CONTROL OF SYSTEMS IN THE PRESENCE OF UNKNOWN BUT BOUNDED DISTURBANCES

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CONTROL OF SYSTEMS IN THE PRESENCE OF UNKNOWN

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SUMMARY

This dissertation considers the problem of control of dynamical systems operating in the presence of uncertainty. A control procedure is presented in which uncertainty, which may be either a deterministic function of time or a stochastic process, is described by only its bounds. No statistical modeling is used for this noise. The systems under consideration are described by a set of nonlinear differential equations. These equations are linearized about a specified nominal trajectory resulting in a linearized perturbation model. The region around the nominal trajectory containing the perturbed state is described by bounding ellipsoids. Using the trace of these ellipsoids as a performance index, this index is minimized subject to a constraint which is an ellipsoid generation algorithm. This minimization procedure results in a control algorithm.

In order to evaluate the control algorithm the procedure is applied to the problem of a vehicle re-entering earth's atmosphere. The re-entry problem fits the class of problems under consideration in this work because it is described by a set of nonlinear differential equations and the uncertainty is unknown but bounded. The control algorithm is developed for this example and the re-entry is simulated on the digital computer. The re-entry is simulated using deterministic perturbations, Gaussian white noise, and uniform noise. The ellipsoidal bounding region is generated and by comparison with the statistical techniques is shown to be a conservative bound. The character of the bounding ellipsoid and

the sample variance, however, are very similar. The bounding ellipsoid is computationally more efficient to compute than the statistical bound and can therefore be recommended as a tool to obtain qualitative information about the performance of a given closed-loop control system.

The major contribution of this research is an alternate design approach to the control of systems operating in the presence of uncertainty. This design procedure yields a time-varying controller that is a linear function of the system state. The controller is implemented on-line by using a digital computer to store the nominal trajectory, nominal control, and the controller gains. The system state is measured and compared to the nominal state, thus giving the perturbed state. The perturbation control can then be generated using the appropriate gain matrix. This perturbation control and the nominal control form the on-line system control.

CHAPTER I

INTRODUCTION

Motivation

The basic problem in control systems engineering involves the control of a physical process in the presence of uncertainty. This uncertainty can take many different forms such as lack of knowledge about the process itself, unknown inputs to the process, or errors in attempting to measure the true state of the system. By far, the most widely used technique of representing this uncertainty in the mathematical system model is to consider the uncertainty as a stochastic process with known statistics. Frequently, the uncertain parameters are modeled as white Gaussian noise. In many problems the uncertainty may be adequately represented by Gaussian noise processes. There are, on the other hand, many physical systems in which the disturbances cannot be accurately characterized in this manner. For example, in the guidance and control of aircraft, ships, rockets, and space stations the external forces such as updrafts, wind gusts, waves, ocean currents, gravity gradients, and crew motion are not Gaussian in nature. In many adversary situations resulting in the game theory type of problem formulations, the tracker has no a priori knowledge of the evasive pattern of the target. The movement of the target reflects itself as an unknown disturbance input to the tracker which is not likely to be Gaussian. In any case, a priori statistics are not available. However, bounds on the magnitude of the disturbances can frequently be estimated. Another example is the control of a system subjected to a random bias disturbance. In all of these control problems it is reasonable to seek an alternative to the white Gaussian model. One such alternative, although relatively undeveloped, is to model the uncertainty as a set-constrained process. In this approach the unknown disturbance is regarded as a stochastic or deterministic process that is contained within a specified region. No statistical properties are assumed to be known about the process.

Another aspect of the general control problem which is related to the disturbance characterization is the overall function of the controller. In such problems where control in the presence of an unknown disturbance is required, a possible method of approach is to require that the control action result in acceptable performance for any possible disturbance. That is, in many control problems it is imperative that the control action be such that the state of the system be confined to a bounded region in the state space. Guaranteed performance is more important than "on the average" performance in these problems.

Therefore it is the objective of this dissertation to develop and evaluate a controller which results in guaranteed performance in the face of disturbances which are characterized only in terms of absolute amplitudes.

Background

The two distinct modeling techniques mentioned above make it convenient to classify the historical background into two areas: Stochastic Control Theory and Set Constrained Control Theory.

Stochastic Control Theory

The basic approach in stochastic control theory is to represent the disturbances or uncertainties in the system to be controlled as stochastic processes. These processes can be described mathematically in many different ways; but the characterization that has proven most useful is to treat the stochastic process as a white Gaussian noise process. The control of linear plants or processes subjected to Gaussian disturbances has been the subject of much past and present research and is commonly referred to as the Linear-Quadratic-Gaussian Problem (LQG)[1]. Because this dissertation also considers a linear process model, this background material on stochastic theory will focus on just the LQG problem.

The LQG design approach can be summarized as follows [1]. The deterministic nominal trajectory for a process to be controlled is determined, uncertainty in the process is modeled as Gaussian white noise, and the system equations are linearized about the nominal. Next the deterministic optimal control is generated. This control is optimal in the sense that it minimizes an artificial cost functional that depends quadratically on the state deviation. Now because all of the state variables are generally not available for measurement and because the ones that are available usually are considered corrupted by noise, a Kalman-Bucy filter is employed to estimate the system state [4,5]. The filter is driven by the output of the system and furnishes an estimate of the system state as output. The control structure is then specified using the deterministic optimal controller gain acting on the estimated state vector as indicated by the well known Separation Theorem [6]. A paper by Witsenhausen [7] pulls together the many results relating to separation of estimation and control in discrete time stochastic control theory. The separation principle provides a complete solution to the problem from a theoretical viewpoint. From a practical viewpoint, however, there is still a lot of work to be done with regard to system modeling.

To implement the design resulting from the separation principle, the engineer must know the mean and covariance matrices of the random variables modeling the plant initial state, and the mean and covariance matrices of the measurement noise. In addition, the state variable weighting matrix and control variable weighting matrix to be used in the quadratic performance criterion must be selected. There is no systematic procedure available to use in the selection of these matrices. The designer, of course, must also be concerned about the accuracy of the plant or process model and whether or not the plant and measurement noises are actually white and Gaussian.

Some research has been done on these problems. It has been observed that after an extended period of operation of the Kalman-Bucy filter, the errors in the state estimates eventually diverge from the expected error values. This phenomenon is known as "divergence." In a definitive paper by Fitzgerald [8], model mismatch, incorrect selection of process and measurement noise matrices, process bias, measurement bias, and numerical inaccuracies are all considered as causes of filter divergence. The most commonly accepted solution for the divergence problem is to increase the intensity of the process noise assumed in the model. Fitzgerald shows that this fix may or may not correct the divergence. Additional work on the filter divergence problem has been done by Sage and Melsa [9] and Jazwinski [10].

In the LQG design problem, application of the separation theorem is generally interpreted to mean the following: the optimal feedback gain matrix of the deterministic linear regulator and the optimal filter are designed separately and then the filter and gain are cascaded for optimal system performance. In a recent paper by Mendel [11] it is claimed that this procedure generally leads to an unsatisfactory design due to the fact that the optimal stochastic control law is a function of the estimated state. The estimated state contains the true state as well as an estimation error term. When the loop is closed this estimation error term appears as an unknown disturbance to the plant. Mendel's solution to this problem adds another step to the LQG design procedure. This additional step consists of selecting the quadratic criterion weighting matrixes such that some meaningful performance measure is optimized. This results in a separate optimization problem.

In summary then, a satisfactory LQG design is still heavily dependent on Monte Carlo simulations, and on the designer's engineering experience. If the assumption on linearity of the system or the Gaussian noise or the quadratic criterion is withdrawn there is no unified approach to the stochastic control problem. Therefore, if the noise is in reality non-gaussian, the designer usually has to make the "Gaussian assumption," proceed with an LQG design and then tune the Kalman filter to give the desired results.

<u>Set Constrained Control Theory</u>

An alternative approach for problems where the noise is not white and Gaussian is afforded by recently developed techniques for modeling the disturbance as a set constrained process. A set constrained process is represented by only its bounds. No statistical properties are assumed to be known about the process. This representation for the disturbance has been used by several investigators in their work. The following paragraphs describe the results to date.

Witsenhausen [12] considered the worst-case design of controllers for a linear differential system subjected to a bounded control and a bounded disturbance. The viewpoint in this paper is that given a control law, there is a maximum cost over all perturbations, the guaranteed performance for this control law. An algorithm is developed to find the minimum of this number over all control laws. Delfour and Mitter [13] considered the problem of reachability for control processes operating in the presence of set constrained disturbances. Specifically they considered the problem of finding the best open loop control in the presence of the worst disturbance.

In a 1968 paper by Schweppe [14] an approach also based on the ideas of reachability and set constrained disturbances was taken. In this paper a method for estimating the state of a linear dynamic system using noisy observations was developed. The input to the system and the observation errors were completely unknown except for bounds on their magnitude or energy. The state estimate was actually a set in state space rather than a single vector. This set was bounded by a time-varying ellipsoid which was generated with a recursive algorithm. This algorithm was based on the concept of reachability. Bertsekas and Rhodes [15], using a set-membership description of uncertainty, also developed a recursive state estimator. This estimator was very similar to Schweppe's with one exception; the gain matrix in the algorithm was independent of the observations and therefore precomputable. This feature made their algorithm look very similar to the corresponding stochastic linear minimum-variance estimator (Kalman filter).

Bertsekas and Rhodes [16] also examined the problem of keeping the state trajectory in a specified target tube. Necessary and sufficient conditions for reachability of a target set and a target tube are given in the case where the system state can be measured exactly. Their results [16] give sufficient conditions for reachability for the case when only disturbance corrupted output measurements are available. An algorithm is suggested that leads to linear control laws. Unfortunately, these control laws will fail for certain bounded disturbances and the exact relationships that lead to the failure are not specified. Glover and Schweppe [17] also formulated the problem of keeping the state of a linear system in a specified region of the state space. Again the uncertainties were constrained to be in specified sets. A bounding ellipsoid algorithm was developed and two arbitrary control laws were suggested.

The above mentioned work by Witsenhausen and by Delfour and Mitter developed control strategies but were not applied to any specific problem. Furthermore, the techniques outlined appear to be applicable to only scalar systems. Thus their results are interesting but strictly theoretical in nature. The papers by Schweppe [14] and by Bertsekas and Rhodes [16] are of a more practical nature but again the control strategies are not specifically evaluated. Furthermore, an arbitrary and consequently not very appealing procedure for selecting the control is embodied in these results. Therefore, the control schemes set forth in the literature on set constrained disturbances still remain to be thoroughly evaluated.

In summary then, it appears that while the LQG design approach is applicable to many control problems there are large classes of problems for which the set constrained theory is possibly a more realistic approach. The LQG approach, however, has received a great deal of attention from researchers and consequently offers to the control engineer a much more systematic design procedure than the set constrained theory does. It is possible, however, that additional research in the bounded disturbance area can make the solving of this problem much more systematic and, therefore, allow the designer to choose from the two approaches, the approach most appropriate to the problem at hand.

The Problem

The problem investigated in this research is the development and evaluation of a controller for a noisy dynamical system. The mathematical model for the system is a set of nonlinear differential equations with an additive noise term

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) + \underline{N} \tag{1.1}$$

No statistical information is assumed regarding $\underline{N}$. Rather, the noise, $\underline{N}$, is modeled as a function of time with known bounds. The noise is modeled in this manner for several reasons. First, in many problems a probabilistic noise description is not as readily available as the bounds are. Second, this noise treatment offers an alternative to the designer faced with a non-gaussian problem. And third, in many problems a system must be confined to a critical region or specified tolerances must be met with certainty so the bounds on the driving noise are the critical factors.

The basic objective of the controller to be developed is to keep the state, $\underline{x}$, of the system (1.1) close to a specified nominal or desired trajectory. If the state of the system is confined to a region around the nominal trajectory, a valid mathematical representation of the system is obtained by expanding $\underline{f}(\underline{x},\underline{u})$ in a Taylor Series about the nominal trajectory and truncating the higher order terms. This resulting linearized perturbation model is then used to represent the system and a perturbation controller is developed for this model. In an effort to make the controller implementation reasonable, it is assumed that the control can be realized as a linear function of the state. Because the major thrust here is the development and evaluation of the controller, the problem of noisy measurement is not considered.

It is useful in this thesis to think of the space containing the state of the system as a region or tube centered about the nominal trajectory. This region is described mathematically by a set of bounding ellipsoids. The desired controller, therefore, is the one that minimizes this bounded region and thus meets the objective of keeping the state close to the nominal trajectory. Selection of this minimizing control thus specifies the desired controller.

The evaluation of the controller is performed by applying it to the re-entry problem. The problem of spacecraft re-entering the earth's atmosphere is chosen because it is of current research interest. Also, the problem is one in which it is imperative that the state of the system be confined to a region or tube centered about the nominal trajectory. In addition, the disturbances or uncertainties present such as vehicle characteristics, winds, and atmospheric density are unknown but estimated bounds can be determined from physical considerations.

Approach to the Problem

Because of this unknown but bounded disturbance representation assumed in this research, it is clearly impossible to derive a controller that can maintain the state of the system exactly on the nominal trajectory. An approach alternative to the best "average" control determined by the separation principle approach is used in this research. In this approach it is recognized that since the controller cannot keep the state of the system exactly on the nominal trajectory, the real purpose of the controller is to keep the system "close" to the desired trajectory for all possible disturbances. Minimization of a performance index is a secondary consideration. The controller sought here, then, is the one that minimizes the region around the nominal trajectory in which it is possible for the state to lie. This is analogous to seeking directly a "guaranteed performance" controller rather than the usual "average" controller. To realize this controller, then, this region or tube around the trajectory must be formulated. This region at any point in time may be thought of as two concentric sets in state space. The first set is the set of reachable states. The second set is made up of the additional states the system can reach if the worst case noise disturbs the system. Once a description is obtained for these sets an optimization technique is applied that directly minimizes this region or tube. The controller results from this optimization step. The detailed mathematical formulation of this approach is presented in Chapter II.

CHAPTER II

DEVELOPMENT OF THE CONTROLLER

Introduction

It is the objective of the first phase of this research to develop a controller for a dynamical system operating in the presence of uncertainty. This uncertainty is mathematically modeled by only its bounds. No statistical modeling is used.

Before the algorithms describing the controller can be presented, several preliminary ideas are developed. First the class of system for which the control scheme is applicable is described. Then, the region or tube around the nominal trajectory containing the system state is characterized. Control algorithms are then defined by finding the control which minimizes this region.

Problem Formulation

Consider the plant modeled by the nonlinear vector differential equation

$$\dot{\underline{x}}(t) = \underline{f}(\underline{x}(t),\underline{u}(t)) \quad , \; \underline{x}(0) \tag{2.1}$$

where $\underline{x}(t)$ is the n-dimensional plant state vector with components $x_1(t)$, $x_2(t)$, . . ., $x_n(t)$, $\underline{u}(t)$ is the m-dimensional control vector with components $u_1(t)$, $u_2(t)$, . . ., $u_m(t)$, $\underline{x}(0)$ is the initial state vector at $t = 0$ and $\underline{f}(\underline{x}(t),\underline{u}(t))$ is a nonlinear function with components

$f_1(\underline{x}(t),\underline{u}(t))$, $f_2(\underline{x}(t),\underline{u}(t))$, . . ., $f_n(\underline{x}(t),\underline{u}(t))$. This function $\underline{f}(\underline{x}(t),\underline{u}(t))$ is assumed to be continuous and at least twice differentiable with respect to $\underline{x}(t)$ and $\underline{u}(t)$. The notation $\dot{\underline{x}}(t)$ is defined to be $d/dt\ \underline{x}(t)$. The model of this system is shown in Figure 1.

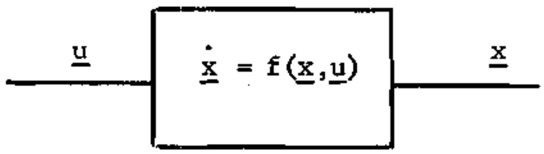


Figure 1. Deterministic Model

If a systematic approach, using nonlinear deterministic optimal control theory, is applied to this plant, an optimal control $\underline{u}_d(t)$ and resulting trajectory $\underline{x}_d(t)$ can be determined. This control is optimal in the sense that it minimizes a given performance criterion of the form

$$J = \phi(\underline{x}(T)) + \int_0^T L(\underline{x}(t),\underline{u}(t))dt$$

where the interval 0 to T is the duration of the trajectory. The determination of this optimal control $\underline{u}_d(t)$ is, in general, a non-trivial problem in its own right and will not be considered in this research. It is assumed, however, that the control term $\underline{u}_d(t)$ and the state $\underline{x}_d(t)$ are defined and represent the desired control and trajectory. It is recognized, however, that the true trajectory $\underline{x}(t)$ of the system will not coincide identically with $\underline{x}_d(t)$. This is true both because of errors in attempting to model the physical process and because of unknown disturbance inputs acting on the process or plant. Therefore, a more realistic model of the process is

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) + \underline{N}(t) \tag{2.2}$$

where $\underline{N}(t)$ represents the uncertainty in the plant and the external disturbances. The uncertainty vector $\underline{N}(t)$ may be a deterministic function of time, a stochastic process, or a constant. The only characteristic that is assumed known about $\underline{N}(t)$, however, is that it is contained within a specified bound. That is, $\underline{N}(t)$ is contained in the set $\Omega_N(t)$. No probability distributions or statistics are assumed known.

In the presence of these disturbances it is still assumed desirable to maintain the trajectory of the system "close" to the desired or nominal trajectory $\underline{x}_d(t)$. In fact, in many problems it is mandatory that the state be confined to a specified region of the state space. To do this a control correction term must be generated. This term can then be added to the nominal control term to generate the real-time control function which will drive the state of the system closer to the nominal state $\underline{x}_d(t)$.

Expanding the nonlinear function $\underline{f}(\underline{x}(t),\underline{u}(t))$ in a Taylor Series about the known desired trajectory and control gives [22]

$$\underline{f}(\underline{x},\underline{u}) = \underline{f}(\underline{x}_d,\underline{u}_d) + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\substack{\underline{x}_d \\ \underline{u}_d}} (\underline{x} - \underline{x}_d) + \left.\frac{\partial \underline{f}}{\partial \underline{u}}\right|_{\substack{\underline{x}_d \\ \underline{u}_d}} (\underline{u} - \underline{u}_d) + \underline{R} \tag{2.3}$$

where $\underline{R}$ represents the higher order terms in the Series. For a scalar case the remainder term in the Series is given by [23]

$$R = \frac{(x - x_d)^{n+1}}{(n+1)!} f^{(n+1)}(\alpha) \qquad x_d < \alpha < x$$

where n = 1. For the vector case comparable terms can be formed. It is assumed at this point the available control $\underline{u}(t)$ maintains the true state of the system close to the nominal state $\underline{x}_d(t)$. That is, $\|\underline{x}(t) - \underline{x}_d(t)\|$ is small, in which case the control correction term $\|\underline{u}(t) - \underline{u}_d(t)\|$ is small. Under this assumption, the motion of the system may be represented by the linear terms in the expansion (2.3). Defining

$$\delta\underline{x} = \underline{x} - \underline{x}_d \qquad \text{(state perturbation vector)}$$

$$\delta\underline{u} = \underline{u} - \underline{u}_d \qquad \text{(control correction vector)}$$

$$A(t) = \left.\frac{\partial f}{\partial \underline{x}}\right|_{\substack{\underline{x}_d \\ \underline{u}_d}}$$

$$B(t) = \left.\frac{\partial f}{\partial \underline{u}}\right|_{\substack{\underline{x}_d \\ \underline{u}_d}}$$

and substituting into (2.3) gives

$$\delta\dot{\underline{x}} = A(t)\delta\underline{x} + B(t)\delta\underline{u} + \underline{N} + \underline{R} \tag{2.4}$$

The remainder term, $\underline{R}$, from the Taylor Series is bounded and so is $\underline{N}$. Therefore, it is reasonable to combine these two terms into a term of the form $G(t)\underline{w}(t)$. Therefore (2.4) becomes

$$\delta\dot{\underline{x}} = A(t)\delta\underline{x} + B(t)\delta\underline{u} + G(t)\underline{w}(t) \tag{2.5}$$

where A, B, and G are known matrices of proper dimension. Because the solution of the control problem considered here involves a digital computer, this linearized model is discretized and takes the form

$$\delta\underline{x}(n+1) = \phi(n)\delta\underline{x}(n) + H(n)\delta\underline{u}(n) + G(n)\underline{w}(n) \tag{2.6}$$

This is the perturbation model for which the controller will be developed. The sample size for the discrete model is obtained by examination of the eigenvalues or transient frequencies of A(t). The sample size selected must be small compared to the corresponding time constant of the highest frequency present in the system (2.5).

Before the control scheme is developed, however, the region around the nominal trajectory containing the perturbed state $\delta\underline{x}(n)$ will be described mathematically. After this region is characterized, the control scheme will be developed by minimizing this region with respect to a selected performance index. The following sections, then, describe the region or tube around the nominal trajectory, justify the selection of a performance criterion, and develop the control algorithms.

The Bounding Ellipsoids

The purpose of the controller that is to be developed for the system (2.2) is to keep the state of the system close to the nominal or desired trajectory. That is, at any point in time the state must be contained in a certain region around the nominal trajectory. The control scheme to be developed is based on the idea of minimizing this region around the trajectory. Before this minimization can be performed, however, this region or tube must be formulated mathematically. First this region will be described for the linearized, deterministic, unforced system under consideration here (2.7). Then the region will be described for the more general case--the system with an input. After these regions or sets of states have been formulated, it will be possible to select a performance criterion and perform a minimization with respect to this criterion that will result in the desired controller.

Consider, first, a description of the region surrounding the nominal trajectory that will contain the state of the unforced perturbation

$$\delta\underline{x}(n+1) = \phi(n)\delta\underline{x}(n) \tag{2.7}$$

If the initial state, $\delta\underline{x}(0)$, is known, then in one time increment there is only one state the system can reach. However, if the initial state is only known to lie within a certain region, there is a set of states the system can reach. From a practical point of view it is important that this reachable set be characterized by a finite set of numbers. One way to do this is to specify the initial condition region to be an ellipsoid. An ellipsoid can be completely described by its center and a weighting matrix. With this description of the initial state region, then, the set of reachable states for the system (2.7) is also an ellipsoid. The following example illustrates these ideas.

Example:

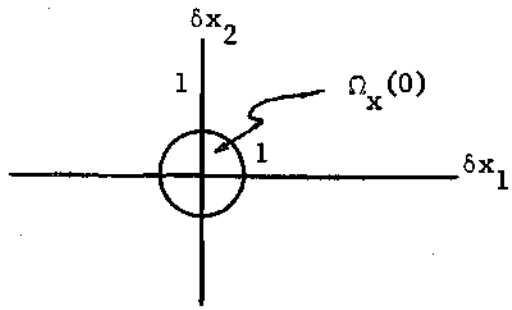
Given: the system - $\delta\underline{x}(n+1) = \phi(n)\delta\underline{x}(n)$

$$\delta\underline{x}(0) \in \Omega_{\mathbf{x}}(0)$$

where

$$\Omega_x(0) = \{\delta\underline{x} \in R^2 : (\delta\underline{x}(0))^T\Gamma_x^{-1}(0)\delta\underline{x}(0) \le 1\}$$

and
$$\Gamma_x^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Determine: the region the state of the system is in at the next time increment. At $n = 0$ the state is contained in

$$(\delta\underline{x}(0))^T\Gamma_x^{-1}(0)\delta\underline{x}(0) \le 1 \tag{2.8}$$

Now $\quad \delta\underline{x}(1) = \phi(0)\delta\underline{x}(0)$

or $\quad \delta\underline{x}(0) = \phi^{-1}(0)\delta\underline{x}(1)$

substituting into (2.8) gives

$$(\phi^{-1}(0)\delta\underline{x}(1))^T\Gamma^{-1}(0)(\phi^{-1}(0)\delta\underline{x}(1)) \le 1$$

or

$$(\delta\underline{x}(1))^T\phi^{-T}(0)\Gamma^{-1}(0)\phi^{-1}(0)(\delta\underline{x}(1)) \le 1$$

$$(\delta\underline{x}(1))^T\Gamma^{-1}(1)(\delta\underline{x}(1)) \le 1$$

where
$$\Gamma^{-1}(1) = \phi^{-T}(0)\Gamma^{-1}(0)\phi^{-1}(0)$$

and
$$\Gamma(1) = \phi(0)\Gamma(0)\phi^T(0)$$

Therefore at $n = 1$ the state is in the region

$$\delta\underline{x}(1) \in \Omega_x(1) = \{\delta\underline{x} \in R^2 \colon (\delta\underline{x}(1))^T \Gamma_x^{-1}(1)(\delta\underline{x}(1)) \le 1\}$$

In general for the unforced perturbation system model

$$\delta\underline{x}(n+1) = \emptyset(n)\delta\underline{x}(n)$$

with initial state known to be contained in the region

$$\delta\underline{x}(0) \in \Omega_x(0) = \{\delta\underline{x} \in R^n \colon (\delta\underline{x} - \underline{c}_x(0))^T \Gamma^{-1}(0)(\delta\underline{x} - \underline{c}_x(0)) \le 1\}$$

the system state will be contained in the region

$$\delta\underline{x}(n) \in \Omega_x(n) = \{\delta\underline{x} \in R^n \colon (\delta\underline{x} - \underline{c}_x(n))^T \Gamma^{-1}(n)(\delta\underline{x} - \underline{c}_x(n)) \le 1\} \quad (2.9)$$

where

$$\Gamma_x(n+1) = \emptyset(n)\Gamma_x(n)\emptyset^T(n)$$

$$\underline{c}_x(n+1) = \emptyset(n)\underline{c}_x(n)$$

$\Gamma_x(n)$ is positive definite and symmetric.

With this formulation it is possible now to extend these ideas to the forced system and thus characterize the region around the nominal for the system (2.7) under consideration here.

If an input is applied to the perturbation model, the model is of the form

$$\delta\underline{x}(n+1) = \emptyset(n)\delta\underline{x}(n) + G(n)\underline{w}(n) \quad (2.10)$$

Again it is assumed that the initial state $\delta\underline{x}(0)$ is contained in an ellipsoid and in addition it is assumed that the input $\underline{w}(0)$ is known to be contained in an ellipsoid. The state of the system $\delta\underline{x}(n)$ can be bounded by an ellipsoid by making use of the following information [17]. Given two sets described by

$$\Omega_1 = \{z \in R^n:\ (z - c_1)^T \Gamma_1^{-1} (z - c_1) \le 1\}$$

$$\Omega_2 = \{z \in R^n:\ (z - c_2)^T \Gamma_2^{-1} (z - c_2) \le 1\}$$

The sum of these two sets is contained in the ellipsoidal set

$$\Omega_s = \{z \in R^n:\ (z - (c_1+c_2))^T \Gamma_s^{-1} (z - (c_1+c_2)) \le 1\}$$

where Γ_s is given by

$$\Gamma_s = \frac{1}{\beta}\Gamma_1 + \frac{1}{1-\beta}\Gamma_2 \qquad 0 < \beta < 1$$

and Γ_1, Γ_2, Γ_s are positive definite symmetric nxn matrices. β is a scalar parameter.

By adding dynamical characteristics to these ellipsoids, as was done in the unforced case, the bounding ellipsoid containing the state $\delta\underline{x}(n)$ is

$$\Gamma_x(n+1) = \frac{1}{1-\beta(n)}\,\phi(n)\Gamma_x(n)\phi(n)^T + \frac{1}{\beta(n)}\,G(n)Q(n)G^T(n) \qquad (2.11)$$

$$\delta\underline{x}(n) \in \Omega_x(n) = \{\delta\underline{x} \in R^n:\ (\delta\underline{x} - \underline{c}_x(n))^T \Gamma_x^{-1}(n)(\delta\underline{x} - \underline{c}_x(n)) \le 1\} \qquad (2.12)$$

where

$$\delta\underline{x}(0) \in \Omega_x(0) = \{\delta\underline{x} \in R^n\colon (\delta\underline{x} - \underline{c}_x(0))^T\Gamma^{-1}(0)(\delta\underline{x} - \underline{c}_x(0)) \leq 1\}$$

and

$$\underline{w}(n) \in \Omega_w(n) = \{\underline{w} \in R^\ell\colon (\underline{w} - \underline{c}_w(n))^T Q^{-1}(\underline{w} - \underline{c}_w(n)) \leq 1\}$$

$$\underline{c}_x(n+1) = \phi(n)\underline{c}_x(n) + G(n)\underline{c}_w(n)$$

At this point the region containing the state $\delta\underline{x}(n)$ has been characterized. After the performance criterion is selected in the next section, it will be possible to develop the control scheme by minimizing this region described by the bounding ellipsoid (2.11).

The Performance Criterion

To minimize the region around the nominal trajectory in which the system state can lie, a performance criterion must be chosen. This criterion must be a reasonable mathematical representation of the region as well as be mathematically tractable. For these reasons, the trace of the appropriate ellipsoid matrix is chosen as the performance index. The following paragraph relates the trace to the space to be minimized.

At any point in the trajectory, the state will lie in a region described by

$$\Omega_x(n) = \{\delta\underline{x} \in R^n\colon (\delta\underline{x} - \underline{c}_x(n))^T\Gamma_x^{-1}(n)(\delta\underline{x} - \underline{c}_x) \leq 1\}$$

where $\Gamma_x^{-1}(n)$ is the nxn positive definite symmetric weighting matrix of an ellipsoid. By rotation of the co-ordinate system (similarity transformation) the matrix Λ shown below can be obtained from the matrix Γ.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \; 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & & 0 & \lambda_n \end{bmatrix}, \quad \Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{\lambda_2} & \cdots & 0 & 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 0 & \cdots & & 0 & \frac{1}{\lambda_n} \end{bmatrix}$$

This rotation results in an ellipsoid in so-called standard form. The eigenvalues of Γ are assumed to be distinct. This assumption is not restrictive and does not effect the idea being explained here. If this rotated ellipsoid is expanded it takes the form

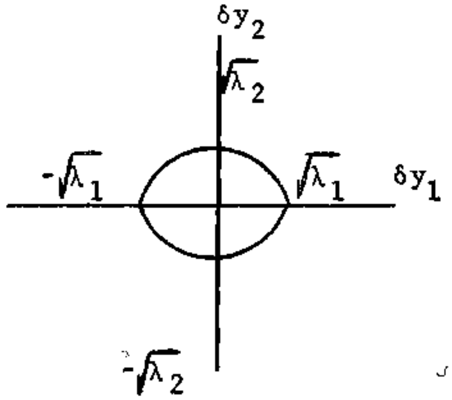
$$\delta \underline{y}^T \Lambda^{-1} \delta \underline{y} = \frac{1}{\lambda_1} \delta y_1^2 + \frac{1}{\lambda_2} \delta y_2^2 + \ldots + \frac{1}{\lambda_n} \delta y_n^2 \leq 1$$

with no loss in generality by assuming $c_x(n) = 0$. Therefore, the axes of the ellipse are the square roots of the eigenvalues. Furthermore, the trace of a square matrix is equal to the sum of its eigenvalues.

$$\text{Trace } [\Gamma] = \text{Trace } [\Lambda] = \sum_{i=1}^{n} \lambda_i$$

For purposes of illustration consider a two dimensional example.

$$\delta \underline{y}^T \Lambda^{-1} \delta \underline{y} = \frac{1}{\lambda_1} \delta y_1^2 + \frac{1}{\lambda_2} \delta y_2^2 \leq 1$$



From this discussion and example it is seen that the trace of the ellipsoid Γ is a measure of the region to be minimized. The trace is also easy to compute and does lead to reasonable control laws as shown in the next sections.

Development of the Control Algorithms

The previous sections of this chapter have described the model under consideration in this thesis, the ellipsoidal region containing the state, and a performance criterion that measures the size of this region. It is now possible to formulate the control problem being considered and to proceed to develop the control algorithms for the problem.

The linearized discrete time perturbation model being considered here is

$$\delta \underline{x}(n+1) = \emptyset(n)\delta \underline{x}(n) + H(n)\delta \underline{u}(n) + G(n)\underline{w}(n) \tag{2.13}$$

The perturbation controller for this model is assumed to be a linear function of the state

$$\delta \underline{u}(n) = L(n)\delta \underline{x}(n) \tag{2.14}$$

and the initial state $\delta\underline{x}(0)$ is in the region

$$\delta\underline{x}(0) \in \Omega_x(0) = \{\delta\underline{x} \in R^n\colon\ [\delta\underline{x} - \underline{c}_x(0)]^T T_x^{-1}(0)[\delta\underline{x} - \underline{c}_x(0)] \leq 1\}$$

and the noise is contained in

$$\underline{w}(n) \in \Omega_w(n) = \{\underline{w} \in R^\ell\colon\ [\underline{w} - \underline{c}_w(n)]^T Q^{-1}[\underline{w} - \underline{c}_w(n)] \leq 1\}$$

The problem, therefore, is to determine the perturbation controller, $L(n)$, for $n = 0,1,\ldots N$ that minimizes a measure of the size of the ellipsoidal region bounding the state $\underline{x}(n)$. This problem is solved in the following manner. Substitution of (2.14) into (2.13) gives

$$\delta\underline{x}(n+1) = \phi(n)\delta\underline{x}(n) + H(n)L(n)\delta\underline{x}(n) + G(n)\underline{w}(n)$$

or

$$\delta\underline{x}(n+1) = [\phi(n) + H(n)L(n)]\delta\underline{x}(n) + G(n)\underline{w}(n) \tag{2.15}$$

Defining

$$\hat{\phi}(n) = \phi(n) + H(n)L(n)$$

(2.15) becomes

$$\delta\underline{x}(n+1) = \hat{\phi}(n)\delta\underline{x}(n) + G(n)\underline{w}(n)$$

From the previous section on the bounding ellipsoids it is seen that $\delta\underline{x}(n)$ is contained in the region

$$\Omega_x(n) = \{\delta\underline{x} \in R^n\colon\ [\delta\underline{x} - \underline{x}_x(n)]^T T_x^{-1}(n)[\delta\underline{x} - \underline{c}_x(n)] \leq 1\}$$

described by the ellipsoid

$$\Gamma_x(n+1) = \frac{1}{1-\beta(n)}\,\hat{\emptyset}(n)\Gamma_x(n)\hat{\emptyset}^T(n) + \frac{1}{\beta(n)}\,G(n)Q(n)G^T(n) \tag{2.16}$$

$$\underline{c}_x(n+1) = \hat{\emptyset}(n)\underline{c}_x(n) + G(n)\underline{c}_w(n)$$

$$0 < \beta(n) < 1$$

At this point the matrix L(n) that minimizes the performance criterion of $\Gamma(n)$ must be determined. This is analogous in the vector case to determining the control $\underline{u}(n)$ that minimizes a performance index of the state variables, $\underline{x}(n)$. The problem has now been formulated so that the Matrix Minimum Principle (Appendix I) may be applied to determine the controller. This is done in the following manner.

The performance index is

$$J = \mathrm{Tr}[\Gamma(N)] + \sum_{n=0}^{N-1} \mathrm{Tr}[\Gamma(n)] \tag{2.17}$$

with the constraint

$$\Gamma(n+1) = \frac{1}{1-\beta}\,[\emptyset(n) + H(n)L(n)]\Gamma(n)[\emptyset(n) + H(n)L(n)]^T \tag{2.18}$$

$$+ \frac{1}{\beta}\,G(n)Q(n)G^T(n)$$

The Hamiltonian is

$$H = \mathrm{Tr}[\Gamma(n+1)P^T(n+1)] + \mathrm{Tr}[\Gamma(n)] \tag{2.19}$$

where the elements, p_{ij}, of P are the co-state variables corresponding to the element x_{ij} of Γ. Now

$$\Gamma(n+1) = \frac{1}{1-\beta} (\emptyset(n)\Gamma(n)\emptyset^T(n) + H(n)L(n)\Gamma(n)\emptyset^T(n) \tag{2.20}$$

$$+ \emptyset(n)\Gamma(n)(H(n)L(n))^T + H(n)L(n)\Gamma(n)(H(n)L(n))^T$$

$$+ \frac{1}{\beta} G(n)Q(n)G^T(n)$$

substituting (2.20) into the Hamiltonian (2.19) yields

$$H = \mathrm{Tr}\left[\frac{1}{1-\beta(n)} \emptyset(n)\Gamma(n)\emptyset^T(n)P^T(n+1)\right] + \mathrm{Tr}\left[\frac{1}{1-\beta(n)} H(n)L(n)\Gamma(n)\emptyset^T(n)P^T(n+1)\right]$$

$$+ \mathrm{Tr}\left[\frac{1}{1-\beta(n)} \emptyset(n)\Gamma(n)(H(n)L(n))^T P^T(n+1)\right] + \mathrm{Tr}\left[\frac{1}{1-\beta(n)} H(n)L(n)\Gamma(n)\right.$$

$$\left. \times\ (H(n)L(n))^T P^T(n+1)\right] + \mathrm{Tr}\left[\frac{1}{\beta(n)} G(n)Q(n)G^T(n)P^T(n+1)\right] + \mathrm{Tr}[\Gamma(n)]$$

Application of the Matrix Minimum Principle requires that, for the optimum controller

$$\frac{\partial H}{\partial L(n)} = 0 \tag{2.21}$$

$$\frac{\partial H}{\partial \Gamma(n)} = P(n) \tag{2.22}$$

with

$$P(N) = I \tag{2.23}$$

Using the relations in Appendix I, the following relationships can be obtained.

$$\frac{\partial H}{\partial \Gamma(n)} = P(n) = (\phi^T(n)P^T(n+1)\phi(n))^T + (\phi^T(n)P^T(n+1)H(n)L(n))^T$$

$$+ (L^T(n)H^T(n)P^T(n+1)\phi(n))^T + (L^T(n)H^T(n)P^T(n+1)H(n)L(n))^T + I$$

or

$$P(n) = (\phi(n) + H(n)L(n))^T P(n+1)(\phi(n) + H(n)L(n))\left(\frac{1}{1-\beta}\right) + I \qquad (2.24)$$

and that $P = P^T$.

Now

$$\frac{\partial H}{\partial L(n)} = 0 = H^T(n)P(n+1)\phi(n)\Gamma^T(n) + H^T(n)P^T(n+1)\phi(n)\Gamma(n)$$

$$+ H^T(n)P(n+1)H(n)L(n)\Gamma^T(n) + H^T(n)P^T(n+1)H(n)L(n)\Gamma(n)$$

or

$$H^T(n)P(n+1)\phi(n)\Gamma(n) + H^T(n)P(n+1)H(n)L(n)\Gamma(n) = 0 \qquad (2.25)$$

$\Gamma(n)$ is positive definite and symmetric so $\Gamma^{-1}(n)$ exists. Equation (2.22) can then be written

$$H^T(n)P(n+1)\phi(n) + H^T(n)P(n+1)H(n)L(n) = 0$$

solving for L(n)

$$L(n) = -(H^T(n)P(n+1)H(n))^{-1}H^T(n)P(n+1)\phi(n) \qquad (2.26)$$

The quantity $H^T(n)P(n+1)H(n)$ is a scalar. The matrix Ricatti equation (2.24) with initial condition (2.23) can be solved backwards in time and thus specify the controller

$$\delta\underline{u}(n) = L(n)\delta\underline{x}(n)$$

As expected, the resulting controller is of the Ricatti form. In particular, this controller is the same as the Ricatti controller with quadratic performance matrices R = R, Q = 0, where R is the weight on the state and Q is the weight on the control. In the procedure presented here, however, the performance index is the weighted trace of the ellipsoid that bounds the system state. Changing R can therefore be thought of as changing the size of the tube containing the state. This design procedure is therefore an alternate approach for this control problem. This control scheme can be implemented as shown in Figure 2.

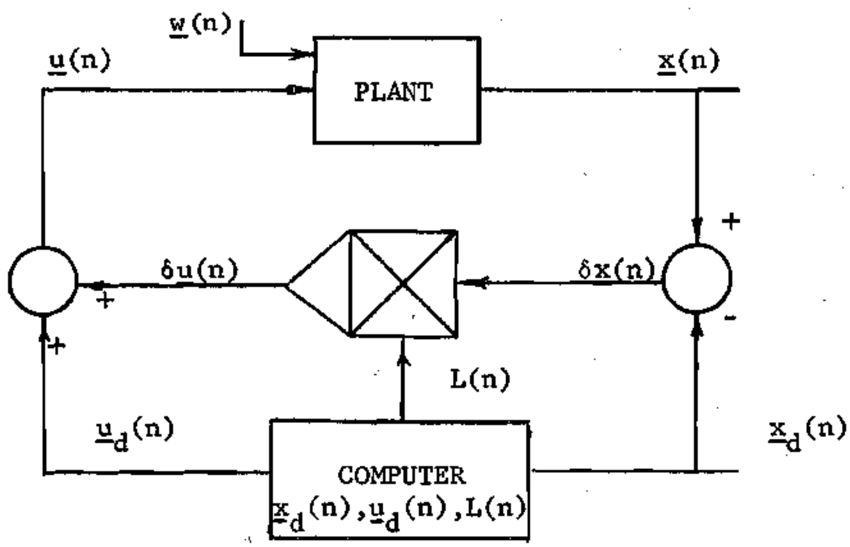


Figure 2. Controller Implementation

This scheme works as follows.

The matrix L(n), the nominal trajectory $\underline{x}_d(n)$, and the nominal control $\underline{u}_d(n)$ are stored in the computer. When the state $\underline{x}(n)$ is measured the appropriate nominal state $\underline{x}_d(n)$ is referenced and $\delta\underline{x}(n)$ is generated.

This perturbation vector $\delta\underline{x}(n)$ is then multiplied by $L(n)$ giving the perturbation control $\delta\underline{u}(n)$. The nominal control $\underline{u}_d(n)$ is then added to $\delta\underline{u}(n)$ to form the control $\underline{u}(n)$ that is applied to the plant.

In this section the control scheme for the perturbation model was developed. The basic idea in this development is the concept of maintaining the state in a region or tube around the nominal trajectory. This region is described by a set of bounding ellipsoids and the controller is assumed to be a linear function of the state. The objective of the controller therefore is to minimize this region containing the state and thus keep the state close to the nominal trajectory. This minization is performed using the Matrix Minimum Principle with the trace of the bounding ellipsoid as a performance index. The resulting control scheme is computed off-line and stored in a computer for use on-line as shown in Figure 2.

Selection of the Parameter $\beta(n)$

In the previous sections on the bounding ellipsoids and the development of the control algorithms, the parameter $\beta(n)$ appeared in the algorithm that generated the ellipsoids. This parameter is free within the range $0 < \beta(n) < 1$. However, this parameter does affect the size of the bounding ellipsoid at each step in time. Therefore, it is desirable that $\beta(n)$ be selected in some optimal manner. In this research the following technique is used.

The bounding ellipsoid is given by

$$\Gamma(n+1) = \frac{1}{1-\beta(n)} \hat{\phi}(n)\Gamma(n)\hat{\phi}^T(n) + \frac{1}{\beta(n)} G(n)Q(n)G^T(n) \tag{2.27}$$

By letting

$$C(n) = \hat{\phi}(n)\Gamma(n)\hat{\phi}^T(n)$$

and
$$D(n) = G(n)Q(n)G^T(n)$$

(2.25) becomes

$$\Gamma(n+1) = \frac{1}{1-\beta(n)} C(n) + \frac{1}{\beta(n)} D(n)$$

Because the trace of $\Gamma(n+1)$ is equal to the sum of the eigenvalues of $\Gamma(n+1)$, the parameter $\beta(n)$ is chosen so as to minimize the trace of $\Gamma(n+1)$ at each step. This is done in the following manner.

$$\mathrm{Tr}[\Gamma(n+1)] = \frac{1}{1-\beta} (c_{11} + c_{22} + \ldots + c_{nn}) + \frac{1}{\beta} (d_{11} + d_{22} + \ldots + d_{nn})$$

$$\mathrm{Tr}[\Gamma(n+1)] = \frac{1}{1-\beta}\left(\sum_{i=1}^{n} c_{ii}\right) + \frac{1}{\beta}\left(\sum_{i=1}^{n} d_{ii}\right) \tag{2.28}$$

At $\beta = 1,0$ the trace is a maximum. This implies there is a minimum between 0,1. Differentiating (2.28) with respect to β gives

$$\frac{d\mathrm{Tr}[\Gamma(n+1)]}{d\beta} = \frac{1}{(1-\beta)^2}\left(\sum_{i=1}^{n} c_{ii}\right) - \frac{1}{\beta^2}\left(\sum_{i=1}^{n} d_{ii}\right) = 0$$

$$\frac{d\mathrm{Tr}[\Gamma(n+1)]}{d\beta} = \beta^2\left(\sum_{i=1}^{n} c_{ii}\right) - (1-\beta)^2\left(\sum_{i=1}^{n} d_{ii}\right) = 0$$

$$= \beta^2\left(\sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n} d_{ii}\right) + 2\beta\left(\sum_{i=1}^{n} d_{ii}\right) - \sum_{i=1}^{n} d_{ii} = 0$$

Solving for β gives

$$\beta = \frac{-\sum_{i=1}^{n} c_{ii}}{\left(\sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n} d_{ii}\right)} \pm \frac{\left[\left(\sum_{i=1}^{n} (d_{ii})\right)^2 + \left(\sum_{i=1}^{n} d_{ii}\right)\left(\sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n} d_{ii}\right)\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n} d_{ii}\right)}$$

$$\beta = \frac{-\sum_{i=1}^{n} d_{ii} \pm \left[\left(\sum_{i=1} d_{ii}\right)\left(\sum_{i=1} c_{ii}\right)\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{n} c_{ii} - \sum_{i=1}^{n} d_{ii}\right)} \tag{2.29}$$

Because $0 < \beta < 1$, the positive sign is used. Therefore, at each step $\beta(n)$ is calculated using (2.29).

In the generation of the ellipsoid that bounds the state, any β in the range $0 < \beta < 1$ can be used. However, to obtain the most conservative estimate possible of the region containing the state, $\beta(n)$ is chosen using the relationship (2.29).

Summary

In summary this chapter has presented the class of systems under consideration in this thesis, the idea of bounding the state space with ellipsoids, and the development and implementation of the system controller. The control procedure is applicable to systems described by nonlinear differential equations that operate in the presence of uncertainty. To use this procedure the system equations are linearized about a known nominal trajectory and the bounds on the noise or model uncertainty are specified. The controller is developed to minimize the region around the nominal trajectory that contains the system state. In this work this region is described by ellipsoids and the controller minimizes the trace of the ellipsoids. The resultant controller is specified by a linear time-varying gain matrix and is computed off-line. This controller matrix, the nominal trajectory, and the nominal control are stored in a digital computer for use in on-line systems control. The controller operation is shown in Figure 2.

CHAPTER III

THE RE-ENTRY CONTROL PROBLEM

Introduction

To evaluate the controller developed in Chapter II, the problem of control of a vehicle re-entering the earth's atmosphere is considered. The mathematical model for this problem is described by a set of nonlinear differential equations and the uncertainty in the problem is most readily described as unknown but bounded. Therefore, the re-entry problem is in the class of problem described in Chapter II and in this chapter it is so formulated that the control scheme developed in the previous chapter can be applied. The controller structure is determined and re-entry of the controlled vehicle is then simulated on the digital computer. This simulated performance is used to evaluate the controller.

Background

There are many problems associated with sending a vehicle into space and returning it to earth. The most critical problem, however, is the guidance and control of the spacecraft while it is re-entering the earth's atmosphere. For a vehicle to successfully re-enter the earth's atmosphere and land, its trajectory must stay within certain bounds or tolerances. The general form of this bounded region or tube is shown in Figure 3 [18]. The skip-out boundary defines the region where the vehicle is traveling too fast and too high and, therefore, will skip out of the atmosphere uncontrolled. The recovery boundary defines the region where the vehicle is traveling too high and too slowly. In this area the vehicle will soon dive into a steep trajectory which will exceed the deceleration boundary. The lower boundaries keep the vehicle from traveling at too high a speed for a given altitude. In this instance the dense atmosphere causes excessive heating or excessive deceleration.

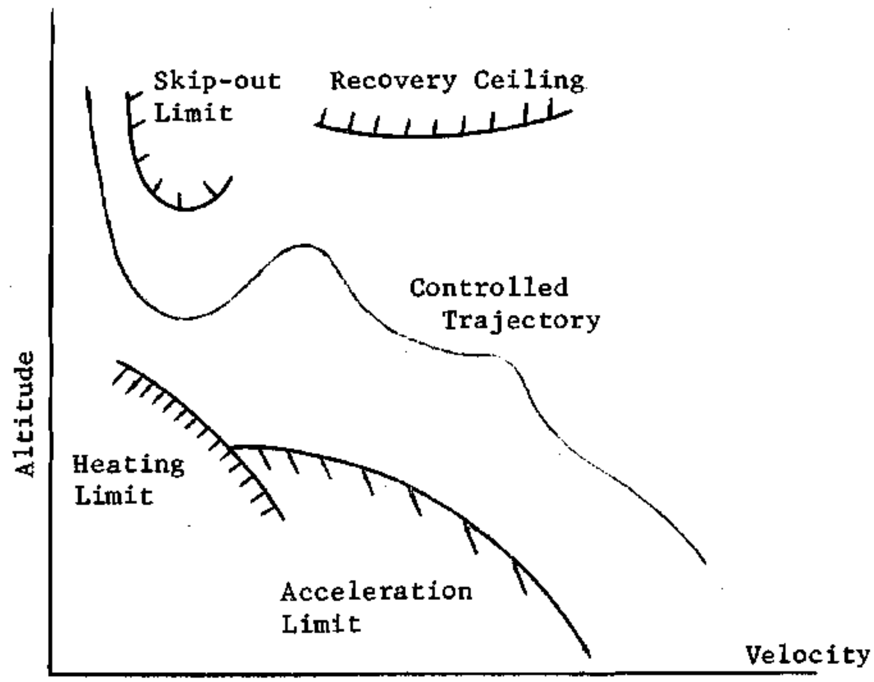


Figure 3. Re-entry Bounding Region

The prime consideration in re-entry guidance, therefore, has to be to keep the vehicle confined to a certain region at all times. This must be done in the presence of external disturbances acting on the vehicle as well as uncertainty concerning the vehicle's characteristics. The vehicle's characteristics are generally not measured very accurately prior to flight and, in fact, can actually change during the flight. Therefore, the uncertainty is characterized by a priori bounds.

Re-entry Model

In this section the nonlinear system of equations describing the motion of the spacecraft are presented [18,19]. These equations assume that the vehicle is approaching a non-rotating, spherical earth and that the motion of the vehicle is planar. The inertial coordinate system is shown in Figure 4. During the re-entry, the major forces acting on the craft are gravity and aerodynamic effects. The force due to the earth's gravity acts towards the center of the earth. The resistance of the atmosphere to the motion of the vehicle is aerodynamic drag (D). The aerodynamic force which tends to deflect the vehicle from its velocity direction is lift (L). These forces are shown acting on the vehicle in Figure 4. The gravitational acceleration is denoted by g, m is the vehicle mass, L and D are the lift and drag accelerations, R_o is the radius of the earth, h is the altitude of the vehicle measured from the earth's surface, and V is the vehicle velocity. The angles are defined in Figure 4. If the forces acting on the vehicle are summed using the Cartesian inertial coordinates, the following equations are obtained

$$F_x = - D \cos \phi + L \sin \phi - mg \sin \psi = m \frac{d(V \cos \phi)}{dt} \tag{3.1}$$

$$F_y = L \cos \phi + D \sin \phi - mg \cos \psi = - m \frac{d(V \sin \phi)}{dt} \tag{3.2}$$

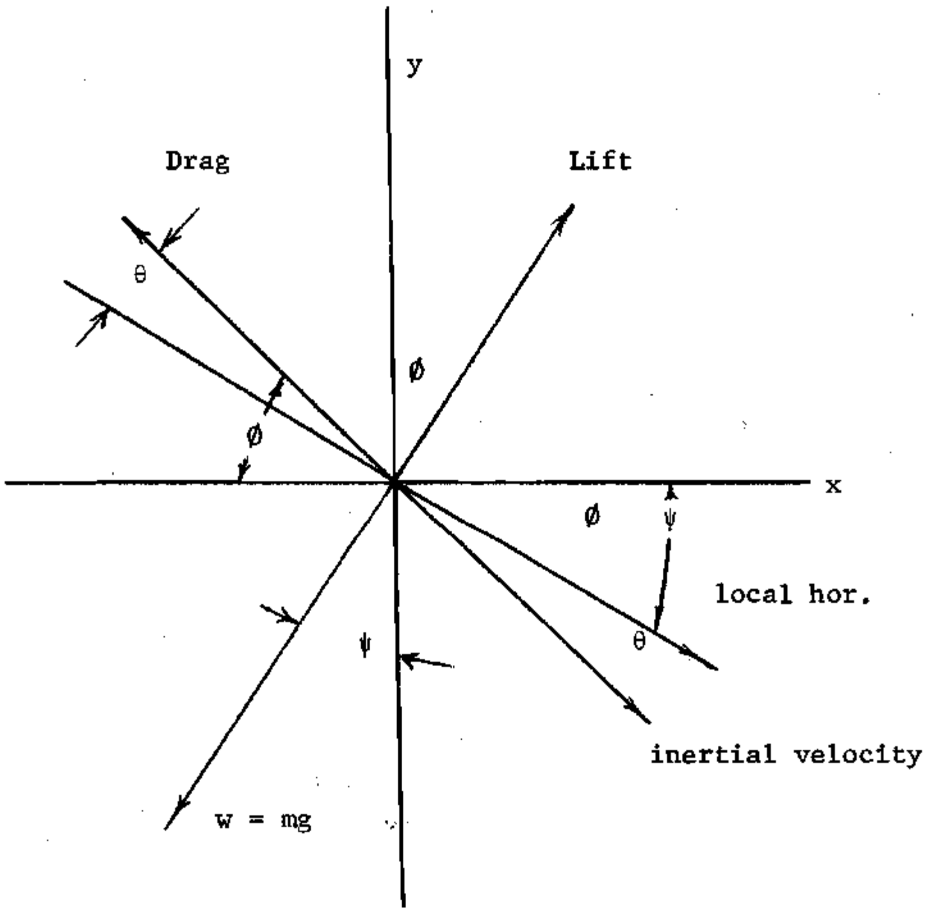


Figure 4. Vehicle Inertial Coordinate System

By resolving these forces in the velocity direction and recognizing that $\theta = \phi - \psi$, (3.1) and (3.2) become

$$- D + mg \sin \theta = m \frac{dV}{dt} \tag{3.3}$$

$$L - mg \cos \theta = - m \frac{dV}{dt} \sin \phi - mV \cos \phi \frac{d\phi}{dt}$$

Now

$$\frac{d\phi}{dt} = \frac{d\theta}{dt} + \frac{d\psi}{dt} = \frac{d\theta}{dt} + \frac{V \cos \theta}{R_o + h}$$

so (3.3) becomes

$$L - mg \cos \theta = - mV \left(\frac{d\theta}{dt} + \frac{V \cos \theta}{R_o + h}\right)$$

The equations of motion are, therefore

$$\frac{dh}{dt} = - V \sin \theta$$

$$m \frac{dV}{dt} = mg \sin \theta - D$$

$$mV \frac{d\theta}{dt} = mg \cos \theta - mV \left(\frac{V \cos \theta}{R_o + h}\right) - L$$

or

$$\frac{dh}{dt} = - V \sin\theta \tag{3.4}$$

$$\frac{dV}{dt} = g \sin \theta - D/m$$

$$\frac{d\theta}{dt} = \frac{g \cos \theta}{V} - \frac{V \cos \theta}{R_o + h} - \frac{L}{mV}$$

The lift (L) and drag (D) are dependent on the atmospheric density, velocity of the vehicle relative to the air, and the physical characteristics of the vehicle. This dependency is

$$L = \frac{1}{2} V^2 \rho \; C_L \; S$$

$$D = \frac{1}{2} V^2 \rho \; C_D \; S$$

where ρ is the density of the air, S is the wing plan-form area, and C_L and C_D are the lift and drag coefficients.

In this work an exponential model is used for the atmosphere.

$$\rho = \rho_o \exp(\beta h)$$

ρ_o is the air density at sea level.

The lift and drag coefficients are functions of velocity, vehicle shape, and the angle of attack, α. The angle of attack is the angle between the direction of velocity and the direction of the zero lift axis of the vehicle. The vehicle is controlled by varying this angle. It is assumed here that these coefficients are functions only of the angle of attack. The lift-drag polar used here is

$$C_L = C_{LO} \sin\alpha \cos\alpha$$

$$C_D = C_{DO} + C_{DL} \sin^2\alpha$$

Substitution of these expressions into (3.4) gives

$$\frac{dh}{dt} = - V \sin\theta \tag{3.5}$$

$$\frac{dV}{dt} = g \sin\theta - \frac{1}{2m} V^2 \rho_o e^{\beta h} S(C_{DO} + C_{DL} \sin^2\alpha)$$

$$\frac{d\theta}{dt} = \frac{g \cos\theta}{V} - \frac{V \cos\theta}{R_o + h} - \frac{1}{2m} V \rho_o e^{\beta h} S(C_{LO} \sin\alpha \cos\alpha)$$

These are the deterministic equations that describe the motion of the vehicle. That is, these equations describe the vehicle motion if all assumptions in their derivation are satisfied and if the parameters are exactly known.

In reality, however, these parameters are not known exactly. For example, consider the vehicle characteristics. The lift and drag coefficients are seldom established very accurately prior to flight and can undergo significant changes while the vehicle is re-entering the atmosphere [21]. The atmospheric density affects the vehicle motion and is known to vary from day to day as well as month to month. All of these uncertain elements must somehow be reflected in the system equations (3.5). Therefore, as an effort to more accurately model the vehicle's motion during the re-entry, the equations must have a term added to them that mathematically reflects these unknown effects acting on the system. This is the purpose of the terms $n_1(t)$ and $n_2(t)$ in the equations shown below.

$$\frac{dh}{dt} = - V \sin \theta$$

$$\frac{dV}{dt} = g \sin \theta - \frac{1}{2m} V^2 \rho_o e^{\beta h} S(C_{DO} + C_{DL} \sin^2 \alpha) + n_1(t)$$

$$\frac{d\theta}{dt} = \frac{g \cos \theta}{V} - \frac{V \cos \theta}{R_o + h} - \frac{1}{2m} V \rho_o e^{\beta h} S(C_{LO} \sin \alpha \cos \alpha) + n_2(t)$$

Now the question of how to model $n_1(t)$ and $n_2(t)$ remains. They are certainly not deterministic functions of time. On the other hand, they may or may not be stochastic processes. If they are stochastic processes, the statistics are not generally known a priori. However, these two terms representing the uncertainty in the re-entry process do share two common characteristics. They are unknown but can be bounded by the designer. Available to the designer are published data and information that can be used to determine these bounds. Using [20] as a guide, it is seen that ± 10 percent density variations can be used to bound this source of uncertainty. The available literature [21] indicates that a vehicle's characteristics can be determined to within ± 10 percent of their true value. Therefore, the uncertainty in the re-entry problem is treated as a set constrained process. That is, only the bounds on the disturbances are assumed to be known.

Re-entry Controller

Before the controller developed in Chapter II is applied to the re-entry system model developed in the previous section, several steps must be taken. First the state variables must be selected, second the nominal or desired trajectory and control must be specified, third the nonlinear system of equations must be linearized about this nominal trajectory, and fourth the resulting linearized equations must be discretized.

In this problem the state variables are the altitude, h, of the vehicle measured from the earth's surface; the velocity, V, of the vehicle; and the angle, θ, between the velocity direction and the local horizontal. The control variable is the angle of attack which is the angle between the direction of velocity and the direction of the zero lift axis of the vehicle.

Letting $x_1 = h$, $x_2 = V$, $x_3 = \theta$, and $u = \alpha$, the motion equations are

$$\dot{x}_1 = - x_2 \sin x_3 \tag{3.6}$$

$$\dot{x}_2 = g \sin x_3 - \frac{1}{2m} x_2^2 \rho_o e^{\beta x_1} S(C_{DO} + C_{DL} \sin^2 u) + n_1(t)$$

$$\dot{x}_3 = \frac{g \cos x_3}{x_2} - \frac{x_2 \cos x_3}{R_o + x_1} - \frac{1}{2m} x_2 \rho_o e^{\beta x_1} S(C_{LO} \sin u \cos u) + n_2(t)$$

or

$$\underline{\dot{x}} = \underline{f}(\underline{x},u) + \underline{n}$$

This system of equations (3.6) can now be linearized by expanding in a Taylor Series about a given nominal trajectory. The difficult problem of calculating or selecting this trajectory and control is not considered here. It is assumed the nominal trajectory and control are specified. The expansion is:

$$\underline{\dot{x}} = \underline{f}(\underline{x},u) + \underline{n} = \underline{f}(\underline{x}_d,u_d) + \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\substack{\underline{x}_d \\ u_d}} (\underline{x} - \underline{x}_d) + \left.\frac{\partial \underline{f}}{\partial u}\right|_{\substack{\underline{x}_d \\ u_d}} (u - u_d) + \underline{R} + \underline{n}$$

where R represents the higher order terms in the Series.

Defining

$$\delta\underline{\dot{x}} = \underline{\dot{x}} - \underline{\dot{x}}_d$$

$$\delta\underline{x} = \underline{x} - \underline{x}_d$$

$$\delta u = u - u_d$$

$$A(t) = \left.\frac{\partial \underline{f}}{\partial \underline{x}}\right|_{\substack{\underline{x}_d \\ u_d}}$$

(continued)

$$B(t) = \left.\frac{\partial \underline{f}}{\partial u}\right|_{\substack{\underline{x}_d \\ u_d}}$$

and performing the indicated operations gives:

$$A_{11} = \frac{\partial f_1}{\partial x_1} = 0$$

$$A_{12} = \frac{\partial f_1}{\partial x_2} = - \sin x_3$$

$$A_{13} = \frac{\partial f_1}{\partial x_3} = - x_2 \cos x_3$$

$$A_{21} = \frac{\partial f_2}{\partial x_1} = - \frac{1}{2m} x_2^2 \rho_o \beta e^{\beta x_1} S(C_{DO} + C_{DL} \sin^2 u)$$

$$A_{22} = \frac{\partial f_2}{\partial x_2} = - \frac{1}{m} x_2 \rho_o e^{\beta x_1} S(C_{DO} + C_{DL} \sin^2 u)$$

$$A_{23} = \frac{\partial f_2}{\partial x_3} = g \cos x_3$$

$$A_{31} = \frac{\partial f_3}{\partial x_1} = \frac{x_2 \cos x_3}{(R_o + x_1)^2} - \frac{1}{2m} x_2 \rho_o \beta e^{\beta x_1} S(C_{LO} \sin u \cos u)$$

$$A_{32} = \frac{\partial f_3}{\partial x_2} = - \left(\frac{g \cos x_3}{x_2^2} + \frac{\cos x_3}{R_o + x_1} + \frac{1}{2m} \rho_o e^{\beta x_1} S(C_{LO} \sin u \cos u)\right)$$

$$A_{33} = \frac{\partial f_3}{\partial x_3} = - \frac{g \sin x_3}{x_2} + \frac{x_2 \sin x_3}{R_o + x_1}$$

$$B_{11} = \frac{\partial f_1}{\partial u} = 0$$

(continued)

$$B_{21} = \frac{\partial f_2}{\partial u} = -\frac{1}{m} x_2^2 \rho_o e^{\beta x_1} S(C_{DL} \sin u \cos u)$$

$$B_{31} = \frac{\partial f_3}{\partial u} = -\frac{1}{2m} x_2 \rho_o e^{\beta x_1} SC_{LO}(\cos^2 u - \sin^2 u)$$

and

$$\begin{bmatrix} \delta\dot{x}_1 \\ \delta\dot{x}_2 \\ \delta\dot{x}_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{bmatrix} + \begin{bmatrix} B_{11} \\ B_{21} \\ B_{31} \end{bmatrix} \delta u + \underline{n} + \underline{R}$$

where the A and B matrices are evaluated at $\underline{x} = \underline{x}_d$ and $u = u_d$ and are, therefore, functions of time. The bounded disturbance term and the Taylor Series remainder term can be combined to give

$$\delta\dot{\underline{x}} = A(t)\delta\underline{x} + B(t)\delta u + \underline{N} \tag{3.7}$$

Next the linearized system (3.7) is discretized. The linear continuous system model $\delta\dot{\underline{x}} = A(t)\delta\underline{x} + B(t)\delta u$ must be represented by the discrete model

$$\delta\underline{x}(n+1) = \phi(n)\delta\underline{x}(n) + H(n)\delta u(n) \tag{3.8}$$

The computation of the discrete model is performed on the digital computer.

At this point the discrete perturbation model (3.8) is recognized as the same model (2.6) for which the controller in Chapter II was developed using the performance criterion (2.17). Therefore, the re-entry system (3.8) is controlled such that the performance measure of the bounding ellipsoids containing the state is minimized. This performance criterion is

$$J = \mathrm{Tr}[\Gamma(N)] + \sum_{n=0}^{N-1} \mathrm{Tr}[\Gamma(n)]$$

where $\Gamma^{-1}(n)$ is the weighting matrix describing the ellipsoid that contains the re-entry state. The re-entry controller for the system (3.8) is given by

$$\delta u(n) = L(n)\delta \underline{x}(n) \tag{3.9}$$

where

$$L(n) = -\left(H^T(n)P(n+1)H(n)\right)^{-1}H^T(n)P(n+1)\phi(n) \tag{3.10}$$

and $$P(n) = \left(\phi(n) + H(n)L(n)\right)^T P(n+1)\left(\phi(n) + H(n)L(n)\right) + I \tag{3.11}$$

$$P(N) = I \tag{3.12}$$

The relationships (3.9 - 3.12) specify the re-entry control scheme.

With the re-entry controller specified, the re-entry process can now be simulated on the digital computer. The details of the simulation and the results are presented in the next chapter.

CHAPTER IV

RE-ENTRY CONTROLLER PERFORMANCE

Introduction

At this point the controller for the re-entry problem has been developed. The important practical consideration of how well the controlled spacecraft performs must now be considered The controlled system is evaluated by simulating the re-entry on the digital computer. The re-entry is simulated under several different conditions. The details of the simulations, the results, and the analysis of the results are presented in this chapter.

As noted in Chapter III, controller development requires that the nonlinear system model (3.5) be linearized and then discretized. To linearize the model for a simulation requires the knowledge of certain constant and vehicle parameters as well as a nominal trajectory and control about which to linearize. Since the re-entry is into the earth's atmosphere the following constants are known:

earth's radius	$R_o = 2.09 \times 10^7$ feet
gravity constant	$g = 32.2$ ft/sec
air density at sea level	$\rho_o = 2.70 \times 10^{-4}$ lb - sec^2/ft^4

The vehicle parameters for these simulations are taken to be [19]:

mass of vehicle	$m = 250$ lb - sec^2/ft
wing-plan form area	$S = 66.5$ ft^2

lift-drag polar parameters $C_{DO} = 0.274$

$$C_{DL} = 1.8$$

$$C_{LO} = 1.2$$

The nominal trajectory and control that is needed is shown in Table 1 [19].

Using these constants, vehicle parameters, and Table 1, the linearized system matrices A(t) and B(t) are evaluated. This is done in the following manner. Table 1 is linearly interpolated over five second intervals and these resulting values are substituted into A(t) and B(t). This gives a model sample size of 0.5 second.

With the A(t) and B(t) matrices known, the discrete model

$$\delta \underline{x}(n + 1) = \emptyset(n)\delta \underline{x}(n) + H(n)\delta u(n)$$

is generated in the computer. A sample time of T = .05 seconds is used to compute the discrete model. Using $\emptyset(n)$ and H(n), the control algorithms developed in Chapter II are used to generate the control for the vehicle.

A weighting matrix is used in the generation of the controller for the linearized re-entry model. This linearized model is obtained from truncation of second-order and higher terms in a Taylor Series expansion of the nonlinear system equations. Therefore, the weighting matrix is chosen to reduce the effects of this truncation. The second derivations $\frac{\delta^2 f_i}{\delta x_i^2}$, of the nonlinear system are:

Table 1. Nominal Trajectory and Control

Time (seconds)	Altitude-x_1 (feet)	Velocity-x_2 (ft/second)	Flight angle-x_3 (degrees)	Control (degrees)
0	221227.00	35677.00	5.83	40.12
5	204219.00	35177.00	5.11	35.72
10	190269.00	34396.00	4.00	29.35
15	180462.00	33437.00	2.57	22.60
20	175244.00	32469.00	1.06	16.99
25	174163.00	31611.00	- .02	12.78
30	176240.00	30915.00	-1.22	9.37
35	180421.00	30385.00	-1.85	6.02
40	185795.00	2996.00	-2.18	2.13
45	191652.00	29704.00	-2.28	- 2.42
50	197493.00	29470.00	-2.22	- 7.21
55	203015.00	29267.00	-2.08	-11.64
60	208085.00	29081.00	-1.90	-15.38
65	212671.00	28911.00	-1.72	-18.40
70	216794.00	28754.00	-1.55	-20.81
75	220498.00	28611.00	-1.40	-22.74
80	223829.00	28479.00	-1.27	-24.28
85	226831.00	28359.00	-1.15	-25.54
90	229542.00	28248.00	-1.04	-26.58
95	231997.00	28146.00	- .95	-27.45
100	234223.00	28051.00	- .86	-28.20
105	236244.00	27963.00	- .79	-28.83
110	238080.00	27880.00	- .72	-29.38
115	239747.00	27803.00	- .65	-29.86
120	241260.00	27729.00	- .59	-30.28

$$\frac{\partial^2 f_1}{\partial x_1^2} = 0 \tag{4.1}$$

$$\frac{\partial^2 f_2}{\partial x_2^2} = -\frac{1}{m}\rho_o e^{\beta x_1} S(C_{DO} + C_{DL}\sin^2 u)$$

$$\frac{\partial^2 f_3}{x_3^2} = -\frac{g\cos x_3}{x_2} + \frac{x_2\cos x_3}{R_o + x_1}$$

The terms in the matrix are chosen to be inversely proportional to these second derivations. Substituting typical values in (4.1) the matrix is

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1000 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.2}$$

The values are inversely proportional because $\Gamma_x(n)$ is used in the performance index and $\Gamma_x^{-1}(n)$ is the bounding ellipsoid matrix. Experimentation with several other weighting matrices indicates that (4.2) gives the most satisfactory performance. Figure 5 gives a flowchart for generating the re-entry controller.

Using the constants, nominal trajectory, and the controller described in the preceding paragraphs, the re-entry process is simulated on the digital computer. The simulations can be classified as deterministic and stochastic. The performance of the system and the bounding ellipsoid is studied for both types of simulations.

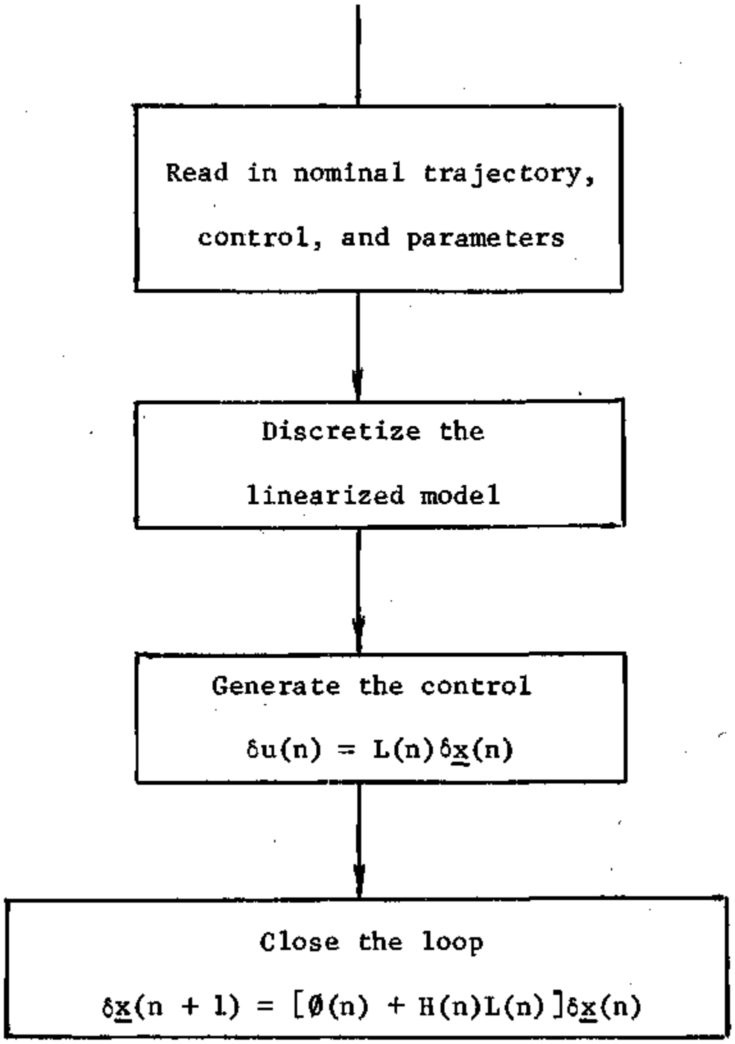


Figure 5. Computation of Re-entry Controller

Deterministic Performance

The re-entry is first simulated assuming that no noise is acting on the system. The purpose of this simulation is to study the transient and steady-state response of the controlled system. The simulation is performed in the following manner. With the model

$$\delta \underline{x}(n+1) = \emptyset(n)\delta \underline{x}(n) + H(n)\delta u(n) \tag{4.3}$$

and control

$$\delta u(n) = L(n)\delta \underline{x}(n) \tag{4.4}$$

stored in the computer, the loop is closed by substituting (4.4) into (4.3) to obtain

$$\delta \underline{x}(n+1) = [\emptyset(n) + H(n)L(n)]\delta \underline{x}(n) \tag{4.5}$$

or

$$\delta \underline{x}(n+1) = \hat{\emptyset}(n)\delta \underline{x}(n) \tag{4.6}$$

where $$\hat{\emptyset}(n) = \emptyset(n) + H(n)L(n)$$

By selecting initial conditions, $\delta \underline{x}(0)$, the deterministic re-entry process (4.6) is simulated. A representative trajectory is shown in Figure 6.

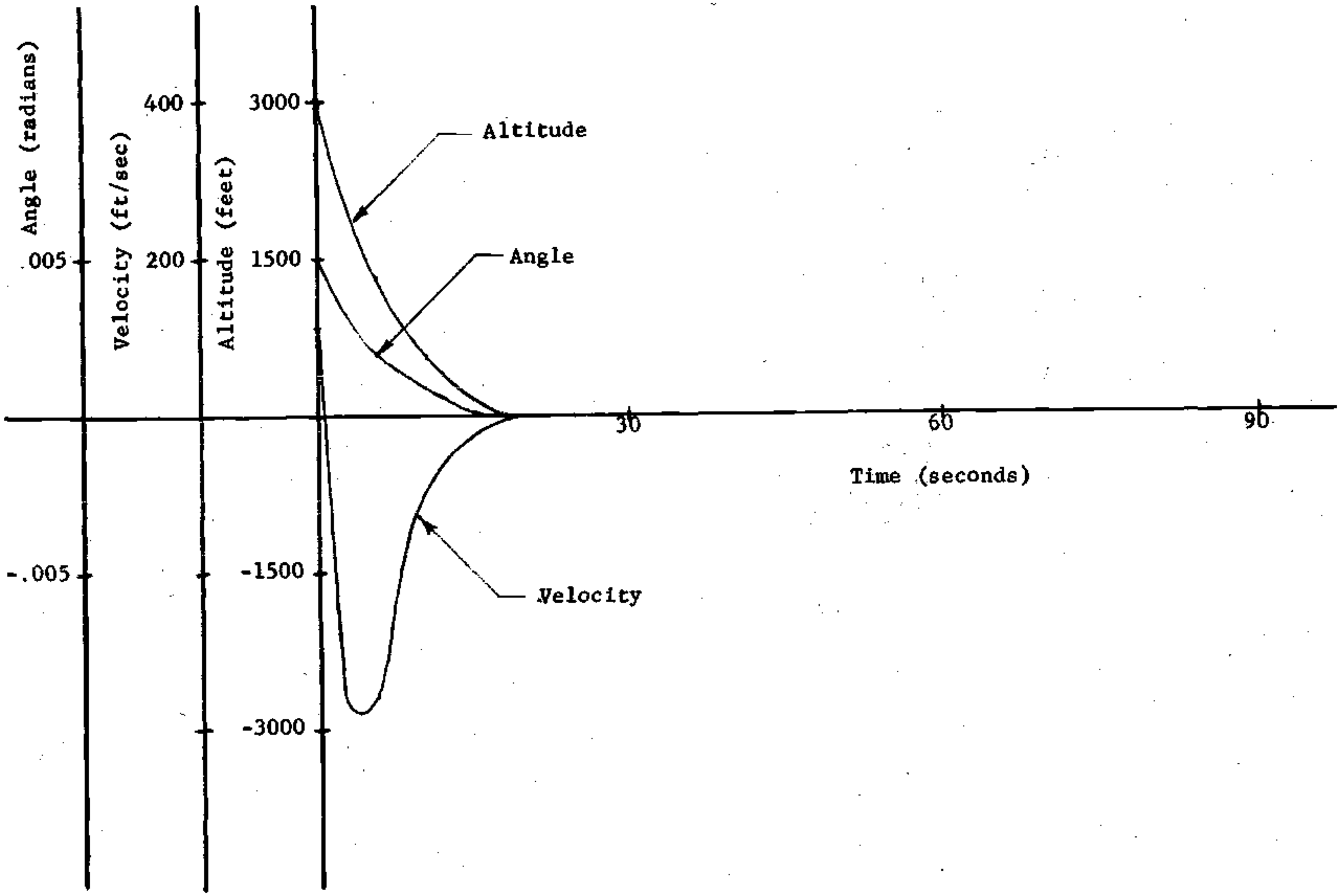


Figure 6. Deterministic Perturbations

In this simulation, the following observations are made. First the re-entry perturbation controller developed in Chapter III does work. That is, the closed-loop system reduces deviations from the nominal trajectory to zero. Second the transient response is reasonable, in that, there are no undue oscillations on the one hand while on the other the perturbed state variables reach zero in a reasonable time.

While these observations apply to the re-entry problem developed in the previous chapter and simulated in this chapter, there are broader implications. The re-entry problem is a practical and difficult control problem. The design procedure developed in Chapter II is applied to this problem in a straightforward manner and results in a workable control scheme. That is, the concept of finding the linear controller that minimizes directly the region around the nominal trajectory results in a valid design procedure. Because this procedure works for a difficult problem, the re-entry problem, it is reasonable to assume it can be applied to many other control problems.

<u>Bounding Ellipsoid Performance--Deterministic Case</u>

It was shown in Chapter II for the deterministic case that the state of the system

$$\delta\underline{x}(n + 1) = \hat{\phi}(n)\delta\underline{x}(n)$$

where

$$\hat{\phi}(n) = \phi(n) + H(n)L(n)$$

and initial state

$$\delta\underline{x}(0) \in \Omega_x(0) = \{\delta x \in R^n: (\delta\underline{x} - \underline{c}_x(0))^T \Gamma_x^{-1}(0)(\delta\underline{x} - \underline{c}_x(0)) \le 1\}$$

is contained in the region

$$\Omega_x(n) = \{\delta\underline{x} \in R^n: (\delta\underline{x} - \underline{c}_x(n))^T \Gamma_x^{-1}(n)(\delta\underline{x} - \underline{c}_x(n)) \le 1\}$$

described by the ellipsoid $\Gamma_x^{-1}(n)$.

To determine the performance of this bounding ellipsoid, $\Gamma_x(n)$ is generated using

$$\Gamma_x(n + 1) = \hat{\emptyset}(n)\Gamma_x(n)\hat{\emptyset}^T(n) \tag{4.7}$$

where $\hat{\emptyset}(n)$ is the closed-loop re-entry system matrix. The trace of $\Gamma_x(n)$ is used to give a measure of the ellipsoid performance and the region $\Omega_x(n)$ containing $\delta\underline{x}(n)$ during re-entry. Shown in Figure 7 is a plot of the trace of $\Gamma_x(n)$ versus time into the trajectory. The trace initially increases and then decays to zero. This implies that the bounding ellipsoid also increases and then decreases to essentially a point.

The deterministic ellipsoid performs in this manner because of the following reasoning. The ellipsoid is generated from (4.7). Let $\Gamma_x(0)$ be

$$\Gamma_x(0) = \begin{bmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{bmatrix}$$

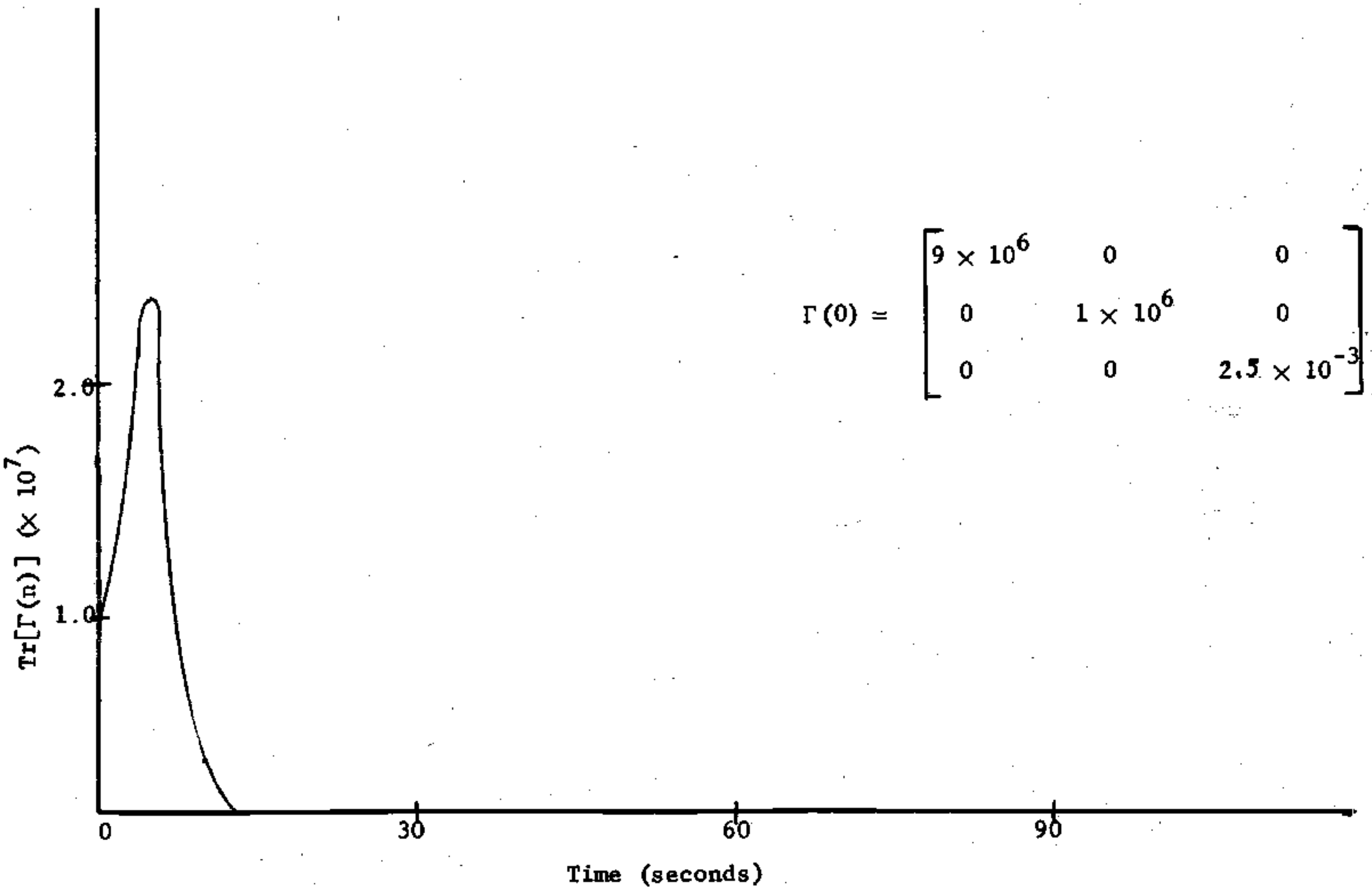


Figure 7. Ellipsoidal Trace versus Time (Q = 0)

If the eigenvalues of $\hat{\emptyset}(n)$ are real and distinct, the state coordinate system can be rotated such that $\hat{\emptyset}(n)$ is similar to

$$\Lambda(n) = \Lambda^T(n) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Then the ellipsoid at the first time increment is

$$\Gamma_x(1) = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & 0 \\ 0 & 0 & y_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$$\Gamma_x(1) = \begin{bmatrix} \lambda_1^2 y_{11} & 0 & 0 \\ 0 & \lambda_2^2 y_{22} & 0 \\ 0 & 0 & \lambda_3^2 y_{33} \end{bmatrix}$$

The $|\lambda_i(n)| < 1$ for all $n = 0,1,2, \ldots, N$ so the traces in the rotated system are as follows:

$$Tr[\Gamma(0)] = y_{11} + y_{22} + y_{33}$$

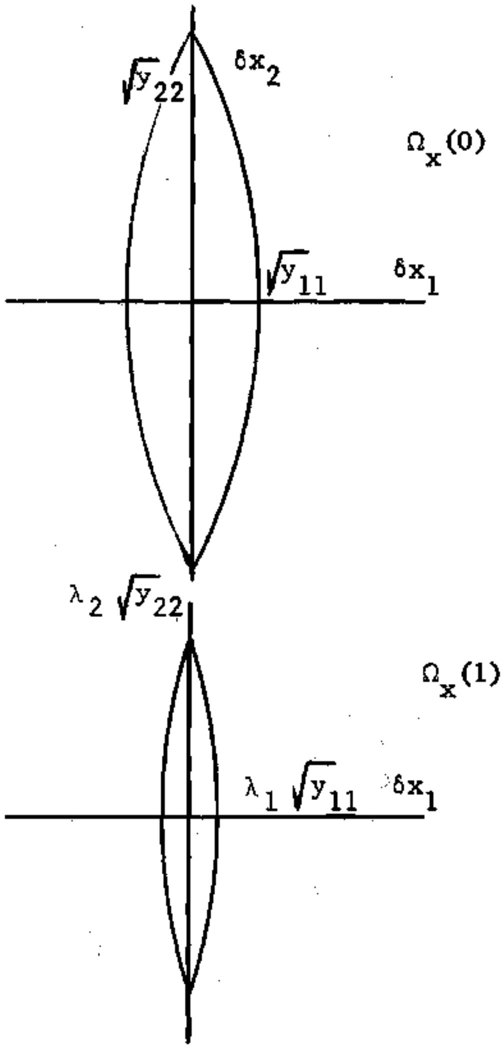
$$Tr[\Gamma(1)] = \lambda_1^2 y_{11} + \lambda_2^2 y_{22} + \lambda_3^2 y_{33}$$

$$Tr[\Gamma(1)] < Tr[\Gamma(0)]$$

In two dimensions the ellipsoids look like

$$\delta x^T \Gamma_x^{-1}(0)\delta x \le 1$$

$$\delta x^T \Gamma_x^{-1}(1)\delta x \le 1$$



If the eigenvalues are complex and distinct the same reasoning as above can be followed, but it is a little more difficult to illustrate graphically. The initial trace is

$$\mathrm{Tr}[\Gamma(0)] = y_{11} + y_{22} + y_{33}$$

and

$$\mathrm{Tr}[\Gamma(1)] = \lambda_1^2 y_{11} + \lambda_1^{*2} y_{22} + \lambda_2^2 y_{22}$$

where λ_1 and λ_1^* are complex conjugates. The magnitude of the trace is

$$|\mathrm{Tr}[\Gamma(1)]| = |\lambda_1^2 y_{11} + \lambda_1^{*2} y_{22} + \lambda_2^2 y_{22}| \leq |\lambda_1^2 y_{11}| + |\lambda_1^{*2} y_{22}| + |x_2^2 y_{22}|$$

Letting $\lambda_1 = a + jb$ then $\lambda_1^* = a - jb$

and
$$|\lambda_1| = |\lambda_1^*| = \sqrt{a^2+b^2} \leq 1$$

$$|\lambda_1^2| = |\lambda_1^{*2}| \leq 1 , \quad |\lambda_2^2| \leq 1$$

Therefore $\mathrm{Tr}[\Gamma(1)]| < |\mathrm{Tr}[\Gamma(0)]|$ and the trace in this case also decays to zero as shown in Figure 7.

Monte Carlo Simulations

An important consideration in the re-entry problem is the performance of the controller in the presence of noise. For the nominal trajectory used in this problem, the principle source of noise is parameter uncertainty. This uncertainty can be represented mathematically by

$$\underline{\dot{x}} = \underline{f}(\underline{x}, \underline{a}, u) \tag{4.8}$$

where $\underline{a}$ is the vector of parameters containing the uncertainty. If (4.8) is expanded in a Taylor Series about the nominal characteristics and is truncated after the linear terms, (4.8) becomes

$$\delta\underline{\dot{x}} = \left.\frac{\partial \underline{f}}{\partial \underline{a}}\right|_N \delta\underline{a}$$

The uncertain parameters are the lift coefficient, C_L, and the drag coefficient, C_D. That is,

$$\underline{a} = \begin{bmatrix} C_L \\ C_D \end{bmatrix}$$

so the $\frac{\partial \underline{f}}{\partial \underline{a}}$ is

$$\frac{\partial \underline{f}}{\partial \underline{a}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2m} \rho x_2^2 S \\ \frac{1}{2m} \rho x_2 S & 0 \end{bmatrix}$$

Typical values for these parameters give

$$\left.\frac{\partial \underline{f}}{\partial \underline{a}}\right|_N = \begin{bmatrix} 0 & 0 \\ 0 & 50 \\ .002 & 0 \end{bmatrix}$$

Using a 10 percent change in the nominal characteristics $\pm .1\ C_{LN}$ and $\pm .1\ C_{DN}$ results in

$$\left.\frac{\partial \underline{f}}{\partial \underline{a}}\right|_N \underline{a} = \begin{bmatrix} 0 \\ \pm 5 \\ \pm .0001 \end{bmatrix}$$

This is the basis for letting the noise distribution matrix be

$$G = \begin{bmatrix} 0 \\ 5 \\ .0001 \end{bmatrix}$$

With this G matrix and the $\hat{\emptyset}(n)$ matrix, the system model is

$$\delta\underline{x}(n + 1) = \hat{\emptyset}(n)\,\delta\underline{x}(n) + Gw(n)$$

where w(n) is a stochastic process.

Two different noise models are used in these Monte Carlo simulations. In the first simulation white Gaussian noise with zero mean and a variance of one is used as input In the second simulation uniform noise with a mean of zero and variance of one is the input. One hundred runs are made using each noise model. The performance of the system in this environment is shown in Figures 5 to 11. Figures 5 and 6 are the sample mean versus time and Figures 7 and 8 are the sample variances versus time. The sample mean and variance are calculated using

$$\delta\overline{x}_i = \frac{1}{N} \sum_{i=1}^{N} \delta x_i$$

$$\sigma^2_{\delta x_i} = \frac{1}{N} \sum_{i=1}^{N} \delta x_i^2 - \left(\frac{1}{N} \sum_{i=1}^{N} \delta x_i\right)^2$$

where N = 100.

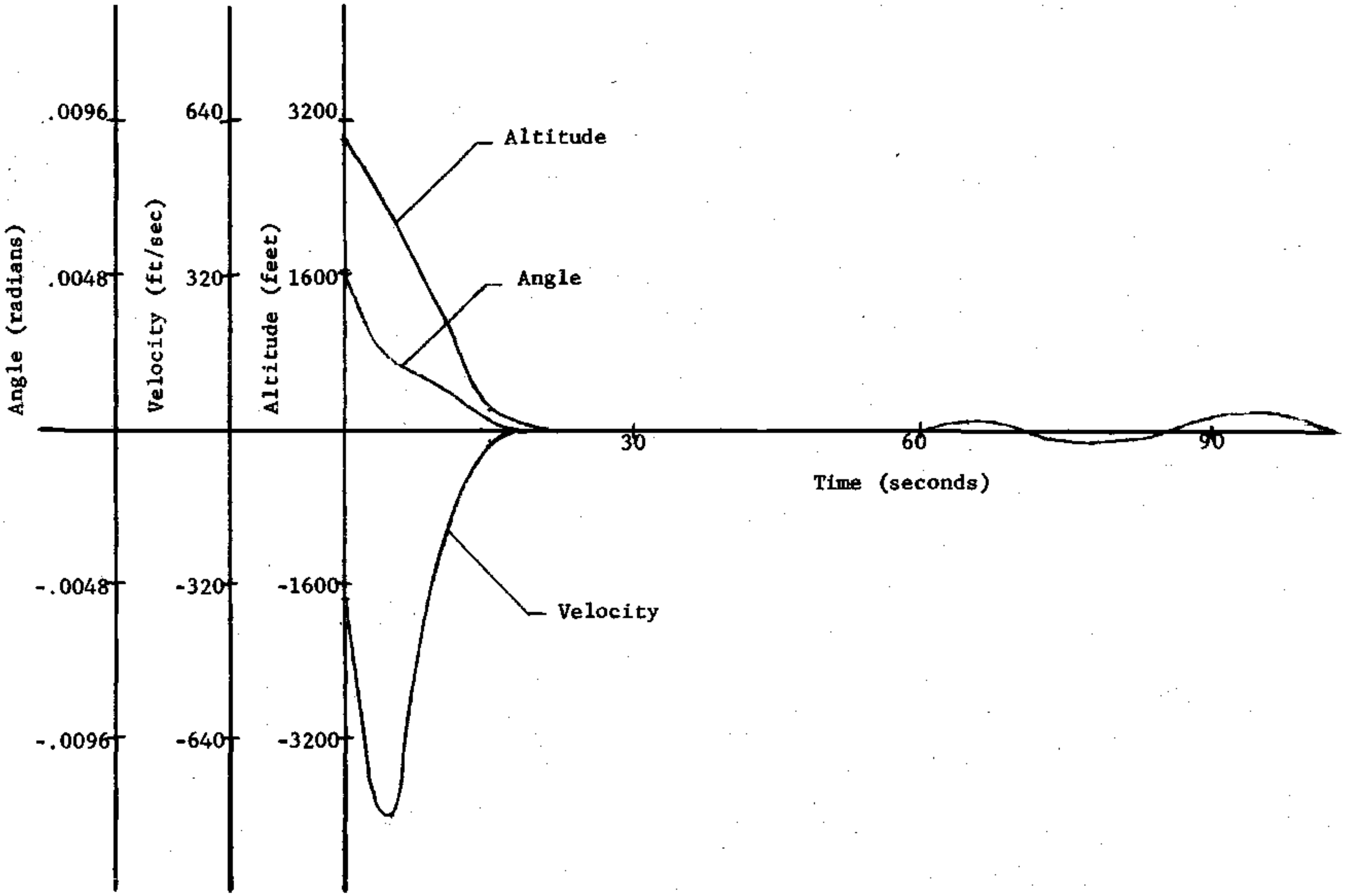


Figure 8. Sample Means, N[0,1]

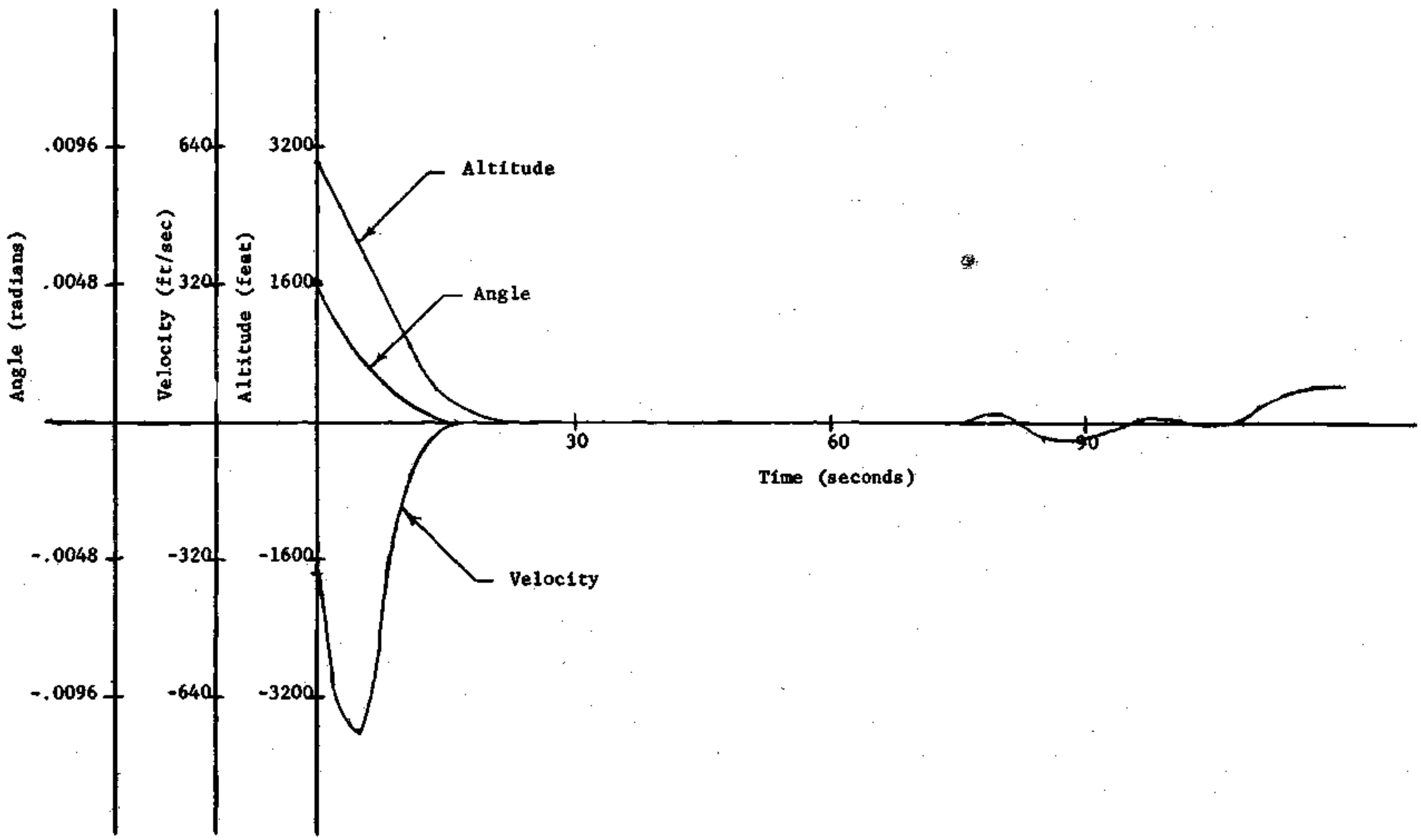


Figure 9 . Sample Means, Uniform Noise [-1.73,1.73]

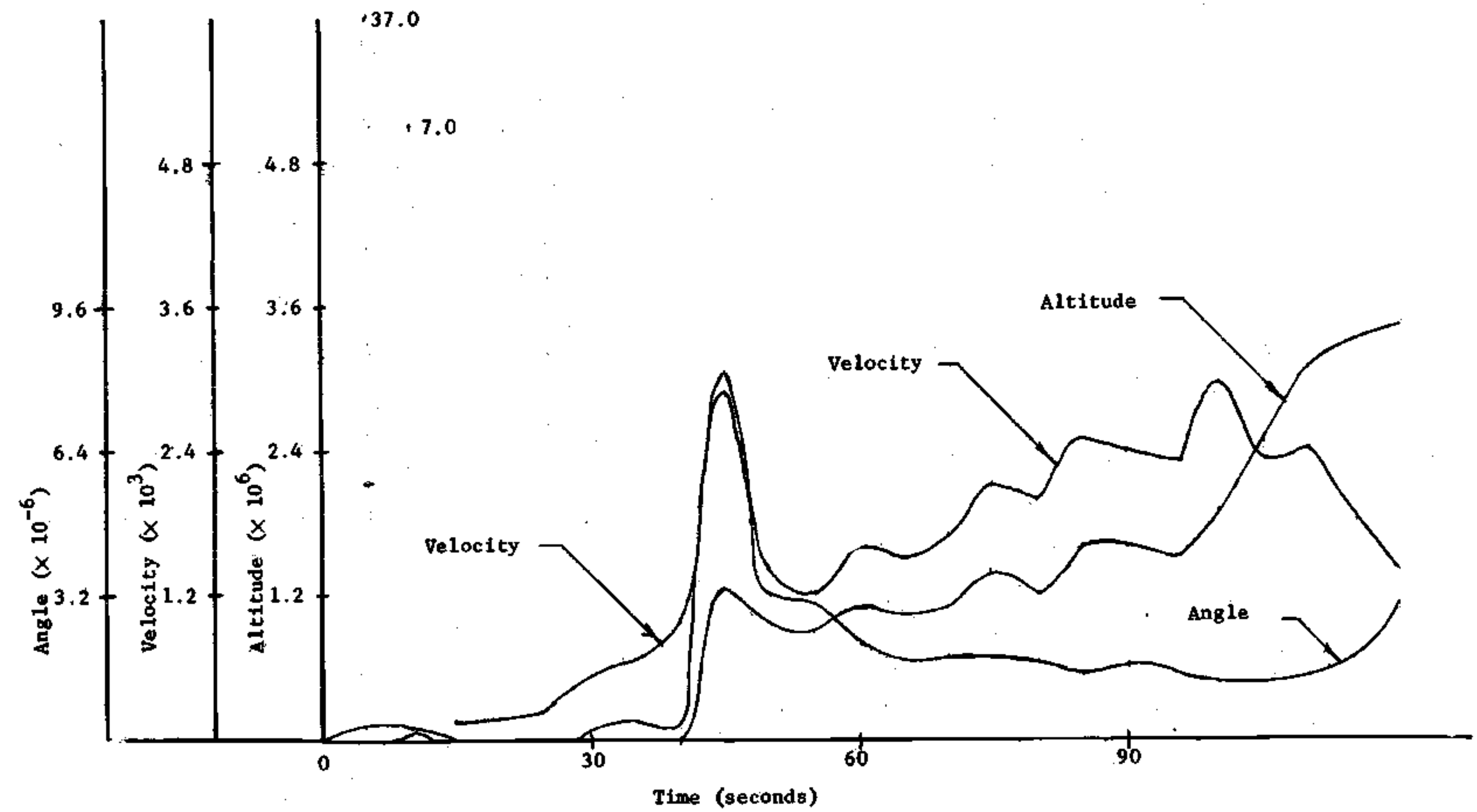


Figure 10. Sample Variance, N[0,1] versus Time

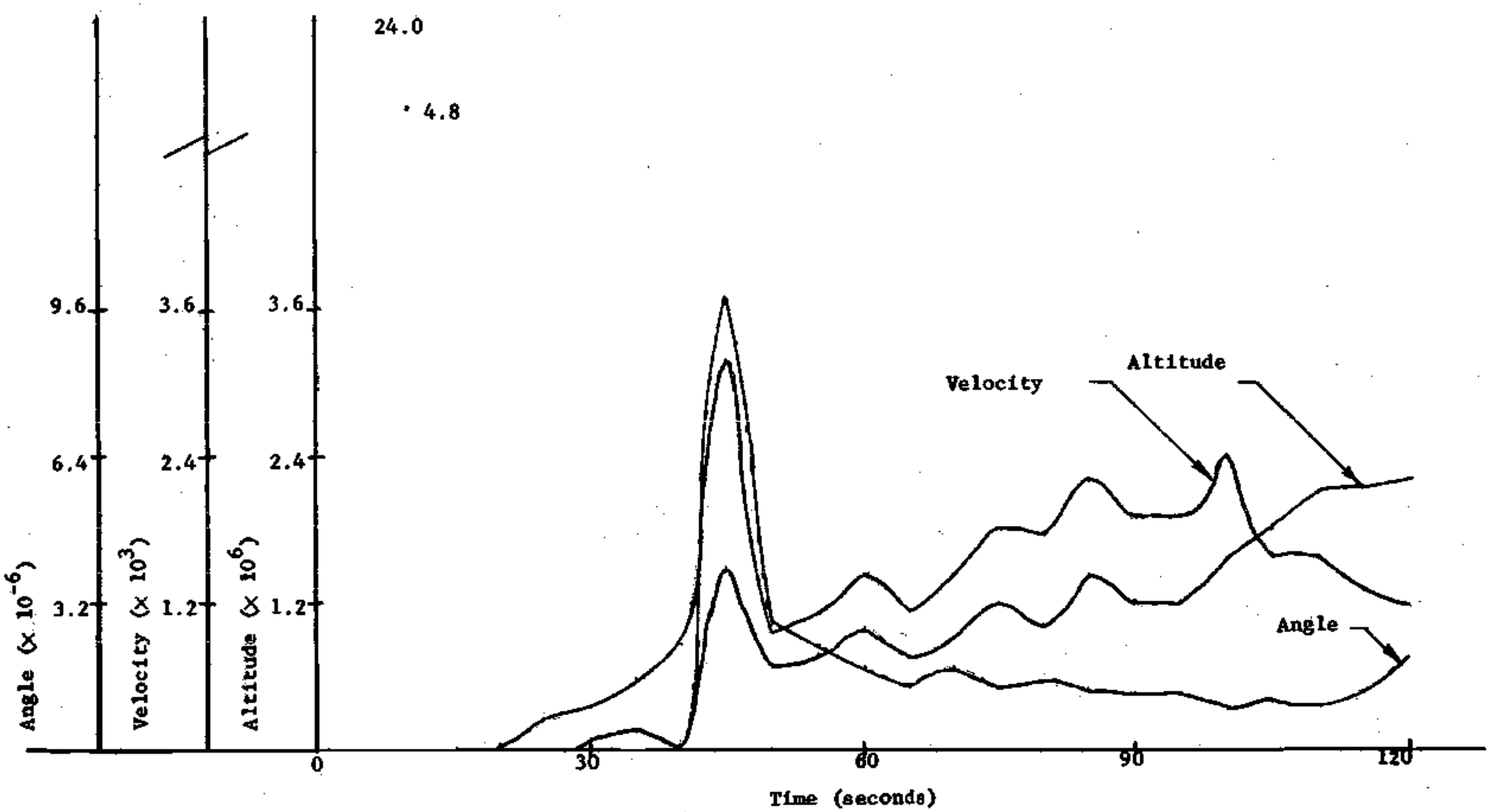


Figure 11. Sample Variance, Uniform Noise [-1.73,1.73] versus Time

In these stochastic simulations, it is evident from Figures 5 and 6 that the sample mean and the deterministic trajectories look very similar.

The sample variance graphs are very interesting. The variance increases initially and then begins to decrease. At approximately 45 seconds into the trajectory, however, the variance increases sharply, decreases somewhat and then continues to increase. The explanation for this behavior is obtained from an examination of the nominal trajectories shown in Table 1. From 0 to 45 seconds the flight angle, x_3, is decreasing. At 45 seconds the nominal x_3 stops decreasing and starts to increase. The system is sensitive to changes in this angle and this is reflected in the graphs of the variances.

Bounding Ellipsoid Performance--Stochastic Case

For the stochastic re-entry system model

$$\delta\underline{x}(n+1) = \hat{\phi}(n)\delta\underline{x}(n) + Gw(n)$$

it was shown in Chapter II that with the initial state

$$\delta\underline{x}(0) \in \Omega_x(0) = \{\delta\underline{x} \in R^n\colon (\delta\underline{x} - \underline{c}_x(0))^T \Gamma_x^{-1}(0)(\delta\underline{x} - \underline{c}_x(0)) \le 1\}$$

the state $\delta\underline{x}(n)$ is contained in the region

$$\Omega_x(n) = \{\delta\underline{x} \in R^n\colon (\delta\underline{x} - \underline{c}_x(n))^T \Gamma_x^{-1}(n)(\delta\underline{x} - \underline{c}_x(n)) \le 1\}$$

described by the ellipsoid

$$\Gamma_x(n + 1) = \frac{1}{1 - \beta(n)} \hat{\phi}(n)\Gamma_x(n)\hat{\phi}^T(n) + \frac{1}{\beta(n)} GQG^T$$

where $\qquad w^2(n) \le Q$

The ellipsoid described by $\Gamma_x(n)$ is generated on the computer and again the trace of $\Gamma_x(n)$ is used as a measure of the region containing $\delta\underline{x}(n)$ during re-entry. In Figure 12 the trace of $\Gamma_x(n)$ for this noisy case is shown versus time.

The shape of this trace plot is similar to the shape of the variance plots shown in Figures 10 and 11. The trace increases initially, decreases and then at approximately 45 seconds begins to increase again. This is explained by examining the algorithm used to generate the bounding ellipsoid. The algorithm is

$$\Gamma_x(n + 1) = \frac{1}{1 - \beta(n)} \hat{\phi}(n)\Gamma(n)\hat{\phi}^T(n) + \frac{1}{\beta(n)} GQG^T$$

and if this is written out in detail for this problem it becomes

$$\Gamma(1) = \frac{1}{1 - \beta(0)} \begin{bmatrix} \lambda_1^2 y_{11} & 0 & 0 \\ 0 & \lambda_2^2 y_{22} & 0 \\ 0 & 0 & \lambda_3^2 y_{33} \end{bmatrix} + \frac{Q}{\beta(n)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & g_2^2 & g_2 g_3 \\ 0 & g_2 g_3 & g_3^2 \end{bmatrix}$$

$$\Gamma(1) = \begin{bmatrix} \frac{1}{1-\beta}\lambda_1^2 y_{11} & 0 & 0 \\ 0 & \frac{1}{1-\beta} x_2^2 y_{22} + \frac{Q}{\beta} g_2^2 & \frac{Q}{\beta} g_2 g_3 \\ 0 & \frac{Q}{\beta} g_2 g_3 & \frac{1}{1-\beta}\lambda_3^2 y_{33} + \frac{Q}{\beta} g_3^2 \end{bmatrix}$$

and

$$\mathrm{Tr}[\Gamma(1)] = \frac{1}{1-\beta}(\lambda_1^2 y_{11} + \lambda_2^2 y_{22} + \lambda_3^2 y_{33}) + \frac{Q}{\beta}(g_2^2 + g_3^2)$$

As was shown previously, the term $\lambda_1^2 y_{11} + \lambda_2^2 y_{22} + \lambda_3^2 y_{33}$ is smaller than $y_{11} + y_{22} + y_{33}$ because $|\lambda_i| < 1$. $\beta(n)$ is generated using the relationship developed in Chapter II and is a small number

$$\beta(n) \ll 1$$

so

$$\frac{1}{1 - \beta(n)} \simeq 1$$

and the first term in the $\mathrm{Tr}[\Gamma(1)]$ is approximately the same as in the $\mathrm{Tr}[\Gamma(1)]$ for the deterministic case. Here, however, is the additional term $\frac{Q}{\beta(n)}(g_2^2 + g_3^2)$ which increases $\mathrm{Tr}[\Gamma(1)]$. If the eigenvalues, λ_i's, are small it is still possible for the trace to decrease with time. On the other hand if the eigenvalues are close to one, the trace increases. This is the reason the trace varies as shown in Figure 12.

In Figure 13 is plotted the ratio of the semi-axis of the bounding ellipsoid to 3σ where σ is the standard deviation obtained from the Gaussian re-entry simulation. From examinations of this graph several

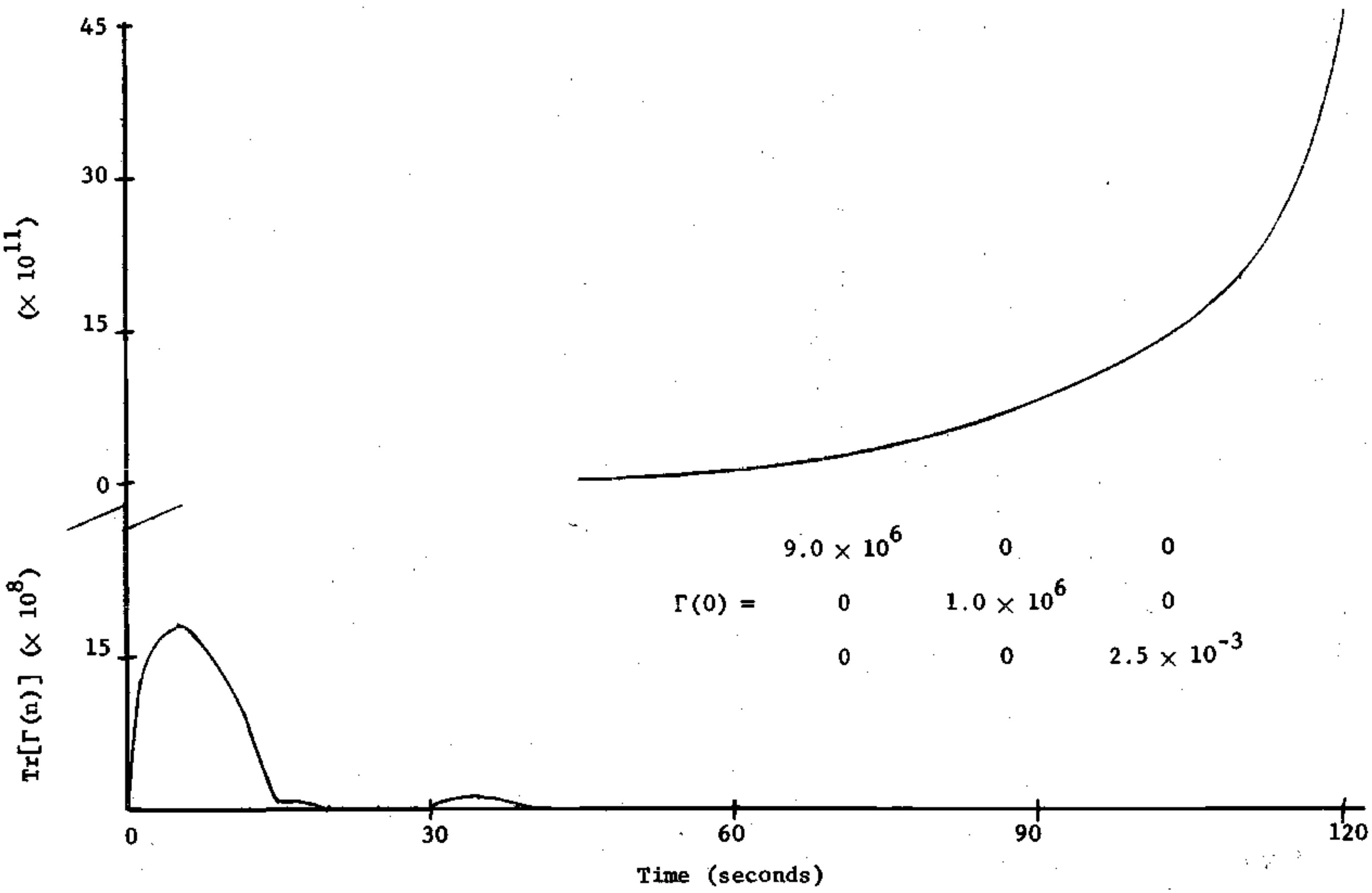


Figure 12. Ellipsoidal Trace versus Time (Q = 6.25)

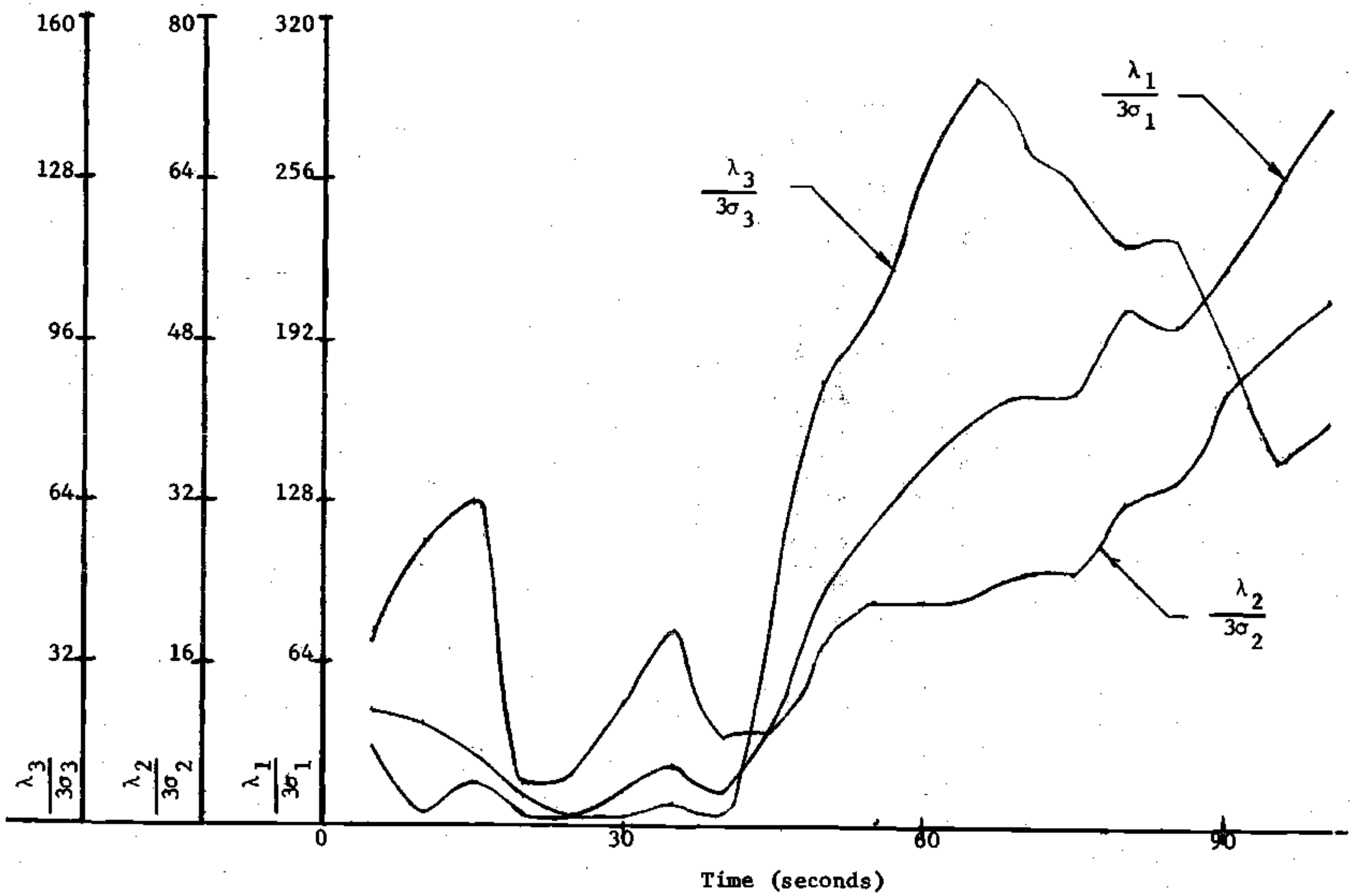


Figure 13. Ratio of Square Root of Ellipsoid Eigenvalue to 3σ versus Time ($Q = 6.25$, $N(0,1)$)

points are evident. First, the bounding ellipsoid is very conservative. Second, there is a region where the ellipsoid performs reasonably well. This region is the 20 to 30 second interval on the graph. This is also approximately the same interval where the variance (Figure 10) for the Gaussian simulation is very small. It appears, therefore, that the ellipsoid bound is quite good if the variance is small and extremely conservative if the variance is large.

Ellipsoid Parameter β

In the computer runs that are used to generate the ellipsoids and the ellipsoid traces, the parameter $\beta(n)$ is calculated using the relationship (2.29) developed in Chapter II. A plot of β versus time is shown in Figure 14.

Sensitivity to Model Parameters

In the introduction to this chapter the nominal values for the model constants and parameters were specified. These are the values used in the re-entry simulations and in generating the bounding regions $\Gamma_x(n)$. If the re-entry controller is to be useful it must not be sensitive to variations in these mathematical model parameters. This is because these parameters are not known exactly prior to the re-entry. Therefore it is worthwhile to vary them and study the effect they have on the re-entry process. Accordingly, the lift coefficient, C_L, the drag coefficient, C_D, and the vehicle mass, m, are varied one at a time in the linearized model. The discrete model is computed and the bounding ellipsoid is generated using

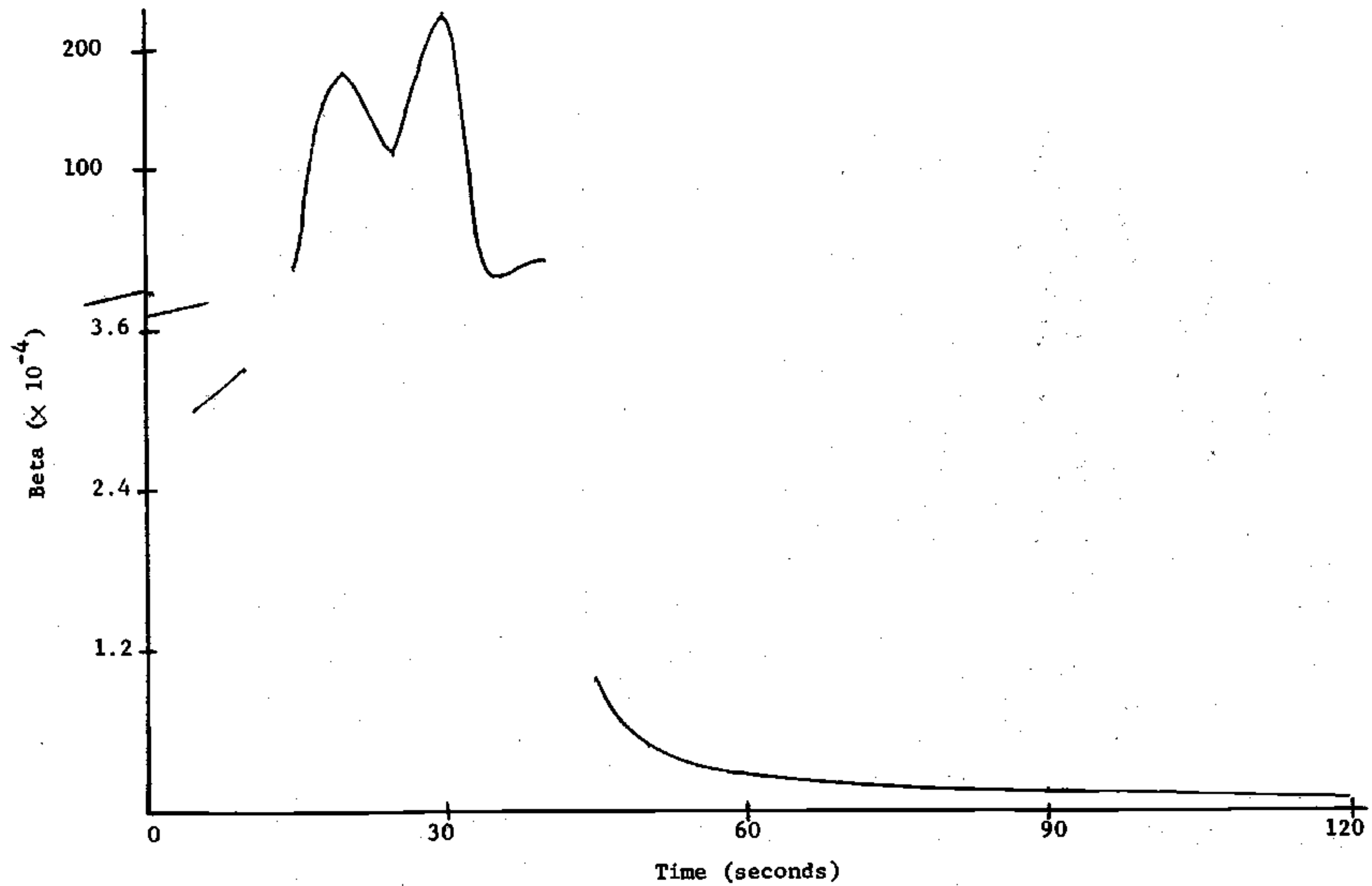


Figure 14. The Ellipsoid Parameter Beta (β) versus Time

$$\Gamma_x(n + 1) = \hat{\phi}(n)\Gamma_x(n)\hat{\phi}^T(n)$$

The control used in closing the loop is generated using nominal values of C_L,C_D, and m and is not changed. The trace of $\Gamma_x(n)$ versus time for the different cases is shown in Figures 15, 16, and 17.

In Figure 15 is shown the variation in performance due to changes in the drag coefficient. As the drag coefficient is increased this allows more control to be applied to the velocity equation so deviations from the nominal are reduced to zero faster. This effect is shown in Figure 15. When the lift coefficient is increased more control can be applied to the flight angle. This tighter angle control is shown in Figure 16. The perturbations from the nominal are reduced to zero in the same time interval with less flare in the performance. The effect of varying the lift and drag coefficients together is shown in Figure 17. The increased drag coefficient reduces perturbations to zero faster and the increased lift coefficient reduces the flare.

The re-entry controller, therefore, performs as predicted and is not overly sensitive to vehicle parameter variations. This is a very desirable characteristic in any controller not just the re-entry controller. Once again these results have a broader implication. That is, the controller developed in Chapter II is not sensitive to parameter variations in the system model.

Summary

In this Chapter the performance of the re-entry controller is studied by performing various simulations on the digital computer. These

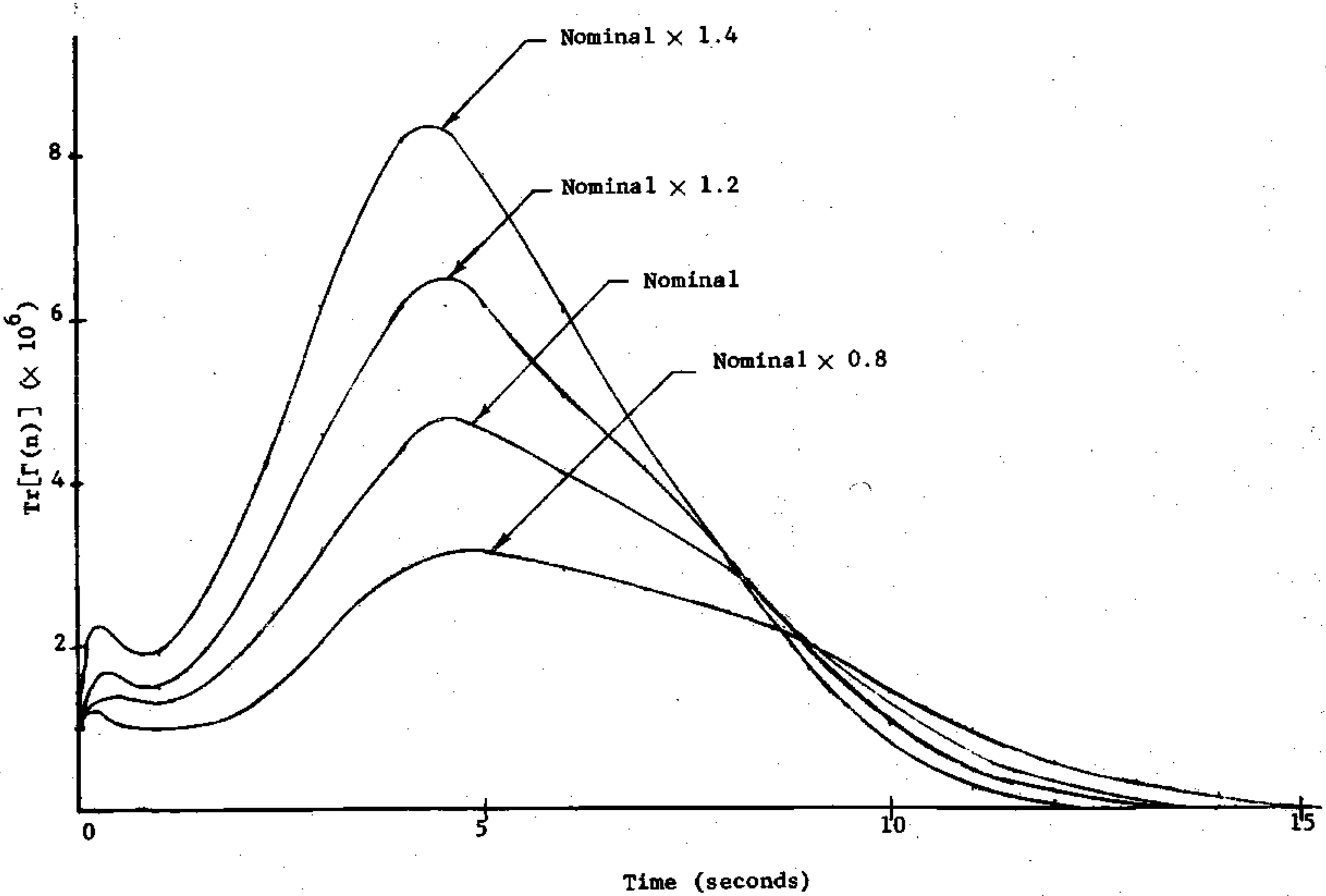


Figure 15. Effect of Variation of Drag Coefficient on Ellipsoid Trace versus Time

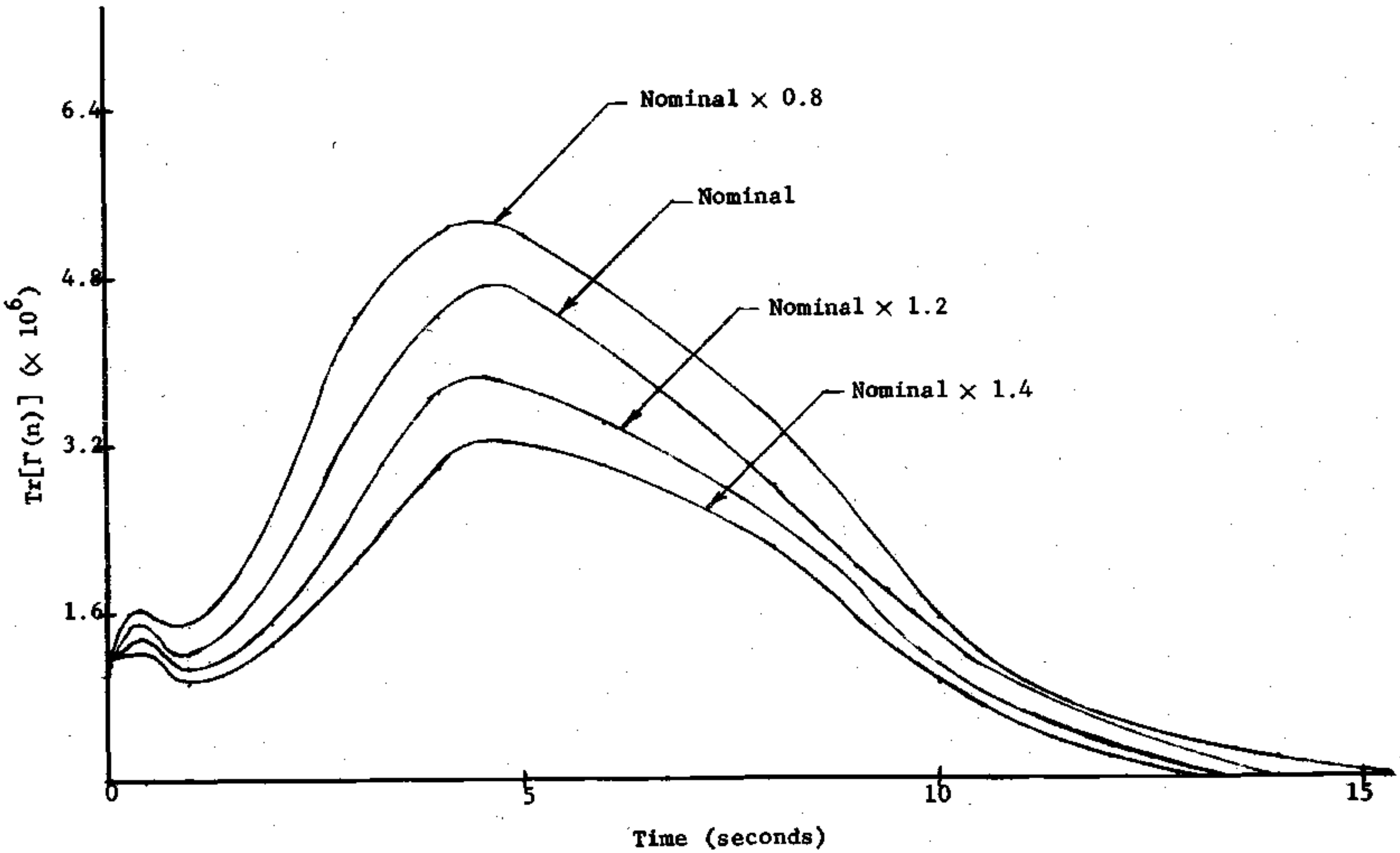


Figure 16. Effect of Variation of Lift Coefficient on Ellipsoid Trace versus Time

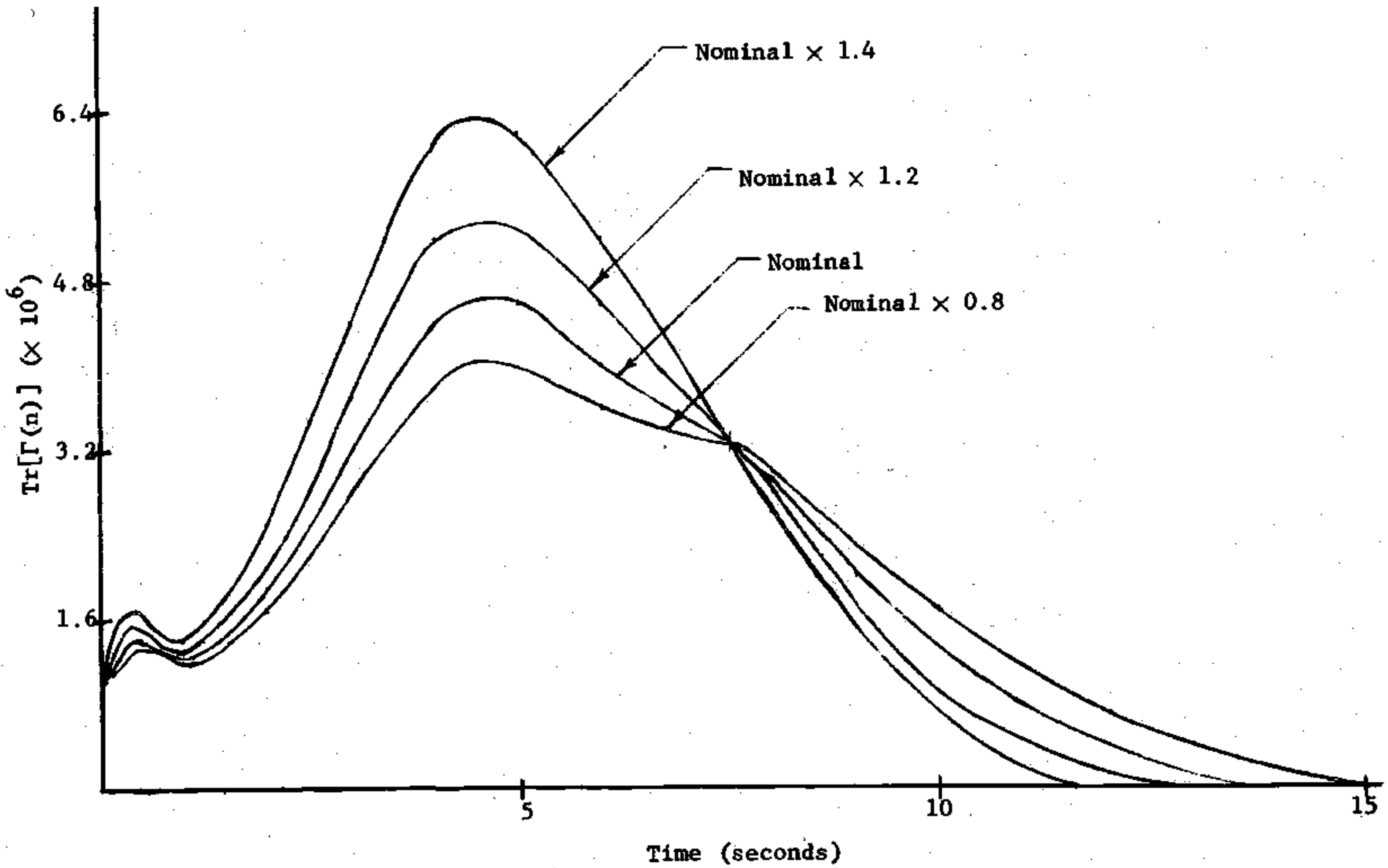


Figure 17. Effect of Variation of Lift and Drag Coefficient on Ellipsoid Trace versus Time

simulations assumed both a deterministic re-entry process and a noisy re-entry process. The results of the simulations are presented in Figures 6 to 17. These results are interpreted in terms of just the re-entry problem. Many of these results, however, have much broader implications than this one example. The following chapter discusses the implied results.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

This thesis has considered the control of dynamical systems operating in the presence of uncertainty. These systems are described by

$$\dot{\underline{x}} = \underline{f}(\underline{x},\underline{u}) + \underline{N}$$

where $\underline{N}$ represents any uncertainty in the plant and external disturbances acting on the plant. The only characteristic assumed known about $\underline{N}$ is that it is contained within a specified bound. No statistical modeling is used.

It is assumed that the purpose of the controller for this system is to maintain the state of the system close to a given or specified trajectory. With this objective in mind it is reasonable to expand the nonlinear function, $\underline{f}(\underline{x},\underline{u})$, in a Taylor Series about the known nominal trajectory and truncate the higher order terms in the Series. This step results in the linearized perturbation model

$$\delta\dot{\underline{x}} = A(t)\delta\underline{x} + B(t)\delta\underline{u} + \underline{N}$$

This linearized model is discretized on the digital computer. This step results in the linearized, discrete-time, perturbation model

$$\delta\underline{x}(n+1) = \emptyset(n)\delta\underline{x}(n) + H(n)\delta\underline{u}(n) + G(n)\underline{w}(n)$$

The controller is assumed to be of the form

$$\delta\underline{u}(n) = L(n)\delta\underline{x}(n)$$

which leads to the model

$$\delta\underline{x}(n+1) = [\emptyset(n) + H(n)L(n)]\delta\underline{x}(n) + G(n)\underline{w}(n) \qquad (5.1)$$

At this point the region around the nominal trajectory containing $\delta\underline{x}(n)$ must be characterized. In this work, this region is described by the bounding ellipsoids

$$\delta\underline{x}^T(n)\Gamma^{-1}(n)\delta\underline{x}(n) \le 1$$

That is, the state $\delta\underline{x}(n)$ is always contained within the ellipsoid described by the positive definite weighting matrix $\Gamma^{-1}(n)$. These ellipsoids are generated using the algorithm

$$\Gamma(n+1) = \frac{1}{1-\beta(n)}\hat{\emptyset}(n)\Gamma(n)\hat{\emptyset}^T(n) + \frac{1}{\beta(n)}GQG^T \qquad (5.2)$$

where $\hat{\emptyset} = \emptyset + HL$ is the closed-loop system matrix, $\beta(n)$ is a free parameter selected according to the relationship developed in Chapter II, and Q is the bound on the noise.

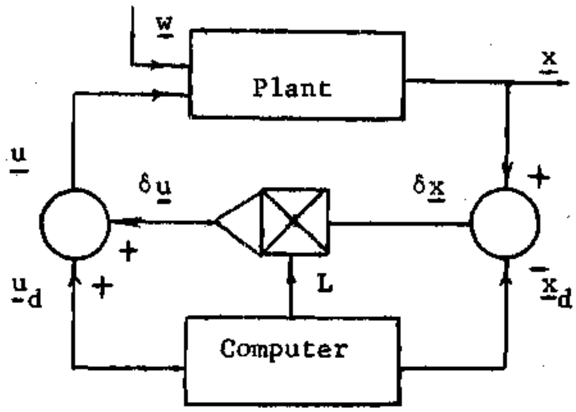
Using the model (5.1) and the ellipsoidal bounded state space, the general approach to this control problem is to find the controller that minimizes this bounded region that the state $\delta\underline{x}(n)$ can lie in. In more specific terms, in this research the trace of $\Gamma(n)$, the bounding ellipsoid, is taken as a measure of the "size" of the region containing the state of the system. With this performance measure the control problem can be stated as follows. Determine the control matrix $L(n)$ that minimizes the performance index

$$J = \mathrm{Tr}[\Gamma(N)] + \sum_{n=0}^{N-1} \mathrm{Tr}[\Gamma(n)]$$

subject to the constraint

$$\Gamma(n+1) = \frac{1}{1-\beta(n)} \hat{\phi}(n)\Gamma(n)\hat{\phi}^T(n) + \frac{1}{\beta(n)} GQG^T$$

This basic control problem formulation allows the Matrix Minimum Principle to be applied. In this matrix formulation the elements of the control matrix, ℓ_i, play the role of the control variables in the standard vector form of the Minimum Principle and the elements of the bounding ellipsoid, y_{ij}, correspond to the components of the state vector. The result of applying the Matrix Minimum Principle is a control algorithm, or in other words, the result is the specification of the control matrix, $L(n)$, for all $n = 0,1,2,\ldots,N$, and consequently the controller $\delta\underline{u}(n) = L(n)\delta\underline{x}(n)$. This control is calculated off-line and stored for use in an on-line environment as shown below.



To evaluate this control scheme, a specific control problem was selected and the controller was developed as outlined above. The problem selected was the re-entry problem. This controlled re-entry process was simulated on the digital computer. This problem was selected because it is a member of the class of systems to which this procedure applies and because it is an interesting problem. The re-entry system dynamics are described by a set of nonlinear differential equations and the uncertainty in the modeling is unknown but bounded. This system was linearized about a known nominal re-entry trajectory. The performance index for this application was weighted and took the form

$$J = \mathrm{Tr}[\Gamma(N)] + \sum_{n=0}^{N-1} \mathrm{Tr}[R\Gamma(n)] \tag{5.3}$$

where R was selected to minimize the effects of truncating the Taylor Series expansion of the nonlinear re-entry dynamics. This weighting matrix adds flexibility to this design process. In a different control problem, in all likelihood, a different weighting matrix would be selected based on physical considerations. With this performance index (5.3), the controller for the re-entry problem was calculated on the digital computer. The bounding region containing the state of the system was calculated using the ellipsoid generation algorithm. The shape of this bounding region can be varied by the weighting matrix R.

In summary, then, given a nonlinear system operating in the presence of uncertainty the following procedure is followed to generate the system control. A nominal trajectory for the system must be specified and the system linearized about this trajectory. The bounds on the noise must be specified and the weighting matrix in the performance index must be selected. The control and the bounded region containing the state can then be generated. If desirable, at this point the shape of the bounded region can be changed by changing the weighting matrix.

These simulations showed that the control scheme developed in Chapter II is a valid scheme. The re-entry problem is a difficult and demanding control problem. It is believed therefore, that the control scheme can be applied to many other problems.

Both the deterministic and stochastic simulations showed the re-entry to be well controlled. In the stochastic runs the sample variance indicated that tighter control should be generated in certain intervals of the trajectory. This could be accomplished by using a time-varying weighting matrix in the performance index.

The performance of the bounding ellipsoid was interesting. The bounding ellipsoid algorithm generated a bound for the re-entry system operating in the presence of noise. This bound was generated using a deterministic algorithm and while the bound was conservative in absolute magnitude the character of the bounding region was the same as the statistically generated bound--the sample variance. The sample variance was generated from the Monte Carlo simulations consisting of 100 separate passes through the trajectory. The bounding ellipsoid with one pass through the trajectory also suggested the generation of tighter control in certain intervals. Therefore the deterministic bounding ellipsoid can be used to give qualitative information about the performance of a given closed-loop control system.

Recommendations

There are several areas associated with this thesis research that are recommended for further study. The first area is the technique used for describing the region that bounds the state. In this work ellipsoids were used to bound the state. This is a straightforward technique but an approximate technique. The simulation results indicate that it is also a very conservative technique. There needs to be an error analysis performed on this ellipsoidal bounding technique. Another suggestion in this same area is the method of describing the bounded region containing the state. Ellipsoids were used in this work, but it is quite possible another technique might yield an algorithm that generates the exact bounding surface. For example if the initial condition sets are polyhedra then a polyhedral algorithm can be used to describe the region.

In this research the performance index used in the controller derivation was the trace of the bounding ellipsoid. It is quite possible other performance indices will yield the same or different controllers. Therefore in future work the formulation of different indices could lead to new and interesting results.

APPENDIX I

A systematic notational approach for the problem dealing with the time evolution of matrices is available in terms of the Matrix Minimum Principle [24]. The purpose of this appendix is to present the pertinent information from [24] that is used in this research.

The Hamiltonian for a problem can be written

$$H = F[\Gamma(n), L(n)] + \sum_{i=1}^{n} \sum_{j=1}^{n} y_{ij}(n+1) p_{ij}(n+1) \tag{A.1}$$

where $P_{ij}(n+1)$ is the costate variable associated with $y_{ij}(n+1)$. This Hamiltonian can also be written as

$$H = F[\Gamma(n), L(n)] + \mathrm{Tr}[\Gamma(n+1) P^T(n+1)]$$

where $P(n+1)$ is the costate matrix associated with the state matrix $\Gamma(n+1)$. That is, y_{ij} and p_{ij} are the elements of $\Gamma(n+1)$ and $P(n+1)$, respectively.

The key to the Matrix Minimum Principle [24] is in the use of gradient matrices. A gradient matrix is defined as follows: $f(\Gamma)$ is a scalar-valued function of the elements y_{ij} of Γ. The gradient matrix of $f(\Gamma)$ is denoted by

$$\frac{\partial f(\Gamma)}{\partial(\Gamma)}$$

and the ijth element is given by

$$\left[\frac{\partial f(\Gamma)}{\partial(\Gamma)}\right]_{ij} = \frac{\partial f(\Gamma)}{\partial y_{ij}}$$

With this definition, the costate equations associated with the Hamiltonian H

$$p_{ij}(n) = \frac{\partial H}{\partial y_{ij}}$$

can be written as

$$P(n) = \frac{\partial H}{\partial \Gamma(n)}$$

Using this notation the necessary conditions for optimality can be stated for matrix problems.

$$P(n) = \frac{\partial H}{\partial \Gamma(n)}$$

$$\Gamma(n+1) = \frac{\partial H}{\partial P(n+1)}$$

$$\frac{\partial H}{\partial L(n)} = 0$$

The gradient matrices needed in this work are given below.

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{X}] = \underline{I} \tag{A.2}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{AX}] = \underline{A}' \tag{A.3}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{AX}'] = \underline{A} \tag{A.4}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{A}\underline{X}\underline{B}] = \underline{A}'\underline{B}' \tag{A.5}$$

$$\frac{\partial}{\partial X} \mathrm{Tr}[\underline{A}\underline{X}'\underline{B}] = \underline{B}\underline{A} \tag{A.6}$$

$$\frac{\partial}{\partial \underline{X}'} \mathrm{Tr}[\underline{A}\underline{X}] = \underline{A} \tag{A.7}$$

$$\frac{\partial}{\partial \underline{X}'} \mathrm{Tr}[\underline{A}\underline{X}'] = \underline{A}' \tag{A.8}$$

$$\frac{\partial}{\partial \underline{X}'} \mathrm{Tr}[\underline{A}\underline{X}\underline{B}] = \underline{B}\underline{A} \tag{A.9}$$

$$\frac{\partial}{\partial \underline{X}'} \mathrm{Tr}[\underline{A}\underline{X}'\underline{B}] = \underline{A}'\underline{B}' \tag{A.10}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{X}\underline{X}] = 2\underline{X}' \tag{A.11}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{X}\underline{X}'] = 2\underline{X} \tag{A.12}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{X}^n] = n(\underline{X}^{n-1})' \tag{A.13}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{A}\underline{X}^n] = \left(\sum_{i=0}^{n-1} \underline{X}^i\underline{A}\underline{X}^{n-1-i}\right)' \tag{A.14}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[\underline{A}\underline{X}\underline{B}\underline{X}] = \underline{A}'\underline{X}'\underline{B}' + \underline{B}'\underline{X}'\underline{A}' \tag{A.15}$$

$$\frac{\partial}{\partial \underline{X}} \mathrm{Tr}[AXBX'] = \underline{A}'\underline{X}\underline{B}' + \underline{A}\underline{X}\underline{B} \tag{A.16}$$

The reader is cautioned that in making these gradient computations the element x_{ij} of X must be assumed independent.

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