Formulating Invariant Heat-Type Curve Flows [†]

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Abstract

We describe a geometric method for formulating planar curve evolution equations which are invariant under certain transformation group. The approach is based on concepts from the classical theory of differential invariants. The flows we obtain are geometric analogues of the classical heat equation, and can be used to define invariant scale-spaces. We give a "high-level" general procedure for the construction of these flows. Examples are presented for viewing transformations.

1 Introduction

Curve evolution theory has recently become a major topic of research, and indeed has appeared in such diverse fields as differential geometry [13, 14, 21, 25], parabolic equations theory [3], numerical analysis [19], computer vision [16, 17, 23], viscosity solutions [7, 8, 11], and image processing [2, 24]. In particular, evolution equations which are geometric non-linear versions of the classical heat equation have received much attention, since these equations have both theoretical and practical importance.

In what follows, we describe a "high-level" general procedure for obtaining invariant *geometric heat flows*, and related invariant evolution equations. The approach is based on concepts from the classical theory of differential invariants. The obtained flows can be used for the definition of invariant geometric multiscale representation of planar shapes, from which we may derive invariant hierarchical shape representations (see [1, 17, 23]). As examples, we describe the corresponding flows for the Euclidean and affine groups. We show that the theory holds for any Lie group as well. For details see [21, 22, 23, 25].

2 Invariant Flows: A General Approach

In this section, a general approach for formulating invariant flows is described. Hence, given a certain transformation (Lie) group $\mathcal{L}$, we show how to obtain the corresponding invariant geometric heat flow. We also show how to formulate this flow just in terms of Euclidean parameters such as the Euclidean curvature. This formulation permits us to employ already existing results and techniques for the analysis of such flows.

[†]This paper is a short version of the work presented in [25].

[*]This work was supported in part by grants from the National Science Foundation DMS-8811084 and ECS-9122106, by the Air Force Office of Scientific Research AFOSR-90-0024, and by the Army Research Office DAAL03-91-G-0019.

 0-8194-1280-5/93/$6.00

2.1 Differential Invariants

We present now basic concepts of differential geometry and invariant theory which are necessary in the sequel.

Let $\mathcal{C} : S^1 \to \mathbf{R}^2$ denote a (parametrized) closed plane curve. We take p to be the plane curve parameter. We assume throughout this paper that all of our mappings are sufficiently smooth, so that all the relevant derivatives may be defined. We also assume that our curves are embedded, and so have no self-intersections. $\mathcal{C}$ can be written in Cartesian coordinates as $\mathcal{C}(p) = [x(p), y(p)]^T$, where $x(\cdot)$ and $y(\cdot)$ are maps from S^1 to $\mathbf{R}$.

Next recall that an *invariant descriptor* [9] is a property of an object, which does not change when the object undergoes certain transformations. More precisely, a quantity Q is called an *invariant* of a Lie group $\mathcal{L}$ if whenever Q transforms into $\tilde{Q}$ by any transformation $\mathbf{L} \in \mathcal{L}$, we obtain $\tilde{Q} = \Theta Q$, where Θ is a function of $\mathbf{L}$ alone. If $\Theta \equiv 1$ for all $\mathbf{L} \in \mathcal{L}$, Q is called an *absolute invariant* [9]. What we call "invariant" here is sometimes referred to in the literature as "relative invariant." (For more detailed and rigorous discussions, see [9, 15].)

A special class of invariants is given by the *differential invariants*, which are based only on local information. The theory of differential invariants is classical and goes back at least to the work of Gauss on Euclidean geometry. The texts of Wilczynski [28] and Blaschke [4] contain extensive treatments of projective and affine invariants respectively. These invariants were found to be very useful for invariant shape representation and recognition under partial occlusions [6, 12].

In order to separate the geometric concept of a plane curve from its parametric description, it is useful to consider the *image* (or *trace*) of $\mathcal{C}(p)$, denoted by $\mathrm{Img}[\mathcal{C}(p)]$. Therefore, if the curve $\mathcal{C}(p)$ is parametrized by a new parameter w such that $w = w(p)$, $\frac{\partial w}{\partial p} > 0$, we obtain

$$\mathrm{Img}[\mathcal{C}(p)] = \mathrm{Img}[\mathcal{C}(w)].$$

In general, the parametrization gives the "velocity" of the trajectory. Given a transformation group $\mathcal{L}$, the curve can be parametrized using what is called the *group arc-length.* This parametrization, which is an invariant of the group, is useful for defining differential invariant descriptors [15]. In order to perform this re-parametrization, the *group metric* is defined. This group metric, which we denote by g, must be a differential invariant of the group. The group arc-length r is then obtained via the relation (after fixing an arbitrary initial point)

$$r(p) := \int_0^p g(\xi) d\xi, \tag{1}$$

and the re-parametrization is given by $\mathcal{C} \circ r$. We have of course,

$$\mathrm{Img}[\mathcal{C}(p)] = \mathrm{Img}[\mathcal{C}(r)].$$

For example, in the Euclidean case we have

$$g_{euc} := \| \frac{\partial \mathcal{C}}{\partial p} \|, \tag{2}$$

and the Euclidean arc-length is given by

$$v := \int_0^p \| \frac{\partial \mathcal{C}}{\partial \xi} \| d\xi.$$

This parametrization is Euclidean invariant (since the norm is invariant), and implies that the curve $\mathcal{C}(s)$ is traversed with constant velocity ($\| \frac{\partial \mathcal{C}}{\partial v} \| \equiv 1$). For examples of other groups, see Section 4 and [15, 28].

Based on the group metric and arc-length, the *group curvature* χ, is computed. (Note that g, r, and χ can be computed using the Cartan method [15].) The group curvature, as a function of the arc-length, is defined as the simplest non-trivial differential invariant of the group. General theorems [15] show the exact number of derivatives involved in the curvature as a function of the number of group parameters.

For example, in the Euclidean case, since

$$\| \frac{\partial \mathcal{C}}{\partial v} \| \equiv 1.$$

we have that $C_v \perp C_{vv}$, and the Euclidean curvature is defined as

$$\kappa := \| C_{vv} \| .$$

κ is also the rate of change of the angle between the tangent to the curve and a fixed direction. The corresponding invariants of the affine group will be presented below in Section 4.1.

2.2 Geometric Invariant Flow Formulation

We are now ready to describe the type of evolution equation that will be the main mathematical object of study in this paper.

First let $\mathcal{C}(p,t) : S^1 \times [0,\tau) \to \mathbf{R}^2$ be a family of smooth curves, where p parametrizes the curve and t the family. (In this case, we take p to be independent of t.) Assume that this family evolves according to the following evolution equation:

$$\begin{cases} \frac{\partial \mathcal{C}(p,t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p,t)}{\partial p^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{cases} \tag{3}$$

which is the classical heat equation. If $\mathcal{C}(p,t) = [x(p,t), y(p,t)]^T$, then $[x(p,t), y(p,t)]$ satisfying (3) can also be obtained by convolution of $[x_0(p), y_0(p)]$ with a Gaussian filter whose variance depends on t. Equation (3) has been studied by the computer vision community, and is used for the definition of a linear scale-space for planar shapes [29, 30].

Assume that we want to formulate an intrinsic *geometric heat flow* for plane curves which is invariant under certain transformation group $\mathcal{L}$. Let r denote the group arc-length. Then, the *invariant geometric heat flow* is given by

$$\begin{cases} \frac{\partial \mathcal{C}(p,t)}{\partial t} = \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{4}$$

If $\mathcal{L}$ acts linearly, then it is easy to see that since r is an invariant of the group, so is $\mathcal{C}_{rr}$. More precisely, if $\mathbf{L}[\cdot]$ represents a group transformation, and "$\tilde{}$" stands for the transformed curve (e.g., $\tilde{\mathcal{C}} = \mathbf{L}[\mathcal{C}]$), then

$$\mathcal{C}_{rr} = \Theta \tilde{\mathcal{C}}_{\tilde{r}\tilde{r}}, \tag{5}$$

where Θ is a function of $\mathbf{L}[\cdot]$. $\mathcal{C}_{rr}$ is called the *group normal.* If the group normal is an absolute invariant ($\Theta = 1$ for all $\mathbf{L}[\cdot] \in \mathcal{L}$ in (5)), then $\mathcal{C}$ and $\tilde{\mathcal{C}}$ satisfy exactly the same flow (4), and the flow is absolutely invariant in this sense. Otherwise, the velocity $\mathcal{C}_{rr}$ in (4) is multiplied by a constant, depending only on the specific $\mathbf{L}[\cdot]$, making the flow relatively invariant. Of course, this doesn't change the basic geometry of the flow.

When $\mathcal{L}$ acts nonlinearly, then the flow (4) is still *geometrically invariant* even though the parametrization may not be invariant [18, 25, 24]. This follows form the fact that the differential D_r with respect to the group arc-length is a differential invariant of the group action in the sense of [18, 25, 24]. By change of parametrization, such flows may be made equivariant as well [18, 25, 24]. Once again, this does not change the basic geometry of the flow.

We have just formulated the invariant geometric heat flow in terms of concepts intrinsic to the group itself, i.e., based on the group arc-length. For different reasons, which we will explain shortly, it is useful to formulate the group velocity $\mathcal{C}_{rr}$ in terms of Euclidean notions such as the Euclidean normal and Euclidean curvature. In order to do this, we need to calculate

$$< \mathcal{C}_{rr}, \vec{\mathcal{N}} >,$$

where $\vec{\mathcal{N}}$ is the Euclidean unit (inward) normal, and $< \cdot, \cdot >$ is the standard inner product in $\mathbf{R}^2$. In this way, we will be able to decompose the group normal $\mathcal{C}_{rr}$ into its Euclidean unit normal $\vec{\mathcal{N}}$ and Euclidean unit tangential $\vec{\mathcal{T}}$ components, and to re-write the flow (4) as

$$\frac{\partial \mathcal{C}}{\partial t} = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}. \tag{6}$$

In order to calculate α and β, assume for the moment that the curve $\mathcal{C}$ is parametrized by the Euclidean arc-length v. Then,

$$\frac{\partial^2 \mathcal{C}}{\partial r^2} = \frac{1}{g^2}\frac{\partial^2 \mathcal{C}}{\partial v^2} - \frac{g_v}{g^3}\frac{\partial \mathcal{C}}{\partial v}, \tag{7}$$

where g is the group metric defined in Section 2.1. (In this case, g is a function of v.) Now, using the relations

$$\mathcal{C}_{vv} = \kappa \vec{\mathcal{N}}, \quad \mathcal{C}_v = \vec{\mathcal{T}},$$

we obtain

$$\alpha = -\frac{g_v}{g^3}, \quad \beta = \frac{\kappa}{g^2}. \tag{8}$$

In general, $g(v)$ in (8) is written as a function of κ and its derivatives (see Section 4).

The importance of the formulation (6) can be seen from the following:

Lemma 1 ([10]) *Let β be a geometric quantity for a curve, i.e., a function whose definition is independent of a particular parametrization. Then a family of curves which evolves according to*

$$\mathcal{C}_t = \alpha \vec{\mathcal{T}} + \beta \vec{\mathcal{N}}$$

can be converted into the solution of

$$\mathcal{C}_t = \bar{\alpha} \vec{\mathcal{T}} + \bar{\beta} \vec{\mathcal{N}}$$

for any continuous function $\bar{\alpha}$, by changing the space parametrization of the original solution. Since β is a geometric function, $\beta = \bar{\beta}$ when the same point in the (geometric) curve is considered.

In particular, the above lemma shows that $\mathrm{Img}[\mathcal{C}(p,t)] = \mathrm{Img}[\hat{\mathcal{C}}(w,t)]$, where $\mathcal{C}(p,t)$ and $\hat{\mathcal{C}}(w,t)$ are the solutions of

$$\mathcal{C}_t = \alpha\vec{\mathcal{T}} + \beta\vec{\mathcal{N}},$$

and

$$\hat{\mathcal{C}}_t = \bar{\beta}\vec{\mathcal{N}},$$

respectively. For proofs of the lemma, see [10, 23].

In other words, Lemma 1 says that if the normal component of the velocity is a geometric function of the curve, then Img[$\mathcal{C}$] (which represents the "geometry" of the curve) is only affected by its normal component. The tangential component affects only the parametrization, and not Img[$\mathcal{C}$] (which is independent of the parametrization by definition). Therefore, assuming that the normal component β of $\vec{\nu}$ (the curve evolution velocity) does not depend on the curve parametrization, we can consider the evolution equation

$$\frac{\partial \mathcal{C}}{\partial t} = \beta\vec{\mathcal{N}}, \tag{9}$$

where $\beta = < \vec{\nu}, \vec{\mathcal{N}} >$, i.e., the projection of the velocity vector in the Euclidean normal direction. Since C_{rr} is a geometric quantity, equation (6) can be reduced to (9).

The formulation (4) gives a very intuitive formulation of the invariant geometric heat flow. On the other hand, the formulation given by equation (6), together with (8), gives an Euclidean-type flow which also allows us to simplify the flow using the result of Lemma 1. This type of analysis is crucial, since it allows one to understand the partial differential equation underlying the flow and to study its essential properties (such as short and long term existence, convergence, etc.). This will be a key technique when we study affine invariant flows in Section 4. Finally, reduction of equation (4) to (9) allows one to numerically implement the flow on computer. In fact, there is now available an efficient numerical algorithm due to Osher and Sethian [19] to do this.

The flow given by (4) is non-linear, since the group arc-length r is a function of time. This flow gives the invariant geometric heat-type flow of the group, and provides the invariant direction of the deformation. More general invariant flows are obtained if the group curvature χ is incorporated to the flow:

$$\begin{cases} \frac{\partial \mathcal{C}(p,t)}{\partial t} = \Psi(\chi)\,\frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p), \end{cases} \tag{10}$$

where $\Psi(\cdot)$ is a given function. (We discuss the existence of possible solutions of (10) in [25].) Since the group arc-length and group curvature are the basic invariants of the group transformations, it is natural to formulate (10) as the most general geometric invariant flow.

Since we have expressed the flow (4) in terms of Euclidean properties (equations (6), (8)), we can do the same for the general flow (10). All what we have to do is to express χ as a function of κ and it derivatives. This is done by expressing the curve components $x(p)$ and $y(p)$ as a function of κ, and then computing χ.

3 Euclidean Group

We now show how to use the general theory presented in previous section, for the computation of the invariant heat flows corresponding to the Euclidean group. In the next section, we will discuss the affine group.

A general Euclidean transformation in the plane is given by

$$\tilde{\mathcal{X}} = R\mathcal{X} + V,$$

where $\mathcal{X} \in \mathbf{R}^2$, R is a 2×2 rotation matrix, and V is a 2×1 translation vector. The Euclidean transformations constitute a group, and give some of the basic shape deformations which appears in computer vision applications.

We proceed to find, based on the above developed method, an Euclidean invariant geometric heat equation. From (4), we obtain that the *Euclidean geometric heat flow* is given by

$$\begin{cases} \mathcal{C}_t = \mathcal{C}_{vv}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{11}$$

(Recall that v is the Euclidean arc-length.) The Euclidean metric is defined by equation (2), and if the curve is already parametrized by arc-length, then of course $g_{euc}(v) \equiv 1$. Therefore, using equation (8) we obtain

$$\alpha_{euc} = 0, \quad \beta_{euc} = \kappa.$$

Then, the "Euclidean type" equation equivalent to (11) is (see equation (6))

$$C_t = \kappa \vec{\mathcal{N}}. \tag{12}$$

Equation (12) has a large research literature devoted to it. Gage and Hamilton [13] proved that any smooth, embedded convex curve, converges to a round point when deforming according to it. Grayson [14] proved that any non-convex embedded curve converges to a convex one. Hence, any simple curve converges to a round point when evolving according to the Euclidean geometric heat equation. The flow is also known as the *Euclidean shortening flow*, since the Euclidean perimeter shrinks as fast as possible when the curve evolves according to (12) [14]. Equation (12) was also found to be very important for image enhancement applications [2, 24].

Euclidean Constant Motion

There is a second flow which has been found to be very useful is computer vision applications, and especially for shape theory [5, 17]. Assume now that in (10), $\Psi(\kappa) = \frac{1}{\kappa}$ (we take $r \equiv v$ here). Then, combining equations (10) and (12), we obtain the *Euclidean constant motion flow*:

$$\begin{cases} \mathcal{C}_t = \vec{\mathcal{N}}, \\ \mathcal{C}(p,0) = \mathcal{C}_0. \end{cases} \tag{13}$$

This classical flow models the Hüygens principle. As is well-known, an initial smooth curve can develop singularities (shocks) when evolving according to (13). The question is how to continue the evolution after the singularities appear. The natural way is to choose the solution which agrees with the Hüygens principle [26]. If the front is viewed as a burning flame, this solution is based on the *entropy condition* that *once a particle is burnt, it stays burnt* [26, 27].

4 Affine Invariant Flows

In this section, we present the affine flow corresponding to equation (4) [21, 22, 25]. We first make some remarks about classical affine differential geometry.

4.1 Basic Concepts in Affine Differential Geometry

An affine transformation, transforms disks into ellipses, and rectangles into parallelograms. The general *affine transformation* in the plane ($\mathbf{R}^2$) is defined by

$$\tilde{\mathcal{X}} = A\mathcal{X} + B, \tag{14}$$

where $\mathcal{X} \in \mathbf{R}^2$ is a vector, $A \in \mathrm{GL}_2^+(\mathbf{R})$ (the group of invertible real 2×2 matrices with positive determinant) is the affine matrix, and $B \in \mathbf{R}^2$ is a translation vector. It is easy to show that transformations of the type (14) form a real algebraic group $\mathcal{A}$, called the *group of proper affine motions.* We will also consider the case of when we restrict $A \in \mathrm{SL}_2(\mathbf{R})$ (i.e., the determinant of A is 1), in which case (14) gives us the *group of special affine motions*, $\mathcal{A}_{sp}$.

In the case of Euclidean motions (in which case A in (14) is a rotation matrix), we have that the Euclidean curvature κ of a given plane curve, is a differential invariant of the transformation. In the case of general affine transformations, in order to keep the invariance property, a new definition of curvature is necessary. In this section, this *affine* curvature is presented [4, 15, 21]. See [4] for general properties of affine differential geometry.

As above, let $\mathcal{C} : S^1 \to \mathbf{R}^2$ be an embedded curve with parameter p (where S^1 denotes the unit circle). We now make the invariant re-parametrization of $\mathcal{C}(p)$ by defining a new parameter s such that

$$[\mathcal{C}_s, \mathcal{C}_{ss}] = 1, \tag{15}$$

where $[\mathcal{X}, \mathcal{Y}]$ stands for the determinant of the 2×2 matrix whose columns are given by the vectors $\mathcal{X}$, $\mathcal{Y} \in \mathbf{R}^2$. This relation is invariant under proper affine transformations, and the parameter s is the *affine arc-length.* Setting

$$g_{aff}(p) := [\mathcal{C}_p, \mathcal{C}_{pp}]^{1/3}, \tag{16}$$

the parameter s is explicitly given by

$$s(p) = \int_0^p g_{aff}(\xi) d\xi. \tag{17}$$

Note, we have assumed (of course) that g_{aff} (the *affine metric*) is different from zero at each point of the curve, i.e., the curve has no inflection points. In general, affine differential geometry is defined just for convex curves [4, 15]. In Section 4.2, we will show how to overcome this problem for the evolution of non-convex curves.

By differentiating (15) we obtain that the two vectors $\mathcal{C}_s$ and $\mathcal{C}_{sss}$ are linearly dependent and so there exists μ such that

$$\mathcal{C}_{sss} + \mu \mathcal{C}_s = 0. \tag{18}$$

The last equation implies

$$\mu = [\mathcal{C}_{ss}, \mathcal{C}_{sss}], \tag{19}$$

and μ is the *affine curvature*, i.e., the simplest non-trivial differential affine invariant function of the curve $\mathcal{C}$ [4]. Moreover, one can easily show [21] that ds, $\mathcal{C}_s$, $\mathcal{C}_{ss}$, μ, and the area enclosed by a closed curve, are absolute invariants of the group $\mathcal{A}_{sp}$ of special affine motions and relative invariants of the group $\mathcal{A}$ of proper affine motions.

4.2 Affine Geometric Heat Equation

With s the *affine* arc-length, the affine-invariant geometric heat flow is given by [21, 25]

$$\begin{cases} \mathcal{C}_t = \mathcal{C}_{ss}, \\ \mathcal{C}(p,0) = \mathcal{C}_0(p). \end{cases} \tag{20}$$

Since s is only defined for convex curves, the flow (20) is defined *a priori* for such curves only. However, in fact the evolution can be extended to the non-convex case, in the following natural manner. Observe that if $\mathcal{C}$ is parametrized by the Euclidean arc-length, then

$$g_{aff}(v) \quad = \quad [\mathcal{C}_v, \mathcal{C}_{vv}]^{1/3} = [\vec{\mathcal{T}}, \kappa\vec{\mathcal{N}}]^{1/3} = \kappa^{1/3},$$

and we obtain

$$\alpha_{aff} = -\frac{(\kappa^{1/3})_v}{\kappa}, \quad \beta_{aff} = \kappa^{1/3}.$$

Using Lemma 1, we obtain that the following flow is geometric equivalent to the affine invariant flow (20):

$$\mathcal{C}_t = \kappa^{1/3}\vec{\mathcal{N}}. \tag{21}$$

If we extend the evolution (20) to [22, 25]

$$\mathcal{C}_t = \begin{cases} \mathcal{C}_{ss} & \text{non-inflection points}, \\ 0 & \text{inflection points}, \end{cases} \tag{22}$$

we obtain an affine invariant flow which is also well-defined for non-convex curves [22] (the inflection points are affine invariant). Since $\kappa = 0$ at inflection points, the Euclidean-based geometric flow equivalent to (22) is also given by equation (21). Hence, we obtain that the flow given by (21) is an affine invariant flow. Note that the flow of the geometric curves Img[$\mathcal{C}$] is affine invariant, not the one of the parametrized curves. We should also add that recently Alvarez *et al.* [1] derived (21) using a completely different approach.

In summary, despite the fact that we cannot define the basic differential invariants of affine differential geometry on non-convex curves, nevertheless an affine invariant heat-type flow can be formulated. This is possible due to the possibility to "ignore" the tangential component of the deformation velocity, together with the invariant property of inflection points. Also note that $\mathcal{C}_{ss}$ contains three derivatives, but its normal component contains only two. This allows one to write the geometric affine heat flow as a function of κ.

We conclude this section by pointing out that in [21], we proved that any simple convex curve converges to an ellipse when evolving according to the affine heat flow (20). In [22] (see also [25]) we extended the results for non-convex curves using the flow (22) (or (21)). We showed that as in the Euclidean case, a non-convex curve deforms into a convex one, and from there into an ellipse according to the results in [22]. Figure 1 presents an example of curves, related by affine transformations, evolving according to the affine geometric heat flow.

Remark. Despite the above results for the affine and Euclidean groups, it is important to note that the flow (4) may not be a diffusive smoothing process relative to an arbitrary Lie group. In fact, in [25] we give examples of scale-invariant flows which actually develop singularities. (Some key scale-invariant groups are the Euclidean similarity group, full affine group, or $SL_3(\mathbf{R})$).

5 Multiscale Representations

Multiscale descriptions of signals have been studied for several years already. In computer vision, they have been employed in connection with the problem of representing the shape of a planar curve that has been extracted from an image.

A possible formalism for this topic comes from the idea of multiscale filtering which was introduced by Witkin [29], and developed in several different frameworks by a number of researchers in the past decade. The idea of scale-space filtering is very simple and can be formulated as follows: Given an initial signal $\Phi_0(\vec{X}) : \mathbf{R}^n \to \mathbf{R}^m$, the scale-space is obtained by filtering it with a kernel $\mathcal{K}(\vec{X}, t) : \mathbf{R}^n \to \mathbf{R}^m$, where $t \in \mathbf{R}^+$ represents the scale. In other words, the scale-space is given by $\Phi(\vec{X}, t)$ defined as

$$\Phi(\vec{X}, t) := \Omega_{\mathcal{K}(\vec{X},t)}[\Phi_0(\vec{X})], \tag{23}$$

where $\Omega_{\mathcal{K}(\cdot,t)}[\cdot]$ represents the action of the filter $\mathcal{K}(\cdot, t)$. Larger values of t correspond to signals at coarser resolutions.

A famous example of a kernel is the Gaussian kernel. In this case, the scale-space is linear, and the filter in (23) is just defined by convolution. The Gaussian kernel is one of the most studied in the theory of scale-spaces. It has some very interesting properties, one of them being that the signal Φ obtained from it, is the solution of the heat equation (with Φ_0 as initial condition) given by

$$\frac{\partial \Phi}{\partial t} = \Delta \Phi.$$

Therefore, equation (3) gives a Gaussian (linear) scale-space for plane curves. One of the key facts that can be gleaned from the Gaussian example, is that the scale-space can be obtained as the solution of a partial differential equation called an *evolution equation*. This idea was developed in a number of papers [1, 2, 16, 17, 23] for evolution equations different from the classical heat equation.

In general, a number of properties are required in a scale-space. One of these is the *causality criterion.*, which means that "information" is not added when moving from a finer to a coarser scale. The "information" is in general characterized by zero-crossings, extremum, and so on. The causality criteria is usually also related to the *semi-group property.* Alvarez *et al.* [1] give a characterization of the evolution equations for which these and other properties hold.

In general, equation (10) will define a scale-space or multiresolution representation of signals. (For examples when it does not define a scale-space, at least in its classical definition, see [25].) Since the scale-space is obtained as the solution of a PDE, the original signal is obtained for $t = 0$, and the semi-group property (causality criterion) holds. Other properties hold depending on the function Ψ, and principally on the number of spatial derivatives involved in the flow. It is important to note that the scale-spaces obtained via (10) are geometric, invariant, and intrinsic to the curve properties.

Important examples of scale-spaces obtained via (10) are the Euclidean heat flow (11) and the affine heat flow (22). (The affine heat flow, as a scale-space, is studied in depth in [23].) In this case, the evolving curves also preserve inclusions, i.e., if an initial curve $\hat{\mathcal{C}}_0$ is included in an initial curve $\mathcal{C}_0$, the same is true for the corresponding evolving curves [1, 23]. These flows perform invariant curve smoothing, and permit us to obtain invariant hierarchical representations. Scale spaces obtained from (10) can be implemented using the numerical algorithms for curve evolution developed by Osher and Sethian [19]. Figure 2 presents an example of two noisy ellipses, related by an unimodular affine transformation, evolving according to the affine heat flow.

Another important scale-space derived from (10) is the Euclidean constant motion (13) [20]. As pointed out in Section 3, a curve can develop singularities when evolving with constant Euclidean velocity. This singularities give the *medial axis representation*, first defined by Blum [5], and frequently used in different computer vision problems as pattern recognition and geometric image coding.

Kimia *et al.* [16, 17] studied the combination of the Euclidean shortening flow with the Euclidean constant motion, i.e., $\Psi(\kappa) = \left(1 + \frac{1}{\kappa}\right)$ in (10). Based on this flow, they defined a reaction-diffusion scale-space, where the smoothing property of the Euclidean heat equation competes with the Euclidean constant motion, which develops the curve toward singularities. Further, the authors presented (among other topics) a geometric segmentation of plane curves.

6 Discussion and Concluding Remarks

In this note, we have presented a general approach for the formulation of invariant curve evolution flows. The approach is based on basic concepts of the classical theory of differential invariants. The equations which are obtained are geometric versions of the classical heat equation and can be used for the definition of invariant multiscale representations. We presented examples for the Euclidean and affine groups. For other groups of interest, see [25, 24].

Two related frameworks for the flow were presented. The first one is based just on the group arc-length and group curvature, and is very intuitive. This gives the notion of "geometric" heat equation. Based on this formulation, a second one, equivalent to it, was derived. This second formulation is based only on concepts of Euclidean differential geometry. The necessity for this formulation is that the properties of the underlying partial differential equation can be analyzed using techniques such as those described in [3, 8, 14, 19].

In conclusion, the theory presented here not only unifies the underlying structure of invariant geometric flows and their corresponding multiscale representation, but it also allows one to define new and useful invariant evolutions using similar ideas.

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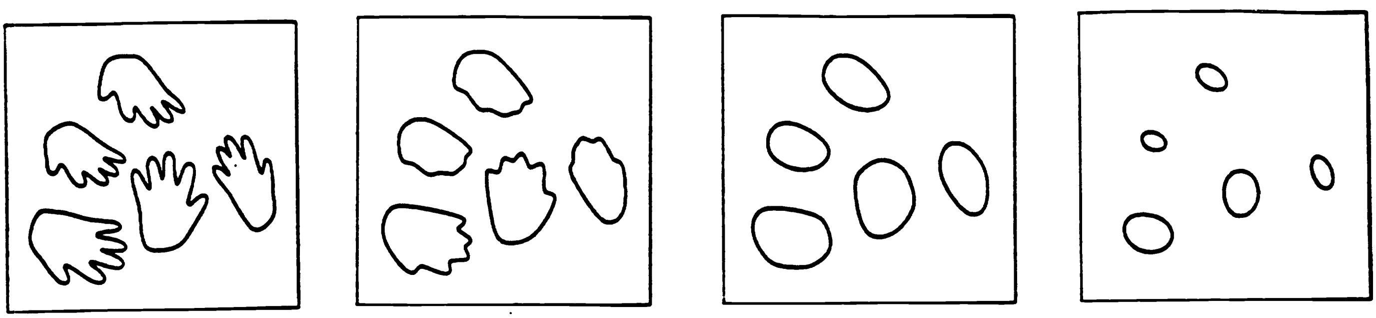
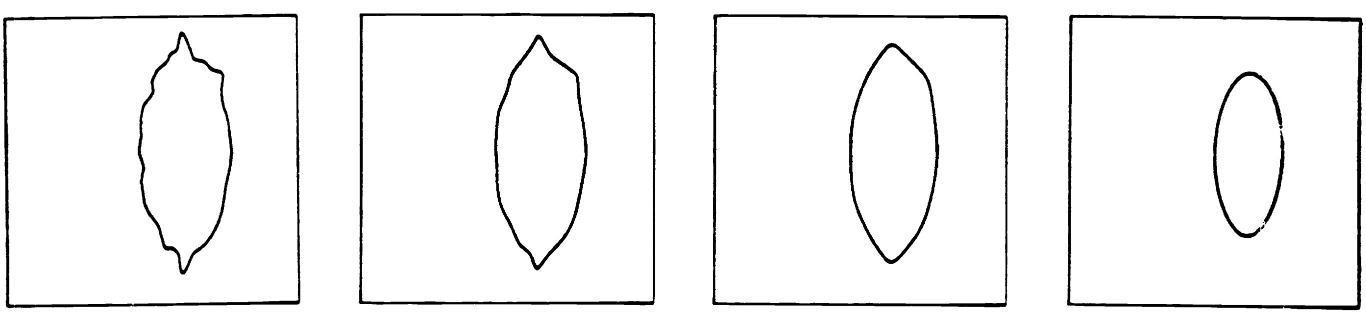


1. An example of the affine geometric heat flow.



2. The affine geometric heat equation as an affine invariant smoothing process.