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A MULTIPLE ITEM, PROCUREMENT-INVENTORY SYSTEM UNDER A PROLONGED PRICE INCREASE

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SUMMARY

A procurement-inventory system operating under a prolonged price increase is analyzed. A model that represents such a system is developed.

This system is considered to be a multiple item system. The system parameters, that is the items' unit costs, carrying costs, ordering costs, backorder costs, lead times and demands, for each time period from the present to the planning horizon, are known. The items' salvage values at the horizon are also known.

The profit of all items over the planning horizon is selected as the measure of effectiveness for constructing the model. The policy decisions for each time period are those concerning the items' order quantities and sale prices. The system has a restricted warehouse space which creates a constrained optimization problem.

The dynamic programming solution is formulated for both deterministic and probabilistic demands. This solution requires great computational effort. Therefore, an approximate model for which the dynamic programming solution requires less computation, is developed. A numerical example shows how the approximate model can be applied and solved.

Some general conclusions which may be stated on the basis of the findings are the following:

1. Item cost increases and ordering cost increases affect the ordering policy differently than the carrying cost increases. In order to minimize the item cost as well as the ordering cost, it is necessary to order in greater quantities and less frequently.

2. In a period, the sale price to be selected for an item is not necessarily the price that maximizes the item's revenue in this period.

3. The optimal order quantities calculated using the approximate model are smaller than they would be if the exact model were used.

CHAPTER I

INTRODUCTION

The main objectives of this study are to analyze the procurement-inventory system in an inflationary economy, to develop a model that represents this situation and to show a solution for this model.

During periods of inflation, wage rates and prices of goods and services are never decreasing. The direct consequences of this fact in a procurement-inventory system are:

1. The unit cost of each item, as well as other important costs of the system, increase from time to time.

2. The sale price of each item has to be increased, at some moments, by the system.

The unit cost of an item is increased due to the wage increase of the related labor and to the price increases of the required raw materials, component parts, machines, tools, and overhead.

In a great number of the inventory systems studied, there is no mention of the items' sale prices. The reason for this is that sale prices are considered as constants in time and they do not influence optimal ordering decisions. In our problem, since some important costs are increasing with time and the business can exist only if there is profit, the system has to increase the sale price of individual items intermittently.

Brief Survey of Inflation

About fifteen years ago, the demand-pull theory of inflation was the only theory that had been developed to explain a general price level increase in an economy. This theory states that once the economy is operating at the full-employment level and the total output becomes fixed, an excess of total demand will necessarily have the effect of pulling up the price level. This theory can still be applied to many underdeveloped countries experiencing inflation today.

After 1956 the American economy began to exhibit peculiar behavior; employment and output were declining, while at the same time the general price level was rising. Therefore, this inflation could not be explained by the demand-pull theory. McConnell[1] presents two theories that explain inflation in the absence of full employment: the cost-push theory and the structural inflation theory. With respect to cost-push inflation, he states the following:

> Unions have considerable control over wage rates; that is, they possess considerable market power. Indeed, they have so much market power that even with a moderate deficiency of total demand, some unemployment, and some excess industrial capacity, the stronger unions can demand and obtain wage increases. Large employers, faced now with increased costs but also in the possession of considerable market power, push their increased wage costs and "something extra" on to consumers by raising the prices of their products. This theory is obviously based on the presumption that both unions and businesses typically possess some significant degree of market power and therefore can within limits manipulate wages and prices independent of overall conditions of total demand.

Concerning structural inflation, McConnell states the following:

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1. C. R. McConnell, Economics: Principles, Problems and Policies, McGraw-Hill Book Co., p. 391.

Briefly stated, the rationale is based on the fact that for a number of reasons - a basic one of which is the market power of businesses and unions - prices and wages tend to be flexible upward but inflexible downward. Now let us suppose that total demand is not excessive; as a matter of fact, let us assume that it is slightly deficient, resulting in, say, 5 percent unemployment. Now a rather sharp change in the structure or composition of this total demand occurs. This structural change in demand means that prices and wages will rise in those segments of the economy experiencing an expanding demand. However, because of their downward stickiness, wages and prices will not fall, or at least will not fall by much, in those sectors of the economy witnessing a declining demand. The result is a net increase in the price and wage levels; that is, inflation will occur. Remember: This inflation arises despite the fact that there is less than full employment and the economy is failing to realize its growth potential.

<u>Problems of Procurement-Inventory Systems in an Inflationary Economy</u>

In an inflationary economy, if we take a good or a service and examine its price during previous short time intervals, we can observe that the price increase is discontinuous. Sometimes, the good or service remains at the same price for several intervals before an increase occurs. Also, the amount of increase in price of this good or service may not be the same at different time intervals. In general, the increase in price of goods and services does not occur at the same time.

As mentioned before, the unit cost of each item in inventory increases from time to time. Generally, these increases follow and are proportional either to wage increases for those who produce the good, or to price increases for raw materials, component parts, machines, tools and everything else required in the production of that good. Also, the price increases for the raw materials, component parts, machines, tools, and overhead, follow and are proportional to the wage increases for other employees.

If a decision maker is cognizant of the potential wage increases that unions may cause as well as the governmental economic policy, he can predict with certain accuracy, when prices will increase and by what amount. With this in mind, if the decision maker must order items and keep them in inventory in order to satisfy future demands, it will be cost effective to order an increased quantity of an item just before its cost increases. However, if the decision maker orders a greater amount than usual, he will have more units stocked, increasing the inventory carrying cost. But by ordering in greater quantity, he will order fewer times, decreasing the ordering cost. Therefore, a trade-off exists between the item cost savings, the ordering cost savings and the differential carrying cost.

We have been discussing the costs of the items for the inventory system. At this point, let us examine the sale prices of these items. After a cost increase, or at any other moment, the decision maker can increase the sale price of an item. As a result of this increase in the sale price, the demand of the item will remain constant or decrease depending on the amount the price is increased. A decrease in the demand indicates that some consumers cannot meet the increased price until salaries are increased as well. Hence this decrease will be temporary; with new salary increases the demand reacts progressively.

It is known that the correction of the purchasing power of the salaries and each increase in the price of the goods or services do not occur simultaneously. There is a time lapse between consecutive salary increases for the same group of workers. Additionally, the various unions' salary agreements are approved at different times. For these reasons, demand reacts progressively.

Therefore, the decision maker often faces the same problem: he can increase the sale price by an amount that makes him certain the demand will remain constant, or he can raise the sale price beyond this point by different amounts and obtain decreased demands.

Bach[2] states the following about this subject:

> The economy has never been perfectly competitive with prices and wages responding only to impersonal market forces of supply and demand. Both wages and prices have long been administered to varying degrees in different markets. And with nearly all administered wages and prices there is a margin, large or small, within which the price is set mainly according to the judgment of the price setter. If competitive pressures are strong, this discretionary margin is small; but if the seller has a substantial monopoly position, it may be quite large.
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> But no seller, no matter how administered his prices, can long escape the test of the market. He can raise his wage or price; but if his price moves far beyond customers' willingness or ability to buy, he will lose sales. If many prices are moving up at the same time, widespread sales losses may occur as prices across the board begin to outrun consumer incomes.

The Specific Problem

We will study a multiple item, procurement-inventory system. The system has a warehouse with limited space in which the items are stocked. The ordering cost, the inventory carrying cost and the backorder costs of each item in each future time interval over a planning horizon are known. The unit cost of an item at any time is dependent on its order quantity. The relationship between order quantity and unit cost of each item in each future time interval over the horizon are known. All possible sale prices and their related demands are also known for each item in each future time interval over the horizon. In each time interval where no change occurs, all the costs are considered to be deterministic parameters.

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2. G. L. Bach, "Inflation in Perspective" published in the book by M. L. Joseph, N. C. Seeker and G. L. Bach, Economic Analysis and Policy, Prentice-Hall, Inc., p. 33.

The demands are considered to be either deterministic or stochastic parameters. When they are stochastic parameters, their probability distributions are given. The present inventory levels of the items are known. The salvage prices of the items at the horizon are known with certainty. The lead times of the items at each decision point are also known with certainty.

Based on the revenue from sales, the item cost, the ordering cost, the inventory carrying cost and the backorder cost of all items in all time intervals over the horizon, and the salvage value of all items at the horizon, our objective is to maximize the profit. In order to achieve this purpose, at the decision points during the planning interval the following policy elements must be implemented:

1. If an order is placed at that time, how much to order.

2. If a sale price increase is warranted, how much should the price be increased.

CHAPTER II

LITERATURE SURVEY

The literature that pertains to inventory theory is greatly diversified. Inventory systems have been classified in many forms. A criterion for classifying inventory systems is according to the number of decision points in the system; these systems can be classified as multiperiod or single period systems. A multiperiod system can be classified as a dynamic or static system, depending on the nature of the inputs. Static systems can be classified as transactions reporting or periodic review systems, depending on the type of procedure used to report the inventory levels to the decision maker.

We are particularly interested in the multiperiod dynamic inventory system in which all important costs, the lead time, and the demand are time variant. Important costs, in this definition, refers to the item cost, the cost of placing an order, the cost of carrying a unit in inventory during a time interval and the cost of a unit short. The variation in the process generating demands is considered most in the literature.

The first work related to dynamic inventory systems is a paper written by Arrow, Harris and Marschak[3]. The authors developed a model assuming that the inventory system will last indefinitely. Also, all important costs and the probability distribution of the demand will be

3. K. J. Arrow, T. Harris and J. Marschak, "Optimal Inventory Policy," Econometrica, Vol. 19, No. 3, pp. 250-272.

the same in each period. The authors' objective is to minimize a long run expected cost. Due to extreme generalization in this model, solutions cannot be obtained in explicit form, except in rare cases. Another model with a finite number of periods was shown as a means of approximating the infinite period model. In this other model, both the costs and the demand distribution were permitted to vary over time. The expression of this model relates the expected cost of a period with the expected cost of the preceding period, in a manner similar to the recurrence relationship of dynamic programming.

Dvoretzky, Kiefer and Wolfowitz[4] studied a dynamic inventory system where the demand distribution in each period is dependent on the values of the demand and the ordering quantity of each preceding period. Their objective is to minimize the expected cost in the last period. The principle of dynamic programming is used to solve this model. Even in a simple example, this model requires a great computational effort, so that its use is impracticable.

The basic dynamic inventory model, as shown by Hadley and Whitin[5], represents the inventory system of a single item where some parameters vary from time to time. This problem was analyzed for both a deterministic and a probabilistic demand. Let us treat first the deterministic demand problem. The cost of placing an order, the inventory

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4. A. Dvoretzky, J. Kiefer and J. Wolfowitz, "The Inventory Problem," Econometrica, Vol. 20, No. 2, pp. 187-222.

5. G. Hadley and T. M. Whitin, Analysis of Inventory Systems, Prentice-Hall, Inc., pp. 336-350.

carrying cost, the lead time and the demand are considered to be time variants. Their values in each future time interval over the planning horizon are known. The values of the lead time are assumed to be deterministic and to be such that orders cannot cross. In other words, for two orders of an item the first to arrive will be the first placed. The item cost is considered to be constant in time and its value is known. The values of the present inventory level and the required inventory level at the horizon are also known. We have the restriction that the order quantities must permit a demand to be satisfied when it occurs. The problem is to select the order quantity at each decision point in order to minimize the variable cost of all time intervals over the horizon. In this model, it is assumed that the sale price is constant in time, as is the cost of the item. Therefore a minimization of variable cost will give the same result as a maximization of profit.

The natural tool for solving dynamic models in order to avoid exhaustive enumeration is dynamic programming. In the basic model with deterministic demands, the stages of the dynamic programming type of problem are the decision points. The decisions in a stage are the possible amounts of the item to be ordered in this decision point. Since the ending inventory is specified as well as the beginning inventory, it is possible to use either a forward or a backward reasoning. The states of a stage are the feasible closing or opening inventory levels of the respective period, depending on whether we are working forward or backward. Wagner and Whitin[6] developed some simplifications in the computational

6. H. M. Wagner and T. M. Whitin, "Dynamic Version of the Economic Lot Size Model," Management Science, Vol. 5, No. 1, pp. 89-96.

procedure of dynamic programming for this model. These simplifications yield reductions in the number of states and decisions at each stage. If a person is going to solve a problem of this type manually, he can save time and effort by using these simplifications.

In the probabilistic demand case, shortages may occur; a backorder cost is then introduced in the model. All considerations made for the deterministic demand case are still valid, unless it is stated to the contrary. In addition to the parameters mentioned in the deterministic demand case, the item cost and the backorder cost are now also considered to be time variants. Their values and the probability distribution of the demand are also known, in each future time interval over the horizon. As a matter of fact, the item cost in each future time interval is known as a dependent value of the order quantity. Due to the stochastic nature of the process, the inventory at the horizon cannot be chosen in advance by the decision maker. It is assumed that all items remaining on hand at the horizon will be sold. The unit salvage value of each item is known. The objective is to minimize the subtraction of the expected salvage value at the horizon, from the expected variable cost over the horizon of procurement, carrying and backorders.

Dynamic programming can be used in a similar fashion as in the deterministic demand case, except that on the right hand side of the recurrence relationship we will have some expected values instead of deterministic values. For example, we will have the expected return of some states of the preceding stage instead of the return of a particular state of the preceding stage. We have to work backward since we only know the beginning inventory. It is interesting to note that the order quantity of the first decision point is the only policy variable obtained explicitly. Each of the other order quantities is dependent on its own preceding stochastic demands.

In addition to the basic system, a great number of studies about very specific situations can be found in the literature of dynamic inventory systems. There are some works somehow related to the situation that we are going to study. The rest of this chapter will be devoted to presenting these works.

Whitin[7] studied an inventory system of a single item in two situations. In the first, the demand will exist indefinitely and it will be the same in all periods. In the second, the demand will exist only in one period. For both situations the demand is a function of the sale price. In the first case, the item cost, the ordering cost, the inventory carrying cost and the demand curve are known. The demand curve represents the relation between the sale price and the demand. Using the search of extreme points by calculus, Whitin derived the optimal sale price and the optimal order quantity to be used in all periods in order to maximize the profit. He did not consider the possibility of changes in the parameters and in the demand curve as the system moves in time. In the second case, we have a single period problem. The demand is stochastic and for each price there is a probability distribution of the demand. The objective is to balance the liquidation loss, when the demand is smaller than the order quantity, with the goodwill loss plus the profit loss, when the demand is greater than the order quantity, in order to obtain the maximum expected profit. Using graphs and the marginal analysis

7. T. M. Whitin, "Inventory Control and Price Theory," Management Science, Vol. 2, No. 1, pp. 61-68.

of economics, he explained how to obtain the optimal price and the optimal quantity.

Wagner and Whitin[8] considered two situations similar to the preceding ones. However, the demand exists for a known number of periods. Wagner and Whitin first studied the situation where the demand curve, the production cost function, the inventory carrying cost and the ordering cost are constant in time. By means of a three-dimensional graph and marginal analysis, they showed how to obtain the optimal level of sales and the optimal order quantity. Next, they presented a situation where the demand curve and all the costs vary over time. First, they found the marginal cost and revenue curves for each period. The possibilities of production during the periods were then enumerated. For each possibility, they constructed the aggregate marginal cost and revenue curves taking into consideration the inventory carrying costs. The optimal output level is at the junction point of these curves. With this output level, Wagner and Whitin returned to the marginal cost and revenue curves and found the demand and the order quantity to be chosen for each period. Having these values and the ordering costs, they calculated the profit related to each possibility of production. Finally, the values that yield the maximum profit, constitute the optimal solution.

Fabian, Fisher, Sasieni and Yardeni[9] analyzed the problem of a company that uses a raw material, the price of which is subject to

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8. H. M. Wagner and T. M. Whitin, "Dynamic Problems in the Theory of the Firm," Naval Research Logistics Quarterly, Vol. 5, No. 1, pp. 53-74.

9. T. Fabian, J. L. Fisher, M. W. Sasieni and A. Yardeni, "Purchasing Raw Material on a Fluctuating Market," Operations Research, Vol. 7, No. 1, pp. 107-122.

considerable fluctuation. The analysis was made for only one item. At the decision point, the company knows the existing inventory, the current market price, the cost of holding inventory, the cost of shortage and for each period until the horizon, the probability density functions for the price and the requirement of the raw material. The decision problem is the timing of the purchases and the decision of how much to purchase when the time arises, in order to balance purchasing costs, inventory costs and shortage costs. They developed a dynamic programming model and its analytic solution.

Bellman and Dreyfus[10] solved the warehousing problem by dynamic programming. This problem can be explained in the following way: given a warehouse with fixed capacity and an initial stock of a certain product which is subject to known seasonal price and cost variations, find the optimal pattern of purchasing, storage and sales, in order to maximize the profit. In this problem, the idea is to buy the product at minimum cost and to keep it in inventory until it can be sold at a maximum price. The authors assumed that the market was unlimited. They did not consider the inventory carrying cost and the ordering cost.

Eastman[11] solved this last problem for the multi-commodity case. The warehousing problem requires the determination of a sequence of buy and sell decisions at discrete points in time. By equating points in time and points in space, Eastman identified this problem with a special case of the shortest-route problem. He considered the inventory carrying

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10. R. E. Bellman and S. E. Dreyfus, Applied Dynamic Programming, Princeton University Press, pp. 125-131

11. W. L. Eastman, "A Note on the Multi-Commodity Warehouse Problem," Management Science, Vol. 5, No. 3, pp. 327-331.

cost in his model.

Naddor[12] presented the problem of inventory systems in which the cost of the commodity being replenished is expected to change. There was only one cost change and, therefore, only one decision point. He knew the present cost, the cost change, and the time at which this change would occur. He also knew the cost of placing an order, the cost of carrying one unit in inventory during a period, and the demand in each period. He treated the demand as deterministic. The general approach to the development of the appropriate model was to compare the cost of not taking advantage of the anticipated cost change with the cost of purchasing an increased amount just before the cost change. The increased amount which maximizes the difference between these costs, gives the optimal solution. Naddor solved this problem for both a deterministic cost and a probabilistic cost change.

Bell[13] studied a specific situation of an inventory system in which the firm has monopoly of the product. Its sale price is inversely related to the amount delivered to the market. If the product is not sent to the market it will have only salvage value. The demand is variable. If the number of products delivered is not sufficient to satisfy the demand, an emergency shipment entailing a higher ordering cost must be made. The firm has no control over the rate of arrivals into inventory and it seeks to minimize its expected losses over a distant horizon.

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12. E. Naddor, Inventory Systems, John Wiley and Sons, Inc., pp. 96-102.

13. D. Bell, "An Inverse Warehousing Problem for Imperfect Markets," Management Science, Vol. 14, No. 9, pp. 536-542.

CHAPTER III

DEVELOPMENT OF THE MODEL

The model that we are going to develop represents a procurement-inventory system in an inflationary economy. Since the prices of goods and services are always rising, the important costs of this system will be rising too. As a consequence, the sale prices of the products in the system will have to be raised occasionally. The decision maker is facing a dynamic type of problem. It is dynamic in the sense that the best decision to be taken at a decision point will seldom be the same as the decision made at the preceding point. This is because some of the parameters of the system have different values from one decision point to another.

Description of the System

The system that we are studying contains a warehouse where items are stored in order to meet future demands. The warehouse may be connected to a plant. In this case, the item cost will be its production cost. The warehouse may be a separate entity; then the item cost will be its purchase price.

The warehouse has a maximum capacity which limits the number of items' units that can be stored at one time. A unit of each item occupies a different amount of warehouse space. We know the warehouse capacity and the space occupied by a unit of each item.

The parameters of the system are the unit costs, the inventory carrying costs, the ordering costs, the backorder costs, the lead times and the demands, for all items. As in the basic dynamic model, the parameters will have time dependent values. Our model differs from the basic model in that in addition to being a multiple item model, it considers the sale prices of the items as forming a second group of decision variables, and the demands are dependent on these sale prices. Therefore, the order quantities and the sale prices must cause the maximum profit over the horizon by balancing the revenue from sales, including the one at the horizon; the cost of the items; the holding cost; the ordering cost and the backorder cost.

Another difference between the two models is that we have a constraint limiting the space destined to store the items. This constraint was included based on the fact that if a decision maker knows about a future cost increase of an item, and if the savings obtained by ordering more units at a lower cost compensates the consequent changes in the other costs, then he will order the maximum profitable amount. Depending on the situation, this amount is far beyond the capability of the system to store it.

We can suppose that the system has either been in existence for some time in the past or that it is starting to operate now. Basically, the reasoning is the same for both cases. In the first case, we will probably have a positive inventory level entering our planning interval. However, we have to point out that the decisions taken in the past cannot be accounted for in our model since these decisions, good or bad, can no longer be changed.

The decision maker can somehow predict the changes in the parameters of the items that will occur from the present moment until the planning horizon. The changes in all the costs are predicted in a deterministic way. The demands of the items in each future time period until the horizon are predicted using either deterministic values or probability distributions. The planning horizon will be the end of the last time period where all values of the parameters of the items, including their salvage values, can still be predicted.

The unit cost of an item in any time period is dependent on its order quantity. This dependence may not be linear. In practice it is shown in the quantity discount structures. There will be a quantity discount structure for each time period where no change in this dependence occurs. In this structure the possible order quantities are divided into intervals. For each quantity interval there is a different unit cost. Of course, as the quantity increases, so that it moves from one interval to another, the unit cost will decrease. We will assume that the discount given in each interval is for all units ordered and not only for the units in that interval. This type of discount is usually called an "all units discount".

We stated before that the unit cost of each item increases from time to time. We need to clarify that the unit costs of an item, corresponding to the quantity intervals, will all be increased at the same moment, and these simultaneous increases will occur from time to time. We also stated that the decision maker can predict the changes in the parameters of the items. Specifically, the changes in the unit costs are predicted in a deterministic way. We also need to clarify that the decision maker will predict the changes in the quantity discount structures of each item, and for each quantity interval of an item in a time period, the unit cost is predicted in a deterministic way.

The sale price and the demand of an item in a time period are related by a demand curve. Despite the classical name "demand curve," in reality we do not have a curve between them. Both are discrete variables; their correspondence is such that all prices in a price interval will yield the same demand. Since we are ultimately trying to maximize profits, we will choose in each price interval the maximum price that yields the demand of this interval. Therefore, we will have a one-to-one correspondence between maximum prices of the intervals and demands, i.e., for each maximum price only one demand will correspond and vice-versa. It is possible to construct a table for an item in a time period showing these pairs of values, and the decision maker will have to decide which pair to select from the table.

In the last paragraph, we talked about the demand as a deterministic value. If we treat it as probabilistic, for the maximum price of each interval we will have a probability distribution of the demand. In this situation, the decision maker will still have to choose a price among some selected values instead of among all possible prices. However, associated with each choice there will be a probability distribution of the demand.

As mentioned in the first chapter, the decision maker can increase the sale price of an item at any moment by an amount that makes him certain the demand will remain constant, or he can raise the sale price beyond this point by different amounts and obtain decreased demands. We also mentioned that this decrease in demand will be temporary; with new salary increases, the demand reacts progressively. All these facts are expressed when the decision maker constructs tables of correspondence between sale price and demand of an item in a time interval. If we examine the tables of an item, we will verify that in any two consecutive time intervals for a fixed sale price the demand is increasing or reacting, as we said before. We will also verify that for a constant demand the sale price will never be decreasing.

The lead times of the items may be treated either as deterministic or as probabilistic values. This investigation will only consider situations where lead time is treated as a deterministic value. From one decision point to another, it is possible to have different values for the lead time of an item, but there is an assumption that they are such that orders cannot cross. A result of this assumption is that we are certain that no orders of an item placed in the past will arrive after an order of this item placed during our planning interval has arrived.

A set of decision points will be chosen to guarantee no change in the tables relating unit cost and order quantity, and no change in the ordering costs of items between two consecutive decision points. After obtaining the arrival times of the items' orders, by adding the decision times to the respective lead times, the decision points also guarantee no change in the inventory carrying cost, the backorder costs and the demand curve of an item, during any interval between two consecutive arrival times of this item. All this can be accomplished by chosing convenient small time intervals between the decision points so that all changes will occur at the decision points or at the arrival times.

Note that neither the decision points nor the arrival times need to be equally spaced. A period of an item is the time interval between two consecutive orders' arrivals of this item. Note that the periods of an item do not necessarily have the same value. Note also that the periods of different items related to the arrivals of orders that were placed in the same two decision points, are not necessarily coincident. Furthermore, due to the differences in the items' lead times in a decision point, decisions for all items in the last decision points may not be required. If such decisions were required, some orders might arrive after the planning horizon. Of course, it is possible to place an order for at least one item at the last decision point and this order will arrive before the horizon. Also, at least the order placed in the first decision point for each item, will arrive before the horizon.

In reality, if no item becomes obsolete at the horizon and the system continues to exist after this time, there will be decisions for all items in the last decision points. Nevertheless, with the forecast of the items' parameters that we now have, it is only possible to plan for this horizon, even if for some items we know parameter values beyond this point. It is quite probable that after solving this model, only the decisions of the first decision points will be used. After some time, with new forecasts, better knowledge of the future will be obtained, and with data representing the new situation the model will be solved again.

In the case where the system continues to exist as it is now after the horizon, the salvage value, in spite of its name, does not represent the price at which the item will be discarded at the horizon, since no units will be discarded. In this case, a minimum value of the salvage value can be the item's cost at the last decision point at which an order for it was placed.

Model with Deterministic Demands

First, let us construct a model for the case where the items' demands, in addition to all the costs in each time interval where no change occurs, are considered to be deterministic values. In this model, instead of using a backorder cost, we will assume that the decision maker wants no item to be out of stock when a demand for it occurs.

We will find that there are n times $u_1, u_2, \ldots, u_n$ during our planning period that satisfy the conditions imposed on the decision points. The time u ($u > u_n$) satisfies the requirement for a planning horizon. We have m items in the system. The lead time of an order of item i placed at the decision point u_j will be v_j^i and the correspondent arrival time will be $t_j^i = u_j + v_j^i$. Note that v_j^i is a deterministic value, but the lead time of an item is permitted to vary from one decision point to another. For the first decision point u_1, we will have m lead times: $v_1^1, v_1^2, \ldots, v_1^m$, which create the arrival times: $t_1^1, t_1^2, \ldots, t_1^m$. All these arrival times are smaller than the horizon. As we mentioned before, in some decision points excluding the first, it will probably not be possible to place orders for all items. An order for item i can only be placed until the decision point u_{n_i}, because the orders' arrival times for this item placed in this point and in its successor, have the following relationship with the horizon: $t_{n_i}^i < u < t_{n_i+1}^i$. Recall that we made the assumption that the orders cannot cross. We shall point out that, even when it is possible to place an order for an item in a decision point, it may not be profitable to do so, and the order will not be placed. From the preceding explanations, it becomes apparent that n is equal to the maximum n_i for all i.

The order quantity of item i to be placed at the decision point u_j will be called Q_j^i. The unit cost, as a function of the order quantity, and the cost of placing an order of item i at the decision point u_j will be $C_j^i(Q_j^i)$ and A_j^i, respectively. We shall remember that $C_j^i(Q_j^i)$ and A_j^i are deterministic values, but the unit cost for a fixed order quantity, and the ordering cost of an item are allowed to vary with j.

We shall define the period j of item i as the time from t_j^i to t_{j+1}^i; it is represented by $T_j^i=t_{j+1}^i-t_j^i$. Suppose that $t_{n_i+1}^i=u$, for all i. Note that there are n_i periods for item i. Note also that the order quantity Q_j^i will arrive at the beginning of period j, since its arrival time is t_j^i. Let L^i be the unit salvage value of item i at the horizon. The inventory carrying cost of item i in period j will be H_j^i. The demand of an item in a period will be a function of the chosen sale price. If we call the sale price of item i in period j P_j^i then the associated demand will be $D_j^i(P_j^i)$. It will be recalled that H_j^i, and the pairs of P_j^i and $D_j^i(P_j^i)$ are deterministic values. However, the inventory carrying cost, and the pairs of the sale price and the correspondent demand of an item may vary from one period to the next. Let I_j^i be the opening inventory level of item i in period j, before the arrival of the order expected for this period, and considering only the inventory on hand. This level is the same closing inventory level of the preceding period. Hence the material balance equation for item i in period j is: $I_{j+1}^i=I_j^i+Q_j^i-D_j^i(P_j^i)$. The values of $I_1^1,I_1^2,\ldots,I_1^m$ are known deterministic values. Consider $I_{n_i+1}^i$ as the inventory on hand of item i at time u. This inventory will be sold at the unit salvage value of item i. For the deterministic demand situation, the fact that the decision maker requires that all units of the items at the horizon be sold, implies that the inventories on hand of the items at this time may vary. It can be possible that the decision maker, instead of requiring that all units of the items at the horizon be sold, will stipulate an inventory level for each item at this time, i.e., the value of $I^i_{n_i+1}$ for each i will be fixed.

The demand rate of item i in period j will be written $d^i_j(P^i_j,t)$. Observe that the demand rate is a function of time, as well as a function of the sale price. The demand of item i in period j is related to this demand rate in the following way:

$$D^i_j(P^i_j) = \int_{t^i_j}^{t^i_{j+1}} d^i_j(P^i_j,t)\ dt \tag{1}$$

Since the demand is a function of the sale price, then the demand rate will be too. The demand of item i from the beginning of period j until any time t within this period, will be given by

$$\int_{t^i_j}^{t} d^i_j(P^i_j,y)\ dy$$

Since the inventory on hand of an item at the beginning of a period is equal to its opening inventory in this period plus its order quantity expected for this period, then the inventory on hand of item i at any time t within period j will be

$$I^i_j + Q^i_j - \int_{t^i_j}^{t} d^i_j(P^i_j,y)\ dy$$

If we call $\bar{I}_j^i$ the average inventory on hand of item i during period j, we will have

$$\bar{I}_j^i = \frac{1}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \left[I_j^i + Q_j^i - \int_{t_j^i}^{t} d_j^i(P_j^i, y)\ dy \right] dt$$

$$= I_j^i + Q_j^i - \frac{1}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \int_{t_j^i}^{t} d_j^i(P_j^i, y)\ dy\ dt \qquad (2)$$

Therefore the inventory carrying cost of item i in period j will be

$$H_j^i\ \bar{I}_j^i = H_j^i \left[I_j^i + Q_j^i - \frac{1}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \int_{t_j^i}^{t} d_j^i(P_j^i, y)\ dy\ dt \right]$$

If we apply the material balance equation, we obtain

$$H_j^i\ \bar{I}_j^i = H_j^i\ I_{j+1}^i + H_j^i \left[D_j^i(P_j^i) - \frac{1}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \int_{t_j^i}^{t} d_j^i(P_j^i, y)\ dy\ dt \right]$$

The term $H_j^i\ I_{j+1}^i$ is the inventory carrying cost in period j for those units of item i carried into period j+1. The other term in the expression above represents the cost of carrying $D_j^i(P_j^i)$ units of item i, each one during a different time period smaller than period j. This is explained because these units will be demanded at different points within this period. In the basic dynamic model mentioned before, this last carrying cost was not included in the cost expression that was going to be minimized. The reason for this was that this cost is independent of the order quantities and it cannot be avoided, since the $D_j^i(P_j^i)$ units of item i demanded in period j must be on hand at the beginning of this period. However in our case, since the demand is a function of the sale price, this part of the carrying cost must be considered. For different sale prices we will have different demands and different demand rates, and therefore, different values for this term.

The decisions to be made concerning the order quantities and the sale prices of item i, will have no influence over the inventory carrying costs incurred from the first decision point u_1 until the related arrival time t_1^i. Since these costs are independent of Q_j^i and P_j^i, they need not be included in the carrying cost expression of this item. The only carrying costs of item i that are relevant are those incurred between t_1^i and u, or in other words, those incurred in the periods of this item. Remember that:

$$u - t_1^i = (u - t_{n_i}^i) + (t_{n_i}^i - t_{n_i-1}^i) + \dots + (t_2^i - t_1^i) = T_{n_i}^i + T_{n_i-1}^i + \dots + T_1^i$$

Therefore, the inventory carrying cost of all items in their periods will be the carrying cost to appear in our model:

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} H_j^i \left[I_j^i + Q_j^i - \frac{1}{T_j^i} \int_{t_j^i}^{t_{j+1}^i} \int_{t_j^i}^{t} d_j^i(P_j^i, y)\, dy\, dt \right]$$

The cost of the items to be ordered during the planning period is

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} Q_j^i\, C_j^i(Q_j^i)$$

The ordering cost of these items is

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} A_j^i\, R_j^i$$

where $R_j^i = 0$, if $Q_j^i = 0$; and $R_j^i = 1$, if $Q_j^i > 0$. Analogous to the inventory carrying costs, the revenue from sales of an item in our interest are those incurred in the periods of this item. The decisions to be found for an item will not influence the revenue obtained before the first period of this item. The revenue from sales of all items in their periods is

$$\sum_{i=1}^{m} \sum_{j=1}^{n_i} P_j^i \, D_j^i(P_j^i)$$

The salvage value of all items left over at the horizon is

$$\sum_{i=1}^{m} L^i \, I_{n_i+1}^i$$

Calling G the profit of all items, we obtain the expression of our model with deterministic parameters:

$$G = \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \left\{ P_j^i \, D_j^i(P_j^i) - Q_j^i \, C_j^i(Q_j^i) - A_j^i \, R_j^i - H_j^i \left[I_j^i + Q_j^i - \frac{1}{T_j^i} \int_{t_j^i}^{t_{j+1}^i} \int_{t_j^i}^{t} d_j^i(P_j^i, y) \, dy \, dt \right] \right\} + L^i \, I_{n_i+1}^i \right) \tag{3}$$

Note that the expression inside the exterior set of parentheses represents the contribution of item i to the profit. Our objective is to maximize the profit G by making decisions about the sale prices and the order quantities of the items in their periods. This maximization is subject to the following groups of constraints.

1. The non-negativity constraints of the order quantities:

   $Q_j^i \geq 0$, for $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n_i$

2. The no stockout constraints of the items in their periods:

$$I^i_j + Q^i_j \geqslant D^i_j(P^i_j), \text{ for } i = 1,2,\ldots,m, \text{ and } j = 1,2,\ldots,n_i$$

3. The last group of constraints are due to the restricted warehouse space. Let us call w^i the space occupied by one unit of item i in cubic units, and W the warehouse capacity also in cubic units. The most probable times at which the warehouse capacity could be exceeded are the arrival times of orders of the items. If we are sure that at these times this constraint is not violated then it will never be violated. Imagine that at time $t^{i'}_{j'}$ an order of item i' will arrive. Time $t^{i'}_{j'}$ is the beginning of period j' of this item. At this same moment, each of the other items will be within one of their periods. Let us call j_i the period in which item i will be when the system is at time $t^{i'}_{j'}$. Note that the value of j_i may vary as i varies, what means that different items may be in periods of different numbers at time $t^{i'}_{j'}$. Different items may also be at different points within their periods. One may be near the beginning of its period while other may be near the end. By coincidence, another may even be at the beginning point of a period as item i' is. Then the inventory on hand of item i at time $t^{i'}_{j'}$ will be

$$I^i_{j_i} + Q^i_{j_i} - \int_{t^i_{j_i}}^{t^{i'}_{j'}} d^i_{j_i}(P^i_{j_i},t)\, dt$$

Hence the warehouse capacity constraints are

$$\sum_{i=1}^{m} w^i \left[I^i_{j_i} + Q^i_{j_i} - \int_{t^i_{j_i}}^{t^{i'}_{j'}} d^i_{j_i}(P^i_{j_i},t)\, dt \right] \leqslant W \qquad (4)$$

for $i' = 1,2,\ldots,m$ and $j' = 1,2,\ldots,n_{i'}$. Observe in the summation that one of the values that i will take is i'. In this case $j_{i'}$ will be the same j'.

Some simplifications will be obtained in this model if we assume that the demand rate of an item at any period is a value independent of the time within the period. It will be called $d^i_j(P^i_j)$. In this situation the demand of item i in period j from (1)[14] will be given by $D^i_j(P^i_j) = d^i_j(P^i_j)\ T^i_j$. Observe that with this assumption the demand is a linear function of the time. The average inventory on hand of item i during period j from (2) will be

$$\bar{I}^i_j = I^i_j + Q^i_j - \frac{d^i_j(P^i_j)\ T^i_j}{2} = I^i_j + Q^i_j - \frac{D^i_j(P^i_j)}{2}$$

Therefore, the total profit from (3) will become

$$G = \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \left\{ P^i_j\ D^i_j(P^i_j) - Q^i_j\ C^i_j(Q^i_j) - A^i_j\ R^i_j - H^i_j\ \left[I^i_j + Q^i_j - \frac{D^i_j(P^i_j)}{2} \right] \right\} + L^i\ I^i_{n_i+1} \right) \tag{5}$$

The constraints due to the warehouse capacity from (4) will be modified too. They will now be given by

$$\sum_{i=1}^{m} w^i \left\{ I^i_{j_i} + Q^i_{j_i} - d^i_{j_i}(P^i_{j_i}) \left[t^{i'}_{j'} - t^i_{j_i} \right] \right\} \leq W \tag{6}$$

for i' = 1,2,...,m, and j' = 1,2,...,$n_{i'}$.

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14. Number in parentheses refers to an expression already shown.

Model with Probabilistic Demands

The model that we are going to develop in this section considers item demand as probabilistic. All the costs in each time period in which no change occurs are still treated as deterministic values.

When the demand for an item cannot be satisfied by the system due to shortage of this item it is assumed that backorders occur. However the system will incur an extra cost. The cost of a backorder has two components. One is a fixed cost and the other is proportional to the length of time for which the backorder exists.

As before, the decision points will be called $u_1, u_2, \ldots u_n$. By changing the origin in the time scale it is advisable to make $u_1 = 0$. The planning horizon will be time u. There are m items in the system. The lead time of an order of item i placed at the decision point u_j will be called v_j^i, and the corresponding arrival time: $t_j^i = u_j + v_j^i$. The lead times of an item are such that orders cannot cross. The last decision point at which an order of item i can be placed so that it will arrive before the horizon, will be u_{n_i}. We know that at least the order placed in the first decision point for any item, will arrive before the horizon. Period j of item i will be the time between the arrival times t_j^i and t_{j+1}^i; it will be represented by $T_j^i = t_{j+1}^i - t_j^i$. Assume that $t_{n_i+1}^i = u$, for all i. Observe that there are n_i periods for item i.

The order quantity of item i to be ordered at the decision point u_j will be called Q_j^i. This order quantity will arrive at the beginning of period j. The unit cost as a function of the order quantity, and the ordering cost of item i at the decision point u_j will be denoted by $C_j^i(Q_j^i)$ and A_j^i, respectively. The unit salvage value of item i at the horizon will be L^i. All units left over at the horizon will be sold at this price. The holding cost of item i in period j will be written H_j^i. The fixed cost of a backorder of item i incurred in period j will be K_j^i and the cost of a unit year of shortage will be called J_j^i. Suppose P_j^i will be the sale price of item i to be chosen for period j. The demand of item i during period j will be a function of the chosen sale price; it will be written $D_j^i(P_j^i)$. However, for a fixed value of P_j^i, $D_j^i(P_j^i)$ is now a random variable following a probability distribution function. The expected value of this variable will be represented by $\hat{D}_j^i(P_j^i)$. The decision maker knows the demand distribution for each sale price of an item in a period.

Different from its meaning in the deterministic demand case, I_j^i now represents the inventory position of item i at the decision point u_j prior to placing an order of this item. Inventory position means the amount on hand plus the amount on order minus the backorders. The values of I_1^i are known for all i. Because of the stochastic demands of the items in each period, the ending inventories at time u cannot be stipulated.

The expected demand rate of item i in period j will be denoted by $\hat{d}_j^i(P_j^i)$. Observe that we consider the expected demand rate of an item in a period as a function of the sale price only. It will not be a function of the time within the period. Therefore the expected demand of item i in period j will be

$$\hat{D}_j^i(P_j^i) = \hat{d}_j^i(P_j^i)\ T_j^i$$

Since the expected demand of each item in each of its periods will be predicted, the expected demand rates of the items in their periods will automatically be predicted too. We then know the expected demand rates of item i for the time interval from its first period to its last period, that is to say from time t_1^i to time u. It will also be necessary to predict the expected demand rates for the interval from time u_1 to time t_1^i. This interval will be divided in a certain number of periods, in each of which the expected demand rate can be considered constant. This number of periods is called q_i. Observe that for different items these periods may be completely different, and even the number of them may be different. The first of these periods of item i, the one beginning at time u_1, will be called period $-q_i$; the following will be called period $-(q_i-1)$; until the last period which ends at time t_1^i and it will be called period -1. The beginning of period -k of item i will be called time t_{-k}^i. Of course $t_{-q_i}^i = u_1$. Period -k of item i will be represented by $T_{-k}^i = t_{-(k-1)}^i - t_{-k}^i$. The expected demand rate of item i in period -k will be called $\hat{d}_{-k}^i$.

At this point, the same reasoning used by Hadley and Whitin[15] will be applied in order to obtain the mean demand rate of an item in the time interval from a decision point to any time within a certain period of this item. The order for this item which was placed at that decision point arrives at the beginning of this period. First pick two consecutive decision points, for example, u_j and u_{j+1}. The lead times of the orders of item i placed at these decision points are v_j^i and v_{j+1}^i which create the arrival times t_j^i and t_{j+1}^i. The decision point u_j is within period -k

15. G. Hadley and T. M. Whitin, "A Family of Dynamic Inventory Models," Management Science, Vol. 8, No. 4, pp. 458-469.

of item i. The expected demand of item i from time u_j to time t^i_{j+1} is given by

$$\hat{d}^i_{-k}\left[t^i_{-(k-1)} - u_j\right] + \hat{d}^i_{-(k-1)}\,T^i_{-(k-1)} + \ldots + \hat{d}^i_{-1}\,T^i_{-1} + \hat{d}^i_1(P^i_1)\,T^i_1 +$$

$$+ \ldots + \hat{d}^i_{j-1}(P^i_{j-1})\,T^i_{j-1} + \hat{d}^i_j(P^i_j)\,T^i_j$$

If we divide the expected demand shown above by $(t^i_{j+1} - u_j)$, we will obtain $\bar{d}^i_j$. This value represents the mean rate of demand of item i which if maintained constant from time u_j to time t^i_{j+1} would yield a probability distribution for the demand identical to the distribution obtained by the time varying rate of this item over the interval. Note that $\bar{d}^i_j$ is dependent on the sale prices of item i for periods 1 through j.

Using the same situation just explained, suppose now that time t is a time within period j of item i. Similarly we will define $\bar{d}^i_t$ by the following relationship:

$$\bar{d}^i_t = \hat{d}^i_{-k}\,\frac{t^i_{-(k-1)} - u_j}{t - u_j} + \hat{d}^i_{-(k-1)}\,\frac{T^i_{-(k-1)}}{t - u_j} + \ldots + \hat{d}^i_{-1}\,\frac{T^i_{-1}}{t - u_j} + \hat{d}^i_1(P^i_1)\,\frac{T^i_1}{t - u_j} +$$

$$+ \ldots + \hat{d}^i_{j-1}(P^i_{j-1})\,\frac{T^i_{j-1}}{t - u_j} + \hat{d}^i_j(P^i_j)\,\frac{t - t^i_j}{t - u_j}$$

Unfortunately, as we vary t within period j the value of $\bar{d}^i_t$ varies too. We shall assume that the value of $\bar{d}^i_j$ may be used over the whole period j. In the same way in which the cited authors defended this assumption, we will write that if u_j and u_{j+1} are chosen to be sufficiently close together, then we will have a good enough approximation. Then, the expected demand of item i from time u_j to any time t within period j will be given by $\bar{d}_j^i\,(t - u_j)$. Let us define by $p\left[x;\ \bar{d}_j^i\,(t - u_j)\right]$ the probability that x units of item i will be demanded in the interval from the decision point u_j to any time t within period j.

The inventory position of item i at time u_j after any order is placed will be $I_j^i + Q_j^i$. Observe that all units on order of item i at the decision point u_j prior to placing the order quantity Q_j^i, must arrive before this order quantity, in other words before the beginning of period j of item i. This happens because we made the assumption that the orders cannot cross. Therefore, if x units of item i were demanded from time u_j to a time t within period j, then the inventory on hand of this item at time t will be equal to $I_j^i + Q_j^i - x$, if $x \leqslant I_j^i + Q_j^i$; and equal to zero, if $x > I_j^i + Q_j^i$. Also, the number of backorders of this item at time t will be equal to zero, if $x \leqslant I_j^i + Q_j^i$; and equal to $x - I_j^i - Q_j^i$, if $x > I_j^i + Q_j^i$. The expected inventory on hand of item i at any time t within period j will be

$$\sum_{x=0}^{I_j^i+Q_j^i} (I_j^i + Q_j^i - x)\ p\left[x;\ \bar{d}_j^i\,(t - u_j)\right] \tag{7}$$

The expected cost of carrying inventory of item i in its period j is then

$$\frac{H_j^i}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \sum_{x=0}^{I_j^i+Q_j^i} (I_j^i + Q_j^i - x)\ p\left[x;\ \bar{d}_j^i\,(t - u_j)\right] dt =$$

$$= \frac{H_j^i}{T_j^i}\int_{t_j^i}^{t_{j+1}^i} \left\{ I_j^i + Q_j^i - \bar{d}_j^i\,(t - u_j) + \sum_{x=I_j^i+Q_j^i+1}^{\infty} (x - I_j^i - Q_j^i)\ p\left[x;\ \bar{d}_j^i\,(t - u_j)\right] \right\} dt$$

$$= H_j^i \left[I_j^i + Q_j^i + \bar{d}_j^i \, u_j - \frac{\bar{d}_j^i}{2} (t_{j+1}^i + t_j^i) + \frac{1}{T_j^i} B_j^i(I_j^i + Q_j^i, \, t_{j+1}^i, \, t_j^i) \right]$$

where

$$B_j^i(I_j^i + Q_j^i, \, t_{j+1}^i, \, t_j^i) = \int_{t_j^i}^{t_{j+1}^i} \sum_{x=I_j^i+Q_j^i+1}^{\infty} (x - I_j^i - Q_j^i) \, p\left[x; \, \bar{d}_j^i \, (t - u_j)\right] dt$$

Observe that

$$\sum_{x=I_j^i+Q_j^i+1}^{\infty} (x - I_j^i - Q_j^i) \, p\left[x; \, \bar{d}_j^i \, (t - u_j)\right] \tag{8}$$

is the expected number of backorders of item i standing on the books at a time t within period j. Therefore $B_j^i(I_j^i + Q_j^i, \, t_{j+1}^i, \, t_j^i)$ is the expected unit years of shortage of item i incurred in period j.

As in the deterministic demand case, the decisions to be made concerning the order quantities and the sale prices of item i, will have no influence over the inventory carrying costs incurred from the first decision point u_1 until the related arrival time t_1^i. Then, these costs need not be included in the carrying cost expression of this item. The only relevant carrying costs of item i are those incurred between t_1^i and u. Hence, the expected inventory carrying cost of item i to appear in our model will be

$$\sum_{j=1}^{n_i} H_j^i \left[I_j^i + Q_j^i + \bar{d}_j^i \, u_j - \frac{\bar{d}_j^i}{2} (t_{j+1}^i + t_j^i) + \frac{1}{T_j^i} B_j^i(I_j^i + Q_j^i, \, t_{j+1}^i, \, t_j^i) \right]$$

The expected number of backorders of item i incurred during period j is equal to the difference between the expected numbers of backorders of this item at the times t^i_{j+1} and t^i_j. Hence, the expected number of back-orders of item i incurred during period j is given by

$$E^i_j(I^i_j + Q^i_j, t^i_{j+1}, t^i_j) = \sum_{x=I^i_j+Q^i_j+1}^{\infty} (x - I^i_j - Q^i_j)\ p\left[x;\ \bar{d}^i_j\ (t^i_{j+1} - u_j)\right] -$$

$$- \sum_{x=I^i_j+Q^i_j+1}^{\infty} (x - I^i_j - Q^i_j)\ p\left[x;\ \bar{d}^i_j\ (t^i_j - u_j)\right] =$$

$$= \sum_{x=I^i_j+Q^i_j+1}^{\infty} (x - I^i_j - Q^i_j) \left\{ p\left[x;\ \bar{d}^i_j\ (t^i_{j+1} - u_j)\right] - \right.$$

$$\left. - p\left[x;\ \bar{d}^i_j\ (t^i_j - u_j)\right] \right\}$$

The expected backorder cost of item i for period j will be written

$$J^i_j\ B^i_j(I^i_j + Q^i_j, t^i_{j+1}, t^i_j) + K^i_j\ E^i_j(I^i_j + Q^i_j, t^i_{j+1}, t^i_j)$$

With the same reasoning used for the carrying cost, we can conclude that the only relevant backorder costs of item i are those incurred between t^i_1 and u. Thus, the expected backorder costs of item i to appear in the model will be

$$\sum_{j=1}^{n_i} \left[J^i_j\ B^i_j(I^i_j + Q^i_j, t^i_{j+1}, t^i_j) + K^i_j\ E^i_j(I^i_j + Q^i_j, t^i_{j+1}, t^i_j)\right]$$

The cost of item i and of its order during the planning period is

$$\sum_{j=1}^{n_i} \left[Q^i_j\ C^i_j(Q^i_j) + A^i_j\ R^i_j\right]$$

where $R_j^i = 0$, if $Q_j^i = 0$; and $R_j^i = 1$, if $Q_j^i > 0$.

The revenues from sales of item i in our interest are also those incurred from time t_1^i to time u. A change in the sale price of an item is possible between the annotation of a backorder of this item and the time at which the backorder is satisfied. It will be assumed that the sale price of a backordered item will be the price when the respective demand occurred. This assumption is quite reasonable because, if the customer knows that by backordering he may have to purchase the item at a higher price while he can buy now elsewhere at the present price, then he will never place a backorder. If there were no backorders of item i standing on the books at the horizon, then the expected revenue from sales of this item would be $\sum_{j=1}^{n_i} P_j^i \hat{D}_j^i(P_j^i)$. However, there is a chance of having some backorders of this item at the horizon, and then not all of its demands will be satisfied by the horizon. The part of the shown expected revenue that corresponds to these unsatisfied demands must be subtracted. Since the expected number of backorders of item i on the books at the horizon can be obtained by (8) and accepting $P_{n_i}^i$ as a good representation of the price that these unsatisfied demands would have bought this item, then the expected revenue from sales of item i will be

$$\sum_{j=1}^{n_i} P_j^i \hat{D}_j^i(P_j^i) - P_{n_i}^i \sum_{x=I_{n_i}^i+Q_{n_i}^i+1}^{\infty} (x - I_{n_i}^i - Q_{n_i}^i)\, p\left[x;\ \bar{d}_{n_i}^i\ (u - u_{n_i})\right]$$

We referred to $P_{n_i}^i$ as being an approximation because it may be possible that some of the backorders at the horizon will not come from the last period. However, because a backorder will continue to exist during the next period only if the order quantity which has just arrived is not sufficient to satisfy all backorders at this time, there is a great probability that all backorders at the horizon will originate from demands of the last period.

The expected salvage value of item i at the horizon will be

$$L^i \sum_{x=0}^{I^i_{n_i}+Q^i_{n_i}} (I^i_{n_i} + Q^i_{n_i} - x)\ p\left[x;\ \bar{d}^i_{n_i}\ (u - u_{n_i})\right]$$

Observe that as it is shown by (7), the summation represents the expected inventory on hand of item i at the horizon.

If we call z^i_j the demand of item i from the decision point u_j to the other decision point u_{j+1}, then the material balance equation of item i for these decision points will be $I^i_{j+1} = I^i_j + Q^i_j - z^i_j$. The probability that z^i_j units will be demanded is $p\left[z^i_j;\ \bar{b}^i_j\ (u_{j+1} - u_j)\right]$, where $\bar{b}^i_j$ is the mean demand rate of item i in the time interval from time u_j to time u_{j+1}. The determination of $\bar{b}^i_j$ is analogous to the determination of $\bar{d}^i_j$. The difference is that $\bar{b}^i_j$ is related to the time from u_j to u_{j+1}, and $\bar{d}^i_j$ is related to the time from u_j to t^i_{j+1}. If we assign a value for the demand of item i in each time period between two consecutive decision points, then we will obtain a set of demands for this item. Since item i will use only the n_i first decision points, there will be n_i-1 values in its set of demands. The probability that a given set of item i, represented by $z^i_1, z^i_2, \ldots, z^i_{n_i-1}$, will occur is given by

$$\prod_{j=1}^{n_i-1} p\left[z^i_j;\ \bar{b}^i_j\ (u_{j+1} - u_j)\right]$$

Observe that for a given set of demands of item i the values of $I_2^i, I_3^i, \ldots, I_{n_i}^i$ become functions of the order quantities only.

The sum of the products of a set's probability and the item profit related to this set, for all possible sets of demands for item i, is the expected profit of item i. Finally, calling G the expected profit of all items, we write the expression of our model with probabilistic demands:

$$G = \sum_{i=1}^{m} \left\{ \sum_{\substack{\text{all } z_j^i \geq 0 \\ j=1,2,\ldots,n_i-1}} \left(\prod_{j=1}^{n_i-1} p\left[z_j^i;\ \bar{b}_j^i\ (u_{j+1} - u_j)\right] \right) \cdot \right.$$

$$\left(\sum_{j=1}^{n_i} \left\{ P_j^i\ \hat{D}_j^i(P_j^i) - Q_j^i\ C_j^i(Q_j^i) - A_j^i\ R_j^i - H_j^i \left[I_j^i + Q_j^i + \bar{d}_j^i\ u_j - \frac{\bar{d}_j^i}{2}\ (t_{j+1}^i + \right. \right. \right.$$

$$\left. \left. + t_j^i) \right] - \left(\frac{H_j^i}{T_j^i} + J_j^i \right) B_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i) - K_j^i\ E_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i) \right\} +$$

$$+ L^i \sum_{x=0}^{I_{n_i}^i + Q_{n_i}^i} (I_{n_i}^i + Q_{n_i}^i - x)\ p\left[x;\ \bar{d}_{n_i}^i\ (u - u_{n_i}\ )\right] - P_{n_i}^i \sum_{x=I_{n_i}^i + Q_{n_i}^i + 1}^{\infty} (x -$$

$$\left. \left. - I_{n_i}^i - Q_{n_i}^i)\ p\left[x;\ \bar{d}_{n_i}^i\ (u - u_{n_i})\right] \right) \right\} \tag{9}$$

Note that the expression inside the exterior set of brackets represents the contribution of item i for the expected profit. If the probabilities $p\left[x;\ \bar{d}_j^i\ (t - u_j)\right]$ for each t within period j are assumed to be Poisson probabilities, then the expressions containing them as $B_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i)$ and $E_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i)$ can be transformed using some properties of the Poisson distribution. These transformations are well accepted because the Poisson distribution is a good representation of the real world demand distribution. They also have the advantage that the expressions come ready to be evaluated needing only the values of the parameters. An expression similar to $B_j^i(I_j^i + Q_j^i, t_{j+1}^i, t_j^i)$ when the demands are assumed to be Poisson distributed can be found in Hadley and Whitin's book[16].

Our objective is to maximize the expected profit G by making decisions concerning the sale prices and the order quantities of the items. This maximization is subject to the following groups of constraints.

1. The non-negativity constraints of the order quantities:

   $Q_j^i \geqslant 0$, for $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n_i$.

2. The warehouse capacity constraints. In the present case, these constraints will be slightly modified. Since we are now dealing with stochastic demands we can work with expected values; however, it is impossible to guarantee that the warehouse capacity will not be exceeded. In reality, there is another way to guarantee this, but its use is far beyond economic justification. This way would prevent an overflow of the capacity even for the worst demand situation. The worst situation is one in which the demands of all items from the first decision point u_1 to the horizon u are equal to zero, because all orders will be accumulating on the shelves without being touched. However, it is not necessary to work with this restriction. It is avoided by accepting the following premise: the decision maker never wants the space occupied by the average inventory on hand of all items to be greater than the warehouse capacity. Suppose that there is an extra space not included in the warehouse capacity that the decision maker reserves for cases in which the demands,

16. G. Hadley and T. M. Whitin, <u>Analysis of Inventory Systems</u>, p. 347.

due to their uncertainty, cause an overflow in this warehouse capacity.

At this point, we can apply the same reasoning used in the deterministic demand case. Let us call w^i the space occupied by one unit of item i in cubic units, and W the warehouse capacity, also in cubic units. If we assure that at the arrival times of orders of the items, this constraint is not violated then it will never be violated. Imagine that at time $t^{i'}_{j'}$ an order of item i' will arrive. This time is the beginning of period j' of this item. At this same time, item i will be at some point within its period j_i. From (7) the expected inventory on hand of item i at time $t^{i'}_{j'}$ will be

$$\sum_{x=0}^{I^i_{j_i}+Q^i_{j_i}} (I^i_{j_i} + Q^i_{j_i} - x)\ p\left[x;\ \bar{d}^i_{j_i}\ (t^{i'}_{j'} - u_{j_i})\right]$$

Hence the warehouse capacity constraints are

$$\sum_{i=1}^{m} w^i \sum_{x=0}^{I^i_{j_i}+Q^i_{j_i}} (I^i_{j_i} + Q^i_{j_i} - x)\ p\left[x;\ \bar{d}^i_{j_i}\ (t^{i'}_{j'} - u_{j_i})\right] \leqslant W$$

for $i' = 1,2,\ldots,m$ and $j' = 1,2,\ldots,n_{i'}$.

An Approximate Model

If we make the assumption that it is possible to choose equally spaced decision points and that the lead times of all items are multiple values of the interval between consecutive decision points, then a model that simplifies the solution procedure will be obtained. With these two conditions satisfied, the arrival time of an order placed at a certain decision point will be coincident with a future decision point. Also the arrival times of different items' orders, not necessarily placed at the same decision point, will be the same.

Due to these assumptions, we can construct the model working only with the arrival times. At the times $t_1, t_2, \ldots, t_n$ orders of all items can arrive. We are sure that it is still possible to place n orders of an item and these orders will arrive at the n arrival times. As in the preceding sections when we selected the decision points, when we now choose the arrival times we must guarantee no change in the inventory carrying charge and the demand-price table of an item during the time between two consecutive arrivals. The arrival times must also guarantee, after finding the related decision points, that there will be no change in the tables relating unit cost and order quantity and no change in the ordering charges of the items during the time between two consecutive decision points.

We will call the time between the arrival times t_j and t_{j+1} period j. The planning horizon is the end of the last period in which the parameter values of all items can still be forecasted. Note that the length of the periods is constant. The period will be represented by $T = t_2 - t_1 = \ldots = t_{n+1} - t_n$, where t_{n+1} is the horizon. Note also that now the periods are commom to all items, so that during the planning period the items will have the same number of periods, n.

In this model, the items' demands in each period, in addition to all the costs, will be certain known values. It will be assumed that the decision maker wants no item to be out of stock when a demand for it occurs.

Q_j^i is the order quantity of item i, to be placed at the decision point, which assures that this order will arrive at the beginning of period j. The unit cost as a function of the order quantity, and the cost of placing an order of item i at the decision point which assures this order's arrival at the beginning of period j, will be $C_j^i(Q_j^i)$ and A_j^i, respectively. The carrying cost of item i in period j will be denoted by H_j^i. The sale price of item i in period j and its associated demand will be represented by P_j^i and $D_j^i(P_j^i)$. The unit salvage value of item i at the horizon will be written L^i. Let I_j^i be the inventory on hand of item i at the beginning of period j, before the arrival of the order of this item expected for this period. Consider I_{n+1}^i as the inventory on hand of item i at the horizon. This inventory will be sold at the unit salvage price of item i. The space occupied by one unit of item i will be w^i cubic units. The warehouse capacity will be W cubic units. Let us call G the profit of all items in all periods.

The demand rate of an item at any period is independent of the time within the period. The model that we must develop here, is quite similar to the one shown by (5). Following the same steps that were followed in the deterministic demand model, we will arrive at almost the same expression. The difference, which is interesting to note, is that the upper limit of the second summation sign is now independent of the item. Due to this independence we can interchange these summation signs, obtaining the following equation which represents the approximate model.

$$G = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \left\{ P_j^i\, D_j^i(P_j^i) - Q_j^i\, C_j^i(Q_j^i) - A_j^i\, R_j^i - H_j^i \left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2} \right] \right\} \right) + \sum_{i=1}^{m} L^i\, I_{n+1}^i \qquad (10)$$

Observe that the expression inside the exterior set of parentheses represents the profit of all items in period j.

Our objective is to maximize G. The decision variables are Q_j^i and P_j^i, for all i and all j. This maximization is subject to the following groups of constraints:

1. The non-negativity constraints of the order quantities:

   $Q_j^i \geq 0$, for $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n$.

2. The no stockout constraints:

   $I_j^i + Q_j^i \geq D_j^i(P_j^i)$, for $i = 1,2,\ldots,m$, and $j = 1,2,\ldots,n$.

3. The warehouse capacity constraints. From (6) we obtain

$$\sum_{i=1}^{m} w^i \ (I_j^i + Q_j^i) \leq W, \text{ for } j = 1,2,\ldots,n.$$

In this situation the maximum inventory on hand of the items in a period will occur at the same moment, i.e., at the very beginning of the period.

The assumptions on which this approximate model is based, are not difficult to accept. The simplification in the solution procedure made possible by accepting the assumptions is enormous, as is shown in the next chapter.

CHAPTER IV

DYNAMIC PROGRAMMING SOLUTION

The principles of dynamic programming can be used to solve the problem presented in the preceding chapter. The use of dynamic programming to solve dynamic problems, including inventory dynamic problems, is well known. First, a procedure is shown for solving the approximate model. This model is the easiest to solve and its solution helps to acquaint us with some features of the dynamic programming approach.

Solution of the Approximate Model

The stages of the dynamic programming type of problem represent the periods. The decision variables of a stage are the order quantities and the sale prices of the items related to the period corresponding to this stage. The states of a stage represent all combinations of the opening inventories on hand of the items in the period corresponding to this stage, before the arrivals of the orders of these items expected for that period. The backward procedure is applied in this solution since we know the opening inventories on hand of the items in the first period. By backward procedure, we mean that the first stage represents the last period, the second stage represents the period before the last, and so on. A forward procedure cannot be applied because we do not know the inventories on hand at the horizon.

The profit of all items in period j is a function of the inventories on hand entering this period, and a function of the decisions to be made concerning order quantities and sale prices, related to this period. Let us call the profit of all items in period j $G_j(I_j^1,\ldots,I_j^m,Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m)$. Then as we observed following (10), we have

$$G_j(I_j^1,\ldots,I_j^m,Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m) = = \sum_{i=1}^{m}\left\{P_j^i\ D_j^i(P_j^i) - Q_j^i\ C_j^i(Q_j^i) - A_j^i\ R_j^i - H_j^i\left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2}\right]\right\} \tag{11}$$

Let us also call the gain from sales of the items at the horizon $G_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)$. Then we have

$$G_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m) = \sum_{i=1}^{m} L^i I_{n+1}^i \tag{12}$$

Now let us define the following function, for k = 1,2,...,n.

$$F_k(I_k^1,\ldots,I_k^m) = \underset{\substack{Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m \\ \text{for } j = k,\ldots,n}}{\text{maximum}} \left\{\sum_{j=k}^{n} G_j(I_j^1,\ldots,I_j^m,Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m) + G_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)\right\}$$

This maximization is subject to the following groups of constraints.

1. $Q_j^i \geqslant 0$, for $i = 1,\ldots,m$, and $j = k,\ldots,n$.
2. $I_j^i + Q_j^i \geqslant D_j^i(P_j^i)$, for $i = 1,\ldots,m$, and $j = k,\ldots,n$.
3. $\sum_{i=1}^{m} w^i\ (I_j^i + Q_j^i) \leqslant W$, for $j = k,\ldots,n$.

The interpretation of $F_k(I_k^1,\ldots,I_k^m)$ is the maximum profit of all items for periods k through n, including the gain from sales at the horizon, if the inventories on hand of the items at the beginning of period k are $I_k^1,\ldots,I_k^m$. The values of I_j^i for $i = 1,\ldots,m$ and $j = k,\ldots,n$, must satisfy the material balance equation $I_{j+1}^i = I_j^i + Q_j^i - D_j^i(P_j^i)$. The values of $I_1^1,\ldots,I_1^m$ are known. Note that the maximum profit of all items in all periods, which is called G^*, is equal to $F_1(I_1^1,\ldots,I_1^m)$.

At this point, we will obtain the recurrence relationship. For $k = 1,\ldots,n-1$

$$F_k(I_k^1,\ldots,I_k^m) = \underset{Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m}{\text{maximum}} \Bigg\{ G_k(I_k^1,\ldots,I_k^m,Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m) +$$

$$+ \underset{\substack{Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m \\ \text{for } j = k+1,\ldots,n}}{\text{maximum}} \Bigg(\sum_{j=k+1}^{n} G_j(I_j^1,\ldots,I_j^m,Q_j^1,\ldots,Q_j^m,P_j^1,\ldots,P_j^m) +$$

$$+ G_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)\Bigg)\Bigg\} =$$

$$= \underset{Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m}{\text{maximum}} \Bigg\{ G_k(I_k^1,\ldots,I_k^m,Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m) +$$

$$+ F_{k+1}(I_{k+1}^1,\ldots,I_{k+1}^m)\Bigg\} \tag{13}$$

If we define a function $F_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)$ as being identical to $G_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)$, then the final equation above also holds for $k = n$. A set of values of $I_k^1,\ldots,I_k^m$ forms a state of period k. A set of values of $Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m$ forms a decision of period k. When solving the model at period k, we know all states of period k + 1. We also know for each state, the maximum profit from period k + 1 to period n, and the decisions that bring this profit. Using these data we must find for each state of period k, the maximum profit from period k to period n, and the related decision at period k. This is the point at which recursive reasoning is applied. A state of period k is connected with some states of period k + 1 by means of the material balance equation. Showing this connection explicitly in (13) we obtain the final form of the recurrence relationship. For $k = 1,\ldots,n$

$$F_k(I_k^1,\ldots,I_k^m) = \underset{Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m}{\text{maximum}} \Big\{ G_k(I_k^1,\ldots,I_k^m,Q_k^1,\ldots,Q_k^m,P_k^1,\ldots,P_k^m) + F_{k+1}\left[I_k^1+Q_k^1-D_k^1(P_k^1),\ldots,I_k^m+Q_k^m-D_k^m(P_k^m)\right]\Big\} \tag{14}$$

This maximization is subject to the following groups of constraints.

1. $Q_k^i \geqslant 0$, for $i = 1,\ldots,m$
2. $I_k^i + Q_k^i \geqslant D_k^i(P_k^i)$, for $i = 1,\ldots,m$
3. $\sum_{i=1}^{m} w^i\,(I_k^i + Q_k^i) \leqslant W$

These constraints limit the number of decisions and the number of states. Without constraint 3, for example, the values of $Q_k^1,\ldots,Q_k^m$ would have no upper bound. These values could even be infinite. Those decisions and states which satisfy these constraints are called feasible decisions and feasible states.

The computational procedure follows these steps:

1. Find $F_{n+1}(I_{n+1}^1,\ldots,I_{n+1}^m)$ for all feasible combinations of inventories on hand of the items at the horizon, by using (12).

2. Compute $F_n(I_n^1,\ldots,I_n^m)$ for all feasible states of this period by using (14). The sets of values of $Q_n^1,\ldots,Q_n^m,P_n^1,\ldots,P_n^m$, which yield

the values of $F_n(I_n^1,\dots,I_n^m)$ corresponding to the feasible states, are also tabulated.

3. The same procedure used in step 2 for period n, is now repeated for the periods n-1,...,2.

4. In period 1, we know the values of $I_1^1,\dots,I_1^m$; now it is only necessary to compute $F_1(I_1^1,\dots,I_1^m)$ for this state. This value of $F_1(I_1^1,\dots,I_1^m)$, is the maximum profit G^*. The optimal values of $Q_1^1,\dots,Q_1^m,P_1^1,\dots,P_1^m$, i.e., the values $Q_1^{1*},\dots,Q_1^{m*},P_1^{1*},\dots,P_1^{m*}$ are also obtained at this point.

5. Working in the forward direction, we obtain the item's optimal order quantities and sale prices from the tables (constructed in steps 2 and 3) for each period following the first. For example, in period 2 we have $I_2^{i*} = I_1^i + Q_1^{i*} - D_1^i(P_1^{i*})$, for $i = 1,\dots,m$. Referring to the second period's table with the state $I_2^{1*},\dots,I_2^{m*}$, we find the values of $Q_2^{1*},\dots,Q_2^{m*},P_2^{1*},\dots,P_2^{m*}$.

An example problem is solved in Chapter V by applying this dynamic programming formulation.

Solution of the Deterministic Demand Model

The stages of dynamic programming represent the decision points rather than the periods as they did in the preceding solution. This is because each item now has its own periods not necessarily coincident with other items' periods. Since for a decision point there is a period of each item, then for a stage there are many periods, each one belonging to a different item. The decision variables at a stage are the order quantities and the sale prices of the items related to the decision point corresponding to this stage. The states of a stage represent all combinations of the items' opening inventories on hand in the periods corresponding to this stage, before the orders' arrivals of these items expected for these periods. For the same reason explained in the preceding section, backward procedure will be applied in this solution.

Remember that in some decision points, excluding the first, it is not possible to place orders for all items, as some of these orders would arrive after the horizon. Thus, the number of periods for different items may vary. This can be observed in the model by checking as the upper limit of the second summation sign of equation (5) is dependent on the item. Due to this fact, if we want to apply dynamic programming then a small arrangement must be made in this model. Let us define the variable y, dependent on the item and the decision point, in the following way: $y_j^i = 1$, if $j \leqslant n_i$, and $y_j^i = 0$, if $j > n_i$. Hence the expression of the profit given by (5) is equivalent to

$$G = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} y_j^i \left\{ P_j^i \, D_j^i(P_j^i) - Q_j^i \, C_j^i(Q_j^i) - A_j^i \, R_j^i - H_j^i \left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2} \right] \right\} + L^i \, I_{n_i+1}^i \right)$$

Since the upper limit of the second summation sign is now independent of the item, we can interchange the summation signs obtaining

$$G = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_j^i \left\{ P_j^i \, D_j^i(P_j^i) - Q_j^i \, C_j^i(Q_j^i) - A_j^i \, R_j^i - H_j^i \left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2} \right] \right\} \right) + \sum_{i=1}^{m} L^i \, I_{n_i+1}^i$$

The expression inside the exterior set of parentheses represents the profit of some or of all items related to decision point u_j. It will be called $G_j(y_j^1 I_j^1,\ldots,y_j^m I_j^m, y_j^1 Q_j^1,\ldots,y_j^m Q_j^m, y_j^1 P_j^1,\ldots,y_j^m P_j^m)$. We say "the profit of some or of all items" because it may be that orders of some items cannot be placed at this decision point. If this is true for item i, then the value of y_j^i is equal to zero, otherwise it is equal to one. Note that G_j will not be a function of I_j^i, Q_j^i and P_j^i if a decision concerning item i cannot be placed at the decision point u_j; that is why G_j is function of the products $y_j^i I_j^i$, $y_j^i Q_j^i$ and $y_j^i P_j^i$.

It will not be necessary to define a function G_{n+1} representing the gain from sales of the items at the horizon. This gain will be accounted for in the solution in a different way.

Following the same line of reasoning as in the solution of the approximate model, let us define the following function, for k = 1,2,..., n.

$$F_k(y_k^1 I_k^1,\ldots,y_k^m I_k^m) = \underset{\substack{y_j^1 Q_j^1,\ldots,y_j^m Q_j^m, y_j^1 P_j^1,\ldots,y_j^m P_j^m \\ \text{for } j = k,\ldots,n}}{\text{maximum}} \left\{ \sum_{j=k}^{n} G_j(y_j^1 I_j^1,\ldots, y_j^m I_j^m, y_j^1 Q_j^1,\ldots,y_j^m Q_j^m, y_j^1 P_j^1,\ldots,y_j^m P_j^m) + \sum_{i=1}^{m} y_k^i \, L^i \, I_{n_i+1}^i \right\}$$

The values of y_j^i for a fixed i, as we vary j from 1 to n, form a sequence of ones continued by a sequence of zeros, or they form a sequence of ones only. When y_k^i is equal to zero, we are sure that y_j^i is also equal to zero, for any $j > k$. If y_k^i is equal to one then y_j^i for $j > k$, is either one or zero. However, when y_k^i is equal to one, we know that the number of periods of item i is greater than or equal to k. If y_k^i is one and y^i_{k+1} is zero, then the number of periods of item i is k and the inventory to be sold at the horizon will be I^i_{k+1}. The summation of $y^i_k L^i I^i_{n_i+1}$ for all i represents the gain from sales at the horizon, of those items for which the number of periods is greater than or equal to k. When k = 1 we obtain the gain from sales of all items at the horizon.

The maximization in the last formula is subject to the following groups of constraints.

1. $y^i_j Q^i_j \geqslant 0$, for $i = 1,\dots,m$, and $j = k,\dots,n$
2. $y^i_j (I^i_j + Q^i_j) \geqslant y^i_j D^i_j(P^i_j)$, for $i = 1,\dots,m$, and $j = k,\dots,n$
3. From (6): $y^{i'}_{j'} \sum_{i=1}^{m} w^i \left\{ I^i_{j_i} + Q^i_{j_i} - d^i_{j_i}(P^i_{j_i}) \left[t^{i'}_{j'} - t^i_{j_i} \right] \right\} \leqslant W,$

   for $i' = 1,\dots,m$, and $j' = k,\dots,n$

The values of $y^i_j I^i_j$, for $i = 1,\dots,m$ and $j = k,\dots,n$, must satisfy the material balance equation $I^i_{j+1} = I^i_j + Q^i_j - D^i_j(P^i_j)$. The values of $I^1_1,\dots,I^m_1$ are known. The maximum profit of all items in their periods, i.e., G^* is equal to $F_1(I^1_1,\dots,I^m_1)$.

At this point, we will obtain the recurrence relationship. For $k = 1,\dots,n-1$

$$F_k(y^1_k I^1_k,\dots,y^m_k I^m_k) = \underset{y^1_k Q^1_k,\dots,y^m_k Q^m_k, y^1_k P^1_k,\dots,y^m_k P^m_k}{\text{maximum}} \Bigg\{ G_k(y^1_k I^1_k,\dots,y^m_k I^m_k,$$

$$y^1_k Q^1_k,\dots,y^m_k Q^m_k, y^1_k P^1_k,\dots,y^m_k P^m_k) + \sum_{i=1}^{m} (y^i_k - y^i_{k+1})\, L^i\, I^i_{n_i+1} +$$

$$+ \underset{\substack{y^1_j Q^1_j,\dots,y^m_j Q^m_j, y^1_j P^1_j,\dots,y^m_j P^m_j \\ \text{for } j = k+1,\dots,n}}{\text{maximum}} \left(\sum_{j=k+1}^{n} G_j(y^1_j I^1_j,\dots,y^m_j I^m_j, \right.$$

$$y_j^1Q_j^1,\ldots,y_j^mQ_j^m,y_j^1P_j^1,\ldots,y_j^mP_j^m) + \sum_{i=1}^{m} y_{k+1}^i \; L^i \; I_{n_i+1}^i\Big)\Big\} =$$

$$= \underset{y_k^1Q_k^1,\ldots,y_k^mQ_k^m,y_k^1P_k^1,\ldots,y_k^mP_k^m}{\text{maximum}} \Big\{ G_k(y_k^1I_k^1,\ldots,y_k^mI_k^m,y_k^1Q_k^1,\ldots,$$

$$y_k^mQ_k^m,y_k^1P_k^1,\ldots,y_k^mP_k^m) + \sum_{i=1}^{m} (y_k^i - y_{k+1}^i) \; L^i \; I_{n_i+1}^i +$$

$$+ F_{k+1}(y_{k+1}^1I_{k+1}^1,\ldots,y_{k+1}^mI_{k+1}^m)\Big\}$$

The summation of $(y_k^i - y_{k+1}^i) \; L^i \; I_{n_i+1}^i$ for all i represents the gain from sales at the horizon, of those items for which the number of periods equals k.

For k = n, the relationship will be

$$F_n(y_n^1I_n^1,\ldots,y_n^mI_n^m) = \underset{y_n^1Q_n^1,\ldots,y_n^mQ_n^m,y_n^1P_n^1,\ldots,y_n^mP_n^m}{\text{maximum}} \Big\{ G_n(y_n^1I_n^1,\ldots,y_n^mI_n^m,$$

$$y_n^1Q_n^1,\ldots,y_n^mQ_n^m,y_n^1P_n^1,\ldots,y_n^mP_n^m) + \sum_{i=1}^{m} y_n^i \; L^i \; I_{n_i+1}^i \Big\}$$

The maximization in the recurrence relationship is subject to the following groups of constraints

1. $y_k^i \; Q_k^i \geqslant 0$, for $i = 1,\ldots,m$
2. $y_k^i \; (I_k^i + Q_k^i) \geqslant y_k^i \; D_k^i(P_k^i)$, for $i = 1,\ldots,m$
3. $y_k^{i'} \sum_{i=1}^{m} w^i \left\{ I_{j_i}^i + Q_{j_i}^i - d_{j_i}^i(P_{j_i}^i) \left[t_k^{i'} - t_{j_i}^i \right] \right\} \leqslant W$

for $i' = 1,\ldots,m$.

This last group of constraints creates great difficulty when solving the model. The problem is that when making decisions about $y_k^1Q_k^1,\dots,y_k^mQ_k^m,y_k^1P_k^1,\dots,y_k^mP_k^m$, in order to find $F_k(y_k^1I_k^1,\dots,y_k^mI_k^m)$ we do not yet know the optimal values of $y_j^1Q_j^1,\dots,y_j^mQ_j^m,y_j^1P_j^1,\dots,y_j^mP_j^m$, for $j<k$. In the mentioned constraints, it is possible that a certain item will be in a period preceding its period k when the order of item i' arrives at the beginning of its own period k. Therefore, the optimal values of $I_{j_i}^i$, $Q_{j_i}^i$ and $P_{j_i}^i$ for this certain item must be used, but they are not known yet. It is also possible that more than one item will be in periods preceding their periods k, when the order of item i' arrives at the beginning of its period k. The only way to solve this situation is to calculate for fixed i' and k, some values of $F_k(y_k^1I_k^1,\dots,y_k^mI_k^m)$ each one corresponding to a set of values of those $I_{j_i}^i$, $Q_{j_i}^i$ and $P_{j_i}^i$, that are not yet known. However this procedure greatly amplifies the computation, and it is questionable if dynamic programming, with the inclusion of this additional procedure, will require less computational effort than exhaustive enumeration. This third group of constraints is the point at which the solution of the approximate model is much simpler and easier to compute. Remember that we did not have this difficulty in the solution of the approximate model.

Solution of the Probabilistic Demand Model

The stages of dynamic programming represent the decision points. The decision variables at a stage are the order quantities and the sale prices of the items related to the decision point corresponding to this stage. The states of a stage represent all combinations of the items' inventory positions at the decision point corresponding to this stage, before the placement of these items' orders at this decision point. Once more, the backward procedure will be applied in the solution.

Let us define the variable y, dependent on the item and the decision point, in the following way: $y_j^i = 1$, if $j \leqslant n_i$; and $y_j^i = 0$, if $j > n_i$. The demand for item i in the time interval between the decision points u_j and u_{j+1}, which is called z_j^i, is defined only until $j = n_i - 1$. Therefore there is no probability distribution of z_j^i for $j > n_i - 1$. For the future application of dynamic programming, if $j > n_i - 1$ then let us define $p\left[z_j^i;\ \overline{b}_j^i(u_{j+1} - u_j)\right] = 1$ for all $z_j^i \geqslant 0$. Of course in this case, $p\left[z_j^i;\ \overline{b}_j^i\ (u_{j+1} - u_j)\right]$ loses its meaning of a probability. The point is that, since a number multiplied by one is equal to itself, we define these unreal probabilities as equal to one, just to preserve the products of the existent probabilities, as will be seen.

With reasoning similar to that used in the preceding section, we rearrange the expression of the profit given by (9), thus arriving at

$$G = \sum_{\substack{\text{all defined } z_j^i \geqslant 0,\\ \text{for } j = 1,\ldots,n-1,\\ \text{and } i = 1,\ldots,m}} \left(\prod_{j=1}^{n-1} \prod_{i=1}^{m} p\left[z_j^i;\ \overline{b}_j^i\ (u_{j+1} - u_j)\right]\right) \left(\sum_{j=1}^{n} G_j(y_j^1 I_j^1,\right.$$

$$\ldots, y_j^m I_j^m, y_j^1 Q_j^1, \ldots, y_j^m Q_j^m, y_j^1 P_j^1, \ldots, y_j^m P_j^m) + \sum_{i=1}^{m} L^i \sum_{x=0}^{I_{n_i}^i + Q_{n_i}^i} (I_{n_i}^i + Q_{n_i}^i -$$

$$- x)\ p\left[x;\ \overline{d}_{n_i}^i\ (u - u_{n_i})\right] - \sum_{i=1}^{m} P_{n_i}^i \sum_{x=I_{n_i}^i + Q_{n_i}^i + 1}^{\infty} (x - I_{n_i}^i - Q_{n_i}^i)\ p\left[x;\right.$$

$$\left.\left.\overline{d}_{n_i}^i\ (u - u_{n_i})\right]\right),$$

where $G_j(y_j^1 I_j^1, \ldots, y_j^m I_j^m, y_j^1 Q_j^1, \ldots, y_j^m Q_j^m, y_j^1 P_j^1, \ldots, y_j^m P_j^m) = \sum_{i=1}^{m} y_j^i \left\{P_j^i\ \hat{D}_j^i(P_j^i) -\right.$

$$- Q_j^i\, C_j^i(Q_j^i) - A_j^i\, R_j^i - H_j^i \left[I_j^i + Q_j^i + \bar{d}_j^i\, u_j - \frac{\bar{d}_j^i}{2}\,(t_{j+1}^i + t_j^i) \right] - \left(\frac{H_j^i}{T_j^i} +\right.$$

$$\left. + J_j^i\right) B_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i) - K_j^i\, E_j^i(I_j^i + Q_j^i,\ t_{j+1}^i,\ t_j^i) \Bigg\}$$

The values of $y_j^i I_j^i$, for $i = 1,\ldots,m$, and $j = 1,\ldots,n$, must satisfy the material balance equation $I_{j+1}^i = I_j^i + Q_j^i - z_j^i$.

As before, let us define the following function, for $k = 1,\ldots,n$.

$$F_k(y_k^1 I_k^1,\ldots,y_k^m I_k^m) = \underset{\substack{y_j^1Q_j^1,\ldots,y_j^mQ_j^m,y_j^1P_j^1,\ldots,y_j^mP_j^m \\ \text{for } j = k,\ldots,n}}{\text{maximum}} \left\{ \sum_{\substack{\text{all defined } z_j^i \geqslant 0, \\ \text{for } j = k,\ldots,n-1, \\ \text{and } i = 1,\ldots,m}} \left(\prod_{j=k}^{n-1} \prod_{i=1}^{m} p\left[z_j^i;\ \bar{b}_j^i\,(u_{j+1} - u_j) \right] \right) \left(\sum_{j=k}^{n} G_j(y_j^1 I_j^1,\ldots,y_j^m I_j^m, y_j^1 Q_j^1,\ldots, \right. \right.$$

$$y_j^m Q_j^m, y_j^1 P_j^1,\ldots,y_j^m P_j^m) + \sum_{i=1}^{m} y_k^i\, L^i \sum_{x=0}^{I_{n_i}^i + Q_{n_i}^i} (I_{n_i}^i + Q_{n_i}^i - x)\; p\left[x;\ \bar{d}_{n_i}^i\,(u -\right.$$

$$\left.\left.\left. - u_{n_i})\right] - \sum_{i=1}^{m} y_k^i\, P_{n_i}^i \sum_{x=I_{n_i}^i + Q_{n_i}^i + 1}^{\infty} (x - I_{n_i}^i - Q_{n_i}^i)\; p\left[x;\ \bar{d}_{n_i}^i\,(u - u_{n_i})\right] \right) \right\}$$

The maximization in the last expression is subject to the following groups of constraints.

1. $y_j^i\, Q_j^i \geqslant 0$, for $i = 1,\ldots,m$, and $j = k,\ldots,n$
2. $y_{j'}^{i'} \sum_{i=1}^{m} w^i \sum_{x=0}^{I_{j_i}^i + Q_{j_i}^i} (I_{j_i}^i + Q_{j_i}^i - x)\; p\left[x;\ \bar{d}_{j_i}^i\,(t_{j'}^{i'} - u_{j_i})\right] \leqslant W$

   for $i' = 1,\ldots,m$, and $j' = k,\ldots,n$.

The maximum profit of all items in their periods is equal to $F_1(I_1^1,\ldots,I_1^m)$. The values of $I_1^1,\ldots,I_1^m$ are known.

The recurrence relationship for this stochastic model is more difficult to obtain. Following the reasonings used by Hadley and Whitin[17] in the stochastic case of their basic dynamic inventory model, the recurrence relationship of our model will be, for k = 1,...,n-1

$$F_k(y_k^1 I_k^1,\ldots,y_k^m I_k^m) = \underset{y_k^1 Q_k^1,\ldots,y_k^m Q_k^m, y_k^1 P_k^1,\ldots y_k^m P_k^m}{\text{maximum}} \Bigg\{ G_k(y_k^1 I_k^1,\ldots,y_k^m I_k^m,$$

$$y_k^1 Q_k^1,\ldots,y_k^m Q_k^m, y_k^1 P_k^1,\ldots,y_k^m P_k^m) + \sum_{i=1}^{m} (y_k^i - y_{k+1}^i)\, L^i \sum_{x=0}^{I_{n_i}^i + Q_{n_i}^i} (I_{n_i}^i +$$

$$+ Q_{n_i}^i - x)\, p\left[x;\, \bar{d}_{n_i}^i\, (u - u_{n_i})\right] - \sum_{i=1}^{m} (y_k^i - y_{k+1}^i)\, P_{n_i}^i \sum_{x=I_{n_i}^i + Q_{n_i}^i + 1}^{\infty} (x -$$

$$- I_{n_i}^i - Q_{n_i}^i)\, p\left[x;\, \bar{d}_{n_i}^i\, (u - u_{n_i})\right] + \sum_{\substack{\text{all defined } z_k^i \geq 0 \\ \text{for } i = 1,\ldots,m}} \left(\prod_{i=1}^{m} p\left[z_k^i;\, \bar{b}_k^i\, (u_{k+1} -\right.\right.$$

$$\left.\left.- u_k)\right]\right) F_{k+1}(y_{k+1}^1 I_{k+1}^1,\ldots,y_{k+1}^m I_{k+1}^m) \Bigg\}$$

For k = n, the relationship will be

$$F_n(y_n^1 I_n^1,\ldots,y_n^m I_n^m) = \underset{y_n^1 Q_n^1,\ldots,y_n^m Q_n^m, y_n^1 P_n^1,\ldots,y_n^m P_n^m}{\text{maximum}} \Bigg\{ G_n(y_n^1 I_n^1,\ldots,y_n^m I_n^m,$$

$$y_n^1 Q_n^1,\ldots,y_n^m Q_n^m, y_n^1 P_n^1,\ldots,y_n^m P_n^m) + \sum_{i=1}^{m} y_n^i\, L^i \sum_{x=0}^{I_{n_i}^i + Q_{n_i}^i} (I_{n_i}^i + Q_{n_i}^i - x)$$

$$p\left[x;\, \bar{d}_{n_i}^i\, (u - u_{n_i})\right] - \sum_{i=1}^{m} y_n^i\, P_{n_i}^i \sum_{x=I_{n_i}^i + Q_{n_i}^i + 1}^{\infty} (x - I_{n_i}^i - Q_{n_i}^i)\, p\left[x;\right.$$

17. G. Hadley and T. M. Whitin, <u>Analysis of Inventory Systems</u>, pp. 334-335.

$$\bar{d}^{i}_{n_i} \; (u - u_{n_i})\Big]\Big\}$$

The maximization in the recurrence relationship is subject to the following groups of constraints

1. $y^i_k \; Q^i_k \geqslant 0$, for $i = 1,\ldots,m$

2. $y^{i'}_k \sum_{i=1}^{m} w^i \sum_{x=0}^{I^i_{j_i}+Q^i_{j_i}} (I^i_{j_i} + Q^i_{j_i} - x) \; p\left[x;\; \bar{d}^i_{j_i} \; (t^{i'}_k - u_{j_i})\right] \leqslant W$

for $i' = 1,\ldots,m$.

This last group of constraints creates the same difficulty in the solution of this model as the corresponding group, related to the deterministic demand model, creates in the solution of that model. This difficulty is a great obstacle in solving these non approximate models.

As a final comment in this chapter, we recall that, for the probabilistic demand model, a definite value can be obtained only for $Q^{1*}_1,\ldots,Q^{m*}_1, P^{1*}_1,\ldots,P^{m*}_1$. The values of $Q^{1*}_j,\ldots,Q^{m*}_j, P^{1*}_j,\ldots,P^{m*}_j$, for $j = 2,\ldots,n_i$ are functions of the values of $I^{1*}_j,\ldots,I^{m*}_j$, for $j = 2,\ldots,n_i$ which are dependent on the demands.

CHAPTER V

A NUMERICAL EXAMPLE

In this chapter we will consider an example of a procurement-inventory system operating under a prolonged price increase. This example characterizes the situation of procurement-inventory systems in an inflationary economy.

The system has a warehouse where three items must be stocked. The warehouse has a maximum capacity of 100 cubic feet which limits the number of units of each item that can be stored at one time. A unit of each item occupies a different amount of warehouse space. The space occupied by items 1, 2 and 3 is 5, 3 and 2 cubic feet per unit, respectively.

We can assume that the decision points are equally spaced, and that the lead times are multiple values of the interval between consecutive decision points. Therefore, the system is well represented by the approximate model. In this model, the arrivals of different items' orders, not necessarily placed at the same decision point, occur at the same time. Also the time interval between any two consecutive arrivals of the orders, is constant. This time interval is called a period.

Using the forecasts presently available, the decision maker can only plan for three future periods. The first of these periods begins at the end of the last period for which decisions have already been made.

The lead times are known. They assure that it is still possible to place orders for the items, and these orders will arrive at the beginning of the three periods.

Table 1 shows the items' unit costs, if these items are ordered at the decision points guaranteeing the orders' arrival at the beginning of each future period.

Table 1. Unit Cost as a Function of the Order Quantity

Item	Order Quantity	Unit Cost Related to Period 1 in Dollars	Unit Cost Related to Period 2 in Dollars	Unit Cost Related to Period 3 in Dollars
1	$0<Q_j^1\leq 5$	6.95	7.95	8.95
1	$5<Q_j^1\leq 10$	6.85	7.85	8.85
1	$10<Q_j^1$	6.75	7.75	8.75
2	$0<Q_j^2\leq 5$	4.95	4.95	5.95
2	$5<Q_j^2\leq 10$	4.90	4.90	5.90
2	$10<Q_j^2$	4.85	4.85	5.85
3	$0<Q_j^3\leq 6$	5.95	6.95	6.95
3	$6<Q_j^3\leq 12$	5.90	6.90	6.90
3	$12<Q_j^3$	5.85	6.85	6.85

Table 2 shows the costs of placing item orders, if these orders are placed at the decision points guaranteeing the orders' arrival at the beginning of each future period.

Table 2. Ordering Cost

Item	Ordering Cost Related to Period 1 in Dollars	Ordering Cost Related to Period 2 in Dollars	Ordering Cost Related to Period 3 in Dollars
1	1.95	1.95	2.00
2	2.00	2.05	2.05
3	2.00	2.00	2.00

The costs of carrying one unit of each item in inventory during the three periods, are shown in table 3.

Table 3. Inventory Carrying Cost

Item	Carrying Cost during Period 1 in Dollars per Unit per Period	Carrying Cost during Period 2 in Dollars per Unit per Period	Carrying Cost during Period 3 in Dollars per Unit per Period
1	0.30	0.35	0.35
2	0.20	0.25	0.30
3	0.45	0.45	0.50

Observe that all costs of an item are either increasing or remaining constant, from one period to the next.

The alternative sale prices for each item in periods 1,2,3, and the corresponding demands, are presented in tables 4,5,6.

Observe that the demands corresponding to the alternative sale prices of an item are deterministic values. The decision maker wants no item to be out of stock when a demand for it occurs. Observe also that an item's sale price related to a fixed demand, is increasing from one period to the next.

Table 4. Sale Price - Demand Relationship in Period 1

Item	Alternative	Sale Price in Dollars per Unit	Demand
1	1	8.35	8
1	2	8.55	7
1	3	8.70	6
2	1	5.50	10
2	2	5.65	9
2	3	5.75	8
3	1	7.35	12
3	2	7.50	11
3	3	7.65	10

Table 5. Sale Price - Demand Relationship in Period 2

Item	Alternative	Sale Price in Dollars per Unit	Demand
1	1	8.75	8
1	2	8.95	7
1	3	9.10	6
2	1	5.75	10
2	2	5.90	9
2	3	6.00	8
3	1	7.75	12
3	2	7.90	11
3	3	8.00	10

Table 6. Sale Price - Demand Relationship in Period 3

Item	Alternative	Sale Price in Dollars per Unit	Demand
1	1	9.15	8
1	2	9.35	7
1	3	9.50	6
2	1	6.10	10
2	2	6.25	9
2	3	6.35	8
3	1	8.05	12
3	2	8.20	11
3	3	8.30	10

The system will be in operation before the first period. Therefore, the initial inventories of items 1,2,3, before the arrival of the orders expected for this period, will be 6,8,10 units, respectively.

The salvage values of items 1,2,3 at the horizon, will be 8.75, 5.85, 6.85 dollars per unit, repectively.

The decision variables, which have the objective of maximizing the profit related to the three periods, are:

1. The order quantities of each item arriving at the beginning of

the three periods (it is possible that no order of an item will arrive at the beginning of a period), and

2. The sale prices of each item in the three periods.

Using the nomenclature presented in chapter III we can write: $m = 3$; $n = 3$; $W = 100$; $w^1 = 5$; $w^2 = 3$; $w^3 = 2$; $A_1^1 = 1.95$; $A_1^2 = 2.00$; $A_1^3 = 2.00$; $A_2^1 = 1.95$; $A_2^2 = 2.05$; $A_2^3 = 2.00$; $A_3^1 = 2.00$; $A_3^2 = 2.05$; $A_3^3 = 2.00$; $H_1^1 = 0.30$; $H_1^2 = 0.20$; $H_1^3 = 0.45$; $H_2^1 = 0.35$; $H_2^2 = 0.25$; $H_2^3 = 0.45$; $H_3^1 = 0.35$; $H_3^2 = 0.30$; $H_3^3 = 0.50$; $I_1^1 = 6$; $I_1^2 = 8$; $I_1^3 = 10$; $L^1 = 8.75$; $L^2 = 5.85$; $L^3 = 6.85$. If $0 < Q_1^1 \leqslant 5$, then $C_1^1(Q_1^1) = 6.95$. If $5 < Q_1^1 \leqslant 10$, then $C_1^1(Q_1^1) = 6.85$. If $10 < Q_1^1$, then $C_1^1(Q_1^1) = 6.75$.... If $0 < Q_3^3 \leqslant 6$, then $C_3^3(Q_3^3) = 6.95$. If $6 < Q_3^3 \leqslant 12$, then $C_3^3(Q_3^3) = 6.90$. If $12 < Q_3^3$, then $C_3^3(Q_3^3) = 6.85$. If $P_1^1 = 8.35$, then $D_1^1(P_1^1) = 8$. If $P_1^1 = 8.70$, then $D_1^1(P_1^1) = 6$.... If $P_3^3 = 8.05$, then $D_3^3(P_3^3) = 12$. If $P_3^3 = 8.30$, then $D_3^3(P_3^3) = 10$.

The profit of all items related to the three periods from expression (10), is:

$$G = \sum_{j=1}^{3} \left(\sum_{i=1}^{3} \left\{ P_j^i \, D_j^i(P_j^i) - Q_j^i \, C_j^i(Q_j^i) - A_j^i \, R_j^i - H_j^i \left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2} \right] \right\} \right) + \sum_{i=1}^{3} L^i \, I_4^i$$

Our objective is to maximize G. The decision variables are Q_j^i and P_j^i, for $i = 1,2,3$ and $j = 1,2,3$. This maximization is subject to the following groups of constraints:

1. $Q_j^i \geqslant 0$, for $i = 1,2,3$, and $j = 1,2,3$
2. $I_j^i + Q_j^i \geqslant D_j^i(P_j^i)$, for $i = 1,2,3$, and $j = 1,2,3$
3. $\sum_{i=1}^{3} w^i \, (I_j^i + Q_j^i) \leqslant W$, for $j = 1,2,3$

The profit of all items in period j from expression (11), is:

$$G_j(I_j^1,I_j^2,I_j^3,Q_j^1,Q_j^2,Q_j^3,P_j^1,P_j^2,P_j^3) = \sum_{i=1}^{3}\left\{P_j^i\, D_j^i(P_j^i) - Q_j^i\, C_j^i(Q_j^i) - A_j^i\, R_j^i - \right.$$
$$\left. - H_j^i\left[I_j^i + Q_j^i - \frac{D_j^i(P_j^i)}{2}\right]\right\}, \text{ for } j = 1,2,3$$

The salvage value of all items at the horizon given by expression (12), is:

$$G_4(I_4^1,I_4^2,I_4^3) = \sum_{i=1}^{3} L^i\, I_4^i$$

Since the inventories on hand of the items at the horizon are not specified, we must apply the backward solution procedure of dynamic programming.

As defined in chapter IV, if the inventories on hand of the items at the beginning of period k are I_k^1, I_k^2, I_k^3, then $F_k(I_k^1,I_k^2,I_k^3)$ is the maximum profit of all items for periods k,...,3, including the salvage value at the horizon.

Defining $F_4(I_4^1,I_4^2,I_4^3)$ as being identical to $G_4(I_4^1,I_4^2,I_4^3)$, and using expression (14); the recurrence relationship for our example is:

$$F_k(I_k^1,I_k^2,I_k^3) = \underset{Q_k^1,Q_k^2,Q_k^3,P_k^1,P_k^2,P_k^3}{\text{maximum}}\left\{G_k(I_k^1,I_k^2,I_k^3,Q_k^1,Q_k^2,Q_k^3,P_k^1,P_k^2,P_k^3) + \right.$$
$$\left. + F_{k+1}\left[I_k^1+Q_k^1-D_k^1(P_k^1), I_k^2+Q_k^2-D_k^2(P_k^2), I_k^3+Q_k^3-D_k^3(P_k^3)\right]\right\}, \qquad (15)$$

for k = 1,2,3.

The maximization in the recurrence relationship is subject to the following groups of constraints:

1. $Q_k^i \geqslant 0$, for i = 1,2,3
2. $I_k^i + Q_k^i \geqslant D_k^i(P_k^i)$, for i = 1,2,3

3. $\sum_{i=1}^{3} w^i \; (I_k^i + Q_k^i) \leq W$

Substituting the values of the parameters in the expression of the salvage value, we obtain

$$G_4(I_4^1, I_4^2, I_4^3) = (8.75)\; I_4^1 + (5.85)\; I_4^2 + (6.85)\; I_4^3 \qquad (16)$$

Since we are working backward, the first stage of the dynamic programming solution corresponds to the third period. If we substitute the values of the parameters in the recurrence relationship of the first stage, we have

$$F_3(I_3^1, I_3^2, I_3^3) = \underset{Q_3^1, Q_3^2, Q_3^3, P_3^1, P_3^2, P_3^3}{\text{maximum}} \left\{ \sum_{i=1}^{3} P_3^i \; D_3^i(P_3^i) - \sum_{i=1}^{3} Q_3^i \; C_3^i(Q_3^i) - (2.00)\; R_3^1 - \right.$$

$$- (2.05)\; R_3^2 - (2.00)\; R_3^3 - (0.35) \left[I_3^1 + Q_3^1 - \frac{D_3^1(P_3^1)}{2} \right] - (0.30) \left[I_3^2 + \right.$$

$$\left. + Q_3^2 - \frac{D_3^2(P_3^2)}{2} \right] - (0.50) \left[I_3^3 + Q_3^3 - \frac{D_3^3(P_3^3)}{2} \right] + F_4\Big[I_3^1 + Q_3^1 - D_3^1(P_3^1),\, I_3^2 + Q_3^2 - D_3^2(P_3^2),$$

$$\left. I_3^3 + Q_3^3 - D_3^3(P_3^3) \Big] \right\} \qquad (17)$$

This maximization is subject to the following groups of constraints:

1. $Q_3^1 \geq 0; \quad Q_3^2 \geq 0; \quad Q_3^3 \geq 0;$
2. $I_3^1 + Q_3^1 \geq D_3^1(P_3^1); \; I_3^2 + Q_3^2 \geq D_3^2(P_3^2); \; I_3^3 + Q_3^3 \geq D_3^3(P_3^3);$
3. $5\;(I_3^1 + Q_3^1) + 3\;(I_3^2 + Q_3^2) + 2\;(I_3^3 + Q_3^3) \leq 100$

With analogous substitutions we can obtain the recurrence relationships of the second and the third stages, and the corresponding groups of constraints.

Adapting the computational steps shown in the solution of the approximate model to our example, we obtain the following steps:

1. Find $F_4(I_4^1,I_4^2,I_4^3)$ for all feasible combinations of inventories on hand of the items at the horizon, using expression (16).

2. Compute $F_3(I_3^1,I_3^2,I_3^3)$ for all feasible states of this period by using (17). The sets of values of $Q_3^1,Q_3^2,Q_3^3,P_3^1,P_3^2,P_3^3$, which yield the values of $F_3(I_3^1,I_3^2,I_3^3)$ corresponding to the feasible states, are also tabulated.

3. The same procedure used in step 2 for period 3, is now repeated for period 2. The recurrence relationship formed by setting k = 2 in (15), is used to compute $F_2(I_2^1,I_2^2,I_2^3)$.

4. In period 1, we know that I_1^1,I_1^2,I_1^3 are equal to 6,8,10 units, respectively. Therefore, it is only necessary to compute $F_1(6,8,10)$. This value is the maximum profit G^*. In order to obtain this value, the recurrence relationship of period 1 is applied. The values of the decision variables related to period 1 that yield this maximum profit, i.e., the values of $Q_1^{1*},Q_1^{2*},Q_1^{3*},P_1^{1*},P_1^{2*},P_1^{3*}$, are also obtained at this point.

5. Working in the forward direction, we obtain the optimal values of the decision variables related to periods 2 and 3 from the tables constructed in steps 3 and 2, respectively. For example, in period 2 we have $I_2^{i*} = I_1^i + Q_1^{i*} - D_1^i(P_1^{i*})$, for i = 1,2,3. Referring to the second period's table with the state $I_2^{1*},I_2^{2*},I_2^{3*}$, we find the values of $Q_2^{1*},Q_2^{2*},Q_2^{3*},P_2^{1*},P_2^{2*},P_2^{3*}$.

A computer program which performs these computational steps was written specifically for this example. The program written in ALGOL, and the respective output are shown in the Appendix.

From the computer output, we obtain table 7 which shows the salvage values of all feasible combinations of inventories on hand at the horizon. This table was constructed by the program during the execution of step 1.

We also obtain tables 8 and 9 from the computer output. These tables represent the final tables for periods 3 and 2, respectively. Table 8 shows the values of the function $F_3(I_3^1, I_3^2, I_3^3)$ for all feasible states of period 3, and the values of the decision variables in this period that yield the mentioned values of the function $F_3(I_3^1, I_3^2, I_3^3)$. Table 8 was constructed by the program during the execution of step 2. Similarly, table 9 shows the values of the function $F_2(I_2^1, I_2^2, I_2^3)$ for all feasible states of period 2, and the values of the decision variables in this period that yield the mentioned values of the function $F_2(I_2^1, I_2^2, I_2^3)$. Table 9 was constructed during the execution of step 3.

In the execution of step 4, the program calculated $F_1(6,8,10)$ to be 213.275 dollars. This is the maximum profit G^*. At the same step, the values of $Q_1^{1*}, Q_1^{2*}, Q_1^{3*}, P_1^{1*}, P_1^{2*}, P_1^{3*}$ were also found. The values of $Q_2^{1*}, Q_2^{2*}, Q_2^{3*}, P_2^{1*}, P_2^{2*}, P_2^{3*}, Q_3^{1*}, Q_3^{2*}, Q_3^{3*}, P_3^{1*}, P_3^{2*}, P_3^{3*}$ were obtained in the execution of step 5. All these values of the decision variables obtained from the computer output are arranged in table 10. This optimum solution is unique. The program which includes an alternative solution check, confirmed the absence of such solutions.

Table 7. Salvage Values at the Horizon

I_4^1	I_4^2	I_4^3	$F_4(I_4^1,I_4^2,I_4^3)$
0	0	0	0.00
0	0	1	6.85
0	0	2	13.70
0	0	3	20.55
0	0	4	27.40
0	0	5	34.25
0	0	6	41.10
0	0	7	47.95
0	0	8	54.80
0	0	9	61.65
0	0	10	68.50
0	0	11	75.35
0	0	12	82.20
0	0	13	89.05
0	1	0	5.85
.	.	.	.
.	.	.	.
.	.	.	.
5	0	0	43.75

Table 8. Final Table for Period 3

I_3^1	I_3^2	I_3^3	$F_3(I_3^1,I_3^2,I_3^3)$	Q_3^1	Q_3^2	Q_3^3	P_3^1	P_3^2	P_3^3
0	0	0	10.75	6	8	11	9.50	6.35	8.20
0	0	1	17.65	6	8	10	9.50	6.35	8.20
0	0	2	24.55	6	8	9	9.50	6.35	8.20
0	0	3	31.45	6	8	8	9.50	6.35	8.20
0	0	4	38.35	6	8	7	9.50	6.35	8.20
0	0	5	44.95	6	8	5	9.50	6.35	8.30
0	0	6	51.90	6	8	4	9.50	6.35	8.30
0	0	7	58.85	6	8	3	9.50	6.35	8.30
0	0	8	65.80	6	8	2	9.50	6.35	8.30
0	0	9	72.75	6	8	1	9.50	6.35	8.30
0	0	10	81.70	6	8	0	9.50	6.35	8.30
0	0	11	88.65	6	8	0	9.50	6.35	8.20
0	0	12	95.00	6	8	0	9.50	6.35	8.20
0	0	13	101.35	6	8	0	9.50	6.35	8.20
0	1	0	16.65	6	7	11	9.50	6.35	8.20
.	.	.	.	.	.	.	.	.	.

Table 8. Continuation

.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
5	0	0	54.90	1	8	11	9.50	6.35	8.20

Table 9. Final Table for Period 2

I_2^1	I_2^2	I_2^3	$F_2(I_2^1,I_2^2,I_2^3)$	Q_2^1	Q_2^2	Q_2^3	P_2^1	P_2^2	P_2^3
0	0	0	36.60	6	16	10	9.10	6.00	8.00
0	0	1	43.50	6	16	9	9.10	6.00	8.00
0	0	2	50.40	6	16	8	9.10	6.00	8.00
0	0	3	57.30	6	16	7	9.10	6.00	8.00
0	0	4	63.975	6	16	7	9.10	6.00	7.90
0	0	5	70.85	6	16	5	9.10	6.00	8.00
0	0	6	77.80	6	16	4	9.10	6.00	8.00
0	0	7	84.75	6	16	3	9.10	6.00	8.00
0	0	8	91.70	6	16	2	9.10	6.00	8.00
0	0	9	98.65	6	16	1	9.10	6.00	8.00
0	0	10	107.60	6	16	0	9.10	6.00	8.00
0	0	11	114.275	6	16	0	9.10	6.00	7.90
0	0	12	117.825	6	15	0	9.10	6.00	7.90
0	0	13	123.425	6	14	0	9.10	6.00	7.90
0	1	0	41.45	6	15	10	9.10	6.00	8.00
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
5	0	0	75.75	1	16	10	9.10	6.00	8.00

Table 10. Optimum Values of the Decision Variables

	Item 1	Item 2	Item 3
Order Quantity to arrive at the beginning of Period 1	0	0	13
Sale Price to be chosen for Period 1 in Dollars	8.70	5.75	7.35
Order Quantity to arrive at the beginning of Period 2	6	16	0

Table 10. Continuation

Sale Price to be chosen for Period 2 in Dollars	9.10	6.00	7.90
Order Quantity to arrive at the beginning of Period 3	6	0	11
Sale Price to be chosen for Period 3 in Dollars	9.50	6.35	8.20

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

During the elaboration of this investigation, some conclusions related to procurement-inventory systems operating under a prolonged price increase, were made. Also, some additional points beyond the scope of this study were observed as deserving more attention. Based on these facts, the conclusions and recommendations for further studies are presented.

Conclusions

The following conclusions may be drawn from the methodology developed in previous chapters:

1. Item cost increases and ordering cost increases have an effect on the ordering policy different from that caused by carrying cost increases. In order to minimize the item cost as well as the ordering cost, it is necessary to order in greater quantities and less frequently. As a result, if the item cost savings plus the ordering cost savings obtained by ordering in greater quantities is greater than the differential carrying cost, then the optimum quantities are so large that they often are limited by the warehouse capacity.

2. In a particular period, the sale price to be selected for an item is not necessarily the price that maximizes the item's revenue in that period. It is possible that the sale price which maximizes the revenue yields a larger demand, and the units that will satisfy this

demand must occupy a larger warehouse space. Therefore, this sale price restricts the space remaining for extra units; it is these extra units which make the item cost savings possible.

3. The dynamic programming solution of the models, other than the approximate treatment, requires so much computational effort that it is impractical to solve them. The dynamic programming solution of the approximate model requires less computation and can be accomplished in a reasonable amount of computer time. The assumptions on which the approximate model are based, are not difficult to accept.

4. Another peculiarity of the approximate model is that the optimum order quantities obtained by using this model are smaller than they would be if the exact model were used. This is due to the fact that in the approximate model, the items' maximum inventories on hand occur at the same time. That is, the arrival time of the item's orders. Also the warehouse space constraint applied to the inventories at that arrival time forces the order quantities to be smaller.

Recommendations

The following recommendations for future studies are made with the purpose of extending the knowledge of the procurement-inventory system under a prolonged price increase:

1. Study should be devoted to the situation in which each item's unit costs in future periods are predicted using probability distributions.

2. As a simple extension, an approximate model with stochastic demands should be constructed, and its dynamic programming solution should be formulated.

3. Studies should be undertaken to develop an efficient solution technique for the more complex models.

4. Simplifications in the computational procedure of the approximate model should be obtained by limiting the number of decisions at each stage. For example, if a relation between item costs, carrying costs and ordering costs in a period is satisfied, then the optimal order quantities will completely occupy the space available at the arrival time of this period. Therefore, it is necessary to consider only those decisions that result in full usage of the warehouse capacity at that arrival time.

APPENDIX

COMPUTER PROGRAM AND OUTPUT

```
BEGIN
FILE IN  FI    (2,10);
FILE OUT FO 16(2,15);
INTEGER J,K;
REAL TEMP,FA;
SAVE INTEGER ARRAY QOPT[2:3,0:5,0:8,0:13,0:3];
SAVE REAL ARRAY POPT[2:3,0:5,0:8,0:13,0:3];
INTEGER ARRAY Q[0:3,0:3],D[0:3,0:3],I[0:4,0:3],R[0:3,0:3],QA[0:3];
REAL ARRAY P[0:3,0:3],F[1:4,0:5,0:8,0:13],PA[0:3],C[0:3,0:3];
LABEL LB1,LB2,LB3,LB4,LB10,LB11;
FORMAT FT1(///,"THE OPTIMAL SOLUTION IS:",//);
FORMAT FT2("ORDER",I3," UNITS OF ITEM",I2," TO ARRIVE AT THE BEGINNING O
F PERIOD",I2,/);
FORMAT FT3("CHOOSE",F5.2," DOLLARS AS THE SALE PRICE OF ITEM",I2," IN PE
RIOD",I2,/);
FORMAT FT4(/,"THE MAXIMUM PROFIT OBTAINED BY THIS SOLUTION IS",F10.5," D
OLLARS.");
FORMAT FT5("TIE IN THE FIRST STAGE, BETWEEN",3(I3),3(F5.2)," AND",3(I3),
3(F5.2),", GIVEN THAT THE STATE IS",3(I3));
FORMAT FT6("TIE IN THE SECOND STAGE, BETWEEN",3(I3),3(F5.2)," AND",3(I3)
,3(F5.2),", GIVEN THAT THE STATE IS",3(I3));
FORMAT FT7("TIE IN THE THIRD STAGE, BETWEEN",3(I3),3(F5.2)," AND",3(I3),
3(F5.2),", GIVEN THAT THE STATE IS",3(I3));
FORMAT FT8(///,"SALVAGE VALUES OF ALL FEASIBLE ENDING INVENTORIES",/);
FORMAT FT9("I[4,1]",X2,"I[4,2]",X2,"I[4,3]",X2,"F[4,I[4,1],I[4,2],I[4,3]
]");
FORMAT FT10(I4,X4,I4,X4,I4,X4,F15.5);
FORMAT FT11(//,"VALUES OF THE FUNCTION F FOR ALL FEASIBLE STATES OF PERI
OD",I2,", AND THE CORRESPONDENT OPTIMAL DECISIONS",/);
FORMAT FT12("I[3,1]",X2,"I[3,2]",X2,"I[3,3]",X2,"F[3,I[3,1],I[3,2],I[3,3
]]",X2,"Q[3,1]",X2,"Q[3,2]",X2,"Q[3,3]",X2,"P[3,1]",X2,"P[3,2]",X2,"P[3,
3]");
FORMAT FT13(I4,X4,I4,X4,I4,X4,F15.5,X12,I4,X4,I4,X4,I4,X4,F5.2,X3,F5.2,X
3,F5.2);
FORMAT FT14("I[2,1]",X2,"I[2,2]",X2,"I[2,3]",X2,"F[2,I[2,1],I[2,2],I[2,3
]]",X2,"Q[2,1]",X2,"Q[2,2]",X2,"Q[2,3]",X2,"P[2,1]",X2,"P[2,2]",X2,"P[2,
3]");
WRITE(FO,<"EXAMPLE OF PROCUREMENT-INVENTORY SYSTEM UNDER PROLONGED PRICE
 INCREASE - SOLUTION BY DYNAMIC PROGRAMMING - PAULO C METRI">);
WRITE(FO,FT8);
WRITE(FO,FT9);
LB1:F[4,I[4,1],I[4,2],I[4,3]]←8.75×I[4,1]+5.85×I[4,2]+6.85×I[4,3];
F[1,I[4,1],I[4,2],I[4,3]]←1.0;
WRITE(FO,FT10,FOR J←1,2,3 DO I[4,J],F[4,I[4,1],I[4,2],I[4,3]]);
I[4,3]←I[4,3]+1;
IF I[4,3]≤(26-5×I[4,1]-3×I[4,2])/2 THEN GO TO LB1;
I[4,3]←0;
I[4,2]←I[4,2]+1;
IF I[4,2]≤(26-5×I[4,1]-2×I[4,3])/3 THEN GO TO LB1;
I[4,2]←0;
I[4,1]←I[4,1]+1;
IF I[4,1]≤(26-3×I[4,2]-2×I[4,3])/5 THEN GO TO LB1;
FOR I[3,1]←0 STEP 1 UNTIL 5 DO
FOR I[3,2]←0 STEP 1 UNTIL 8 DO
FOR I[3,3]←0 STEP 1 UNTIL 13 DO BEGIN
IF F[1,I[3,1],I[3,2],I[3,3]]≠1.0 THEN GO TO LB4;
      IF I[3,1]≥6  THEN BEGIN Q[3,1]←0;R[3,1]←0;END  ELSE BEGIN Q[3,1]←6-
I[3,1];R[3,1]←1;END;
      IF I[3,2]≥8  THEN BEGIN Q[3,2]←0;R[3,2]←0;END  ELSE BEGIN Q[3,2]←8-
I[3,2];R[3,2]←1;END;
      IF I[3,3]≥10 THEN BEGIN Q[3,3]←0;R[3,3]←0;END  ELSE BEGIN Q[3,3]←10
-I[3,3];R[3,3]←1;END;
IF Q[3,1]≤5 THEN C[3,1]←8.95 ELSE IF Q[3,1]<10 THEN C[3,1]←8.85 ELSE C[3
```

```
,1]←8.75;
IF Q[3,2]≤5 THEN C[3,2]←5.95 ELSE IF Q[3,2]<10 THEN C[3,2]←5.90 ELSE C[3
,2]←5.85;
IF Q[3,3]≤6 THEN C[3,3]←6.95 ELSE IF Q[3,3]≤12 THEN C[3,3]←6.90 ELSE C[3
,3]←6.85;
LB2:FOR P[3,1]←9.15,9.35,9.50 DO
FOR P[3,2]←6.10,6.25,6.35 DO
FOR P[3,3]←8.05,8.20,8.30 DO BEGIN
IF P[3,1]=9.15 THEN D[3,1]←8 ELSE IF P[3,1]=9.35 THEN D[3,1]←7 ELSE D[3,
1]←6;
IF P[3,2]=6.10 THEN D[3,2]←10 ELSE IF P[3,2]=6.25 THEN D[3,2]←9 ELSE D[3
,2]←8;
IF P[3,3]=8.05 THEN D[3,3]←12 ELSE IF P[3,3]=8.20 THEN D[3,3]←11 ELSE D[
3,3]←10;
FOR J←1,2,3 DO I[4,J]←I[3,J]+Q[3,J]-D[3,J];
IF I[4,1]<0 THEN GO TO LB3;
IF I[4,2]<0 THEN GO TO LB3;
IF I[4,3]<0 THEN GO TO LB3;
TEMP←D[3,1]×P[3,1]+D[3,2]×P[3,2]+D[3,3]×P[3,3]-Q[3,1]×C[3,1]-Q[3,2]×C[3,
2]-Q[3,3]×C[3,3]-2.00×R[3,1]-2.05×R[3,2]-2.00×R[3,3]-0.35×(I[3,1]+Q[3,1]
-D[3,1]/2)-0.30×(I[3,2]+Q[3,2]-D[3,2]/2)-0.50×(I[3,3]+Q[3,3]-D[3,3]/2)+F
[4,I[4,1],I[4,2],I[4,3]];
IF ABS(TEMP-F[3,I[3,1],I[3,2],I[3,3]])<0.001
                                          THEN WRITE(FO,FT5,FOR J←1,2,3 DO QOPT[
3,I[3,1],I[3,2],I[3,3],J],FOR J←1,2,3 DO POPT[3,I[3,1],I[3,2],I[3,3],J],
FOR J←1,2,3 DO Q[3,J],FOR J←1,2,3 DO P[3,J],FOR J←1,2,3 DO I[3,J]);
IF TEMP>F[3,I[3,1],I[3,2],I[3,3]] THEN BEGIN
F[3,I[3,1],I[3,2],I[3,3]]←TEMP;
FOR J←1,2,3 DO QOPT[3,I[3,1],I[3,2],I[3,3],J]←Q[3,J];
FOR J←1,2,3 DO POPT[3,I[3,1],I[3,2],I[3,3],J]←P[3,J];
END;
LB3:END;
Q[3,3]←Q[3,3]+1;
IF Q[3,3]>(100-5×(Q[3,1]+I[3,1])-3×(Q[3,2]+I[3,2]))/2-I[3,3] THEN GO TO
LB10;
R[3,3]←1;
IF Q[3,3]≤6 THEN C[3,3]←6.95 ELSE IF Q[3,3]<12 THEN C[3,3]←6.90 ELSE C[3
,3]←6.85;
GO TO LB2;
LB10:IF I[3,3]≥10 THEN BEGIN Q[3,3]←0;R[3,3]←0;END  ELSE BEGIN Q[3,3]←10
-I[3,3];R[3,3]←1;END;
IF Q[3,3]≤6 THEN C[3,3]←6.95 ELSE IF Q[3,3]<12 THEN C[3,3]←6.90 ELSE C[3
,3]←6.85;
Q[3,2]←Q[3,2]+1;
IF Q[3,2]>(100-5×(Q[3,1]+I[3,1])-2×(Q[3,3]+I[3,3]))/3-I[3,2] THEN GO TO
LB11;
R[3,2]←1;
IF Q[3,2]≤5 THEN C[3,2]←5.95 ELSE IF Q[3,2]≤10 THEN C[3,2]←5.90 ELSE C[3
,2]←5.85;
GO TO LB2;
LB11:IF I[3,2]≥8  THEN BEGIN Q[3,2]←0;R[3,2]←0;END  ELSE BEGIN Q[3,2]←8-
I[3,2];R[3,2]←1;END;
IF Q[3,2]≤5 THEN C[3,2]←5.95 ELSE IF Q[3,2]<10 THEN C[3,2]←5.90 ELSE C[3
,2]←5.85;
Q[3,1]←Q[3,1]+1;
IF Q[3,1]>(100-3×(Q[3,2]+I[3,2])-2×(Q[3,3]+I[3,3]))/5-I[3,1] THEN GO TO
LB4;
R[3,1]←1;
IF Q[3,1]≤5 THEN C[3,1]←8.95 ELSE IF Q[3,1]≤10 THEN C[3,1]←8.85 ELSE C[3
,1]←8.75;
GO TO LB2;
LB4:END;
K←3;
WRITE(FO,FT11,K);
WRITE(FO,FT12);
FOR I[3,1]←0 STEP 1 UNTIL 5 DO
FOR I[3,2]←0 STEP 1 UNTIL 8 DO
FOR I[3,3]←0 STEP 1 UNTIL 13 DO
IF F[1,I[3,1],I[3,2],I[3,3]]=1.0 THEN
WRITE(FO,FT13,FOR J←1,2,3 DO I[3,J],F[3,I[3,1],I[3,2],I[3,3]],FOR J←1,2,
3 DO QOPT[3,I[3,1],I[3,2],I[3,3],J],FOR J←1,2,3 DO POPT[3,I[3,1],I[3,2],
I[3,3],J]);
BEGIN INTEGER DUMMY;
LABEL LB5,LB6,LB7,LB12,LB13;
FOR I[2,1]←0 STEP 1 UNTIL 5 DO
FOR I[2,2]←0 STEP 1 UNTIL 8 DO
FOR I[2,3]←0 STEP 1 UNTIL 13 DO BEGIN
IF F[1,I[2,1],I[2,2],I[2,3]]≠1.0 THEN GO TO LB7;
      IF I[2,1]≥6  THEN BEGIN Q[2,1]←0;R[2,1]←0;END  ELSE BEGIN Q[2,1]←6-
I[2,1];R[2,1]←1;END;
      IF I[2,2]≥8  THEN BEGIN Q[2,2]←0;R[2,2]←0;END  ELSE BEGIN Q[2,2]←8-
I[2,2];R[2,2]←1;END;
      IF I[2,3]≥10 THEN BEGIN Q[2,3]←0;R[2,3]←0;END  ELSE BEGIN Q[2,3]←10
```

```
-I[2,3];R[2,3]←1;END;
IF Q[2,1]≤5 THEN C[2,1]←7.95 ELSE IF Q[2,1]≤10 THEN C[2,1]←7.85 ELSE C[2
,1]←7.75;
IF Q[2,2]≤5 THEN C[2,2]←4.95 ELSE IF Q[2,2]<10 THEN C[2,2]←4.90 ELSE C[2
,2]←4.85;
IF Q[2,3]≤6 THEN C[2,3]←6.95 ELSE IF Q[2,3]≤12 THEN C[2,3]←6.90 ELSE C[2
,3]←6.85;
LB5:FOR P[2,1]←8.75,8.95,9.10 DO
FOR P[2,2]←5.75,5.90,6.00 DO
FOR P[2,3]←7.75,7.90,8.00 DO BEGIN
IF P[2,1]=8.75 THEN D[2,1]←8 ELSE IF P[2,1]=8.95 THEN D[2,1]←7 ELSE D[2,
1]←6;
IF P[2,2]=5.75 THEN D[2,2]←10 ELSE IF P[2,2]=5.90 THEN D[2,2]←9 ELSE D[2
,2]←8;
IF P[2,3]=7.75 THEN D[2,3]←12 ELSE IF P[2,3]=7.90 THEN D[2,3]←11 ELSE D[
2,3]←10;
FOR J←1,2,3 DO I[3,J]←I[2,J]+Q[2,J]-D[2,J];
IF I[3,1]<0 THEN GO TO LB6;
IF I[3,2]<0 THEN GO TO LB6;
IF I[3,3]<0 THEN GO TO LB6;
TEMP←D[2,1]×P[2,1]+D[2,2]×P[2,2]+D[2,3]×P[2,3]-Q[2,1]×C[2,1]-Q[2,2]×C[2,
2]-Q[2,3]×C[2,3]-1.95×R[2,1]-2.05×R[2,2]-2.00×R[2,3]-0.35×(I[2,1]+Q[2,1]
-D[2,1]/2)-0.25×(I[2,2]+Q[2,2]-D[2,2]/2)-0.45×(I[2,3]+Q[2,3]-D[2,3]/2)+F
[3,I[3,1],I[3,2],I[3,3]];
IF ABS(TEMP-F[2,I[2,1],I[2,2],I[2,3]])<0.001
                                        THEN WRITE(FO,FT6,FOR J←1,2,3 DO QOPT[
2,I[2,1],I[2,2],I[2,3],J],FOR J←1,2,3 DO POPT[2,I[2,1],I[2,2],I[2,3],J],
FOR J←1,2,3 DO Q[2,J],FOR J←1,2,3 DO P[2,J],FOR J←1,2,3 DO I[2,J]);
IF TEMP>F[2,I[2,1],I[2,2],I[2,3]] THEN BEGIN
F[2,I[2,1],I[2,2],I[2,3]]←TEMP;
FOR J←1,2,3 DO QOPT[2,I[2,1],I[2,2],I[2,3],J]←Q[2,J];
FOR J←1,2,3 DO POPT[2,I[2,1],I[2,2],I[2,3],J]←P[2,J];
END;
LB6:END;
Q[2,3]←Q[2,3]+1;
IF Q[2,3]>(100-5×(Q[2,1]+I[2,1])-3×(Q[2,2]+I[2,2]))/2-I[2,3] THEN GO TO
LB12;
R[2,3]←1;
IF Q[2,3]≤6 THEN C[2,3]←6.95 ELSE IF Q[2,3]<12 THEN C[2,3]←6.90 ELSE C[2
,3]←6.85;
GO TO LB5;
LB12:IF I[2,3]≥10 THEN BEGIN Q[2,3]←0;R[2,3]←0;END  ELSE BEGIN Q[2,3]←10
-I[2,3];R[2,3]←1;END;
IF Q[2,3]≤6 THEN C[2,3]←6.95 ELSE IF Q[2,3]≤12 THEN C[2,3]←6.90 ELSE C[2
,3]←6.85;
Q[2,2]←Q[2,2]+1;
IF Q[2,2]>(100-5×(Q[2,1]+I[2,1])-2×(Q[2,3]+I[2,3]))/3-I[2,2] THEN GO TO
LB13;
R[2,2]←1;
IF Q[2,2]≤5 THEN C[2,2]←4.95 ELSE IF Q[2,2]<10 THEN C[2,2]←4.90 ELSE C[2
,2]←4.85;
GO TO LB5;
LB13:IF I[2,2]≥8  THEN BEGIN Q[2,2]←0;R[2,2]←0;END  ELSE BEGIN Q[2,2]←8-
I[2,2];R[2,2]←1;END;
IF Q[2,2]≤5 THEN C[2,2]←4.95 ELSE IF Q[2,2]<10 THEN C[2,2]←4.90 ELSE C[2
,2]←4.85;
Q[2,1]←Q[2,1]+1;
IF Q[2,1]>(100-3×(Q[2,2]+I[2,2])-2×(Q[2,3]+I[2,3]))/5-I[2,1] THEN GO TO
LB7;
R[2,1]←1;
IF Q[2,1]≤5 THEN C[2,1]←7.95 ELSE IF Q[2,1]≤10 THEN C[2,1]←7.85 ELSE C[2
,1]←7.75;
GO TO LB5;
LB7:END;
K←2;
WRITE(FO,FT11,K);
WRITE(FO,FT14);
FOR I[2,1]←0 STEP 1 UNTIL 5 DO
FOR I[2,2]←0 STEP 1 UNTIL 8 DO
FOR I[2,3]←0 STEP 1 UNTIL 13 DO
IF F[1,I[2,1],I[2,2],I[2,3]]=1.0 THEN
WRITE(FO,FT13,FOR J←1,2,3 DO I[2,J],F[2,I[2,1],I[2,2],I[2,3]],FOR J←1,2,
3 DO QOPT[2,I[2,1],I[2,2],I[2,3],J],FOR J←1,2,3 DO POPT[2,I[2,1],I[2,2],
I[2,3],J]);
END;
BEGIN INTEGER GHOST;
LABEL LB8,LB9,LB14,LB15,LB16;
I[1,1]←6;
I[1,2]←8;
I[1,3]←10;
Q[1,1]←0;
Q[1,2]←0;
Q[1,3]←0;
```

```
R[1,1]←0;
R[1,2]←0;
R[1,3]←0;
C[1,1]←6.95;
C[1,2]←4.95;
C[1,3]←5.95;
LB8:FOR P[1,1]←8.35,8.55,8.70 DO
FOR P[1,2]←5.50,5.65,5.75 DO
FOR P[1,3]←7.35,7.50,7.65 DO BEGIN
IF P[1,1]=8.35 THEN D[1,1]←8 ELSE IF P[1,1]=8.55 THEN D[1,1]←7 ELSE D[1,
1]←6;
IF P[1,2]=5.50 THEN D[1,2]←10 ELSE IF P[1,2]=5.65 THEN D[1,2]←9 ELSE D[1
,2]←8;
IF P[1,3]=7.35 THEN D[1,3]←12 ELSE IF P[1,3]=7.50 THEN D[1,3]←11 ELSE D[
1,3]←10;
FOR J←1,2,3 DO I[2,J]←I[1,J]+Q[1,J]−D[1,J];
IF I[2,1]<0 THEN GO TO LB9;
IF I[2,2]<0 THEN GO TO LB9;
IF I[2,3]<0 THEN GO TO LB9;
TEMP←D[1,1]×P[1,1]+D[1,2]×P[1,2]+D[1,3]×P[1,3]−Q[1,1]×C[1,1]−Q[1,2]×C[1,
2]−Q[1,3]×C[1,3]−1.95×R[1,1]−2.00×R[1,2]−2.00×R[1,3]−0.30×(I[1,1]+Q[1,1]
−D[1,1]/2)−0.20×(I[1,2]+Q[1,2]−D[1,2]/2)−0.45×(I[1,3]+Q[1,3]−D[1,3]/2)+F
[2,I[2,1],I[2,2],I[2,3]];
IF ABS(TEMP−FA)<0.001
              THEN WRITE(FO,FT7,FOR J←1,2,3 DO QA[J],FOR J←1,2,3 DO PA[J],F
OR J←1,2,3 DO Q[1,J],FOR J←1,2,3 DO P[1,J],FOR J←1,2,3 DO I[1,J]);
IF TEMP>FA THEN BEGIN
FA←TEMP;
FOR J←1,2,3 DO QA[J]←Q[1,J];
FOR J←1,2,3 DO PA[J]←P[1,J];
END;
LB9:END;
Q[1,3]←Q[1,3]+1;
IF Q[1,3]>(26−5×Q[1,1]−3×Q[1,2])/2 THEN GO TO LB14;
R[1,3]←1;
IF Q[1,3]≤6 THEN C[1,3]←5.95 ELSE IF Q[1,3]≤12 THEN C[1,3]←5.90 ELSE C[1
,3]←5.85;
GO TO LB8;
LB14:Q[1,3]←0;
R[1,3]←0;
C[1,3]←5.95;
Q[1,2]←Q[1,2]+1;
IF Q[1,2]>(26−5×Q[1,1]−2×Q[1,3])/3 THEN GO TO LB15;
R[1,2]←1;
IF Q[1,2]≤5 THEN C[1,2]←4.95 ELSE IF Q[1,2]<10 THEN C[1,2]←4.90 ELSE C[1
,2]←4.85;
GO TO LB8;
LB15:Q[1,2]←0;
R[1,2]←0;
C[1,2]←4.95;
Q[1,1]←Q[1,1]+1;
IF Q[1,1]>(26−3×Q[1,2]−2×Q[1,3])/5 THEN GO TO LB16;
R[1,1]←1;
IF Q[1,1]≤5 THEN C[1,1]←6.95 ELSE IF Q[1,1]<10 THEN C[1,1]←6.85 ELSE C[1
,1]←6.75;
GO TO LB8;
LB16:
IF PA[1]=8.35 THEN D[1,1]←8  ELSE IF PA[1]=8.55 THEN D[1,1]←7  ELSE D[1,
1]←6;
IF PA[2]=5.50 THEN D[1,2]←10 ELSE IF PA[2]=5.65 THEN D[1,2]←9  ELSE D[1,
2]←8;
IF PA[3]=7.35 THEN D[1,3]←12 ELSE IF PA[3]=7.50 THEN D[1,3]←11 ELSE D[1,
3]←10;
FOR J←1,2,3 DO I[2,J]←I[1,J]+QA[J]−D[1,J];
IF POPT[2,I[2,1],I[2,2],I[2,3],1]=8.75 THEN D[2,1]←8  ELSE IF POPT[2,I[2
,1],I[2,2],I[2,3],1]=8.95 THEN D[2,1]←7  ELSE D[2,1]←6;
IF POPT[2,I[2,1],I[2,2],I[2,3],2]=5.75 THEN D[2,2]←10 ELSE IF POPT[2,I[2
,1],I[2,2],I[2,3],2]=5.90 THEN D[2,2]←9  ELSE D[2,2]←8;
IF POPT[2,I[2,1],I[2,2],I[2,3],3]=7.75 THEN D[2,3]←12 ELSE IF POPT[2,I[2
,1],I[2,2],I[2,3],3]=7.90 THEN D[2,3]←11 ELSE D[2,3]←10;
FOR J←1,2,3 DO I[3,J]←I[2,J]+QOPT[2,I[2,1],I[2,2],I[2,3],J]−D[2,J];
WRITE(FO,FT1);
K←1;
FOR J←1,2,3 DO WRITE(FO,FT2,QA[J],J,K);
FOR J←1,2,3 DO WRITE(FO,FT3,PA[J],J,K);
FOR K←2,3 DO BEGIN
FOR J←1,2,3 DO WRITE(FO,FT2,QOPT[K,I[K,1],I[K,2],I[K,3],J],J,K);
FOR J←1,2,3 DO WRITE(FO,FT3,POPT[K,I[K,1],I[K,2],I[K,3],J],J,K);
END;
WRITE(FO,FT4,FA);
END;
END.
```

SALVAGE VALUES OF ALL FEASIBLE ENDING INVENTORIES

I[4,1]	I[4,2]	I[4,3]	F[4,I[4,1],I[4,2],I[4,3]]
0	0	0	0.00000
0	0	1	6.85000
0	0	2	13.70000
0	0	3	20.55000
0	0	4	27.40000
0	0	5	34.25000
0	0	6	41.10000
0	0	7	47.95000
0	0	8	54.80000
0	0	9	61.65000
0	0	10	68.50000
0	0	11	75.35000
0	0	12	82.20000
0	0	13	89.05000
0	1	0	5.85000
0	1	1	12.70000
0	1	2	19.55000
0	1	3	26.40000
0	1	4	33.25000
0	1	5	40.10000
0	1	6	46.95000
0	1	7	53.80000
0	1	8	60.65000
0	1	9	67.50000
0	1	10	74.35000
0	1	11	81.20000
0	2	0	11.70000
0	2	1	18.55000
0	2	2	25.40000
0	2	3	32.25000
0	2	4	39.10000
0	2	5	45.95000
0	2	6	52.80000
0	2	7	59.65000
0	2	8	66.50000
0	2	9	73.35000
0	2	10	80.20000
0	3	0	17.55000
0	3	1	24.40000
0	3	2	31.25000
0	3	3	38.10000
0	3	4	44.95000
0	3	5	51.80000
0	3	6	58.65000
0	3	7	65.50000
0	3	8	72.35000
0	4	0	23.40000
0	4	1	30.25000
0	4	2	37.10000
0	4	3	43.95000
0	4	4	50.80000
0	4	5	57.65000
0	4	6	64.50000
0	4	7	71.35000
0	5	0	29.25000
0	5	1	36.10000
0	5	2	42.95000
0	5	3	49.80000
0	5	4	56.65000
0	5	5	63.50000
0	6	0	35.10000
0	6	1	41.95000
0	6	2	48.80000
0	6	3	55.65000
0	6	4	62.50000
0	7	0	40.95000
0	7	1	47.80000
0	7	2	54.65000
0	8	0	46.80000
0	8	1	53.65000
1	0	0	8.75000
1	0	1	15.60000
1	0	2	22.45000
1	0	3	29.30000
1	0	4	36.15000
1	0	5	43.00000

1	0	6	49.85000
1	0	7	56.70000
1	0	8	63.55000
1	0	9	70.40000
1	0	10	77.25000
1	1	0	14.60000
1	1	1	21.45000
1	1	2	28.30000
1	1	3	35.15000
1	1	4	42.00000
1	1	5	48.85000
1	1	6	55.70000
1	1	7	62.55000
1	1	8	69.40000
1	1	9	76.25000
1	2	0	20.45000
1	2	1	27.30000
1	2	2	34.15000
1	2	3	41.00000
1	2	4	47.85000
1	2	5	54.70000
1	2	6	61.55000
1	2	7	68.40000
1	3	0	26.30000
1	3	1	33.15000
1	3	2	40.00000
1	3	3	46.85000
1	3	4	53.70000
1	3	5	60.55000
1	3	6	67.40000
1	4	0	32.15000
1	4	1	39.00000
1	4	2	45.85000
1	4	3	52.70000
1	4	4	59.55000
1	5	0	38.00000
1	5	1	44.85000
1	5	2	51.70000
1	5	3	58.55000
1	6	0	43.85000
1	6	1	50.70000
1	7	0	49.70000
2	0	0	17.50000
2	0	1	24.35000
2	0	2	31.20000
2	0	3	38.05000
2	0	4	44.90000
2	0	5	51.75000
2	0	6	58.60000
2	0	7	65.45000
2	0	8	72.30000
2	1	0	23.35000
2	1	1	30.20000
2	1	2	37.05000
2	1	3	43.90000
2	1	4	50.75000
2	1	5	57.60000
2	1	6	64.45000
2	2	0	29.20000
2	2	1	36.05000
2	2	2	42.90000
2	2	3	49.75000
2	2	4	56.60000
2	2	5	63.45000
2	3	0	35.05000
2	3	1	41.90000
2	3	2	48.75000
2	3	3	55.60000
2	4	0	40.90000
2	4	1	47.75000
2	4	2	54.60000
2	5	0	46.75000
3	0	0	26.25000
3	0	1	33.10000
3	0	2	39.95000
3	0	3	46.80000
3	0	4	53.65000
3	0	5	60.50000
3	1	0	32.10000
3	1	1	38.95000
3	1	2	45.80000
3	1	3	52.65000
3	1	4	59.50000

```
3    2    0    37.95000
3    2    1    44.80000
3    2    2    51.65000
3    3    0    43.80000
3    3    1    50.65000
4    0    0    35.00000
4    0    1    41.85000
4    0    2    48.70000
4    0    3    55.55000
4    1    0    40.85000
4    1    1    47.70000
4    2    0    46.70000
5    0    0    43.75000
TIE IN THE FIRST STAGE, BETWEEN 6 8  5 9.50 6.35 8.30 AND 6 8  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 0 5
TIE IN THE FIRST STAGE, BETWEEN 6 8  4 9.50 6.35 8.30 AND 6 8  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 0 6
TIE IN THE FIRST STAGE, BETWEEN 6 8  3 9.50 6.35 8.30 AND 6 8  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 0 7
TIE IN THE FIRST STAGE, BETWEEN 6 8  2 9.50 6.35 8.30 AND 6 8  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 0 8
TIE IN THE FIRST STAGE, BETWEEN 6 8  1 9.50 6.35 8.30 AND 6 8  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 0 9
TIE IN THE FIRST STAGE, BETWEEN 6 7  5 9.50 6.35 8.30 AND 6 7  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 1 5
TIE IN THE FIRST STAGE, BETWEEN 6 7  4 9.50 6.35 8.30 AND 6 7  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 1 6
TIE IN THE FIRST STAGE, BETWEEN 6 7  3 9.50 6.35 8.30 AND 6 7  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 1 7
TIE IN THE FIRST STAGE, BETWEEN 6 7  2 9.50 6.35 8.30 AND 6 7  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 1 8
TIE IN THE FIRST STAGE, BETWEEN 6 7  1 9.50 6.35 8.30 AND 6 7  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 1 9
TIE IN THE FIRST STAGE, BETWEEN 6 6  5 9.50 6.35 8.30 AND 6 6  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 2 5
TIE IN THE FIRST STAGE, BETWEEN 6 6  4 9.50 6.35 8.30 AND 6 6  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 2 6
TIE IN THE FIRST STAGE, BETWEEN 6 6  3 9.50 6.35 8.30 AND 6 6  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 2 7
TIE IN THE FIRST STAGE, BETWEEN 6 6  2 9.50 6.35 8.30 AND 6 6  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 2 8
TIE IN THE FIRST STAGE, BETWEEN 6 6  1 9.50 6.35 8.30 AND 6 6  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 2 9
TIE IN THE FIRST STAGE, BETWEEN 6 5  5 9.50 6.35 8.30 AND 6 5  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 3 5
TIE IN THE FIRST STAGE, BETWEEN 6 5  4 9.50 6.35 8.30 AND 6 5  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 3 6
TIE IN THE FIRST STAGE, BETWEEN 6 5  3 9.50 6.35 8.30 AND 6 5  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 3 7
TIE IN THE FIRST STAGE, BETWEEN 6 5  2 9.50 6.35 8.30 AND 6 5  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 3 8
TIE IN THE FIRST STAGE, BETWEEN 6 4  5 9.50 6.35 8.30 AND 6 4  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 4 5
TIE IN THE FIRST STAGE, BETWEEN 6 4  4 9.50 6.35 8.30 AND 6 4  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 4 6
TIE IN THE FIRST STAGE, BETWEEN 6 4  3 9.50 6.35 8.30 AND 6 4  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 4 7
TIE IN THE FIRST STAGE, BETWEEN 6 3  5 9.50 6.35 8.30 AND 6 3  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 0 5 5
TIE IN THE FIRST STAGE, BETWEEN 5 8 11 9.50 6.35 8.20 AND 6 8 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 0 0
TIE IN THE FIRST STAGE, BETWEEN 5 8 10 9.50 6.35 8.20 AND 6 8  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 0 1
TIE IN THE FIRST STAGE, BETWEEN 5 8  9 9.50 6.35 8.20 AND 6 8  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 0 2
TIE IN THE FIRST STAGE, BETWEEN 5 8  8 9.50 6.35 8.20 AND 6 8  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 0 3
TIE IN THE FIRST STAGE, BETWEEN 5 8  5 9.50 6.35 8.30 AND 5 8  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 5
TIE IN THE FIRST STAGE, BETWEEN 6 8  5 9.50 6.35 8.30 AND 6 8  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 5
TIE IN THE FIRST STAGE, BETWEEN 5 8  4 9.50 6.35 8.30 AND 5 8  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 6
TIE IN THE FIRST STAGE, BETWEEN 6 8  4 9.50 6.35 8.30 AND 6 8  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 6
TIE IN THE FIRST STAGE, BETWEEN 5 8  3 9.50 6.35 8.30 AND 5 8  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 7
TIE IN THE FIRST STAGE, BETWEEN 6 8  3 9.50 6.35 8.30 AND 6 8  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 7
TIE IN THE FIRST STAGE, BETWEEN 5 8  2 9.50 6.35 8.30 AND 5 8  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 8
TIE IN THE FIRST STAGE, BETWEEN 6 8  2 9.50 6.35 8.30 AND 6 8  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 8
TIE IN THE FIRST STAGE, BETWEEN 5 8  1 9.50 6.35 8.30 AND 5 8  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 9
TIE IN THE FIRST STAGE, BETWEEN 6 8  1 9.50 6.35 8.30 AND 6 8  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 0 9
TIE IN THE FIRST STAGE, BETWEEN 5 7 11 9.50 6.35 8.20 AND 6 7 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 1 0
TIE IN THE FIRST STAGE, BETWEEN 5 7 10 9.50 6.35 8.20 AND 6 7  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 1 1
TIE IN THE FIRST STAGE, BETWEEN 5 7  9 9.50 6.35 8.20 AND 6 7  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 1 2
TIE IN THE FIRST STAGE, BETWEEN 5 7  8 9.50 6.35 8.20 AND 6 7  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 1 3
TIE IN THE FIRST STAGE, BETWEEN 5 7  5 9.50 6.35 8.30 AND 5 7  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 5
TIE IN THE FIRST STAGE, BETWEEN 6 7  5 9.50 6.35 8.30 AND 6 7  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 5
TIE IN THE FIRST STAGE, BETWEEN 5 7  4 9.50 6.35 8.30 AND 5 7  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 6
TIE IN THE FIRST STAGE, BETWEEN 6 7  4 9.50 6.35 8.30 AND 6 7  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 6
TIE IN THE FIRST STAGE, BETWEEN 5 7  3 9.50 6.35 8.30 AND 5 7  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 7
TIE IN THE FIRST STAGE, BETWEEN 6 7  3 9.50 6.35 8.30 AND 6 7  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 7
TIE IN THE FIRST STAGE, BETWEEN 5 7  2 9.50 6.35 8.30 AND 5 7  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 8
TIE IN THE FIRST STAGE, BETWEEN 6 7  2 9.50 6.35 8.30 AND 6 7  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 8
TIE IN THE FIRST STAGE, BETWEEN 5 7  1 9.50 6.35 8.30 AND 5 7  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 9
TIE IN THE FIRST STAGE, BETWEEN 6 7  1 9.50 6.35 8.30 AND 6 7  2 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 1 9
TIE IN THE FIRST STAGE, BETWEEN 5 6 11 9.50 6.35 8.20 AND 6 6 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 2 0
TIE IN THE FIRST STAGE, BETWEEN 5 6 10 9.50 6.35 8.20 AND 6 6  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 2 1
TIE IN THE FIRST STAGE, BETWEEN 5 6  9 9.50 6.35 8.20 AND 6 6  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 2 2
TIE IN THE FIRST STAGE, BETWEEN 5 6  8 9.50 6.35 8.20 AND 6 6  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 2 3
TIE IN THE FIRST STAGE, BETWEEN 5 6  5 9.50 6.35 8.30 AND 5 6  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 5
TIE IN THE FIRST STAGE, BETWEEN 6 6  5 9.50 6.35 8.30 AND 6 6  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 5
TIE IN THE FIRST STAGE, BETWEEN 5 6  4 9.50 6.35 8.30 AND 5 6  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 6
TIE IN THE FIRST STAGE, BETWEEN 6 6  4 9.50 6.35 8.30 AND 6 6  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 6
TIE IN THE FIRST STAGE, BETWEEN 5 6  3 9.50 6.35 8.30 AND 5 6  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 7
TIE IN THE FIRST STAGE, BETWEEN 6 6  3 9.50 6.35 8.30 AND 6 6  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 2 7
TIE IN THE FIRST STAGE, BETWEEN 5 5 11 9.50 6.35 8.20 AND 6 5 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 3 0
TIE IN THE FIRST STAGE, BETWEEN 5 5 10 9.50 6.35 8.20 AND 6 5  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 3 1
TIE IN THE FIRST STAGE, BETWEEN 5 5  9 9.50 6.35 8.20 AND 6 5  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 3 2
TIE IN THE FIRST STAGE, BETWEEN 5 5  8 9.50 6.35 8.20 AND 6 5  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 3 3
TIE IN THE FIRST STAGE, BETWEEN 5 5  5 9.50 6.35 8.30 AND 5 5  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 3 5
TIE IN THE FIRST STAGE, BETWEEN 6 5  5 9.50 6.35 8.30 AND 6 5  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 3 5
TIE IN THE FIRST STAGE, BETWEEN 5 5  4 9.50 6.35 8.30 AND 5 5  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 3 6
TIE IN THE FIRST STAGE, BETWEEN 6 5  4 9.50 6.35 8.30 AND 6 5  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 1 3 6
TIE IN THE FIRST STAGE, BETWEEN 5 4 11 9.50 6.35 8.20 AND 6 4 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 4 0
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TIE IN THE FIRST STAGE, BETWEEN 5 4 10 9.50 6.35 8.20 AND 6 4  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 4 1
TIE IN THE FIRST STAGE, BETWEEN 5 4  9 9.50 6.35 8.20 AND 6 4  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 4 2
TIE IN THE FIRST STAGE, BETWEEN 5 4  8 9.50 6.35 8.20 AND 6 4  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 4 3
TIE IN THE FIRST STAGE, BETWEEN 5 3 11 9.50 6.35 8.20 AND 6 3 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 5 0
TIE IN THE FIRST STAGE, BETWEEN 5 3 10 9.50 6.35 8.20 AND 6 3  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 5 1
TIE IN THE FIRST STAGE, BETWEEN 5 3  9 9.50 6.35 8.20 AND 6 3  8 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 5 2
TIE IN THE FIRST STAGE, BETWEEN 5 3  8 9.50 6.35 8.20 AND 6 3  7 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 5 3
TIE IN THE FIRST STAGE, BETWEEN 5 2 11 9.50 6.35 8.20 AND 6 2 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 6 0
TIE IN THE FIRST STAGE, BETWEEN 5 2 10 9.50 6.35 8.20 AND 6 2  9 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 6 1
TIE IN THE FIRST STAGE, BETWEEN 5 1 11 9.50 6.35 8.20 AND 6 1 10 9.50 6.35 8.30, GIVEN THAT THE STATE IS 1 7 0
TIE IN THE FIRST STAGE, BETWEEN 4 8  5 9.50 6.35 8.30 AND 4 8  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 0 5
TIE IN THE FIRST STAGE, BETWEEN 4 8  4 9.50 6.35 8.30 AND 4 8  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 0 6
TIE IN THE FIRST STAGE, BETWEEN 4 8  3 9.50 6.35 8.30 AND 4 8  4 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 0 7
TIE IN THE FIRST STAGE, BETWEEN 4 8  2 9.50 6.35 8.30 AND 4 8  3 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 0 8
TIE IN THE FIRST STAGE, BETWEEN 4 7  5 9.50 6.35 8.30 AND 4 7  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 1 5
TIE IN THE FIRST STAGE, BETWEEN 4 7  4 9.50 6.35 8.30 AND 4 7  5 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 1 6
TIE IN THE FIRST STAGE, BETWEEN 4 6  5 9.50 6.35 8.30 AND 4 6  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 2 2 5
TIE IN THE FIRST STAGE, BETWEEN 3 8  5 9.50 6.35 8.30 AND 3 8  6 9.50 6.35 8.20, GIVEN THAT THE STATE IS 3 0 5
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VALUES OF THE FUNCTION F FOR ALL FEASIBLE STATES OF PERIOD 3, AND THE CORRESPONDENT OPTIMAL DECISIONS

I[3,1]	I[3,2]	I[3,3]	F[3,I[3,1],I[3,2],I[3,3]]	Q[3,1]	Q[3,2]	Q[3,3]	P[3,1]	P[3,2]	P[3,3]
0	0	0	10.75000	6	8	11	9.50	6.45	8.20
0	0	1	17.65000	6	8	10	9.50	6.35	8.20
0	0	2	24.55000	6	8	9	9.50	6.35	8.20
0	0	3	31.45000	6	8	8	9.50	6.35	8.20
0	0	4	38.35000	6	8	7	9.50	6.35	8.20
0	0	5	44.95000	6	8	5	9.50	6.35	8.30
0	0	6	51.90000	6	8	4	9.50	6.35	8.30
0	0	7	58.85000	6	8	3	9.50	6.35	8.30
0	0	8	65.80000	6	8	2	9.50	6.35	8.30
0	0	9	72.75000	6	8	1	9.50	6.35	8.30
0	0	10	81.70000	6	8	0	9.50	6.35	8.30
0	0	11	88.65000	6	8	0	9.50	6.35	8.20
0	0	12	95.00000	6	8	0	9.50	6.35	8.20
0	0	13	101.35000	6	8	0	9.50	6.35	8.20
0	1	0	16.65000	6	7	11	9.50	6.35	8.20
0	1	1	23.55000	6	7	10	9.50	6.35	8.20
0	1	2	30.45000	6	7	9	9.50	6.35	8.20
0	1	3	37.35000	6	7	8	9.50	6.35	8.20
0	1	4	44.25000	6	7	7	9.50	6.35	8.20
0	1	5	50.85000	6	7	5	9.50	6.35	8.30
0	1	6	57.80000	6	7	4	9.50	6.35	8.30
0	1	7	64.75000	6	7	3	9.50	6.35	8.30
0	1	8	71.70000	6	7	2	9.50	6.35	8.30
0	1	9	78.65000	6	7	1	9.50	6.35	8.30
0	1	10	87.60000	6	7	0	9.50	6.35	8.30
0	1	11	94.55000	6	7	0	9.50	6.35	8.20
0	2	0	22.55000	6	6	11	9.50	6.35	8.20
0	2	1	29.45000	6	6	10	9.50	6.35	8.20
0	2	2	36.35000	6	6	9	9.50	6.35	8.20
0	2	3	43.25000	6	6	8	9.50	6.35	8.20
0	2	4	50.15000	6	6	7	9.50	6.35	8.20
0	2	5	56.75000	6	6	5	9.50	6.35	8.30
0	2	6	63.70000	6	6	4	9.50	6.35	8.30
0	2	7	70.65000	6	6	3	9.50	6.35	8.30
0	2	8	77.60000	6	6	2	9.50	6.35	8.30
0	2	9	84.55000	6	6	1	9.50	6.35	8.30
0	2	10	93.50000	6	6	0	9.50	6.35	8.30
0	3	0	28.20000	6	5	11	9.50	6.35	8.20
0	3	1	35.10000	6	5	10	9.50	6.35	8.20
0	3	2	42.00000	6	5	9	9.50	6.35	8.20
0	3	3	48.90000	6	5	8	9.50	6.35	8.20
0	3	4	55.80000	6	5	7	9.50	6.35	8.20
0	3	5	62.40000	6	5	5	9.50	6.35	8.30
0	3	6	69.35000	6	5	4	9.50	6.35	8.30
0	3	7	76.30000	6	5	3	9.50	6.35	8.30
0	3	8	83.25000	6	5	2	9.50	6.35	8.30
0	4	0	34.15000	6	4	11	9.50	6.35	8.20
0	4	1	41.05000	6	4	10	9.50	6.35	8.20
0	4	2	47.95000	6	4	9	9.50	6.35	8.20
0	4	3	54.85000	6	4	8	9.50	6.35	8.20
0	4	4	61.75000	6	4	7	9.50	6.35	8.20
0	4	5	68.35000	6	4	5	9.50	6.35	8.30
0	4	6	75.30000	6	4	4	9.50	6.35	8.30
0	4	7	82.25000	6	4	3	9.50	6.35	8.30
0	5	0	40.10000	6	3	11	9.50	6.35	8.20
0	5	1	47.00000	6	3	10	9.50	6.35	8.20
0	5	2	53.90000	6	3	9	9.50	6.35	8.20
0	5	3	60.80000	6	3	8	9.50	6.35	8.20
0	5	4	67.70000	6	3	7	9.50	6.35	8.20
0	5	5	74.30000	6	3	5	9.50	6.35	8.30

0	6	0	46.05000	6	2	11	9.50	6.35	8.20
0	6	1	52.95000	6	2	10	9.50	6.35	8.20
0	6	2	59.85000	6	2	9	9.50	6.35	8.20
0	6	3	66.75000	6	2	8	9.50	6.35	8.20
0	6	4	73.65000	6	2	7	9.50	6.35	8.20
0	7	0	52.00000	6	1	11	9.50	6.35	8.20
0	7	1	58.90000	6	1	10	9.50	6.35	8.20
0	7	2	65.80000	6	1	9	9.50	6.35	8.20
0	8	0	60.00000	6	0	11	9.50	6.35	8.20
0	8	1	66.90000	6	0	10	9.50	6.35	8.20
1	0	0	19.15000	6	8	11	9.50	6.35	8.20
1	0	1	26.05000	6	8	10	9.50	6.35	8.20
1	0	2	32.95000	6	8	9	9.50	6.35	8.20
1	0	3	39.85000	6	8	8	9.50	6.35	8.20
1	0	4	46.75000	6	8	7	9.50	6.35	8.20
1	0	5	53.35000	6	8	5	9.50	6.35	8.30
1	0	6	60.30000	6	8	4	9.50	6.35	8.30
1	0	7	67.25000	6	8	3	9.50	6.35	8.30
1	0	8	74.20000	6	8	2	9.50	6.35	8.30
1	0	9	81.15000	6	8	1	9.50	6.35	8.30
1	0	10	90.10000	6	8	0	9.50	6.35	8.30
1	1	0	25.05000	6	7	11	9.50	6.35	8.20
1	1	1	31.95000	6	7	10	9.50	6.35	8.20
1	1	2	38.85000	6	7	9	9.50	6.35	8.20
1	1	3	45.75000	6	7	8	9.50	6.35	8.20
1	1	4	52.65000	6	7	7	9.50	6.35	8.20
1	1	5	59.25000	6	7	5	9.50	6.35	8.30
1	1	6	66.20000	6	7	4	9.50	6.35	8.30
1	1	7	73.15000	6	7	3	9.50	6.35	8.30
1	1	8	80.10000	6	7	2	9.50	6.35	8.30
1	1	9	87.05000	6	7	1	9.50	6.35	8.30
1	2	0	30.95000	6	6	11	9.50	6.35	8.20
1	2	1	37.85000	6	6	10	9.50	6.35	8.20
1	2	2	44.75000	6	6	9	9.50	6.35	8.20
1	2	3	51.65000	6	6	8	9.50	6.35	8.20
1	2	4	58.55000	6	6	7	9.50	6.35	8.20
1	2	5	65.15000	6	6	5	9.50	6.35	8.30
1	2	6	72.10000	6	6	4	9.50	6.35	8.30
1	2	7	79.05000	6	6	3	9.50	6.35	8.30
1	3	0	36.60000	6	5	11	9.50	6.35	8.20
1	3	1	43.50000	6	5	10	9.50	6.35	8.20
1	3	2	50.40000	6	5	9	9.50	6.35	8.20
1	3	3	57.30000	6	5	8	9.50	6.35	8.20
1	3	4	64.20000	6	5	7	9.50	6.35	8.20
1	3	5	70.60000	6	5	5	9.50	6.35	8.30
1	3	6	77.75000	6	5	4	9.50	6.35	8.30
1	4	0	42.55000	6	4	11	9.50	6.35	8.20
1	4	1	49.45000	6	4	10	9.50	6.35	8.20
1	4	2	56.35000	6	4	9	9.50	6.35	8.20
1	4	3	63.25000	6	4	8	9.50	6.35	8.20
1	4	4	70.15000	6	4	7	9.50	6.35	8.20
1	5	0	48.50000	6	3	11	9.50	6.35	8.20
1	5	1	55.40000	6	3	10	9.50	6.35	8.20
1	5	2	62.30000	6	3	9	9.50	6.35	8.20
1	5	3	69.20000	6	3	8	9.50	6.35	8.20
1	6	0	54.45000	6	2	11	9.50	6.35	8.20
1	6	1	61.35000	6	2	10	9.50	6.35	8.20
1	7	0	60.40000	6	1	11	9.50	6.35	8.20
2	0	0	28.05000	4	8	11	9.50	6.35	8.20
2	0	1	34.95000	4	8	10	9.50	6.35	8.20
2	0	2	41.85000	4	8	9	9.50	6.35	8.20
2	0	3	48.75000	4	8	8	9.50	6.35	8.20
2	0	4	55.65000	4	8	7	9.50	6.35	8.20
2	0	5	62.25000	4	8	5	9.50	6.35	8.30
2	0	6	69.20000	4	8	4	9.50	6.35	8.30
2	0	7	76.15000	4	8	3	9.50	6.35	8.30
2	0	8	83.10000	4	8	2	9.50	6.35	8.30
2	1	0	33.95000	4	7	11	9.50	6.35	8.20
2	1	1	40.85000	4	7	10	9.50	6.35	8.20
2	1	2	47.75000	4	7	9	9.50	6.35	8.20
2	1	3	54.65000	4	7	8	9.50	6.35	8.20
2	1	4	61.55000	4	7	7	9.50	6.35	8.20
2	1	5	68.15000	4	7	5	9.50	6.35	8.30
2	1	6	75.10000	4	7	4	9.50	6.35	8.30
2	2	0	39.85000	4	6	11	9.50	6.35	8.20
2	2	1	46.75000	4	6	10	9.50	6.35	8.20
2	2	2	53.65000	4	6	9	9.50	6.35	8.20
2	2	3	60.55000	4	6	8	9.50	6.35	8.20
2	2	4	67.45000	4	6	7	9.50	6.35	8.20
2	2	5	74.05000	4	6	5	9.50	6.35	8.30
2	3	0	45.50000	4	5	11	9.50	6.35	8.20
2	3	1	52.40000	4	5	10	9.50	6.35	8.20
2	3	2	59.30000	4	5	9	9.50	6.35	8.20

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2  3  3  66.20000  4  5   8  9.50  6.35  8.20
2  4  0  51.45000  4  4  11  9.50  6.35  8.20
2  4  1  58.35000  4  4  10  9.50  6.35  8.20
2  4  2  65.25000  4  4   9  9.50  6.35  8.20
2  5  0  57.40000  4  3  11  9.50  6.35  8.20
3  0  0  37.00000  3  8  11  9.50  6.35  8.20
3  0  1  43.90000  3  8  10  9.50  6.35  8.20
3  0  2  50.80000  3  8   9  9.50  6.35  8.20
3  0  3  57.70000  3  8   8  9.50  6.35  8.20
3  0  4  64.60000  3  8   7  9.50  6.35  8.20
3  0  5  71.20000  3  8   5  9.50  6.35  8.30
3  1  0  42.90000  3  7  11  9.50  6.35  8.20
3  1  1  49.80000  3  7  10  9.50  6.35  8.20
3  1  2  56.70000  3  7   9  9.50  6.35  8.20
3  1  3  63.60000  3  7   8  9.50  6.35  8.20
3  1  4  70.50000  3  7   7  9.50  6.35  8.20
3  2  0  48.80000  3  6  11  9.50  6.35  8.20
3  2  1  55.70000  3  6  10  9.50  6.35  8.20
3  2  2  62.60000  3  6   9  9.50  6.35  8.20
3  3  0  54.45000  3  5  11  9.50  6.35  8.20
3  3  1  61.35000  3  5  10  9.50  6.35  8.20
4  0  0  45.95000  2  8  11  9.50  6.35  8.20
4  0  1  52.85000  2  8  10  9.50  6.35  8.20
4  0  2  59.75000  2  8   9  9.50  6.35  8.20
4  0  3  66.65000  2  8   8  9.50  6.35  8.20
4  1  0  51.85000  2  7  11  9.50  6.35  8.20
4  1  1  58.75000  2  7  10  9.50  6.35  8.20
4  2  0  57.75000  2  6  11  9.50  6.35  8.20
5  0  0  54.90000  1  8  11  9.50  6.35  8.20
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 0 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 0 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 1 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 1 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 1 4
TIE IN THE SECOND STAGE, BETWEEN 6  6 7 9.10 6.00 7.90 AND 6  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 2 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 2 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 2 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 2 4
TIE IN THE SECOND STAGE, BETWEEN 6  6 7 9.10 6.00 7.90 AND 6  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 3 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 3 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 3 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 3 4
TIE IN THE SECOND STAGE, BETWEEN 6  6 7 9.10 6.00 7.90 AND 6  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 4 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 4 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 4 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 4 4
TIE IN THE SECOND STAGE, BETWEEN 6  6 7 9.10 6.00 7.90 AND 6  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 5 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 5 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 5 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 5 4
TIE IN THE SECOND STAGE, BETWEEN 6  6 7 9.10 6.00 7.90 AND 6  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 6 4
TIE IN THE SECOND STAGE, BETWEEN 6  7 7 9.10 6.00 7.90 AND 6  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 6 4
TIE IN THE SECOND STAGE, BETWEEN 6  8 7 9.10 6.00 7.90 AND 6  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 6 4
TIE IN THE SECOND STAGE, BETWEEN 6  9 7 9.10 6.00 7.90 AND 6 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 0 6 4
TIE IN THE SECOND STAGE, BETWEEN 5  8 7 9.10 6.00 7.90 AND 5  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 0 4
TIE IN THE SECOND STAGE, BETWEEN 5  9 7 9.10 6.00 7.90 AND 5 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 0 4
TIE IN THE SECOND STAGE, BETWEEN 5  7 7 9.10 6.00 7.90 AND 5  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 1 4
TIE IN THE SECOND STAGE, BETWEEN 5  8 7 9.10 6.00 7.90 AND 5  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 1 4
TIE IN THE SECOND STAGE, BETWEEN 5  9 7 9.10 6.00 7.90 AND 5 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 1 4
TIE IN THE SECOND STAGE, BETWEEN 5  6 7 9.10 6.00 7.90 AND 5  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 2 4
TIE IN THE SECOND STAGE, BETWEEN 5  7 7 9.10 6.00 7.90 AND 5  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 2 4
TIE IN THE SECOND STAGE, BETWEEN 5  8 7 9.10 6.00 7.90 AND 5  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 2 4
TIE IN THE SECOND STAGE, BETWEEN 5  9 7 9.10 6.00 7.90 AND 5 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 2 4
TIE IN THE SECOND STAGE, BETWEEN 5  6 7 9.10 6.00 7.90 AND 5  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 3 4
TIE IN THE SECOND STAGE, BETWEEN 5  7 7 9.10 6.00 7.90 AND 5  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 3 4
TIE IN THE SECOND STAGE, BETWEEN 5  8 7 9.10 6.00 7.90 AND 5  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 3 4
TIE IN THE SECOND STAGE, BETWEEN 5  9 7 9.10 6.00 7.90 AND 5 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 3 4
TIE IN THE SECOND STAGE, BETWEEN 5  6 7 9.10 6.00 7.90 AND 5  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 4 4
TIE IN THE SECOND STAGE, BETWEEN 5  7 7 9.10 6.00 7.90 AND 5  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 4 4
TIE IN THE SECOND STAGE, BETWEEN 5  8 7 9.10 6.00 7.90 AND 5  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 4 4
TIE IN THE SECOND STAGE, BETWEEN 5  9 7 9.10 6.00 7.90 AND 5 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 1 4 4
TIE IN THE SECOND STAGE, BETWEEN 4  8 7 9.10 6.00 7.90 AND 4  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 0 4
TIE IN THE SECOND STAGE, BETWEEN 4  9 7 9.10 6.00 7.90 AND 4 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 0 4
TIE IN THE SECOND STAGE, BETWEEN 4  7 7 9.10 6.00 7.90 AND 4  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 1 4
TIE IN THE SECOND STAGE, BETWEEN 4  8 7 9.10 6.00 7.90 AND 4  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 1 4
TIE IN THE SECOND STAGE, BETWEEN 4  9 7 9.10 6.00 7.90 AND 4 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 1 4
TIE IN THE SECOND STAGE, BETWEEN 4  6 7 9.10 6.00 7.90 AND 4  7 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 2 4
TIE IN THE SECOND STAGE, BETWEEN 4  7 7 9.10 6.00 7.90 AND 4  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 2 4
TIE IN THE SECOND STAGE, BETWEEN 4  8 7 9.10 6.00 7.90 AND 4  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 2 4
TIE IN THE SECOND STAGE, BETWEEN 4  9 7 9.10 6.00 7.90 AND 4 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 2 2 4
TIE IN THE SECOND STAGE, BETWEEN 3  8 7 9.10 6.00 7.90 AND 3  9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 3 0 4
TIE IN THE SECOND STAGE, BETWEEN 3  9 7 9.10 6.00 7.90 AND 3 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 3 0 4
TIE IN THE SECOND STAGE, BETWEEN 3  7 7 9.10 6.00 7.90 AND 3  8 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 3 1 4
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TIE IN THE SECOND STAGE, BETWEEN 3 8 7 9.10 6.00 7.90 AND 3 9 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 3 1 4
TIE IN THE SECOND STAGE, BETWEEN 3 9 7 9.10 6.00 7.90 AND 3 10 6 9.10 5.90 8.00, GIVEN THAT THE STATE IS 3 1 4

VALUES OF THE FUNCTION F FOR ALL FEASIBLE STATES OF PERIOD 2, AND THE CORRESPONDENT OPTIMAL DECISIONS

I[2,1]	I[2,2]	I[2,3]	F[2,I[2,1],I[2,2],I[2,3]]	Q[2,1]	Q[2,2]	Q[2,3]	P[2,1]	P[2,2]	P[2,3]
0	0	0	36.60000	6	16	10	9.10	6.00	8.00
0	0	1	43.50000	6	16	9	9.10	6.00	8.00
0	0	2	50.40000	6	16	8	9.10	6.00	8.00
0	0	3	57.30000	6	16	7	9.10	6.00	8.00
0	0	4	63.97500	6	16	7	9.10	6.00	7.90
0	0	5	70.85000	6	16	5	9.10	6.00	8.00
0	0	6	77.80000	6	16	4	9.10	6.00	8.00
0	0	7	84.75000	6	16	3	9.10	6.00	8.00
0	0	8	91.70000	6	16	2	9.10	6.00	8.00
0	0	9	98.65000	6	16	1	9.10	6.00	8.00
0	0	10	107.60000	6	16	0	9.10	6.00	8.00
0	0	11	114.27500	6	16	0	9.10	6.00	7.90
0	0	12	117.82500	6	15	0	9.10	6.00	7.90
0	0	13	123.42500	6	14	0	9.10	6.00	7.90
0	1	0	41.45000	6	15	10	9.10	6.00	8.00
0	1	1	48.35000	6	15	9	9.10	6.00	8.00
0	1	2	55.25000	6	15	8	9.10	6.00	8.00
0	1	3	62.15000	6	15	7	9.10	6.00	8.00
0	1	4	68.82500	6	15	7	9.10	6.00	7.90
0	1	5	75.70000	6	15	5	9.10	6.00	8.00
0	1	6	82.65000	6	15	4	9.10	6.00	8.00
0	1	7	89.60000	6	15	3	9.10	6.00	8.00
0	1	8	96.55000	6	15	2	9.10	6.00	8.00
0	1	9	103.50000	6	15	1	9.10	6.00	8.00
0	1	10	112.45000	6	15	0	9.10	6.00	8.00
0	1	11	119.12500	6	15	0	9.10	6.00	7.90
0	2	0	46.30000	6	14	10	9.10	6.00	8.00
0	2	1	53.20000	6	14	9	9.10	6.00	8.00
0	2	2	60.10000	6	14	8	9.10	6.00	8.00
0	2	3	67.00000	6	14	7	9.10	6.00	8.00
0	2	4	73.67500	6	14	7	9.10	6.00	7.90
0	2	5	80.55000	6	14	5	9.10	6.00	8.00
0	2	6	87.50000	6	14	4	9.10	6.00	8.00
0	2	7	94.45000	6	14	3	9.10	6.00	8.00
0	2	8	101.40000	6	14	2	9.10	6.00	8.00
0	2	9	108.35000	6	14	1	9.10	6.00	8.00
0	2	10	117.30000	6	14	0	9.10	6.00	8.00
0	3	0	51.15000	6	13	10	9.10	6.00	8.00
0	3	1	58.05000	6	13	9	9.10	6.00	8.00
0	3	2	64.95000	6	13	8	9.10	6.00	8.00
0	3	3	71.85000	6	13	7	9.10	6.00	8.00
0	3	4	78.52500	6	13	7	9.10	6.00	7.90
0	3	5	85.40000	6	13	5	9.10	6.00	8.00
0	3	6	92.35000	6	13	4	9.10	6.00	8.00
0	3	7	99.30000	6	13	3	9.10	6.00	8.00
0	3	8	106.25000	6	13	2	9.10	6.00	8.00
0	4	0	56.00000	6	12	10	9.10	6.00	8.00
0	4	1	62.90000	6	12	9	9.10	6.00	8.00
0	4	2	69.80000	6	12	8	9.10	6.00	8.00
0	4	3	76.70000	6	12	7	9.10	6.00	8.00
0	4	4	83.37500	6	12	7	9.10	6.00	7.90
0	4	5	90.25000	6	12	5	9.10	6.00	8.00
0	4	6	97.20000	6	12	4	9.10	6.00	8.00
0	4	7	104.15000	6	12	3	9.10	6.00	8.00
0	5	0	60.85000	6	11	10	9.10	6.00	8.00
0	5	1	67.75000	6	11	9	9.10	6.00	8.00
0	5	2	74.65000	6	11	8	9.10	6.00	8.00
0	5	3	81.55000	6	11	7	9.10	6.00	8.00
0	5	4	88.22500	6	11	7	9.10	6.00	7.90
0	5	5	95.10000	6	11	5	9.10	6.00	8.00
0	6	0	65.20000	6	10	10	9.10	6.00	8.00
0	6	1	72.10000	6	10	9	9.10	6.00	8.00
0	6	2	79.00000	6	10	8	9.10	6.00	8.00
0	6	3	85.90000	6	10	7	9.10	6.00	8.00
0	6	4	92.57500	6	10	7	9.10	6.00	7.90
0	7	0	70.10000	6	9	10	9.10	6.00	8.00
0	7	1	77.00000	6	9	9	9.10	6.00	8.00
0	7	2	83.90000	6	9	8	9.10	6.00	8.00
0	8	0	75.00000	6	8	10	9.10	6.00	8.00
0	8	1	81.90000	6	8	9	9.10	6.00	8.00
1	0	0	43.95000	5	16	10	9.10	6.00	8.00
1	0	1	50.85000	5	16	9	9.10	6.00	8.00
1	0	2	57.75000	5	16	8	9.10	6.00	8.00
1	0	3	64.65000	5	16	7	9.10	6.00	8.00
1	0	4	71.32500	5	16	7	9.10	6.00	7.90
1	0	5	78.20000	5	16	5	9.10	6.00	8.00

1	0	6	85.15000	5	16	4	9.10	6.00	8.00
1	0	7	92.10000	5	16	3	9.10	6.00	8.00
1	0	8	99.05000	5	16	2	9.10	6.00	8.00
1	0	9	106.00000	5	16	1	9.10	6.00	8.00
1	0	10	114.95000	5	16	0	9.10	6.00	8.00
1	1	0	48.80000	5	15	10	9.10	6.00	8.00
1	1	1	55.70000	5	15	9	9.10	6.00	8.00
1	1	2	62.60000	5	15	8	9.10	6.00	8.00
1	1	3	69.50000	5	15	7	9.10	6.00	8.00
1	1	4	76.17500	5	15	7	9.10	6.00	7.90
1	1	5	83.05000	5	15	5	9.10	6.00	8.00
1	1	6	90.00000	5	15	4	9.10	6.00	8.00
1	1	7	96.95000	5	15	3	9.10	6.00	8.00
1	1	8	103.90000	5	15	2	9.10	6.00	8.00
1	1	9	110.85000	5	15	1	9.10	6.00	8.00
1	2	0	53.65000	5	14	10	9.10	6.00	8.00
1	2	1	60.55000	5	14	9	9.10	6.00	8.00
1	2	2	67.45000	5	14	8	9.10	6.00	8.00
1	2	3	74.35000	5	14	7	9.10	6.00	8.00
1	2	4	81.02500	5	14	7	9.10	6.00	7.90
1	2	5	87.90000	5	14	5	9.10	6.00	8.00
1	2	6	94.85000	5	14	4	9.10	6.00	8.00
1	2	7	101.80000	5	14	3	9.10	6.00	8.00
1	3	0	58.50000	5	13	10	9.10	6.00	8.00
1	3	1	65.40000	5	13	9	9.10	6.00	8.00
1	3	2	72.30000	5	13	8	9.10	6.00	8.00
1	3	3	79.20000	5	13	7	9.10	6.00	8.00
1	3	4	85.87500	5	13	7	9.10	6.00	7.90
1	3	5	92.75000	5	13	5	9.10	6.00	8.00
1	3	6	99.70000	5	13	4	9.10	6.00	8.00
1	4	0	63.35000	5	12	10	9.10	6.00	8.00
1	4	1	70.25000	5	12	9	9.10	6.00	8.00
1	4	2	77.15000	5	12	8	9.10	6.00	8.00
1	4	3	84.05000	5	12	7	9.10	6.00	8.00
1	4	4	90.72500	5	12	7	9.10	6.00	7.90
1	5	0	68.20000	5	11	10	9.10	6.00	8.00
1	5	1	75.10000	5	11	9	9.10	6.00	8.00
1	5	2	82.00000	5	11	8	9.10	6.00	8.00
1	5	3	88.90000	5	11	7	9.10	6.00	8.00
1	6	0	72.55000	5	10	10	9.10	6.00	8.00
1	6	1	79.45000	5	10	9	9.10	6.00	8.00
1	7	0	77.45000	5	9	10	9.10	6.00	8.00
2	0	0	51.90000	4	16	10	9.10	6.00	8.00
2	0	1	58.80000	4	16	9	9.10	6.00	8.00
2	0	2	65.70000	4	16	8	9.10	6.00	8.00
2	0	3	72.60000	4	16	7	9.10	6.00	8.00
2	0	4	79.27500	4	16	7	9.10	6.00	7.90
2	0	5	86.15000	4	16	5	9.10	6.00	8.00
2	0	6	93.10000	4	16	4	9.10	6.00	8.00
2	0	7	100.05000	4	16	3	9.10	6.00	8.00
2	0	8	107.00000	4	16	2	9.10	6.00	8.00
2	1	0	56.75000	4	15	10	9.10	6.00	6.00
2	1	1	63.65000	4	15	9	9.10	6.00	8.00
2	1	2	70.55000	4	15	8	9.10	6.00	8.00
2	1	3	77.45000	4	15	7	9.10	6.00	8.00
2	1	4	84.12500	4	15	7	9.10	6.00	7.90
2	1	5	91.00000	4	15	5	9.10	6.00	8.00
2	1	6	97.95000	4	15	4	9.10	6.00	8.00
2	2	0	61.60000	4	14	10	9.10	6.00	8.00
2	2	1	68.50000	4	14	9	9.10	6.00	8.00
2	2	2	75.40000	4	14	8	9.10	6.00	8.00
2	2	3	82.30000	4	14	7	9.10	6.00	8.00
2	2	4	88.97500	4	14	7	9.10	6.00	7.90
2	2	5	95.85000	4	14	5	9.10	6.00	8.00
2	3	0	66.45000	4	13	10	9.10	6.00	8.00
2	3	1	73.35000	4	13	9	9.10	6.00	8.00
2	3	2	80.25000	4	13	8	9.10	6.00	8.00
2	3	3	87.15000	4	13	7	9.10	6.00	8.00
2	4	0	71.30000	4	12	10	9.10	6.00	8.00
2	4	1	78.20000	4	12	9	9.10	6.00	8.00
2	4	2	85.10000	4	12	8	9.10	6.00	8.00
2	5	0	76.15000	4	11	10	9.10	6.00	8.00
3	0	0	59.85000	3	16	10	9.10	6.00	8.00
3	0	1	66.75000	3	16	9	9.10	6.00	8.00
3	0	2	73.65000	3	16	8	9.10	6.00	8.00
3	0	3	80.55000	3	16	7	9.10	6.00	8.00
3	0	4	87.22500	3	16	7	9.10	6.00	7.90
3	0	5	94.10000	3	16	5	9.10	6.00	8.00
3	1	0	64.70000	3	15	10	9.10	6.00	8.00
3	1	1	71.60000	3	15	9	9.10	6.00	8.00
3	1	2	78.50000	3	15	8	9.10	6.00	8.00
3	1	3	85.40000	3	15	7	9.10	6.00	8.00
3	1	4	92.07500	3	15	7	9.10	6.00	7.90

3	2	0	69.55000	3	14	10	9.10	6.00	8.00
3	2	1	76.45000	3	14	9	9.10	6.00	8.00
3	2	2	83.35000	3	14	8	9.10	6.00	8.00
3	3	0	74.40000	3	13	10	9.10	6.00	8.00
3	3	1	81.30000	3	13	9	9.10	6.00	8.00
4	0	0	67.80000	2	16	10	9.10	6.00	8.00
4	0	1	74.70000	2	16	9	9.10	6.00	8.00
4	0	2	81.60000	2	16	8	9.10	6.00	8.00
4	0	3	88.50000	2	16	7	9.10	6.00	8.00
4	1	0	72.65000	2	15	10	9.10	6.00	8.00
4	1	1	79.55000	2	15	9	9.10	6.00	8.00
4	2	0	77.50000	2	14	10	9.10	6.00	8.00
5	0	0	75.75000	1	16	10	9.10	6.00	8.00

THE OPTIMAL SOLUTION IS:

ORDER 0 UNITS OF ITEM 1 TO ARRIVE AT THE BEGINNING OF PERIOD 1
ORDER 0 UNITS OF ITEM 2 TO ARRIVE AT THE BEGINNING OF PERIOD 1
ORDER 13 UNITS OF ITEM 3 TO ARRIVE AT THE BEGINNING OF PERIOD 1
CHOOSE 8.70 DOLLARS AS THE SALE PRICE OF ITEM 1 IN PERIOD 1
CHOOSE 5.75 DOLLARS AS THE SALE PRICE OF ITEM 2 IN PERIOD 1
CHOOSE 7.35 DOLLARS AS THE SALE PRICE OF ITEM 3 IN PERIOD 1
ORDER 6 UNITS OF ITEM 1 TO ARRIVE AT THE BEGINNING OF PERIOD 2
ORDER 16 UNITS OF ITEM 2 TO ARRIVE AT THE BEGINNING OF PERIOD 2
ORDER 0 UNITS OF ITEM 3 TO ARRIVE AT THE BEGINNING OF PERIOD 2
CHOOSE 9.10 DOLLARS AS THE SALE PRICE OF ITEM 1 IN PERIOD 2
CHOOSE 6.00 DOLLARS AS THE SALE PRICE OF ITEM 2 IN PERIOD 2
CHOOSE 7.90 DOLLARS AS THE SALE PRICE OF ITEM 3 IN PERIOD 2
ORDER 6 UNITS OF ITEM 1 TO ARRIVE AT THE BEGINNING OF PERIOD 3
ORDER 0 UNITS OF ITEM 2 TO ARRIVE AT THE BEGINNING OF PERIOD 3
ORDER 11 UNITS OF ITEM 3 TO ARRIVE AT THE BEGINNING OF PERIOD 3
CHOOSE 9.50 DOLLARS AS THE SALE PRICE OF ITEM 1 IN PERIOD 3
CHOOSE 6.35 DOLLARS AS THE SALE PRICE OF ITEM 2 IN PERIOD 3
CHOOSE 8.20 DOLLARS AS THE SALE PRICE OF ITEM 3 IN PERIOD 3

THE MAXIMUM PROFIT OBTAINED BY THIS SOLUTION IS 213.27500 DOLLARS.

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