II. Rotation velocities and radial electric field in the plasma edge

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1. Introduction

The toroidal and poloidal rotation and related radial electric field observed in the edge (and core) of tokamak plasmas are of interest for several reasons, not least of which is what they reveal about radial momentum transport, but also because of their apparent role in the L-H transition and the edge pedestal. It was recently shown[1] that if the heat transport coefficients and rotation velocities are taken from experiment, then the particle, momentum and energy balance equations and the conductive heat conduction relation are sufficient to determine the observed edge pedestal profile structure in the density and temperature profiles in several DIII-D discharges. Thus, it would seem that understanding the edge pedestal structure is a matter of understanding the edge rotation profiles. We present a practical computational model for the rotation and the radial electric field profiles in the plasma edge that is based on momentum and particle balance, includes both convective (including anomalous) and neoclassical gyroviscous momentum transport, and incorporates atomic physics effects associated with recycling neutrals.

2. Radial Electric Field

An expression for the radial electric field can be derived from the radial component of the momentum balance equation for ion species 'j'

$$\begin{aligned} &n_j m_j\left[\left(\mathbf{V}_j \cdot \nabla\right)\mathbf{V}_j\right]_r + \left[\nabla \cdot \Pi_j\right]_r + \frac{\partial p_j}{\partial r} = n_j e_j\left(E_r + V_{\theta j}B_\phi - V_{\phi j}B_\theta\right) + \\ &F_{rj} + M_{rj} - m_j\left(n_j \nu_{at,j} + \tilde{S}_j\right)V_{rj} \end{aligned} \tag{1}$$

where $M_{\xi j}$ is the $\xi -$component of the momentum input, $\tilde{S}_j \equiv S_j - \left\langle S_j \right\rangle$ is the poloidally varying part of the ionization source (due to recycling and fueling neutral influx and neutral beam injection), F_j is the friction force and $\nu_{at,j} = \nu_{ion,j} + \nu_{cx,j} + \nu_{el,j}$ represents atomic physics processes—ionization, charge exchange, elastic scattering.

The radial component of Eq. (1) yields

$$E_r = \frac{1}{n_j e_j}\frac{\partial p_j}{\partial r} + V_{\phi j}B_\theta - V_{\theta j}B_\phi - \frac{m_j}{e_j}\left(\cos\theta \frac{V_{\phi j}^2}{R} + \frac{V_{\theta j}^2}{r}\right) \tag{2}$$

where the unfamiliar last term results from retention of inertial effects to leading order.

3. Poloidal rotation and density asymmetries

Equations for the poloidal rotation velocities and for the poloidal density asymmetries can be derived from the poloidal components of the momentum balance equations and the particle balance equations for ion species "j"

$$\begin{aligned} &n_j m_j\left[\left(\mathbf{V}_j \cdot \nabla\right)\mathbf{V}_j\right]_\theta + \left[\nabla \cdot \Pi_j\right]_\theta + \frac{1}{r}\frac{\partial p_j}{\partial \theta} - M_{\theta j} - F_{\theta j} + \\ &n_j e_j\left(V_{rj}B_\phi - E_\theta\right) + m_j\left(n_j \nu_{elcx} + \tilde{S}_j\right) = 0 \end{aligned} \tag{3}$$

where the poloidal components of the inertial and viscous terms are

$$n_j m_j\left[\left(\mathbf{V}_j \cdot \nabla\right)\mathbf{V}_j\right]_\theta = n_j m_j\left[V_{rj}\frac{\partial V_{\theta j}}{\partial r} + \frac{V_{rj}V_\theta}{r} + \frac{1}{2}\frac{1}{r}\frac{\partial V_{\theta j}^2}{\partial \theta} + \frac{V_{\phi j}^2}{R}\sin\theta\right] \tag{4}$$

and

$$\left[\nabla\cdot\Pi_{\mathrm{j}}\right]_{\theta}=\eta_{0j}\left(\frac{1}{2}A_{oj}\right)\left\{\frac{1}{r}\frac{\partial\ln\left(\eta_{0j}A_{oj}\right)}{\partial\theta}-3\frac{\sin\theta}{R}\right\}\tag{5}$$

with

$$A_0=2\left\{-\frac{1}{3}\left(\frac{\partial V_p}{\partial l_p}\right)+\left[\left(\frac{1}{R}\right)\frac{\partial R}{\partial l_p}+\frac{1}{3}\left(\frac{1}{B_p}\right)\frac{\partial B_p}{\partial l_p}\right]V_p+f_pR\frac{\partial\left(V_\phi/R\right)}{\partial l_p}\right\}\tag{6}$$

and the neoclassical parallel viscosity coefficient can be represented by[2]

$$\eta_{0j}=\frac{n_jm_j\upsilon_{thj}qR\varepsilon^{-3/2}\nu_{jj}^*}{\left(1+\varepsilon^{-3/2}\nu_{jj}^*\right)\left(1+\nu_{jj}^*\right)}\equiv n_jm_j\upsilon_{thj}qRf_j\left(\nu_{jj}^*\right)\tag{7}$$

where $\nu_{jj}^*=\nu_{jj}qR/\upsilon_{thj}$, $f_p=B_p/B_\phi$ and $\varepsilon=r/R$.

Making low-order Fourier expansions of the form $n_j\left(r,\theta\right)=n_j^0\left(r\right)+n_j^s\sin\theta+n_j^c\cos\theta$ and taking the flux surface average with weighting functions 1, *sinθ* and *cosθ* results in a coupled set of equations (three times the number of ion species) that can be solved for the $V_{\theta j}^0$ and $\tilde{n}_j^{s,c}\equiv n_j^{s,c}/\varepsilon n_j^0$ for all the plasma ion species. Assuming $V_{rj}<<V_{\theta j}<V_{\phi j}$, the resulting equations are

$$\begin{aligned}
&\hat{V}_{\theta j}\Bigg[-q\hat{V}_{\phi j}\,\varepsilon\left(\tilde{n}_j^s+\hat{\Phi}^s\right)-q^2f_jf_p\left(1+\hat{\Phi}^c+\frac{2}{3}\tilde{n}_j^c\right)+f_p\sum_{k\neq j}\nu_{jk}^*+\\
&\frac{q}{\varepsilon}\nu_{elcx,j}^*f_p+\frac{1}{2}f_p\varepsilon\nu_{ionj}^*\left\{\left(1+\tilde{n}_j^c\right)\left(\frac{n_e^0}{n_j^0}\left(\tilde{n}_e^c+\tilde{n}_{oj}^c\right)-\left(\tilde{n}_j^c+\tilde{n}_{oj}^c\right)\right)+\right.\\
&\left.\tilde{n}_j^s\left(\frac{\bar{n}_e}{\bar{n}_j}\left(\tilde{n}_e^s+\tilde{n}_{oj}^s\right)-\left(\tilde{n}_j^s+\tilde{n}_{oj}^s\right)\right)\right\}\Bigg]-\sum_{k\neq j}\hat{V}_{k\theta}\left[f_p\nu_{jk}\sqrt{\frac{m_j}{m_k}}\right]=\\
&-\hat{V}_{rj}-q\varepsilon\frac{1}{4}\tilde{n}_j^s-q\varepsilon\hat{\Phi}_j\left[\frac{1}{4}\left(\hat{\Phi}^s+\tilde{n}_j^c\hat{\Phi}^s-\tilde{n}_j^s\hat{\Phi}^c\right)\right]-q^2f_jf_p\left(\hat{V}_{\phi j}+\hat{P}_j'\right)\hat{\Phi}^c\\
&-q\varepsilon\hat{V}_{\phi j}\left[\left(\hat{V}_{\phi j}+\hat{P}_j'\right)\hat{\Phi}^s+\frac{1}{2}\hat{V}_{\phi j}\tilde{n}_j^s\right]-\frac{n_e^0}{n_j^0}\nu_{ionj}^*q\Bigg[\hat{V}_{\phi j}\varepsilon\left(\tilde{n}_e^c+\tilde{n}_{oj}^c\right)\\
&-\frac{2}{3}qf_j\left(\tilde{n}_e^s+\tilde{n}_{oj}^s\right)\Bigg]
\end{aligned}\tag{8}$$

$$
\begin{aligned}
&\tilde{n}_j^s\left[\frac{1}{3}\frac{q^2}{\varepsilon}f_j f_p \hat{V}_{\theta j}+\frac{1}{2}\varepsilon \hat{V}_{rj}-\frac{1}{2}\varepsilon f_p\sum_{k\neq j}\nu_{jk}^*\hat{V}_{\theta k}\sqrt{\frac{m_j}{m_k}}+\frac{1}{2}q\nu_{ion}^* f_p \hat{V}_{\theta j}\right]\\
&+\tilde{n}_j^c\left[\frac{1}{2}qf_p^2\hat{V}_{\theta j}^2-\frac{1}{4}q+\frac{1}{2}q\nu_{elcx,j}^*\nu_{ionj}^*\right]=-\frac{1}{2}\varepsilon f_p\sum_{k\neq j}\nu_{jk}^*\hat{V}_{\theta j}\tilde{n}_k^s\\
&-\frac{1}{4}q\hat{\Phi}_j\left[-\hat{\Phi}^c\right]-\frac{q^2}{\varepsilon}f_j f_p\left[\frac{1}{2}\left(\hat{V}_{\theta j}-\hat{V}_{\phi j}-\hat{P}'_j\right)\hat{\Phi}^s-\frac{1}{2}qf_p^2\hat{V}_{\theta j}^2-\frac{1}{2}q\hat{V}_{\phi j}^2\right.\\
&-\frac{1}{2}q\nu_{elcx,j}^*\left[f_p\hat{V}_{\theta j}\tilde{n}_{oj}^s+\nu_{ionj}^*\tilde{n}_{oj}^c\right]-q\nu_{ionj}^* f_p\left[\frac{1}{2}\hat{V}_{\theta j}\left\{\tilde{n}_{oj}^s\left(1+\frac{n_e^0}{n_j^0}\right)+\frac{n_e^0}{n_j^0}\tilde{n}_e^s\right\}\right.\\
&\left.\left.+\frac{1}{3}\frac{q}{\varepsilon}f_j\frac{n_e^0}{n_j^0}\left(\tilde{n}_e^c+\tilde{n}_{oj}^c\right)\right]\right]
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
&\tilde{n}_j^c\left[\frac{1}{3}\frac{q^2}{\varepsilon}f_j f_p \hat{V}_{\theta j}+\frac{1}{2}\varepsilon \hat{V}_{rj}-\frac{1}{2}\varepsilon f_p\sum_{k\neq j}\nu_{jk}^*\hat{V}_{\theta k}\sqrt{\frac{m_j}{m_k}}+\frac{1}{2}q\nu_{ionj}^* f_p \hat{V}_{\theta j}\right]\\
&+\tilde{n}_j^s\left[-\frac{1}{2}qf_p\hat{V}_{\theta j}^2+\frac{1}{4}q-\frac{1}{2}q\nu_{elcx,j}^*\nu_{ionj}^*\right]=-\sum_{k\neq j}\tilde{n}_k^c\left[\frac{1}{2}\varepsilon f_p\nu_{jk}^*\hat{V}_{\theta j}\right]\\
&-\frac{1}{4}q\hat{\Phi}_j\left[\hat{\Phi}^s\right]-\frac{q^2}{\varepsilon}f_j f_p\left[\frac{1}{2}\left\{\left(1+\hat{\Phi}^c\right)\hat{V}_{\theta j}-\left(\hat{V}_{\phi j}-\hat{P}'_j\right)\hat{\Phi}^c\right\}\right]\\
&-q\hat{V}_{\phi j}^2\left[\frac{1}{4}\varepsilon\left\{\hat{V}_{\phi j}^s\hat{V}_{\phi j}^c+\tilde{n}_j^c\hat{V}_{\phi j}^s+\tilde{n}_j^s\hat{V}_{\phi j}^c\right\}\right]-\frac{1}{2}q\nu_{elcx,j}^*\left[f_p\hat{V}_{\theta j}\tilde{n}_{oj}^c-\nu_{ionj}^*\tilde{n}_{oj}^s\right]\\
&-qf_p\nu_{ionj}^*\left[\frac{1}{2}\hat{V}_{\theta j}\left\{\tilde{n}_{oj}^c\left(1+\frac{n_e^0}{n_j^0}\right)+\frac{n_e^0}{n_j^0}\hat{n}_e^c\right\}+\frac{1}{3}\frac{q}{\varepsilon}f_j\frac{n_e^0}{n_j^0}\left(\tilde{n}_e^s+\tilde{n}_{oj}^s\right)\right]
\end{aligned}
\tag{10}
$$

where the "e" and "o" subscripts refer to electrons and neutrals, respectively, and

$$
\begin{aligned}
&\hat{V}_{\theta j}\equiv\frac{V_{\theta j}^0}{\left|f_p\right|\upsilon_{thj}},\quad \hat{V}_{\phi j}\equiv\frac{V_{\phi j}^0}{\upsilon_{thj}},\quad \hat{V}_{rj}\equiv\frac{V_{rj}^0}{\left(\dfrac{m_j\upsilon_{thj}}{e_jB_\theta^0}\right)\left|f_p\right|\left(\dfrac{\upsilon_{thj}}{qR}\right)},\quad \hat{\Phi}_j\equiv\frac{e_j\Phi^0}{T_j},\\
&\hat{P}'_j\equiv\frac{1}{B_\theta^0 n_j^0 e_j\upsilon_{thj}}\frac{\partial p_j^0}{\partial r},\quad \tilde{n}_{oj}^{c/s}\equiv\frac{n_{oj}^{c/s}}{\varepsilon n_{oj}^0},\quad \hat{\Phi}^{c/s}\equiv\frac{\Phi^{c/s}}{\varepsilon}=\frac{n_e^{c/s}}{\varepsilon\left(e\Phi^0/T_e\right)},\\
&\hat{V}_{\phi j}^s\equiv V_{\phi j}^s\big/\varepsilon V_{\phi j}^0=-\left(V_{\theta j}^0\big/V_{\phi j}^0\right)f_p^{-1}\left(\tilde{n}_j^s+\hat{\Phi}^s\right)+\hat{\Phi}^s\left(1+\bar{P}'_j\big/V_{\phi j}^0\right)\\
&\hat{V}_{\phi j}^c\equiv V_{\phi j}^c\big/\varepsilon V_{\phi j}^0=1-\left(V_{\theta j}^0\big/V_{\phi j}^0\right)f_p^{-1}\left(2+\tilde{n}_j^c+\hat{\Phi}^c\right)+\hat{\Phi}^c\left(1+\bar{P}'_j\big/V_{\phi j}^0\right)\\
&\nu_{jk}^*\equiv\frac{\nu_{jk}^0}{\upsilon_{thj}/qR},\quad \nu_{ionj}^*\equiv\frac{\left(\nu_{ionj}^0+\nu_{ionj,nb}^0\right)r}{\upsilon_{thj}},\quad \nu_{elcx,j}^*\equiv\frac{\left(\nu_{cx,j}^0+\nu_{el,j}^0\right)r}{\upsilon_{thj}}
\end{aligned}
\tag{11}
$$

The corresponding Fourier components of the poloidal velocity are given by

$$\tilde{V}^{s}_{\theta j} \equiv V^{s}_{j} \big/ \varepsilon V^{0}_{j} = \left(r\nu_{ion} \big/ V^{0}_{\theta j} \right)\left(\tilde{n}^{c}_{j} + \tilde{n}^{c}_{oj} \right) - \tilde{n}^{s}_{j},$$

$$\tilde{V}^{c}_{\theta j} \equiv V^{c}_{j} \big/ \varepsilon V^{0}_{j} = -\left(r\nu_{ion} \big/ V^{0}_{\theta j} \right)\left(\tilde{n}^{s}_{j} + \tilde{n}^{s}_{oj} \right) - \left(1 + \tilde{n}^{s}_{j} \right) \tag{12}$$

4. Toroidal viscous force

The toroidal viscous force (actually torque) can be written in toroidal flux surface coordinates[3]

$$R^2 \nabla\phi \cdot \nabla \cdot \Pi = \frac{1}{Rh_p} \frac{\partial}{\partial l_\psi}(R^2 h_p \Pi_{\psi\phi}) + B_p \frac{\partial}{\partial l_p}\left(\frac{R\Pi_{p\phi}}{B_p}\right) \tag{13}$$

where the $\Pi_{\xi\phi}$ are the stress tensor elements. In this Braginskii decomposition[4] of the rate-of-strain tensor in a flux-surface coordinate system, the neoclassical viscous stress tensors have 'perpendicular' components with coefficients η_2 that are well known to be too small to account for the observed radial momentum transport rate, gyroviscous components

$$\Pi^{gv}_{\psi\phi} = -\eta_4 R \frac{\partial}{\partial l_p}\left(\frac{V_\phi}{R} \right), \Pi^{gv}_{p\phi} = -\eta_4 R \frac{\partial}{\partial l_\psi}\left(\frac{V_\phi}{R} \right) \tag{14}$$

and 'parallel' viscous components

$$\Pi^{\parallel}_{\psi\phi} = 0, \Pi^{\parallel}_{p\phi} = -\frac{3}{2}\eta_0 f_p A_0 \tag{15}$$

The Braginskii values[4] of the viscosity coefficients for a collisional plasma are

$$\eta_0 \simeq nT\tau f_{ne0} \gg \eta_4 \simeq nTm / ZeB = \eta_0 f^{-1}_{neo} / \Omega\tau \gg \eta_2 \simeq \eta_4 / \Omega\tau \simeq \eta_0 f^{-1}_{neo} / \left(\Omega\tau\right)^2 \tag{16}$$

where τ is the self-collision frequency and $\Omega = m/ZeB$ is the gyrofrequency. Since typically $\Omega\tau \approx 10^{-3} - 10^{-4}$, $\eta_0 f_{neo} >> \eta_4 >> \eta_2$. Taking into account lower collisionality should not effect η_4, which has no τ-dependence, and has been shown[5,6] to have very little effect on η_2. However, collisionality has a major effect on η_0, which we represent as indicated in Eq. (7) and as f_{neo} above. It has also been shown[7-9] that it may be necessary to extend the viscous torque to include heat flux terms in steep gradient regions with small rotation velocities, such as are found in the plasma edge.

5. Toroidal Rotation

Equations for the toroidal rotation can be derived from the toroidal component of the angular momentum balance equation and the particle balance equation for species "j"

$$\begin{aligned} &R^2 \nabla\phi \cdot n_j m_j \left(\mathbf{V}_{\mathrm{j}} \cdot \nabla \right) \mathbf{V}_{\mathrm{j}} + R^2 \nabla\phi \cdot \nabla \cdot \Pi_{\mathrm{j}} = n_j e_j R\left(E^A_\phi + V_{rj} B_\theta \right) - \\ &R n_j m_j \nu_{jk}\left(V_{\phi j} - V_{\phi k} \right) + RM_{\phi j} - Rm_j\left(n_j \nu_{at,j} + \hat{S}_j \right) V_{\phi j} \end{aligned} \tag{17}$$

where the toroidal component of the inertial term is

$$R^2 \nabla\phi \cdot n_j m_j \left(\mathbf{V}_{\mathrm{j}} \cdot \nabla \right) \mathbf{V}_{\mathrm{j}} = R n_j m_j \left(V_{rj} \frac{\partial V_{\phi j}}{\partial r} + \frac{V_{rj} V_{\phi j}}{R}\cos\theta + \frac{V_{\theta j}}{r}\frac{\partial V_{\phi j}}{\partial \theta} - \frac{V_{\phi j} V_{\theta j}}{R}\sin\theta \right) \tag{18}$$

A. Gradient Scale Length Formulation

If we can obtain gradient scale lengths (e.g. from experiment), then the flux surface averages of Eq. (17) for all can be written as a coupled set of algebraic equations at each radial point

$$n^0_j m_j \nu^0_{jk}\left(\left(1 + \beta_j \right) V^0_{\phi j} - V^0_{\phi k} \right) = n^0_j e_j E^A_\phi + e_j B^0_\theta \Gamma_j + M^0_{\phi j} \equiv n^0_j m_j \nu^0_{jk} y_j, \tag{19}$$

where $M_{\phi j}$ is the momentum input from the neutral beams, $M^{nb}_{\phi j}$, and possibly from other "anomalous" mechanisms, $M^{anom}_{\phi j}$, and the radial transfer of toroidal momentum by viscous, inertial, and atomic physics and perhaps "anomalous" processes is represented by the parameter

$$\beta_j \equiv \frac{\nu_{dj}^0 + \nu_{nj}^0 + \nu_{ionj,nb}^0 + \nu_{ionj}^0 + \nu_{elcx,j}^0 + \nu_{anom,j}}{\nu_{jk}^0} \equiv \frac{\nu_{dj}^0 + \nu_{nj}^0 + \nu_{atom,j}^0 + \nu_{anom,j}}{\nu_{jk}^0} \equiv \frac{\nu_{dj}^*}{\nu_{jk}^0} \tag{20}$$

where ν_{nj} is the frequency for the radial transport of toroidal angular momentum due to inertial effects, $\nu_{atom,j}^0$ is the frequency for loss of toroidal momentum due to atomic physics processes $\nu_{anom,j}$ is the frequency for loss of toroidal momentum by "anomalous" processes (e.g. turbulent transport, ripple viscosity).

The gyroviscous momentum transport frequency is defined by

$$\left\langle R^2\nabla\phi\cdot\nabla\cdot\boldsymbol{\pi}_j\right\rangle_{gv} \simeq \frac{1}{2}\tilde{\theta}_j G_j \frac{n_j m_j T_j}{e_j B_\phi}\frac{V_{\phi j}^0}{\bar{R}} \equiv R n_j m_j \nu_{dj} V_{\phi j}^0 \tag{21}$$

where

$$\tilde{\theta}_j \equiv \left(4+\tilde{n}_j^c\right)\tilde{V}_{\phi j}^s + \tilde{n}_j^s\left(1-\tilde{V}_{\phi j}^c\right) \tag{22}$$

represents poloidal asymmetries and

$$G_j \equiv -\frac{r}{\eta_{4j}V_{\phi j}}\frac{\partial\left(\eta_{4j}V_{\phi j}\right)}{\partial r} = r\left(L_n^{-1}+L_T^{-1}+L_{V_\phi}^{-1}\right) \tag{23}$$

represent radial gradients. We have used the gyroviscosity coefficient $\eta_{4j} \approx n_j m_j T_j / e_j B$.

The inertial momentum transport frequency is defined by

$$\begin{aligned}&\left\langle R^2\nabla\phi\cdot n_j m_j\left(\mathbf{V}_j\bullet\nabla\right)\mathbf{V}_j\right\rangle \simeq \frac{1}{2}\Bigg(\frac{V_{rj}}{R_o}\left\{\varepsilon\left(1+\tilde{n}_j^c+\tilde{V}_{\phi j}^c\right)-2R_o L_{v\phi,j}^{-1}\right\}-\\ &\varepsilon\frac{V_{\theta j}^0}{R_o}\left\{\tilde{V}_{\phi j}^s\left(1+\tilde{n}_j^c+\tilde{V}_{\theta j}^c\right)-\tilde{V}_{\theta j}^s\left(1+\tilde{V}_{\phi j}^c\right)-\tilde{V}_{\phi j}^c\tilde{n}_j^s\right\}\Bigg)n_j m_j R V_{\phi j}^0 \equiv R n_j m_j \nu_{nj} V_{\phi j}^0\end{aligned} \tag{24}$$

B. *Differential Equation Formulation*

If the radial gradient scale lengths in the ν_{dj} and ν_{nj} (in β_j) in Eqs. (19) are replaced by their definitions $L_x^{-1} \equiv -\left(1/x\right)\left(dx/dr\right)$, then these equations become coupled first order ODEs that must be solved for the $V_{\phi j}^0$, together with similar equations for the density and temperature[1].

Alternatively, it is possible to solve explicitly for the poloidal dependence of the toroidal rotation velocity by expanding the poloidal dependence of the toroidal rotation frequency

$$\Omega_{\phi j}\left(r,\theta\right) \equiv \frac{V_{\phi j}}{R} = \Omega_{\phi j}^0\left(r\right) + \Omega_{\phi j}^s\left(r\right)\sin\theta + \Omega_{\phi j}^c\left(r\right)\cos\theta \tag{26}$$

using similar density and poloidal velocity expansions, and flux surface averaging with weighting functions of 1, *sinθ* and *cosθ* then leads to three equations for each ion species "j"

$$
\begin{aligned}
&\frac{d\Omega_{\phi j}^{0}}{dr}\left[R_o V_{rj}\right]+\frac{d\Omega_{\phi j}^{c}}{dr}\left[\frac{\eta_{4j}^{0}\tilde{n}_j^{s}}{2n_j^0 m_j}\right]-\frac{d\Omega_{\phi j}^{s}}{dr}\left[\frac{\eta_{4j}^{0}\left(\tilde{n}_j^{c}+3\right)}{2n_j^0 m_j}\right]+\\
&\Omega_{\phi j}^{0}\left[\begin{array}{l}V_{rj}\varepsilon\left(2+\tilde{n}_j^{c}\right)+R_o\left(\bar{\nu}_{jk}+\bar{\nu}_{at,j}\right)\\ +\dfrac{r\tilde{S}_j^{c}}{n_j^0}-V_{\theta j}^{0}\varepsilon(\tilde{V}_{\theta j}^{s}+\tilde{n}_j^{s})\end{array}\right]-\Omega_{\phi j}^{c}\left[\begin{array}{l}\dfrac{\eta_{4j}^{0}}{2n_j^0 m_j}\left(\tilde{n}_j^{s}\left(L_{nj}^{-1}+L_{Tj}^{-1}\right)-\dfrac{1}{\varepsilon n_j^0}\dfrac{dn_j^{s}}{dr}\right)-\\ \dfrac{1}{2}\dfrac{R_o\tilde{S}_j^{c}}{n_j^0}+V_{\theta j}^{0}\left(\tilde{V}_{\theta j}^{s}+\tilde{n}_j^{s}\right)-V_{rj}\end{array}\right]+\\
&\Omega_{\phi j}^{s}\left[\frac{\eta_{4j}^{0}}{2n_j^0 m_j}\left(\left(L_{nj}^{-1}+L_{Tj}^{-1}\right)\left(\tilde{n}_j^{c}+3\right)-\frac{4}{r}-\frac{1}{\varepsilon n_j^0}\frac{dn_j^{c}}{dr}\right)+V_{\theta j}^{0}\left(2+\tilde{V}_{\theta j}^{c}+\tilde{n}_j^{c}\right)\right]-\Omega_{\phi k}^{0}\left[R_o\bar{\nu}_{jk}\right]\\
&=\frac{e_j}{m_j}\left[E_\phi^{A}+V_{rj}B_\theta^{0}\right]+\frac{M_{\phi j}}{n_j^0 m_j}
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\frac{d\Omega_{\phi j}^{0}}{dr}\left[\frac{\eta_{4j}^{0}\tilde{n}_j^{s}}{n_j^0 m_j}+rV_{rj}\left(\tilde{n}_j^{c}+3\right)\right]+\frac{d\Omega_{\phi j}^{c}}{dr}\left[R_o V_{rj}\right]+\\
&\Omega_{\phi j}^{0}\left[2V_{rj}+r\bar{\nu}_{jk}\left(2+\tilde{n}_j^{c}+\tilde{n}_k^{c}\right)+r\bar{\nu}_{at,j}\left(2+\tilde{n}_j^{c}+\tilde{n}_{oj}^{c}\right)+\frac{R_o\tilde{S}_j^{c}}{n_j^0}\right]+\\
&\Omega_{\phi j}^{c}\left[R_o\left(\bar{\nu}_{jk}+\bar{\nu}_{at,j}\right)+\frac{3\eta_{0j}}{n_j^0 m_j q^2 R_o}\right]+\\
&\Omega_{\phi j}^{s}\left[\frac{\eta_{4j}^{0}}{n_j^0 m_j}\left(\frac{L_{T_j}^{-1}}{\varepsilon}+\frac{L_{n_j}^{-1}}{\varepsilon}\right)+\frac{V_{\theta j}^{0}}{\varepsilon}\right]-\Omega_{\phi k}^{c}\left[R_o\bar{\nu}_{jk}\right]-\Omega_{\phi k}^{0}\left[r\bar{\nu}_{jk}\left(2+\tilde{n}_j^{c}+\tilde{n}_k^{c}\right)\right]\\
&=\varepsilon\left[\frac{e_j}{m_j}\left(E_\phi^{A}\left(\tilde{n}_j^{c}+1\right)+V_{rj}B_\theta^{0}\tilde{n}_j^{c}\right)+\frac{M_{\phi j}}{n_j^0 m_j}\right]+\frac{\eta_{0j}}{n_j^0 m_j R_o}\left(\frac{f_p}{r}\right)V_{\theta j}^{0}\left[\tilde{V}_{\theta j}^{c}-2\right]
\end{aligned}
\tag{28}
$$

and

$$
\begin{aligned}
&\frac{d\Omega_{\phi j}^{0}}{dr}\left[rV_{rj}\tilde{n}_j^{s}-\frac{\eta_{4j}^{0}\left(\tilde{n}_j^{c}+3\right)}{n_j^0 m_j}\right]+\frac{d\Omega_{\phi j}^{s}}{dr}\left[R_o V_{rj}\right]-\Omega_{\phi j}^{c}\left[\frac{\eta_{4j}^{0}}{n_j^0 m_j}\frac{\left(L_{T_j}^{-1}+L_{n_j}^{-1}\right)}{\varepsilon}+\frac{V_{\theta j}^{0}}{\varepsilon}\right]+\\
&\Omega_{\phi j}^{0}\left[-V_{\theta j}^{0}+r\left(\bar{\nu}_{jk}\left(\tilde{n}_j^{s}+\tilde{n}_k^{s}\right)+\bar{\nu}_{at,j}\left(\tilde{n}_j^{s}+\tilde{n}_{oj}^{s}\right)\right)+\frac{R_o\tilde{S}_j^{s}}{n_j^0}\right]+\\
&\Omega_{\phi j}^{s}\left[V_{rj}+R_o\left(\bar{\nu}_{jk}+\bar{\nu}_{at,j}\right)+\frac{3\eta_{0j}}{n_j^0 m_j q^2 R_o}\right]-\Omega_{\phi k}^{0}\left[r\bar{\nu}_{jk}\left(\tilde{n}_j^{s}+\tilde{n}_k^{s}\right)\right]-\Omega_{\phi k}^{s}\left[R_o\bar{\nu}_{jk}\right]\\
&=\varepsilon\tilde{n}_j^{s}\frac{e_j}{m_j}\left(E_\phi^{A}+V_{rj}B_\theta^{0}\right)+\frac{\eta_{0j}}{n_j^0 m_j R_o}\frac{f_p}{r}V_{\theta j}^{0}\tilde{V}_{\theta j}^{s}
\end{aligned}
\tag{29}
$$

where $S_j^{s,c} \equiv (+,-)\nu_{ion}\left(n_j^{c,s}+n_{oj}^{c,s}\right)$. The radial velocity $V_{rj}=V_{rj}^{class}+V_{rj}^{anom}$, where the classical term can be calculated from particle, momentum and energy balance[1] and any anomalous momentum transport is assumed to be convective.

6. Application to DIII-D

The above formalism, with the gradient scale-length formulation of section 5A, was applied to calculate rotation V_θ and E_r in a few DIII-D shots. Density and temperature profiles and gradient scale lengths were taken from experiment, and the total momentum transfer frequency ν_{dj}^* was inferred from experiment by matching the $V_{\varphi j}$ calculated from Eq. (18) to experiment, and then compared with the calculated gyroviscous and atomic transfer frequencies, as shown in Fig. 1. The poloidal velocities are compared with measured values in Fig. 2.

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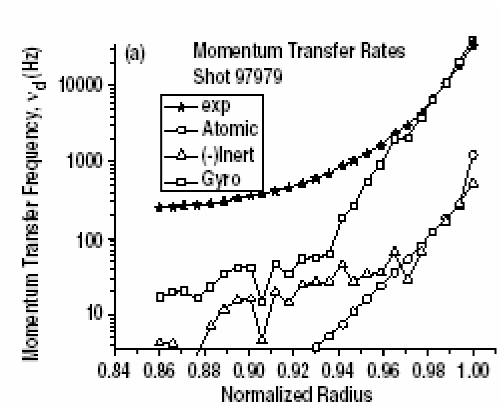

Fig. 1 Experimentally inferred and calculated angular momentum transfer frequencies in DIII-D.

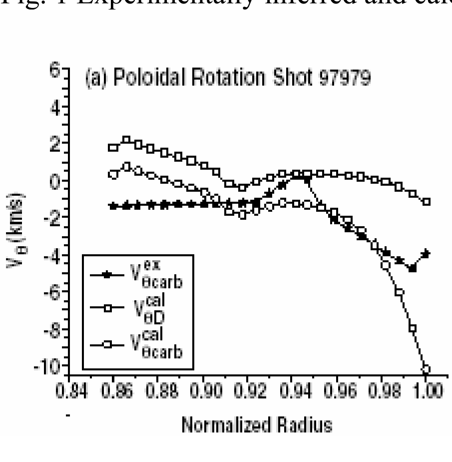

Fig. 2 Measured and calculated poloidal rotation velocities in DIII-D.