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NETS WITH WELL-ORDERED DOMAINS

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CHAPTER I

INTRODUCTION

Most of the limit processes of elementary analysis can be studied in terms of sequences. For example, if f is a real-valued function defined on the real line, $\lim_{x \to a} f(x) = b$ if and only if the sequence $\{f(x_n)\}$ converges to b whenever $\{x_n\}$ is a sequence converging to a.

Indeed, a quite satisfactory theory of convergence in metric spaces can be based on sequences. Thus, a metric space X is compact if and only if every sequence in X has a convergent subsequence. There is also the well-known result that a sequence in a metric space has a point x as a cluster point if and only if it has a subsequence converging to x. Furthermore, closures, accumulation points, and continuity can all be characterized in metric space topology in terms of sequences.

In fact, except for the characterization of compactness, these results hold in all first countable spaces. However, in spaces which are not first countable sequences do not generally provide a fruitful theory of convergence.

One well-known way to construct a theory of convergence valid for general topological spaces (including those which are not first countable) is to employ <u>nets</u>, which are a generalization of sequences. A net is defined to be a function from a <u>directed set</u> into a topological space, a directed set being a set S with a relation $\leq$ which is reflexive and transitive and has the property that for each pair of elements x and y in S there exists an element z in S such that $x \leq z$ and $y \leq z$. Clearly, any sequence in a space may be viewed as a net since any linearly ordered set, in particular the positive integers with the usual order, is a directed set. The definitions of subnet, convergence of a net, and cluster point of a net are in varying degrees motivated by the corresponding notions relating to sequences.

The primary object of investigation in this volume is the extent to which it is possible to establish a theory of convergence employing only those nets whose domains are linearly ordered. We shall, in fact, limit our attention to nets with well-ordered domains; for, as will be indicated in Theorem 4.3, theorems involving nets with linearly ordered domains can generally be translated into theorems involving nets with well-ordered domains. By "well-ordered set" we mean, of course, any linearly ordered set each of whose nonempty subsets contains a smallest element. A net whose domain is well-ordered will henceforth be called a <u>well-ordered net</u>.

In Chapter III we shall discover that compactness, closures, accumulation points, and continuity can be characterized in general spaces in terms of the cluster points of well-ordered nets, though no such characterization holds in terms of the cluster points of sequences. However, it will also be found that no satisfactory general theory of convergence can be constructed in terms of well-ordered nets.

Chapter IV will investigate those spaces in which closures, accumulation points, and continuity can be characterized in terms of convergent well-ordered nets and the points to which they converge.

Chapter V will explore possible extensions to well-ordered nets and to other spaces of the theorem that in a first countable space a point x is a cluster point of a sequence Φ if and only if Φ has a subsequence converging to x.

In Chapter VI we shall introduce a generalization of metric spaces and use it to characterize some of the spaces discussed in Chapter IV, thus generalizing a theorem of Arhangel'skiĭ which characterizes those Hausdorff spaces in which the closure of each subset is the collection consisting of each point to which some sequence in that subset converges. We shall also generalize in connection with well-ordered nets certain facts about compactness such as the equivalence of compactness and sequential compactness in metric spaces.

Chapter II will provide necessary background material concerning transfinite numbers and general topology. There we shall define the term <u>filter</u> and describe briefly how filters are used to construct a second well-known theory of convergence in general topology. Filters and nets generate essentially equivalent theories of convergence in the sense that any theorem involving nets can be translated into a theorem involving filters [4]. We shall introduce <u>well-ordered filters</u> and, in Chapters III-VI, illustrate by an occasional example how one may obtain from the theorems concerning well-ordered nets analogous results on well-ordered filters.

CHAPTER II

PRELIMINARIES

This chapter will provide a résumé of those facts concerning transfinite numbers, nets, and filters which will be needed in later chapters. No proofs will be given for the better known propositions. For these the reader can consult Abian [1] on transfinite numbers, Kelley [6] on nets, and Bartle [4] or Kowalsky [7] on filters.

Cardinal and Ordinal Numbers

There exists a class of sets known as the ordinal numbers with the properties that each element of an ordinal number is an ordinal number and that a set S of ordinal numbers is an ordinal number if and only if $w \subseteq S$ for every $w \in S$. Also, if S is any set of ordinal numbers, the relation $\subseteq$ is a well-ordering of S. Any ordinal number w is equal to $\{\text{ordinals } w' : w' < w\}$, where $<$ means $\subsetneq$.

Definitions. A partial ordering is a reflexive and transitive relation. Let S_1 and S_2 be sets with partial orderings $\leq$ and $\lesssim$, respectively. A function $\Phi : S_1 \to S_2$ is said to be an order homomorphism if $x \leq y$ implies that $\Phi(x) \lesssim \Phi(y)$. Φ is said to be an order isomorphism if Φ is a bijective order homomorphism and $\Phi^{-1} : S_2 \to S_1$ is an order homomorphism. It is easily shown that a bijective order homomorphism whose domain and range are linearly ordered is an order isomorphism. If there is an order isomorphism from S_1 to S_2, S_1 is said to be order isomorphic to S_2. It is easily verified that the relation of being order isomorphic is an equivalence relation.

Every well-ordered set is order isomorphic to a unique ordinal number. Thus, distinct ordinal numbers cannot be order isomorphic.

Definition. Two sets are equipollent if there is a bijective function from one to the other. This, too, is an equivalence relation.

Definitions. If w is an ordinal number, there is a set consisting of all ordinal numbers equipollent to w. The smallest ordinal number in this set is called the cardinal number corresponding to w. If S is any set, there is a cardinal number equipollent to S since S can be well-ordered. This cardinal number is unique since, as can be easily shown, two cardinal numbers are equal if and only if they are equipollent. The unique cardinal number equipollent to S is called the cardinal number of S, or the cardinality of S.

Any set of cardinal numbers is a set of ordinal numbers and is therefore well-ordered by the relation $\subseteq$. If S_1 and S_2 are sets and there exists an injective function $\Phi:S_1 \to S_2$, then the cardinality of S_1 is less than or equal to the cardinality of S_2. If there exists a surjective function $\Psi:S_1 \to S_2$, then the cardinality of S_1 is greater than or equal to the cardinality of S_2.

There is a commutative and associative addition of cardinal numbers whereby $c+d$ is the cardinality of $(c\times\{0\})\cup(d\times\{1\})$. There is also a commutative and associative multiplication of cardinal numbers under which $c\cdot d$ is the cardinality of $c\times d$. If either c or d is infinite, $c+d = \max\{c,d\}$. If, in addition, neither c nor d is zero, then $c\cdot d = \max\{c,d\}$.

Under the usual Cartesian products notation, c^d represents the collection of all functions from d into c. The symbol c^d is also used to denote the cardinality of this collection. If S is a set of cardinality c, the power set of S (set of all subsets of S) has cardinality 2^c, which can be shown to be strictly greater than c.

The cardinality of the set of non-negative integers is denoted by $\aleph_0$. In fact, if we regard the non-negative integers as ordinal numbers, $\aleph_0$ is equal to the set of all non-negative integers. It is easily shown that $\aleph_0$ is the smallest infinite cardinal. The cardinality of the set of real numbers is $2^{\aleph_0}$.

There is a more sophisticated form of addition of cardinal numbers than the binary operation mentioned above. Let $\{c_a : a \in A\}$ be an indexed collection of cardinal numbers. The sum of this collection is defined to be the cardinality of $\cup\{c_a \times \{a\} : a \in A\}$.

Definition. An infinite cardinal number K is called a regular aleph if, for every indexed collection $\{c_a : a \in A\}$ of cardinal numbers with sum $\geq K$, either the cardinality of A is $\geq K$ or there is an $a \in A$ such that the cardinality of c_a is $\geq K$.

$\aleph_0$ is an example of a regular aleph. If for each non-negative integer we inductively define $\aleph_{n+1}$ to be the smallest cardinal greater than $\aleph_n$, then the sum of $\{\aleph_n : n \geq o\}$ exemplifies a non-regular infinite aleph.

Definition. Let D be a directed set and let E be a subset of D. E is said to be a cofinal subset of D if for every $d \in D$ there exists $e \in E$ so that $d \leq e$.

Definition. An ordinal number is called an irreducible ordinal if it is order isomorphic to each of its cofinal subsets.

The finite irreducible ordinals are clearly 0 and 1. Theorem 2.2 below equates the infinite irreducible ordinals with the regular alephs. The proof employs the principle of transfinite inductive definition, which we now state.

Let w be a non-zero ordinal number and let S be a set. Let k be a function whose range is S and whose domain is the set of all functions having an ordinal number less than w as domain and having S as range. Let s be an element of S. Then there exists exactly one function $h:w \to S$ such that $h(o) = s$ and $h(w') = k(h|w')$ for all $w' \in w-\{0\}$.

For Theorem 2.2 we shall also need the following lemma.

Lemma 2.1. If w is an ordinal number and S is a subset of w, then the ordinal number corresponding to S is less than or equal to w.

Proof. Suppose, to the contrary, there is some ordinal number for which Lemma 2.1 fails to hold. Let w be the smallest such ordinal number. Then there is a subset $S \subseteq w$ so that the ordinal number w' corresponding to S is greater than w. Let $\Phi:w' \to S$ be an order isomorphism. Now $\Phi|w$ is an order isomorphism of w onto $\{s \in S : s < \Phi(w)\} = S \cap \Phi(w)$. Since $\Phi(w) \in S \subseteq w$, $\Phi(w) < w$. From the way in which w was chosen we conclude that Lemma 2.1 holds for $\Phi(w)$. Hence, the ordinal number w'' corresponding to $S \cap \Phi(w)$ is less than or equal to $\Phi(w)$. But $w = w''$ since $\Phi|w$ is an order isomorphism onto $S \cap \Phi(w)$. We thus have $w = w'' \leq \Phi(w) < w$. This is the desired contradiction. $\|$

Theorem 2.2. An infinite ordinal is irreducible if and only if it is a regular aleph.

Proof. Let w be an infinite irreducible ordinal. Let K be the cardinality of w. We first show that $K = w$. Let $f:K \to w$ be a bijection. For each $w' \in K$ and function $g:w' \to w$, we choose an element $h(g)$ in w as follows. Since w' has cardinality less than K, $\{g(w''):w'' \in w'\} \cup \{f(w''):w'' \leq w'\} = S_g$ is a subset of w of cardinality less than K. Hence, the ordinal number corresponding to S_g is less than K which is less than or equal to w. Since w is irreducible, S_g cannot be cofinal in w. Hence, w has elements which are greater than any element in S_g. Define k(g) to be the smallest such element of w. By the principle of transfinite inductive definition, there exists a function $h:K \to w$ so that $h(o) = 0$ and $h(w') = k(h|w')$ for each $w' \in K-\{0\}$. It is easily verified that h is an order isomorphism onto a cofinal subset of w. Since w is irreducible, we conclude that K = w.

We now show that K must be a regular aleph. If not, then there is an indexed set of cardinal numbers $\{c_a : a \in A\}$ such that A and each c_a have cardinality less than K and such that the sum of $\{c_a : a \in A\}$ is greater than or equal to K. Since K is irreducible and A has cardinality less than K, $\{c_a : a \in A\}$ cannot be a cofinal subset of K. Let w' be an ordinal in K which exceeds every element of $\{c_a : a \in A\}$. Let K' be the maximum of the cardinality of A and the cardinality of w'. It is easily seen that $K' < K$, and it can be shown that the sum of $\{c_a : a \in A\}$ does not exceed $K' \cdot K'$, which is certainly less than K. This contradicts the previous statement that the sum of $\{c_a : a \in A\}$ is greater than or equal to K. Thus, K must be a regular aleph.

Now we assume that K is a regular aleph and show that it must be an irreducible ordinal. Let S be any cofinal subset of K. The ordinal number corresponding to S is less than or equal to K by Lemma 2.1. We thus need only to show that the ordinal number corresponding to S is greater than or equal to K. It clearly will suffice to show that the cardinality of S is greater than or equal to K.

Consider the indexed set of cardinal numbers $\{C_s : s \in S\}$ where C_s is the cardinality of s. For each $s \in S$, let $g_s : s \to C_s \times \{S\}$ be a bijection. Then for each $w \in K$ let $\Phi(w)$ be the smallest element of S which is greater than or equal to w. Define $f:K \to \bigcup\{C_s \times \{s\} : s \in S\}$ by letting $f(w) = g_{\Phi(w)}(w)$.

It is easily verified that f is an injection. Hence the sum of $\{C_s : s \in S\}$ is greater than or equal to K. Since each C_s is less than K, the cardinality of S must be greater than or equal to K, for K is a regular aleph. This completes the proof that K must be an irreducible ordinal. ||

Theorem 2.4 will reveal the principal importance of irreducible ordinals for our purposes. We shall need the following lemma.

Lemma 2.3. Every linearly ordered set has a cofinal well-ordered subset.

Proof. Let L be a linearly ordered set. Let W = {subsets of L which are well-ordered}. Clearly W is nonempty since it contains the singleton subsets of L. Define the relation $\leq$ on W by $w \leq w'$ if:

either i)

$$w = w'$$

or ii)

there exists $x \in w'$ such that $w = \{y \in w' : y \text{ is less than } x \text{ in } L\}$.

It is easily checked that this relation is a partial ordering of W.

Now let C be any chain (linearly ordered subset) of W. We seek to show that C is bounded above in W. Let $w' = \bigcup\{w : w \in C\}$. If $w' \in C$, then it is easily seen that w' is a bound for C.

Now suppose that $w' \notin C$. We wish to show that $w' \in W$. Let T be a nonempty subset of w'. Let w be a member of C such that $T \cap w$ is nonempty. Let t be the smallest element of $T \cap w$. We assert that t is the smallest element of T. Suppose $t_1 \in T$. Choose $w_1 \in C$ such that $t_1 \in w_1$. Since C is a chain, either $w_1 \leq w$ or $w < w_1$. If $w_1 \leq w$, then $t_1 \in w_1 \subseteq w$ and thus $t_1 \in T \cap w$ so that $t \leq t_1$. If $w < w_1$, then there exists $x \in w_1$ so that $w = \{y \in w_1 : y < x\}$. Now, if $t_1 < x$, $t_1 \in T \cap w$ and $t \leq t_1$. If $x \leq t_1$, then $t < x \leq t_1$. Hence t is the smallest element of T. Thus, w' is well-ordered and belongs to W.

Still working on the assumption that $w' \notin C$, we now show that w' is an upper bound for C. Let $w \in C$ be given. Since $w \subsetneq w'$, $w'-w$ is nonempty. Let x be the smallest element of $w'-w$. Choose $w_1 \in C$ so that $x \in w_1$. Since C is a chain, $w < w_1$ for w_1 is not a subset of w and thus w_1 is not less than or equal to w. So there exists $z \in w_1$ such that $w = \{y \in w_1 : y < z\}$. Since $x \in w_1-w$, $z \le x$. Since $z \in w'-w$ and x is the smallest element of $w'-w$, $x = z$. So $w = \{y \in w_1 : y < x\} \subseteq \{y \in w' : y < x\}$. Now let $u \in \{y \in w' : y < x\}$ be given. Since x is the smallest element of $w'-w$, $u \in w$. Thus, $w = \{y \in w' : y < x\}$. Hence, $w \le w'$. Hence, w' is an upper bound for C.

By Zorn's Lemma, W has a maximal element m. Because m is a member of W, it is a well-ordered subset of L. We need only show that m is cofinal in L. Suppose, to the contrary, that there is an element $x \in L$ which is greater than every element of m. Then $m \cup \{x\}$ belongs to W. But $m < m \cup \{x\}$. This denies the maximality of m, and thus m is cofinal in L. ||

Theorem 2.4. For every linearly ordered set L there is a unique irreducible ordinal w such that L has a cofinal subset order isomorphic to w.

Proof. Let L be a linearly ordered set. Let w be the smallest ordinal number order isomorphic to a cofinal subset of L. Let C be a cofinal subset of L such that there is an order isomorphism $h: w \to C$. Let S be any cofinal subset of w. Then $h|S$ is an order isomorphism onto a cofinal subset of L. Hence, by the way w was chosen, the ordinal number corresponding to S is greater than or equal to w. By Lemma 2.1 the ordinal number corresponding to S is less than or equal to w. Hence, every cofinal subset of w is order isomorphic to w, and thus w is an irreducible ordinal.

Now we wish to show that w is the only irreducible ordinal order isomorphic to a cofinal subset of L. Let w' be any ordinal number not equal to w such that w' is order isomorphic to C', a cofinal subset of L. Since $w < w'$, we need only show that w' has a cofinal subset order isomorphic to an ordinal number less than or equal to w in order to prove that w' is not irreducible.

For each $v \in w$, let $\Psi(v)$ be the smallest element of C' such that $\Psi(v) \geq h(v)$. If $v \leq v'$, then $h(v) \leq h(v') \leq \Psi(v')$, and hence $\Psi(v) \leq \Psi(v')$. We now show that the image set of the function $\Psi: w \to C'$ is cofinal in C'. Let $c' \in C'$ be given. Since C is cofinal in L, there exists $c \in C$ so that $c' \leq c$. Hence, $c' \leq c \leq \Psi(h^{-1}(c))$. Now we define a function $\Phi: \Psi(w) \to w$ by letting $\Phi(c')$ be the smallest element of w such that $\Psi(\Phi(c')) = c'$. Φ is clearly injective. Now let $c_1', c_2' \in \Psi(w)$ be given where $c_1' < c_2'$. Suppose $\Phi(c_1') \geq \Phi(c_2')$. Then $c_1' = \Psi(\Phi(c_1')) \geq \Psi(\Phi(c_2')) = c_2'$, which is a contradiction. So $\Phi(c_1') < \Phi(c_2')$. Thus, Φ is an order isomorphism from $\Psi(w)$ to a subset of w. By Lemma 2.1 $\Psi(w)$ is order isomorphic to an ordinal less than or equal to w. Since $\Psi(w)$ is cofinal in C', and since C' is order isomorphic to w', w' has a cofinal subset order isomorphic to an ordinal less than or equal to w. ||

Definition. Let L be a linearly ordered set. Then the irreducible ordinal which is order isomorphic to a cofinal subset of L is called the final order of L.

It is easily seen that a linearly ordered set and each of its cofinal subsets have the same final order.

Definition. A limit ordinal is an ordinal number having no largest element.

Any regular aleph is a limit ordinal since it is an infinite irreducible ordinal and cannot, therefore, have a singleton cofinal subset. Our last lemma of this chapter gives a property of limit ordinals which will be used in an example in Chapter V.

Lemma 2.5. Every limit ordinal is a union of two disjoint cofinal subsets.

Proof. Let w be a limit ordinal. Let w' be an element of w. w'' is said to be the predecessor of w' if w'' is the largest ordinal less than w'. By using the principle of transfinite inductive definition, we can clearly define a function $f:w \to \{0,1\}$ so that $f(w') = 0$ if w' has no predecessor or if f evaluated at the predecessor is 1 and $f(w') = 1$, otherwise. It is clear that $f^{-1}(0)$ and $f^{-1}(1)$ are disjoint cofinal subsets whose union is w. ||

Nets and Filters

Definitions. Let $\Phi:D \to X$ be a net and let A be a subset of X. Φ is said to be frequently in A if $\Phi^{-1}(A)$ is a cofinal subset of D. Φ is eventually in A if there exists $d \in D$ so that $\Phi(d') \in A$ for all $d' \geq d$. Φ is in A if $\Phi(D) \subseteq A$. A point $x \in X$ is called a cluster point of Φ if Φ is frequently in every neighborhood of x. Φ is said to converge to x if Φ is eventually in every neighborhood of x.

Definitions. Let D and E be directed sets. A function $\Psi:E \to D$ is said to be cofinal if for every $d \in D$ there exists $e \in E$ so that $\Psi(e') \geq d$ for every $e' \geq e$. If $\Phi:D \to X$ is a net and $\Psi:E \to D$ is a cofinal function, then $\Phi \cdot \Psi$ is said to be a subnet of Φ.

Theorem 2.6. A space X is compact if and only if every net in X has a convergent subnet.

Proof. See Kelley [6], p. 136. ||

Thorem 2.7. A point x of a space X belongs to the closure of a subset A of X if and only if there is a net in A converging to x.

Proof. See Kelley [6], p. 66. ||

Theorem 2.8. A point x of a space X is an accumulation point of a subset A of X if and only if there exists a net in $A-\{x\}$ converging to x.

Proof. See Kelley [6], p. 66. ||

Theorem 2.9. Let X and Y be spaces and let x be a point in X. A function $f:X \to Y$ is continuous at x if and only if $f\circ\Phi$ converges to $f(x)$ whenever Φ is a net converging to x.

Proof. See Kelley [6], p. 86. ||

Theorem 2.10. A net Φ clusters at a point x if and only if it has a subnet converging to x.

Proof. See Kelley [6], p. 71. ||

Definitions. Let S be a set. A nonempty collection $\underline{F}$ of subsets of S is a filter in S if,

i) $A\cap B \in \underline{F}$ for every $A, B \in \underline{F}$;

ii) if $A \in \underline{F}$ and $A\subseteq B\subseteq S$, then $B \in \underline{F}$; and,

iii) the empty set is not a member of $\underline{F}$.

If $A \subseteq S$ and A meets every member of $\underline{F}$, then $\underline{F}$ is said to be frequently in A. If $A \in \underline{F}$, then we say that $\underline{F}$ is eventually in A. Another filter $\underline{G}$ in S is called a refinement of $\underline{F}$ if $\underline{F}\subseteq\underline{G}$.

Definitions. If $\underline{F}$ is a filter in a topological space X, then a point $x \in X$ is called a cluster point of $\underline{F}$ if $\underline{F}$ is frequently in every neighborhood of x. $\underline{F}$ is said to converge to x if $\underline{F}$ is eventually in every neighborhood of x.

It is easily seen that the neighborhood system of a point x in a space X is a filter and that a filter $\underline{F}$ in X converges to x if and only if $\underline{F}$ is a refinement of the neighborhood system at x.

Theorems 2.6 through 2.10 form the core of the theory of convergence based on nets. There are corresponding theorems relating to filters such as the following analogue of Theorem 2.10.

Theorem 2.11. A space X is compact if and only if every filter in X has a convergent refinement.

Proof. See Bartle [4]. ||

Definition. A subset $\underline{B}$ of a filter $\underline{F}$ is called a base for $\underline{F}$ if for every $F \in \underline{F}$ there exists $B \in \underline{B}$ so that $B \subseteq F$.

It is easily verified that a nonvoid collection $\underline{B}$ of nonempty subsets of a set S is a base for some filter in S if and only if for every $A, B \in \underline{B}$ there exists $C \in \underline{B}$ so that $C \subseteq A \cap B$.

Definitions. Let $\underline{B}$ be a filter base. If we define the relation $\leq$ on the set $\underline{B}$ by requiring that $A \leq B$ if and only if $B \subseteq A$, it is clear that $\underline{B}$ with the relation $\leq$ is a directed set. We shall refer to the $\leq$ as the natural order of $\underline{B}$. A filter will be called a well-ordered filter if it has a base which is well-ordered under the natural order.

We are now ready to begin our investigations into the use of well-ordered nets and filters in general topological spaces.

CHAPTER III

THE CLUSTER POINTS OF WELL-ORDERED NETS

Theorems 2.6 - 2.9 in the preceding chapter are classical results of general topology characterizing compactness, closures, accumulation points, and continuity in terms of convergent nets. The theorems in this chapter show how these same things can be characterized by means of the cluster points of well-ordered nets. Following the theorems examples are given to show that these characterizations cannot be made in a natural way in terms of either convergent well-ordered nets or the cluster points of sequences.

The following lemma is needed in the proof of Theorem 3.2.

Lemma 3.1. If U is an open cover of a space X and U has infinite cardinality K and if every net $\Phi:K \to X$ has a cluster point, then U has a subcover of cardinality less than K.

Proof. Let $f:K \to U$ be a bijection. For $k \in K$, define $U_k = \{f(k'):k' \in K$ and $k' \leq k\}$. Assume that U has no subcover of cardinality less than K. Now the cardinality of U_k is less than K. Thus, by the assumption that U has no subcover of cardinality less than K, U_k is not a subcover. Hence, we may choose an element $\Phi(k)$ of $X - (\cup\{f(k'):k' \leq k\})$, thus constructing a net $\Phi:K \to X$. By hypothesis, Φ must have a cluster point x. Since U is a cover of X, there exists some $\bar{k} \in K$ such that $x \in f(\bar{k})$. Since x is a cluster point of Φ, there exists some $k \geq \bar{k}$ such that $\Phi(k) \in f(\bar{k})$. Then, since $f(\bar{k}) \in U_k$, $\Phi(k) \in \cup\{f(k'):k' \leq k\}$. But this contradicts the way $\Phi(k)$ was chosen. Hence U must have a subcover of cardinality less than K. ||

Theorem 3.2. A space X is compact if and only if every well-ordered net in X has a cluster point.

Proof. Necessity. Since X is compact, by Theorems 2.6 and 2.10 every net in X has a cluster point.

Sufficiency. Let U be any infinite open cover of X. Let K be the least infinite cardinal such that U has a subcover of cardinality K. By hypothesis every net $\Phi:K \to X$ has a cluster point. Hence, by Lemma 3.1, U has a subcover of cardinality $K' < K$. By the minimality of K, K' is finite. Hence X is compact. ||

Theorem 3.3. A point x of a space X belongs to the closure of a subset A of X if and only if there is a well-ordered net in A clustering at x.

Proof. Necessity. If x belongs to the closure of A, then every neighborhood of x meets A. Let K be the smallest cardinal number such that x has a neighborhood basis of that cardinality, and let N be a neighborhood basis for x of cardinality K. Let $f:K \to N$ be a bijection, and define $\Phi:K \to X$ by choosing $\Phi(k) \in f(k) \cap A$ for each $k \in K$. Let V be any member of N and let k' be an element of K. We wish to show that there exists $k \geq k'$ such that $\Phi(k) \in V$. If not, then for each $k \geq k'$ $f(k)$ is not a subset of V. But $\{V' \in N:N:V' \subseteq V\}$ is a neighborhood basis for x. Note that $\{V' \in N:V' \subseteq V\} \subseteq \{f(k):k < k'\}$. This contradicts the minimality of K since $\{f(k):k < k'\}$ has cardinality less than K. Hence, there exists $k \geq k'$ such that $\Phi(k) \in V$. Thus, Φ is a net in A clustering at x.

Sufficiency. By Theorems 2.7 and 2.10, x is in the closure of A if there is a net in A clustering at x. ||

Theorem 3.4. A point x of a space X is an accumulation point of a subset A of X if and only if there exists a well-ordered net in $A-\{x\}$ clustering at x.

Proof. Clearly, x is an accumulation point of A if and only if x is in the closure of $A-\{x\}$. The conclusion then follows from an application of Theorem 3.3. ||

Theorem 3.5. Let X and Y be spaces and let x be a point in X. A function $f:X \to Y$ is continuous at x if and only if $f\cdot\Phi$ clusters at f(x) whenever Φ is a well-ordered net clustering at x.

Proof. Necessity. Let U be an open neighborhood of f(x). Since f is continuous at x, there exists an open set $V \subseteq X$ such that $x \in V$ and $f(V) \subseteq U$. If Φ is a well-ordered net clustering at x, for every element d in the domain of Φ there exists $e \geq d$ such that $\Phi(e) \in V$. Then $f\cdot\Phi(e) \in f(V) \subseteq U$. Hence, $f\cdot\Phi$ clusters at f(x).

Sufficiency. Suppose f is not continuous at x. Then there exists an open set U containing f(x) so that $f(V) \cap (Y-U) \neq \phi$ whenever V is a neighborhood of x. Hence, for any neighborhood V of x, V meets $f^{-1}(Y-U)$. Thus x is in the closure of $f^{-1}(Y-U)$. By Theorem 3.3, there is a well-ordered net Φ in $f^{-1}(Y-U)$ clustering at x. It is clear that $f\cdot\Phi$ does not cluster at f(x) since $f\cdot\Phi$ is never in U. ||

It might seem reasonable to conjecture that the preceding theorems would hold even if the term "sequence" were substituted for "well-ordered net." Example 3.6 is a counterexample to such a modification of Theorem 3.2, and Example 3.7 is a counterexample to the corresponding modifications of Theorems 3.3, 3.4, and 3.5.

Definition. Let S be a linearly ordered set. For each $s \in S$, let L_s be the set of all elements of S less than s, and let R_s be the set of all elements of S greater than s. The order topology on S is the topology having $\{L_s : s \in S\} \cup \{R_s : s \in S\}$ as a subbasis.

Example 3.6. Let X be the space of all ordinals less than the least uncountable ordinal Ω with the order topology. By Theorem 3.2, X fails to be compact since the identity map on X is a well-ordered net which has no cluster point. We shall show, however, that every sequence in X has a cluster point.

Let $\{x_n\}$ be a sequence in X. Recalling that $x_n = \{x : x$ is an ordinal and $x < x_n\}$, let $T_n = \cup\{x_m : m \geq n\}$. T_n is a countable set since it is a union of countably many countable sets. Since X is uncountable, $X - T_n$ is nonempty. Let y be the least element of $\cup\{X - T_n : n$ is a positive integer$\} = X - \cap\{T_n : n$ is a positive integer$\}$. It is easily shown that y is a cluster point of $\{x_n\}$. ||

Example 3.7. Let Y be the space of all ordinals less than or equal to the least uncountable ordinal Ω with the order topology. Clearly, Ω is in the closure of $Y - \{\Omega\}$ and is an accumulation point of Y. Yet, as we shall show presently, no sequence in $Y - \{\Omega\}$ clusters at Ω. Hence, Theorems 3.3 and 3.4 do not hold if "well-ordered net" is replaced by "sequence."

Since no sequence in $Y - \{\Omega\}$ clusters at Ω, if a sequence in Y clusters at Ω it must be frequently in $\{\Omega\}$. Thus, if f is any function with domain Y and Φ is a sequence in Y clustering at Ω, the sequence $f \cdot \Phi$ is frequently in $\{f(\Omega)\}$ and, hence, clusters at $f(\Omega)$. Yet the characteristic function of $\{\Omega\}$, the function which is 1 at Ω and 0 elsewhere in Y, is discontinuous at Ω. Consequently, Y provides a counterexample to Theorem 3.5 with "sequence" substituted for "well-ordered net."

We now give the easy proof that no sequence in $Y - \{\Omega\}$ clusters at Ω. Let $\{x_n\}$ be a sequence in $Y - \{\Omega\}$ and let $T = \cup\{x_n : n$ is a positive integer}. Just as in Example 3.6 $(Y-\{\Omega\})-T$ is nonempty, for $Y-\{\Omega\}$ is uncountable and T is countable. Let x be a point in $(Y-\{\Omega\})-T$. The set of all ordinals in Y greater than x is a neighborhood of Ω not meeting $\{x_n\}$. ||

Another reasonable conjecture stemming from the theorems in this chapter is that Theorems 2.6 through 2.9 would hold even if the terms "well-ordered net" and "well-ordered subnet" were substituted for "net" and "subnet" respectively. Example 3.8 is a counterexample to such modifications of Theorems 2.6 through 2.9.

Example 3.8. Let I be the closed unit interval, and let I^I be the set of functions from I into I with the Tychonoff product topology. Let $A = \{g\epsilon I^I$: For finitely many values of $x\epsilon I$ $g(x)=1$, and for all other values of x $g(x)=0.\}$. Let f be the function on I which is constantly equal to 1. Clearly, f is an accumulation point of A. We shall later show that there is no well-ordered net in A converging to f. Hence, neither Theorem 2.7 nor Theorem 2.8 holds if the term "net" is changed to "well-ordered net."

Consider the function $h:I^I \rightarrow I$ such that $h(g) = 1$ if $g \in A$ and $h(g) = 0$ if $g \notin A$. Since $h(f) = 0$ and $h(A) = \{1\}$, h is clearly discontinuous at f. If Φ is any well-ordered net in I^I converging to f, then Φ must be eventually in I^I-A. For $\Phi^{-1}(A)$ must fail to be a cofinal subset of the domain of Φ, since otherwise $\Phi|\Phi^{-1}(A)$ would be a well-ordered net in A converging to f. As Φ is eventually in I^I-A, $h\cdot\Phi$ is eventually in $h(I^I-A) = \{0\}$ and thus $h\cdot\Phi$ converges to $h(f) = 0$. So Theorem 2.9 does not hold if "net" is replaced by "well-ordered net."

We now show that there is no well-ordered net in A converging to f. Suppose, on the contrary, that the net $\Phi:W \rightarrow I^I$ with W well-ordered and $\Phi(W) \subseteq A$ converges to f. For each $x \in I$, let $U_x = \{g \in I^I : g(x) > \frac{1}{2}\}$. As U_x is a neighborhood of f in I^I, Φ is eventually in U_x. Now define the function $F: I \to W$ by requiring $F(x)$ to be the smallest element of W such that $\{\Phi(w): w \geq F(x)\} \subseteq U_x$. We shall show that for each $w \in W$, $F^{-1}(w)$ is a finite set; and we shall also show that $F(I)$ is a countable subset. But then $I = F^{-1}(W) = \cup\{F^{-1}(w): w \in F(I)\}$ would be a countable union of finite sets, which is the desired contradiction since I is uncountable.

Suppose $x \in F^{-1}(w)$. Then $F(x) = w$ and, by the definition of F, $\Phi(w) \in U_x$. Since it is assumed that $\Phi(w) \in A$, then, by the definition of A, $[\Phi(w)]\ (x) > \frac{1}{2}$ for at most finitely many values of x. Hence, $\Phi(w) \in U_x$ for at most finitely many values of x. $F^{-1}(w)$ is thus a finite set.

Let w be an element in $F(I)$. Let V be the set of all elements $v \in F(I)$ such that $v \leq w$. For each $v \in V$, choose $G(v) \in I$ so that $F(G(v)) = v$. Clearly, G is injective. Now since $F(G(v)) = v \leq w$, then, by the definition of F, $\Phi(w) \in U_{G(v)}$. Since $\Phi(w) \in U_x$ for at most finitely many values of x, $G(V)$ is a finite set. Since G is injective, V is then finite. So w has finitely many predecessors in $F(I)$, and $F(I)$ is thus countable. This completes the proof that there is no well-ordered net in A converging to f.

In Example 5.5 it will be shown that although $I^{P(\aleph_0)}$ is compact, where $P(\aleph_0)$ is the power set of $\aleph_0$, there exists a sequence in $I^{P(\aleph_0)}$ having no convergent well-ordered subnet. Now $P(\aleph_0)$ has the same cardinality as I. Since the Tychonoff topology is not affected by any property of the exponent except its cardinality, $I^{P(\aleph_0)}$ and I^I are homeomorphic. Thus, I^I also provides a counterexample to Theorem 2.6 modified by substituting "well-ordered net" and "well-ordered subnet" for "net" and "subnet." ||

In summary, this chapter has shown how certain important topological properties and relations can be characterized by the cluster points of well-ordered nets though they cannot be characterized in a natural way by the cluster points of sequences. However, Example 3.8 uses I^I to show that a satisfactory theory of convergence cannot be based on well-ordered nets. The importance of I^I, particularly in applications of topology to analysis, dramatizes this failure. The remaining chapters will concentrate primarily on the properties of those particular spaces for which it is indeed possible to construct a satisfactory theory of convergence based on the class of well-ordered nets or subclasses thereof.

But before we proceed to Chapter IV, it is appropriate to mention that analogues of the theorems of this chapter are available involving the cluster points of well-ordered filters. Furthermore, our counter-examples to possible extensions of Theorems 3.2 - 3.5 are also counter-examples to the corresponding extensions of the filter analogues of these theorems.

For illustrative purposes we shall prove the analogue to Theorem 3.2.

Theorem 3.9. A space X is compact if and only if every well-ordered filter in X has a cluster point.

Proof. Necessity. Let $\underline{F}$ be a filter in X with a well-ordered base $\underline{B}$. Define a net $\Phi:\underline{B} \to X$ by choosing $\Phi(B) \in B$ for each $B \in \underline{B}$. By Theorem 3.2, Φ has a cluster point x, which is then easily shown to be a cluster point of $\underline{F}$.

Sufficiency. Let $\Phi:W \to X$ be a well-ordered net in X. For each $w \in W$, let $C_w = \{\Phi(w'):w' \geq w\}$. Clearly, each C_w is nonempty and $C_w \cap C_{w'} = C_{w''}$ where w'' is the larger of w and w'. If $\underline{B} = \{C_w:w\epsilon W\}$, then $\underline{B}$ is clearly a filter base for a filter $\underline{F}$. $\underline{F}$ is a well-ordered filter since $\underline{B}$ is clearly well-ordered by set inclusion. Thus, $\underline{F}$ has a cluster point x. So every neighborhood of x meets every member of $\underline{F}$, and in particular, each C_w meets every neighborhood of x. Clearly, then, Φ clusters at x, and by Theorem 3.2 X is compact. ||

Theorem 3.9 is actually very close to the classical result that a space X is compact if and only if every nest of nonempty closed subsets of X has a nonempty intersection. (A <u>nest</u> is a nonempty collection of sets which is linearly ordered by set inclusion.) Each of the two results may be easily proven from the other by using the fact that every linearly ordered set has a cofinal well-ordered subset and observing that a filter $\underline{F}$ has a point x as one of its cluster points if and only if x is in the closure of each element of $\underline{F}$.

CHAPTER IV

SEMI-ERECT, ERECT, AND K-FRÉCHET SPACES

Our objective in this chapter is to investigate those spaces in which closures, accumulation points, and continuity can be characterized in a straightforward way in terms of the convergence of well-ordered nets. A number of definitions and the first three theorems of this chapter, none of which deal with well-ordered nets *per se*, will enable us to give a unified exposition.

Definition. Let D_1 and D_2 be directed sets. A function $h:D_1 \to D_2$ will be called an order homomorphism if $d \leq d'$ implies that $h(d) \leq h(d')$.

Definitions. Let $\underline{D}$ be a class of directed sets. A net will be called a *$\underline{D}$-net* if its domain belongs to $\underline{D}$. A filter will be said to be a *$\underline{D}$-filter* if it has a base which under its natural order is an order-homomorphic image of a member of $\underline{D}$. It is clear that a filter is well-ordered if and only if it is a $\{W\}$-filter for some well-ordered set W.

Definition. Let X be a space and let x be an element of X. A class of directed sets $\underline{D}$ will be said to *describe the topology of X at x* if for every subset A of X such that x is in the closure of A there is a *$\underline{D}$-net* in A converging to x.

Theorem 4.1. Let X be a space and let x be an element of X. Let $\underline{D}$ be a nonempty class of directed sets. Then the following statements are equivalent:

(a) $\underline{D}$ describes the topology of X at x.

(b) For every subset A of X such that x is an accumulation point of A, there is a D-net in A-{x} converging to x.

(c) For every space Y and function $f:X \to Y$, f is continuous at x if and only if $f \cdot \Phi$ converges to f(x) whenever Φ is a D-net in X converging to x.

(d) For every subset A of X such that x is in the closure of A, there is a D-filter in X which is eventually in A and converges to x.

(e) For every subset A of X such that x is an accumulation point of A, there is a D-filter in X which is eventually in A-{x} and converges to x.

(f) For every space Y and function $f:X \to Y$, f is continuous at x if and only if the filter in Y for which $\{f(F):F \in \underline{F}\}$ is a base converges to f(x) whenever $\underline{F}$ is a D-filter in X converging to x.

(g) The neighborhood system of x in X is an intersection of D-filters.

Proof. (a) Implies (b). Since x is an accumulation point of A, x is in the closure of A-{x}. By (a), there is a D-net in A-{x} converging to x. (b) Implies (c). If $f:X \to Y$ is continuous at x, then for every neighborhood U of f(x) in Y there is a neighborhood V of x in X such that $f(V) \subseteq U$. If Φ is a D-net in X converging to x, then Φ is eventually in V and thus $f \cdot \Phi$ is eventually in U. Since U was arbitrary, $f \cdot \Phi$ converges to f(x).

Suppose $f:X \to Y$ is not continuous at x. Then there is a neighborhood U of f(x) in Y such that f(V)-U is nonempty whenever V is a neighborhood of x in X. Thus, x is an accumulation point of $f^{-1}(Y-U)$. By (b), there is a D-net Φ in $f^{-1}(Y-U)$ converging to x. Clearly, $f \cdot \Phi$ does not converge to f(x)since it is never in U.

<u>(c) Implies (d)</u>. Suppose x is in the closure of A. If $x \in A$, then (d) holds trivially. If $x \notin A$, then $f:X \to I$, defined by $f(y) = 0$ for $y \in A$ and $f(y) = 1$ for $y \notin A$, is clearly discontinuous at x. By (c), there exists $D \in \underline{D}$ and a net $\Phi:D \to X$ converging to x such that $f\cdot\Phi$ does not converge to $f(x)$. Clearly $f\cdot\Phi$ must be frequently at 0 and, hence, Φ must be frequently in A. If $\underline{F}$ is the filter in X generated by the filter base $\{\{\Phi(d'):d'\geq d\}:d\in D\}$, then clearly every element of $\underline{F}$ meets A and $\{F\cap A: F\in\underline{F}\}$ is a base for a $\underline{D}$-filter in X which is eventually in A and converges to x.

<u>(d) Implies (e)</u>. If x is an accumulation point of A, then x is in the closure of $A-\{x\}$ and by (d) there is a $\underline{D}$-filter in X which is eventually in $A-\{x\}$ and converges to x.

<u>(e) Implies (f)</u>. This is quite similar to the proof that (b) implies (c).

<u>(f) Implies (g)</u>. Let $\underline{N}$ be the neighborhood system of x in X and let $\underline{N}'$ be the intersection of all $\underline{D}$-filters in X converging to x. We clearly need only show that $\underline{N} = \underline{N}'$. If $V \in \underline{N}$ then V is a neighborhood of x and thus any $\underline{D}$-filter converging to x is eventually in V. Hence, $V \in \underline{N}'$. Now suppose $V \notin \underline{N}$. Then x is in the closure of $X-V$. If $x \in X-V$, let $\underline{F}$ be the filter in X having as a base the single set $\{x\}$. It is easily seen that $\underline{F}$ is a $\underline{D}$-filter converging to x. Since $V \notin \underline{F}$, it follows that $V \notin \underline{N}'$. If $x \notin X-V$, consider the function $f:X \to I$ defined by $f(y) = 0$ if $y \in X-V$ and $f(y) = 1$ if $y \notin X-V$. f is clearly discontinuous at x. By (f), there is a $\underline{D}$-filter $\underline{F}$ in X converging to x such that the filter in Y having $\{f(F):F\in\underline{F}\}$ as a base does not converge to $f(x)$. Clearly $0 \in f(F)$ for every $F \in \underline{F}$, and hence every $F \in \underline{F}$ meets $X-V$. If it were true that $V \in \underline{N}'$, then we would have that $V \in \underline{F}$. Since this contradicts the fact that every

element of $\underline{F}$ meets X-V, then $V \notin \underline{N}'$.

<u>(g) Implies (a)</u>. We are supposing that the neighborhood system $\underline{N}$ at x is an intersection of a set $\underline{S}$ of $\underline{D}$-filters in X. Suppose that x is in the closure of a subset A of X. Now $A \cap V$ is nonempty for every $V \in \underline{N}$. Hence, $X\text{-}A \notin \underline{N}$. Thus, there is a $\underline{D}$-filter $\underline{F}$ in $\underline{S}$ such that $X\text{-}A \notin \underline{F}$. So $A \cap F$ is nonempty for every $F \in \underline{F}$. Let $\underline{B}$ be a base for $\underline{F}$ such that there is a directed set $D \in \underline{D}$ and a surjective order homomorphism $h:D \to \underline{B}$. Define a net $\Phi:D \to X$ by choosing $\Phi(d)$ to be a point in $A \cap h(d)$. Φ is then a $\underline{D}$-net in A converging to x. Hence, $\underline{D}$ describes the topology of X at x. ||

The equivalence of (a) to (g) is an approximate generalization of a theorem of Kowalsky [7], p. 74, stating that $\aleph_0$ describes the topology of a space X at each of its points if and only if each neighborhood system in X is an intersection of filters generated by sequences. (A filter $\underline{F}$ is said to be generated by a sequence Φ if $\{\{\Phi(m):m \geq n\}:n$ is a positive integer$\}$ is a filter base for $\underline{F}$.)

It is interesting to note that for any space X there exists a directed set D so that $\{D\}$ describes the topology of X at each of its points. In fact, if K is the cardinality of the open sets in X, we may choose D to be the collection of finite nonempty subsets of K with $S_1 \leq S_2$ if and only if $S_1 \subseteq S_2$. If x is in the closure of a subset A of X, we choose a neighborhood basis B for x of cardinality not greater than K and let $f:K \to B$ be a surjection. We then define $\Phi:D \to X$ by choosing $\Phi(\{w_1, w_2, \ldots, w_n\} \in f(w_1) \cap f(w_2) \cap \ldots \cap f(w_n) \cap A$. Φ is then a $\{D\}$-net in A converging to x. This is somewhat similar to Theorem 1 of Venkataramen's paper [9].

<u>Theorem 4.2</u>. Suppose that $\underline{D}$ and $\underline{D}'$ are classes of directed sets such

that for every member D of $\underline{D}$ there is a member D' of $\underline{D}'$ which is order-isomorphic to a cofinal subset of D. If $\underline{D}$ describes the topology of X at x, then $\underline{D}'$ describes the topology of X at x.

Proof. Suppose x is in the closure of the subset A of X. Then there exists a directed set $D \in \underline{D}$ such that there is a net $\Phi : D \to A$ converging to x. By hypothesis there is a directed set $D' \in \underline{D}'$ such that there is an order isomorphism $\Psi : D' \to D$ onto a cofinal subset of D. Clearly, $\Phi \cdot \Psi$ is a $\underline{D}'$-net in A converging to x. ||

Theorem 4.3. If $\underline{D}$ is a class of linearly ordered sets, then there is a class $\underline{D}'$ of regular alephs such that for every space X and point $x \in X$ $\underline{D}'$ describes the topology of X at x if $\underline{D}$ describes the topology of X at x. If $\underline{D}$ has only one non-zero element, $\underline{D}'$ can be chosen to have only one element.

Proof. Let $\underline{D}''$ be the class of non-zero final orders of elements of $\underline{D}$. By Theorem 4.2, $\underline{D}''$ describes the topology of X at x if $\underline{D}$ does. If 1 is not an element of $\underline{D}''$, let $\underline{D}' = \underline{D}''$. If 1 is an element of $\underline{D}''$, let $\underline{D}'$ be $\underline{D}''$ with $\aleph_0$ substituted for 1. Clearly, $\underline{D}'$ has one element if $\underline{D}$ has one non-zero element. ||

Because of Theorem 4.3, we may restrict our attention to regular alephs when considering the construction of a theory of convergence based on linearly ordered nets.

Definitions. If the class of all regular alephs describes the topology of X at each of its points, then X will be said to be semi-erect.

If for each $x \in X$ there is a regular aleph K_x such that $\{K_x\}$ describes the topology of X at x, then X will be said to be erect.

If K is a regular aleph and $\{K\}$ describes the topology of X at each of its points, then X will be called K-Fréchet.

In Example 3.8 we showed that I^I contains a point f such that the class of all well-ordered sets does not describe the topology of I^I at f. (In fact, the class of all well-ordered sets does not describe the topology of I^I at any of its points.) Thus, I^I is an example of a space that fails to be semi-erect.

Theorem 4.4. Every linearly ordered set L with the order topology is semi-erect.

Proof. Let x be a point in L which is in the closure of a subset A of L. We need only show that there is a net in A which has a linearly ordered domain and converges to x. Let $A_1 = \{a \in A : a \leq x\}$ and let $A_2 = \{a \in A : a \geq x\}$. Now $Cl(A) = Cl(A_1 \cup A_2) = Cl(A_1) \cup Cl(A_2)$. So either $x \in Cl(A_1)$ or $x \in Cl(A_2)$.

If $x \in Cl(A_1)$, then the inclusion map $\Phi : A_1 \to L$ clearly is a net in A which has a linearly ordered domain and converges to x.

If $x \in Cl(A_2)$, let $<$ be the reverse order of $\leq$ on A_2. That is, for each pair of elements a and b in A_2, we require that $a < b$ if and only if $b \leq a$. Then the inclusion map $\Phi : A_2 \to L$, using $<$ as the ordering on A_2, clearly is a net having the desired properties. ||

From the definitions it is evident that every K-Fréchet space, for any regular aleph K, is erect and that every erect space is semi-erect. We now wish to provide examples showing that these concepts are in fact distinct from one another. We shall need the following lemma.

Lemma 4.5. If K is a regular aleph such that $\{K\}$ describes the topology of X at x and if x is in the closure of a subset A of X, then either there exists $a \in A$ so that x is in the closure of $\{a\}$ or the cardinality of A is not less than K.

Proof. Suppose that A contains no point a such that x is in the closure of $\{a\}$. Then, for every $a \in A$, $X-\{a\}$ is a neighborhood of x. Since $\{K\}$ describes the topology of X at x, there is a net $\Phi: K \to A$ converging to x. Now, for each $a \in A$, let $\Psi(a)$ be the smallest element in K such that $\Phi(w) \in X-\{a\}$ for all $w \geq \Psi(a)$. The existence of $\Psi(a)$ follows from the fact that Φ converges to x and $X-\{a\}$ is a neighborhood of x. From the definition of Ψ it is clear that $w < \Psi(\Phi(w))$. Thus, $\Psi(A)$ is a cofinal subset of K. Since K is a regular aleph, we know by Theorem 2.2 that $\Psi(A)$ must have cardinality K. Hence, the cardinality of A is not less than K. ||

Example 4.6. We now provide an example of a linearly ordered set with the order topology which is not erect, though by Theorem 4.4 it is semi-erect.

Let $X = ((\Omega+1) \times \{0\}) \cup (\aleph_0 \times \{1\})$ where $\Omega+1$ is the set of all ordinals less than or equal to the smallest uncountable ordinal Ω. We define the relation $\prec$ on X by $(x_1,i) \prec (x_2,j)$ if:

1) $i = 0$ and $j = 1$

or

2) $i = j = 0$ and $x_1 \leq x_2$

or

3) $i = j = 1$ and $x_2 \geq x_1$.

It is easily shown that $\prec$ is a linear order. Let $A = \{x \in X : x \prec (\Omega,0)$ and $x \neq (\Omega,0)\}$, and let $B = \{x \in X : (\Omega,0) \prec x$ and $(\Omega,0) \neq x\}$. It is clear that $(\Omega,0)$ is in the closure of A and is also in the closure of B. Suppose that a regular aleph K describes the topology of X at $(\Omega,0)$. By an argument similar to that in Example 3.7, there is no sequence in A converging to $(\Omega,0)$. Hence, $K \neq \aleph_0$. Now it is clear that there is no element $b \in B$

such that $(\Omega,0)$ is in the closure of $\{b\}$. Thus, by Lemma 4.5 the cardinality of B is not less than K. Since the cardinality of B is $\aleph_0$ we have $\aleph_0 \geq K$. But this contradicts the fact that $\aleph_0$ is the smallest regular aleph. X then fails to be erect. ||

Theorem 4.7. Every well-ordered set with the order topology is erect.

Proof. Let X be a well-ordered set with the order topology and let x be a point in X. Now $W = \{\{y \in X : y \leq x\}\} \cup \{\{y \in X : a < y \leq x\} : a < x\}$ is a base, well-ordered by set inclusion, for the neighborhood system at x. So the neighborhood system at x in X is a $\{W\}$-filter. Hence, by Theorem 4.1(g), $\{W\}$ describes the topology of X at x. By Theorem 4.3 there is a regular aleph K_x such that $\{K_x\}$ describes the topology of X at x. Hence, x is erect. ||

Example 4.8. We now give an example of a space that is erect but is not K-Fréchet for any regular aleph K.

Let X be the set of all ordinals less than or equal to the least uncountable ordinal Ω with the order topology. By the preceding theorem X is erect. By an argument similar to the one employed in Example 4.6 no regular aleph greater than $\aleph_0$ describes the topology of X at $\aleph_0$ and $\aleph_0$ does not describe the topology of X at Ω. Hence, no regular aleph describes the topology of X at each of its points and X consequently fails to be K-Fréchet for any regular aleph K. ||

Theorem 4.9. If K is a regular aleph and X is a space such that every neighborhood system is a $\{K\}$-filter, then X is K-Fréchet.

Proof. Let x be any point in X. By Theorem 4.1(g), $\{K\}$ describes the topology of X at x. Hence, X is K-Fréchet. ||

Corollary 4.10. Every first countable space is $\aleph_0$-Fréchet.

Proof. Let $\{U_n\}$ be a neighborhood basis at a point $x \in X$. For each positive integer n, let $V_n = \bigcap_{j=1}^{n} U_j$. Clearly, $\{V_n\}$ is a neighborhood basis at x which is an order-homomorphic image of the positive integers. Thus, the neighborhood system at x is an $\{\aleph_0\}$-filter, and by Theorem 4.9 X is $\aleph_0$-Fréchet. ||

Example 4.11. We are now in a position to give examples of K-Fréchet spaces for each regular aleph K.

If K is a regular aleph, let X be the set of all ordinals less than or equal to K. A basis for the topology of X will consist of all subsets of the form $\{x \in X : w < x\}$ as well as all subsets not containing the element K. Thus, a neighborhood basis at a point $x \neq K$ is $\{x\}$, and a neighborhood basis at K is $\{\{x \in X : w < x\} : w < K\}$. Clearly, each neighborhood basis is a $\{K\}$-filter. By Theorem 4.9, X is K-Fréchet.

It should be noted that X is not K′-Fréchet for any regular aleph $K' < K$. If, on the contrary, X were K′-Fréchet there would be a net $\Phi : K' \to X - \{K\}$ converging to K. But then $\Phi(K')$ would be a cofinal subset of $X - \{K\} = K$. This cannot happen since every cofinal subset of K is order-isomorphic to K, whereas $\Phi(K')$ has cardinality less than K. ||

Theorem 4.1 yields a good deal of information about the properties of semi-erect, erect, and K-Fréchet spaces. For example, from (a), (b), and (c) of this theorem, along with Corollary 4.10, we obtain the classical results on first countable spaces mentioned in the third paragraph of Chapter I. The remainder of this chapter will investigate a few additional properties of semi-erect, erect, and K-Fréchet spaces.

Theorem 4.12. Every subspace of a semi-erect (erect)(K-Fréchet) space is semi-erect (erect)(K-Fréchet).

Proof. We shall prove only the semi-erect, the proofs of the erect and K-Fréchet being quite similar.

Let X' be a subspace of a semi-erect space X. Suppose a point $x \in X'$ lies in the closure of $A \subseteq X'$ relative to the topology of X'. Let U be an open set in X containing x. Since A meets $U \cap X'$, A meets U. Thus, x is in the closure of A relative to the topology of X. Thus, there is a regular aleph K and a net $\Phi: K \to A$ such that Φ converges to x in the topology of X. Let V' be any open neighborhood of x in X'. Then there is a set V which is open in X such that $V' = V \cap X'$. Now Φ is eventually in $V \cap A \subseteq V \cap X' = V'$. Thus, Φ converges to x in the topology of X'. Hence, X' is semi-erect. ||

Definition. Let $\{X_m : m \in M\}$ be an indexed collection of spaces. Then let $X = \bigcup_{m \in M} X_m$. We define a topology $\underline{T}$ for X by requiring that $U \in \underline{T}$ if and only if $U \cap X_m$ is open in X_m for every $m \in M$. Then $(X, \underline{T})$, denoted by $\sum_{m \in M} X_m$, is called the sum of the collection $\{X_m : m \in M\}$ of spaces.

Theorem 4.13. The sum of a pairwise disjoint indexed collection of semi-erect (erect)(K-Fréchet) spaces is semi-erect (erect)(K-Fréchet).

Proof. We shall prove only the semi-erect, since the proofs of the erect and K-Fréchet are similar.

Let $\{X_m : m \in M\}$ be a pairwise disjoint collection of semi-erect spaces indexed by M. Let x be any point in X. Let m be the unique member of M such that $x \in X_m$. If x is in the closure of a subset A of X, A must meet every neighborhood of x in X_m. Thus, x is in the closure of $A \cap X_m$ relative to the topology of X_m. Hence, there is a regular aleph K and a net $\Phi: K \to A \cap X_m$ which converges to x in the topology of X_m. Since the neighborhoods of x in X_m form a neighborhood basis for x in X,

Φ converges to x in the topology of X. Hence, X is semi-erect. ||

Example 4.14. We now show that for each regular aleph K there exist K-Fréchet spaces X and Y so that X x Y fails to be semi-erect. Thus, none of the properties of being semi-erect, erect, or K-Fréchet extend even to finite products.

Let $X = (K \times K) \cup \{x\}$ where x is an object not in K x K. For each i and j in K, let $B_{ij} = \{(i,k) \in K \times K: k > j\} \cup \{x\}$. Let $\underline{C}$ be the collection of all subsets of K x K, and let $\underline{D}$ be the family of all sets of the form $\cup\{B_{ij(i)}: i \in K\}$ where j(i) is a member of K, depending on i. We shall let X have the topology for which $\underline{C} \cup \underline{D}$ is a basis. X is clearly discrete everywhere except at x. For each $i \in K$, let $\underline{F}_i$ be the filter in X having $\{B_{ij}: j \in K\}$ as a base. It is easily seen that each $\underline{F}_i$ is a {K}-filter and that the neighborhood system of x in X is the intersection of $\{\underline{F}_i: i \in K\}$. By Theorem 4.1(g), {K} describes the topology of X at x and, thus, X is K-Fréchet.

Let $Y = K \cup \{y\}$ where y is an object not in K. For each $i \in K$, let C_i be $\{k \in K: k > i\} \cup \{y\}$. We shall give Y the topology having $\{C_i: i \in K\}$ united with the family of all subsets of K as a basis. Clearly, Y is K-Fréchet with $\{C_i: i \in K\}$ forming a base for the neighborhood system at y.

Let A be set of all points in X x Y of the form ((i,j),i) where i and j are arbitrary members of K. We shall show that (x,y) is in the closure of A. Let U be any neighborhood of (x,y) in X x Y. U must have a subset of the form $(\cup\{B_{ij(i)}: i \in K\}) \times C_k$. This subset meets A at the point ((k+1,j(k+1) + 1), k+1).

Now we show that there is no well-ordered net in A converging to (x,y). Suppose, to the contrary, that there is a regular aleph K′ and a net $\Phi:K' \to A$ converging to (x,y). Then, if p is the projection from $X \times Y$ into Y, $p\cdot\Phi$ is a net in K converging to y. By Lemma 4.5 $K' \leq K$. Since $p\cdot\Phi(K')$ must be cofinal in K, and since every cofinal subset of K has cardinality K, $K' \geq K$. Hence, $K' = K$. For each $k \in K$, let $\Psi(k)$ be the smallest element of K such that $p\cdot\Phi(k') > k$ for every $k' \geq \Psi(k)$. The existence of $\Psi(k)$ is insured by the fact that $p\cdot\Phi$ is eventually in C_k for every k. Now, by the definition of Ψ, we know that for any $i \in K$ $k < \Psi(i)$ if $p\cdot\Phi(k) = i$. Hence, for a fixed $i \in K$, $\{k\in K: p\cdot\Phi(k) = i\} \subseteq \{k\in K: k < \Psi(i)\}$ which has cardinality less than K. Thus, $\{j\in K: ((i,j),i) \in \Phi(K)\}$ has cardinality less than K. Hence, there exists $j(i) \in K$ so that $j < j(i)$ for all j such that $((i,j),i) \in \Phi(K)$. Thus, Φ is never in the neighborhood $(\cup\{B_{ij(i)}: i\in K\}) \times C_1$ of (x,y). Hence, Φ cannot converge to (x,y) and we have thus completed the proof that $X \times Y$ is not semi-erect. ||

Under very restrictive hypotheses it is sometimes possible to conclude that a product space is K-Fréchet. For example, if X and Y are spaces in which every neighborhood system is a $\{K\}$-filter, then $X \times Y$ is clearly of the same type and is thus K-Fréchet. This result does not extend to infinite products except that it applies to countable products when $K = \aleph_0$. This last reference is, of course, to the well-known fact that the countable product of first countable spaces is first countable.

CHAPTER V

WELL-ORDERED SUBNETS

Suppose that the analogue of Theorem 2.5 for well-ordered nets and well-ordered subnets could be proven. That is, suppose it were true that for each cluster point x of any well-ordered net Φ there should exist a well-ordered subnet of Φ converging to x. By Theorems 3.3 and 4.3 and the definition of a semi-erect space we would then have the (false) result that every space is semi-erect. Thus, from one point of view, the failure of well-ordered nets to yield a satisfactory theory of convergence in general spaces is due to the fact that a well-ordered net does not necessarily have well-ordered subnets converging to each of its cluster points. In this chapter we shall investigate conditions under which a well-ordered net Φ with a cluster point x will have a well-ordered subnet converging to x.

Definition. If $\Phi: D \to X$ is a net and D' is a cofinal subset of D, then $\Phi|D'$ is called a cofinal restriction of Φ.

The following theorem shows that we may restrict our attention to cofinal restrictions when studying well-ordered subnets of well-ordered nets.

Theorem 5.1. If W is a well-ordered set and $\Phi: W \to X$ has a well-ordered subnet converging to a point $x \in X$, then Φ has a cofinal restriction converging to x.

Proof. By hypothesis there exists a well-ordered set W' and a cofinal function $\Psi: W' \to W$ such that $\Phi \cdot \Psi$ converges to x. Let us define $\Psi': W' \to W$

by letting $\Psi'(w') = \min\{\Psi(w''):w'' \geq w'\}$. We now show that $\Phi\cdot\Psi'$ converges to x. Let U be any neighborhood of x. Since $\Phi\cdot\Psi$ converges to x, there exists $w' \in W'$ so that $\Phi\cdot\Psi(w'') \in U$ for all $w'' \geq w'$. Since $\Psi'(w'') = \Psi(w)$ for some $w \geq w''$, then for each $w'' \geq w'$ $\Phi\cdot\Psi'(w'') \in U$.

Let $V = \Psi'(W')$. Now let U be a neighborhood of x. Choose $w' \in W'$ so that $\Phi\cdot\Psi'(w'') \in U$ for all $w'' \geq w'$. Denote $\Psi'(w')$ by v'. Now for $v'' \in V$ such that $v'' \geq v'$, there exists $w'' \in W'$ so that $w'' \geq w'$ and $\Psi'(w'') = v''$. The fact that w'' can be chosen to be greater than or equal to w' follows from the monotonicity of Ψ'. By the way in which w' was chosen, $\Phi(v'') = \Phi\cdot\Psi'(w'') \in U$. Hence, $\Phi|V$ converges to x. Since Ψ is a cofinal function, Ψ' is also a cofinal function and V is thus a cofinal subset of W. ||

The next theorem gives a sufficient condition for a $\{K\}$-net with a cluster point x to have a cofinal restriction converging to x.

<u>Theorem 5.2</u>. Let K be a regular aleph, and suppose that $\{K\}$ describes the topology of X at x. Then every $\{K\}$-net in X having x as a cluster point has a cofinal restriction converging to x.

<u>Proof</u>. Let Φ be a $\{K\}$-net in X clustering at x. For each $k \in K$, let $C_k = \{k'\in K:k'\geq k\}$. Clearly, x is in the closure of $\Phi(C_k)$ for every k. Suppose that for every k there exists y_k in $\Phi(C_k)$ such that x is in the closure of $\{y_k\}$. Then y_k is an element of every neighborhood of x. Thus, if $K' = \{k\in K:x$ is in the closure of $\Phi(k)\}$, then K' is a cofinal subset of K and $\Phi|K'$ converges to x.

Suppose, on the other hand, that there exists some k in K so that $\Phi(C_k)$ contains no element y such that x is in the closure of $\{y\}$. Since x is in the closure of $\Phi(C_k)$ and since $\{K\}$ describes the topology of X at x, there is a net $\Psi:K \to \Phi(C_k)$ converging to x. Clearly, x is in the closure of $\Psi(K)$. Since $\Psi(K) \subseteq \Phi(C_k)$, $\Psi(K)$ contains no element y such that x is in the closure of $\{y\}$. By Lemma 4.5, $\Psi(K)$ has cardinality K. For each $a \in \Psi(K)$, choose $\gamma(a) \in K$ such that $\Psi\cdot\gamma(a) = a$. Clearly, γ is injective and, if $W = \gamma(\Psi(K))$, W has cardinality K. W is thus a cofinal subset of K. It is easily seen that $\Psi|W$ is injective and converges to x.

For each $w \in W$, choose f(w) in K so that $\Phi(f(w)) = \Psi(w)$. Since $\Psi|W$ is injective, f is injective. We wish to show that $f:W \to K$ is a cofinal function. For any $k \in K$, $\{w\in W:f(w) < k\}$ has cardinality less than K since f is injective and $\{k'\in K:k'< k\}$ has cardinality less than K. Since W is cofinal in K and is therefore order-isomorphic to K, $\{w\in W:f(w) < k\}$ cannot be cofinal in W. Thus, there exists $w'\in W$ such that $f(w) \geq k$ for all $w \geq w'$. So f is a cofinal function and $\Psi|W = \Phi\cdot f$ is a well-ordered subnet of Φ converging to x. By Theorem 5.1, Φ has a cofinal restriction converging to x. ||

Corollary 5.3. Let K be a regular aleph and let X be a K-Fréchet space. Suppose that Φ is a $\{K\}$-net in X. For each cluster point x of Φ, there is a cofinal restriction of Φ converging to x.

Proof. Since X is K-Fréchet, $\{K\}$ describes the topology of X at each of its points. ||

Corollaries 4.10 and 5.3 yield the classical result that in a first countable space a point x is a cluster point of a sequence Φ only if Φ has a subsequence converging to x.

It should be noted that the converses of Theorem 5.2 and Corollary 5.3 are trivially true, for if any net Φ has a cofinal restriction converging to a point x then x must be a cluster point of Φ.

Theorem 5.2 and Corollary 5.3 are the main results of this chapter. In Example 5.4 we shall provide a counterexample to a possible strengthening of the conclusion of Corollary 5.3. Example 5.5 is a counterexample to a possible weakening of its hypotheses.

Example 5.4. It might reasonably be conjectured that in a "sufficiently well-behaved" space every $\underline{C}$-net (where $\underline{C}$ is the class of regular alephs) would have a cofinal restriction converging to each of its cluster points. Indeed it is easily seen that discrete spaces have this property. However, we shall now show that in as nicely behaved a space as the real numbers R there is a $\underline{C}$-net having a cluster point to which no cofinal restriction converges.

Let $W = \aleph_0 \times \Omega$. Well-order W by requiring that $(n_1,w_1) \leq (n_2,w_2)$ if (1) $w_1 < w_2$ or (2) $w_1 = w_2$ and $n_1 \leq n_2$. It is rather easily seen that W is order-isomorphic to Ω since W has cardinality greater than or equal to Ω whereas every initial segment of W is countable. Define $\Phi: W \rightarrow R$ by $\Phi((n,w)) = \frac{1}{n}$. Φ clearly clusters at 0. Suppose there is a cofinal subset V of W such that $\Phi|V$ converges to 0. Clearly, for each positive integer n there exists $v_n \in V$ such that $\Phi(v) < \frac{1}{n}$ for every $v \in V$ such that $v \geq v_n$. Clearly, $\{v_n\text{:n a positive integer}\}$ must be cofinal in V, for if there existed $v \in V$ such that $v_n < v$ for every positive n then $\Phi(v)$ would be less than $\frac{1}{n}$ for each positive integer n. But then $\{v_n\text{:n a positive integer}\}$ is a countable cofinal subset of W. This, however, is impossible since W is order-isomorphic to Ω. Hence, Φ has no cofinal restriction converging to 0. ||

Example 5.5. We now show that there is no regular aleph K such that every {K}-net, in any space whatsoever, has a cofinal restriction converging to each cluster point. This example is a generalization of one appearing in Schubert [8], p. 65.

Let K be a regular aleph and let $X = I^{P(K)}$ with the Tychonoff product topology where I is the closed unit interval and P(K) is the power set of K. Since X is compact by the Tychonoff Product Theorem, every {K}-net in X has a cluster point. We shall complete this example by producing a {K}-net in X which has no convergent cofinal restriction. By Lemma 2.5, K is the union of two disjoint cofinal subsets, which we shall label A and B. We define a net $\Phi: K \to X$ as follows. Let k be any element of K and let S be any member of P(K). If S is not cofinal in K, let the Sth coordinate of $\Phi(k)$ be 0. If S is cofinal in K, then there is an order isomorphism i_S from K onto S. Define the Sth coordinate of $\Phi(k)$ to be 0 if k is not a member of S or if $k \in i_S(A)$ and to be 1 if $k \in i_S(B)$. Now let V be any cofinal subset of K. We shall show that $\Phi|V$ cannot converge to any point in X. Consider $p \cdot (\Phi|V)$ where p is the projection from X into its Vth coordinate space. $p \cdot \Phi(i_V(A)) = \{0\}$ and $p \cdot \Phi(i_V(B)) = \{1\}$. Since $i_V(A)$ and $i_V(B)$ are cofinal subsets of V, $p \cdot (\Phi|V)$ is frequently at 0 and frequently at 1 and thus fails to converge. Hence, $\Phi|V$ fails to converge. ||

It has been mentioned previously that the theory of convergence with nets is equivalent to the theory of convergence with filters in the sense that every true theorem concerning nets can be translated into a true theorem concerning filters. However, the statement of the true theorem concerning filters is sometimes different from what one might have expected at first thought. For example, because of Corollary 5.3 one might suspect that in a K-Fréchet space each {K}-filter has, corresponding to each cluster point, a $\{K\}$-refinement converging to that cluster point. However, a counterexample to this conjecture will be given in Example 5.7, the last result of this chapter. The true theorem concerning filters which corresponds to Corollary 5.3 is that in a K-Fréchet space each filter generated by a $\{K\}$-net has, corresponding to each cluster point, a $\{K\}$-refinement converging to that cluster point. (A filter $\underline{F}$ is said to be generated by a $\{K\}$-net Φ if $\{\{\Phi(k'):k' \geq k\}:k \in K\}$ is a filter base for $\underline{F}$.) The proof is easy and will not be given here.

We shall utilize the following lemma in Example 5.7.

Lemma 5.6. Let K be a regular aleph. Suppose that some point $x \in X$ has the property that for every $\{K\}$-filter $\underline{F}$ in X, $\underline{F}$ clusters at x only if $\underline{F}$ has a $\{K\}$-refinement converging to x. Then, if W is any well-ordered set having final order K and if $\Phi:W \to X$ is a net clustering at x, Φ has a cofinal restriction converging to x.

Proof. By Theorem 5.1 it is sufficient to show that Φ has a well-ordered subnet converging to x. Let K' be a cofinal subset of W order-isomorphic to K under the isomorphism $i:K \to K'$. For each $k \in K$ let $B_k = \{\Phi(w):w \in W$ and $w \geq i(k)\}$. Now $\{B_k:k \in K\}$ is clearly a base for a $\{K\}$-filter $\underline{F}$ clustering at x. By hypothesis, $\underline{F}$ has a $\{K\}$-refinement $\underline{G}$ converging to x. Let C be a base for $\underline{G}$ such that there is a surjective order homomorphism $h:K \to C$. Since $\underline{F} \subseteq \underline{G}$, $\{B_k \cap h(k):k \in K\}$ is a base for $\underline{G}$. In particular, $B_k \cap h(k)$ is nonempty for each $k \in K$, and for each neighborhood U of x there exists a $k' \in K$ such that $B_{k'} \cap h(k') \subseteq U$. We now define a function $\Psi:K \to W$ by choosing, for each $k \in K$, $\Psi(k)$ to be an element of W such that $\Psi(k) \geq i(k)$ and $\Phi(\Psi(k)) \in B_k \cap h(k)$. This can be done because of the definition of B_k and the fact that $B_k \cap h(k)$ is nonempty. It is easily seen that Ψ is a cofinal function. If U is a neighborhood of x there exists, as noted above, a $k' \in K$ such that $B_{k'} \cap h(k') \subseteq U$. Now for $k \geq k'$, $\Phi(\Psi(k)) \in B_k \cap h(k) \subseteq B_{k'} \cap h(k') \subseteq U$. Hence, $\Phi \cdot \Psi$ converges to x and thus Ψ determines a well-ordered subnet of Φ converging to x. ||

Example 5.7. We now produce an example of a K-Fréchet space in which there is a $\{K\}$-filter $\underline{F}$ clustering at a point x though there is no $\{K\}$-refinement of $\underline{F}$ converging to x. We shall proceed by showing that there is a well-ordered set W of final order K and a net $\Phi: W \to X$ clustering at x such that there is no cofinal restriction of Φ converging to x. The existence of $\underline{F}$ will then follow from Lemma 5.6.

Consider the space X defined in Example 4.14. Let $W = K \times K$ with the well-ordering $\leq$ defined by requiring that $(i,j) \leq (i',j')$ if (1) $i < i'$ or (2) $i = i'$ and $j \leq j'$. Let $\Phi: W \to X$ be the inclusion map. It is easily seen that Φ clusters at x.

Let S be any cofinal subset of W. For each $k \in K$, choose k' as follows:

(1) if $(\{k\} \times K) \cap S$ is empty, let k' be any element of K, and

(2) if $(\{k\} \times K) \cap S$ is nonempty, choose k' so that $(k,k') \in S$.

Now let $N = \cup\{B_{k,k'}: k \in K\}$. N is a neighborhood of x. But $\{(k,k'): k \in K\}$ is a cofinal subset of S not meeting N. Hence, $\Phi|S$ does not converge to x. ||

CHAPTER VI

K-METRIC SPACES

The following definition is motivated by the fact that a uniform space is pseudometrizable if and only if the uniformity is an $\{\aleph_0\}$-filter in X x X. This is a consequence of the well-known theorem that a uniform space is pseudometrizable if and only if its uniformity has a countable base.

Definition. Let K be a regular aleph. If X is a uniform space whose uniformity is a {K}-filter in X x X, then X will be called a K-pseudometric space. If X is also Hausdorff, it will be called a K-metric space.

The first portion of this chapter will be devoted to a characterization of K-Fréchet spaces in terms of K-metric spaces. This result generalizes Arhangel'skiĭ's [3] characterization of Hausdorff $\aleph_0$-Fréchet spaces in terms of metric spaces.

In the remainder of the chapter we shall prove a theorem generalizing a number of facts concerning compactness and associated properties. Part of the results relate to K-pseudometric spaces.

Theorem 6.1. A K-pseudometric space is K-Fréchet.

Proof. Let X be a K-pseudometric space and let B be a base for its uniformity such that there is a surjective order-homomorphism $h:K \to B$. Since $\{B[x]:B\in B\}$ forms a neighborhood base for any point x, the neighborhood system at each point in X is a {K}-filter. By Theorem 4.9, X is K-Fréchet.

Definition. A surjective continuous function $f:X \rightarrow Y$ is said to be pseudo-open if for each $y \in Y$ and for any neighborhood U of $f^{-1}(y)$, y is in the interior of $f(U)$.

Theorem 6.2. If a space Y is the image of a K-Fréchet space X under a pseudo-open map f, then Y is K-Fréchet.

Proof. Let y be a point in Y and let A be a subset of Y such that y is in the closure of A. We need only show that there is a K-net in A converging to y.

We first show that $f^{-1}(y)$ meets $[f^{-1}(A)]^{-}$. If not, then $X-[f^{-1}(A)]^{-}$ is a neighborhood of $f^{-1}(y)$. Since f is pseudo-open, $f(X-[f^{-1}(A)]^{-})$ is a neighborhood of y. Since $A \cap f(X-[f^{-1}(A)]^{-})$ is empty, this contradicts the fact that y is in the closure of A.

Let x be some point in $f^{-1}(y) \cap [f^{-1}(A)]^{-}$. Since X is K-Fréchet, there is a $\{K\}$-net Φ in $f^{-1}(A)$ converging to x. $f \cdot \Phi$ is a $\{K\}$-net in A which, by the continuity of f, converges to $f(x) = y$. ||

Corollary 6.3. Every pseudo-open image of a K-metric space is K-Fréchet.

Proof. This follows from Theorem 6.1. ||

We now prove the converse of Corollary 6.3.

Theorem 6.4. Every K-Fréchet space is a pseudo-open image of a K-metric space.

Proof. Let X be a K-Fréchet space. Let S be the collection of all ordered pairs (Φ,x) such that Φ is a $\{K\}$-net in X, x is a point in X, and Φ converges to x. Give S the discrete topology and let $W = Y \times S$ with the Tychonoff product topology, where Y is the space Y defined in Example 4.14. Recall that $Y = K \cup \{y\}$ where y is an object not in K. Now we define a function $f:W \rightarrow X$ by $f(k,(\Phi,x)) = \Phi(k)$, for each $k \in K$, and

$f(y,(\Phi,x)) = x$. It is easily seen that f is surjective. We shall show that f is a pseudo-open map and that W is a K-metric space.

For each $k \in K$ and $(\Phi,x) \in S$, $\{(k,(\Phi,x))\}$ is an open set in W and, hence, f must be continuous at $(k,(\Phi,x))$. We need now to show that f is continuous at any point of the form $(y,(\Phi,x))$. Now $f(y,(\Phi,x)) = x$. Let N be any neighborhood of x in X. Since Φ converges to x, there exists a $k \in K$ such that $\Phi(k') \in N$ for all $k' \geq k$. Recall from Example 4.14 that C_k is a neighborhood of y in Y where $C_k = \{k' \in K: k' > k\} \cup \{y\}$. So $C_k \times \{(\Phi,x)\}$ is a neighborhood of $(y,(\Phi,x))$ in W. From the definition of f it is clear that $f(C_k \times \{(\Phi,x)\}) \subseteq N$. Hence, f is continuous at $(y,(\Phi,x))$.

Now we wish to show that f is pseudo-open. Suppose not. Then for some point $x \in X$ and some neighborhood U of $f^{-1}(x)$ in W, x is not an interior point of f(U). Hence, x is in the closure of X-f(U). Since X is K-Fréchet, there is a {K}-net Φ in X-f(U) converging to x. Thus, $(\Phi,x) \in S$. So $(y,(\Phi,x)) \in f^{-1}(x) \subseteq U$. Since U is an open set and since every neighborhood of y in Y contains C_k for some $k \in K$, we have $(y,(\Phi,x)) \in C_k \times \{(\Phi,x)\} \subseteq U$. Choose any $k' > k$. Clearly $\Phi(k') = f(k',(\Phi,x)) \in f(U)$. Thus, Φ could not have been in X-f(U). Hence, f must be pseudo-open.

Now we show that W is K-metric by exhibiting a uniformity for W which has a {K}-filter as a base and which induces the topology of W. Let Δ be the diagonal in $W \times W$. For each $k \in K$, let $B_k = \Delta \cup (\bigcup \{(C_k \times \{(\Phi,x)\}) \times (C_k \times \{(\Phi,x)\}) : (\Phi,x) \in S\})$. Since $B_k \cap B_{k'} = B_{k''}$ where $k'' = \max\{k,k'\}$, $\{B_k : k \in K\}$ is a base for a {K}-filter in $W \times W$. Since $\Delta \subseteq B_k$, $B_k^{-1} = B_k$, and $B_k \circ B_k = B_k$ for each $k \in K$, the {K}-filter is a uniformity for W. It remains to show that the uniformity induces the topology of W. For that, it is sufficient to show that the neighborhood system at each point of W is exactly the neighborhood system induced by the uniformity. First we consider a point of the form $(k,(\Phi,x))$ where $k \in K$. Now $\{(k,(\Phi,x))\}$ is a base for its neighborhood system in W. A base for the neighborhood system induced by the uniformity is $\{B_{k'}[(k,(\Phi,x))]:k'\in K\}$. If $k' > k$, $B_{k'}[(k,(\Phi,x))] = \{(k,(\Phi,x))\}$. Thus, $\{(k,(\Phi,x))\}$ is also a base for the neighborhood system induced by the uniformity. Now we consider a point of the form $(y,(\Phi,x))$. A base for its neighborhood system in W is $\{C_k \times \{(\Phi,x)\}:k\in K\}$. Since $B_k[(y,(\Phi,x))] = C_k \times \{(\Phi,x)\}$, this is exactly the neighborhood system induced by the uniformity. Since W is clearly Hausdorff, it is then K-metric. ||

We thus have the following characterization of K-Fréchet spaces.

Corollary 6.5. A space is K-Frechet if and only if it is a pseudo-open image of a K-metric space.

Proof. Combine Corollary 6.3 and Theorem 6.4. ||

Arhangelśkii's result [3] was that a Hausdorff space is $\aleph_0$-Fréchet if and only if it is a pseudo-open image of a metric space. Franklin [5] gives a proof. We have adopted here some of Franklin's methodology.

It is interesting to note that what we have done up to now in this chapter could have been done for all directed sets rather than just the regular alephs. More specifically, suppose that D is any directed set and we define a space X to be D-Fréchet if D describes the topology of X at each of its points. Suppose further that we define a Hausdorff uniform space to be D-metric if and only if the uniformity is a $\{D\}$-filter. We could then prove by the methods of this chapter that a space is D-Fréchet if and only if it is a pseudo-open image of a D-metric space. In the paragraph preceding Theorem 4.2 we have already noted that every topological space is D-Fréchet for some directed set D.

The following theorem and its corollary are our final results. They generalize a number of propositions related to compactness. We shall first need a few definitions.

Definitions. Let K be a regular aleph. A space X will be said to be K-Lindelöf if every open cover of X has a subcover of cardinality $\leq$ K. X will be said to be K^*-compact if every open cover of X has a subcover of cardinality < K.

Note that an "$\aleph_0$-Lindelöf" space is what is usually called a "Lindelöf" space. Similarly, "$\aleph_0^*$-compact" simply means "compact." The star in "K^*-compact" is used to differentiate our term from Kowalsky's [7] "k-compact" which denotes the property that every open cover of cardinality $\leq$ k has a finite subcover.

Definition. Let K be any cardinal number. If X is a space, then $x \in X$ is said to be a K-accumulation point of $A \subseteq X$ if the cardinality of $U \cap A$ is greater than or equal to K whenever U is a neighborhood of x.

Theorem 6.6. Let K be a regular aleph. Then, if X is a space, the statements below have the following relationships: For all spaces (a) is equivalent to (b), and (d) implies (a). If X is K-Fréchet then (a), (b), and (c) are all equivalent. If X also is K-Lindelöf then all four conditions are equivalent. If X is a K-pseudometric space then each of the four conditions implies that X is K-Lindelöf and all four are equivalent.

(a) Every subset of X of cardinality $\geq$ K has a K-accumulation point.

(b) Every {K}-net in X has a cluster point.

(c) For each K-net in X there is a cofinal restriction converging to a point of X.

(d) The space X is K^*-compact.

Proof. We first show that for all spaces (a) implies (b) for any space X. Let Φ be a $\{K\}$-net in X. If there is a point $x_o \in \Phi(K)$ such that $\Phi^{-1}(x_o)$ has cardinality K, then $\Phi^{-1}(x_o)$ is a cofinal subset of K and thus Φ clusters at x_o. On the other hand, suppose that for each $x \in X$ the cardinality of $\Phi^{-1}(x)$ is less than K. Then, for each $x \in \Phi(K)$, $\Phi^{-1}(x)$ fails to be cofinal in K and we can choose an element $f(x) \in K$ such that $k < f(x)$ for each $k \in \Phi^{-1}(x)$. So we have a function $f:\Phi(K) \to K$. Clearly, $f(\Phi(K))$ is a cofinal subset of K and is thus of cardinality K, and hence $\Phi(K)$ is of cardinality K. By (a), $\Phi(K)$ thus has a K-accumulation point y. Let U be a neighborhood of y. Let $k \in K$ be given. Now $U \cap \Phi(K)$ must have cardinality $\geq K$ since y is a K-accumulation point of $\Phi(K)$. It follows that $\Phi(k') \in U$ for some $k' \geq k$ and hence Φ is frequently in U. Thus, Φ clusters at y.

Now we show that (b) implies (a) for any space X. Let A be a subset of X of cardinality $\geq K$. Let $\Phi:K \to A$ be any injective function. By (b), Φ has a cluster point $x \in X$. Let U be any neighborhood of x. Since Φ clusters at x, $\Phi^{-1}(U)$ must be cofinal in K and hence must have cardinality K. Since Φ is injective $\Phi(\Phi^{-1}(U))$ must have cardinality K. But $\Phi(\Phi^{-1}(U)) = \Phi(K) \cap U \subseteq A \cap U$. Hence, $A \cap U$ has cardinality $\geq K$. Thus, x is a K-accumulation point of A.

To show that (d) implies (a), it is sufficient to show that (d) implies (b) since (a) and (b) are equivalent. Let Φ be a $\{K\}$-net in X. Suppose that Φ has no cluster point. Then, for each $k \in K$, let C_k be the closure of $\{\Phi(k'):k' \geq k\}$. Since Φ has no cluster point, $\bigcap\{C_k:k\in K\}$ is empty. Hence, $\{X-C_k:k\in K\}$ is an open cover of X. By (d), there is a subset $A \subseteq K$ of cardinality less than K such that $X = \bigcup\{X-C_k:k\in A\} =$

$X-\bigcap\{C_k : k \in A\}$. This implies that $\bigcap\{C_k : k \in A\}$ is empty. It is easily seen that A must then be a cofinal subset of K. But this contradicts the fact that K is a regular aleph while A has cardinality less than K. Hence, Φ must have a cluster point.

We can easily see that (b) and (c) are equivalent if X is K-Fréchet, for (c) implies (b) in any space and Corollary 5.3 shows that (b) implies (c) if X is K-Fréchet.

Now we show that all four conditions are equivalent if X is K-Lindelöf as well as K-Fréchet. From the preceding we know that the first three are equivalent and that (d) implies (a). Hence, it will be sufficient to show that (b) implies (d). Let U be any open cover of X. Since X is K-Lindelöf, U has a subcover V of cardinality $\leq K$. If V has cardinality $< K$, we are through. If V has cardinality K, we can appeal to (b) and Lemma 3.1 to get the result that V has a subcover W of cardinality less than K. Thus, X must be K^*-compact.

Finally, we show that if X is K-pseudometric each of the four conditions implies that X is K-Lindelöf. It will then follow from the preceding paragraph and the fact that K-pseudometric spaces are K-Fréchet that all four conditions will be equivalent. We shall actually show that each of the four conditions implies that X has a basis for its topology of cardinality $\leq K$. We will then show that this latter property implies that X is K-Lindelöf.

Suppose that X satisfies (a). This is equivalent to assuming that X satisfies (b) or (c), since K-pseudometric spaces are K-Fréchet and in K-Fréchet spaces (a), (b), and (c) are equivalent. Since X is K-pseudometric there is a base $\underline{B}$ for its uniformity such that there exists a surjective order homomorphism $\Phi:K \to \underline{B}$. $\underline{B}$ can clearly be chosen so that each of its elements is symmetric. Now, for each k, we can choose a symmetric member V_k of the uniformity of X so that $V_k \circ V_k \subseteq \Phi(k)$. By Zorn's Lemma it is easily shown that there exists a maximal subset A_k of X such that whenever $a_1, a_2 \in A_k$, and $a_2 \in \Phi(k)[a_1]$ then $a_1 = a_2$. Now let $x \in X$ be given. Suppose $a_1, a_2 \in V_k[x]$ where $a_1, a_2 \in A_k$. Then $(a_1,x), (x,a_2) \in V_k$. So $(a_1,a_2) \in V_k \circ V_k \subseteq \Phi(k)$. Thus, $a_2 \in \Phi(k)[a_1]$, and hence $a_1 = a_2$. So $V_k[x]$ is a neighborhood of x which meets at most one point in A_k. Since x was arbitrary, A_k has no K-accumulation points. Thus, by (a), A_k has cardinality less than K. Hence, $A = \cup\{A_k : k \in K\}$ has cardinality $\leq K$. By the maximality of A_k, $\Phi(k)[x]$ meets A_k for each $k \in K$ and $x \in X$. Hence, A is dense in X. Now let $\underline{C} = \{\Phi(k)[a] : k \in K \text{ and } a \in A\}$. $\underline{C}$ clearly has cardinality $\leq K$. We shall show that $\underline{C}$ is a basis for the topology of X. Let U be an open set containing a given point x. There is a $k \in K$ so that $\Phi(k)[x] \subseteq U$. Choose $k' \in K$ so that $\Phi(k') \subseteq V_k$. Now there exists a point $a \in A$ so that $a \in \Phi(k')[x]$, since A is dense in X. If $y \in \Phi(k')[a]$, then $(a,y) \in \Phi(k')$ and thus $(x,y) \in \Phi(k') \circ \Phi(k') \subseteq V_k \circ V_k \subseteq \Phi(k)$. So $y \in \Phi(k)[x] \subseteq U$. Hence, $x \in \Phi(k')[a] \subseteq U$. Since $\Phi(k')[a] \in \underline{C}$, we have that $\underline{C}$ is a basis for the topology of X.

Suppose now that X satisfies (d). We showed earlier that (d) implies (a). By the previous paragraph, X has a basis of cardinality $\leq K$.

All that remains is to show that X is K-lindelöf if X has a basis for its topology of cardinality $\leq K$. Let $\underline{B}$ be such a basis. Let $\underline{U}$ be an open cover of X. For each $x \in X$, we can find $U_x \in \underline{U}$ and $B_x \in \underline{B}$ so that $x \in B_x \subseteq U_x$. Let $\underline{C} = \{B_x : x \in X\}$. For each $B \in \underline{C}$, choose $U_B \in \underline{U}$ so that $B \subseteq U_B$. Let $\underline{V} = \{U_B : B \in \underline{C}\}$. It is easily verified that $\underline{V}$ is a subcover of

$\underline{U}$ of cardinality $\leq K$. Hence, X is K-Lindelöf. ||

The space Y defined in Example 4.14 is an example of a K-Fréchet (in fact, K-pseudometric) space which is K^*-compact. K^*-compactness is, in a sense, the "best" compactness property that a non-trivial T_1 K-Fréchet space can satisfy. More specifically, if K and K_1 are regular alephs with $K_1 < K$, and if X is a K-Fréchet space which is K_1^*-compact, then X must be discrete. To prove this, we note that if X is not discrete then there is a point x so that x is in the closure of $X-\{x\}$. By Lemma 4.5 and the fact that X is T_1, we have that $X-\{x\}$ has cardinality $\geq K$. Thus, there exists an injective function $\Phi:K_1 \to X$. By Theorem 6.6(b), Φ must have a cluster point y. Clearly y must be in the closure of $\Phi(K_1)-\{y\}$. But Lemma 4.5 implies that $\Phi(K_1)-\{y\}$ must have cardinality $\geq K$. This is the desired contradiction. Hence, X must have been discrete. In particular, if $K > \aleph_0$ and X is a T_1 compact K-Fréchet space, then X must be a finite discrete space.

We close with a corollary to Theorem 6.6.

Corollary 6.7. Theorem 6.6 holds if "is K-Fréchet" is replaced by "has a $\{K\}$-filter as its neighborhood system at each point" and "is K-Lindelöf" is replaced by "has a basis for its topology of cardinality $\leq K$."

Proof. If X has a $\{K\}$-filter as its neighborhood system at each point, then X is K-Fréchet by Theorem 4.9 and thus (a), (b), and (c) are all equivalent. If X also has a basis for its topology of cardinality $\leq K$, then, as shown in the last paragraph of the proof of Theorem 6.6, X is K-Lindelöf and hence all four conditions {(a),(b),(c), and (d)} are equivalent. We showed in the proof to Theorem 6.6 that if X is a K-pseudometric space, each of the four conditions implies that X has a basis for its topology of cardinality $\leq K$; it follows from the preceding sentence that all four conditions are then equivalent. $\|$

Note that in the special case $K = \aleph_0$ Corollary 6.7 becomes Theorem 5 on page 138 of Kelley [6].

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