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STUDY OF NONEQUILIBRIUM TURBULENT WALL FLOWS
BY THE STATISTICAL METHOD

by

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SUMMARY

The model equation proposed by Lundgren for turbulent distribution functions based on the continuity and the Navier-Stokes equation is solved by the discrete ordinate method. The technique has been applied to turbulent channel flow and turbulent boundary-layer flow. The numerical results have been found to be in fairly good agreement with available experimental data. Of particular significance is the fact that the technique yields the turbulent distribution function from which the mean velocity, the root mean square velocity fluctuations, the Reynolds stress, the turbulent energy flux, and the local integral scale of the turbulence can be directly calculated.

INTRODUCTION

Since turbulent flows and their effects are encountered in nearly every flight case where fluid motion is involved, it becomes very important to know their behavior. Most of the existing methods of calculating turbulent boundary layer have been discussed and reviewed in the AFOSR-IFP-Stanford Conference[1].

The basic assumption made in most present-day methods of calculating the development of turbulent boundary layers is that the shear-stress profiles at a given distance from the origin of the boundary layer are uniquely related to the mean-flow conditions at that station. The simplest version of this assumption is the "mixing length" or "eddy viscosity" assumption that the shear stress at a point depends on the mean velocity gradient at that point[2]. It was argued by Bradshaw and Ferriss[3] that the shear stress $\tau_{xy} \equiv - \rho\, \overline{u'v'}$ is closely related to the turbulent kinetic energy $\frac{1}{2}\rho(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$ and that the latter, being governed by the turbulent energy equation, is certainly not determined uniquely by the local mean flow conditions (Here u', v', and w' are turbulent fluctuation velocity components in x, y, and z directions, respectively). This argument is certainly not new; one of the first to propose it was Dryden[4]. The poor performance of the above mentioned calculation method in practice lends support to the view that assumption of a close relation between the shear-stress profile and the mean velocity profile is not a realistic one for boundary layers in arbitrary pressure gradients, because it ignores the effect of the past history of the boundary layer, and that it is therefore

unsuitable as the basis fo the empirical correlations which must inevitably be used in any method of calculating turbulent flows.

One method suggested by Bradshaw, Ferriss and Atwell[5] is that the turbulent energy equation is converted into a differential equation for the turbulent shear stress by defining three empirical functions relating the turbulent intensity, diffusion and dissipation to the shear stress profile. This equation, the mean momentum equation and the mean continuity equation form a hyperbolic system which can be solved by the method of characteristics with preliminary choices of the three empirical functions. This method yields numerical solutions which compare favorably with results of conventional calculation methods over a wide range of pressure gradients, however this method relies heavily on the information of three empirical functions which have to be known a priori.

In short, most of the existing methods of calculating the development of turbulent boundary layers have to rely heavily upon empiricism and phenomenological arguments guided by experimental data. Because of the empiricism, the generalization of these methods is severely limited.

A system of equations for turbulent distribution functions was derived by Lundgren from the continuity and the Navier-Stokes equation[6]. The hierarchy of equations was closed at the one-point level by approximating the pressure term by a term of the Boltzmann type[7]. The model equation which is quite similar to the Boltzmann equation in kinetic theory can be solved accurately by the discrete ordinate method[10-13]. This method has successfully been used by the author and his co-workers in solving the Boltzmann type equations for rarefied gasdynamic and nonequilibrium plasma flow problems. Therefore, it will be proposed to investigate several nonhomogeneous turbulent flow problems using the model

equation as the fundamental governing relation and the discrete ordinate method as a tool. In fact, the model equation was solved by Lundgren[7] for a class of homogeneous rectilinear flows and for decay of a periodic wake. Good agreement with available experimental data was found[7]. It will be seen that the advantages of the proposed technique are: (1) the technique yields turbulent distribution functions from which the mean velocity, the root mean square velocity fluctuations, the Reynolds stress, the turbulent energy flux, and the local integral scale of turbulence can be directly calculated without making any assumption or modeling, (2) the accurate solution of the model equation can be obtained using the discrete ordinate method, and (3) the degree of roughness of the wall can be simulated accordingly through the boundary conditions used in the present technique.

THE MODEL EQUATION

A system of equations for turbulent distribution functions was derived by Lundgren from the continuity and the Navier-Stokes equation[6]. The hierarchy of equations was closed at the one-point level by approximating the pressure term by a term of the Boltzmann type[7]. Following Lundgren[2], the model equation takes the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \left(-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial \vec{r}} + \nu \frac{\partial}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{r}} \vec{u} \right) \cdot \frac{\partial f}{\partial \vec{v}}$$

$$= \frac{1}{\tau} (F - f) + \frac{1}{3} \frac{\epsilon}{U^2} \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} - \vec{u}) f \qquad (1)$$

where $f(\vec{r}, \vec{v}, t) d\vec{v}$ is the probability that the velocity of fluid element at point $\vec{r}$ and time t is in the range $\vec{v}$, $\vec{v} + d\vec{v}$, $\bar{p}$ is the mean pressure, and $\vec{u}$ is the mean velocity. Here, the pressure term has been approximated by a relaxation term (the so-called Bhatnagar-Gross-Krook model in kinetic theory[8])

$$\frac{1}{\tau} (F - f) \qquad (2)$$

where τ is a relaxation time and

$$F = \frac{1}{(2_\pi U^2)^{3/2}} e^{-(\vec{v}-\vec{u})^2/2U^2} \qquad (3)$$

is a Gaussian distribution with $\vec{u}\,(\vec{r},\,t)$ and $U\,(\vec{r},\,t)$ the mean velocity and the root mean square velocity fluctuation, respectively. The relaxation time is taken to be

$$\frac{1}{\tau} = \frac{K\left(\epsilon + \frac{3}{2}\frac{DU^2}{Dt}\right)}{U^2} \tag{4}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u}\cdot\frac{\partial}{\partial\vec{r}}$$

where K is a constant that can be adjusted for the model, and ϵ is the turbulent dissipation rate.

Equation (1) is quite similar to the Boltzmann equation in kinetic theory. The first and second terms on the left-hand side are the unsteady term and the convective term, respectively; the third term is the acceleration term due to the pressure gradient; the fourth term is the acceleration term due to the viscous effect. The first term in the right-hand side is the pressure fluctuation term which is mainly responsible for describing the nonequilibrium phenomenon and which has been approximated by the BGK model. This is in agreement with the intuitive feeling that pressure fluctuations have a randomizing effect. The last term will yield the structure of turbulence.

After Equation (1) is solved for distribution functions $f(\vec{r},\,\vec{v},\,t)$, the flow properties can be obtained by taking the moments of distribution functions as follows:

$$\vec{u}(\vec{r},t) = \int \vec{v} \, f \, (\vec{r}, \vec{v}, \, t) \, d\vec{v} \tag{5a}$$

$$3U^2(\vec{r},t) = \int (\vec{v} - \vec{u})^2 \, f(\vec{r},\vec{v},t) \, d\vec{v} \tag{5b}$$

pressure tensor:

$$P_{ij} = \int (v_i - u_i)(v_j - u_j) \, f(\vec{r}, \vec{v}, \, t) \, d\vec{v} \tag{5c}$$

turbulent energy flux:

$$q_i = \frac{1}{2} \int (v_i - u_i)(\vec{v} - \vec{u})^2 \, f(\vec{r}, \vec{v}, \, t) \, d\vec{v} \tag{5d}$$

The quantity $- \rho P_{ij}$ is the Reynolds stress and ρq_i is turbulent energy flux.

To complete the model equation in Equation (1), the relaxation time τ must be specified. To this end, the well-known Chapman-Enskog method[9] is applied to Equation (1). This application leads to

$$P_{ij} = U^2 \delta_{ij} - 2\nu_T D_{ij} \tag{6}$$

where ν_T is the eddy viscosity coefficient which is

$$\nu_T = \frac{U^4}{K \left(\epsilon + \frac{3}{2} \frac{DU^2}{Dt} \right)} = U^2 \tau \tag{7}$$

and D_{ij} is the rate of strain tensor for the mean flow,

$$D_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$ (8)

The application also leads to

$$q_i = - k_T \frac{\partial}{\partial x_i} \frac{3}{2} U^2$$ (9)

where k_T is the energy flux coefficient.

Thus, combining Equations (6), (7), and (8) yields

$$\frac{1}{\tau} = \frac{U^2 \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}{U^2 \delta_{ij} - P_{ij}} = \frac{KU}{L}$$ (10)

where τ is assumed to be proportional to L/U, L is the integral scale of the turbulence, and K is a constant which can be adjusted to fit with experimental data.

The model equation was solved by Lundgren[7] for a class of homogeneous rectilinear flows and for decay of a periodic wake. Good agreement with available experiments was found[7]. Further test must be made for non-homogeneous turbulent flows in order to prove the applicability of the model equation for practical flow problems.

In the following, the model equation is used to solve the one-dimensional plane Couette flow and channel flow, and the two-dimensional boundary-layer flow.

NONHOMOGENEOUS COUETTE AND CHANNEL FLOW

A. GOVERNING EQUATION

Consider the steady turbulent flow between two parallel plates, $y = -d$ and $y = d$. For the Couette flow case the plate $y = -d$ is translating in its own plane to the left with a constant speed u_w and the plate $y = d$ to the right with the same speed. For the channel flow case both plates are stationary.

For these one-dimensional steady turbulent flows, Equation (1) becomes

$$v_y \frac{\partial f}{\partial y} - \bar{K} \cdot \frac{\partial f}{\partial v_x} + \nu \frac{\partial^2 u}{\partial y^2} \cdot \frac{\partial f}{\partial v_x} = -\frac{KU}{L}(f - F) + \frac{U}{3L} \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} - \vec{u})\, f \qquad (11)$$

The instantaneous velocity $\vec{v}$ has components v_x, v_y and v_z, and solutions for the distribution function $f(y, \vec{v})$ are sought. $\bar{K} \equiv 1/\rho \cdot \partial p/\partial x$, which is the pressure gradient in the x-direction. For the channel flow, $\bar{K} =$ constant, which is the main driving force of the flow; for the Couette flow case, $\bar{K} = 0$.

B. DISCRETIZATION OF DISTRIBUTION FUNCTIONS

The distribution function $f(y, \vec{v})$ in Equation (11) is one-dimensional in physical space but three-dimensional in the instantaneous velocity space. In order to reduce the computer storage requirement the following reduced distribution functions are defined:

$$g(y;v_y) = \int_{-\infty}^{\infty}\!\!\int f(y;\vec{v})\ dv_x dv_z \tag{12a}$$

$$j(y;v_y) = \int_{-\infty}^{\infty}\!\!\int v_x f(y;\vec{v})\ dv_x dv_z \tag{12b}$$

$$h(y;v_y) = \int_{-\infty}^{\infty}\!\!\int (v_x^2 + v_z^2)\ f(y;\vec{v})\ dv_x dv_z \tag{12c}$$

If Equation (11) is multiplied by unity and integrated over $dv_x dv_z$, one obtains

$$v_y \frac{\partial g}{\partial y} = -\frac{KU}{L}\left[(g - G) - \frac{1}{3K}\ (g + v_y \frac{\partial g}{\partial v_y})\right] \tag{13a}$$

If the multiplying factor is, respectively, v_x and $v_x^2 + v_z^2$ and the integration then performed, the following equations are obtained:

$$v_y \frac{\partial j}{\partial y} + \bar{K}g - \nu \frac{\partial^2 u}{\partial y^2}\ g = -\frac{KU}{L}\left[(j - J) - \frac{1}{3K}\ (ug + v_y \frac{\partial j}{\partial v_y})\right] \tag{13b}$$

$$v_y \frac{\partial h}{\partial y} + 2\bar{K}j - 2\nu \frac{\partial^2 u}{\partial y^2}\ j = -\frac{KU}{L}\left[(h - H) - \frac{1}{3K}\ (- h + 2uj + v_y \frac{\partial h}{\partial v_y})\right] \tag{13c}$$

The local equilibrium reduced distribution functions are

$$G = \frac{1}{(2\pi U^2)^{1/2}} \exp(-v_y^2/2U^2) \tag{14a}$$

$$H = 2U^2 G + u^2 G \tag{14b}$$

$$J = uG \tag{14c}$$

The equations are nondimensionalized by the following scheme:

1. **Couette flow:**

$$\hat{y} = \frac{y}{d}; \quad \hat{u} = \frac{u}{u_w}; \quad \hat{v} = \frac{v}{u_w}; \quad \hat{L} = \frac{L}{d}; \quad \hat{g} = gu_w;$$

$$\hat{j} = j; \quad \hat{h} = \frac{h}{u_w}; \quad \hat{U} = \frac{U}{u_w}; \quad \hat{P}_{xy} = \frac{P_{xy}}{u_w^2}; \quad R_e = \frac{u_w d}{\nu}$$

where u_w is the flow velocity at the wall and R_e is Reynolds number.

2. **Channel flow:**

$$\hat{y} = \frac{y}{d}; \quad \hat{u} = \frac{u}{u_o}; \quad \hat{v} = \frac{v}{u_o}; \quad \hat{L} = \frac{L}{d}; \quad \hat{g} = gu_o;$$

$$\hat{j} = j; \quad \hat{h} = \frac{h}{u_o}; \quad \hat{U} = \frac{U}{u_o}; \quad \hat{P}_{xy} = \frac{P_{xy}}{u_o^2}; \quad R_e = \frac{u_o d}{\nu};$$

$$\hat{K} = \frac{d}{\rho u_o^2} \frac{d\bar{p}}{dx}$$

where u_o is the flow velocity at $\hat{y} = 0$.

This gives the equations

$$\hat{v}_y = \frac{\delta\hat{g}}{\delta\hat{v}} = -\frac{\hat{KU}}{\hat{L}}\left[(\hat{g} - \hat{G}) - \frac{1}{3K}(\hat{g} + \hat{v}_y \frac{\delta\hat{g}}{\delta\hat{v}_y})\right]$$ (15a)

$$\hat{v}_y \frac{\delta\hat{J}}{\delta\hat{v}} + \hat{K}\hat{g} - \frac{1}{R_e}\frac{\delta^2\hat{u}}{\delta\hat{y}^2}\hat{g} = -\frac{\hat{KU}}{\hat{L}}\left[(\hat{J} - \hat{J}) - \frac{1}{3K}(\hat{u}\hat{g} + \hat{v}_y \frac{\delta\hat{J}}{\delta\hat{v}_y})\right]$$ (15b)

$$\hat{v}_y \frac{\delta\hat{h}}{\delta\hat{v}} + 2\hat{K}\hat{J} - \frac{2}{R_e}\frac{\delta^2\hat{u}}{\delta\hat{y}^2}\hat{J} = -\frac{\hat{KU}}{\hat{L}}\left[(\hat{h} - \hat{H}) - \frac{1}{3K}(-\hat{h} + 2\hat{u}\hat{J} + \hat{v}_y \frac{\delta\hat{h}}{\delta\hat{v}_y})\right]$$ (15c)

$$\hat{G} = \frac{1}{(2\pi\hat{u}^2)^{1/2}}\exp(-\hat{v}_y^2/2\hat{u}^2)$$ (16a)

$$\hat{J} = \hat{u}\hat{G}$$ (16b)

$$\hat{H} = 2\hat{U}^2\hat{G} + \hat{u}^2\hat{G}$$ (16c)

where

$$\hat{u} = \int_{-\infty}^{\infty}\hat{J}\,d\hat{v}_y$$ (17a)

$$\hat{u}^2 = \frac{1}{3}\left[\int_{-\infty}^{\infty}\hat{h}\,d\hat{v}_y + \int_{-\infty}^{\infty}\hat{v}_y^2\hat{g}\,d\hat{v}_y - \hat{u}^2\right]$$ (17b)

$$\hat{P}_{xy} = \int_{-\infty}^{\infty}\hat{J}\hat{v}_y\,d\hat{v}_y - \hat{u}\int_{-\infty}^{\infty}\hat{g}\hat{v}_y\,d\hat{v}_y$$ (17c)

Equations (15) are nonlinear integro-differential equation to be solved for $\hat{g}$, $\hat{J}$, and $\hat{h}$. The macroscopic flow properties can be calculated through Equations (17). It is noted that in Equations (15), $\hat{K} = 0$, for the Couette flow case.

The discrete ordinate method which has been used by the author in solving the Boltzmann equation for rarefied gasdynamic problems[5-8] is now applied to Equations (15). The continuous dependency of the $\hat{g}$, $\hat{j}$, and $\hat{h}$ functions in instantaneous velocity space $\hat{v}_y$ is replaced by a point function dependency; that is, the function $\hat{g}(\hat{y}, \hat{v}_y)$ which is continuous in velocity space is replaced by N functions $\hat{g}_i(\hat{y}, \hat{v}_y = \hat{v}_i)$ which are continuous in physical space but are point functions in velocity space. Thus, the partial differential equations are approximated by systems of N ordinary differential equations which are then solved numerically using an iterative scheme. The macroscopic flow properties are determined from Equations (17) using the same quadrature which is being used in solving Equations (15). Thus, one has

$$\hat{u} = \int_{-\infty}^{\infty} \hat{j} \, d\hat{v}_y \cong \sum_{i=1}^{N} w_i \hat{j}_i \tag{18a}$$

$$\hat{u}^2 = \frac{1}{3}\left[\sum_{i=1}^{N} w_i \hat{h}_i + \sum_{i=1}^{N} \hat{v}_i^2 \hat{g}_i w_i - \hat{u}^2 \right] \tag{18b}$$

$$\hat{P}_{xy} = \sum_{i=1}^{N} w_i \hat{v}_i \hat{j}_i - \hat{u} \sum_{i=1}^{N} w_i \hat{v}_i \hat{g}_i \tag{18c}$$

where the w_i are the weighting coefficients corresponding to the quadrature being used.

C. THE BOUNDARY CONDITIONS

Since it is important to distinguish particles traveling toward a plate from those moving away from it within a distance of the order of a micro-scale of turbulence from the plate, it is convenient to define the half-range distribution function as follows:

$$\hat{f} = \hat{f}^+ + \hat{f}^-$$ (19)

where

$$\hat{f}^+(\hat{y}; \hat{v}_y) = 0, \text{ for } \hat{v}_y < 0$$

$$\hat{f}^-(\hat{y}; \hat{v}_y) = 0, \text{ for } \hat{v}_y > 0$$

The completely diffuse reflection is assumed here for the particle-surface interaction. Thus, the normalized boundary conditions for Equations (15) can be written as follows:

1. Couette flow:

$$\hat{g}_i^+ (\hat{y} = -1) = \lim_{\Delta\hat{U}\to 0} \frac{1}{\sqrt{2\pi}\,(\Delta\hat{U})} e^{-\hat{v}_i^2/2(\Delta\hat{U})^2}$$

$$= \hat{g}_i^- (\hat{y} = 1)$$ (20a)

$$\hat{J}_i^+ (\hat{y} = -1) = - \hat{g}_i^+ (\hat{y} = -1)$$ (20b)

$$\hat{J}_i^- (\hat{y} = 1) = \hat{g}_i^- (\hat{y} = 1)$$ (20c)

$$\hat{h}_i^+ (\hat{y} = -1) = \lim_{\Delta\hat{U}\to 0} \sqrt{\frac{2}{\pi}}\,\hat{\Delta U}\, e^{-\hat{v}_i^2/2(\Delta\hat{U})^2} + \hat{g}_i^+ (y = -1)$$

$$= \hat{h}_i^- (\hat{y} = 1)$$ (20d)

2. <u>Channel flow:</u>

$$\hat{g}_i^+ (\hat{y} = -1) = \lim_{\Delta\hat{U}\to 0} \frac{1}{\sqrt{2\pi}\,(\Delta\hat{U})} e^{-\hat{v}_i^2/2(\Delta\hat{U})^2}$$

$$= \hat{g}_i^- (\hat{y} = 1) \tag{21a}$$

$$\hat{\jmath}_i^+ (\hat{y} = -1) = \hat{\jmath}_i^- (\hat{y} = 1) = 0 \tag{21b}$$

$$\hat{h}_i^+ (\hat{y} = -1) = \lim_{\Delta\hat{U}\to 0} \sqrt{\frac{2}{\pi}}\,\Delta\hat{U}\, e^{-\hat{v}_i^2/2(\Delta\hat{U})^2}$$

$$= \hat{h}_i^- (\hat{y} = 1) \tag{21c}$$

Thus, the problem leads to solving the discrete ordinate forms of Equations (15) subject to Equations (20) or Equations (21) for the Couette flow or channel flow, respectively.

D. NUMERICAL PROCEDURE

The discrete ordinate forms of Equations (15) are approximated by forward differences in the numerical solution and an initial guess is made for the profiles of the macroscopic properties, $\hat{u}$, $\hat{U}$, and $\hat{L}$. This initial guess is used to evaluate $\hat{G}_i(\hat{y})$, $\hat{H}_i(\hat{y})$, and $\hat{J}_i(\hat{y})$ through the channel. Equations (15) are solved numerically for values of $\hat{g}_i(\hat{y})$, $\hat{j}_i(\hat{y})$, and $\hat{h}_i(\hat{y})$. These are then used to calculate new macroscopic flow properties using Equations (18). Unless the initial guess has been extremely clever, the new values will differ from the first ones. Thus, hopefully better approximations to $\hat{G}_i$, $\hat{J}_i$, and $\hat{H}_i$ are now computed and the iterative process is repeated. Convergence was assumed to have occured when the differences in macroscopic properties at every physical station between successive iterations was less than 0.001.

The selection of the discrete velocities $\hat{v}_i$ at which the distribution functions are evaluated must be such that (i) the spacing between the $\hat{v}_i$'s must be close enough to accurately define the highly peaked distribution functions at and near the boundaries and (ii) the range of discrete points must be sufficient to cover the spread of the distribution functions across the channel. It is noted that the normalized distribution functions at the boundaries are delta functions (because $\hat{\Delta U} = 0$ at the boundaries). Numerically, it is difficult to start with the delta functions. Therefore, in the calculations two different values were used for $\hat{\Delta U}$ at the boundaries, i.e., $\hat{\Delta U} = 0.0001$ and $\hat{\Delta U} = 0.0001$. On several test cases no graphical distinction could be observed between results obtained using 0.0001 and those obtained using 0.0001. For this reason, the profiles reported here

are based on this latter figure used for the root mean square fluctuation
at the boundaries. The constant K in Equations (15) is taken to be 5 in
the present calculations.

E. RESULTS

Calculations for both the Couette flow and channel flow were performed at the flow conditions for which experimental data were available. Figures 1 and 2 present the flow velocity profiles for the Couette flow case at three different Reynolds numbers (2900, 18000 and 36000). The experimental data of Reichardt[14] are also shown for comparisons. It should be mentioned that all the results presented here are those obtained after nine iterations which took about 56 seconds on the UNIVAC 1108 machine of the Rich's Electronic Computer Center, Georgia Institute of Technology. It is seen that the comparison between the calculated results and experimental data is quite good. Figure 3 presents the calculated mean square velocity fluctuation and the normalized Reynolds stress. Since there is no existing experimental data to be compared, the results are compared with the results of Chung's theory[15].

Figure 4 presents reduced distribution functions of the Couette flow at five different locations in the flow field. Since root mean square fluctuation velocities correspond to mean widthes of distribution functions, it is seen that the rms fluctuation velocity has a maximum at the location which is very close to the wall ($\hat{y} \cong 0.98$) and its magnitude decreases when the distance from the wall increases. It is also noted that the distribution function is not Maxwellian near the wall (see the distribution function at $\hat{y}$ = 0.98) and becomes more and more Maxwellian when the distance from the wall gets further.

Figure 5 shows a comparison of the calculated mean flow velocity, the y-component fluctuation velocity, and the root-mean-square velocity

fluctuation for the channel flow with the experimental data of Laufer[16].
The comparison is seen to be quite satisfactory. Again, all the calculated
results presented here are obtained after nine iterations which took about
58 seconds on the UNIVAC 1108 machine. In these calculations K has been
taken to be 5.

TWO-DIMENSIONAL TURBULENT BOUNDARY-LAYER FLOW

For steady two-dimensional turbulent flows, Equation (1) may be written as

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + \left[-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial f}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial f}{\partial v_x} \right]$$

$$= \frac{1}{\tau} (F - f) + \frac{1}{3\tau K} \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} - \vec{u}) f \tag{22}$$

where

$$\frac{1}{\tau} = \frac{U^2}{\nu_T} = -\frac{U^2 \frac{\partial u_x}{\partial y}}{P_{xy}} = \frac{KU}{L}$$

and the Reynolds stress

$$\tau_{xy} = -\rho P_{xy}$$

The particle velocity $\vec{v}$ has components v_x, v_y, and v_z, and the flow velocity $\vec{u}$ has components u_x, u_y, and u_z ($u_z = 0$ for the two-dimensional case considered and u_y is small except for the case of very adverse pressure gradient), and $\partial \bar{p}/\partial x$ is the pressure gradient in the x-direction. Solutions for the distribution function $f(x, y; \vec{v})$ are sought.

The distribution function $f(x, y; \vec{v})$ in Equation (22) is two-dimensional in physical space but three-dimensional in the particle velocity space. In order to reduce the computer storage requirement the following reduced distribution functions are defined:

$$g(x,\ y;\ v_x,\ v_y) = \int_{-\infty}^{\infty} f(x,\ y;\ \vec{v})\ dv_z \qquad (23a)$$

$$h(x,\ y;\ v_x,\ v_y) = \int_{-\infty}^{\infty} v_z^{\ 2} f(x,\ y;\ \vec{v})\ dv_z \qquad (23b)$$

Multiplying Equation (22) by 1 and integrating with respect to v_z yields

$$v_x \frac{\partial g}{\partial x} + v_y \frac{\partial g}{\partial y} + \left[-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial g}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial g}{\partial v_x} \right]$$

$$= -\frac{1}{\tau}(g - G) + \frac{1}{3}\frac{1}{\tau K} \left[2g + (v_x - u_x) \frac{\partial g}{\partial v_x} + (v_y - u_y) \frac{\partial g}{\partial v_y} \right] \qquad (24a)$$

Multiplying Equation (11) by v_z and integrating with respect to v_z yields

$$v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} + \left[-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial h}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial h}{\partial v_x} \right]$$

$$= -\frac{1}{\tau}(h - H) + \frac{1}{3}\frac{1}{\tau K} \left[(v_x - u_x) \frac{\partial h}{\partial v_x} + (v_y - u_y) \frac{\partial h}{\partial v_y} \right] \qquad (24b)$$

The local equilibrium reduced distribution functions are

$$G = \frac{1}{2\pi U^2} \exp \left\{ -\frac{1}{2U^2} \left[(v_x - u_x)^2 + (v_y - u_y)^2 \right] \right\} \qquad (25a)$$

and

$$H = U^2 G \qquad (25b)$$

After the reduced distribution functions g and h are solved from Equations (24) subjected to the appropriate boundary conditions, the macroscopic properties can be found by taking the moments of g and h.

$$u_x(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x \, g \, dv_x dv_y \tag{26a}$$

$$u_y(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y \, g \, dv_x dv_y \tag{26b}$$

$$3U^2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h \, dv_x dv_y + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(v_x - u_x)^2 + (v_y - u_y)^2 \right] g \, dv_x dv_y \tag{26c}$$

$$P_{ij} = - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x v_y g \, dv_x dv_y + u_x u_y \tag{26d}$$

$$q_i = \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y v_x^2 g \, dv_x dv_y - u_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x v_y g \, dv_x dv_y$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y^3 g \, dv_x dv_y - u_y \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y^2 g \, dv_x dv_y$$

$$+ \frac{1}{2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y h \, dv_x dv_y - \frac{3}{2} U^2 u_y + \frac{1}{2} u_y (u_x^2 + u_y^2) \tag{26e}$$

The accurate numerical solution of Equations (24) has been obtained through the discrete ordinate method similar to that outlined in the previous section.

Figure 6 presents the calculated mean velocity profile of Klebanoff's boundary layer[17] (zero pressure gradient). Klebanoff's experimental data[17] are also shown for comparisons. It is seen that the comparisons are quite satisfactory.

CONCLUDING REMARKS

A statistical method has been described for the determination of nonhomogeneous turbulent flows. The method has been for the first time applied to calculate the one-dimensional Couette flow and channel flow and the two-dimensional boundary-layer flow. The results as compared with available experimental data indicate that the technique is quite powerful in view of the fact that the computational time needed for obtaining reasonable solutions is quite short.

It is interesting to note that, in contrast to the conventional moment equation approach, the method yields turbulent distribution functions from which the mean velocity, the root mean square fluctuation velocity, the Reynolds stress, and the turbulent energy flux can be directly calculated without making any assumption or modeling.

The most important assumption in the model equation is that pressure fluctuations have a randomizing effect, or more specifically that they drive the distribution function toward an isotropic Gaussian distribution.

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APPENDIX

BOUNDARY CONDITIONS AND NUMERICAL CALCULATIONS

Boundary conditions are applied for g^+ at $\hat{y} = 0$ and for g^- at $\hat{y} = 2$. As $\hat{v} = 0$ at both the walls, the Maxwellian distribution function $\hat{g}$ takes the form of dirac Delta function. $\hat{g}^+$ and $\hat{g}^-$ were assumed to be zero at $\hat{y} = 0$ and $\hat{y} = 2$, respectively, except at $\hat{v}_y = 0$.

Another set of boundary conditions tried was to assume Maxwellian distribution for $\hat{g}$ at the edge of the laminar sublayer. This avoids the difficulty of large gradients at the walls but requires an assumption of the values of $\hat{u}$ and $\hat{v}$ at these points where the boundary conditions are applied.

Boundary conditions on J^+ and h^+ (at $\hat{y} = 0$ or at the edge of the laminar layer near $\hat{y} = 0$) and on J^- and h^- (at $\hat{y} = 2$ or at the edge of the laminar layer near $\hat{y} = 2$) were

$$\hat{J} = \hat{J} = \hat{u}\hat{g}$$

$$\hat{h} = \hat{H} = (\hat{u}^2 + 2\hat{U}^2)\,\hat{G}$$

$$\hat{u} = -1 \quad \text{at} \quad \hat{y} = 0$$

$$\hat{u} = +1 \quad \text{at} \quad \hat{y} = 2$$

Once the system of o.d.e.'s was solved, the distribution functions were used to get the following macroscopic moments at all points in the physical space

(1) Density $= \int_{-\infty}^{\infty} \hat{g} \, d\hat{v}_y = 1.0$

(2) $\hat{u}_y = \int_{-\infty}^{\infty} \hat{v}_y \, \hat{g} \, d\hat{v}_y = 0$

(3) $\overline{\hat{u}_y^{12}} = \int_{-\infty}^{\infty} \hat{v}_y^2 \, \hat{g} \, d\hat{v}_y$

(4) $\hat{u}_x = \int_{-\infty}^{\infty} \hat{\jmath} \, d\hat{v}_y$

(5) $\hat{P}_{xy} = \int_{-\infty}^{\infty} \hat{v}_y \, \hat{\jmath} \, d\hat{v}_y$ shear stress

(6) $\hat{U}^2 = \frac{1}{3} \left[\int_{-\infty}^{\infty} \hat{h} \, d\hat{v}_y + \int_{-\infty}^{\infty} \hat{v}_y^2 \, \hat{g} \, d\hat{v}_y - \hat{u}^2 \right]$

(7) $Q_y = \frac{1}{2} \left[\int_{-\infty}^{\infty} \hat{v}_y \, \hat{h} \, d\hat{v}_y + \int_{-\infty}^{\infty} \hat{v}_y^3 \, \hat{g} \, d\hat{v}_y - 2\hat{u}_x \, \hat{P}_{xy} \right]$ Turbulent energy flux in y direction.

Simpson's rule was used to evaluate the integrals above.

Out of the seven moments calculated above only u_x and U^2 are utilized in the subsequent iterations. $\overline{\hat{u}_y^{12}}$ is used to calculate $\hat{U}^2$. The rest of them are not used directly in the calculations. $\overline{\hat{u}_y^{12}}$, $\hat{P}_{xy}$, $\hat{Q}_y$, $\hat{u}_x$ and $\hat{U}^2$ are useful results though $\hat{P}_{xy}$ and $\hat{Q}_y$ are not used in the calculations.

$$\int_{-\infty}^{\infty} \hat{g} \, dv_y = 1.0$$

$$\int_{-\infty}^{\infty} \hat{v}_y \, g \, dv_y = 0$$

These two moments serve as two usefule checks. The first equation above is the continuity equation. Both of these equations were satisfied well, and

there was an improvement as more and more iterations were performed. The error in the continuity equation was about 0.5% near the wall and much less than that away from the wall. The error in the second equation cannot be expressed in the percentage basis. This equation verifies that the macroscopic velocity in the y-direction is zero. The calculated macroscopic velocity in the $\hat{y}$-direction, $\hat{u}_y$, was found to be less than 1% of fluctuation velocity $\sqrt{u_y^{'2}}$ in the same direction at all points except for a couple of points near the wall. At these points it was about 5% - 10% of the local fluctuation velocity $\sqrt{\hat{u}_y^2 1}$ which itself is very small.
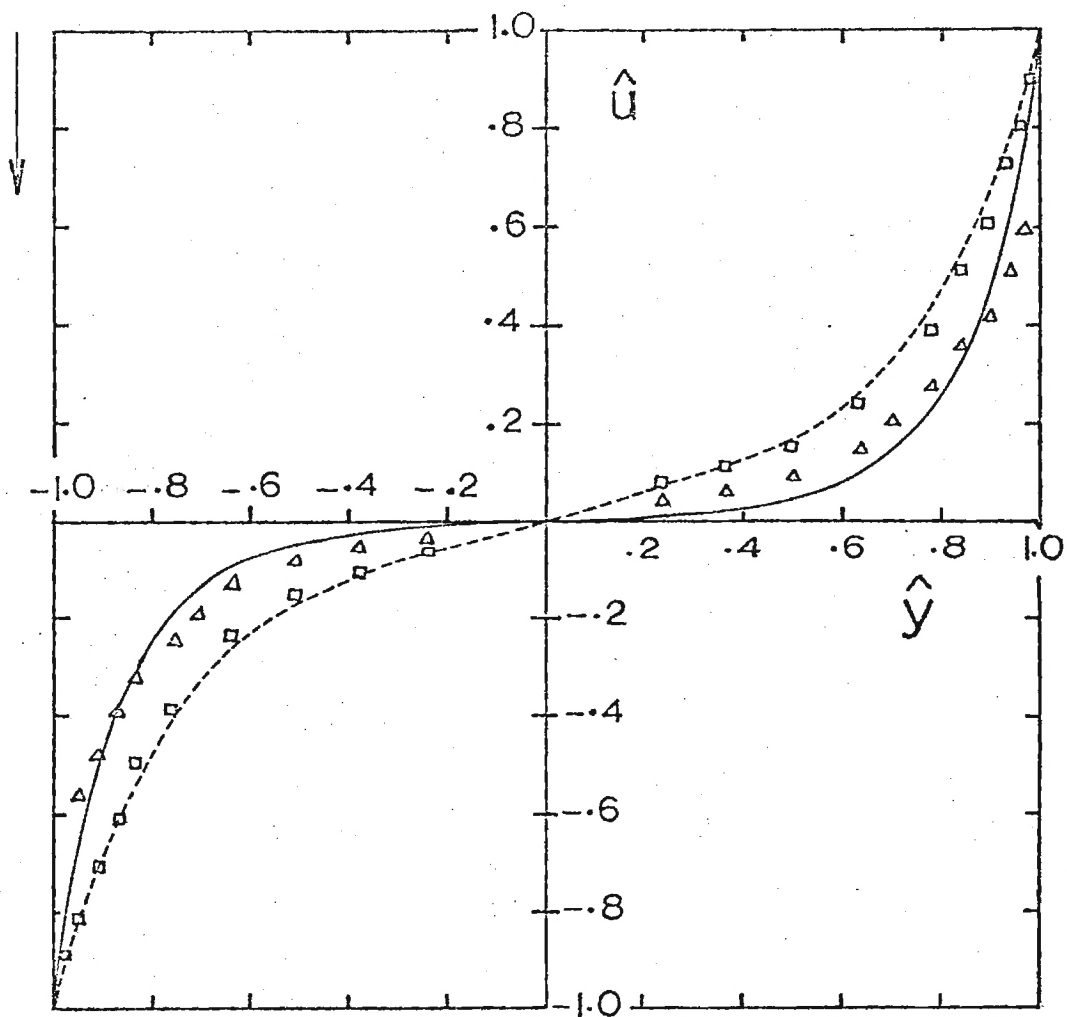
——— Re=18000	} PRESENT RESULTS
-------- Re=2900	
△ Re=18000	} EXPERIMENT
▢ Re=2900	(REICHARDT)

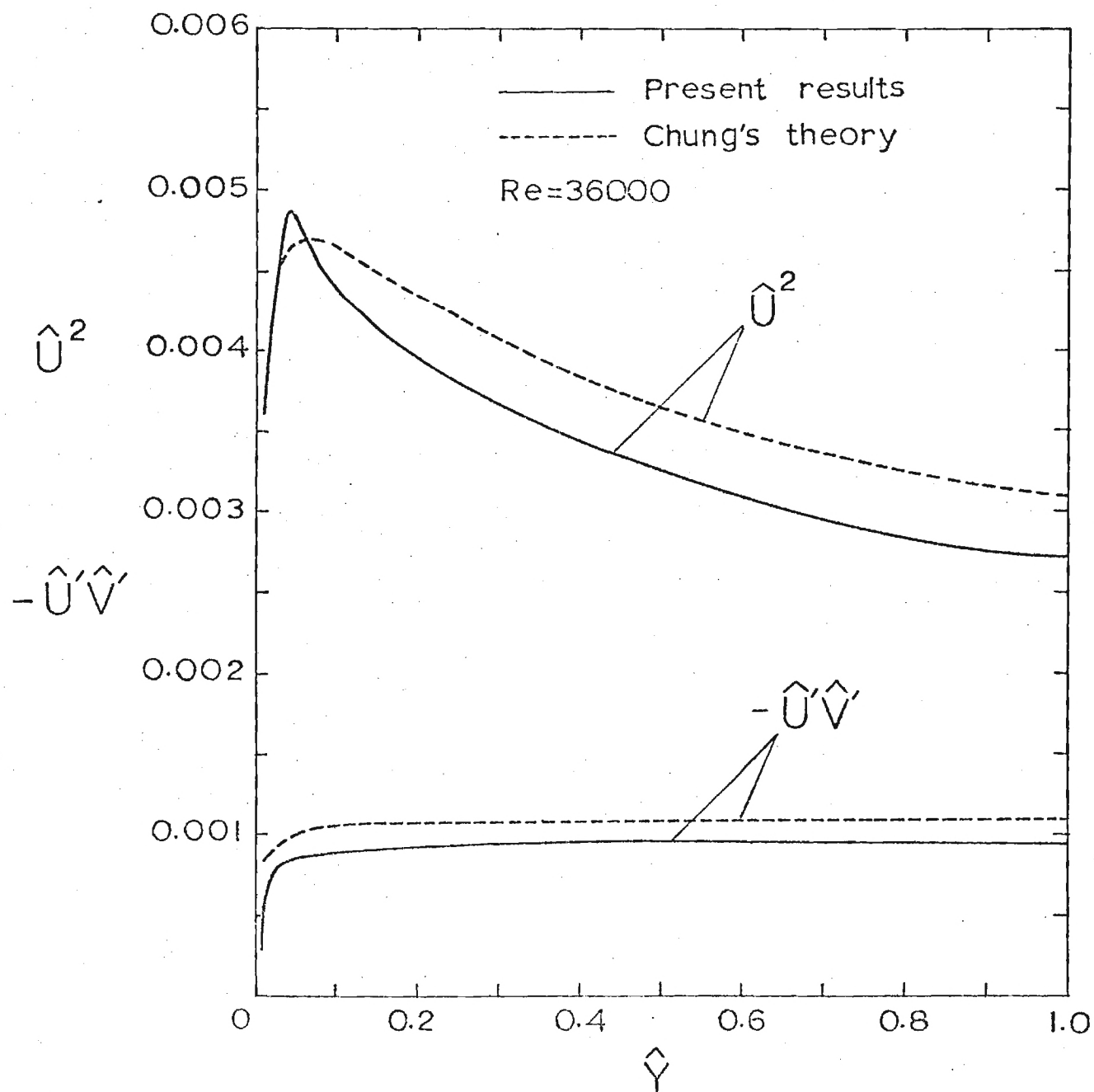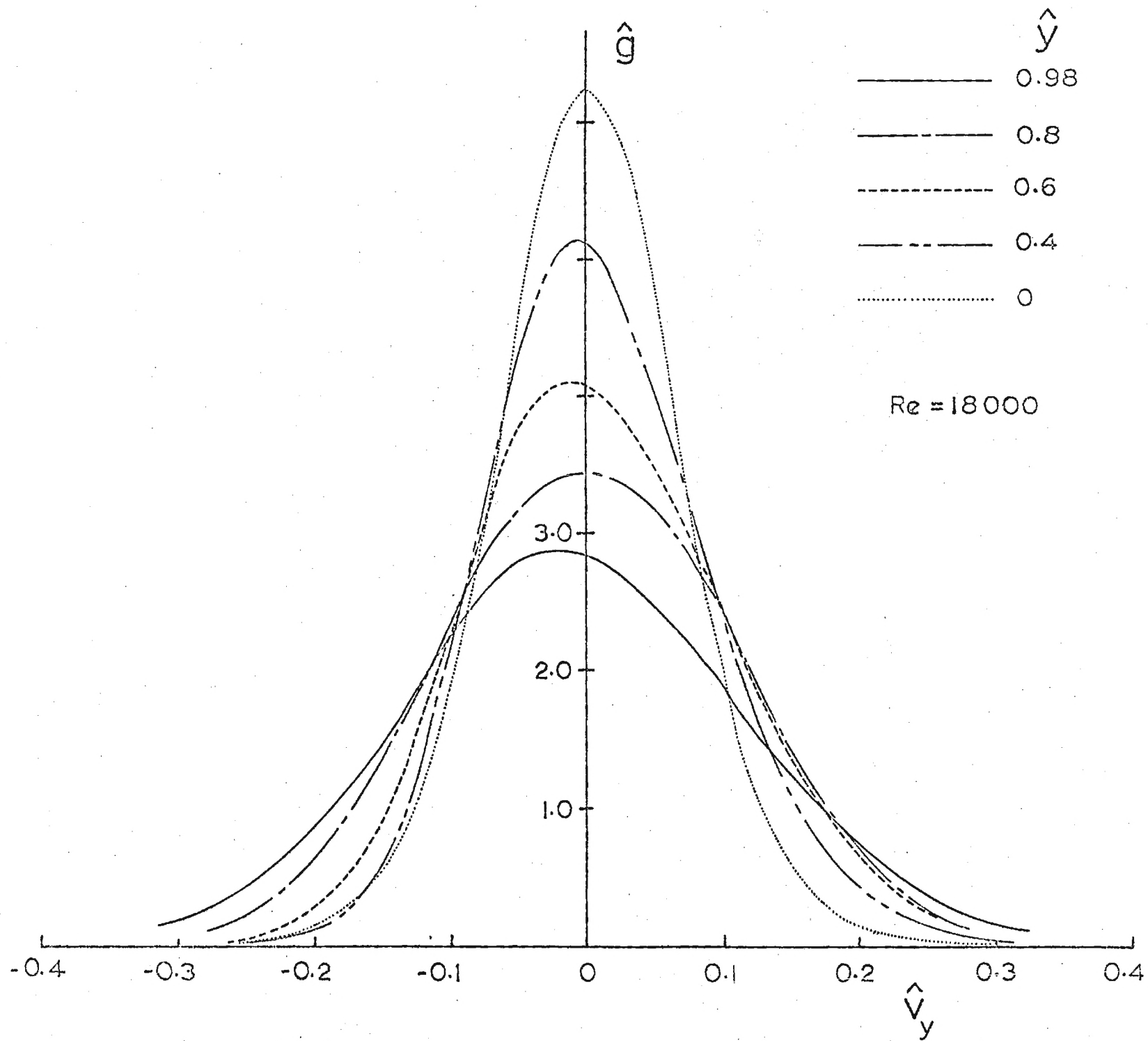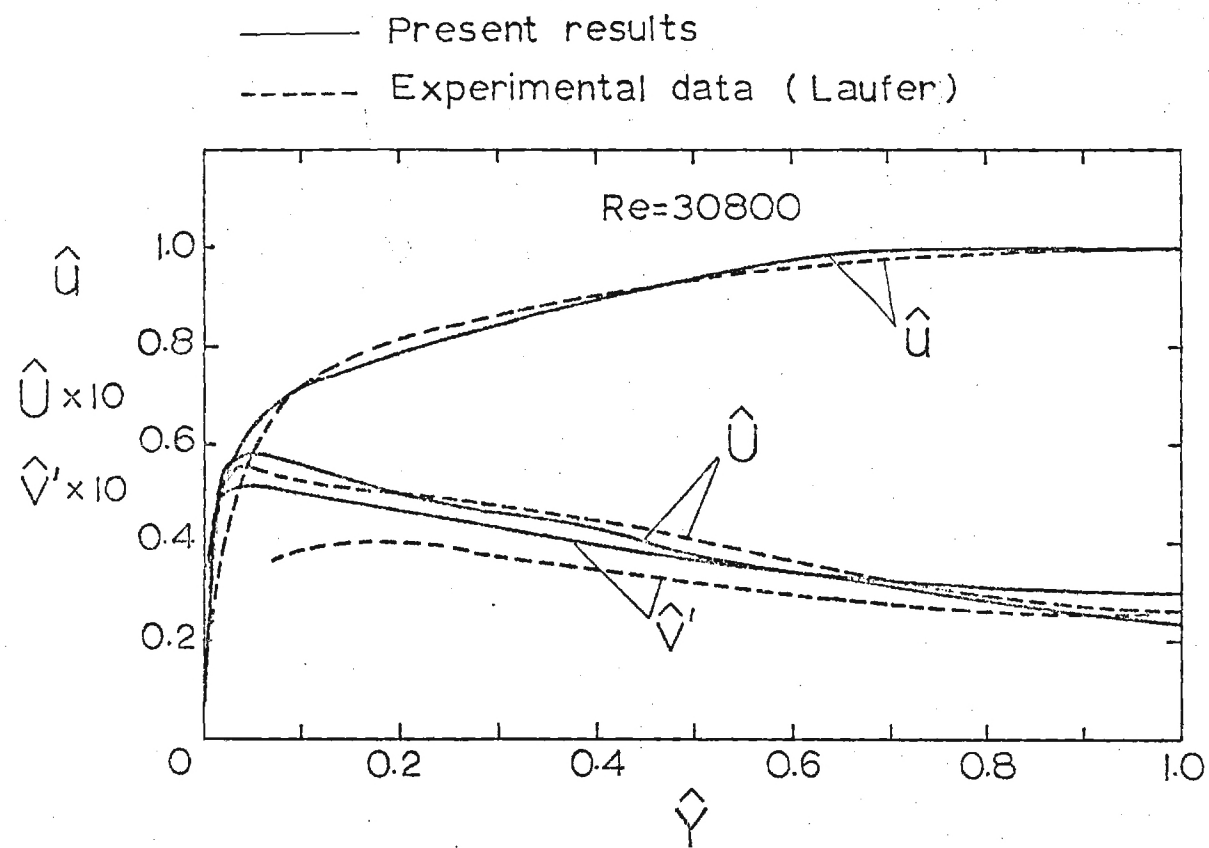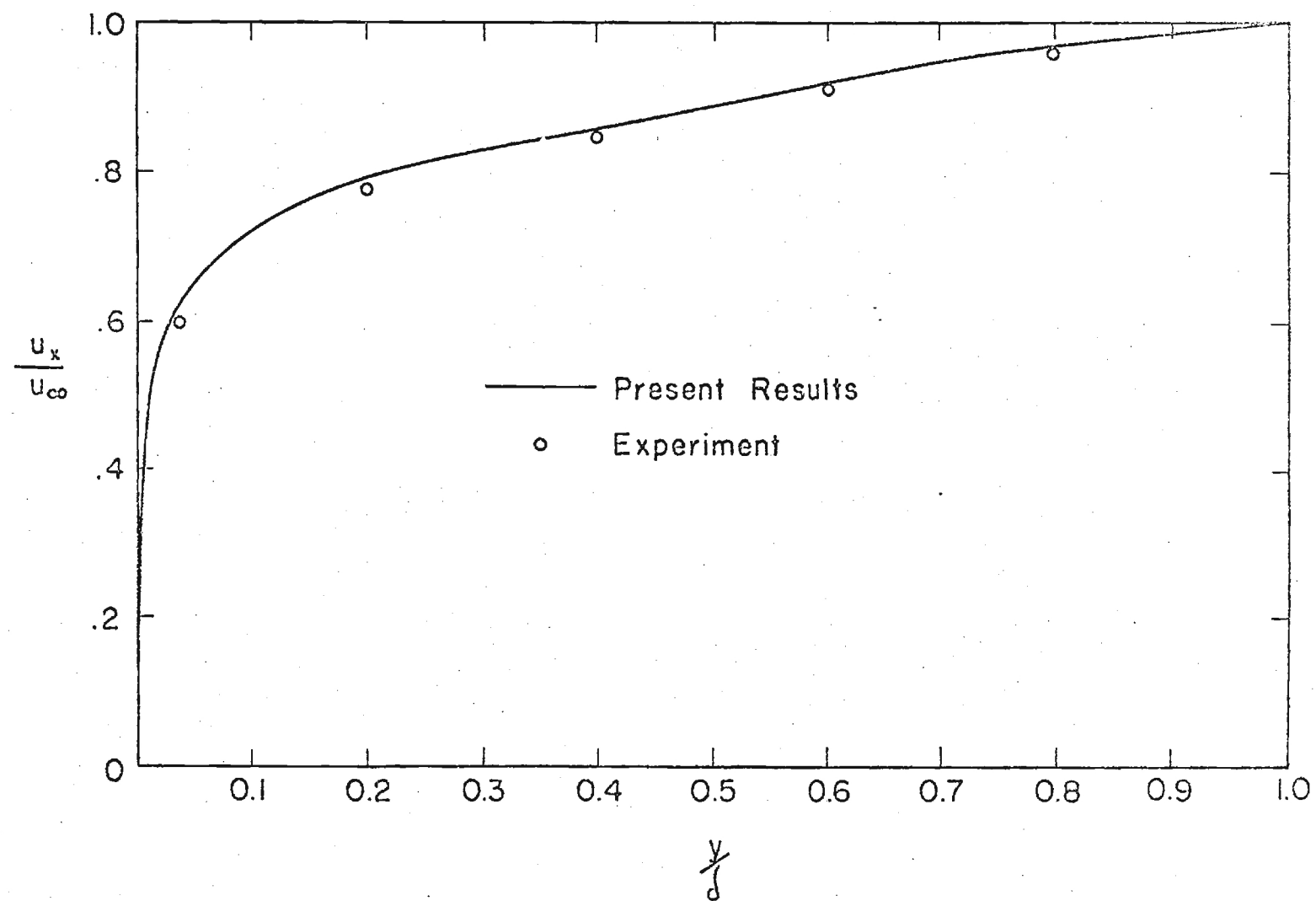Fig. 1

Fig. 2

Fig. 3

Fig. 4

Fig. 5
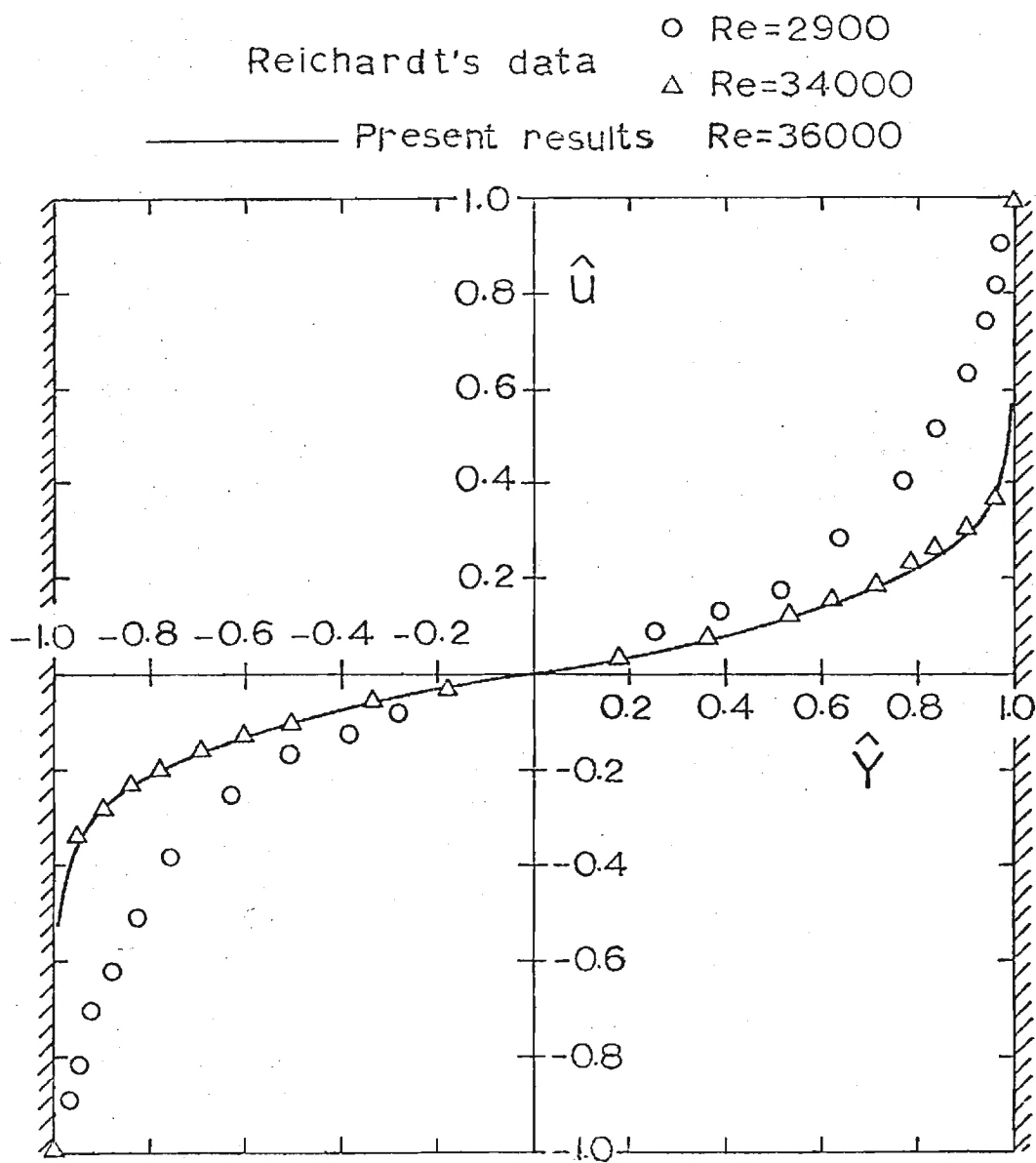
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RESEARCH PROJECT INITIATION

Date: __July 27, 1972__

Project Title: Study of Nonequilibrium Turbulent Wall Flows by the Statistical Method

Project No: E-16-625

Principal Investigator Dr. A. B. Huang

Sponsor: NASA - Langley Research Center; Hampton, Virginia

Agreement Period: From ____August 1, 1972____ Until ____July 31, 1973____

Type Agreement: Grant No. NGR 11-002-159

Amount: $18,700 NASA Funds (E-16-625)
       2,502 GIT Contrib. (E-16-325)
      $21,202 Total

Reports Required:
    Semi-Annual Status Report; Final Technical Report

Sponsor Contact Person (s): Technical Matters

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RA-3 (6-71)

RESEARCH PROJECT TERMINATION

Date: __December 3, 1973__

Project Title: Study of Nonequilibrium Turbulent Wall Flows by the Statistical Method

Project No: E-16-625

Principal Investigator: Dr. A. B. Huang

Sponsor: NASA - Langley Research Center; Hampton, Virginia

Effective Termination Date: ___9/30/73 (Grant Expiration)___

Clearance of Accounting Charges: ___by 10/31/73 (Last day available w/o formal extenxion)___

Grant Closeout Items Remaining: Final Fiscal Report w/certification
Final Report of Inventions
Final Equip. Inventory Report

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RA–4 (5/70)

NASA Research Grant NGR 11-002-159
Semi-Annual Progress Report
March, 1973

STUDY OF NONEQUILIBRIUM TURBULENT WALL FLOW
BY THE STATISTICAL METHOD

by

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

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SUMMARY

The statistical model equation proposed by Lundgren for turbulent distribution functions based on the continuity and the Navier-Stokes equation has been solved by the discrete ordinate method. The technique has been applied to turbulent channel flow and turbulent boundary-layer flow. The numerical results on mean flow velocity, root mean square fluctuation velocities, the Reynolds stress, the shear correlation coefficient, turbulent production rate, and energy dissipation rate have been found to be in very good agreement with the experimental data of Reichardt, and of Laufer for the channel flow problems, and with that of Klebanoff for the boundary-layer flow. It is concluded that the proposed physical model is good for the development of these turbulent flows. The cases of Bradshaw's boundary-layer (with pressure gradient) and the very nonequilibrium boundary-layer are presently under investigation. Preliminary results on these cases have been very encouraging.

INTRODUCTION

As mentioned in the research proposal[1] earlier, the purpose of this research is to develop a basic theoretical capability for calculating the development of nonequilibrium turbulent boundary layers for the purpose of contributing to the understanding and solution of current nonequilibrium turbulent wall flow problems. Several important advantages of the proposed technique over the existing methods are: (1) the technique yields turbulent distribution functions from which the mean velocity, the root mean squre velocity fluctuations, the Reynolds stress, the turbulent energy flux, and the local integral scale of turbulence can be directly calculated without making an assumption or modeling, (2) the accurate solution of the model equation can be obtained using the discrete ordinate method which has been successfully used in the rarefied gasdynamic research by the author, and (3) the degree of roughness of the wall can be simulated through the boundary conditions used in the present technique. In the following the progress of the research program between August, 1972 through February, 1973 is presented.

THE MODEL EQUATION

A system of equations for turbulent distribution functions was derived by Lundgren from the continuity and the Navier-Stokes equation[2]. The hierarchy of equations was closed at the one-point level by approximating the pressure term by a term of the Boltzmann type[3]. Following Lundgren[3], the model equation takes the form

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \frac{\partial f}{\partial \vec{r}} + \left(- \frac{1}{\rho} \frac{\partial \bar{p}}{\partial \vec{r}} + \nu \frac{\partial}{\partial \vec{r}} \cdot \frac{\partial}{\partial \vec{r}} \vec{u} \right) \cdot \frac{\partial f}{\partial \vec{v}}$$

$$= \frac{1}{\tau} (F - f) + \frac{1}{3} \frac{\epsilon}{U^2} \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} - \vec{u}) f \qquad (1)$$

where $f(\vec{r}, \vec{v}, t) \, d\vec{v}$ is the probability that the velocity of fluid element at point $\vec{r}$ and time t is in the range $\vec{v}$, $\vec{v} + d\vec{v}$, $\bar{p}$ is the mean pressure, and $\vec{u}$ is the mean velocity. Here, the pressure term has been approximated by a relaxation term (the so-called Bhatnagar-Gross-Krook model in kinetic theory[4])

$$\frac{1}{\tau} (F - f) \qquad (2)$$

where τ is a relaxation time and

$$F = \frac{1}{(2\pi U^2)^{3/2}} \exp \left[- (\vec{v} - \vec{u})^2 / 2U^2 \right] \qquad (3)$$

is a Gaussian distribution with $\vec{u}(\vec{r}, t)$ and $U(\vec{r}, t)$ the mean velocity and the root mean square velocity fluctuation, respectively. The

relaxation time is taken to be

$$\frac{1}{\tau} = \frac{K \left(\epsilon + \frac{3}{2} \frac{DU^2}{Dt} \right)}{U^2} \qquad (4)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \frac{\partial}{\partial \vec{r}}$$

and K is a constant that can be adjusted for the model, and ϵ is the turbulent dissipation rate.

Equation (1) is quite similar to the Boltzmann equation in kinetic theory. The first and second terms on the left-hand side are the unsteady term and the convective term, respectively; the third term is the acceleration term due to the pressure gradient; the fourth term is the acceleration term due to the viscous effect. The first term in the right-hand side is the pressure fluctuation term which is mainly responsible for describing the nonequilibrium phenomenon and which has been approximated by the BGK model. This is in agreement with the intuitive feeling that pressure fluctuations have a randomizing effect. The last term will yield the structure of turbulence.

After Equation (1) is solved for distribution functions f $(\vec{r}, \vec{v}, t)$, the flow properties can be obtained by taking the moments of distribution functions as follows:

$$\vec{u}(\vec{r}, t) = \int \vec{v} \, f \, (\vec{r}, \vec{v}, t) \, d\vec{v} \qquad (5a)$$

$$3U^2(\vec{r},t) = \int (\vec{v} - \vec{u})^2 f(\vec{r}, \vec{v}, t)\, d\vec{v} \tag{5b}$$

pressure tensor: $\quad P_{ij} = \int (v_i - u_i)(v_j - u_j)\, f(\vec{r}, \vec{v}, t)\, d\vec{v} \tag{5c}$

turbulent energy flux:

$$q_i = \frac{1}{2} \int (v_i - u_i)(\vec{v} - \vec{u})^2\, f(\vec{r}, \vec{v}, t)\, d\vec{v} \tag{5d}$$

The quantity $- \rho P_{ij}$ is the Reynolds stress and ρq_i is turbulent energy flux.

To complete the model equation in Equation (1), the relaxation time τ must be specified. To this end, the well-known Chapman-Enskog method[5] is applied to Equation (1). This application leads to

$$P_{ij} = U^2 \delta_{ij} - 2\nu_T D_{ij} \tag{6}$$

where ν_T is the eddy viscosity coefficient which is

$$\nu_T = \frac{U^4}{K\left(\epsilon + \frac{3}{2} \frac{DU^2}{Dt}\right)} = U^2 \tau \tag{7}$$

and D_{ij} is the rate of strain tensor for the mean flow,

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{8}$$

The application also leads to

$$q_i = - k_T \frac{\partial}{\partial x_i} \frac{3}{2} U^2 \tag{9}$$

where k_T is the energy flux coefficient.

Thus, combining Equations (6), (7), and (8) yields

$$\frac{1}{\tau} = \frac{U^2\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}{U^2 \delta_{ij} - P_{ij}} = \frac{KU}{L} \tag{10}$$

where τ is assumed to be proportional to L/U, L is the integral scale of the turbulence, and K is a constant which can be adjusted to fit with experimental data.

The model equation was solved by Lundgren[3] for a class of homogeneous rectilinear flows and for decay of a periodic wake. Good agreement with available experiments was found[3]. Further test must be made for non-homogeneous turbulent flows in order to prove the applicability of the model equation for practical flow problems.

In the past year, the model equation has been used to solve the one-dimensional plane Couette flow and channel flow and the two-dimensional boundary-layer flow in order to establish a method for calculating the development of nonequilibrium turbulent boundary layers. The progress of this research is reported in the following.

NONHOMOGENEOUS COUETTE AND CHANNEL FLOW

Calculations for both the Couette flow and channel flow were per-
formed using the model equation and the discrete ordinate method[6-9].
The calculated results on the mean flow velocity, the turbulent intensity,
and Reynolds stress are found to be in very good agreement with existing
experimental data of Reichardt[10] and of Laufer[11]. It should be mentioned
that many important aspects of the physical model as well as the numerical
technique have been learned through the calculations of these flow problems.
For instance, the selection of the discrete velocity $\hat{v}_i$ at which the dis-
tribution functions are evaluated must be such that (i) the spacing
between the $\hat{v}_i$'s must be close enough to accurately define the highly
peaked distribution functions at and near the boundaries, i.e., the
discrete velocity points are chosen to be closely spaced near $\hat{v}_y = 0$
and farther apart for large $\hat{v}_y$, and (ii) the range of discrete points
must be sufficient to cover the spread of the distribution functions
across the channel. The spacings used are 0.001 near $\hat{v}_y = 0$ and 0.01
for large $\hat{v}_y$ and is varied in between. The discrete velocity points
cover up to $\hat{v}_y = \pm 0.5$. Seventy-five points on $\hat{v}_y$ were taken. The grid
points on the physical space are chosen such that they are closely spaced
near the wall, where the gradients are very large, in order to obtain the
necessary accuracy.

The profile of the local integral scale of turbulence plays a very
important role in the computations. The shear-stress profile is rather
sensitive to that of local integral scale of turbulence. Also, application
of van Driest's modification[12] on the local integral scale of turbulence
near the wall has a favorable effect on the shear stress profile. Complete

results including the turbulent intensity, shear stress, turbulent production rate, energy dissipation rate, and the local distribution functions on the channel flow problems will be presented in detail in the annual research report.

TWO-DIMENSIONAL TURBULENT BOUNDARY-LAYER FLOW

For steady two-dimensional turbulent flows, Equation (1) may be written as

$$v_x \frac{\partial f}{\partial x} + v_y \frac{\partial f}{\partial y} + \left[- \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial f}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial f}{\partial v_x} \right]$$

$$= \frac{1}{\tau} (F - f) + \frac{1}{3\tau K} \frac{\partial}{\partial \vec{v}} \cdot (\vec{v} - \vec{u}) f \qquad (11)$$

where

$$\frac{1}{\tau} = \frac{U^2}{\nu_T} = - \frac{U^2 \frac{\partial u_x}{\partial y}}{P_{xy}} = \frac{KU}{L}$$

and the Reynolds stress $\tau_{xy} = - \rho P_{xy}$.

The particle velocity $\vec{v}$ has components v_x, v_y, and v_z, and the flow velocity $\vec{u}$ has components u_x, u_y, and u_z ($u_z = 0$ for the two-dimensional case considered and u_y is small except for the case of very adverse pressure gradient), and $\partial \bar{p}/\partial x$ is the pressure gradient in the x-direction. Solutions for the distribution function $f(x, y; \vec{v})$ are sought.

The distribution function $f(x, y; \vec{v})$ in Equation (11) is two-dimensional in physical space but three-dimensional in the particle velocity space. In order to reduce the computer storage requirement the following reduced distribution functions are defined:

$$g(x, y; v_x, v_y) = \int_{-\infty}^{\infty} f(x, y; \vec{v}) \, dv_z \qquad (12a)$$

$$h(x, y; v_x, v_y) = \int_{-\infty}^{\infty} v_z^2 f(x, y; \vec{v}) \, dv_z \qquad (12b)$$

Multiplying Equation (11) by 1 and integrating with respect to v_z yields

$$v_x \frac{\partial g}{\partial x} + v_y \frac{\partial g}{\partial y} + \left[-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial g}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial g}{\partial v_x} \right]$$

$$= -\frac{1}{\tau} (g - G) + \frac{1}{3} \frac{1}{\tau K} \left[2g + (v_x - u_x) \frac{\partial g}{\partial v_x} + (v_y - u_y) \frac{\partial g}{\partial v_y} \right] \qquad (13a)$$

Multiplying Equation (11) by v_z and integrating with respect to v_z yields

$$v_x \frac{\partial h}{\partial x} + v_y \frac{\partial h}{\partial y} + \left[-\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} \frac{\partial h}{\partial v_x} + \nu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} \right) \frac{\partial h}{\partial v_x} \right]$$

$$= -\frac{1}{\tau} (h - H) + \frac{1}{3} \frac{1}{\tau K} \left[(v_x - u_x) \frac{\partial h}{\partial v_x} + (v_y - u_y) \frac{\partial h}{\partial v_y} \right] \qquad (13b)$$

The local equilibrium reduced distribution functions are

$$G = \frac{1}{2\pi U^2} \exp\left\{ -\frac{1}{2U^2} \left[(v_x - u_x)^2 + (v_y - u_y)^2 \right] \right\} \qquad (14a)$$

and

$$H = U^2 G \qquad (14b)$$

After the reduced distribution functions g and h are solved from Equations (13) subjected to the appropriate boundary conditions, the macroscopic properties can be found by taking the moments of g and h.

$$u_x(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x \, g \, dv_x dv_y \tag{15a}$$

$$u_y(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y \, g \, dv_x dv_y \tag{15b}$$

$$3U^2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h \, dv_x dv_y + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[(v_x - u_x)^2 + (v_y - u_y)^2\right] g \, dv_x dv_y \tag{15c}$$

$$P_{iJ} = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x v_y g \, dv_x dv_y + u_x u_y \tag{15d}$$

$$q_i = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y v_x^2 g \, dv_x dv_y - u_x \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_x v_y g \, dv_x dv_y$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y^3 g \, dv_x dv_y - u_y \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y^2 g \, dv_x dv_y$$

$$+ \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_y h \, dv_x dv_y - \frac{3}{2} U^2 u_y + \frac{1}{2} u_y (u_x^2 + u_y^2) \tag{15e}$$

The accurate numerical solution of Equations (13) has been obtained through the discrete ordinate method using the 2 points boundary- value approach. The detailed procedure will be included in the complete annual report. Here only some representative results are summarized in the following.

(1) The case of Klebanoff's boundary layer[13] (zero pressure gradient) has been tested. The calculated mean velocity profile, the root mean square velocity fluctuations, the Reynolds stress, and the shear correlation coefficient have been found to be in very good agreement with Klebanoff's experimental data. This serves a

very important test case for the physical model used in the present calculations. The van Driest modification[12] on the local integral scale of turbulence near the wall seems to give very good results and thus improve the applicability of the physical model near the wall. It is extremely interesting and important to observe that the local integral scale of turbulence of the present theory is found to be in the following relation with the dissipation length parameter used by Bradshaw, et al[14,15];

$$L = L_B \left(\frac{1}{3a_1}\right)^{3/2} \tag{16}$$

where L_B and a_1 are defined by Bradshaw as

$$L_B = \frac{\left(\frac{\tau}{\rho}\right)^{3/2}}{\epsilon}$$

$$a_1 = \frac{\tau}{\rho q^2}$$

The integral scale found in the present theory, L, is the combination of the two empirical parameters used by Bradshaw, et al, in their turbulence computations.

It can be concluded that the physical model with the van Driest modification is very good for this type of boundary-layer development.

(2) The case of Bradshaw's boundary layer[14] (with pressure
gradient in the direction of flow) is being tested. Pre-
liminary results indicate that the results on mean flow
velocity, root mean square fluctuation velocities, and the
Reynolds stress are in very good agreement with Bradshaw's
experimental data. At present, the computer program is
being modified in order to improve the computation efficiency.
It is expected that final results will be obtained within a
few weeks and will be presented to our program manager at that
time.

(3) The technique is being applied to calculate the response of a
retarded equilibrium turbulent boundary layer to the sudden
removal of pressure gradient[16]. This extremely nonequilibrium
turbulent boundary layer will be a very critical test for the
technique. The computer program for this case is still under
development. Some results are expected within a month or so.

(4) The ellipsoidal model is also being used to calculate the
boundary-layer developments. Preliminary calculations indicate
that the computer program is very stable and yields pretty
good results.

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