An Approach for Mapping Kinematic Task Specifications into a Manipulator Design

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Abstract

The Reconfigurable Modular Manipulator System (RMMS) consists of modular links and joints which can be assembled into many manipulator configurations. This capability allows the RMMS to be rapidly reconfigured in order to custom tailor it to specific tasks. An important issue, related to the RMMS, is the determination of the optimal manipulator configuration for a specific task. In this paper, we address the problem of mapping *kinematic* task specifications into a *kinematic* manipulator configuration. For the design of 2 degrees-of-freedom planar manipulators, an *analytical* solution is derived. Since, for problems with more than 2 design parameters, analytical solutions become impractical, we have also developed a numerical approach for the design of 6 degrees-of-freedom manipulators.

1 Introduction

There has been an increasing interest in the kinematic properties of robot manipulators such as workspace features [3, 6, 19, 11], dexterity criteria [1, 17, 20] and inverse kinematics [10, 14]. In parallel with the research efforts on kinematic analysis of manipulators, the problem of kinematic *design* has been addressed. It is concluded in [8, 15, 18], that an elbow manipulator with zero link offsets, is optimal with respect to working volume and dexterity. The optimality measures considered in the above mentioned articles are task independent. They don't guarantee that there is no better manipulator for a specific task. In [16], task specificity is included, assuming though that an elbow configuration is optimal for any task. Only the size of the manipulator and its base position are considered as design parameters.

The work reported in this paper is motivated by our research on the development of a Reconfigurable Modular Manipulator System [13], [4]. This system consists of a set of joints and links of varying specifications, with consistent mechanical and electrical interfaces. The joints and links can be connected rapidly to obtain manipulators with varying performance specification.

To effectively utilize the capabilities provided by the RMMS concept, it is important to address the issue of 'Configuring a manipulator from Task Requirements'. Different tasks require different manipulator configurations. Therefore, it is important to establish methodologies that generate an appropriate manipulator configuration, given a set of kinematic and dynamic task specifications. Kinematic requirements are those task requirements that affect only the kinematic structure of the manipulator, while dynamic requirements affect both the kinematics and dynamics of the manipulator. Examples of kinematic requirements are workspace volume, maximum reach and maximum positional error. Examples of dynamic requirements are maximum payload, maximum joint velocities and maximum joint accelerations.

In this paper, we present both analytical and numerical approaches for determining the kinematic structure (or Denavit-Hartenberg parameters) of a manipulator given a set of task specifications.

2 Problem Statement

The exact problem solved in this paper is the determination of the kinematic structure of a manipulator that satisfies the given kinematic requirements. Since the kinematic structure of a manipulator can be described unambiguously by its Denavit-Hartenberg parameters[1], we can translate the problem into the determination of the DH-parameters. The kinematic requirements taken into consideration are:

1. Reachability: the specified set, W_R, of positions/orientations, $\mathbf{p} = (x, y, z, \vartheta, \varphi, \psi) \in W_R$, has to be inside the reachable workspace of the manipulator.

2. Joint angles: the joint angles must be within the specified limits.

In order to make the problem solvable, we make the following assumptions:

1. All the design parameters can vary continuously.

2. Only non-redundant serial link rotary manipulators are considered.

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[1] henceforth abbreviated to DH-parameters.

3. The robot base position is fixed and known.

4. The last three axes of 6-DOF spatial manipulators intersect in a point.

5. No self-collision is considered.

In this paper, the problem statement is limited to reachability and joint limit specifications. This enables us to clearly explain the basic concepts of the proposed design method. However, the approach is more general and has been extended to include obstacle avoidance and manipulability specifications. Furthermore, the assumption of non-redundancy has been relaxed, so that also the kinematic structure of *redundant* manipulators can be designed [9].

3 General Mathematical Formulation

3.1 Kinematic Space

According to the Denavit-Hartenberg convention, 3 parameters are needed to describe each link of a rotary manipulator [2]. These 3 parameters are the link length a_{i-1}, the link twist α_{i-1} and the link offset d_i. A fourth DH-parameter, the joint angle θ_i, does not characterize any dimensional parameter of a rotary serial link manipulator but merely its position, and therefore is not a design parameter.

To facilitate the development of our design approach, we define the *DH-configuration space* as a $3n$-dimensional space with the $3n$ DH-parameters as base elements (3 DH-parameters for each of the n links). A manipulator with n rotary joints can be represented by a vector, $v_{dh} = [a_0, \alpha_0, d_1, \ldots, a_{n-1}, \alpha_{n-1}, d_n]^T$, of $3n$ DH-parameters, which corresponds to one specific point in the DH-configuration space. And vice versa, each point in the DH-configuration space corresponds to one specific n-DOF rotary manipulator. The vector, $q = [\theta_1, \ldots, \theta_n]^T$ of n joint variables, can be viewed as a point in the n-dimensional *joint space*. Each point in this space represents one specific posture of the manipulator and conversely, each manipulator posture corresponds to exactly one point in the joint space. A point $p = [x, y, z, \vartheta, \varphi, \psi]^T$ in the 6-dimensional *task space* (or cartesian space), corresponds to a physical point and direction in the base frame of the manipulator.

To make the kinematic relation[2] easier to use in different forms (e.g. as in inverse kinematics), it is useful to introduce the idea of the *kinematic space*. The $(4n + 6)$-dimensional kinematic space is the cartesian product of the DH-configuration space, the joint space and the task space. The base-elements of the kinematic space thus include the $3n$ DH-parameters, the n joint variables and the 6 position/orientation parameters of the end effector. This implies that a manipulator in a certain posture at

[2]By *kinematic relation* we mean the relationship between joint positions and end effector position, and not the Jacobian relations between joint velocities and end effector velocities.

a certain point, can be represented as one point in the kinematic space.

3.2 Mathematical Formulation Using the Concept of the Kinematic Hyperplane

As mentioned before, the kinematic relation can be viewed as a mapping from the joint space into the task space. The disadvantage of this representation is that the mapping is different for each manipulator. The concept of kinematic space can be exploited to overcome this disadvantage, since we can formulate, in this space, all the kinematic relations of all the n-DOF rotary manipulators. Indeed, the kinematics of any n-DOF manipulator can be described by the following equation, which corresponds to a $4n$-dimensional hyperplane (henceforth called *kinematic hyperplane*) in the kinematic space:

$$ {}^0_1T \, {}^1_2T \, {}^2_3T \cdots {}^{n-1}_{n}T = {}^0_nT \tag{1} $$

where ${}^{i-1}_iT$ is the transformation from the $(i-1)$-th frame to the i-th frame [2].

An example of the use of the kinematic hyperplane, is the determination of the workspace of a specific manipulator. Specifying the $3n$ DH-parameters α_{i-1}, a_{i-1}, d_i $(i = 1, \ldots, n)$, corresponds to intersecting the $4n$-dimensional kinematic hyperplane with $3n$ of $(4n + 5)$-dimensional hyperplanes. This yields an n-dimensional solution (the dimensionality of the solution corresponds to the n joint angles that are still free to be chosen). To determine the reachable workspace we project this n-dimensional hyperplane on the (x, y, z)-space.

Instead of obtaining the workspace given the kinematic parameters of a manipulator, we are interested in exactly the opposite problem: given a required workspace W_R, determine the kinematic configuration. This problem can be formulated mathematically, using the concept of the kinematic hyperplane. We achieve this in two steps. First, consider the set of all the kinematic configurations that can reach *one* specific point/orientation, p_i. This set is the intersection of the kinematic hyperplane with six $(4n + 5)$-dimensional hyperplanes, given by the following equations:

$$
\begin{aligned}
x &= x_i & \vartheta &= \vartheta_i \\
y &= y_i & \varphi &= \varphi_i \\
z &= z_i & \psi &= \psi_i
\end{aligned}
\tag{2}
$$

The solution of this set of equations will, in general, be an infinite set of manipulators. We can find such a set of manipulators for each point in W_R.

In the second step we take the intersection of all the solutions (each for a different position p_i) found in step one. This yields the set of manipulators that can reach every point in W_R. An exact mathematical formulation of the problem is thus:

Find $v_{dh} = [a_0, \alpha_0, d_1, \ldots, a_{n-1}, \alpha_{n-1}, d_n]^T$
such that for each $p_i \in W_R$: $\exists \theta_i (i = 1, n)$,

557
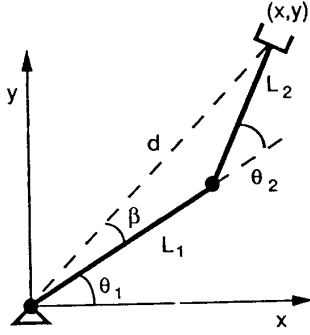
Figure 1: A Simple 2-DOF Example

Figure 2: Graphical depiction of the problem statement.

for which :

$$
{}^0_1T^1_2T^2_3T \cdots {}^{n-1}_n T = \begin{bmatrix} & & & x_i \\ & R(\vartheta_i, \varphi_i, \psi_i) & & y_i \\ & & & z_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$
(3)

and where θmin_i and θmax_i are the joint limits of joint i. In order to achieve our goal, we will exploit the availability of the closed form inverse kinematics solution [7, 10, 14]. This solution appears as a 16th order polynomial in the tangent of the half-angle of one of the joint variables. To account for the case when the point p_i is outside the reach of the manipulator, we have developed a computation scheme called "Generalized Inverse Kinematics" which, in the case of non-reachable points, generates *complex* joint angles [9].

4 Analytical Solution for a 2-DOF Planar Manipulator

4.1 Example

We consider the problem of designing a 2-DOF planar manipulator, that has only two design (or DH-) parameters: the link lengths l_1 and l_2. Figure 1 shows the variables and the notation.

The exact design problem addressed is: Given a finite set of reachable points $(x, y) \in W_R$ and limits on the joint angles θ_1 and θ_2, determine a pair of link lengths (l_1, l_2) such that the planar manipulator is able to reach all the points in W_R. More specifically, the joints limits are $\pm \pi/4$ for θ_1 and $\pm 5\pi/6$ for θ_2. The points $(x, y) \in W_R$ are depicted with an 'x' in Figure 2.

4.2 The Inverse Kinematics

The inverse kinematics of this planar manipulator are expressed as follows:

$$
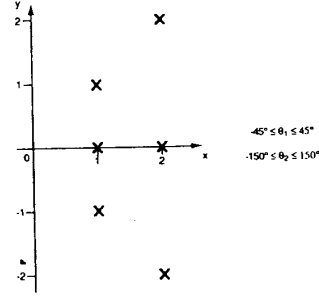\theta_1 = \arctan 2(y, x) - \arccos\left(\frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}}\right) (4)
$$

$$
\theta_2 = \arccos\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1 l_2}\right)
$$
(5)

It is more convenient to rewrite (4) and (5) as a function of u and v, where:

$$
v = \tan(\frac{\theta_1}{2}), and\, u = \tan(\frac{\theta_2}{2})
$$
(6)

From Equation (5) we obtain expressions for u and v:

$$
u^2 = -\frac{(x^2 + y^2) - (l_1 + l_2)^2}{(x^2 + y^2) - (l_1 - l_2)^2}
$$
(7)

$$
v = \frac{y - k_2}{x + k_1} = \frac{k_1 - x}{k_2 + y}
$$
(8)

where:

$$
k_1 = l_1 + l_2 \cos \theta_2
$$
(9)

$$
k_2 = l_2 \sin \theta_2
$$
(10)

4.3 Mathematical Formulation of the Task Specifications

The task specifications considered in this example are *reachability* and *joint limits*. The Generalized Inverse Kinematics, as given by Equations (7) and (8), will in general yield complex solutions for u. Indeed, u^2 is negative when

$$
\sqrt{x^2 + y^2} < |l_1 - l_2| \quad (11)
$$

$$
or \quad \sqrt{x^2 + y^2} > l_1 + l_2 \quad (12)
$$

This corresponds to the cases for which the point (x, y) is outside the reach of the manipulator. The reachability constraint can be expressed as:

$$
\text{Imag}(u) = 0
$$
(13)

The mathematical formulation of the whole problem is then:

558

Find (l_1, l_2) such that for each point $(x, y) \in W_R : \exists(\theta_1, \theta_2)$ and (v, u), for which:

$$\begin{cases} \text{Imag}(u) = 0 \\ \theta_1 \geq \theta min_1 \\ \theta_2 \geq \theta min_2 \\ \theta_1 \leq \theta max_1 \\ \theta_2 \leq \theta max_2 \end{cases} \qquad (14)$$

4.4 Analytical Solution

In this section, we develop expressions for the bounds on the feasible region in the $\{l_1, l_2\}$-plane (i.e. the region containing the admissible solution sets (l_1, l_2)). We derive analytical expressions for:

1. Bounds due to reachability constraints.

2. Bounds due to joint limits on θ_1.

3. Bounds due to joint limits on θ_2.

The reachability bound is described analytically by Equations (11) and (12), or Equation (13). These equations define the admissible solution pairs (l_1, l_2), that describe a manipulator that can reach the point (x, y) *without* joint limits. This region is bounded by the three straight lines:

$$l_1 + l_2 = d \qquad (15)$$

$$l_1 - l_2 = d \qquad (16)$$

$$l_2 - l_1 = d \qquad (17)$$

In order to find an expression for the bounds due to the joint limits on θ_1, we first derive the equation that describes the admissible solution pairs (l_1, l_2) that can reach the point (x, y) with the joint angle θ_1 fixed. If we define the angle β as in Figure 1, the following expression holds:

$$\left(\frac{l_2}{d \sin \beta} \right)^2 - \left(\frac{l_1 - d \cos \beta}{d \sin \beta} \right)^2 = 1 \qquad (18)$$

Equation (18) represents a hyperbola with both asymptotes at 45 degrees, focal axis parallel to the l_2-axis, center at $(d \cos \beta, 0)$ and a vertex distance of $d \sin \beta$. The bound due to joint limits on θ_1 can now be found by choosing the maximum permissible β-value. This hyperbola is labeled 'b' in Figure 3.

The starting point for the deduction of the equation representing the bounds due to joint limits on θ_2 is Equation (7). If we perform a rotation of 45 degrees, transforming the $\{l_1, l_2\}$ DH-configuration space into the space with base variables $\{p = \frac{l_1 + l_2}{\sqrt{2}}, q = \frac{l_1 - l_2}{\sqrt{2}}\}$, we obtain:

$$u^2 = \frac{\sin^2(\frac{\theta_2}{2})}{\cos^2(\frac{\theta_2}{2})} = -\frac{d^2 - 2p^2}{d^2 - 2q^2} \qquad (19)$$

The above expression can be written as:

$$2p^2 \cos^2(\frac{\theta_2}{2}) + 2q^2 \sin^2(\frac{\theta_2}{2}) = d^2 \cos^2(\frac{\theta_2}{2}) + d^2 \sin^2(\frac{\theta_2}{2}) \qquad (20)$$

or:

$$\left(\frac{p}{\frac{d}{\sqrt{2}\cos(\frac{\theta_2}{2})}} \right)^2 + \left(\frac{q}{\frac{d}{\sqrt{2}\sin(\frac{\theta_2}{2})}} \right)^2 = 1 \qquad (21)$$

This is the equation of an ellipse with an axis of $d/(\sqrt{2}\cos(\theta_2/2))$ in the p-direction and an axis of $d/(\sqrt{2}\sin(\theta_2/2))$ in the q-direction. The bound, labeled 'c' in Figure 3, is found for the maximum permissible value of $|\theta_2|$.

For each point (x, y), we thus obtain 5 bounding curves. The bounds that border the feasible region are called "active" (e.g. bound "c"), while "passive" bounds do not touch the feasible region at all, i.e., there exists an active bound that is more restrictive. For problems with multiple points in W_R, the number of bounds can become very large, but the set of active bounds will usually remain small.

One can now pick any pair (l_1, l_2) inside the feasible region, as a solution to the design problem. However, in general, for spatial manipulators, the DH-configuration space has $3n$ dimensions and the analytical bounds are highly non-linear hyperplanes. Therefore, it becomes very hard to construct the set of active bounds and to find a tuple, that is "inside" all the active bounds, since we cannot visualize the solution anymore. In that case, a numerical solution procedure is more appropriate. In the next section, we develop a numerical approach that can be used for the design of spatial manipulators. We show its efficacy by applying it to the 2-DOF design problem and comparing the results with the analytical approach.

5 Numerical Solution

In the previous sections, we formulated the design problem in terms of the Generalized Inverse Kinematics. At first sight, the constraints look like a simple set of nonlinear equations for which standard solution methods are applicable. However, as we mentioned earlier, the GIK have multiple solutions (8 for a 6R serial manipulator that has the last three axes intersecting at a point). Each solution corresponds to a possible posture of the manipulator. Thus, for each point $\mathbf{p} \in W_R$ and a given set of DH-parameters, the GIK yield eight sets of joint variables: $(\theta_1^{(1)}, \ldots, \theta_6^{(1)}), (\theta_1^{(2)}, \ldots, \theta_6^{(2)}), \ldots (\theta_1^{(8)}, \ldots, \theta_6^{(8)})$, where the superscript denotes the posture number. For each posture, a separate set of (in)equalities representing the task requirements, is formulated. The manipulator fulfills all the task requirements, if the set of (in)equalities is satisfied for at least one set $(\theta_1^{(j)}, \ldots, \theta_6^{(j)})$. For instance, the reachability task specification is satisfied when a point, $\mathbf{p}_i$, is reachable in at least one posture but does not require that $\mathbf{p}_i$ has to be reachable in all eight postures possible. This implies that the problem is not formulated as a set of (in)equalities that have to be satisfied simultaneously, but merely as a juxtaposition of subsets of simultaneous (in)equalities for which only at least one subset has to be satisfied.

The algorithms found in the literature to solve a set

of non-linear equalities and inequalities cannot handle the above formulation, however, there is one approach 'penalty function method' [12] which can be adapted to this more complex formulation.

The basic idea of the penalty function approach is to translate the problem of finding a feasible solution for a set of constraints into a minimization problem. For instance, consider the following set of constraints:

$$\begin{cases} c_i(\mathbf{x}) &=& 0 & (i = 1, l) \\ c_j(\mathbf{x}) &\leq& 0 & (j = 1, m) \end{cases} \quad (22)$$

This set of constraints is transformed into the penalty function:

$$F(\mathbf{x}) = \sum_{i=1}^{l} [c_i(\mathbf{x})]^2 + \sum_{j=1}^{m} [\max(0, c_j(\mathbf{x}))]^2 \quad (23)$$

with the property that all the constraints are satisfied simultaneously if and only if the function $F(\mathbf{x})$ is equal to zero. Otherwise, $F(\mathbf{x})$ has a strictly positive value.

This approach is not directly applicable to our problem due to the fact that the constraints only have to be satisfied for one of the eight postures. This can be included in the penalty function formulation by evaluating the function $F(\mathbf{q}^{(j)})$ for each posture, j, and considering only the minimum of these eight values:

$$F_{point\ p}(\mathbf{v}_{dh}) = \min\left\{F(\mathbf{q}^{(1)}), F(\mathbf{q}^{(2)}), \ldots, F(\mathbf{q}^{(8)})\right\} \quad (24)$$

where $\mathbf{v}_{dh}$ is the vector of DH-parameters. Indeed, this function is zero when at least one of the postures has a zero penalty and thus satisfies all the constraints. The total penalty for a certain manipulator is obtained by summing over all the points $p_i \in W_R$:

$$F_{tot}(\mathbf{v}_{dh}) = \sum_{p \in W_R} \min\left\{F(\mathbf{q}^{(1)}), F(\mathbf{q}^{(2)}), \ldots, F(\mathbf{q}^{(8)})\right\} \quad (25)$$

The disadvantage of this approach is that local minima are introduced. In order to find a solution that satisfies all the constraints, we need to find the *global* minimum, using a global optimization procedure.

5.1 Numerical Solution

In Section 4, it was illustrated that analytical bounds for the feasible region can be found when the number of design parameters is small. Here, we continue the 2-DOF example and compare the analytical solution with the numerical solution, obtained as explained above.

Equations (14) express the task requirements as a set of equalities and inequalities. The problem of solving this set of (in)equalities is converted into an optimization prob-
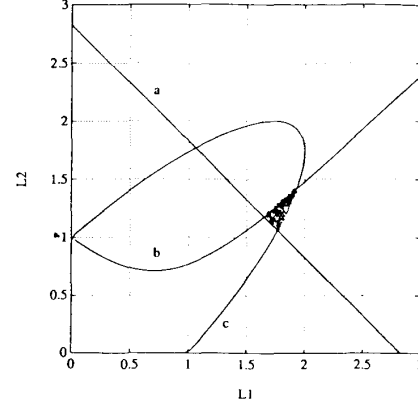

Figure 3: Numerical solutions for example 1.
Numerical solutions (indicated with 'x').
Analytical bounds:
  a) Bound due to reachability constraints.
  b) Bound due to joint limits on θ_1, $|\theta_1| \leq \pi/4$.
  c) Bound due to joint limits on θ_2, $|\theta_2| \leq 5\pi/6$

lem, with the following penalty function:

$$F(l_1, l_2) = \sum_{(x,y) \in W_R} \min_{j=1,2} \left\{ \begin{array}{l} [\text{Imag}(v^{(j)})]^2 \\ + [\text{Imag}(u^{(j)})]^2 \\ + [\max(0, \theta min_1 - \theta_1^{(j)})]^2 \\ + [\max(0, \theta_1^{(j)} - \theta max_1)]^2 \\ + [\max(0, \theta min_2 - \theta_2^{(j)})]^2 \\ + [\max(0, \theta_2^{(j)} - \theta max_2)]^2 \end{array} \right\} \quad (26)$$

The penalty function in general has local minima. Therefore, a global optimization procedure is needed, even for this simple example. We chose simulated annealing [5] and the results for 100 runs of the simulated annealing algorithm are shown in Figure 3.

Each 'x' represents a final solution of the optimization procedure. The initial guesses for the 100 runs are generated using a random number generator and are uniformly distributed over the region $(0 \leq l_1 \leq 3, 0 \leq l_2 \leq 3)$. The solutions of the optimization procedure are all inside the feasible region bordered by the analytical bounds. This implies that the approach presented in this section works well for this simple example. In [9], more complex examples are presented to prove the validity and generality of the approach.

6 Summary

In this paper, we have proposed an approach to solve the kinematic design problem, i.e., the determination of the

Denavit-Hartenberg parameters of a non-redundant manipulator with joint limits, that can reach a set of specified points/orientations. This problem was first formulated mathematically, using the concept of the kinematic hyperplane in the kinematic space. The problem statement was further concretized through a generalized formulation of the inverse kinematics, which resulted in an analytical solution for the design for 2 degrees-of-freedom planar manipulators. In order to extend the capabilities of the solution procedure to spatial problems with six degrees-of-freedom, a numerical approach was developed. Using a global optimization procedure, the penalty of a manipulator design was minimized, resulting in an optimal kinematic configuration to perform the specified task. A simple 2-DOF planar manipulator example shows that the numerical solutions correspond perfectly with the analytical solution.

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