PRODUCT INTEGRAL SOLUTIONS OF STOCHASTIC VOLTERRA-STIELTJES INTEGRAL EQUATIONS WITH DISCONTINUOUS INTEGRATORS

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PRODUCT INTEGRAL SOLUTIONS OF STOCHASTIC
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SUMMARY

Let $[a,b]$ be a closed interval in a linearly ordered set, let (Ω,A,P) be some probability space, and let $\{A_t | a \le t \le b\}$ be a family of σ-subalgebras of A such that $A_s \subseteq A_t$ when $s \le t$. Let Γ_0 denote the class of functions $f:[a,b] \to L^2(\Omega,A,P)$ such that $f(t) \in L^2(\Omega,A_t,P)$ for $a \le t \le b$ for which there exists a non-decreasing function $F:[a,b] \to R$ such that the expectation $E([f(t)-f(s)]^2) \le F(t) - F(s)$ for $s \le t$. Let Γ_3 denote the subclass of functions in Γ_0 for which there exist non-decreasing functions $G_i:[a,b] \to R$, $i = 1,2$, such that the conditional expectations $E([g_i(t) - g_i(s)]^i | A_s)$ do not exceed $G_i(t) - G_i(s)$ in absolute value for $i = 1,2$ and $s \le t$.

If $f,g:[a,b] \to L^2(\Omega,A,P)$, then $(L)\int_a^b fdg$ (the <u>left</u> stochastic Cauchy-Stieltjes integral) is defined as the L^2-norm limit of the net $\{\sum_{k=1}^{n} f(r_{k-1})[g(r_k) - g(r_{k-1})] | \{r_k\}_{k=0}^{n}$ is a subdivision of $[a,b]\}$, where the subdivisions of $[a,b]$ are ordered by inclusion. We show that if $f \in \Gamma_0$ and $g \in \Gamma_3$, then $(L)\int_a^b fdg$ exists.

If $g:[a,b] \to L^2(\Omega,A,P)$, then $\prod_a^b [1+dg]$ (the stochastic product integral) is defined as the L^2-norm limit of the net $\{\prod_{k=1}^{n} [1+g(r_k) - g(r_{k-1})] | \{r_k\}_{k=0}^{n}$ is a subdivision of $[a,b]\}$. We show that if $g \in \Gamma_3$, then $\prod_a^b [1+dg]$ exists. Moreover,

the function $u:[a,b] \to L^2(\Omega,A,P)$ defined by $u(t) = \prod_a^t [1+dg]$ is the almost surely unique solution in class Γ_o for the stochastic Stieltjes-Volterra integral equation

$$u(t) = 1 + (L)\int_a^t udg.$$

We give a characterization of the class of functions generated as product integrals of functions in Γ_3 and a one-to-one correspondence between the product integrals and the members of Γ_3 which map a into $0 \in L^2(\Omega,A,P)$. Under additional hypotheses involving separability, we show the almost sure uniform convergence of the approximations given above to the left Cauchy-Stieltjes and product integrals on the interval $[a,b]$.

CHAPTER I

INTRODUCTION

1. Introduction

Mathematical models in the physical sciences, economic systems, and operations research frequently involve solutions of differential, difference, or integral equations. These models are generally classified as "deterministic" or "stochastic" according to whether the solutions of the equations are deterministic functions or stochastic processes.

Most work on stochastic differential and integral equations has been concerned with processes which were inherently continuous; the prototype being Brownian motion, whose paths are almost surely continuous. These processes are used in models of physical phenomena, usually involving some sort of "white noise," in which the continuity of the processes is both intuitive and reasonable. For example, an elementary description of the use of the Brownian motion process to account for thermal noise in a simple electrical network appears in Chapter 6 of Hoel, Port and Stone [1]. Stochastic processes with continuous paths may also be used to obtain reasonable models of phenomena which are intuitively discontinuous, such as the growth of a population involving a large number of individuals (see Padgett and Tsokos [2]).

The most common models of discontinuous random phenomena fall into one of two categories. In the first, many systems are modelled by discrete-time processes which satisfy some system of difference equations. One of the most common models of this type is the Markov chain storage model in stochastic reservoir theory (see, e.g., Moran [3]). A second type of discontinuous system has a continuous time model where the processes of interest are of pure jump type; for example, a birth-and-death process, or simply a Poisson process. A standard tactic in analyzing this type of system is to shift the concentration of interest to an imbedded discrete-time process and then apply standard results from other areas, such as Markov chain theory and renewal theory. For examples of this type of analysis in queueing theory, see, e.g., Kendall [4], Cohen [5], Cooper [6], etc.

Recently attention has turned to stochastic models of systems in continuous time whose paths are discontinuous, but not of pure jump type, and are governed by stochastic integral equations. One area of activity has been the series of papers by Çinlar and Pinsky [7], [8], and Çinlar [9] which consider a reservoir storage model with stochastic input and a deterministic release mechanism. These models involve the solution of an integral equation with a random, non-decreasing input function and an integral operator from a deterministic integral equation.

In this work, a theory of stochastic Volterra-Stieltjes integral equations is presented which is sufficiently general to accommodate certain processes with both discontinuous input and discontinuous solutions of the integral equation. The solutions of the integral equations are exhibited as stochastic product integrals, which might be thought of as random evolutions on the time-axis or, alternatively, as a generalization of an exponential formula. Also shown is a one-to-one correspondence between stochastic Stieltjes integrators (thought of as generators of the evolution) of a certain class and stochastic product integrals of a corresponding class, together with an integral formula for the integrator corresponding to a given product integral process.

We now give a short sketch of why the solution of a Volterra-Stieltjes integral equation is a product integral. Let [a,b] be some interval of real numbers, let $\{s_k\}_{k=0}^{n}$ be a subdivision of [a,b], and let g be a function from [a,b] to some normed linear space in which the multiplication indicated below is permitted. We abbreviate $g(s_k) - g(s_{k-1})$ by $dg(s_{k-1}, s_k)$. Using the telescoping sum of MacNerney [10, Lemma 1.1], we have

(1.1) $$\prod_{k=1}^{n} [1 + dg(s_{k-1}, s_k)] - 1 =$$

$$\sum_{k=1}^{n} \{\prod_{j=1}^{k} [1 + dg(s_{j-1}, s_j)] - \prod_{j=1}^{k-1} [1 + dg(s_{j-1}, s_j)]\} =$$

$$\sum_{k=1}^{n} dg(s_{k-1}, s_k)\{\prod_{j=1}^{k-1} [1 + dg(s_{j-1}, s_j)]\}.$$

The crux of the theory is this: if (Ω,A,P) is a probability measure space and g is a function from $[a,b]$ to $L^2(\Omega,A,P)$ which satisfies certain additional hypotheses given in Definition II.1.1, then there is a function u, defined on each pair (s,t) with $s \leq t$ in $[a,b]$ with range in $L^2(\Omega,A,P)$, such that $u(s,t)$ is the L^2-norm limit of the net $\{\prod_{k=1}^{n} [1 + dg(r_{k-1}, r_k)] \mid \{r_k\}_{k=0}^{n}$ is a subdivision of $[s,t]\}$, where the directed set of subdivision of $[s,t]$ is ordered by refinement. Thus the first term in (1.1) is an approximation of $u(a,b)$ while the last term of (1.1) is an approximation of

$$(1.2) \qquad \sum_{k=1}^{n} dg(s_{k-1}, s_k)u(a, s_{k-1}).$$

If each of f and g is a function from $[a,b]$ to $L^2(\Omega,A,P)$ and satisfy certain additional hypotheses, then $(L)\int_a^b fdg$ is defined as the L^2-norm limit of the net $\{\sum_{k=1}^{n} f(r_{k-1})dg(r_{k-1}, r_k) \mid \{r_k\}_{k=0}^{n}$ is a subdivision of $[a,b]\}$. Thus the term in (1.2) is an approximation of $(L)\int_a^b u(a,\cdot)dg$, and we have that $u(a,\cdot)$ satisfies the following integral equation:

$$(1.3) \qquad u(a,b) = 1 + (L)\int_a^b u(a,\cdot)dg.$$

The theory presented here follows in part from recent work of Professor E. J. McShane [11], [12], and [13] in the area of stochastic integral equations and from earlier work by

Professor J. S. MacNerney [10] and [14] in the area of deterministic integral equations.

Chapter II develops the theory of the left Cauchy-Stieltjes integral $(L)\int_a^b fdg$ in a stochastic setting and explores convergence of the approximations shown above to the integral process. Chapter III introduces the product integral, investigates convergence of approximations to the product integral process, exhibits a pairing between the product integrals in Chapter III and the integrators in Chapter II, and, finally, shows that in the limit equation (1.1) becomes (1.3). Chapter IV presents examples of the processes considered in Chapters II and III.

2. Deterministic Volterra-Stieltjes Integral Equations

Succeeding chapters will make use of the following deterministic results from MacNerney's paper [14]. Let S denote some nonempty set with a linear ordering, denoted by $\leq$, and some least element, denoted by 0. To study difference equations one might choose $S = Z^+$, and to study integral equations one might choose $S = [0,1]$ or $S = [0,\infty)$, but there are other possible choices, some of them "larger" than $[0,\infty)$. Let $\Delta = \{(s,t) \in S\times S | s \leq t\}$. Let OA^+ [14, p. 150] denote the class of functions $V:\Delta \to [0,\infty)$ such that $V(s,t) + V(t,u) = V(s,u)$ whenever $s \leq t \leq u$ in S. We note that there is a one-to-one correspondence between class OA^+ and the class of functions $F:S \to [0,\infty)$ such that $F(0) = 0$ and F is non-decreasing: if V is in class OA^+, let F be defined by $F(s) = V(0,s)$, $s \in S$; and if F satisfies the conditions above, let $V = dF$, i.e., $V(s,t) = F(t) - F(s)$ whenever $s \leq t$ in S. Let OM^+ denote the class of functions $W:\Delta \to [1,\infty)$ such that $W(s,t)W(t,u) = W(s,u)$ whenever $s \leq t \leq u$ in S.

If (s,t) is in Δ, then a subdivision of (s,t) is a sequence $\rho = \{r_k\}_{k=0}^n$ such that $s = r_0$, $t = r_n$, and $r_{k-1} \leq r_k$ for $1 \leq k \leq n$. If σ and τ are two subdivisions of the same pair in Δ, then τ is said to be a refinement of σ if σ is a subsequence of τ, denoted $\tau >> \sigma$. Since $\{s,t\}$ is a subdivision of (s,t), any subdivision ρ of (s,t) is a refinement of $\{s,t\}$. If (a,b) is in Δ, V is in OA^+ and $\rho = \{r_k\}_{k=0}^n$ is a subdivision of (a,b), then $\Pi_\rho[1+V]$ denotes the product $\prod_{k=1}^n [1 + V(r_{k-1}, r_k)]$, and $\prod_a^b [1+V]$ denotes the limit, in the sense of refinements of subdivisions of (a,b), of the approximations $\{\Pi_\rho[1+V] | \rho >> \{a,b\}\}$. Indeed, $\prod_a^b [1+V] =$ L.U.B. $\{\Pi_\rho[1+V] | \rho >> \{a,b\}\} \leq \exp(V(a,b))\}$ [14, Lemma 2.1]. Similarly, if (a,b) is in Δ, W is in OM^+ and ρ is a subdivision of (a,b), then $\Sigma_\rho[W-1]$ denotes the sum $\sum_{k=1}^n [W(r_{k-1}, r_k) - 1]$, and $\sum_b^a [W-1]$ denotes the limit, in the sense of refinements of subdivisions, of the approximations $\{\Sigma_\rho[W-1] | \rho >> \{a,b\}\}$. Indeed, $\sum_a^b [W-1] =$ G.L.B. $\{\Sigma_\rho[W-1] | \rho >> \{a,b\}\}$ [14, Lemma 2.2].

Furthermore, there is a one-to-one function E^+ [14, Theorem 2.2] from OA^+ to OM^+ such that each of the following is equivalent:

(i) $(V,W) \in E^{+}$

(ii) $V(a,b) = \sum_a^b [W-1]$ for each pair (a,b) in Δ.

(iii) $W(a,b) = \prod_a^b [1+V]$ for each pair (a,b) in Δ.

3. Lemmas on Sums and Products

Sums and products which approximate integrals and product integrals appear frequently in the following chapters. Certain identities and inequalities are collected here as lemmas in order to avoid repetition elsewhere. We shall adopt the convention that if $\{x_i\}_{i=1}^n$ is a sequence of numbers or elements in some other space in which the operations below are defined, then

$$\sum_{i=k}^{j} x_i = 0 \qquad \text{and} \qquad \prod_{i=k}^{j} x_i = 1$$

whenever $1 \leq j < k \leq n$. The following two lemmas are well-known. The first is a special case of Lemma 1.1 of MacNerney [10]. The second is a discrete Gronwall inequality and may be proved by induction.

Lemma I.2.1

Suppose that $\{A_i\}_{i=1}^n$ is a sequence of numbers, then

(i) $$\prod_{i=1}^{n} [1+A_i] - 1 = \sum_{i=1}^{n} A_i \prod_{j=1}^{i-1} [1+A_j] \qquad \text{and}$$

(ii) $$\prod_{i=1}^{n} [1+A_i] - 1 - \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} A_i \{\prod_{j=1}^{i-1} [1+A_j] - 1\}.$$

<u>Lemma I.2.2</u>

Suppose that each of $\{A_k\}_{k=0}^n$, $\{B_k\}_{k=1}^n$, and $\{C_k\}_{k=1}^n$ is a sequence of non-negative numbers and, for $1 \leq k \leq n$,

$$(1.4) \qquad A_k \leq A_{k-1}\,(1+B_k) + C_k.$$

Then

$$(1.5) \qquad A_n \leq A_o \prod_{k=1}^{n} (1+B_k) + \sum_{k=1}^{n} \{C_k \prod_{j=k+1}^{n} (1+B_k)\}.$$

Frequently one is faced with the problem of finding a bound for a conditional expectation $E(|fg|\,|A_o)$, where f and g are in $L^2(\Omega,A,P)$ and A_o is a σ-subalgebra of A. (For the definitions of conditional expectations, see Chapter II.) One inequality which might be used is the Cauchy-Jordan inequality for conditional expectations (see Chung [15], p. 339).

$$(1.6) \qquad E(|fg|\,|A_o) \leq (E(f^2|A_o)E(g^2|A_o))^{1/2}.$$

An alternative approach is to use the inequality $2|xy| \leq x^2 + y^2$ for real numbers x and y to obtain

$$(1.7) \qquad E(|fg|\,|A_o) \leq (1/2)\{E(f^2|A_o) + E(g^2|A_o)\}.$$

Inequality (1.7), Lemma I.2.2 and Lemma II.2.1 of the next chapter provide most of the estimates for bounds of conditional expectations used in following chapters. A crucial point in the proofs of Lemma II.2.1 and succeeding lemmas is to use (1.7) instead of (1.6) in estimating the conditional expectation of terms in a sum and thereby avoid the problem of showing that a sum of square roots is small.

CHAPTER II

STOCHASTIC CAUCHY-STIELTJES INTEGRALS

Chapter II contains definitions, a theorem of existence, and an investigation of convergence for left Cauchy-Stieltjes integrals in a stochastic setting. The left Cauchy-Stieltjes integral $(L)\int_a^b fdg$ is defined as the limit (with respect to an appropriate norm) of approximations of the form

$$\sum_{k=1}^{n} f(r_{k-1})[g(r_k)-g(r_{k-1})], \quad a = r_0 \leq r_1 \leq \cdots \leq r_n = b$$

taken over the directed set of refinements of (a,b).

In the deterministic case, the reason for defining the limit in the sense of refinements of subdivisions rather than the mesh (= max $\{|r_k - r_{k-1}| \, | 1 \leq k \leq 1\}$) is that $\int_a^b fdg$ will not be well defined if f and g have the same type of jump discontinuity at the same point. Considering the integral equation

$$(2.1) \qquad z(t) = 1 + (L)\int_a^t zdg$$

one can see that any solution would (in general) have the same type of discontinuity as g. It is our desire to find discontinuous solutions to an integral equation with a discontinuous integrator; this requires a stochastic integral which can accommodate an integrator and integrand with the same discontinuities (see, e.g., the hypotheses of [11, Theorem 2] and the remark following).

1. Definitions

We now establish the setting for Chapters II and III. The field of real numbers is denoted by R; S denotes some non-empty set with a linear ordering denoted by $\leq$ and a least element denoted by o; (Ω,A,P) denotes a probability measure space; and $\{A_t \mid t \in S\}$ denotes a family of σ-subalgebras of A with the property that $A_s \subseteq A_t$ whenever $s \leq t$ in S. The expectation operator is denoted by $E(\cdot)$, thus $E(f) = \int_\Omega f dP$ for any measurable function $f:\Omega \to R$ such that the integral is finite. If B is a σ-subalgebra of A, then, whenever E(f) exists, $E(f|B)$ denotes the conditional expectation of f relative to B; thus $E(f|B) = g$ means that $g:\Omega \to R$ is B-measurable, and $\int_Q f dP = \int_Q g dP$ whenever Q is in B.

Recall (Chung [15] pp. 277-282) that if B and C are σ-subalgebras of A such that $C \subseteq B \subseteq A$, and E(f) exists, then $E(E(f|B)|C) = E(f|C) = E(E(f|C)|B)$ and $E(E(f|B)) = E(f)$. Moreover, if $g:\Omega \to R$ is C-measurable, then $E(fg|C) = gE(f|C)$. Let $L^2(\Omega,A,P)$, or simply $L^2(A)$, denote the class of A-measurable functions from Ω to R for which $E(f^2)$ is finite. Let $||\cdot||$ denote the L^2 (semi-) norm; thus $||F|| = \{\int_\Omega f^2 dP\}^{1/2}$. As usual, we consider two functions $f,g:\Omega \to R$ equivalent if they are A-measurable and are equal on the complement of a set of P-measure zero. Any particular element of such an equivalence class is called a version of the class.

Again, $\Delta = \{(s,t)\varepsilon S\times S \mid s \leq t\}$, a monotone sequence $\{r_k\}_{k=0}^{n}$ in S is a subdivision of $(a,b)\varepsilon\Delta$ if $r_0 = a$ and $r_n = b$, and subdivision τ refines subdivision σ, denoted $\tau >> \sigma$, if σ is a subsequence of τ. If h is a function from Δ to $L^2(\Omega,A,P)$, then whenever $\sigma = \{s_k\}_{k=0}^{n}$ is a subdivision of some pair in Δ, $\Sigma_\sigma h$ denotes $\sum_{k=1}^{n} h(s_{k-1},s_k)$. The statement "$\sum_a^b h$ exists" for some pair (a,b) in Δ means that there is an element f in $L^2(\Omega,A,P)$ such that, for every $\varepsilon > 0$, there is a subdivision σ of (a,b) such that if $\tau >> \sigma$, then

$$||\Sigma_\tau h - f|| < \varepsilon.$$

If the situation in the preceding sentence holds, then we write $\sum_a^b h = f$. There is some ambiguity since if f = g except on a set of P-measure zero, then $\sum_a^b h$ is equivalent to g.

We now introduce the classes of functions which will become the integrators and integrands of the stochastic integrals.

<u>Definition II.1.1</u>

(i) Let Γ_1 denote the class to which a function $g:S \to L^2(\Omega,A,P)$ belongs only in case g(t) is an element of $L^2(\Omega,A_t,P)$ for each t in S.

(ii) Let Γ_2 denote the subclass of Γ_1 to which g belongs only in case there is a non-decreasing function $G_2:S \to R$ such that $G_2(o) = 0$ and whenever $s \leq t$ in S, the condition

$$E([g(t)-g(s)]^2|A_s) \leq G_2(t)-G_2(s) \tag{2.2}$$

holds except on a set of P-measure zero.

(iii) Let Γ_3 denote the subclass of Γ_2 to which g belongs only in case there is a non-decreasing function $G_1:S \to R$ such that $G_1(o) = 0$ and whenever $s \leq t$ in S, the condition

$$|E(g(t)-g(s)|A_s)| \leq G_1(t)-G_1(s) \tag{2.3}$$

holds except on a set of P-measure zero.

(iv) Let Γ_o denote the subclass of Γ_1 to which f belongs only in case there is a non-decreasing function $F:S \to R$ such that $F(o) = 0$ and the condition

$$E([f(t)-f(s)]^2) \leq F(t)-F(s) \tag{2.4}$$

holds whenever $s \leq t$ in S.

Remark II.1.2

The statement $(g, G_1, G_2)\varepsilon\Gamma_3$ means that g is in class Γ_3 and G_1 and G_2 are non-decreasing functions from S to R such that conditions (2.3) and (2.2) hold, respectively, and $G_1(o) = G_2(o) = 0$. The statement $(f,F)\varepsilon\Gamma_o$ means that f is in class Γ_o and f is a non-decreasing function from S to R such that F(o) = 0 and condition (2.4) holds. We now define the left and right stochastic Cauchy-Stieltjes integrals.

Definition II.1.3

If each of f and g is a function from S to $L^2(\Omega,A,P)$ then for $a \leq b$ in S, $(L)\int_a^b fdg$ and $(R)\int_a^b fdg$ denote $\Sigma_a^b h$, where h(s,t) is f(s)(g(t)-g(s)) and f(t)(g(t)-g(s)), respectively, whenever $s \leq t$ in S.

Lemma II.1.4

If each of f, g, and h is a function from S to $L^2(\Omega,A,P)$ and all except possibly one of the integrals in each statement below exist, then the remaining integral exists and the statement holds.

(i) $(L)\int_a^b fdg + (R)\int_a^b gdf = f(b)g(b) - f(a)g(a)$ for $a \leq b$ in S.

(ii) $(L)\int_a^b fdg + (L)\int_b^c fdg = (L)\int_a^c fdg$ for $a \leq b \leq c$ in S.

(iii) $(L)\int_a^b (f+g)dh = (L)\int_a^b fdh + (L)\int_a^b gdh$ for $a \leq b$ in S.

$$\text{(iv)}\quad (L)\int_a^b fd(g+h) = (L)\int_a^b fdg + (L)\int_a^b fdh$$

for $a \leq b$ in S.

Moreover, the conclusion holds if (L) is replaced throughout by (R) in the last three statements.

Indication of Proof:

Let $(a,b) \in \Delta$ be given and let $\rho = \{r_k\}_{k=o}^n$ be a subdivision of (a,b). Then $f(b)g(b) - f(a)g(a) =$

$$\sum_{k=1}^{n} f(r_k)g(r_k) - f(r_{k-1})g(r_{k-1}) =$$

$$\sum_{k=1}^{n} f(r_{k-1})[g(r_k)-g(r_{k-1})] + \sum_{k=1}^{n} g(r_k)[f(r_k)-f(r_{k-1})].$$

Part (i) follows immediately. Now suppose, for example, that $(L)\int_a^b fdg$ and $(L)\int_a^c fdg$ exist in (ii). Let $\varepsilon > o$ be given, and find partitions $\sigma >> \{a,b\}$ and $\rho >> \{a,c\}$ so that

$$||(L)\Sigma_{\sigma_1} fdg - (L)\int_a^b fdg|| < \frac{\varepsilon}{2} \quad \text{and}$$

$$||(L)\Sigma_{\rho_1} fdg - (L)\int_a^c fdg|| < \frac{\varepsilon}{2}$$

whenever $\sigma_1 >> \sigma$ and $\rho_1 >> \rho$. Let ρ_1 denote the subdivision of ρ obtained by inserting b and any points of σ which are not in ρ. Let τ be the restriction of ρ_1 to (b,c). Let τ_1 be a refinement of τ, and let ρ_2 be the refinement of ρ_1 obtained by inserting any points in τ_1 which were not in ρ_1.

Let σ_1 be the restriction of ρ_2 to (a,b). Then

$$(L)\Sigma_{\tau_1} fdg = (L)\Sigma_{\rho_2} fdg - (L)\Sigma_{\sigma_1} fdg, \quad \text{and}$$

$$\|(L)\Sigma_{\tau_1} fdg - (L)\int_a^c fdg + (L)\int_a^b fdg\|$$

$$\leq \|(L)\Sigma_{\rho_2} fdg - (L)\int_a^c fdg\| + \|(L)\Sigma_{\sigma_1} fdg - (L)\int_a^b fdg\| < \varepsilon.$$

Hence $(L)\int_b^c fdg = (L)\int_a^c fdg - (L)\int_a^b fdg$. The other cases of (ii) are similar. The right-hand version of (ii) is proved with (ii) and (i). Parts (iii) and (iv) follow from a careful investigation of the partial sum approximations. The right-hand versions of (iii) and (iv) follow from their left-hand versions and part (i).

2. Existence of the Stochastic Left Cauchy-Stieltjes Integral

The following lemma is useful in providing estimates for approximating sums for the left Cauchy-Stieltjes integrals considered below and for approximations of product integrals in Chapter III. The conclusion and the proof are both modifications of Lemma (1.1), p. 59, in McShane [13] (see also Lemma 1, p. 290, McShane [11].

Lemma II.2.1

Suppose that $\{A_i\}_{i=1}^n$ is a family of σ-subalgebras of A such that $A_i \subseteq A_j$ if $i \leq j$, and each of $\{x_i\}_{i=1}^n$ and

$\{y_i\}_{i=1}^n$ is a sequence with range in $L^2(\Omega,A,P)$ such that x_{i-1} and y_i are A_k-measurable for $i \leq k$. Suppose that each of $\{X_i\}_{i=1}^n$, $\{Y_i\}_{i=1}^n$ and $\{Z_i\}_{i=1}^n$ is a sequence of numbers such that, except on a set of measure zero, the following estimates hold for $1 \leq i \leq n$:

(i) $E(x_i^2|A_i) \leq X_i$,

(ii) $E(y_i^2) \leq Y_i$,

(iii) $|E(x_i|A_i)| \leq Z_i$.

Then

$$(2.5) \quad E(\{\sum_{i=1}^n x_i y_i\}^2) \leq \prod_{j=1}^n [1+Z_j] \cdot \sum_{i=1}^n Y_i\{X_i+Z_i\}.$$

Corollary II.2.2

If condition (ii) of the preceding lemma is replaced by

(ii') $E(y_i^2|A_o) \leq Y_i$, except on a set of measure zero,

where A_o is a σ-subalgebra of A_1, and the remainder of the hypotheses of the lemma are unchanged, then the left-hand side of (2.5) can be replaced by

$$E(\{\sum_{i=1}^n x_i y_i\}^2|A_o)$$

and the modified inequality holds except on a set of measure zero.

<u>Proof of Lemma II.2.1</u>

Let $D_k = \sum_{i=1}^{k} x_i y_i$ for $1 \leq k \leq n$. Define $D_o = 0$. Then for $k > 1$,

$$||D_k||^2 = ||D_{k-1} + x_k y_k||^2$$

$$= ||D_{k-1}||^2 + 2E(D_{k-1} x_k y_k) + E((x_k y_k)^2)$$

$$= ||D_{k-1}||^2 + 2E(D_{k-1} y_k E(x_k | A_k)) + E(y_k^2 E(x_k^2 | A_k))$$

$$\leq ||D_{k-1}||^2 + 2E(|D_{k-1}\ y_k|) Z_k + E(y_k^2) X_k$$

$$\leq ||D_{k-1}||^2 + E(D_{k-1}^2 + y_k^2) Z_k + E(y_k^2) X_k$$

$$\leq ||D_{k-1}||^2 (1 + Z_k) + Y_k (X_k + Z_k).$$

Hence by the discrete Gronwall lemma (Lemma I.2.2),

$$||D_n||^2 \leq \sum_{i=1}^{n} \prod_{j=i+1}^{n} [1 + Z_i] Y_i (X_i + Z_i).$$

The lemma follows immediately. The proof of the corollary is similar.

<u>Lemma II.2.3</u>

Suppose that (f,F) and (g,G_1,G_2) are members of classes Γ_o and Γ_3 respectively, and $a \le b$ in S. Let $M = \exp(G_1(b) - G_1(a))$ and $G = G_1+G_2$. Then

$$||(L)\Sigma_\tau fdg - (L)\Sigma_\sigma fdg||^2 \le M\{(L)\Sigma_\tau FdG - (L)\Sigma_\sigma FdG\}$$

whenever $\tau >> \sigma >> \{a,b\}$.

Proof:

Suppose $\sigma = \{s_k\}_{k=0}^n$, $\tau = \{t_j\}_{j=0}^m$ and $\tau >> \sigma >> \{a,b\}$. For $1 \le j \le m$, let $p(j) = \sup\ \{k|s_k \le t_{j-1}\}$. Then $s_{p(j)} \le t_{j-1} \le t_j \le s_{p(j)+1}$ and $t_{j-1} < s_{p(j)+1}$ for $1 \le j \le m$. Then

$$(L)\Sigma_\tau fdg = \sum_{j=1}^{m} f(t_{j-1})dg(t_{j-1},t_j)$$

and

$$(L)\Sigma_\sigma fdg = \sum_{k=1}^{n} f(s_{k-1})dg(s_{k-1},s_k)$$

$$= \sum_{k=1}^{n} f(s_{k-1}) \sum_{p(j)=k-1} dg(t_{j-1},t_j)$$

$$= \sum_{j=1}^{m} f(s_{p(j)})dg(t_{j-1},t_j).$$

Thus

$$(L)\Sigma_\tau fdg - (L)\Sigma_\sigma fdg = \sum_{j=1}^{m} df(s_{p(j)},t_{j-1})dg(t_{j-1},t_j).$$

For $1 \leq j \leq m$, let $x_j = dg(t_{j-1},t_j)$, $y_j = df(s_{p(j)},t_{j-1})$, $\Lambda_j = A_{t_j}$, $X_j = dG_2(t_{j-1},t_j)$, $Y_j = dF(s_{p(j)},t_{j-1})$, $Z_j = dG_1(t_{j-1},t_j)$ and $W_j = dG(t_{j-1},t_j) = X_j+Z_j$. Now note that

$$\prod_{j=1}^{m} [1+Z_j] \leq \prod_{j=1}^{m} \exp(Z_j) = \exp(\sum_{j=1}^{m} Z_j) = M.$$

The sequences and σ-algebras defined above satisfy the hypotheses of Lemma II.2.1 by Definition II.1.1, hence by that lemma,

$$||(L)\Sigma_\tau fdg - (L)\Sigma_\sigma fdg||^2 \leq \prod_{j=1}^{n} (1+Z_j)\cdot\sum_{i=1}^{m} Y_iW_i$$

$$\leq M \sum_{i=1}^{m} dF(s_{p(j)},t_{j-1})dG(t_{j-1},t_j)$$

$$= M\{(L)\Sigma_\tau FdG - (L)\Sigma_\sigma FdG\}.$$

The last equality follows from reversing the steps in the manipulations of sums above.

<u>Theorem II.2.4</u>

Suppose that f and g are members of classes Γ_o and Γ_3, respectively, and $a \leq b$ in S. Then $(L)\int_a^b fdg$ exists and has an A_b-measurable version.

<u>Proof</u>:

Let F, G_1 and G_2 be the non-decreasing real-valued functions associated with f and g in Definition II.1.1, and let $G = G_1 + G_2$. J. S. MacNerney has shown [14, Lemma 4.3] that $(L)\int_a^b FdG$ exists (indeed, $(L)\int_a^b FdG = \text{L.U.B.}\{(L)\Sigma_\tau FdG \mid \tau >> \{a,b\}\}$). Hence, for each $\varepsilon > o$, there is a subdivision $\sigma >> \{a,b\}$ such that if $\tau >> \sigma$, then

$$|(L)\Sigma_\tau FdG - (L)\int_a^b FdG| < \frac{\varepsilon^2}{2M},$$

where M is as in Lemma II.2.2. Thus if $\tau >> \sigma$ and $\tau' >> \sigma$, then by Lemma II.2.2,

$$||(L)\Sigma_\tau fdg - (L)\Sigma_{\tau'} fdg|| < \varepsilon.$$

It follows that the directed set of subdivisions of the pair (a,b) generates a Cauchy net

$$\{(L)\Sigma_\tau fdg \mid \tau >> \{a,b\}\}$$

with range in the space $L^2(\Omega, A_b, P)$. In order to see that this net has a measurable limit we proceed as follows: choose a sequence of subdivisions $\{\sigma(n)\}_{n=1}^\infty$ such that $\sigma(n+1) >> \sigma(n) >> \{a,b\}$ and

$$|(L)\Sigma_{\sigma(n)} FdG - (L)\int_a^b FdG| < \frac{1}{2Mn^2}.$$

Thus $\{(L)\Sigma_{\sigma(n)}\ fdg\}_{n=1}^{\infty}$ is a Cauchy sequence in the complete space $L^2(\Omega, A_b, P)$. It follows that there is an element h in $L^2(\Omega, A_b, P)$ such that h is the limit in the L^2-norm of $(L)\Sigma_{\sigma(n)} fdg$. Hence, from the work above,

$$||(L)\Sigma_{\tau} fdg - h|| < \frac{1}{n}$$

for each $\tau >> \sigma(n)$. Thus h is a version of $(L)\int_a^b fdg$.

Corollary II.2.5

Let $N(a,t)[f] = \sup\{||f(s)|| \mid a \leq s \leq t\}$ whenever $a \leq t$ in S and f is in class Γ_o. Suppose (g, G_1, G_2) is in class Γ_3. Let $G = G_1 + G_2$ and let $M = \exp(G_1(b) - G_1(a))$. Then the following inequalities hold for $a \leq b$ in S:

(i) $||(L)\int_a^b fdg||^2 \leq M(L)\int_a^b ||f||^2 dG$ if the integral on the right exists, and

(ii) $||(L)\int_a^b fdg||^2 \leq M(L)\int_a^b \{N(a,t)[f]\}^2 dG$.

Indication of Proof:

If the proof of Lemma II.2.3 is modified such that $y_j = f(t_{j-1})$ for $1 \leq j \leq m$ and $Y_j = ||f(t_{j-1})||^2$ for $1 \leq j \leq m$, then we obtain

$$||(L)\Sigma_{\tau} fdg||^2 \leq M(L)\Sigma_{\tau} ||f||^2 dG$$

whenever $\tau >> \{a,b\}$. The conclusions follow immediately.

3. Uniform Convergence of Approximations

We have now shown the existence of a version of $\int_a^b fdg$ whenever $a \leq b$ in S, $f \in \Gamma_0$ and $g \in \Gamma_3$. Our investigation now turns to the behavior of a stochastic process $h \in \Gamma_0$ such that $h(t) = (L)\int_a^t fdg$ for each $t \geq a$. We make the following assumptions for the remainder of this section.

If f is in class Γ_1, $a \leq b$ in S and $\sigma = \{s_k\}_{k=0}^n >> \{a,b\}$, then f_σ denotes the step-function defined by

$$f_\sigma(t) = f(s_k) \quad \text{if} \quad s_k \leq t < s_{k+1}, \quad 0 \leq k \leq n-1$$

and $f_\sigma(b) = f(b)$.

Note that if f is in class Γ_1 and g is in class Γ_3 and $\sigma >> \{a,b\}$, then

$$(L)\Sigma_\sigma fdg = (L)\int_a^b f_\sigma dg.$$

Suppose $S = [0,a]$ for some a in R. Fix (f,F) and (g,G_1,G_2) in classes Γ_0 and Γ_3 respectively and suppose that each of F, G_1 and G_2 is right-continuous. Let $G = G_1+G_2$ and let $M = (1+G(a))\exp(G_1(a))$.

Lemma II.3.1

For each integer $n \geq 1$, there is a subdivision $\sigma(n)$ of $\{a,b\}$ such that if $\tau >> \sigma(n)$, then

$$F(t) - F_\tau(t) < \frac{1}{4n^6M^2}$$

for each t in [0,a].

Proof:

Let $m = [\![\frac{F(b)}{\varepsilon}]\!] + 1$, where $\varepsilon = \frac{1}{4n^6M^2}$. For $1 \leq k \leq m-1$, let $s_k = \inf\{s | F(s) \geq k\varepsilon\}$, $s_o = 0$, $s_m = a$. Then $F(s_k) \geq k\varepsilon$. If $s_k \leq t \leq s_{k+1}$, then $k\varepsilon \leq F(s_k) \leq F(t) < (k+1)\varepsilon$, so $|F(s_k)-F(t)| < \varepsilon$. If τ is a refinement of $\{s_k\}_{k=1}^m$, then $F(s_k) \leq F_\tau(t) \leq F(t)$, so $|F(t)-F_\tau(t)| < \varepsilon$.

Definition II.3.2

Let $\pi = \{\sigma(n)\}_{n=1}^\infty$ be a sequence of subdivisions of [0,a] such that $(L)\int_0^a FdG = \lim_{n\to\infty} (L)\int_0^a F_{\sigma(n)}dG$. By a version of $(L)\int_a^t fdg$ determined by π, we mean a function $h:S \to L^2(\Omega,A,P)$ for which $(L)\int_a^t f_{\sigma(n)}dg$ converges in the L^2-norm to h(t) for each t in S.

Remark:

There does exist at least one such sequence of partitions: let π be the sequence guaranteed by Lemma II.3.1. Then $0 \leq (L)\int_0^a (F-F_{\sigma(n)})dG \leq \frac{G(a)}{4n^6M^2}$. Then for each t in [0,a],

$$||(L)\int_0^t fdg - (L)\int_0^t f_{\sigma(n)}dg||^2 \leq (L)\int_0^t (F-F_{\sigma(n)}dG \leq \frac{G(t)}{4n^6M^2} .$$

Hence $(L)\int_0^t f_{\sigma(n)}dg$ is a Cauchy sequence in $L^2(\Omega,A_t,P)$.

Before continuing, we introduce a useful result by

Orey [16] concerning F-processes.

Definition II.3.3 (Orey [16], p. 301)

Suppose k is in class Γ_1. Then k is an F-process if there exists a constant K such that for every partition τ of $[0,a]$,

$$E\left(\sum_{i=1}^{m} |E(k(t_i)-k(t_{i-1})|A_{t_{i-1}})|\right) \leq K.$$

Any such K is called an F-bound for k. We may assume, without loss of generality, that τ is a refinement of any prescribed subdivision.

Theorem II.3.4 (Orey [16], Theorem 2.1, p. 303)

Let $k\varepsilon\Gamma_1$ be a separable F-process on $[0,a]$ and let K be an F-bound for k. Then for any $\varepsilon>0$, $\varepsilon P[\sup_{0<s<a} k(s) \geq \varepsilon] \leq E|k(a)|+K$, and $\varepsilon P[\inf_{0\leq s\leq a} k(s) \leq -\varepsilon] \leq E|k(a)|+K$.

Remark:

The following situation will hold for the remainder of this section: Let $\pi = \{\sigma(n)\}_{n=1}^{\infty}$ be a sequence of subdivision of $[0,a]$ which satisfy Definition II.3.2. For each n, let $h_n\varepsilon\Gamma_1$ be a separable version of $(L)\int_0^{\cdot} f_{\sigma(n)}dg$, i.e. $h_n(t)$ is a version of $(L)\int_0^t f_{\sigma(n)}dg$ for each t in $[0,a]$ and $h_n(\cdot)$ is a separable process (see Doob [17], Theorem 2.4, p. 57). There is a problem that h_n (and h defined below) may be an extended-real-valued function, but each h_n is finite on a set of measure 1 (to show this use Corollary II.2.5). Let h be a separable version of $(L)\int_0^{\cdot} fdg$, i.e. $h(t) = (L)\int_0^t fdg$.

Lemma II.3.5

If $k_n = h - h_n$, then k_n is an F-process with F-bound $K_n = \frac{1}{2n^3}$.

Proof:

Let $\tau = \{t_i\}_{i=0}^{n}$ be a sub-division of $[0,a]$. We assume without loss of generality that $\tau >> \sigma(n)$. Let $\varepsilon > 0$ be given. For each i, $1 \leq i \leq m$, let $\rho(i)$ be a subdivision of $[t_{i-1}, t_i]$ such that

$$||(L)\int_{t_{i-1}}^{t_i} f dg - (L)\int_{t_{i-1}}^{t_i} f_{\rho(i)} dg|| \leq \frac{\varepsilon}{m} .$$

Let ρ be the subdivision of $[a,b]$ obtained by "piecing together" the subdivisions $\{\rho(i)\}_{i=1}^{m}$ across the interval $[a,b]$. Then

$$E(|E(dk_n(t_{j-1}, t_j)|A_{t_{j-1}})|)$$

$$\leq E(|(L)\int_{t_{j-1}}^{t_j} (f - f_\rho) dg|) +$$

$$E(|E(L)\int_{t_{j-1}}^{t_j} (f_\rho - f_{\sigma(n)}) dg | A_{t_{j-1}})|).$$

Suppose $\rho(j) = \{r_i\}_{i=0}^{k}$ and $s_{u(i)} \leq t_{j-1} \leq r_o \leq \cdots \leq r_k \leq t_j \leq s_{u(j)+1}$ (since τ refines $\sigma(n)$). Then

$$(L)\int_{t_{j-1}}^{t_j} (f_\rho - f_{\sigma(n)}) dg = \sum_{i=1}^{k} (f(r_{i-1}) - f(s_{u(j)}) dg(r_{i-1}, r_i),$$

and $E(|E((L)\int_{t_{j-1}}^{t_j} (f_\rho - f_{\sigma(n)} dg|A_{t_{j-1}})|)$

$$\leq E \sum_{i=1}^{k} E(|f(r_{i-1})-f(s_{u(j)})||E(dg(r_{i-1},r_i)|A_{r_{i-1}})||A_{t_{j-1}})$$

$$\leq \sum_{i=1}^{k} E(|f(r_{i-1})-f(s_{u(j)})|)dG_1(r_{i-1},r_i)$$

$$\leq \sum_{i=1}^{k} \sqrt{dF(s_{u(j)},r_{i-1})}\, dG_1(r_{i-1},r_i)$$

$$\leq \frac{dG_1(t_{j-1},t_j)}{2n^3M} .$$

Thus $\sum_{j=1}^{m} E(|E(dk_n(t_{j-1},t_j)|A_{t_{j-1}})|)$

$$\leq \sum_{j=1}^{m} \left(\frac{\varepsilon}{m} + \frac{dG_1(t_{j-1},t_j)}{2n^3M}\right)$$

$$\leq \varepsilon + \frac{1}{2n^3} .$$

Since ε was arbitrary, the last estimate must hold for $\varepsilon = 0$ and the lemma is proved.

Theorem II.3.6

If (f,F) and (g,G_1,G_2) are in classes Γ_1 and Γ_3 respectively with F, G_1 and G_2 right continuous on $S = [0,a]$, then versions of the stochastic integral

$$h(t) = (L)\int_0^t fdg, \qquad 0 \leq t \leq a$$

can be chosen in such a way that h is a separable process in class Γ_1. Almost all sample functions will be right continuous on $[0,a)$ and will have left-limits on $(0,a]$. The fixed discontinuities of h will be points of discontinuity of at least one of the functions F, G_1 and G_2. Moreover, there is a sequence $\pi = \{\sigma(n)\}_{n=1}^{\infty}$ of subdivisions of $[0,a]$ such that the sequence of separable processes $\{h_n\}_{n=1}^{\infty}$ satisfying

$$h_n(t) = (L)\int_0^t f_{\sigma(n)}dg$$

for each t in $[0,a]$ converge in the mean to h(t) for each t in $[0,a]$; indeed, the paths of h_n converge uniformly to the paths of h with probability one.

Proof:

Let $\pi = \{\sigma(n)\}_{n=1}^{\infty}$, f_n, h, and h_n be as in the remark following II.3.4. Then h and h_n will have the required separability properties. It follows that

$$E([dh(s,t)]^2) \leq dH(s,t) \quad \text{where}$$

$H(t) = 2\{||f(0)||^2 dG(0,t) + (L)\int_0^t FdG\}$. Then h will be as right continuous and have left-limits because H is right-continuous and has left-limits.

Now note that if $k_n = h-h_n$, $E|k_n(a)| \leq ||(L)\int_0^a (f-f_n)dg||$

$$\leq \sqrt{(L)\int_0^a (F-F_n)\,dGM} \leq \sqrt{\frac{dG(0,a)M}{4n^6M^2}} \leq \frac{1}{2n^3},$$

so by Theorem II.3.4 and Lemma II.3.5,

$$P\{\sup_{0\leq s\leq a} |h(s)-h_n(s)| \geq \tfrac{1}{n}\} \leq 2n(E|k_n(a)| + K_n) \leq \frac{2n}{n^3} = \frac{2}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{2}{n^2} < \infty$ it follows from the Borel-Cantelli lemma that $|h(s)-h_n(s)| < \frac{1}{n}$ for $0 \leq s \leq a$ for sufficiently large n with probability 1.

Remark

(i) The results for stochastic left Cauchy-Stieltjes integrals in Theorem II.3.6 are analagous to the results for the stochastic integrals in Doob [17], Theorem 5.2, p. 445.

(ii) Theorem II.3.6 ends our study of the stochastic left Cauchy-Stieltjes integral. In the next chapter we prove the existence of the stochastic product integral which will be used to represent a solution of

$$u(t) = 1+(L)\int_a^t udg, \quad a \leq t \quad \text{in } S$$

where the integral is a stochastic left Cauchy-Stieltjes integral and g is in class Γ_3.

CHAPTER III

STOCHASTIC PRODUCT INTEGRALS

Product integrals have been used to represent the evolution system generated by deterministic differential and integral equations by many authors. As a (not necessarily representative) selection, we cite J. S. MacNerney [10], [14]; J. W. Neuberger [18]; J. V. Herod [19], [20]; J. A. Reneke [21]; B. W. Helton [22]; G. F. Webb [23], [24]; R. H. Martin [25]; M. G. Crandall and T. M. Liggett [26]; H. Brezis and A. Pazy [27]; J. A. Goldstein [28]; D. L. Lovelady [29]; G. Schmidt [30] and G. Birkhoff [31]. In addition, the Cauchy polygonal process has been used by G. Maruyama [32] and E. J. McShane [13] to represent the solutions of stochastic integral equations of the Itô and belated type with continuous integrators and solutions.

In this chapter we develop a stochastic product integral which will represent the solution of the stochastic integral equation

$$z(t) = 1+(L)\int_{o}^{t} zdg, \quad t \text{ in } s, \tag{3.1}$$

where g is a member of the class Γ_3 described in Chapter II. Moreover, a pairing, similar to J. S. MacNerney's mapping E^+

in [14], is found between the generators, dg, of the evolution system and the evolutions.

Section 1 contains lemmas which provide bounds for the norms of partial products approximating the product integral and a theorem of existence for the product integral. Section 2 shows the pairing between the class T_3 of stochastic evolutions and a subset of generators in class Γ_3. In Section 3 the product integral is seen to generate the almost surely unique solution in class Γ_0 for the integral equation (3.1). Section 4 shows that, with mild additional hypotheses, there is a sequence of approximations of the product integral whose paths converge almost surely uniformly on each bounded interval in S to a version of the stochastic product integral.

1. Existence of the Stochastic Product Integral

Suppose h is a function from Δ to $L^2(\Omega,A,P)$, $a \leq b$ in S, and $\rho = \{r_k\}_{k=0}^{n} >> \{a,b\}$, then define $\Pi_\rho h = \prod_{k=1}^{n} h(r_{k-1},r_k)$. The statement " $\prod_a^b h$ exists" means that there is an element f in $L^2(\Omega,A,P)$ such that for every $\varepsilon>0$, there is a subdivision $\sigma >> \{a,b\}$ such that if $\tau >> \sigma$, then

$$||\Pi_\tau h - f|| < \varepsilon.$$

If the situation in the preceding sentence holds, then we write $\prod_a^b h = f$. As with $\sum_a^b h$, there is some ambiguity since if f = g except on a set of P-measure zero, then $\prod_a^b h$ is also equivalent to g.

Temporarily fix (g,G_1,G_2) in class Γ_3 and for convenience let $G = 2G_1+G_2$. The following lemmas provide estimates for partial products and differences of partial products which estimate $\prod_a^b [1+dg]$.

Lemma III.1.1

Suppose $\rho = \{r_k\}_{k=0}^n >> \{a,b\}$. For convenience let A_k denote A_{r_k} and let $M = \exp(G_1(b)-G_1(a))$. Then the following hold except on a set of measure zero:

(i) $E([1+dg(r_{k-1},r_k)]^2|A_{k-1}) \leq 1+dG(r_{k-1},r_k)$ for $1 \leq k \leq n$,

(ii) $E(\pi_\rho[1+dg]^2|A_0) \leq \pi_\rho[1+dG]$,

(iii) $E(\{\pi_\rho[1+dg]-1\}^2|A_0) \leq \pi_\rho[1+dG]-1$,

(iv) $E(\{\pi_\rho[1+dg]-[1+dg(a,b)]\}^2|A_0) \leq$

$$M\{\pi_\rho[1+dG]-[1+dG(a,b)]\}.$$

Proof:

In what follows, let $x_k = dg(r_{k-1},r_k)$, $X_k = dG(r_{k-1},r_k)$, $y_k = \prod_{i=1}^{k-1} [1+x_i]$, $Y_k = \prod_{i=1}^{k-1} [1+X_i]$, $Z_k = dG_1(r_{k-1},r_k)$, $W_k = dG_2(r_{k-1},r_k)$ for $1 \leq k \leq n$. Then for $1 \leq k \leq n$,

$$\begin{aligned} E([1+x_k]^2|A_{k-1}) &= E(1+2x_k+x_k^2|A_{k-1}) \\ &\leq 1+2Z_k+W_k \\ &= 1+X_k, \end{aligned}$$

where the inequality follows from Definition II.1.1.

Thus (i) is proved. We prove (ii) by induction. Suppose the condition

$$(3.2) \qquad E\Big(\prod_{i=1}^{k} [1+x_i^2] \,|\, A_0\Big) \leq \prod_{i=1}^{k} [1+X_i]$$

holds for some k, $1 \leq k < n$. Then

$$E\Big(\prod_{i=1}^{k+1} [1+x_i^2] \,|\, A_0\Big) = E\Big(\prod_{i=1}^{k} [1+x_i]^2 E([1+x_{k+1}]^2 \,|\, A_k) \,|\, A_0\Big)$$

$$\leq [1+X_{k+1}] E\Big(\prod_{i=1}^{k} [1+x_i]^2 \,|\, A_0\Big)$$

$$\leq \prod_{i=1}^{k+1} [1+X_i].$$

Thus (3.2) holds for k+1. Since (3.2) holds for k=1 by part (i), it must hold for all $1 \leq k \leq n$. To prove (iii), we first use the telescoping series identity (i) of Lemma I.2.1. Thus

$$E\Big(\Big\{\prod_{k=1}^{n} [1+x_k]-1\Big\}^2 \,|\, A_0\Big) = E\Big(\Big\{\sum_{k=1}^{n} x_k \prod_{i=1}^{k-1} [1+x_i]\Big\}^2 \,|\, A_0\Big)$$

$$= E\Big(\Big\{\sum_{k=1}^{n} x_k y_k\Big\}^2 \,|\, A_0\Big).$$

Now we apply Corollary II.2.2 with W_k substituted for X_k,

$1 \leq k \leq n$, to obtain from the last inequality in the proof of Lemma II.2.1 the following inequality:

$$(3.3)\quad E(\{\prod_{k=1}^{n}[1+x_k]-1\}^2|A_0) \leq \sum_{i=1}^{n}\prod_{j=i+1}^{n}[1+Z_j](Z_i+W_i)Y_i$$

except on a set of P-measure zero. Recall that $Y_i = \prod_{j=1}^{i-1}[1+X_j]$, and $Z_i+W_i = X_i-Z_i$; thus the right-hand side of (3.3) is a telescoping sum with value $\prod_{i=1}^{n}[1+X_i] - \prod_{i=1}^{n}[1+Z_i]$. Part (iii) follows from the observation that $\prod_{i=1}^{n}[1+Z_i] \geq 1$.

To show part (iv), use part (ii) of Lemma I.2.1 to write

$$E(\{\pi_\rho[1+dg]-1-dg(a,b)\}^2|A_0) = E(\{\sum_{i=1}^{n} x_i\ (\prod_{j=1}^{i-1}[1+x_j]-1)\}^2|A_0)$$

$$= E(\{\sum_{i=1}^{n} x_i(y_i-1)\}^2|A_0).$$

By part (iii), $E((y_i-1)^2|A_0) \leq \prod_{j=1}^{i-1}[1+X_i]-1$, so by Corollary II.2.2, the following inequality holds except on a set of measure zero.

$$(3.4)\qquad E(\{\sum_{i=1}^{n} x_i(y_i-1)\}^2|A_0) \leq$$

$$\prod_{j=1}^{n}[1+Z_j]\sum_{i=1}^{n}\{Z_i+W_i\}\{\prod_{j=1}^{i-1}[1+X_i]-1\}.$$

Note that $M \geq \prod_{j=1}^{n} [1+Z_j]$, $X_i \geq Z_i + W_i$ for $1 \leq i \leq n$; thus the right-hand side of (3.4) is less than or equal to

$$M \sum_{i=1}^{n} X_i \{ \sum_{j=1}^{i-1} [1+X_j] - 1 \} = M\{ \prod_{i=1}^{n} [1+X_i] - 1 - \sum_{i=1}^{n} X_i \}.$$

Thus (iv) holds.

<u>Lemma III.1.2</u>

With the same assumptions and conventions as Lemma III.1.1, the following condition holds except on a set of measure zero:

(v) $\quad |E(\Pi_\rho [1+dg] - 1 | A_0)| \leq \Pi_\rho [1+dG] - 1$

<u>Proof</u>:

With the notation of the proof of Lemma III.1.1,

$$\Pi_\rho [1+dg] - 1 = \sum_{k=1}^{n} x_k \prod_{j=1}^{k-1} [1+X_j].$$

Then $|E(\sum_{k=1}^{n} x_k \prod_{j=1}^{k-1} [1+x_j] | A_0)| \leq$

$$\sum_{k=1}^{n} E(|E(x_k | A_{k-1})|| \prod_{j=1}^{k-1} [1+x_j]|| A_0) \leq$$

$$\sum_{k=1}^{n} Z_k \, E(| \prod_{j=1}^{k-1} [1+x_j]|| A_0) \leq$$

$$\sum_{k=1}^{n} Z_k \, E(|\prod_{j=1}^{k-1} [1+x_j]| \,|A_0) \leq$$

$$\sum_{k=1}^{n} Z_k (\tfrac{1}{2}) E(\prod_{j=1}^{k-1} [1+x_j]^2 + 1|A_0) \leq$$

$$\sum_{k=1}^{n} Z_k (\tfrac{1}{2}) \{\prod_{j=1}^{k-1} [1+X_i] + 1\} \leq$$

$$\sum_{k=1}^{n} Z_k \prod_{j=1}^{k-1} [1+X_j] \leq \sum_{k=1}^{n} X_k \prod_{j=1}^{k-1} [1+X_j],$$

and the lemma is proved.

Lemma III.1.3

Suppose (g,G_1,G_2) is in class Γ_3, $a \leq b$ in S, and $\tau >> \sigma >> \{a,b\}$. For convenience, let $G = 2G_1+G_2$ and $M = \exp(G(b)-G(a))$. Then except on a set of measure zero,

$$E(\{\Pi_\tau[1+dg] - \Pi_\sigma[1+dg]\}^2|A_a) \leq \tag{3.5}$$

$$M^5\{\Pi_\tau[1+dG] - \Pi_\sigma[1+dG]\}.$$

Hence

$$||\Pi_\tau[1+dg] - \Pi_\sigma[1+dg]||^2 \leq M^5\{\Pi_\tau[1+dG] - \Pi_\sigma[1+dG]\}.$$

Remark

The proof of this lemma involves two applications of Corollary II.2.2 and one application of the discrete Gronwall lemma (Lemma I.2.2). We now give a brief sketch of the proof.

Suppose that $\tau = \{t_j\}_{j=0}^{m}$ is a refinement of $\sigma = \{s_k\}_{k=0}^{n}$. For $1 \leq k \leq n$, we let w_k denote the truncation of the difference of $\Pi_t[1+dg]$ and $\Pi_\sigma[1+dg]$ at the point s_k (see (3.6) below). Then, for $1 \leq k \leq n$, let $W_k = E(w_k^2|A_a)$. The object of the proof is to show that W_n is bounded by the left-hand side of (3.5).

It is necessary to obtain, as an intermediate goal, a difference-inequality for W_k in a form to which we may apply the discrete Gronwall lemma. To obtain difference-inequality (3.9) below, we proceed as follows. Fix k and assume that $s_{k-1} = t_{r(k-1)} \leq \cdots \leq t_{r(k)} = s_k$, and let v_j denote the truncation of $\Pi_\tau[1+dg]$ at t_j minus the truncation of $\Pi_\sigma[1+dg]$ at s_{k-1} for $r(k-1) \leq j \leq r(k)$ (see (3.7)). Then let $V_j = E(v_j^2|A_a)$. We then apply Corollary II.2.2 to obtain a bound for V_j in terms of W_{k-1} (see (3.8)). Corollary II.2.2 may be applied again to obtain difference-inequality (3.9).

We then apply the discrete Gronwall inequality to (3.9) to obtain (3.10). The conclusion of the lemma follows from a simple manipulation of the left-hand side of (3.10).

Proof:

Suppose $\tau = \{t_j\}_{j=0}^{m} >> \sigma = \{s_k\}_{k=0}^{n} >> \{a,b\}$.

$\Pi_\tau[1+dg]-1 = \sum_{j=1}^{m} x_j y_j$ where $x_j = dg(t_{j-1},t_j)$ and

$y_j = \prod_{i=1}^{j-1} [1+x_i]$. Let p be the function from {1, 2, ..., m} to {1, 2, ..., n} such that $p(j) = \max\{k | s_k \leq t_{j-1}\}$. Thus $s_{p(j)} \leq t_{j-1} \leq t_j \leq s_{p(j)+1}$ and $t_{j-1} \leq s_{p(j)+1}$. Now

$$\Pi_\sigma[1+dg]-1 = \sum_{k=1}^{n} dg(s_{k-1},s_k) \prod_{i=1}^{k-1} [1+dg(s_{i-1},s_i)]$$

$$= \sum_{k=1}^{n} \sum_{p(j)=k-1} x_j \prod_{i=1}^{k-1} [1+dg(s_{i-1},s_i)]$$

$$= \sum_{j=1}^{m} x_j z_j .$$

where $z_j = \prod_{k=1}^{p(j)} [1+dg(s_{k-1},s_k)]$.

Hence

$$\Pi_\tau[1+dg] - \Pi_\sigma[1+dg] = \sum_{j=1}^{m} x_j(y_j-z_j).$$

We now assume without loss of generality that σ and τ do not contain repetitions, i.e. $s_{k-1} < s_k$ and $t_{j-1} < t_j$ for $1 \leq k \leq n$ and $1 \leq j \leq m$. Now let r be the function from {1, 2, ..., n} to {1, 2, ..., m} such that $t_{r(k)} = s_k$. Note that if $r(k) < j \leq r(k+1)$, then $p(j) = k$.

Define $w_o = 0$ and for $1 \leq k \leq n$, $w_k = \sum_{j=1}^{r(k)} x_j(y_j-z_j)$. Note that $w_n = \Pi_\tau[1+dg] - \Pi_\sigma[1+dg]$ and

$$(3.6)\quad w_k = \prod_{j=1}^{r(k)} [1+dg(t_{j-1},t_j)] - \prod_{i=1}^{k} [1+dg(s_{i-1},s_i)]$$

$$= y_{r(k)+1} - z_{r(k)+1}.$$

For $1 \le j \le m$, define $X_j = dG_2(t_{j-1},t_j)$, $Z_j = dG_1(t_{j-1},t_j)$, and $M = \exp(G(b)-G(a))$.

Temporarily fix k. For $r(k-1)+1 \le j \le r(k)$, let

$$(3.7)\qquad v_j = y_j - z_j.$$

Thus $v_{r(k-1)+1} = w_{k-1}$. For $r_{(k-1)}+2 \le j \le r(k)$, $v_j - v_{j-1} = z_{j-1}y_{j-1}$. For $r_{(k-1)}+1 \le j \le r(k)$, let

$$V_j = E(v_j^2 | A_a)$$

Then

$$V_j = E(\{w_{k-1} + \sum_{i=r(k-1)+1}^{j-1} x_i y_i\}^2 | A_a).$$

Since $E(x_j^2 | A_{t_{j-1}}) \le X_j$ and $|E(x_j | A_{t_{j-1}})| \le Z_j$ and $E(y_j^2 | A_a) \le M$, it follows from Corollary II.2.2 that

$$V_j \le \prod_{i=r(k-1)+1}^{j-1} [1+Z_i] E(w_{k-1}^2 | A_a) +$$

$$\prod_{i=r(k-1)+1}^{j-1} [1+Z_i] M \sum_{j=r(k-1)+1}^{j-1} (X_j+Z_j).$$

If $s \leq t$ in S, let $M_1(s,t) = \exp(G_1(t)-G_1(s))$ and $M(s,t) = \exp(dG(s,t))$. Thus $M = M(a,b)$. Then

$$(3.8) \qquad V_j \leq M_1(t_{r(k-1)}, t_{j-1})W_{k-1} +$$

$$M_1(t_{r(k-1)}, t_{j-1})MdG_3(t_{r(k-1)}, t_{j-1})$$

$$\leq M_1(t_{r(k-1)}, t_{j-1})W_{k-1} + M^2 dG_3(t_{r(k-1)}, t_{j-1}),$$

where $W_{k-1} = E(w_{k-1}^2|A_a)$ and $G_3 = G_1+G_2$. In order to eliminate k from the expressions above, write $t_{r(k-1)} = s_{p(j)}$ and $W_{k-1} = W_{p(j)}$ to obtain $V_j \leq M_1(s_{p(j)}, t_{j-1})W_{p(j)} + M^2 dG_3(s_{p(j)}, t_{j-1})$ $1 \leq j \leq m$.

Now $w_k = w_{k-1} + \sum_{j=r(k-1)+1}^{r(k)} x_j(y_j-z_j)$. Thus by Corollary II.2.2,

$$W_k \leq W_{k-1} \prod_{j=r(k-1)+1}^{r(k)} [1+Z_j] +$$

$$\prod_{j=r(k-1)+1}^{r(k)} [1+Z_j] \sum_{j=r(k-1)+1}^{r(k)} [X_j+Z_j]V_j.$$

Thus

$$W_k \le W_{k-1}\, M_1(s_{k-1},\ s_k) +$$

$$M_1(s_{k-1},\ s_k) \sum_{j=r(k-1)+1}^{r(k)} dG_3(t_{j-1},\ t_j)\{M_1(s_{k-1},\ t_{j-1})W_{k-1} +$$

$$M^2 dG_3(s_{k-1},\ t_j)\}.$$

So

$$(3.9)\quad W_k \le W_{k-1}\{M_1(s_{k-1},s_k) + M_1^2(s_{k-1},s_k)dG_3(s_{k-1},s_k)\} +$$

$$M^3 \sum_{j=r(k-1)+1}^{r(k)} dG_3(t_{j-1},t_j)dG_3(t_{p(j)},t_{j-1})$$

Note that $a \le b$ implies that $e^a + e^b c \le e^{b+c}$. Thus the coefficient of W_{k-1} in (3.5) is not greater than $\exp(3G_1(s_k)+G_2(s_k)-3G_1(s_{k-1})-G_2(s_{k-1}))$, which is in turn not greater than $M^2(s_{k-1},s_k)$. We may now apply the discrete Gronwall lemma (Lemma I.2.2) to

$$W_k \le W_{k-1}[1+(M^2(s_{k-1},s_k)-1)] +$$

$$M_3 \sum_{j=r(k-1)+1}^{r(k)} dG_3(t_{j-1},t_j)dG_3(t_{p(j)},t_{j-1})$$

to obtain (since $W_o = 0$)

$$W_k \le M^2(a,s_k)M^3 \sum_{j=1}^{r(k)} dG_3(t_{j-1},t_j)dG_3(t_{p(j)},t_{j-1}).$$

Letting k = n, one has

$$(3.10) \qquad E(\{\Pi_\tau[1+dg] - \Pi_\sigma[1+dg]\}^2|A_a) \le$$

$$M^5\{(L)\Sigma_\tau G_3 dG_3 - (L)\Sigma_\sigma G_3 dG_3\}$$

Inequality (3.6) is sufficient to show the existence of the stochastic product integral, but to preserve the symmetry of the situation, we make the following observations.

First, $G_3 \le G$. Then

$$\Pi_\tau[1+dg] - \Pi_\sigma[1+dg] =$$

$$\sum_{j=1}^{m} dG(t_j,t_j)\{\prod_{i=1}^{j-1}[1+dG(t_{i-1},t_i)] - \prod_{k=1}^{p(j)}[1+dG(s_{k-1},s_k)]\}.$$

Also

$$\prod_{i=1}^{j-1}[1+dG(t_{i-1},t_i)] \ge [1+dG(s_{p(j)},t_{j-1})]\prod_{k=1}^{p(j)}[1+dG(s_{k-1},s_k)].$$

Finally $\prod_{k=1}^{p(j)}[1+dG(s_{k-1},s_k)] \ge 1$ and the first conclusion follows. The second conclusion is easily obtained from the first.

Theorem III.1.4

Suppose that g is a member of class Γ_3 and $a \leq b$ in S. Then $\prod_a^b [1+dg]$ exists and has an A_b-measurable version.

Proof:

Let G_1 and G_2 be the non-decreasing real-valued functions associated with g in Definition II.1.1 and let $G = 2G_1+G_2$. J. S. MacNerney has shown [14, Lemma 2.1] that $\prod_a^b [1+dG]$ exists. Hence, for any $\varepsilon>0$, there is a subdivision $\sigma >> \{a,b\}$ such that if $\tau >> \sigma$, then

$$\Pi_\tau[1+dG] - \Pi_\sigma[1+dG] < \frac{\varepsilon^2}{4M^{10}}$$

where M is as in Lemma III.1.3. Thus by Lemma III.1.3,

$$||\Pi_\tau[1+dg] - \Pi_\sigma[1+dg]|| < \frac{\varepsilon}{2} .$$

If $\tau >> \sigma$ and $\tau_1 >> \sigma$, then

$$||\Pi_\tau[1+dg] - \Pi_{\tau_1}[1+dg]|| < \varepsilon .$$

It follows that the directed set of subdivisions of the ordered pair (a,b) generates a Cauchy net

$$\{\Pi_\tau[1+dg] | \tau >> \{a,b\}\}$$

with range in the complete space $L^2(\Omega, A_b, P)$. Hence there is some $h \varepsilon L^2(\Omega, A_b, P)$ such that for every $\varepsilon > 0$ there is a subdivision σ such that for any refinement $\tau >> \sigma$,

$$||h - \Pi_\tau [1+dg]|| < \varepsilon .$$

Thus h is a version of $\Pi_\tau [1+dg]$.

The following lemma will be useful in the sequel.

Lemma III.1.5

Suppose $\{f_n\}_{n=0}^{\infty}$ is a sequence in $L^2(\Omega, A, P)$, B is a σ-subalgebra of A, c is a non-negative number such that $E(f_n|B) \leq c$ for $n \geq 1$ and $\lim_{n\to\infty} E(|f_0 - f_n|) = 0$. Then

$$E(f_0|B) \leq c.$$

Proof:

Let $Q = \{w \varepsilon \Omega | E(f_0|B) > c\}$. Then $Q \varepsilon B$. Suppose that $P(Q) > 0$. Then

$$\int_Q f_0 dP = \int_Q E(f_0|B) dP > cP(Q).$$

On the other hand,

$$\int_Q f_n dP = \int_Q E(f_n|B) dP \leq cP(Q).$$

Thus $\lim_{n\to\infty} \int_Q f_n dP < \int_Q f_0 dP$, a contradiction. Therefore

$P(Q) = 0$ and $E(f_0|B) \leq c$.

Theorem III.1.6

Suppose (g, G_1, G_2) is in class Γ_3 and $a \leq b$ in S. For convenience, let $G = 2G_1+G_2$ and $M = \exp(G(b)-G(a))$. Then for any version of $\prod_a^b [1+dg]$, the following conditions hold except on a set of measure zero:

(i) $E(\{\prod_a^b [1+dg]\}^2|A_a) \leq \prod_a^b [1+dG]$

(ii) $E(\{\prod_a^b [1+dg]-1\}^2|A_a) \leq \prod_a^b [1+dG]-1$

(iii) $E(\{\prod_a^b [1+dg]-[1+dg(a,b)]\}^2|A_a) \leq$

$$M\{\prod_a^b [1+dG] - [1+dG(a,b)]\}$$

(iv) $|E(\prod_a^b [1+dg]-1|A_a)| \leq \prod_a^b [1+dG]-1$

(v) $E(\{\prod_a^b [1+dg] - \Pi_\rho[1+dg]\}^2|A_a) \leq$

$$M^5\{\prod_a^b [1+dG] - \Pi_\rho[1+dG]\}$$ for

any $\rho >> \{a,b\}$.

Moreover, in all cases, the conditional expectation may be replaced by the unconditional expectation.

Proof:

Find a sequence of subdivisions $\{\sigma(n)\}_{n=1}^{\infty}$ such that $\sigma(n+1) >> \sigma(n) >> \{a,b\}$ for $n \geq 1$ and

$$\left|\left| \prod_a^b [1+dg] - \Pi_{\sigma(n)}[1+dg] \right|\right| < \frac{1}{n} .$$

Let $f = \prod_a^b [1+dg]$, $f_n = \Pi_{\sigma(n)}[1+dg]$, $n \geq 1$. Note that $\Pi_{\sigma(n)}[1+dG] \leq \prod_a^b [1+dG]$ for $n \geq 1$. Note also that $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$. Suppose $h \varepsilon L^2(\Omega,A,P)$ is such that $E(h^2|A_a) \leq b < \infty$ for some number b.

Then $E(|(f-h)^2 - (f_n-h)^2|)$

$$= E(|f-f_n|\ |f+f_n-2h|)$$

$$\leq \{E(|f-f_n|^2)E(|f+f_n-2h|^2)\}^{1/2}$$

$$\leq \frac{1}{n} \{E(f^2) + E(f_n^2) + 4E(h^2)\}^{1/2}$$

Thus $\lim_{n\to\infty} E(|f-h)^2 - (f_n-h)^2|) = 0$; so if $E(\{f_n-h\}^2|A_a) \leq c$ for all $n \geq 1$, then $E(\{f-h\}^2|A_a) \leq c$ by Lemma III.1.5. Thus (i), (ii) and (iii) follow from Lemma III.1.1 and part (v) follows from Lemma III.1.3. Part (iv) follows from Lemma III.1.2, noting that $|a| \leq b$ is equivalent to $a \leq b$ and $-a \leq b$. For all except (iv), the unconditional version follows immediately from the conditional case. In (iv), note that

$$|E(f-1)| = |E(E(f-1|A_a))| \leq E(|E(f-1|A_a)|).$$

2. A Pairing of Evolutions with Generators

We now investigate the connection between the classes of stochastic processes of Definition II.1.1 and the classes of stochastic evolutions in R defined below.

Definition III.2.1

(i) Let T_1 denote the class to which a function $u:\Delta\to L^2(\Omega,A,P)$ belongs only in case $u(s,t)$ is an element of $L^2(\Omega,A_t,P)$ whenever (s,t) is in Δ and $u(s_1,s_2)u(s_2,s_3) = u(s_1,s_3)$ except on a set of measure zero whenever each of (s_1,s_2) and (s_2,s_3) is in Δ.

(ii) Let T_2 denote the subclass of T_1 to which u belongs only in case there is a non-decreasing function $H_2:S\to R$ such that $H_2(o) = 0$ and whenever (s,t) is in Δ the condition

$$E([u(s,t)-1]^2|A_s) \leq {}_s\Pi^t\,[1+dH_2]-1$$

holds except on a set of measure zero.

(iii) Let T_3 denote the subclass of T_2 to which u belongs only in case there is a non-decreasing function $H_1:S\to R$ such that $H_1(o) = 0$ and whenever (s,t) is in Δ the condition

$$|E(u(s,t)-1|A_s)| \leq {}_s\Pi^t\,[1+dH_1]-1$$

holds except on a set of measure zero.

Lemma III.2.2

Suppose (g,G_1,G_2) is in class Γ_3 and $u:\Delta\to L^2(\Omega,A,P)$ is given by $u(s,t) = {}_s\Pi^t[1+dg]$ whenever (s,t) is in Δ. Then u is in class T_3.

Proof:

It is clear from III.1.4 that $u(s,t)$ has an A_t-measurable version for each (s,t) in Δ, hence $u(s,t)$ is A_t-measurable. That u possesses the multiplicative property of T_1 is intuitively clear, but a careful proof requires the following estimates.

Suppose $a \leq b \leq c$ in S. Let $\varepsilon>0$ be given. Then there exist subdivisions σ_1,σ_2 of $\{a,b\}$ and $\{b,c\}$ respectively such that if $\tau_1 \gg \sigma_1$ and $\tau_2 \gg \sigma_2$, then

$$||\Pi_{\tau_1}[1+dg] - {}_a\Pi^b[1+dg]|| < \varepsilon,$$

$$||\Pi_{\tau_2}[1+dg] - {}_b\Pi^c[1+dg]|| < \varepsilon, \text{ and}$$

$$||\Pi_{\tau_1}[1+dg]\Pi_{\tau_2}[1+dg] - {}_a\Pi^c[1+dg]|| < \varepsilon.$$

Then

$$||{}_a\Pi^b[1+dg]\,{}_b\Pi^c[1+dg] - {}_a\Pi^c[1+dg]|| <$$

$$\varepsilon+||\Pi_{\tau_1}[1+dg]\Pi_{\tau_2}[1+dg] - {}_a\Pi^b[1+dg]\,{}_b\Pi^c[1+dg]||$$

$$\leq \varepsilon + ||\Pi_{\tau_1}[1+dg]|| \; ||\Pi_{\tau_2}[1+dg] - \prod_b^c [1+dg]||$$

$$+ ||\prod_b^c [1+dg]|| \; ||\Pi_{\tau_1}[1+dg] - \prod_a^b [1+dg]||$$

$$< \varepsilon\{1 + \sqrt{\Pi_{\tau_1}[1+dG]} + \sqrt{\prod_b^c [1+dG]}\} \quad \text{where } G = 2G_1+G_2.$$

The last inequality uses (ii) of Lemma III.1.1 and (i) of Lemma III.1.6. If $M = \exp(G(b)-G(a))$ then the last inequality yields

$$||u(a,b)u(b,c)-u(a,c)|| < \varepsilon(1+2\sqrt{M}).$$

Since ε was arbitrary and M does not depend on the particular subdivisions chosen, it follows that $u(a,b)u(b,c) = u(a,c)$ whenever $a \leq b \leq c$ in S. Now choose $H_1 = H_2 = G$. Then by parts (iv) and (ii) of Theorem III.1.6 respectively, one has

$$|E(u(s,t)-1|A_s)| \leq \prod_s^t [1+dH_1]-1 \quad \text{and}$$

$$E([u(s,t)-1]^2|A_s) \leq \prod_s^t [1+dH_2]-1$$

whenever (s,t) is in Δ. Hence u is in class T_3.

<u>Definition III.2.3</u>

Let ε denote the function from class Γ_3 to class T_3

given by $\varepsilon[g](s,t) = \prod_s^t [1+dg]$ for each pair (s,t) in Δ. Let $\overset{\circ}{\Gamma}_3$ denote the subclass of Γ_3 to which g belongs only in case $g(o) = 0$. Let ε° denote the restriction of ε to $\overset{\circ}{\Gamma}_3$.

Remark

It is the purpose of this section to show that the function ε° is a one-to-one mapping of $\overset{\circ}{\Gamma}_3$ onto T_3 and to derive an integral-like formula for the inverse of ε°. The function ε° is analogous to the functions ε and ε^+ of MacNerney [10], [14] and Herod [19].

Lemma III.2.4

Suppose (u, H_1, H_2) is in class T_3 and $a \leq b$ in S. Then $\sum_a^b (u-1)$ exists. Moreover, the function $g: s \to L^2(\Omega, A, P)$ defined by $g(s) = \sum_o^s (u-1)$ is a member of class Γ_3.

Proof:

Suppose $\tau \gg \sigma \gg \{a,b\}$ for some $a \leq b$ in S. Let $\sigma = \{s_k\}_{k=0}^n$, $\tau = \{t_j\}_{j=0}^m$, and $p(j) = \sup\{k | s_k \leq t_{j-1}\}$. Then $s_{p(j)} \leq t_{j-1} \leq t_j \leq s_{p(j)+1}$ and $t_{j-1} < s_{p(j)+1}$ for $1 \leq j \leq m$. Then

$$\Sigma_\sigma(u-1) = \sum_{k=1}^n (u(s_{k-1}, s_k)-1)$$

$$= \sum_{k=1}^n \sum_{p(j)=k-1} [u(s_{k-1},t_j)-u(s_{k-1},t_{j-1})]$$

$$= \sum_{j=1}^m [u(s_{p(j)},t_j)-u(s_{p(j)},t_{j-1})].$$

And

$$\Sigma_\tau(u-1) = \sum_{j=1}^{m} [u(t_{j-1},t_j)-1].$$

Thus

$$(3.11) \qquad \Sigma_\sigma(u-1)-\Sigma_\tau(u-1) =$$

$$\sum_{j=1}^{m} [u(s_{p(j)},t_j)-u(s_{p(j)},t_{j-1})-u(t_{j-1},t_j)+1].$$

Using the multiplicative property of u, one has

$$\Sigma_\sigma(u-1)-\Sigma_\tau(u-1) = \sum_{j=1}^{m} [u(s_{p(j)},t_{j-1})-1][u(t_{j-1},t_j)-1].$$

For $a \leq s \leq t \leq b$, let $U_i(s,t) = \prod_s^t [1+dH_i]$ for $i = 1,2$.

For $1 \leq j \leq m$, let $x_j = u(t_{j-1},t_j)-1$, $y_j = u(s_{p(j)},t_{j-1})-1$, $X_j = U_2(t_{j-1},t_j)-1$, $Y_j = U_2(s_{p(j)},t_{j-1})-1$, $Z_j = U_1(t_{j-1},t_j)-1$ and $A_j = A_{t_{j-1}}$. These sequences and σ-algebras satisfy the hypotheses of Lemma II.2.1 by Definition III.2.1. Hence, letting $M = \exp(H_1(b)-H_1(a))$, we have

$$(3.12) \quad ||\Sigma_\sigma(u-1)-\Sigma_\tau(u-1)||^2 \leq$$

$$M \sum_{j=1}^{m} [U_2(s_{p(j)},t_{j-1})-1]\{U_1(t_{j-1},t_j)-1 + U_2(t_{j-1},t_j)-1\}.$$

For purposes of this proof, we may assume, without loss of generality, that $H_1 = H_2$, hence $U_1 = U_2$. Thus

$$||\Sigma_\sigma(u-1)-\Sigma_\tau(u-1)||^2 \leq 2M\{\Sigma_\sigma(U_2-1)-\Sigma_\tau(U_2-1)\}$$

J. S. MacNerney has shown [ ] that $\sum_a^b (U_2-1)$ exists, indeed $\sum_a^b (U_2-1) = \text{G.L.B.}\{\Sigma_\sigma(U_2-1)|\sigma >> \{a,b\}\}$. Hence for every $\varepsilon>0$, there is some $\sigma >> \{a,b\}$ such that if $\tau >> \sigma$, then

$$|\Sigma_\tau(U_2-1)-\Sigma_\sigma(U_2-1)| < \frac{\varepsilon^2}{16M^2}, \quad \text{thus}$$

$||\Sigma_\tau(u-1)-\Sigma_\sigma(u-1)|| < \frac{\varepsilon}{2}$, and if τ_1 and τ_2 are both refinements of σ, then $||\Sigma_{\tau_1}(u-1)-\Sigma_{\tau_2}(u-1)||<\varepsilon$. Thus $\{\Sigma_\tau(u-1)|\tau>>\{a,b\}\}$ is a Cauchy net over the directed set of subdivisions of the pair $(a,b)\varepsilon\Delta$. Therefore there is an element $f\varepsilon L^2(\Omega,A_b,P)$ such that $\sum_a^b (u-1) = f$. Now let $g:s\to L^2(\Omega,A,P)$ be such that for each t, g(t) is a version of $\sum_o^t (u-1)$. Then $g\varepsilon\Gamma_1$.

If $\tau = \{t_j\}_{j=0}^m >> \{s,t\}$, then $\Sigma_\tau(u-1) = \sum_{j=1}^m (u(t_{j-1},t_j)-1) = \sum_{j=1}^m x_jy_j$ if $x_j = u(t_{j-1},t_j)-1$ and $y_j = 1$. Let $X_j = U_2(t_{j-1},t_j)-1$, $Y_j = 1$, $Z_j = U_1(t_{j-1},t_j)-1$. Then by Corollary II.2.2,

$$E([\Sigma_\tau(u-1)]^2|A_s) \leq \sum_{j=1}^m U_1(t_j,t)\{U_1(t_{j-1},t_j)-1+U_2(t_{j-1},t_j)-1\}$$

Let $H_3 = H_1 + H_2$, $U_3(a,b) = \prod_a^b [1+dH_3] \geq U_i(a,b)$ for $i = 1,2$.

$$E([\Sigma_\tau(u-1)]^2|A_s) \leq 2\sum_{j=1}^{m}[U_3(t_{j-1},t)-U_3(t_j,t)]$$

$$= 2[U_3(s,t)-1]$$

$$\leq 2[U_3(s,t)-1]U_3(o,s)$$

$$= 2[U_3(o,t)-U_3(o,s)]$$

Choose $G_2 = 2[U_3-1]$. Thus by Lemma III.1.5, except on a set of measure zero,

$$E(\{\sum_s^t (u-1)\}^2|A_s) \leq G_2(t)-G_2(s) \text{ and } g \in \Gamma_2.$$

Again, let $\tau >> \{s,t\}$.

$$|E(\Sigma_\tau(u-1)|A_s)| =$$

$$|E(\sum_{j=1}^{m} (u(t_{j-1},t_j)-1)|A_s)| \leq$$

$$\sum_{j=1}^{n} |E(E(u(t_{j-1},t_j)-1|A_{t_{j-1}})|A_s)| \leq$$

$$\sum_{j=1}^{n} E(|E(u(t_{j-1},t_j)-1|A_{t_{j-1}})|A_s) \leq$$

$$\sum_{j=1}^{n} (U_1(t_{j-1},t_j)-1) \leq$$

$$U_1(s,t)-1 \leq$$

$$(U_1(s,t)-1)U_1(o,s) =$$

$$U_1(o,t)-U_1(o,s).$$

Choose $G_1(t) = U_1(o,t)-1$ and by Lemma III.1.5,

$$|E(\sum_s^t (u-1)|A_s)| \leq G_1(t)-G_1(s)$$

except on a set of measure zero. Thus $g \in \Gamma_3$.

Remark

Following line (3.11) in the previous proof, one could have applied Corollary II.2.2 rather than Lemma II.2.1 to obtain the following useful result.

Corollary III.2.5

Suppose (u,H,H) is in class T_3 and $a \leq b$ in S. Let $U(s,t) = \prod_s^t [1+dH]$ whenever $a \leq s \leq t \leq b$ and let $M = \exp(H(b)-H(a))$. Then whenever $\tau >> \sigma >> \{a,b\}$, the following conditions hold except on a set of measure zero.

(i) $E(\{\Sigma_\sigma(u-1)-\Sigma_\tau(u-1)\}^2|A_a) \leq$

$$2M\{\Sigma_\sigma(U-1)-\Sigma_\tau(U-1)\}$$

(ii) $E(\{\Sigma_\sigma(u-1) - \sum_a^b (u-1)\}^2|A_a) \leq$

$$2M\{\Sigma_\sigma(U-1) - \sum_a^b (U-1)\}.$$

Proof:

With the same notation as the proof of Lemma III.2.4 except that $H_1 = H_2 = H$ from the beginning, line (3.11) yields

$$\Sigma_\sigma(u-1) - \Sigma_\tau(u-1) = \sum_{j=1}^{m} x_j y_j .$$

Using the same sequences and subdivisions, but applying Corollary II.2.2 rather than Lemma II.2.1, one obtains

$$(3.13) \qquad E(\{\Sigma_\sigma(u-1) - \Sigma_\tau(u-1)\}^2 | A_a) \leq$$

$$2M \sum_{j=1}^{m} [U(s_{p(j)}, t_{j-1})-1][U(t_{j-1}, t_j)-1]$$

$$= 2M\{\Sigma_\sigma(u-1) - \Sigma_\tau(u-1)\}$$

rather than (3.11). Hence (i) is proved. Now, for each integer $n \geq 1$, let $\sigma(n)$ be a refinement of σ such that $||\Sigma_{\sigma(n)}(u-1) - \sum_a^b (u-1)|| < \frac{1}{n^2}$. Note that for any subdivision $\tau >> \{a,b\}$,

$$||\Sigma_\tau(u-1)||^2 = ||\sum_{j=1}^{m} x_j y_j||^2 \text{ where } x_j = u(t_{j-1}, t_j)-1, \ y_j = 1.$$

Let X_j, Z_j be as in the proof of Lemma III.2.4 but let $Y_j = 1$. Then by Lemma II.2.1,

$$||\Sigma_\tau(\ \ -1)||^2 \le M\Sigma_\tau[(U-1)+(U-1)] = 2M\Sigma_\tau(U-1)$$

$$\le 2M(U(a,b)-1).$$

Thus, by the reasoning in the proof of Theorem III.1.6, with $f = \sum_a^b (u-1)$, $f_n = \Sigma_{\sigma(n)}(u-1)$, and $h = \Sigma_\sigma(u-1)$, we may apply Lemma III.1.5 to obtain (ii).

Lemma III.2.6

Suppose g is in class Γ_3, u is a member of class T_3 such that $u(s,t) = \prod_s^t [1+dg]$ for each pair (s,t) in Δ. Then $\sum_s^t (u-1) = g(t)-g(s)$ for each pair (s,t) in Δ.

Proof:

Let (g,G_1,G_2) be in class Γ_3. For convenience let $G = 2G_1+G_2$. Suppose the pair (a,b) is in Δ and let $\varepsilon>0$ be given. Let $\sigma \gg \{a,b\}$ be such that if $\tau \gg \sigma$, then

$$||\sum_a^b (u-1) - \Sigma_\tau(u-1)|| < \frac{\varepsilon}{3}$$

and

$$\sum_a^b GdG - \Sigma_\tau GdG < \frac{\varepsilon^2}{18M}$$

where $M = \exp(G_1(b) - G_1(a))$. Now let $\tau \gg \sigma$ be such that if $\tau = \{t_j\}_{j=0}^m$ and $\sigma = \{s_k\}_{k=0}^n$ and $p(j) = \sup\{k|s_k \le t_{j-1}\}$, then

$$||u(s_{k-1},s_k) - \prod_{p(j)=k-1} [1+dg(t_{j-1},t_j)]|| < \frac{\varepsilon}{3n}$$

for $1 \leq k \leq n$. This amounts to choosing a subdivision of each pair (s_{k-1},s_k), then uniting those subdivisions to form τ. Then,

$$||\Sigma_\sigma(u-1) - dg(a,b)|| \leq$$

$$\sum_{k=1}^{n} ||u(s_{k-1},s_k) - \prod_{p(j)=k-1} [1+dg(t_{j-1},t_j)]||$$

$$+ ||\sum_{k=1}^{n} (\prod_{p(j)=k-1} [1+dg(t_{j-1},t_j)]-1) - dg(a,b)||$$

Now, for $1 \leq j \leq m$, let $r(j) = \sup\{i | t_i = s_{p(j)}\}$. Then let

$$x_j = dg(t_{j-1},t_j) \quad \text{and} \quad y_j = \prod_{i=r(j)}^{j-1} [1+dg(t_{i-1},t_i)]-1$$

Then $\sum_{k=1}^{n} \{ \prod_{p(j)=k-1} [1+dg(t_{j-1},t_j)]-1\} = \sum_{k=1}^{n} \sum_{p(j)=k-1} x_j(y_j+1)$.

Thus

$$||\Sigma_\sigma(u-1)-dg(a,b)|| \leq \frac{\varepsilon}{3} + ||\sum_{j=1}^{m} x_jy_j||.$$

Now let $X_j = dG_2(t_{j-1},t_j)$, $Y_j = \prod_{i=r(j)}^{j-1} [1+dG(t_{i-1},t_i)]-1$,

$Z_j = dG_1(t_{j-1},t_j)$ and $A_j = A_{t_{j-1}}$ for $1 \leq j \leq m$. Then by Lemma II.2.1,

$$||\sum_{j=1}^{m} x_j y_j||^2 \leq M \sum_{j=1}^{m} Y_j[X_j+Z_j],\ M = \exp(G(b)-G(a)).$$

$$\text{Now } Y_j = \sum_{i=r(j)}^{j-1} dG(t_{i-1},t_i) \prod_{k=r(j)}^{i-1} [1+dG(t_{k-1},t_k)]$$

$$\leq MdG(t_{r(j)},t_{j-1})$$

and $X_j+Y_j \leq dG(t_{j-1},t_j)$.

$$\text{Thus } ||\sum_{j=1}^{m} x_j y_j||^2 \leq M^2 \sum_{j=1}^{m} dG(t_{r(j)},t_{j-1})dG(t_{j-1},t_j)$$

$$= M^2\{\Sigma_\tau GdG - \Sigma_\sigma GdG\}$$

$$< \frac{\varepsilon^2}{9}.$$

Thus $||\Sigma_\sigma(u-1) - dg(a,b)|| < \frac{2\varepsilon}{3}$, and already

$$||{}_a\Sigma^b(u-1) - \Sigma_\sigma(u-1)|| < \frac{\varepsilon}{3}, \text{ so}$$

$$||\sum_a^b (u-1)-dg(a,b)|| < \varepsilon.$$

Since $\varepsilon > o$ was arbitrary, it follows that $\sum_a^b (u-1) = dg(a,b)$.

Remark

The situation with ε° is now as follows: There is a function, call it δ, from T_3 to $\mathring{\Gamma}_3$ given by

$$\delta[u](t) = \sum_o^t (u-1), \quad \text{for each } t \text{ in } S.$$

We also know that $\delta\cdot\varepsilon^\circ(g) = g$ for any g in $\mathring{\Gamma}_3$. It remains to show that ε° maps $\mathring{\Gamma}_3$ onto T_3. We demonstrate this by showing that $\varepsilon\ \cdot\delta[u] = u$ for each $u \in T_3$.

Lemma III.2.7

Suppose u is in class T_3 and g is a member of class $\mathring{\Gamma}_3$ such that $g(t) = \sum_o^t (u-1)$ for each t in S. Then $\prod_s^t [1+dg] = u(s,t)$ for each pair (s,t) in Δ.

Proof:

Let (u, H_1, H_2) be in class T_3. For convenience, assume $H_1 = H_2 = H$ (if not, let $H = H_1 + H_2$, then $(u,H,H) \in T_3$), and let $U(s,t) = \prod_s^t [1+dH]$ for (s,t) in Δ. Suppose the pair (a,b) is in Δ and let $\varepsilon > o$ be given. Let $\sigma >> \{a,b\}$ be such that if $\tau >> \sigma$, then

$$||\prod_a^b [1+dg] - \prod_\tau [1+dg]|| < \frac{\varepsilon}{3}$$

and

$$M_2\{\Sigma_\sigma(u-1) - \Sigma_\tau(u-1)\} < \frac{\varepsilon^2}{9}$$

where

$$M_1 = \exp(2H(b)-2H(a)) \quad \text{and}$$

$$M_2 = 4M_1^2 \exp((1+2M_1)dH(a,b)).$$

Now let $\tau >> \sigma$ be such that if $\tau = \{t_j\}_{j=0}^m$, $\sigma = \{s_k\}_{k=0}^n$, $p(j) = \sup\{k|s_k \le t_{j-1}\}$ and $r(j) = \sup\{i|t_i = s_{p(j)}\}$, then

$$E(\{\sum_{p(j)=k-1}(u(t_{j-1},t_j)-1)-dg(s_{k-1},s_k)\}^2|A_{s_{k-1}}) \le \frac{\varepsilon^2}{9n^2M_2M_3}$$

where M_3 is the constant guaranteed by Lemma III.1.1 part (i) such that $E(\{\Pi_\rho[1+dg]\}^2) \le M_3$ for any subdivision ρ of (a,b). As in the proof of the preceding lemma, this amounts to choosing a subdivision of each pair (s_{k-1},s_k) (such a subdivision exists by Corollary III.2.5) and uniting those subdivisions to form τ. For $1 \le j \le m$, let $x_j = u(t_{j-1},t_j)-1$ and $y_j = u(a,t_{j-1})$. For $1 \le k \le n$, let $u_k = 1 + \sum_{p(j)=k-1} x_j$. Then let $z_j = \prod_{i=1}^{p(j)} u_i$ for $1 \le j \le m$. Now

$$(3.14)\quad \left\|u(a,b) - \prod_a^b [1+dg]\right\| \leq \left\|u(a,b) - \prod_{k=1}^n u_k\right\| +$$

$$\left\|\prod_{k=1}^n u_k - \Pi_\sigma[1+dg]\right\| + \left\|\Pi_\sigma[1+dg] - \prod_a^b [1+dg]\right\|$$

By the choice of σ, the last term on the right-hand side of (3.10) is less than $\frac{\varepsilon}{3}$. Note that

$$u(a,b)-1 = \sum_{j=1}^m [u(a,t_j)-u(a,t_{j-1})] = \sum_{j=1}^m x_j y_j .$$

Also,

$$\prod_{k=1}^n u_k - 1 = \sum_{k=1}^n \Big(\sum_{p(j)=k-1} x_j\Big) \prod_{i=1}^{k-1} u_i = \sum_{j=1}^n x_j z_j .$$

Thus

$$u(a,b) - \prod_{k=1}^n u_k = \sum_{j=1}^m x_j(y_j - z_j).$$

We suppose, without loss of generality, that σ and τ do not contain repetitions, i.e., $s_{k-1} < s_k$ and $t_{j-1} < t_j$ for $1 \leq j \leq m$ and $1 \leq k \leq n$. Define $w_0 = 0$ and for $1 \leq k \leq n$, $w_k = \sum_{j=1}^{r(k)} x_j(y_j - z_j)$. Note that

$$w_n = u(a,b) - \prod_{k=1}^n u_k \quad \text{and}$$

$$w_k = u(a,s_k) - \prod_{i=1}^{k} u_i$$

$$= y_{r(k)+1} - z_{r(k)+1}.$$

For $1 \leq j \leq m$, define $X_j = Z_j = U(t_{j-1},t_j)-1$, $Y_j = M_1^{1/2}$, and $A_j = A_{t_{j-1}}$ and for $1 \leq k \leq n$ define $W_k = E(w_k^2)$.

Temporarily fix k. For $r(k-1) \leq j \leq r(k)$, let $v_j = y_j - z_j$. Thus $v_{r(k-1)+1} = w_{k-1}$. Also, $z_j - z_{j-1} = 0$ and $y_j - y_{j-1} = x_{j-1}y_{j-1}$, thus $v_j = w_{k-1} + \sum_{i=r(k-1)+1}^{j-1} x_i y_i$.

Let $V_j = E(v_j^2)$. Then by Lemma II.2.1,

$$V_j \leq W_{k-1} \prod_{i=r(k-1)+1}^{j-1} [1+Z_i] +$$

$$\prod_{i=r(k-1)+1}^{j-1} [1+Z_j] \sum_{i=r(k-1)+1}^{j-1} [X_j+Z_j]Y_j$$

$$\leq W_{k-1}U(s_{k-1},t_{j-1}) + 2M_1 \sum_{i=r(k-1)+1}^{j-1} (U(t_{i-1},t_i)-1)$$

$$\leq U(s_{k-1},s_k)W_{k-1} + 2M_1(U(s_{k-1},t_{j-1})-1)$$

We may then apply Lemma II.2.1 to the equation

$$w_k = w_{k-1} + \sum_{j=r(k-1)+1}^{r(k)} x_j v_j \qquad \text{to obtain}$$

$$W_k \le W_{k-1} \prod_{j=r(k-1)+1}^{r(k)} [1+Z_j] + \prod_{j=r(k-1)+1}^{r(k)} [1+Z_j] \cdot$$

$$\sum_{i=r(k-1)+1}^{r(k)} [X_j+Y_j][V_j]$$

$$\le U(s_{k-1},s_k)W_{k-1} + 2U(s_{k-1},s_k)[U(s_{k-1},s_k)-1]M_1W_{k-1}$$

$$+ 4M_1^2 \sum_{j=r(k-1)+1}^{r(k)} (U(s_{k-1},t_{j-1})-1)(U(t_{j-1},t_j)-1)$$

Define $M(s,t) = \exp(1+2M_1)dH(s,t))$. Thus, for any k, $1 \le k \le n$,

(3.15) $$W_k \le M(s_{k-1},s_k)W_{k-1} +$$

$$4M_1^2 \sum_{j=r(k-1)+1}^{r(k)} (U(s_{k-1},t_{j-1})-1)(U(t_{j-1},t_j)-1).$$

Here we have used the fact that if $a \ge 0$ and $b \ge 1$, then $e^{ab}-1 \ge b(e^a-1)$, hence $e^{(1+b)a} \ge e^a b(e^a-1)+e^a$ and $U(s,t) \le \exp(dH(s,t))$ for $s \le t$. We have also used, several times, the fact that if ρ is a subdivision of (s,t), then $\sum_{k=1}^{n} [U(r_{k-1},r_k)-1] \le U(s,t)-1$. We now apply Lemma I.2.2 to inequality (3.15)) to obtain

(3.16)

$$W_n \le \sum_{k=1}^{n} \{4M_1^2 \sum_{j=r(k-1)+1}^{r(k)} (U(s_{k-1},t_{j-1})-1)(U(t_{j-1},t_j)-1)\} M(s_k,b)$$

$$\le 4M_1^2 M(a,b) \sum_{j=1}^{m} (U(t_{p(j)},t_{j-1})-1)(U(t_{j-1}t_j)-1)$$

$$\le M_2\{\Sigma_\sigma(u-1) - \Sigma_\tau(u-1)\}$$

$$< \frac{\varepsilon^2}{9} .$$

But $W_n = E(w_n^2) = ||u(a,b) - \prod_{k=1}^{n} u_k||^2$. Thus the first term on the right-hand side of (3.14) is less than $\frac{\varepsilon}{3}$. Now,

$$||\prod_{k=1}^{n} u_k - \Pi_\sigma[1+dg]|| = ||\sum_{k=1}^{n} a_k b_k c_k||$$

$$\le \sum_{k=1}^{n} ||a_k b_k c_k||$$

here, for $1 \le k \le n$, $a_k = \prod_{i=1}^{k-1} [1+dg(s_{i-1},s_i)]$, $c_k = \prod_{i=k+1}^{n} u_i$, and $b_k = \sum_{j=r(k-1)+1}^{r_k} x_j - dg(s_{k-1},s_k)$. With the notation above,

$$u_k = 1 + \sum_{j=r(k-1)+1}^{r(k)} x_j .$$

Applying Corollary II.2.2 (with $y_j \equiv 1$), one obtains

$$E(u_k^2|A_{s_{k-1}}) \leq 1 \cdot \prod_{j=r(k-1)+1}^{r(k)} Z_j + \sum_{j=r(k-1)+1}^{r(k)} [X_j+Z_j] \prod_{j=r(k-1)+1}^{r(k)} Z_j$$

$$\leq U(s_{k-1},s_k)[1+2(U(s_{k-1},s_k)-1)]$$

Thus $E(u_k^2|A_{s_{k-1}}) \leq M(s_{k-1},s_k)$, and

$$E(c_k^2|A_{s_k}) \leq M(s_k,b) \leq M(a,b) \leq M_2.$$

The subdivision τ was chosen such that

$$E(b_k^2|A_{s_{k-1}}) < \frac{\varepsilon^2}{9n^2 M_2 M_3} ,$$

and M_3 was chosen so that

$$E(c_k^2) \leq M_3.$$

Thus $||a_k b_k c_k||^2 < \frac{\varepsilon^2}{9n^2}$ and

$$\sum_{k=1}^{n} ||a_k b_k c_k|| < \frac{\varepsilon}{3} .$$

Thus the second term on the right-hand side of (3.14) is less than $\frac{\varepsilon}{3}$, and the entire right-hand side is less than ε. Since $\varepsilon > 0$ was arbitrary, the left-hand side of (3.14) must be zero and the lemma is proved.

Remark

The preceding sequence of lemmas can be combined in the following theorem which provides the desired pairing between generators and evolutions:

Theorem III.2.8

The following are equivalent:

(i) The pair (g,u) is a member of the function ε°.

(ii) g is a member of Γ_3° and $u(s,t) = \prod_s^t [1+dg]$ for each pair (s,t) in Δ.

(iii) u is in class T_3 and $g(t) = \sum_o^t (u-1)$ for each t in S.

3. Solution of the Stochastic Integral Equation

In this section we show that the product integral generates a solution of integral equation (3.1) and that the solution is essentially unique.

Theorem III.3.1

Suppose g is a member of Γ_3 and $u(s,t) = \prod_s^t [1+dg]$ for each pair in Δ. Then for each pair (s,t) in Δ, $u(s,t)$ is a version of $1+(L)\int_s^t u(s,\cdot)dg$.

Proof:

Fix (s,t) in Δ and let $\varepsilon>0$ be given. Let $\rho = \{r_k\}_{k=0}^n$ be a subdivision of (s,t) such that $M_1^5\{\prod_a^b [1+dG]-\pi_\rho[1+G]\} < \frac{\varepsilon^2}{4M_2}$, where $(g,G_1,G_2)\varepsilon\Gamma_3$, $G = 2G_1+G_2$, $M_1 = \exp(G(b)-G(a))$ and $M_2 = \exp(G_1(b)-G_1(a))[1+G(b)-G(a)]$. Now for each k, $1 \le k \le n$, define g^k by $g^k(t) = g(t)$ for $t \le r_k$, $g^k(t) =$

$g(r_k)$ for $t \geq r_k$. Note that $(g^k, G_1, G_2) \varepsilon \Gamma_3$ for $1 \leq k \leq n$, $\prod_s^t [1+dg^k] = \prod_s^{r_k} [1+dg]$ and $\prod_{i=1}^{n} [1+dg^k(r_{k-1}, r_i)] = \prod_{i=1}^{k} [1+dg(r_{k-1}, r_i)]$. Applying part (v) of Theorem III.1.6 to each g^k yields

$$(3.17) \quad ||u(s, r_k) - \prod_{i=1}^{k} [1+dg(r_{i-1}, r_i)||^2 < \frac{\varepsilon^2}{4M_2}$$

for $1 \leq k \leq n$.

Note that

$$(3.18) \; ||u(s,t)-1-(L)\int_s^t u(s,\cdot)dg|| \leq ||u(s,t)-\Pi_\rho[1+dg]||$$

$$+ ||\Pi_\rho[1+dg]-1-\sum_{k=1}^{n} dg(r_{k-1}, r_k) \prod_{i=1}^{k-1} [1+dg(r_{i-1}, r_i)]||$$

$$+ ||\sum_{k=1}^{n} dg(r_{k-1}, r_k)\{u(s, r_{k-1}) - \prod_{i=1}^{k-1} [1+dg(r_{i-1}, r_i)]||$$

The first term of the right-hand side of (3.18) is less than $\frac{\varepsilon}{2}$ by inequality (3.17), noting that $M_2 \geq 1$. The second term of the right-hand side of (3.14) is identically zero by Lemma I.2.1, part (i) (see also equation (1.1)). Indeed, it was this last observation which formed the basis of our investigation of the product integral as a solution of the integral equation.

To find a bound for the third term, let $x_k = dg(r_{k-1},r_k)$, $y_k = u(s,r_{k-1}) - \prod_{i=1}^{k-1} [1+dg(r_{i-1},r_i)]$, $X_k = dG_2(r_{k-1},r_k)$, $Z_k = dG_1(r_{k-1},r_k)$ and $Y_k = \frac{\varepsilon^2}{4M_2}$ for $1 \leq k \leq n$. Then these sequences satisfy the hypotheses of Lemma II.2.1. By that lemma, the third term of (3.18) satisfies the bound

$$\|\sum_{k=1}^{n} x_k y_k\|^2 \leq \prod_{k=1}^{n} [1+Z_k] \sum_{k=1}^{n} (X_k+Z_k) \frac{\varepsilon^2}{4M_2}$$

$$\leq \frac{\varepsilon^2}{4} .$$

The last inequality follows from the fact that

$$\prod_{k=1}^{n} [1+Z_k] \sum_{k=1}^{n} (X_k+Z_k) \leq \exp(G_1(b)-G_1(a))[dG_1(a,b)+dG_2(a,b)]$$

$$\leq M_2 .$$

Thus the third term on the right-hand side of (3.18) is less than $\frac{\varepsilon}{2}$. Since $\varepsilon > 0$ was arbitrary, the left-hand side of (3.18) must be zero and the theorem is proved.

Definition III.3.2

If a is in S, z is in $L^2(\Omega, A_a, P)$ and g is in class Γ_3, then a solution of

$$f(t) = z+(L)\int_a^t fdg, \quad t \geq a \tag{3.19}$$

of class Γ_o is a function f in class Γ_o such that for every $t \geq a$, f(t) is a version of $z+(L)\int_a^t fdg$.

Theorem III.3.3

If a is in S and g is in class Γ_3, then any solution of

$$(3.20) \qquad f(t) = (L)\int_a^t fdg, \; t \geq a$$

of class Γ_o is equivalent to zero for each t.

Proof:

Suppose $(g,G_1,G_2)\varepsilon\Gamma_3$ and $(a,b)\ \varepsilon\ \Delta$. Let $G = 2G_1 + G_2$. If f satisfies (3.20) and $N_{a,t}(f)$ is as in Corollary II.2.5, then $||f(t)||^2 \leq M(L)\int_a^t N_{a,s}(f)dG(s)$, $a \leq t \leq b$. For $a \leq s \leq t \leq b$, $||f(s)||^2 \leq M(L)\int_a^s N_{a,\cdot}(f)dG \leq M(L)\int_a^t N_{a,\cdot}(f)dG$, thus

$$N_{a,t}(f) \leq M(L)\int_a^t N_{a,\cdot}(f)dG.$$

Then by a usual Gronwall lemma, e.g. Herod [33], $N_{a,t}(f) \equiv 0$. Thus $||f(t)|| = 0$ for $a \leq t \leq b$. Since b was arbitrary, $||f(t)|| = 0$ for all $t \geq a$.

Theorem III.3.4

Suppose a is in S, z is in $L^2(\Omega,A_a,P)$, g is in class Γ_3 and $u(s,t) = \prod_s^t [1+dg]$ for each pair (s,t) in Δ. Then $f(t) = u(s,t)z$ is a solution of integral equation (3.19).

Moreover, if f_1 is another solution, then $f_1(t) = f(t)$ except on a set of measure zero for each $t \geq a$.

Proof:

Let f be as stated above, then $(L)\int_a^t fdg = [(L)\int_a^t u(s,\ )dg]z$ since this relation holds for every approximating sum $[(L)\sum_\rho u(s,\ )dg]z$. In view of Theorem III.3.1, f must be a solution of (3.19). Now suppose f and f_1 are two solutions of (3.19), then $f_2 = f-f_1$ is a solution of (3.20). Thus $||f(t)-f_1(t)|| = 0$ for each $t \geq a$ and the theorem is proved.

Remark

The fact that any solution of (3.19) can be determined only up to a set of measure zero, possibly a different set for each t, motivates the search in the next section for a "well-behaved" version of the product integral.

4. Separable Versions of the Product Integral and Uniform Convergence of Paths

Using the notion of separability (see Doob [17]) and a theorem of Orey [16] on F-processes (see Theorem II.3.4), we produce a sequence of approximations of the product integral whose path converge uniformly except on a set of measure zero to a separable version of the product integral. For the remainder of this section, fix $S = [a,b]$, (g,G,G) in class Γ_3, let u be the function in class T_3 such that $u(s,t) = \prod_s^t [1+dg]$ whenever $a \leq s \leq t \leq b$ and suppose

(u,G,G) is in class $\mathcal{T}_3$. Let $U(s,t) = \prod_s^t [1+dG]$ whenever $a \leq s \leq t \leq b$. If $\rho = \{r_k\}_{k=0}^n$ is a subdivision of the pair (a,b), let $u_\rho(a,t) = \prod_{k=0}^{m} [1+dg(r_{k-1},r_k)]\cdot[1+dg(r_m,t)]$ whenever $r_m \leq t \leq r_{m+1}$. Define $h_\rho(t) = u(a,t) - u_\rho(a,t)$ and $k_\rho(t) = [h_\rho(t)]^2$ for $a \leq t \leq b$.

Now suppose $\rho >> \{a,b\}$ and fix (s,t) in Δ such that $r_m \leq s \leq t \leq r_{m+1}$. Abbreviate r_m by r. Then

$$|E(h\ (t)-h\ (s)|A_s)| \ =$$

$$|E(u(a,s)[u(s,t)-1] + u_\rho(a,r)dg(s,t)|A_s)|$$

$$\leq |u(a,s)|\ (U(s,t)-1) + |u_\rho(a,r)|dG(s,t)$$

except on a set of measure zero. Let $M_1 = U(a,b)$. Then

$$E(|E(h_\rho(s)dh_\rho(s,t)|A_s)|)$$

$$\leq E(|h_\rho(s)|\{|u(a,s)| + |u_\rho(a,s)|\})(U(s,t)-1+dG(s,t))$$

$$\leq 4\sqrt{M_1}\ \sqrt{E(k_\rho(s))}\ (U(s,t)-1)$$

<u>Lemma III.4.1</u>

Let $\varepsilon>0$ be given. Then there is a subdivision ρ of the pair (a,b) such that if $\tau >> \rho$ and $\tau = \{t_j\}_{j=0}^m$, then

$$(3.21) \quad \sum_{j=1}^{m} E(|E(h_\rho(t_{j-1})[h_\rho(t_j)-h_\rho(t_{j-1})]|A_{t_{j-1}})|) < \varepsilon.$$

Proof:

Let $\varepsilon_1 = \varepsilon/(4\sqrt{M_1}\ [U(a,b)-1])$. By Lemma II.3.3 (see also the proof of Theorem III.3.1), choose ρ such that $(k_\rho(s)) < \varepsilon_1^2$ for each s, $a \leq s \leq b$. Then

$$E(|E(h_\rho(t_{j-1})[h_\rho(t_j)-h_\rho(t_{j-1})]|A_{t_{j-1}})|)$$

$$\leq 4\sqrt{M_1}\ \varepsilon_1[U(t_{j-1},t_j)-1].$$

Thus the left-hand side of (3.17) does not exceed

$$\frac{4\sqrt{M_1}\ \Sigma_\tau(U-1)\varepsilon}{4\sqrt{M_1}\ [U(a,b)-1]}.$$

The conclusion follows from the identity $\Sigma_\tau(u-1) \leq U(a,b)-1$.

Lemma III.4.2

Let $\varepsilon>0$ be given. Then there is a subdivision ρ of the pair (a,b) such that if $\tau >> \rho$ and $\tau = \{t_j\}_{j=0}^{m}$, then

$$(3.22) \quad \sum_{j=1}^{m} E(E([dh_\rho(t_{j-1},t_j)]^2|A_{t_{j-1}})) < \varepsilon.$$

Proof:

Suppose that $\rho >> \{a,b\}$ and $r = r_k \leq s \leq t \leq r_{k+1}$ for some k. Then

$$h_\rho(t)-h_\rho(s) = u(a,r)[u(r,s)-1][u(s,t)-1] +$$

$$[u(a,r)-u_\rho(a,r)][u(s,t)-1]+u_\rho(a,r)[u(s,t)-1-dg(s,t)],$$

and

$$(3.23) \qquad [dh_\rho(s,t)]^2 = u^2(a,r)[u(r,s)-1]^2[u(s,t)-1]^2 +$$

$$h_\rho^2(r)[u(s,t)-1]^2 + u_\rho^2(a,r)[u(s,t)-1-dg(s,t)]^2 +$$

$$2u(a,r)h_\rho(r)[u(r,s)-1][u(s,t)-1]^2 + 2u(a,r)u_\rho(a,r)[u(r,s)-1].$$

$$[u(s,t)-1-dg(s,t)] + 2u(a,r)h_\rho(r)[u(s,t)-1][u(s,t)-1-dg(s,t)]$$

The six terms on the right-hand side of (3.23) will be denoted $T_i(r,s,t)$, i = 1, 2, 3, 4, 5, 6, respectively. Note that

$$E(E([dh_\rho(s,t)]^2|A_s)) = \sum_{i=1}^{6} E(E(T_i(r,s,t)|A_s)).$$

Step 1

$$E(E(T_1(r,s,t)|A_s)) \le E(u^2(a,r)(u(r,s)-1)^2))\{U(s,t)-1\}$$

$$\le E(u^2(a,r))(U(r,s)-1)(U(s,t)-1).$$

(3.24) So $$E(E(T_1(r,s,t)|A_s)) \leq M(U(r,s)-1)(U(s,t)-1)$$

where $M = \exp(G(b)-G(a))$

Step 2

(3.25) $$E(E(T_2(r,s,t)|A_s)) \leq E(h_\rho^2(r))(U(s,t)-1)$$

Step 3

$$E(E(T_3(r,s,t)|A_s)) \leq E(u_\rho^2(a,r))M(U(s,t)-1-dG(s,t))$$

(3.26) $$\leq M^2(U(s,t)-1-dG(s,t))$$

Step 4

$$E(E(T_4(r,s,t)|A_s)) \leq 2E(|u(a,r)h_\rho(r)(u(r,s)-1)|)(U(s,t)-1)$$

$$\leq 2\sqrt{E(h^2(r)}\ \sqrt{E(u^2(a,r)(u(r,s)-1)^2)}(U(s,t)-1).$$

But $E(u^2(a,r)(u(r,s)-1)^2) = E(u^2(a,r)E((u(r,s)-1)^2|A_r))$

$$\leq E(u^2(a,r))(U(r,s)-1)$$

$$\leq M(M-1).$$

(3.27) So $E(E(T_4(r,s,t)|A_s)) \leq \sqrt{E(h^2(r)}\ (2\sqrt{M(M-1)})(U(s,t)-1)$

Step 5

$$E(E(T_5(r,s,t)|A_s)) = E(E(T_5(r,s,t)|A_r))$$

$$\leq 2E(|u(a,r)u_\rho(a,r)|E(|u(r,s)-1||u(s,t)-1-dg(s,t)||A_r))$$

$$\leq E(|u(a,r)u_\rho(a,r)|E(\{(u(r,s)-1)^2(u(s,t)-1)^2+(u(s,t)-1-$$

$$dg(s,t))^2\}|A_r))$$

(3.28)
$$\leq M\{(U(r,s)-1)(U(s,t)-1)+M(U(s,t)-1-dG(s,t))\}.$$

Step 6

$$E(E(T_6(r,s,t)|A_s)) \leq 2E(|u(a,r)h_\rho(r)|E(|u(s,t)-1||u(s,t)-1-$$

$$dg(s,t)||A_s))$$

$$\leq 2E(|u(a,r)h_\rho(r)|.$$

$$\sqrt{E((u(s,t)-1)^2|A_s)E((u(s,t)-1-dg(s,t))^2|A_s))}$$

$$\leq 2\sqrt{M}\sqrt{E(h_\rho^2(r))}\sqrt{U(s,t)-1}\sqrt{(U(s,t)-1-dG(s,t))M}$$

$$\leq 2M\sqrt{E(h^2(r)}\ (U(s,t)-1). \tag{3.29}$$

Now let $\varepsilon>0$ be given. Choose $\rho >> \{a,b\}$ such that if $\tau >> \rho$, then each of the following conditions holds:

$$E(h_\rho^2(s)) < \min\{\varepsilon,\varepsilon^2\}/144\ M^4, \tag{3.30}$$

$$\{\Sigma_\tau(U-1) - \Sigma_\rho(U-1)\} < \varepsilon/12M, \tag{3.31}$$

$$\{\Sigma_\tau(U-1)-dG(a,b)\} < \varepsilon/12M^2. \tag{3.32}$$

Note that $\Sigma_\tau(U-1) \leq U(a,b)-1 < M$. It now follows that if τ is a refinement of ρ with $\tau = \{t_j\}_{j=0}^m$ and $p(j) = \sup\{k|r_k \leq t_{j-1}\}$, then $\sum_{i=1}^{6}\sum_{j=1}^{m} E(E(t_i(r_{p(j)},t_{j-1},t_j)A_{t_{j-1}}))$ $< \sum_{i=1}^{6} \varepsilon/6 = \varepsilon$.

<u>Theorem III.4.3</u>

Suppose $a \leq b$ in S, g is in class Γ_3, and u is the function in class T_3 such that $u(s,t) = \prod_s^t [1+dg]$ whenever (s,t) is in Δ. Then for each positive integer n, there is a subdivision $\sigma(n)$ of the pair (a,b) such that if $\tau >> \sigma(n)$ and $\tau = \{t_j\}_{j=0}^n$, then

$$(3.33) \qquad \sum_{j=1}^{m} E(|E(dk_n(t_{j-1},t_j)|A_{t_{j-1}})|) < 1/n^4 \qquad \text{and}$$

$$(3.34) \qquad E(k_n(b)) < \frac{1}{n^4}, \qquad \text{where } k_n = k_{\sigma(n)}.$$

Proof:

Fix $n > 1$ and let h and k denote $h_{\sigma(n)}$ and $k_{\sigma(n)}$ respectively. Note that for $s \leq t$, $k(t)-k(s) = h^2(t)-h^2(s) = 2h(s)(h(t)-h(s) + (h(t)-h(s))^2$. Also,

$$E(|E(dk(s,t)|A_s)|) = 2E(|E(h(s)dh(s,t)|A_s)|) +$$

$$E(E([dh(s,t)]^2|A_s)).$$

Conclusion (3.33) of the theorem follows immediately from Lemma III.4.1 and Lemma III.4.2. Thus k_n is an F-process with F-bound $1/n^4$. Conclusion (3.34) follows from part (v) of Theorem III.1.6.

Theorem III.4.4

Suppose $a \leq b$ in S, g is in class Γ_3, u is that member of T_3 such that $u(s,t) = \prod_s^t [1+dg]$ whenever $s \leq t$ in S, and $\{\sigma(n)\}_{n=1}^{\infty}$ is a sequence of subdivisions of (a,b) which satisfy the conclusions of Theorem III.4.3. Suppose, without loss of generality that $\sigma(n+1) >> \sigma(n)$ for each n. For each n, let u_n denote a separable version of $u_{\sigma(n)}$, and let u_o denote a separable version of u. Then for each n,

$$P[\sup_{a\leq s\leq b} [u_o(o,s)-u_n(o,s)]^2 \geq \frac{1}{n^2}] \leq 2/n^2.$$

Proof:

Apply Theorem III.4.3 and Theorem II.3.4 (Orey [16], Theorem 2.1, p. 303). Now by an application of the Borel-Cantelli lemma the following theorem holds.

Theorem III.4.5

With the assumptions and definitions of Theorem III.4.4, the paths of the processes $u_n(a,\cdot)$ converge uniformly to the paths of the process $u_o(a,\cdot)$ except on a set of measure zero.

CHAPTER IV

APPLICATIONS AND EXAMPLES

We now offer several examples which demonstrate the applicability of the stochastic left Cauchy-Stieltjes integral and the use of the product integral as the representation of the solution to the linear stochastic integral equation

$$z(t) = z(a) + (L)\int_a^t zdg. \tag{4.1}$$

Section 1 shows that g may be chosen as a separable Brownian motion process and that the product integral generates the same solution of (4.1) (with a stochastic left Cauchy-Stieltjes integral) as previous methods yield in the case where the integral in (4.1) is an Itô or belated integral. Section 2 demonstrates the applicability of the integral equation (4.1) to a situation where g has both fixed and moving discontinuities. Section 3 shows a situation where g has only moving discontinuities, but G_1 and G_2 cannot assume the form G(t) = Kt for some constant K.

1. Brownian Motion

Let S be the interval [0,1] and let b denote a separable Brownian motion process with $\sigma^2 = 1$. Thus

(i) $b(t)-b(s)$ is normally distributed with mean 0 and standard deviation $\sqrt{|t-s|}$.

(ii) A_s is the σ-algebra generated by $b(t)$ for $t \leq s$.

(iii) $b(t)-b(s)$ is independent of A_s for $s \leq t$.

(iv) We assume that $b(0) \equiv 0$.

(v) The paths of b are almost surely continuous.

Let I denote the identity function on [0,1]. Then $(b,0,I)$ is an element of Γ_3. The computation on pages 33 and 34 of McShane [13] is sufficient to show that $(L)\int_s^t bdb = \{b^2(t) - b^2(s) - (t-s)\}/2$ and that $(R)\int_s^t bdb = \{b^2(t) - b^2(s) + (t-s)\}/2$. It is interesting to note that the value of the right integral could have been derived from the value of the left integral by formula (i) (integration-by-parts) of Lemma II.1.4.

Before investigating the product integral, we need to record these well-known facts.

Lemma IV.1.1

With S and b as defined above, the following hold for $s \leq t$ and any positive integer n.

(i) $E(\{db(s,t)\}^{2n}|A_s) = 1\cdot3\cdot5\cdots(2n-1)(t-s)^n$,

(ii) $E(\{db(s,t)\}^{2n-1}|A_s) = 0$,

(iii) $E(\exp(db(s,t))|A_s) = \exp((t-s)/2)$.

It is well known in the theory of Itô stochastic integrals (see McKean [34], page 33) that the solution of the Itô integral equation

$$(4.2) \qquad z(t) = 1 + \int_s^t zdb$$

is $z(t) = \exp(b(t)-b(s)-(t-s)/2)$. We now show that this solution also holds when the integral in (4.2) is a stochastic left Cauchy integral and z is the product integral solution.

For $s \leq t$ in S, define $u(s,t) = \exp(b(t)-b(s)-(t-s)/2)$. Then u is in class T_1 (see definition III.2.1). Lemma IV.1.1 also shows that u is in class T_3. Rather than showing directly that $u(s,t) = \prod_s^t [1+db]$, we take the easier route of showing that $db(s,t) = \sum_s^t (u-1)$ and invoking Theorem III.2.8.

Proposition IV.1.2

With u and b as defined above, $\sum_s^t (u-1) = db(s,t)$ whenever $s \leq t$ in S.

Proof:

Let $a \leq c$ in S be given and let $\tau = \{t_j\}_{j=0}^n$ be a subdivision of the pair (a,c). Then $u(t_{j-1},t_j) = \exp(db(t_{j-1},t_j) - (t_j-t_{j-1})/2)$ for $1 \leq j \leq n$. Let x_j and y_j denote $db(t_{j-1},t_j)$ and $(t_j-t_{j-1})/2$ respectively for $1 \leq j \leq n$. It is sufficient to show that

$$D_\tau = E(\{\sum_{j=1}^{n} [\exp(x_j - y_j) - 1 - x_j]\}^2)$$

converges to zero along the directed set of subdivisions. We note that $\exp(x_j - y_j) - 1 - x_j = \sum_{k=1}^{4} T_j^k$ where

$$T_j^1 = -y_j + x_j^2/2$$

$$T_j^2 = -x_j y_j - x_j^2 y_j/2$$

$$T_j^3 = (1 + x_j + x_j^2/2) \sum_{M=2}^{\infty} (-y_j)^m/m!$$

$$T_j^4 = \exp(-y_j) \sum_{m=3}^{\infty} x_j^m/m!$$

Let $D_\tau^k = E\{\sum_{j=1}^{m} T_j^k\}$ for $k = 1,2,3,4$ and note that

$D_\tau \leq 4 \sum_{k=1}^{4} D_\tau^k$. We now approximate D_τ^k for $k = 1,2,3,4$.

Note that

$$D_\tau^1 = E(\{\sum_{j=1}^{n} x_j^2/2 - y_j\}^2).$$

Noting that $E(x_j^2/2 - y_i) = 0$ and $E([x_j^2/2 - y_j]^2) = E(x_j^4/4) - 2y_j E(x_j^2/2) + y_j^2 = 3(t_j - t_{j-1})^2/4 - y_j(t_j - t_{j-1}) + y_j^2 = 3y_j^2 - 2y_j^2 + y_j^2 = 2y_j^2$, one has, from Lemma II.2.1,

$$D_\tau^1 = \sum_{j=1}^{n} 2y_j^2$$

Now consider

$$D_\tau^2 = E(\{\sum_{j=1}^{n} -y_j(x_j+x_j^2/2)\}^2).$$

Let $Y_j = y_j^2$. Note that $E((x_j+x_j^2/2)^2) = E(x_j^2+x_j^3+x_j^4/4) =$ $(t_j-t_{j-1}) + 3/4(t_j-t_{j-1})^2 \leq 7/4\ (t_j-t_{j-1}) = 7/2\ y_j$. Then choose $X_j = 7/2\ y_j$, $Z_j = y_j$. It follows from Lemma II.2.1 that

$$D_\tau^2 \leq \exp(1/2) \sum_{j=1}^{n} 9/2\ y_j^3.$$

For $k = 3$,

$$D_\tau^3 = E(\{\sum_{j=1}^{n} (1+x_j+x_j^2/2) \sum_{m=2}^{\infty} y_j^m/m!\}^2)$$

Let $Y_j = y_j^4 e^2 \geq y_j^4 \exp(2y_j) \geq (\exp(y_j)-1-y_j)^2$. Let $X_j = 15/4 \geq 1+2(t_j-t_{j-1}) + 3/4(t_j-t_{j-1})^2 = E([1+x_j+x_j^2/2]^2)$. Also, let $Z_j = 3/2 \geq |E(1+x_j+x_j^2/2)|$. Then by Lemma II.2.1,

$$D_\tau^3 \leq 21/4 \exp(2) \sum_{j=1}^{n} y_j^4.$$

Finally, consider

$$D_\tau^4 = E(\{\sum_{j=1}^{n} \exp(-y_j)[\exp(x_j)-1-x_j-x_j^2/2]\}^2).$$

Let $Y_j = 1$ and $Z_j = 1/2\ y_j^2 \geq \exp(y_j)-1-y_j$. We have the following inequality:

$$E(\{\sum_{m=3}^{\infty} [db(t_{j-1},t_j)]^m/m!\}^2)$$

$$= E(\sum_{k=o}^{\infty} \sum_{j=1}^{k} x_j^{k+6}/(j+3)!(k-j+3)!)$$

$$= E(\sum_{k=o}^{\infty} \sum_{j=1}^{zk} x_j^{2k+6}/(j+3)!(2k-j+3)!)$$

$$= \sum_{k=o}^{\infty} \sum_{j=1}^{2k} (1\cdot 3\cdot 5\cdots(2k+5)(t_j-t_{j-1})^{k+3}/(j+3)!(2k-j+3)!$$

$$= \sum_{k=o}^{\infty} \sum_{j=1}^{2k} (2k+6)!\ (t_j-t_{j-1})^{k+3}/(j+3)!(2k-j+3)!2^{k+3}(k+3)!$$

$$\leq \sum_{k=o}^{\infty} 2^{2k+6}(t_j-t_{j-1})^{k+3}/2^{k+3}(k+3)!$$

$$= \exp(2(t_j-t_{j-1}))-1-2(t_j-t_{j-1})-\ 2(t_j-t_{j-1})^2$$

$$\leq 8\ \exp\ (2)(t_j-t_{j-1})^3/6.$$

Choose $X_j = (8e^2/3)y_j^3$. Then by Lemma II.2.1,

$$D_\tau^4 \leq \sum_{i=1}^{n} \{(1/2)y_j^2 + (8e^2/3)y_j^3\}.$$

Since $\sum_{j=1}^{n} (y_j)^m$, $m \geq 2$, can be made arbitrarily small for sufficiently fine subdivisions, it follows that D_τ^k, $k = 1,2,3,4$, and hence D_τ can be made arbitrarily small for sufficiently fine subdivisions. Thus for every $\varepsilon>0$, there is a subdivision of the pair (a,b) such that if τ is a refinement of σ, then

$$||\Sigma_\tau(u-1) - db(a,c)|| < \varepsilon.$$

Hence $\sum_s^t (u-1) = db(s,t)$ whenever $s \leq t$ in [0,1].

Then by Theorem III.2.8, the pair (b,u) is in the mapping ε°. Hence u is the solution of (4.1) where the integral is interpreted as the stochastic left Cauchy integral.

2. A Process with Fixed and Moving Discontinuities

We consider a hypothetical economic example in which an investment receives random increments at fixed times and random decrements at exponentially distributed times. Let $S = [0,\infty)$ and let $\{I_n\}_{n=1}^\infty$ be a sequence of non-negative independent, identically-distributed random variables with $E(I_n) = I_o \varepsilon (0,\infty)$ and $E(I_n^2) = J_o$ for $n \geq 1$. We consider I_n as the interest paid at time $t = n$ for each $n \varepsilon N$. Let $\{S_n\}_{n=1}^\infty$

be a non-decreasing sequence of random variables in S such that the sequence $R_n = S_{n-1} - S_n$, $S_o \equiv 0$ is independent and identically distributed with an exponential distribution of parameter λ. Thus $N(t) = \max\{n \mid S_n \leq t\}$ is a Poisson process with mean $1/\lambda$. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of independent, identically distributed random variables with range in $(0,1)$, $E(D_n) = D_o \in (0,1)$ and $E(D_n^2) = C_o$ for $n \in N$. We consider D_n as the devaluation at time S_n.

Let $g(t) = \sum_{n<t} I_n - \sum_{n=1}^{N(t)} D_n$. If $z(t)$ represents the value of a one dollar investment at time t, then $z(n^+) = (1+I_n)z(n)$ for each $n \in N$ and $z(s) = (1-D_n)z(s^-)$ if $s = S_n$. Thus z is the solution of the integral equation

$$z(t) = 1 + (L)\int_0^t zdg,$$

except on a set of measure zero.

We now show that g is in class Γ_3. Let L be defined by $L(t) = \max\{n\epsilon Z \mid n<t\}$ for $t \in S$. Thus $L(t) = [\![t^-]\!]$. If $s \leq t$ in S, then

$$E(g(t)-g(s) \mid A_s) = E(g(t)-g(s))$$

$$= E(\sum_{s\leq n<t} I_n) - E(\sum_{n=N(s)+1}^{N(t)} D_n)$$

$$= I_o(L(t)-L(s)) - \lambda(t-s)D_o.$$

Thus

$$|E(g(t)-g(s)|A_s)| \leq I_o dL(s,t) + \lambda D_o(t-s).$$

We may set $G_1(t) = I_o L(t) + \lambda D_o t$ and note that G_1 satisfies condition (iii) of Definition III.2.1. The calculation of G_2 is more complicated. For $s \leq t$ in S,

$$E((\sum_{s\leq n<t} I_n)^2) = E(\sum_{s\leq n<t} I_n^2 + 2 \sum_{s\leq m<n<t} I_m I_n)$$

$$= \sum_{s\leq n<t} J_o + 2 \sum_{s\leq m<n<t} I_o^2$$

$$\leq \sum_{n=L(s)+1}^{L(t)} J_o + 2 \sum_{n=L(s)+1}^{L(t)} \sum_{m=1}^{n} I_o^2.$$

Noting that $\sum_{n=i}^{j} n = (j(j+1)-i(i+1))/2$, we have

$$(4.3) \quad E((\sum_{s\leq n<t} I_n)^2) \leq J_o(L(t)-L(s))+I_o^2(L(t)(L(t)+1) - L(s)(L(s)+1)).$$

Let $G_2^1(t) = J_o L(t) + I_o^2 L(t)(L(t)+1)$. Also, for $s \leq t$ in S,

$$(4.4) \quad 2E(\{\sum_{s\leq n<t} I_n\}\cdot\{\sum_{k=N(s)+1}^{N(t)} D_k\}) = 2I_o[L(t)-L(s)]\cdot\lambda|D_o|(t-s)$$

$$\leq 2I_o L(t)\lambda t|D_o| - 2I_o L(s)\lambda s|D_o|.$$

Let $G_2^2(t) = 2I_oL(t)\lambda t|D_o|$ for t in S.

Temporarily fix $s \leq t$ in S and let

(4.5) $\Omega_{m,n} = \{\omega\varepsilon\Omega | N(s) = m,\ N(t)-N(s) = n\}$ and

(4.6) $\Omega_n = \{\omega\varepsilon\Omega | N(t)-N(s) = n\}$ for $m,n \varepsilon N$.

Then we may calculate

$$E\Big(\big(\sum_{k=N(s)+1}^{N(t)} D_k\big)^2\Big) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{\Omega_{m,n}}\big(\sum_{k=m+1}^{m+n} D_k\big)^2 dP$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{\Omega_{m,n}}\sum_{k=1}^{n} D_{m+k}^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} D_{m+k}\cdot D_{m+j}\ dP$$

(4.7)

$$= \sum_{n=0}^{\infty}\int_{\Omega_n}\sum_{k=1}^{n} D_k^2 + 2\sum_{k=1}^{n}\sum_{j=1}^{k-1} D_k D_j\ dP.$$

Since the characteristic function X_{Ω_n} and D_k are independent for $k \leq n$, we have

$$E\Big(\big(\sum_{k=N(s)+1}^{N(t)} D_k\big)^2\Big) = \sum_{n=0}^{\infty}[nC_o+n(n-1)D_o^2]\cdot P(\Omega_n)$$

$$= C_oE(N(t)-N(s))+D_o^2E((N(t)-N(s))^2)-D_o^2E(N(t)-N(s))$$

$$= C_o\lambda(t-s) + D_o^2\lambda^2(t-s)^2$$

$$\leq \lambda C_o(t-s) + \lambda^2\ D_o^2(t^2-s^2).$$

Let $G_2^3(t) = C_o\lambda t + D_o^2\lambda^2 t^2$ for t in S and let

$$G_2(t) = \sum_{j=1}^{3} G_2^j(t)$$

for t in S. Thus g is in class Γ_3.

It now follows that z(t) is given by the product integral $\prod_0^t [1+dg]$, hence

$$z(t) = \prod_{n=1}^{L(t)} [1+I_n] \cdot \prod_{k=1}^{N(t)} [1-D_k].$$

As an estimate of the value of the investment at some later time, we compute E(z(t)). Fix t > 0 and let the sets $\Omega_m = \{w\varepsilon\Omega | N(t) = m\}$ and let χ_m denote the characteristic function of Ω_m. Then

$$E(z(t)) = [1+I_o]^{L(t)} \cdot \sum_{m=0}^{\infty} \int_{\Omega_m} \prod_{k=1}^{m} [1-D_k]dP$$

$$= [1+I_o]^{L(t)} \cdot \sum_{m=0}^{\infty} \int_{\Omega} \chi_m \prod_{k=1}^{m} [1-D_k]dP$$

$$= [1+I_o]^{L(t)} \cdot \sum_{m=0}^{\infty} E(\prod_{k=1}^{m} [1-D_k])P\{N(t) = m\}$$

$$= [1+I_o]^{L(t)} \cdot \sum_{m=0}^{\infty} [1-D_0]^m P\{N(t) = m\}.$$

The summation in the last expression is

$$E((1-D_0)^{N(t)}).$$

The computation on pp. 74-75 of Çinlar [35] shows that the expectation is $e^{-\lambda D_0 t}$. Hence

$$E(z(t)) = [1+I_o]^{L(t)} \cdot e^{-\lambda D_o t}.$$

One can evaluate the long-term expected gain or loss on the investment by noting that for $t > 0$,

$$E(z(t+1))/E(z(t)) = [1+I_o]e^{-\lambda D_o}.$$

3. Jumps Uniformly Distributed in [0,1]

Suppose $S = [0,1)$, $\Omega = S$, P is Lebesgue measure in Ω and g is given by

$$g(t) = g(t,w) = \begin{cases} 0 \text{ if } t < w \\ 1 \text{ if } t \geq w. \end{cases}$$

Then g is in class Γ_3. To see this, note that if A_t is the σ-algebra generated by $\{g(s) | s \leq t\}$, then $E(g(t)|A_s) = E(g(t)|g(s))$ since g is a Markov process. Also note that $g(t)-g(s) \geq 0$ and $g(t) - g(s) = [g(t)-g(s)]^2$ for $s \leq t$. Therefore it is sufficient to show that

$$E(g(t)-g(s)|g(s)) \leq G(t)-G(s)$$

for some non-decreasing $G:S \to R$. If $s \leq t$ in S, then

$$E(g(t)-g(s)|g(s)) = [1-g(s)](t-s)\ (1-s).$$

Since for each s in S, there is a positive probability that $g(s) = 0$, and $g(s) \geq 0$ for s in S, it is necessary and sufficient that $(t-s)/(1-s) \leq G(t)-G(s)$. It follows that $\sum_{i=1}^{n} (t_i - t_{i-1})/(1-t_{i-1}) \leq \sum_{i=1}^{n} G(t_i)-G(t_{i-1})$, whenever $0 = t_o \leq t_1 \leq \ldots \leq t_n = t < 1$. Since the condition above holds for any subdivision of $(0,t)$, it must follow that $G(t) \geq \int_0^t dx/(1-x)$ for $t \leq 1$ (here we assume that $G(0) = 0$). We also note that

$$(t-s)/(1-s) \leq \int_s^t dx/(1-x) \quad \text{for} \quad s \leq t.$$

Therefore the minimal choice of G is

$$G(t) = \int_0^t dx/(1-x), \quad 0 \leq t \leq 1.$$

We note that $\lim_{t \to 1} G(t) = +\infty$. It is clear from thework above that (g,G,G) is in class Γ_3 and u defined by

$$u(s,t) = \begin{cases} 1 \text{ if } s \leq t \leq w \quad \text{or} \quad w \leq s \leq t, \\ 2 \text{ if } s \leq w \leq t \end{cases}$$

is the corresponding product integral in class T_3.

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