell^5} - \lambda^4\,P^2[\frac{20}{11}\,(\frac{x}{\ell})^2 - \frac{10}{11}\,(\frac{x}{\ell})^3 + \frac{1}{11}\,(\frac{x}{\ell})^5\,]\right]\,(a_1\cos\,pt)$$

$$\left[\cos\,pt\,[\frac{20}{11}\,(\frac{x}{\ell})^2 - \frac{10}{11}\,(\frac{x}{\ell})^3 + \frac{1}{11}\,(\frac{x}{\ell})^5\,]\right]\,dx\,dt = 0$$

An evaluation of the displacement terms results in

$$\int_0^{\frac{2\pi}{P}} [3.116 - \lambda^4\,P^2(0.2520)\ell^4]\,a_1\,\cos^2\,pt\,dt = 0$$

Evaluating the time functions, the result is

$$3.116 - 0.252\,\lambda^4\,P^2\,\ell^4 = 0 \tag{4.30}$$

or

$$P^2 = \frac{12.3650}{\ell^4}\cdot\frac{EI}{\rho}$$

The exact value of the lowest natural frequency for the cantilever beam [30] is

$$P^2\,\lambda^4\,\ell^4 = 12.3624$$

Let us now apply Duncan's functions using only ψ_1, Equation (4.28). The approximate solution is

$$\bar{y}_1 = a_1\,\cos\,pt\,[2\,(\frac{x}{\ell})^2 - \frac{4}{3}\,(\frac{x}{\ell})^3 + \frac{1}{6}\,(\frac{x}{\ell})^4] \tag{4.31}$$

Substituting Equation (4.31) into the differential equation of motion (2.15), and applying the orthogonality condition results in

$$P^2 \lambda^4 \ell^4 = 12.4659 \tag{4.32}$$

which compares with the exact value of

$$P^2 \lambda^4 \ell^4 = 12.3624$$

<u>Two-Term Approximations n = 2</u>. A two-term approximation yields

$$\psi_1 = \cos pt \left[\frac{20}{11} \left(\frac{x}{\ell}\right)^2 - \frac{10}{11} \left(\frac{x}{\ell}\right)^3 + \frac{1}{11} \left(\frac{x}{\ell}\right)^5\right] \tag{4.33}$$

$$\psi_2 = \cos pt \left[\frac{20}{7} \left(\frac{x}{\ell}\right)^3 - \frac{15}{7} \left(\frac{x}{\ell}\right)^4 + \frac{2}{7} \left(\frac{x}{\ell}\right)^6\right]$$

and the approximate solution is given by

$$\bar{y}_2 = a_1 \psi_1 + a_2 \psi_2$$

Substituting the expression for $\bar{y}_2$ into the system

$$L (y) = y^{iv} + \lambda^4 \ddot{y} = 0$$

we obtain the Galerkin conditions

$$\int_{0}^{\ell} \int_{0}^{\frac{2\pi}{P}} L(\bar{y}_2)\ \psi_1\ dx\ dt = 0\ , \quad \text{and}$$

$$\int_{0}^{\ell} \int_{0}^{\frac{2\pi}{P}} L(\bar{y}_2)\ \psi_2\ dx\ dt = 0$$

Two homogeneous equations are obtained and the roots of the compatibility equation yield

$$P^2\ \lambda^4\ \ell^4 = -\ 68.883952 \mp 81.246318$$

The lowest root is $P^2\ \lambda^4\ \ell^4 = 12.362326$, or

$$P^2 = \frac{12.362326}{\ell^4} \cdot \frac{EI}{\rho} \tag{4.34}$$

compared to the exact value of

$$P^2 = \frac{12.3624}{\ell^4} \cdot \frac{EI}{\rho}$$

Duncan's functions with the two-term approximations

$$\psi_1 = 2\left(\frac{x}{\ell}\right)^2 - \frac{4}{3}\left(\frac{x}{\ell}\right)^3 + \frac{1}{3}\left(\frac{x}{\ell}\right)^4 \tag{4.35}$$

$$\psi_2 = \frac{10}{3}\left(\frac{x}{\ell}\right)^3 - \frac{10}{3}\left(\frac{x}{\ell}\right)^4 + \left(\frac{x}{\ell}\right)^5$$

presents the approximate solution

$$\bar{y}_2 = a_1 \cos pt \, [2 \, (\tfrac{x}{\ell})^2 - \tfrac{4}{3} \, (\tfrac{x}{\ell})^3 + \tfrac{1}{3} \, (\tfrac{x}{\ell})^4 \,]$$

$$+ \, a_2 \cos pt \, [\tfrac{10}{3} \, (\tfrac{x}{\ell})^3 - \tfrac{10}{3} \, (\tfrac{x}{\ell})^4 + (\tfrac{x}{\ell})^5 \,]$$

Substituting $\bar{y}_2$ into the equation of motion, Galerkin's conditions yield

$$\int_0^{\ell} \int_0^{\frac{2\pi}{P}} L(\bar{y}_2) \, \psi_1 \, dx \, dt = 0 \text{ , and}$$

$$\int_0^{\ell} \int_0^{\frac{2\pi}{P}} L(\bar{y}_2) \, \psi_2 \, dx \, dt = 0$$

After computing the integrals, we get the roots

$$P^2 \lambda^4 \ell^4 = 234.00 \pm 221.6375$$

where the lowest root exhibits the value

$$P^2 = \frac{12.3625}{\ell^4} \cdot \frac{EI}{\rho} \qquad (4.36)$$

whereas the exact value of the lowest root is

$$P^2 = \frac{12.3624}{\ell^4} \cdot \frac{EI}{\rho}$$

Once the roots are known, the ratio a_1 to a_2 can be determined.

Comparing the results of P^2 obtained for one and two-term approximations, we have:

Exact Value of $P^2\ \lambda^4\ \ell^4$	= 12.3624
Equation (4.25), One-Term Approximation	= 12.3650
Equation (4.26), One-Term Approximation	= 12.4659
Equation (4.25), Two-Term Approximations	= 12.3623
Equation (4.26), Two-Term Approximations	= 12.3625

From this we conclude that Equation (4.25) offers a good approximation.

4.3.2 Linearized Curvature and Non-Linear Stress-Strain Relation

Neglecting the term $(\frac{dy}{dx})^2$ in the curvature formula, and retaining the non-linear stress-strain relation, the equation of motion becomes

$$L\ (y) = y^{iv} + \lambda^4\ \ddot{y} - K[6\ y''\ y'''^2 + 3\ y''^2\ y^{iv}] = 0 \qquad (2.16)$$

The assumed approximate solution is

$$\bar{y}_1 = a_1 \cos qt\ [\frac{20}{11}\ (\frac{x}{\ell})^2 - \frac{10}{11}\ (\frac{x}{\ell})^3 + \frac{1}{11}\ (\frac{x}{\ell})^5\ ]$$

Substituting $\bar{y}_1$ into Equation (2.16) and applying the orthogonality condition, the result is

$$\int_0^\ell \int_0^{\frac{2\pi}{q}} \Big[[120x - \lambda^4\ P^2\ (20\ell^3\ x^2 - 10\ell^2\ x^3 + x^5)]\ [\frac{a_1 \cos qt}{11\ell^5}]$$

$$- 6K\,[(40\ell^3 - 60\ell^2 x + 20x^3)(-60\ell^2 + 60x^2)^2$$

$$\left. + \frac{1}{2}(40\ell^3 - 60\ell^2 x + 20x^3)(120x)]\,[\frac{a_1 \cos qt}{11\,\ell^5}]^3 \right] \cdot$$

$$\left[[\frac{\cos qt}{11\,\ell^5}]\,[20\ell^3 x^2 - 10\ell^2 x^3 + x^5] \right] dx\,dt = 0$$

An evaluation of the displacement terms results in

$$\int_0^{\frac{2\pi}{q}} \left[[3.116 - 0.252\,\lambda^4 P^2 \ell^4]\,\frac{a_1}{\ell^3}\cos^2 qt \right.$$

$$\left. - 6K[4.91 + 0.165]\,\frac{a_1^3}{\ell^7}\cos^4 qt \right] dt = 0$$

Evaluating the time functions, we arrive at

$$[3.116 - 0.252\,q^2 \lambda^4 \ell^4]\,\frac{a_1}{\ell^3}\,(\frac{\pi}{q}) - (30.450)K\,\frac{a_1^3}{\ell^7}\,(\frac{3}{4}\,\frac{\pi}{q}) = 0$$

Dividing the above equation by $[3.116\,\frac{a_1}{\ell^3}\,(\frac{\pi}{q})]$, we get

$$1 - \frac{0.252}{3.116}\,q^2 \ell^4 \lambda^4 - 7.32K\,\frac{a_1^2}{\ell^4} = 0$$

from which we obtain the natural frequency of the non-linear system

$$q^2 = \frac{12.36}{\ell^4 \lambda^4}\,[1 - 7.32K\,\frac{a_1^2}{\ell^4}] \text{ , } \quad \text{or}$$

$$q^2 = P^2[1 - 7.32K \frac{a_1^2}{\ell^4}] \tag{4.37}$$

which for K = 0 presents the linearized result. For increasing values of the amplitude, the frequency decreases.

4.3.3 Non-Linear Curvature and Linearized Stress-Strain Relation

Following the same procedure used in the previous section, we have the approximate solution

$$\bar{y}_1 = \frac{a_1}{11\ell^5} [\cos qt] [20\ell^3 x^2 - 10\ell^2 x^3 + x^5]$$

The Galerkin condition corresponding to this solution is

$$\int_0^\ell \int_0^{\frac{2\pi}{q}} \Big[y^{iv} + \lambda^4 \ddot{y} - (\frac{3}{2} y^{iv} y'^2 + 9 y' y'' y''' + 3y''^3)$$

$$+ (\frac{15}{8} y^{iv} y'^4 + \frac{45}{2} y'^3 y''' y'' + \frac{45}{2} y'^2 y''^3) \Big] \cdot$$

$$\left[\frac{\cos qt}{11\ell^5} (20\ell^3 x^2 - 10\ell^2 x^3 + x^5)\right] dx\, dt = 0$$

An evaluation of the displacement functions results in

$$\int_0^{\frac{2\pi}{q}} \Big[[1 - \frac{q^2 \lambda^4 \ell^4}{12.36}] \frac{a_1}{\ell^3} (3.116) \cos^2 qt$$

$$- [\frac{3}{2} (5.393) + 9(0.4343) + 3(9.2234)] \frac{a_1^3}{\ell^5} \cos^4 qt$$

$$+ \left[\frac{15}{8} (93.14) + \frac{45}{2} (393.89) + \frac{45}{2} (-295.96)\right] \frac{a_1^5}{\ell^7} \cos^6 qt \Bigg] dt$$

$$= 0$$

When time functions are evaluated, we arrive at

$$\left[1 - \frac{q^2}{p^2}\right] \left[\frac{a_1}{\ell^3} (3.116) \frac{\pi}{q}\right] - [39.6684] \left[\frac{a_1^3}{\ell^5} \left(\frac{3}{4} \frac{\pi}{q}\right)\right]$$

$$+ [2288.05] \left[\frac{a_1^5}{\ell^7} \left(\frac{\pi}{2q}\right)\right] = 0$$

Dividing the above equation by $\left[\frac{a_1}{\ell^3} (3.116) \left(\frac{\pi}{q}\right)\right]$ we obtain

$$1 - \frac{q^2}{p^2} = 12.73 \frac{a_1^2}{\ell^2} - 367.14 \frac{a_1^4}{\ell^4}$$

from which we have

$$\frac{q^2}{p^2} = 1 - 12.73 \frac{a_1^2}{\ell^2} + 367.14 \frac{a_1^4}{\ell^4} \qquad (4.38)$$

Again the frequency is lower the larger the amplitude.

4.3.4 Non-Linear Curvature and Non-Linear Stress-Strain Relation

Applying the same approximate solution already derived

$$\bar{y}_1 = \frac{a_1}{11\ell^5} \cos qt \, [20\ell^3 x^3 - 10\ell^2 x^3 + x^5]$$

and making use of Galerkin's variational method, we arrive at

$$\int_0^{\ell} \int_0^{\frac{2\pi}{q}} \text{[Terms already evaluated]}$$

$$+ K[27\, y'^2\, y''\, y'''^2 + \frac{27}{2}\, y'^2\, y^{iv}\, y''^2 + 63\, y'\, y''^3\, y''' + 9\, y''^5] \,.$$

$$[\frac{\cos qt}{11\ell^5}\,(20\ell^3\, x^2 - 10\ell^2\, x^3 + x^5)]\; dx\; dt = 0$$

Here the additional term is $K[1449.40]\,\frac{a_1^5}{\ell^9}\,(\frac{\pi}{2q})$ which, if divided by $[\frac{a_1}{\ell^3}\,(3.116)\,\frac{\pi}{q}]$ results in

$$232.57\; K\, \frac{a_1^4}{\ell^6}$$

Thus, the complete frequency equation will be

$$1 - \frac{q^2}{p^2} - 12.73\, \frac{a_1^2}{\ell^2} + 367.14\, \frac{a_1^4}{\ell^4} - 7.32\; K\, \frac{a_1^2}{\ell^4}$$

$$+ 232.57\; K\, \frac{a_1^4}{\ell^6} = 0 \;, \quad \text{or}$$

$$\frac{q^2}{p^2} = 1 - 12.73\, \frac{a_1^2}{\ell^2} + 367.14\, \frac{a_1^4}{\ell^4} \tag{4.39}$$

$$- 7.32\; K\, \frac{a_1^2}{\ell^4} + 232.57\; K\, \frac{a_1^4}{\ell^6} \;.$$

CHAPTER V

RESULTS AND CONCLUSIONS

5.1 Results

The frequency equations obtained in Chapter IV are tabulated below. Response curves for the simply supported beam and the cantilever beam are shown in Figures 11 and 12. These curves are plotted from the numerical example given in the Appendix.

An evaluation of the results reveals the following:

1. The natural frequency of the non-linear system depends on the amplitude.

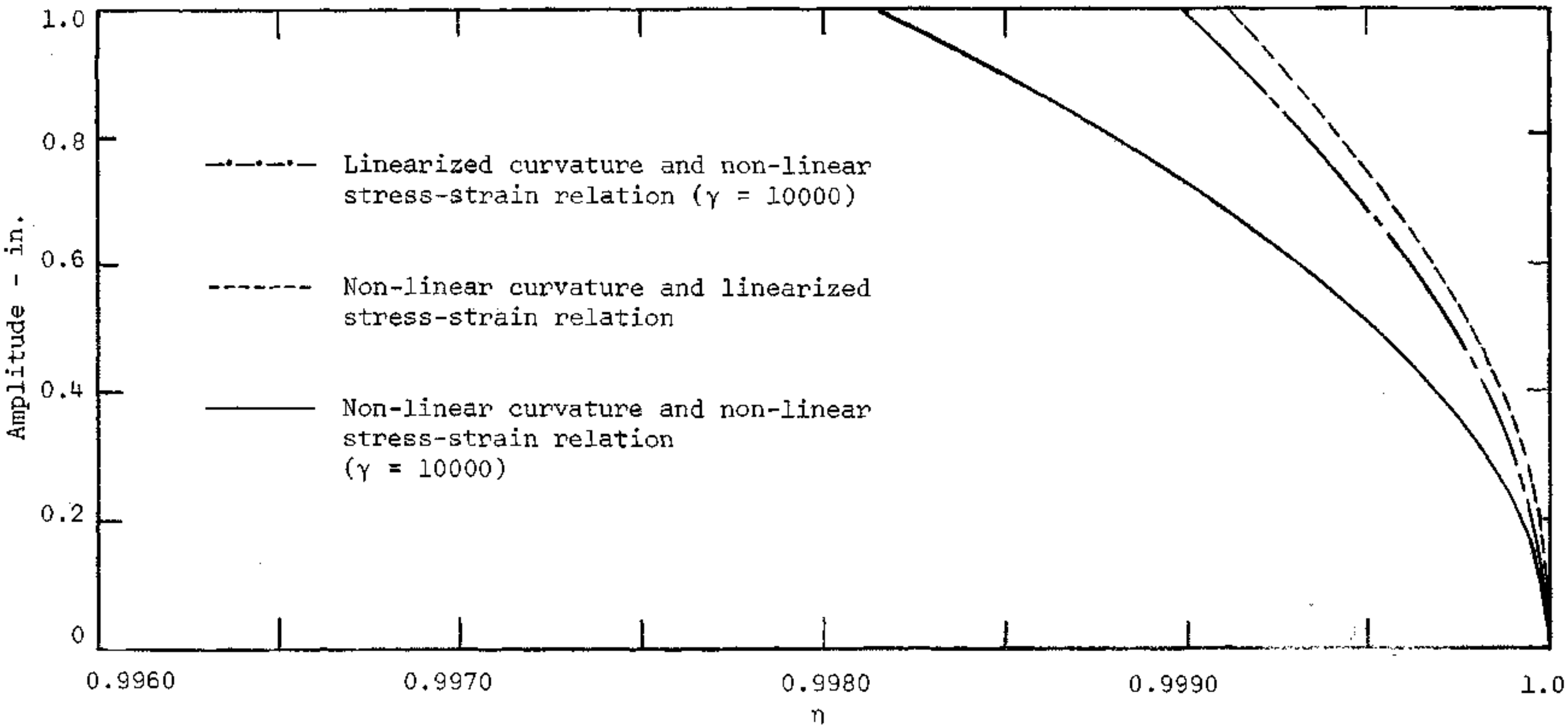
2. The observed behavior of the system exhibits many of the well-known characteristics of a soft spring.

3. For a cantilever, the effect of stress-strain non-linearity is less than the effect of the added non-linear terms connected with the curvature. The curvature has a great effect in determining the shape of the frequency curves whereas the assumed stress-strain relation has a small effect.

4. For simply supported beams, the situation is different. The influence of stress-strain non-linearity is larger than that of the curvature.

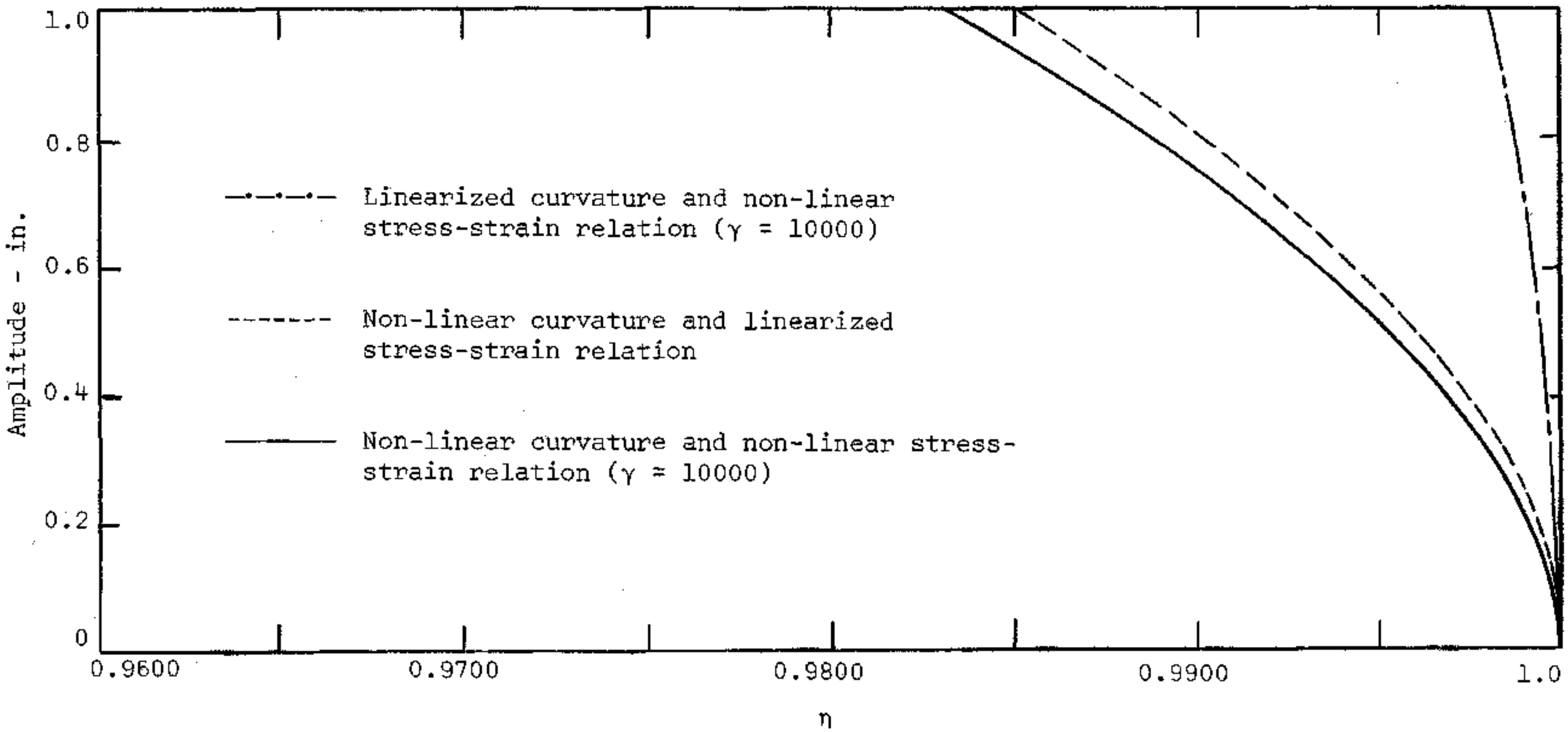
5. The influence of curvature non-linearity on the frequency has a greater effect on the cantilever beam than it does on the simply supported beam.

	Simply Supported Beam	Cantilever Beam
Linearized Curvature and Linearized Stress-Strain Relation	$P^2 = \frac{97.37}{\ell^4 \lambda^4}$	$P^2 = \frac{12.36}{\ell^4 \lambda^4}$
Linearized Curvature and Non-Linear Stress-Strain Relation	$\eta^2 = 1 - 54.78\ K \frac{a_1^2}{\ell^4}$	$\eta^2 = 1 - 7.32\ K \frac{a_1^2}{\ell^4}$
Non-Linear Curvature and Linearized Stress-Strain Relation	$\eta^2 = 1 - 2.77 \frac{a_1^2}{\ell^2} + 11.40 \frac{a_1^4}{\ell^4}$	$\eta^2 = 1 - 12.73 \frac{a_1^2}{\ell^2} + 367.14 \frac{a_1^4}{\ell^4}$
Non-Linear Curvature and Non-Linear Stress-Strain Relation	$\eta^2 = 1 - 2.77 \frac{a_1^2}{\ell^2} + 11.40 \frac{a_1^4}{\ell^4} - 54.78\ K \frac{a_1^2}{\ell^4} + 2672.16\ K \frac{a_1^4}{\ell^6}$	$\eta^2 = 1 - 12.73 \frac{a_1^2}{\ell^2} + 367.14 \frac{a_1^4}{\ell^4} - 7.32\ K \frac{a_1^2}{\ell^4} + 232.57\ K \frac{a_1^4}{\ell^6}$
$\eta = \frac{q}{P}$	$\lambda^4 = \frac{\rho}{EI}$	a_1 Arbitrary Amplitude $K = \frac{E^* J}{EI}$



Beam Length 40 in. Cross-Sectional Dimensions $\frac{1"}{4}$ x $\frac{1"}{4}$

Figure 11. Response Curves for the Simply Supported Beam.



Beam Length 20 in. Cross-Sectional Dimensions $\frac{1''}{4} \times \frac{1''}{4}$

Figure 12. Response Curves for the Cantilever Beam.

6. The effect of fifth order terms in the non-linear differential equation of motion resulted in a very small contribution to the frequency deviations.

5.2 Conclusions

1. By a simple approximate variational method, one is able to determine the non-linear natural frequency of the system.

2. The behavior of beams in lateral bending vibrations exhibits a softening characteristic; indicating a decrease of the natural frequency with increasing amplitude.

3. The resulting non-linearity due to curvature and stress-strain relation has a favorable influence on the frequency-amplitude curve in that it limits the infinite amplitude predicted by the linear theory.

4. No experimental verification of the results is available. Presently very little experimental data exists on the subject.

APPENDIX

APPENDIX

A NUMERICAL EXAMPLE

In order to illustrate the relation between the natural frequency and the arbitrarily chosen amplitude, let us assume the following data:

Beam Cross-Sectional Dimensions	0.25" x 0.25"
Beam Length:	
Simply Supported	40"
Cantilever	20"
Specific Weight of Beam	490 lbs/c ft
Mechanical Properties:	
Tensile Strength	60×10^3 psi
Compression Strength	60×10^3 psi
Modulus of Elasticity	30×10^6 psi

Stress-strain curve is shown in Figure 13.

From the above data, the following is calculated:

Beam Cross-Sectional Area	0.0625 sq in
Weight of Beam Per Unit Length	0.01771875 lbs/in
Mass of Beam per Unit Length	0.00004585 lb $\sec^2/\text{in}^2$

$$I = \int Z^2 \, dA = \frac{(0.25)^4}{12} = \quad 0.00032552 \text{ in}^4$$

$$J = \int Z^4 \, dA = \frac{(0.25)^6}{80} = \quad 0.00000305 \text{ in}^6$$

$$\lambda^4 = \frac{\rho}{EI} = \qquad 0.4695 \times 10^{-8}\ \sec^2/\text{in}^4$$

To calculate γ, let us assume the stress-strain curve shown below.

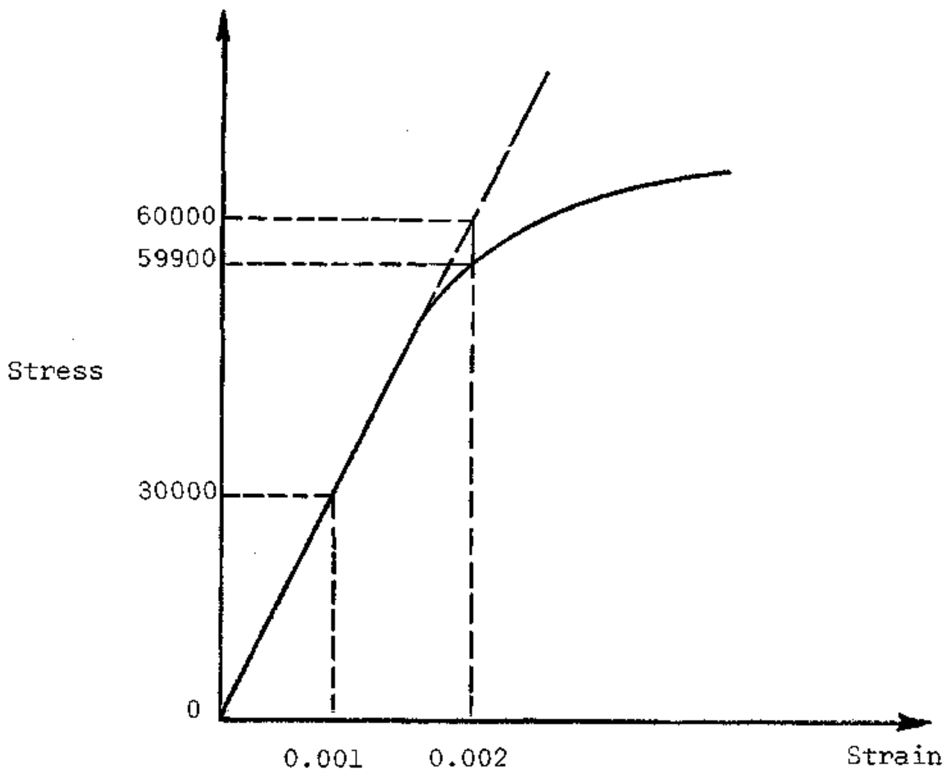


Figure 13. Stress-Strain Diagram

$$\sigma = E(\varepsilon - \gamma\varepsilon^3)\ , \quad \text{or}$$

$$59900 = 30 \times 10^6\ [0.002 - \gamma(0.002)^3]$$

From which we obtain $\gamma = 555.66$

For $\gamma = 555.66$, we have

$$K = \frac{E^{*}J}{EI} = \frac{E\gamma J}{EI} = \frac{\gamma J}{I}$$

$$= 5.206$$

For this study, we are going to consider various values of γ. For each value of γ, K is determined.

γ	K
0	0
500	4.6848
1000	9.3696
5000	46.8481
10000	93.6962

Calculation of P

1. Simply Supported Beam

$$P^2 = \frac{97.39}{\ell^4 \lambda^4} = 8102$$

$$P = 90 \text{ rad. per sec.}$$

2. Cantilever Beam

$$P^2 = \frac{12.36}{\ell^4 \; \lambda^4} = 16453$$

$$P = 128.3 \text{ rad.per sec.}$$

Calculation of q

The next few pages cover the procedure for determining the ratio q to P for the simply supported beam and the cantilever beam.

Simply Supported Beam Linearized Curvature and Non-Linear Stress-Strain Relation

Amplitude a_1	$\frac{a_1^2}{\ell^4}$	$54.78\ K\ \frac{a_1^2}{\ell^4}$			
		γ = 500	1000	5000	10000
in.	x 10^{-8}	x 10^{-8}			
0	0	0	0	0	0
0.2	1.56	400	801	4000	8010
0.4	6.25	1603	3207	16030	32070
0.6	14.06	3608	7217	36080	72170
0.8	25.00	6415	12831	64150	128310
1.0	39.06	10024	20049	100240	200490

Simply Supported Beam
Linearized Curvature and Non-Linear Stress-Strain Relation

Amplitude	η^2			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99999600	0.99999199	0.99995991	0.99991981
0.4	0.99998397	0.99996793	0.99983961	0.99967921
0.6	0.99996392	0.99992783	0.99963911	0.99927822
0.8	0.99993585	0.99987169	0.99935842	0.99871684
1.0	0.99989976	0.99979951	0.99899753	0.99799506

Amplitude	η			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99999799	0.99999599	0.99997995	0.99995990
0.4	0.99999198	0.99998396	0.99991980	0.99983959
0.6	0.99998329	0.99996391	0.99982048	0.99963904
0.8	0.99996792	0.99993584	0.99967915	0.99935821
1.0	0.99994987	0.99989974	0.99950172	0.99899702

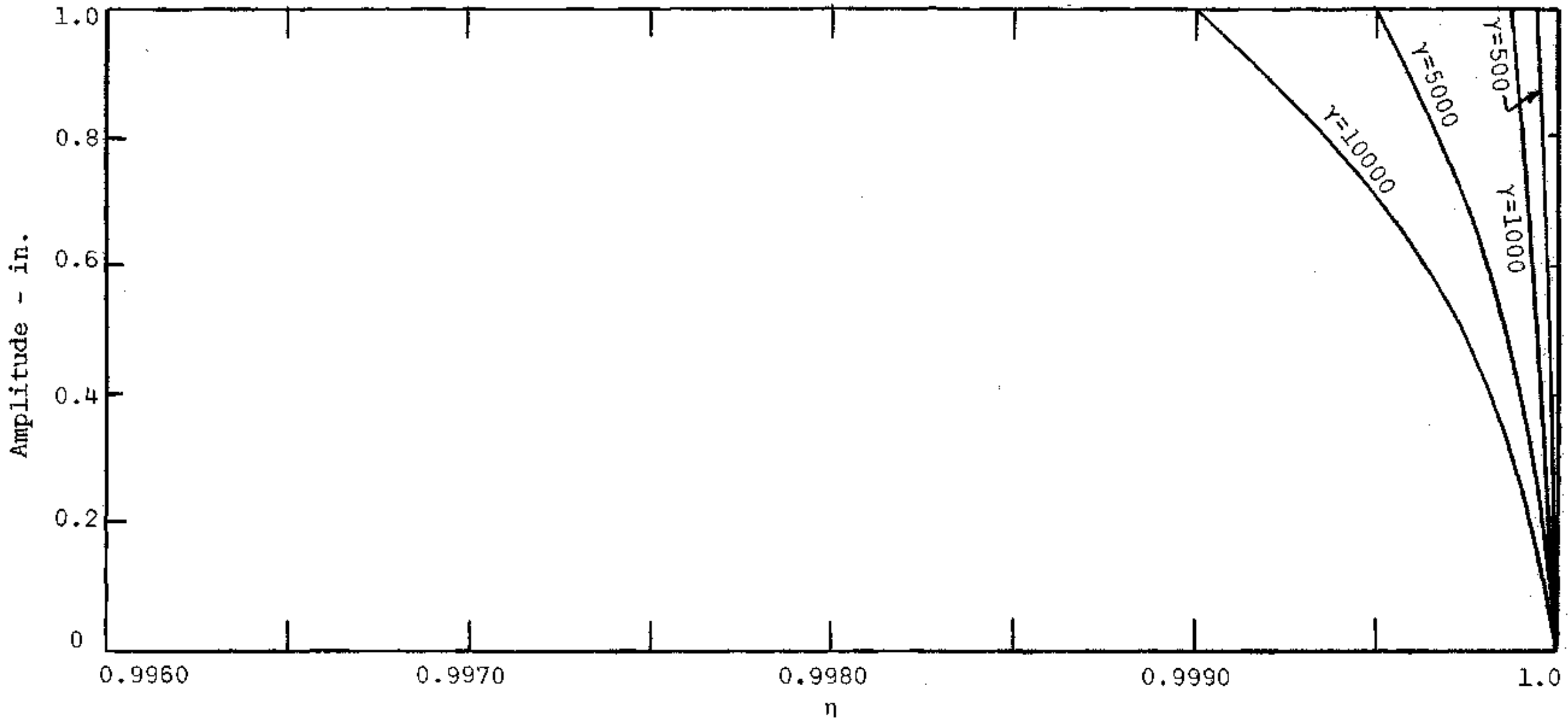


Figure 14. Simply Supported Beam--Response Curves for Linearized Curvature and Non-Linear Stress-Strain Relation.

Simply Supported Beam
Non-Linear Curvature and Linearized Stress-Strain Relation

Amplitude a_1	$\frac{a_1^2}{\ell^2}$	$\frac{a_1^4}{\ell^4}$	$2.77 \frac{a_1^2}{\ell^2}$	$11.40 \frac{a_1^4}{\ell^4}$	$-2.77 \frac{a_1^2}{\ell^2} + 11.40 \frac{a_1^4}{\ell^4}$
in.	$\times 10^{-4}$	$\times 10^{-8}$	$\times 10^{-8}$	$\times 10^{-8}$	$\times 10^{-8}$
0	0	0	0	0	0
0.2	0.25	0.0625	6925	0.7	6924
0.4	1.00	1.0000	27700	11.4	27689
0.6	2.25	5.0600	62325	57.6	62268
0.8	4.00	16.0000	110800	182.4	110618
1.0	6.25	39.1015	173125	445.7	172680

Amplitude	η^2	η
in.		
0	1.0	1.0
0.2	0.99993076	0.99996535
0.4	0.99972311	0.99986154
0.6	0.99937732	0.99968861
0.8	0.99889382	0.99944675
1.0	0.99827320	0.99913622

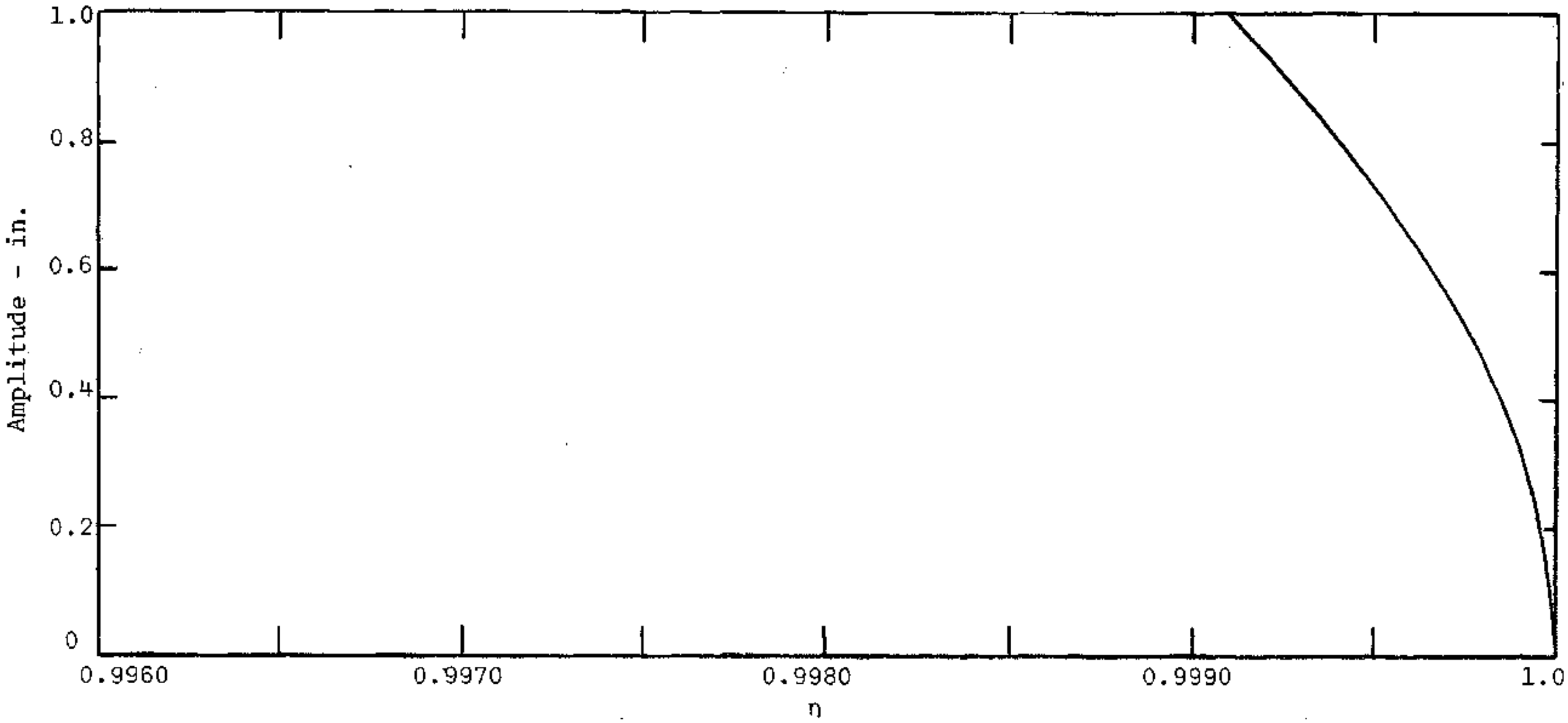


Figure 15. Simply Supported Beam--Response Curve for Non-Linear Curvature and Linearized Stress-Strain Relation. (γ=0)

Simply Supported Beam
Non-Linear Curvature and Non-Linear Stress-Strain Relation

Amplitude a_1	$\frac{a_1^4}{\ell^6}$	$2672.16\ K\ \frac{a_1^4}{\ell^6}$			
		γ = 500	1000	5000	10000
in.	x 10^{-10}	x 10^{-8}			
0	0	0	0	0	0
0.2	0.0039	0.48	0.97	4.86	9.72
0.4	0.0625	7.82	15.64	78.24	156.48
0.6	0.3164	39.60	79.21	396.06	792.13
0.8	1.0000	125.18	250.37	1251.85	2503.70
1.0	2.4414	305.62	611.25	3056.25	6112.51

Amplitude	$\sum \{-2.77\ \frac{a_1^2}{\ell^2} + 11.40\ \frac{a_1^4}{\ell^4} - 58.78K\ \frac{a_1^2}{\ell^4} + 2672.16K\ \frac{a_1^4}{\ell^6}\}$			
	γ = 500	1000	5000	10000
in.	x 10^{-8}			
0	0	0	0	0
0.2	7323	7724	10920	14925
0.4	29285	30881	43641	59603
0.6	65837	69406	97952	113638
0.8	116908	123199	173517	236425
1.0	182399	192118	269864	367058

Simply Supported Beam
Non-Linear Curvature and Non-Linear Stress-Strain Relation

Amplitude	η^2			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99992677	0.99992276	0.99989080	0.99985075
0.4	0.99970715	0.99969119	0.99956359	0.99940397
0.6	0.99934163	0.99930594	0.99902048	0.99886362
0.8	0.99883092	0.99876801	0.99826483	0.99763575
1.0	0.99817601	0.99807882	0.99730136	0.99632942

Amplitude	η			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99996338	0.99996137	0.99994539	0.99992537
0.4	0.99985356	0.99984558	0.99978177	0.99970194
0.6	0.99967076	0.99965290	0.99951012	0.99947851
0.8	0.99941528	0.99938381	0.99913203	0.99881717
1.0	0.99908758	0.99903894	0.99864976	0.99816302

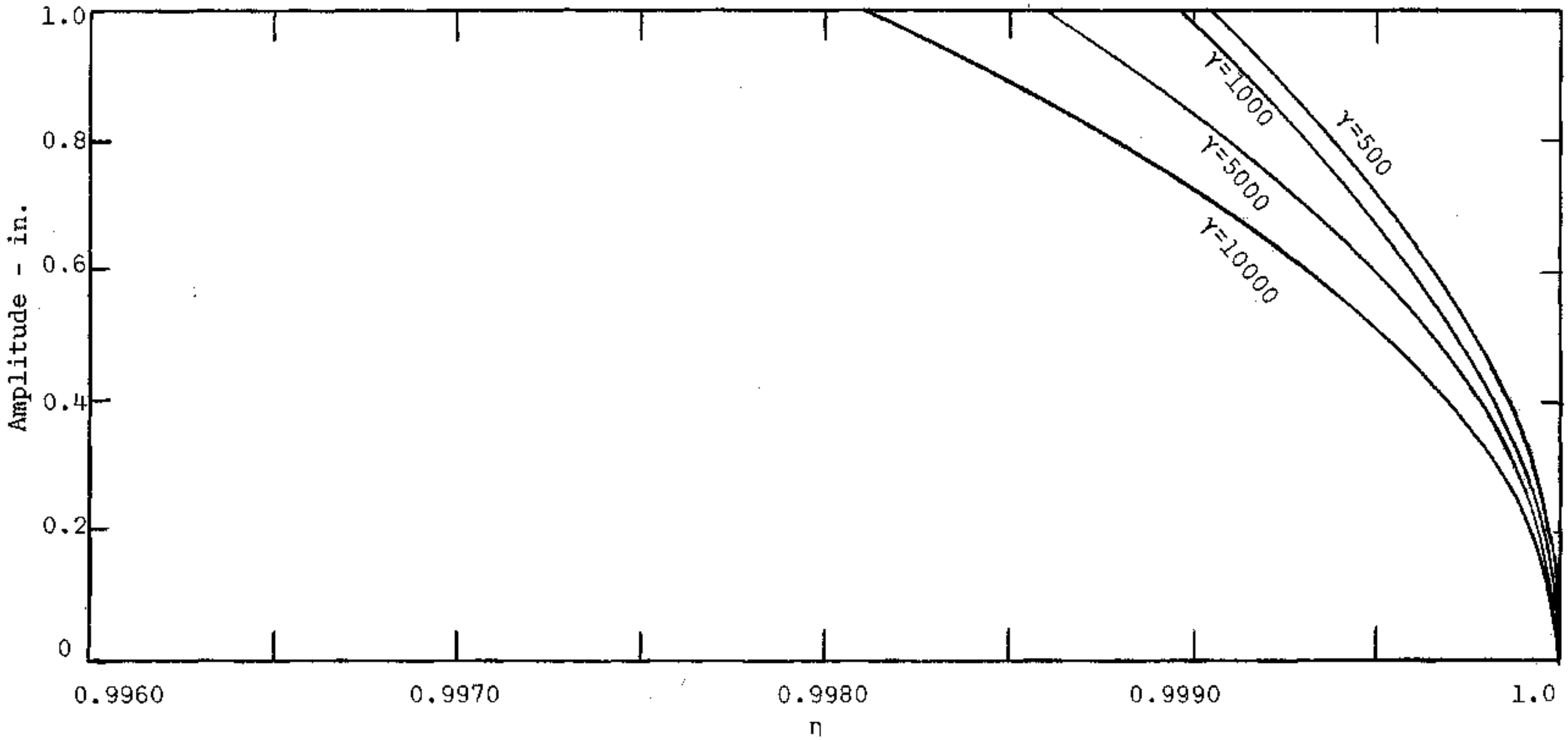


Figure 16. Simply Supported Beam--Response Curves for Non-Linear Curvature and Non-Linear Stress-Strain Relation.

Cantilever Beam
Linearized Curvature and Non-Linear Stress-Strain Relation

Amplitude a	$\frac{a_1^2}{\ell^4}$	$7.32\ K\ \frac{a_1^2}{\ell^4}$			
		$\gamma = 500$	1000	5000	10000
in.	$\times 10^{-6}$	$\times 10^{-6}$			
0	0	0	0	0	0
0.2	0.25	8.5731	17.1462	85.7310	171.4620
0.4	1.00	34.2927	68.5854	342.9270	685.8540
0.6	2.25	77.1586	154.3172	771.5860	1543.1720
0.8	4.00	137.1709	274.3418	1371.7090	1543.1720
1.0	6.25	214.3294	428.6588	2143.2940	4286.5880

Amplitude	η^2			
	$\gamma = 500$	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99999143	0.99998286	0.99991427	0.99982854
0.4	0.99996571	0.99993142	0.99965708	0.99931415
0.6	0.99992285	0.99984569	0.99922842	0.99845683
0.8	0.99986283	0.99972566	0.99862830	0.99725659
1.0	0.99978568	0.99957135	0.99785671	0.99571342

Cantilever Beam
Linearized Curvature and Non-Linear Stress-Strain Relation

Amplitude	η			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99999571	0.99999142	0.99999245	0.99991426
0.4	0.99998285	0.99996570	0.99982852	0.99965701
0.6	0.99996142	0.99992284	0.99961525	0.99922843
0.8	0.99992885	0.99986282	0.99931430	0.99863439
1.0	0.99989283	0.99978565	0.99893053	0.99785440

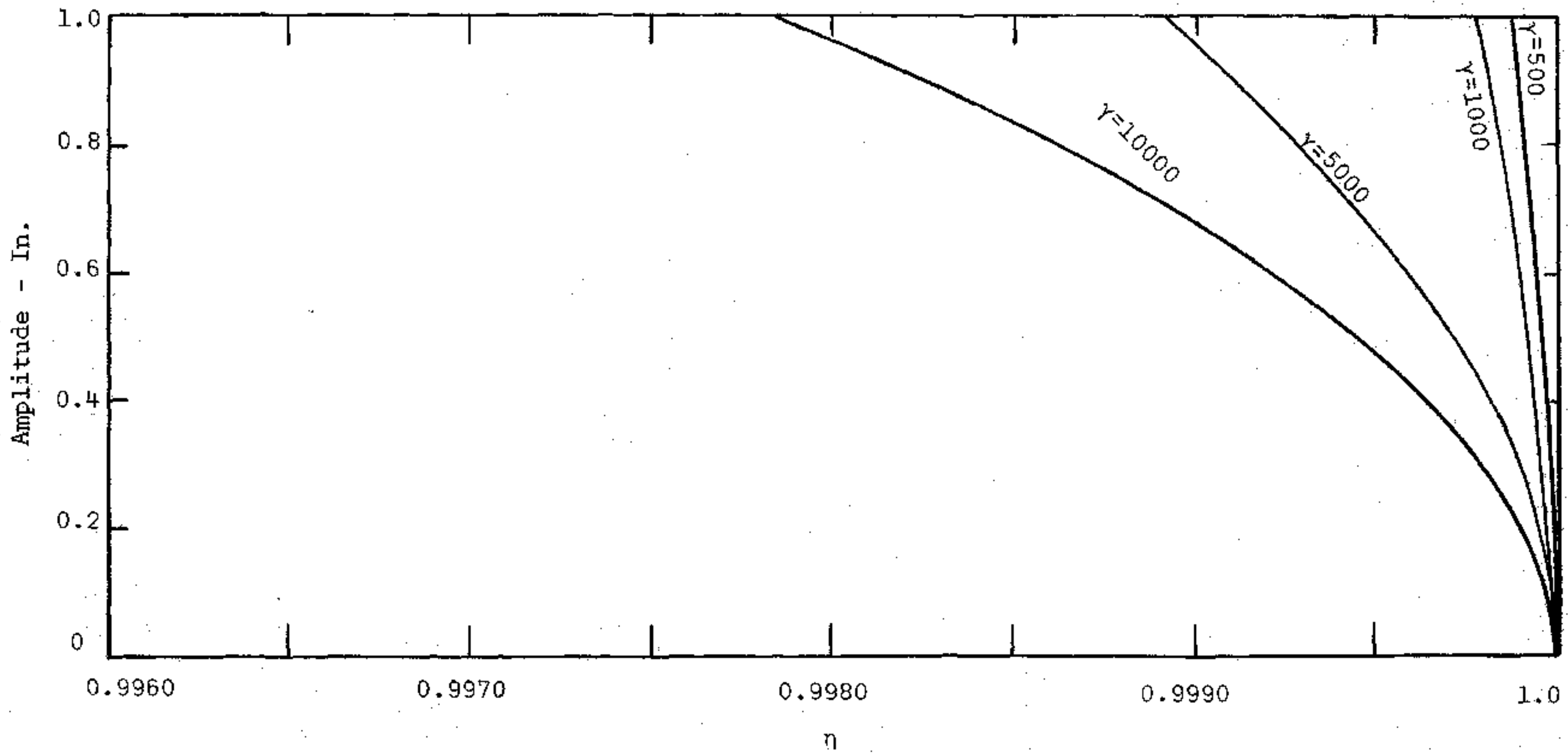


Figure 17. Cantilever Beam--Response Curves for Linearized Curvature and Non-Linear Stress-Strain Relation.

Cantilever Beam
Non-Linear Curvature and Linear Stress-Strain Relation

Amplitude a_1	$\frac{a_1^2}{\ell^2}$	$\frac{a_1^4}{\ell^4}$	$12.73 \frac{a_1^2}{\ell^2}$	$367.14 \frac{a_1^4}{\ell^4}$	$\sum [-12.73 \frac{a_1^2}{\ell^2} + 367.14 \frac{a_1^4}{\ell^4}]$
in.	$\times 10^{-4}$	$\times 10^{-8}$	$\times 10^{-4}$	$\times 10^{-8}$	$\times 10^{-8}$
0	0	0	0	0	0
0.2	1	1	12	367.14	119633
0.4	4	16	50	5874.24	494126
0.6	9	81	114	29738.34	1110262
0.8	16	256	203	93987.84	1936013
1.0	25	625	318	229462.50	2950538

Amplitude	η^2	η
0	1.0	1.0
0.2	0.99880367	0.99940165
0.4	0.99505874	0.99752632
0.6	0.98889738	0.99443319
0.8	0.98063987	0.99027262
1.0	0.97049462	0.98513685

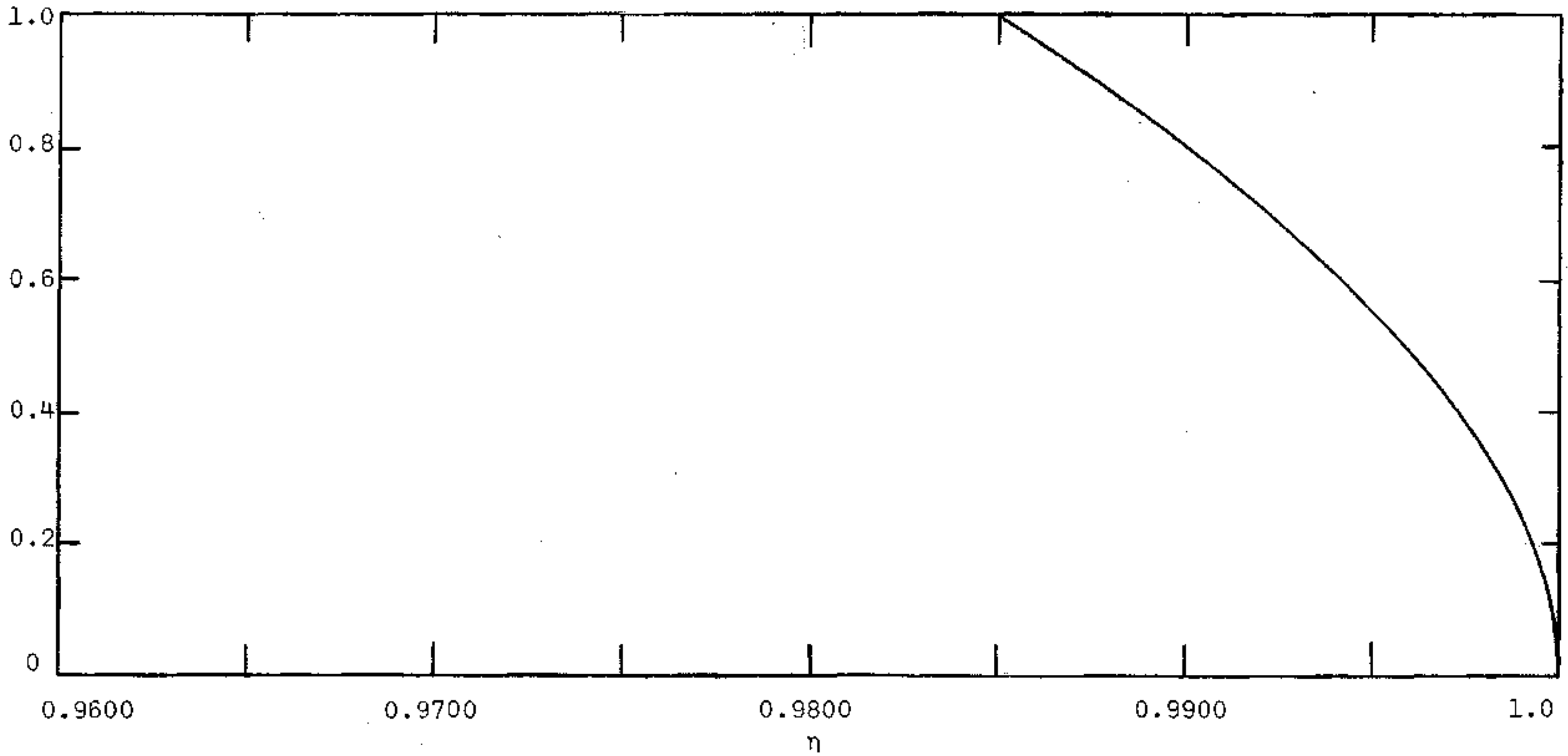


Figure 18. Cantilever Beam--Response Curve for Non-Linear Curvature and Linearized Stress-Strain Relation. (γ=0)

Cantilever Beam
Non-Linear Curvature and Non-Linear Stress-Strain Relation

Amplitude a_1	$\frac{a_1^4}{\ell^6}$	$232.57\ K \frac{a_1^4}{\ell^6}$			
		γ = 500	1000	5000	10000
in.	x 10^{-10}	x 10^{-10}			
0	0	0	0	0	0
0.2	0.25	272.38	544.77	2723.85	5447.71
0.4	4.00	4358.17	8716.35	43581.75	87163.51
0.6	20.25	22063.26	44126.52	220632.64	441265.28
0.8	64.00	69730.81	139461.62	697308.11	1394616.23
1.0	156.25	170241.24	340482.48	1702412.40	3404824.80

Amplitude	$\sum \{-12.73 \frac{a_1^2}{\ell^2} + 367.14 \frac{a_1^4}{\ell^4} - 7.32K \frac{a_1^2}{\ell^4} + 232.57K \frac{a_1^4}{\ell^6}\}$			
	γ = 500	1000	5000	10000
in.	x 10^{-8}			
0	0	0	0	0
0.2	120488	121342	128179	126725
0.4	497512	500897	527983	561840
0.6	1117757	1125252	1185214	1260167
0.8	1949033	1962053	2066210	3196418
1.0	2970268	2989998	3147843	3345148

Cantilever Beam
Non-Linear Curvature and Non-Linear Stress-Strain Relation

Amplitude	η^2			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99879512	0.99878658	0.99871821	0.99873275
0.4	0.99502488	0.99499102	0.99472017	0.99438150
0.6	0.98882243	0.98874748	0.98814785	0.98739833
0.8	0.98050967	0.98038947	0.97933790	0.96803582
1.0	0.97029732	0.97010002	0.96852157	0.96654852

Amplitude	η			
	γ = 500	1000	5000	10000
in.				
0	1.0	1.0	1.0	1.0
0.2	0.99939737	0.99939310	0.99935910	0.99936617
0.4	0.99750933	0.99749251	0.99735659	0.99718679
0.6	0.99439550	0.99435782	0.99405626	0.99367918
0.8	0.99020688	0.99014618	0.98961502	0.98888811
1.0	0.98503670	0.98493655	0.98413493	0.98313199

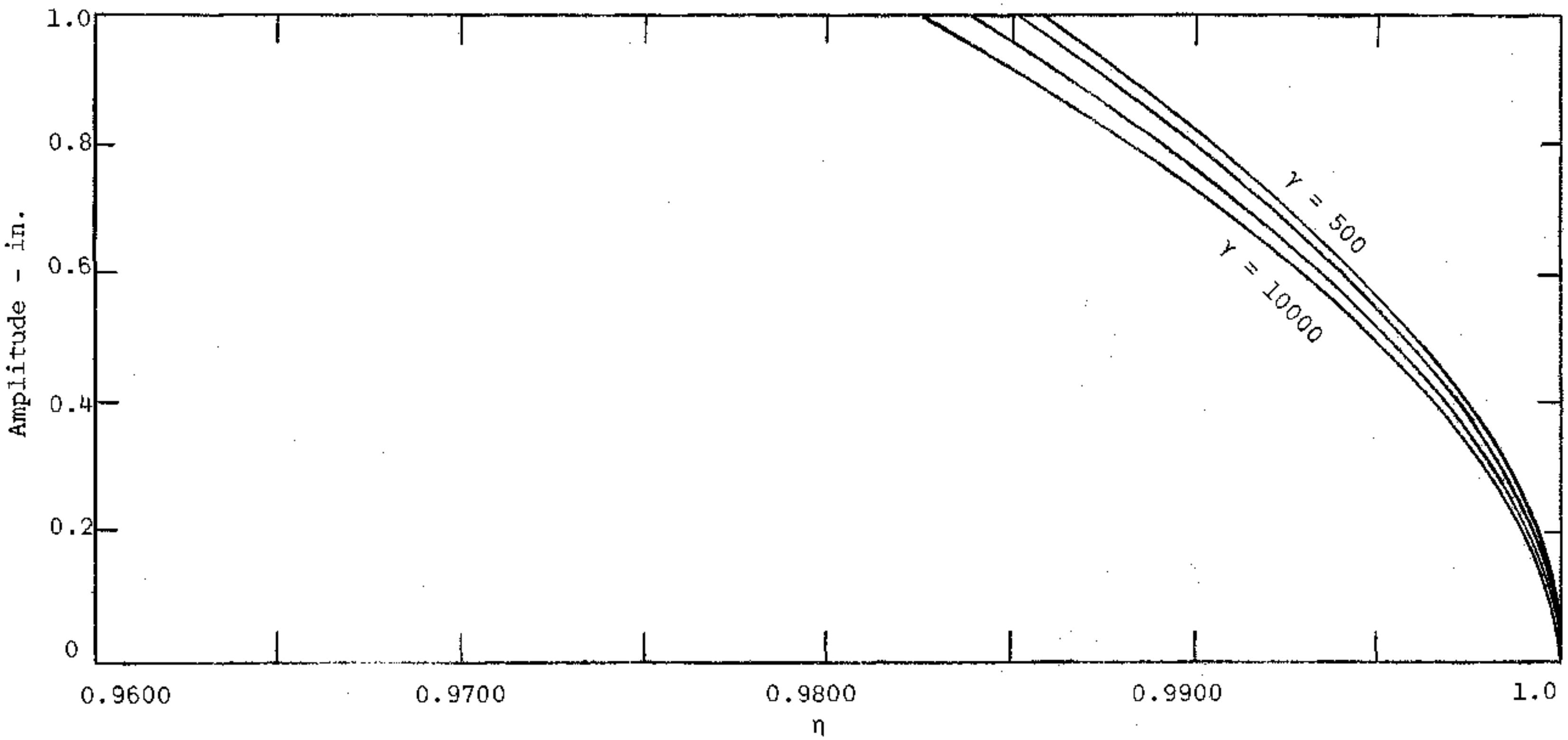


Figure 19. Cantilever Beam--Response Curves for Non-Linear Curvature and Non-Linear Stress-Strain Relation.

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Prescott, J., *Applied Elasticity,* Dover Publications Inc., New York, New York, 1961.

2. L'Hermite, R., *Resistance Des Materiaux,* Dundd, Paris, 1954.

3. Timoshenko, S., *History of Strength of Materials,* McGraw-Hill Book Co., Inc., New York, New York, 1953.

4. Ramberg, W. and Osgood, W. R., *Description of Stress-Strain Curves by Three Parameters,* N.A.C.A.TN. No. 902, 1943.

5. Osgood, W. R., "Stress-Strain Formulas," *Journal of the Aeronautical Sciences,* Vol. 13, 1946, pp. 45-48.

6. Wang, T. K., "Elastic and Plastic Bending of Beams," *Journal of the Aeronautical Sciences,* Vol. 14, 1947, pp. 422-432.

7. Bolotin, V. V., *The Dynamic Stability of Elastic Systems,* Report No. TDR - 169 (3560 - 30) TR-2, Vols. I, II, III, and IV, Aerospace Corp., Inglewood, California, 1962.

8. Kolsky, H., *Stress Waves in Solids,* Clarenden Press, Oxford, 1953.

9. Rayleigh, J., *Theory of Sound,* Dover Publications, Inc., New York, New York, 1945.

10. Den Hartog, J. P., *Mechanical Vibrations,* McGraw-Hill Book Co., Inc. New York, New York, 1956.

11. Myklestad, N. O., *Vibration Analysis,* McGraw-Hill Book Co., Inc., New York, New York, 1944.

12. Waller, E. J. and Reed, R. R., *Non-Linear Systems,* Part I, *Literature Survey,* Part II, *Analytical Methods,* Engineering Research Bulletin No.s 127 and 134, Oklahoma State University, Stillwater, 1963.

13. Cartwright, M. L., "Non-Linear Vibrations," *The Advancement of Science,* Vol. 6, No. 21, 1949, pp. 64-75.

14. Duncan, W. J., *Galerkin's Method in Mechanics and Differential Equations,* R. & M. 1798, 1937.

15. Rayleigh, J., *op. cit.*

16. Timoshenko, S., *Vibration Problems in Engineering,* D. Van Nostrand Co., Inc., New York, 1956.

17. Klotter, K., "Non-Linear Vibration Problems Treated by the Averaging Method of W. Ritz": Part I, *Proceedings of the First U. S. National Congress of Applied Mechanics,* New York, The American Society of Mechanical Engineers, 1952, pp. 125-131.

18. Kantorovich, L. H. and Krylov, V. I., *Approximate Methods of Higher Analysis,* Interscience Publishers, New York, New York, 1958.

19. Duncan, W. J., *The Principles of Galerkin's Method,* R. & M. 1848, 1949.

20. Schwesinger, G., "On One Term Approximations of Forced Non-Harmonic Vibrations," *Journal of Applied Mechanics,* Vol. 17, 1950, pp. 202-208.

21. Klotter, K., *op. cit.*

22. Duncan, W. J., *Mechanical Admittances and Their Applications to Oscillation Problems,* R. & M. 2000, 1946.

23. Duncan, W. J. and Lindsay, D. D., *Methods of Calculating the Frequencies of Overtones,* R. & M. 1888, 1939.

24. Rauscher, M., "Station Functions and Air Density Vibrations in Flutter Analysis," *Journal of the Aeronautical Sciences,* Vol. 16, 1949, pp. 345-353.

25. Klotter, K., *op. cit.*

26. Duncan, W. J., *op. cit.*, 22 and 23.

27. Young, D. and Felgar, R., *Tables of the Characteristic Functions Presenting Normal Modes of Vibration of a Beam,* The University of Texas Publication No. 4913, 1949.

28. Duncan, W. J., *op. cit.,* 22 and 23.

29. Yoshimura, Y. and Eumura, M., "Effect of Amplitude on Lateral Vibrations of Rods," in Japanese, *Rep. Inst. Soc. Tech.,* Vol. 2, Tokyo, 1948, p. 57.

30. Young, D. and Felgar R., *op. cit.*

Additional References Cited

31. Barton, H. J., "On the Deflection of a Cantilever Beam," *Q. Appl. Math.*, Vol. 2, 1944, pp. 168-171.

32. ________, "On the Deflection of a Cantilever Beam," *Q. Appl. Math.*, Vol. 3, 1945, pp. 275-276.

33. Bauer, H., *Methods in Non-Linear Vibration Theory*, George Marshall Space Flight Center, Internal Note No. 40-62, Huntsville, 1962.

34. Biezeno, C. B. and Grammel, R., *Engineering Dynamics*, Vol. I, Blackie and Sons, Glasgow, 1955, p. 170.

35. Bisshop, K. E. and Drucker, D. C., "Large Deflections of Cantilever Beams," *Q. Appl. Math.*, Vol. 3, 1945, pp. 272-275.

36. Davis, H., *Introduction to Non-Linear Differential and Integral Equations*, Dover Publications, Inc., New York, New York, 1962.

37. Hoff, Nicholas, *The Analysis of Structures*, John Wiley, New York, New York, 1956.

38. Karman, T. Von and Biot, M. A., *Mathematical Methods in Engineering*, McGraw-Hill Book Co., Inc., New York, New York, 1940.

39. Lanczos, C., *Variational Principles of Mechanics*, University of Toronto Press, Toronto, 1949.

40. Libby, P. A. and Sauer, R. C., "Comparison of the Rayleigh-Ritz and the Complementary Energy Methods in Vibration Analysis," *Journal of the Aeronautical Sciences*, Vol. 16, 1949, p. 700.

41. Rhode, F. V., "Large Deflections of a Cantilever Beam with Uniformly Distributed Load," *Q. Appl. Math.*, Vol. 6, 1952.

42. Scott, E. J. and Carver, D. E., "On the Non-Linear Differential Equation of Beam Deflection," *Journal of Applied Mech.*, June, 1955, pp. 245-248.

43. Singer, J., "The Equivalence of the Galerkin and Rayleigh-Ritz Methods," *Journal of the Royal Aeron. Soc.*, Vol. 66, 1962, p. 592.

44. Sokolnikoff, I. S., *Mathematical Theory of Elasticity*, McGraw-Hill Book Co., Inc., New York, New York, 1956.

45. Stoker, J. J., *Notes on Applied Mathematics*, New York University, 1942.

46. ________, *Non-Linear Vibrations*, Interscience Publishers, New York, New York, 1950.