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About This Document

This resource contains curriculum for the distance education version of a course offered at the Georgia Institute of Technology, Math 2401, in Spring 2014. This distance education course explored multivariable calculus concepts during lectures and recitations. Recitations are synchronous sessions that offer students an opportunity to apply and review course concepts, which they have been exposed to in lectures.

Contained in this curriculum are materials for 26 recitations and two office hour sessions, available in PDF and presentation slide formats. The slide format is offered for teaching assistants to import directly into web-conferencing software. Slides contain activities that students would solve. The associated notes contain solutions to corresponding activities and are available in PDF format.

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For Further Information

Questions regarding this document can be directed to Greg Mayer (gsmayer@gmail.com), who would be happy to hear your suggestions on how to improve this document.

List of Topics

The following table presents a list of topics that were explored in the recitation and office hour activities. The numbering system got a little strange: there should be a cancelled recitation for Quiz 1 in there, somewhere.

Recitation Number	Topics
1	Introduction to Math 2401, logistical matters, vector derivatives
2	Vector Functions
3	Vector Functions
4	Tangents, Arc Length
5	Arc Length, Acceleration
6	Acceleration, Level Curves
7	Surfaces, Domain, Limits, Gradients
8	Quiz Review, Gradients
9	Quadratic Surfaces, Extreme Values
10	Extreme Values, Lagrange Multipliers
11	Taylor Approximation, Integration
12	Polar Integration
13	Triple Integrals in Cartesian
14	Triple Integrals in Cartesian
15	*No Recitation - Quiz 2*
16	Triple Integrals in Cylindrical Coordinates
17	Triple Integrals in Spherical Coordinates

18	*No Recitation - Quiz 3*
19	Line Integrals and Work
20	Line Integrals: Circulation and Flux
21	Flux, Conservative Fields
22	Green's Theorem
23	*No Recitation - Quiz 4*
24	Surface Integrals
25	Surface Integrals
26	Divergence

Welcome Back!

1. Announcements
2. Vector Derivatives (14.1)

Technologies: same as last semester

Recitations run in Wimba

Wimba technical problems, can contact technical support http://www.wimba.com/services/support

- Recordings of our recitations on Tegrity gatech.tegrity.com
- Tablets, mics: please bring and use them
- All homework on MyMathLab
- Piazza: link in t-square

Grading Weights: Same as Last Semester

	QH6 (%)	All other sections (%)
Homework	10	10
Final	25	25
Quizzes	60	65
Recitations	5	0
Total	100	100

+ random pop quizzes

Questions, Office Hours

Office Hours

Generally held on the night before quizzes (same as last semester)

Questions

email:

phone (office):

phone (cell):

Definition of Torque

Torque, τ, is defined as

$\tau =$

Angular Momentum

If the position of a particle with constant mass *m* is $\mathbf{r}(t)$, its angular momentum is $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$.

Show that $\mathbf{L}'(t)$ is equal to torque.

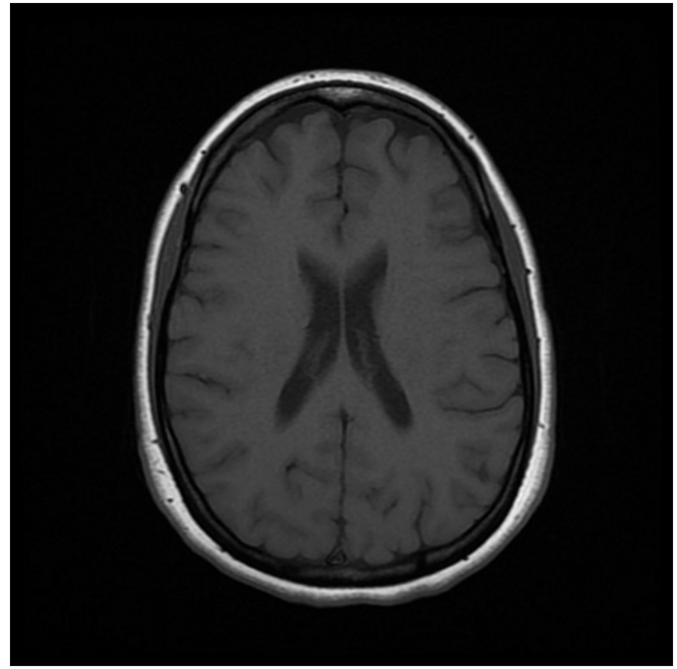
Zero Angular Momentum

Show that if the torque is a zero vector for all t, then the angular momentum of the particle is constant for all t.

Magnetic Resonance Imaging (MRI)

MRI Scanner

MRI Image



applied magnetic field, $\vec{B} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}$, ω = known constant

The applied field creates a measurable signal, $\vec{M}(t)$.

The Bloch Equation

$$\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}, \quad \vec{M} = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}$$

Solve the differential equation, plot the solution.

Recitation 02

Today: Vector Functions (13.1, 13.2)

Describe situations where the following is true for all t.

$$\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$$

HW 1

- HW1 is on MyMathLab
- due next Tuesday at 11:59 pm
- covers 13.1 and 13.2

Velocity, Acceleration

1) The position of particle is **r**(t) = sin(t)**i** + cos(t)**j**.
   a) Show that position is perpendicular to its velocity
   b) For what values of *t* do the position and acceleration have the same direction?
2) Provide another example of a vector function, **s**(t) that satisfies **s**''(t)=**s**(t) for all *t*.

A moving object starts at point (1,0) and its velocity is given by the vector [2, 4t]. Its position is given by:

Group Work

1. Group size is 2 or 3 students
2. Someone is in your group when they write their initials on board
3. Students can create breakout rooms
4. Have 10 minutes
5. Reflect on the problem for a minute before moving into groups

Integration

Consider the conjecture: $\int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt = \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt$

Provide an example to the other members of your group of an f(t) and a g(t) such that

1) the conjecture is **not** satisfied.
2) the conjecture **is** satisfied (for non-zero f and g).

Consider the conjecture: $\int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt = \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt$

Provide an example to the other members of your group of an f(t) and a g(t) such that

1) the conjecture is **not** satisfied.
2) the conjecture **is** satisfied (for non-zero f and g).

---

1) f(t) = g(t) =

$$LHS: \int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt =$$

$$RHS: \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt =$$

Announcements

- Survey: reminder sent last night, only 5 people responded as of yesterday …
- HW2: due Tues Feb 21 at 11:59 pm, covers sections 13.1 to 13.5
- HW1: due tonight, any questions related to the HW?
- Thursday: Graded Group Work: Question 1 from last years Quiz 1, group size 2 or 3

At what point does the twisted cubic
$r_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$,
intersect the plane $x + 2y + 3z = 34$?

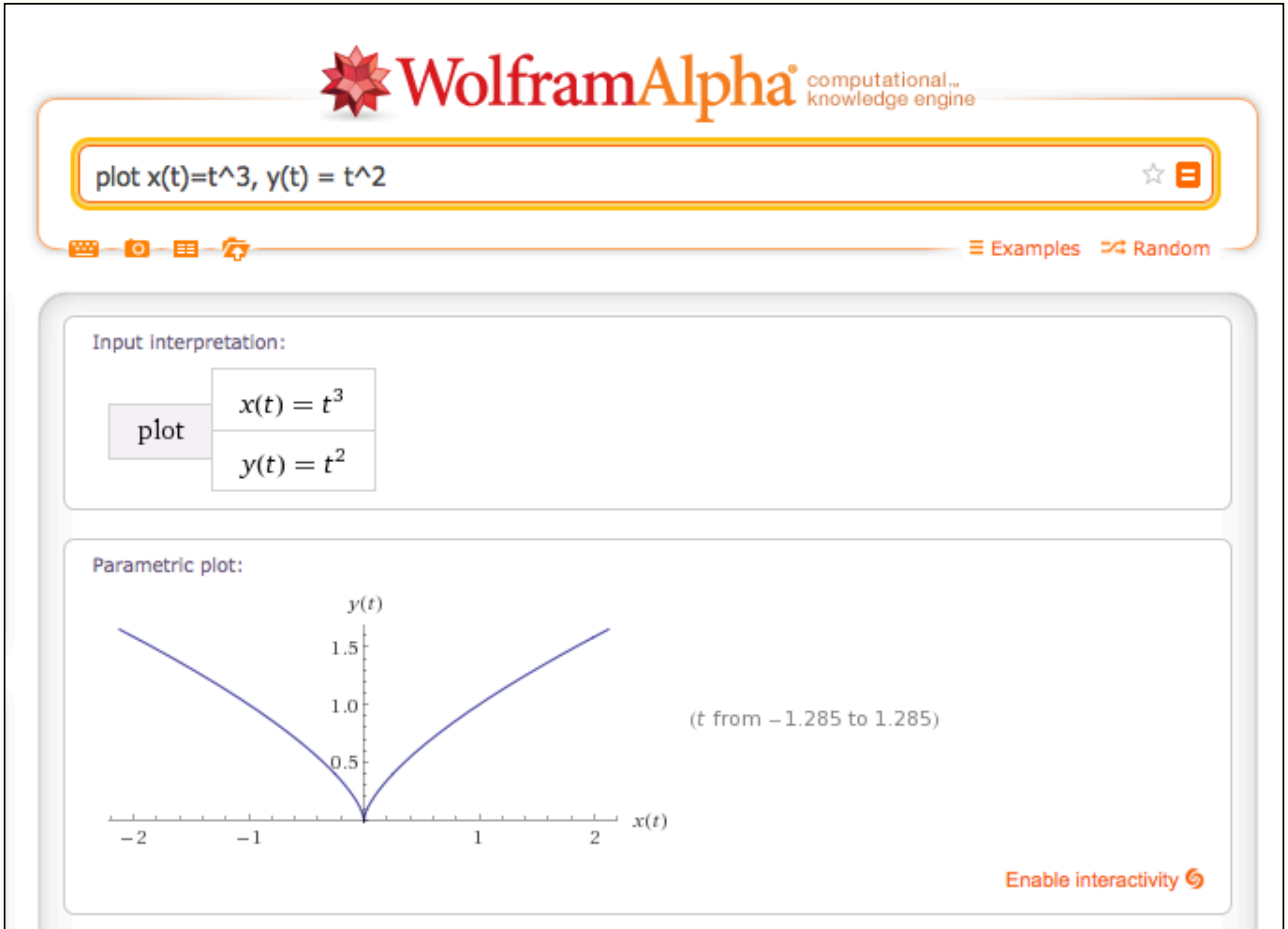
Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

Sketching Vector Functions

Sketch $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$

Wolfram Alpha Syntax for Parametric Curves

This is the syntax you would use for plotting parametric curves in WolframAlpha.

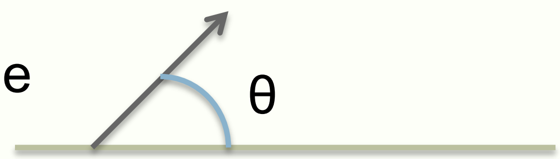


Projectile Motion (13.2)

A projectile is fired at angle θ with speed v_0.

a) derive its horizontal distance as a function of time

b) derive its maximum height



Tangent Vectors

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

a) How is the unit tangent vector, $\mathbf{T}(t)$, defined mathematically?

b) Suppose $x = t^2$, $y = t^3$, $z = t^2$, and *t* is any real number. Then what is the unit tangent vector when $t = 0$?

Position Perpendicular to Tangent

r(t) is the position of a moving particle. Show that $||\mathbf{r}(t)||$ = constant iff $\mathbf{r} \perp \mathbf{r}'$

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

a) How is the unit tangent vector, $\mathbf{T}(t)$, defined mathematically?

b) Suppose $x = t^2$, $y = t^3$, $z = t^2$, and *t* is any real number. Then what is the unit tangent vector when $t = 0$?

Announcements

- Quiz 1 is exactly 3 weeks away
- office hours, night before quiz
- HW2: Tue Feb 21 at 11:59 pm, sections 13.1-13.5 (hard?)
- Today: Graded Group Work: Question 1 from last years Quiz 1, group size 2 or 3

At what point does the twisted cubic
$\mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$,
intersect the plane $x + 2y + 3z = 34$?

Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

Group Work

1. Group size: 2 to 3 students
2. Someone is in your group when they write their initials on board
3. Students can create breakout rooms
4. Colors:
    a) Every student uses a different color
    b) Every student signs initials (or name) on board in their color
5. Only have 10 minutes

6. Press SAVE button to submit your work

At what point does the twisted cubic

$\mathbf{r}_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$,

intersect the plane $x + 2y + 3z = 34$?

Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

Intersection Angle

$r_1(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$
$r_2(u) = \cos(u)\mathbf{j} + \sin(u)\mathbf{k}$

Find the point of intersection, and the angle between their tangent vectors at that point.

Additional Problems (if time permits)

1. A cable is suspended between two poles that are 10 m apart. Find the length of the cable, if the cable's shape is $y(x) = k\,[\cosh(x/k) - 1]$, $-5 \le x \le 5$.
2. Calculate the curvature of
   a) $y = e^{-x}$
   b) $r(t) = 2t\mathbf{i} + t^3\mathbf{j}$
3. Vector $\mathbf{r}(t)$ is the position of a moving particle. Show that $||\mathbf{r}(t)||$ = constant for all t iff $\mathbf{r} \perp \mathbf{r}'$ for all t.
4. From last year's Quiz 1:

Find the arc length between 1 and t for the curve :

$$\mathbf{r}(s) = s\,\mathbf{i} + \left(2 - s^2\right)\mathbf{j} + \left(s^2 - 4\right)\mathbf{k}$$

(Don't evaluate the integral)

Solutions to these problems contained in later recitations.

Today: Arc Length, Acceleration

A cable is suspended between two poles that are 10 m apart. Find the length of the cable, if the cable's shape is $y(x) = k\,[\cosh(x/k) - 1]$, $-5 \le x \le 5$.

Arc Length (2013 Quiz 1, Question 2)

Find the arc length between 1 and t for the curve :

$$\mathbf{r}(s) = s\,\mathbf{i} + \left(2 - s^2\right)\mathbf{j} + \left(s^2 - 4\right)\mathbf{k}$$

(Don't evaluate the integral)

Table of Formulas on page 756, 13.5

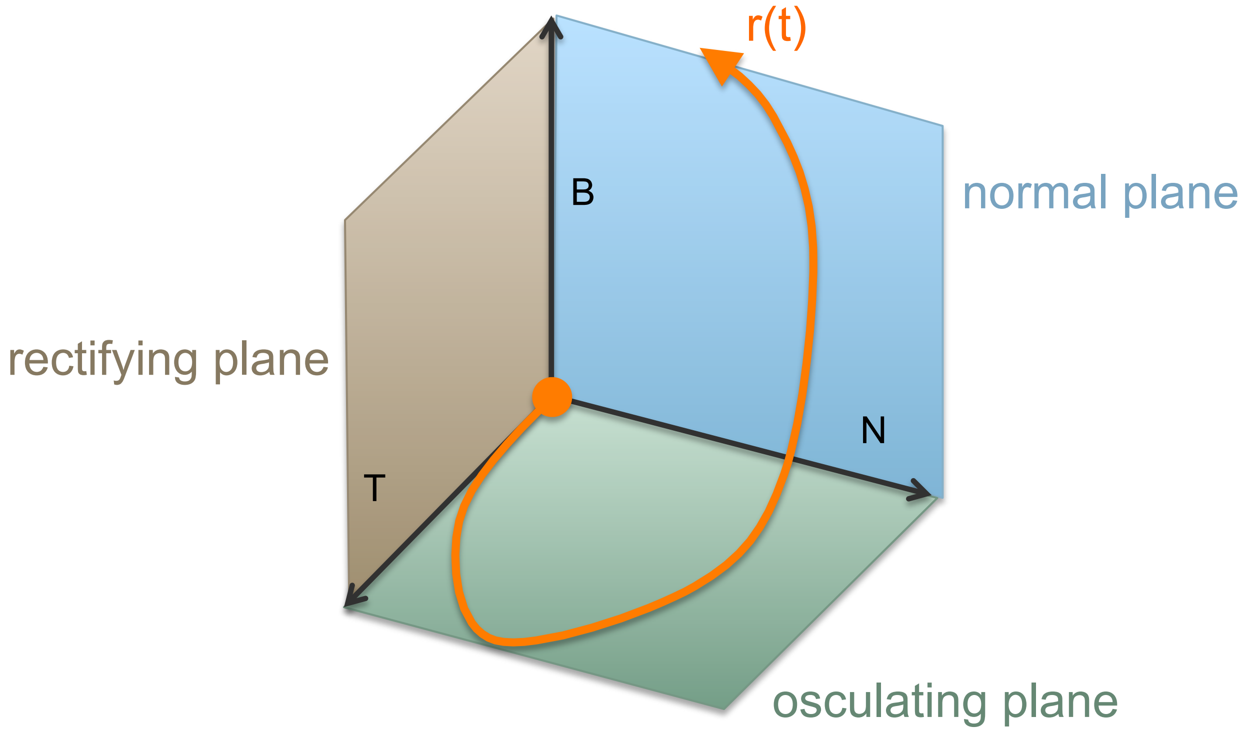
Unit tangent vector	$T =$
Principle unit normal vector	$N =$
Binormal vector	$B =$
Curvature	$\kappa =$
Torsion	$\tau =$

Tangential and normal scalar components of acceleration:

Acceleration (Quiz 1, Q3)

Let $\mathbf{r}(t) = 2t\mathrm{i} + t\mathrm{j} + 2t^2\mathrm{k}$ be a motion. Compute the tangential and normal components of the acceleration.

Rectifying, Normal, and Osculating Planes



The names of the three planes determined by T, N, and B

Rectifying, Normal, and Osculating Planes

Find **r**, **T**, **N**, and **B** at the given value of t. Then find the equations for the osculating, normal, and rectifying planes at that value of t.

$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} - \mathbf{k}$, $t = -\pi/2$.

Position Perpendicular to Tangent

$\mathbf{r}(t)$ is the position of a moving particle.

Show that $||\mathbf{r}(t)||$ = constant, for all t, iff $\mathbf{r} \perp \mathbf{r}'$ for all t.

$$a_T = \frac{d}{dt}|\mathbf{v}|$$

$$a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

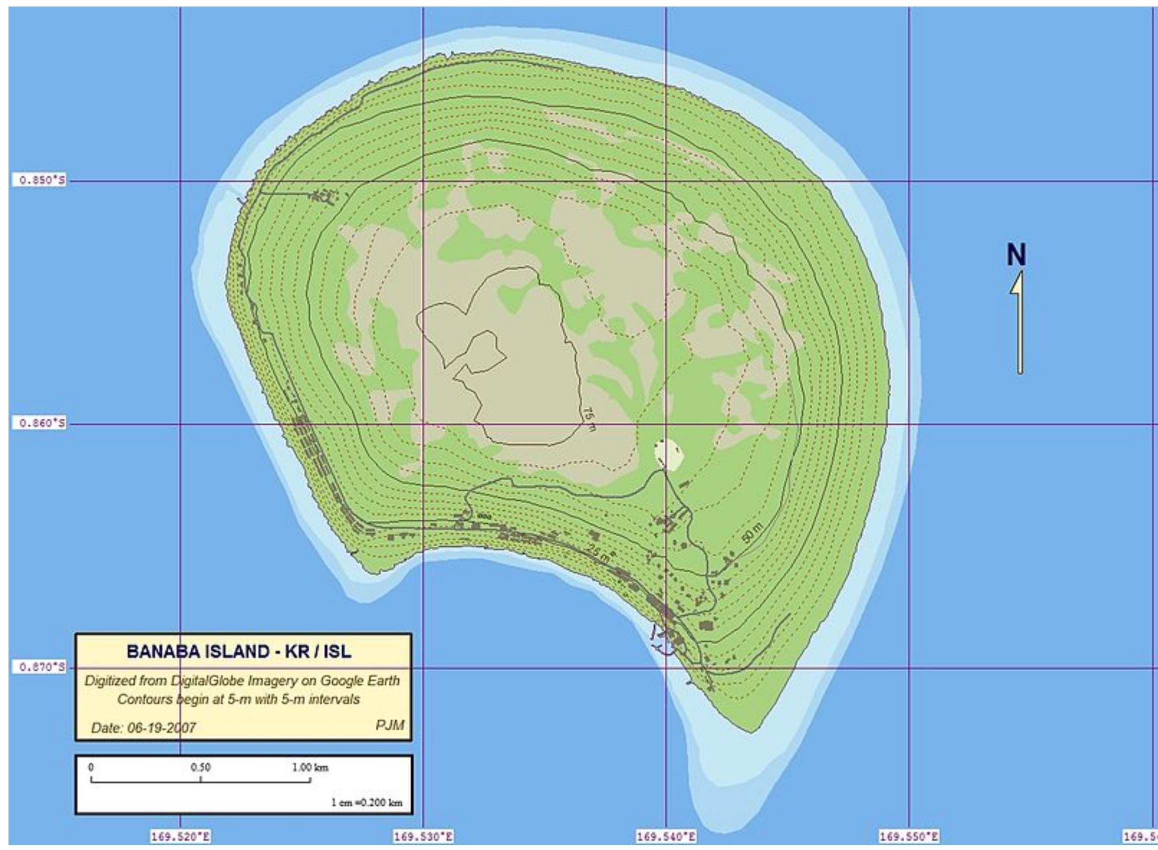
Write **a** in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ at the given value of t without finding **T** and **N**.

$$\mathbf{r}(t) = (e^t\sqrt{2})\mathbf{i} + (e^t\cos t)\mathbf{j} + (e^t\sin t)\mathbf{k}, \quad t = 0$$

Level Curves

1. The level curves of z = f(x,y) are the curves that satisfy the equation:

2. In a topographic map, z describes _______________ , and the level curves describe _____________________.



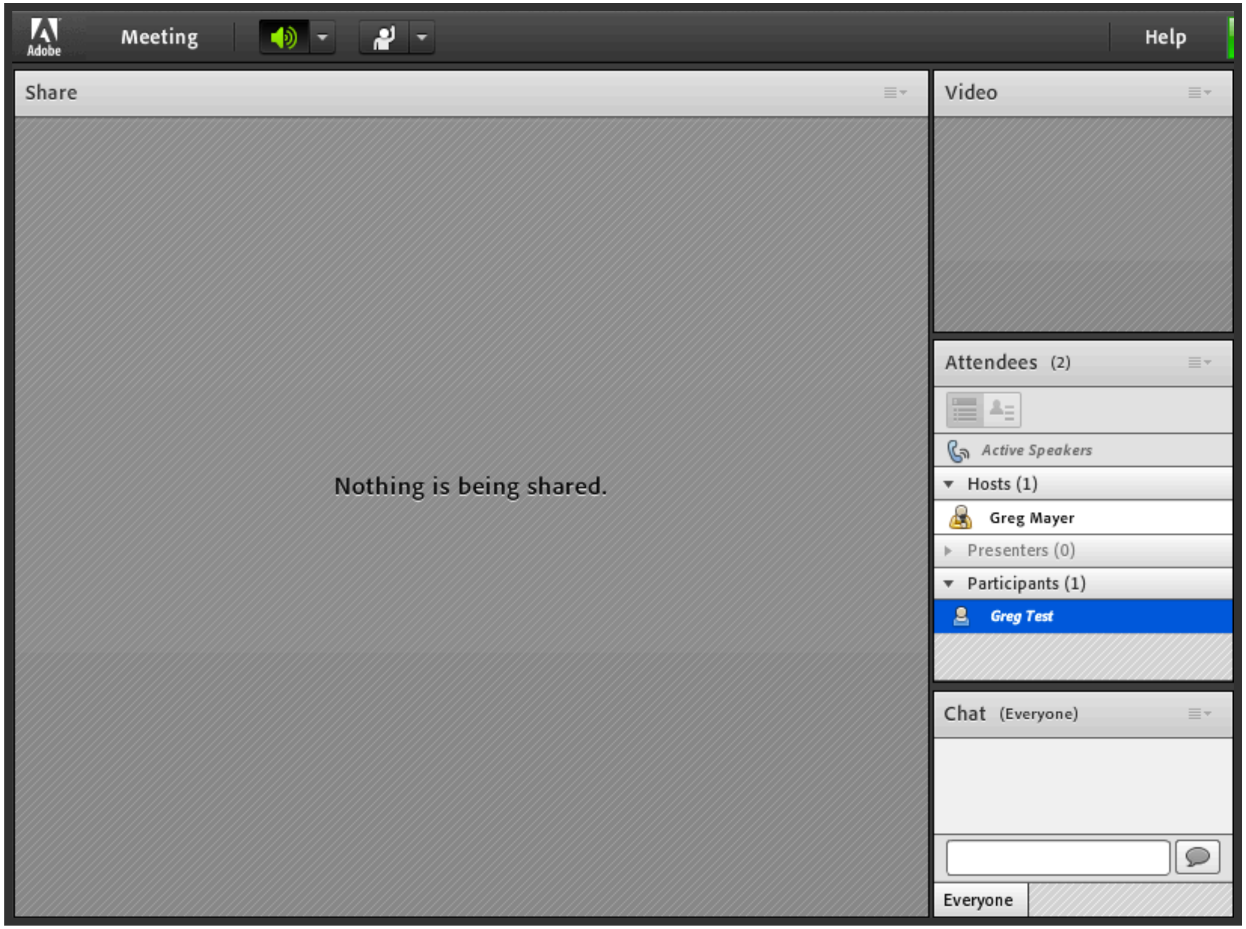
Recitation 07 — Surfaces, Domain, Limits, Gradient

Having trouble with your audio?

- make sure speakers are not muted
- navigate to Meeting >> Audio Setup Wizard

Other issues?

- navigate to Help >>Troubleshooting
- see Quick Start Guide (PDF)



Quadratic Surfaces

Consider $z = Ax^2 + By^2$, A and B are constants. Describe all possible surfaces for the following cases.

i) $A = B = 0$

ii) $AB > 0$

iii) $AB < 0$

Case ii, AB > 0

WolframAlpha computational knowledge engine

plot z = -2*x^2 - 3*y^2

Examples Random

Input interpretation:

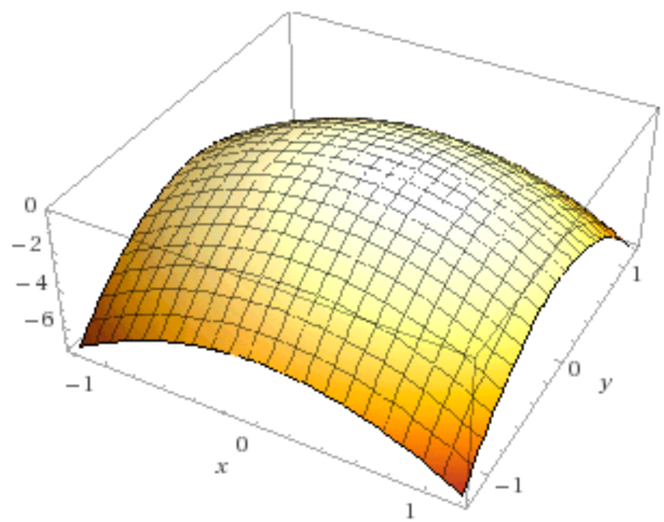
plot	$z = -2x^2 - 3y^2$

Geometric figure:

elliptic paraboloid

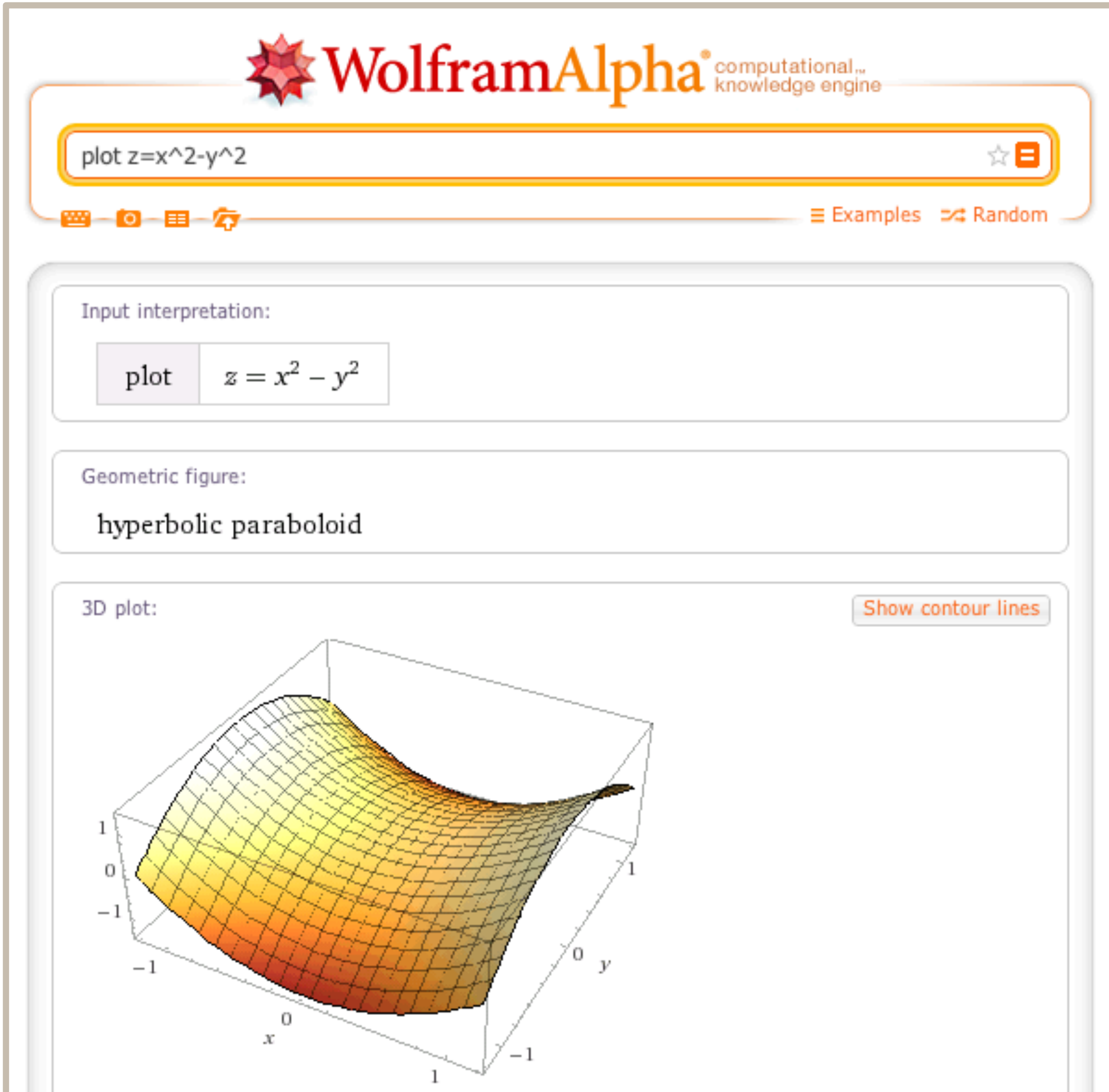
3D plot:

Show contour lines



Enable interactivity

Case iii, AB < 0


WolframAlpha computational knowledge engine
plot z=x^2-y^2
Examples
Random
Input interpretation:
plot $z = x^2 - y^2$
Geometric figure:
hyperbolic paraboloid
3D plot:
Show contour lines
1
0
−1
−1
0
x
1
−1
0
y
1

Domain

Sketch the domain of $g(x,y) = (x + 1)^{1/2} / (yx^2 + xy^2)$ in the xy plane.

Evaluate

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Gradient and Level Curves

If $g(x,y) = K$ defines a curve C in the xy-plane, show that ∇g is perpendicular to curve C.

Let $\mathbf{F} = \nabla f = (x+\sin(y))\mathbf{i} + (x\cos(y)-2y)\mathbf{j}$. Find $f(x,y)$.

Quiz 1

As announced on Friday

- Covers HW1,2,3 + additional problems
- 2 sheet of 8 1/2 x 11 motes (both sides)
- Calculators allowed

Office Hours

- In Adobe Connect at https://georgiatech.adobeconnect.com/distancecalculusofficehours/
- Tuesday and Wednesday 8:00 pm to 9:30 pm

Prepare

- Solve HWs on MyMathLab
- Practice Quiz

During Quiz

- I'll be in Adobe Connect https://georgiatech.adobeconnect.com/distancecalculusquiz/
- Grady HS students: Klaus 2447

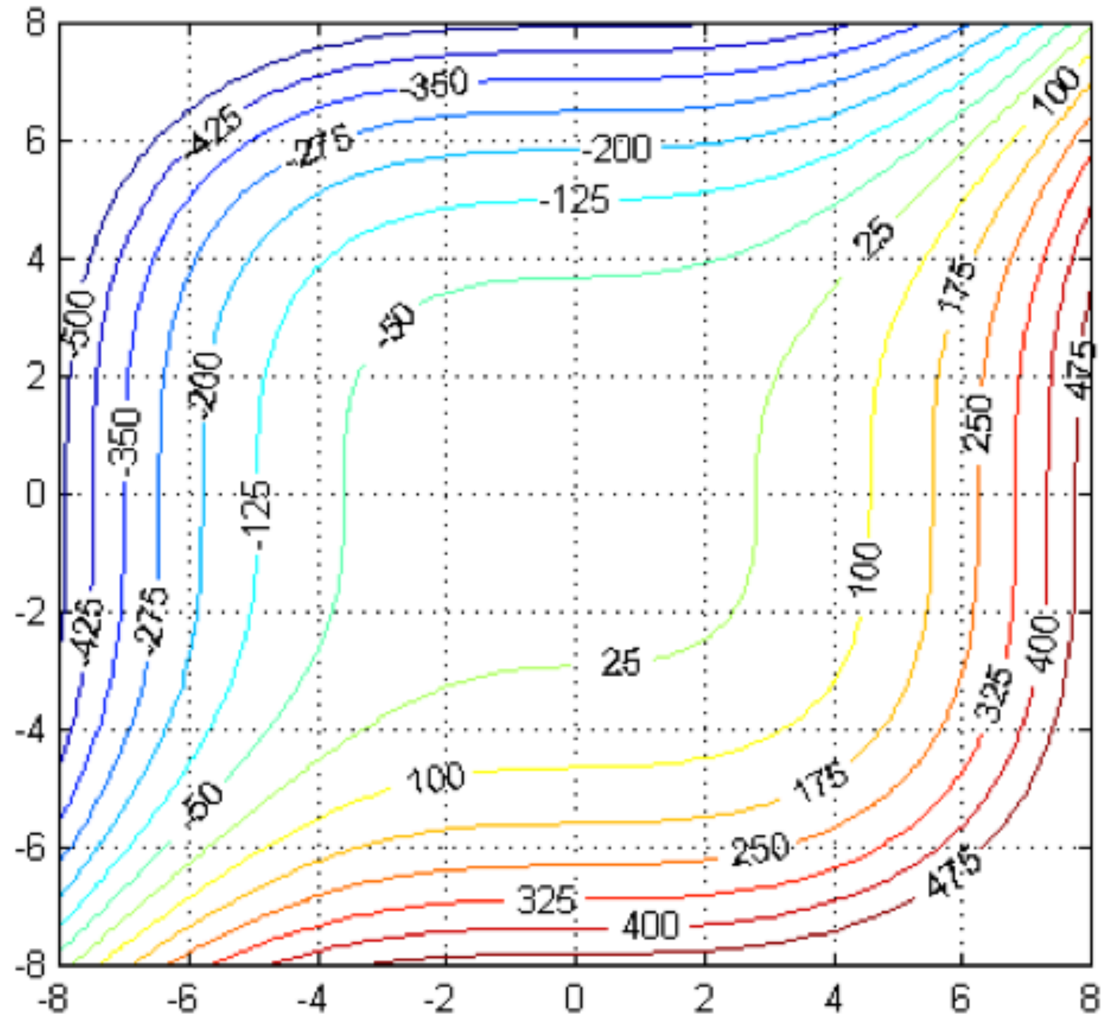
Do You Have Any Questions?

Gradient and Level Curves

At which point will the gradient vector have the largest magnitude?

a) (0,2)
b) (-4,-4)
c) (0,0)
d) (6,-2)

Explain why, and sketch the gradient at that point.



Tangent Planes

Consider $z = 3xy - x^3 - y^3$.

a) Find an equation for the tangent plane at (1,1,1).

b) Determine points where the tangent plane is horizontal.

c) What do points where the tangent plane is horizontal represent?

Wolfram Alpha

WolframAlpha computational knowledge engine

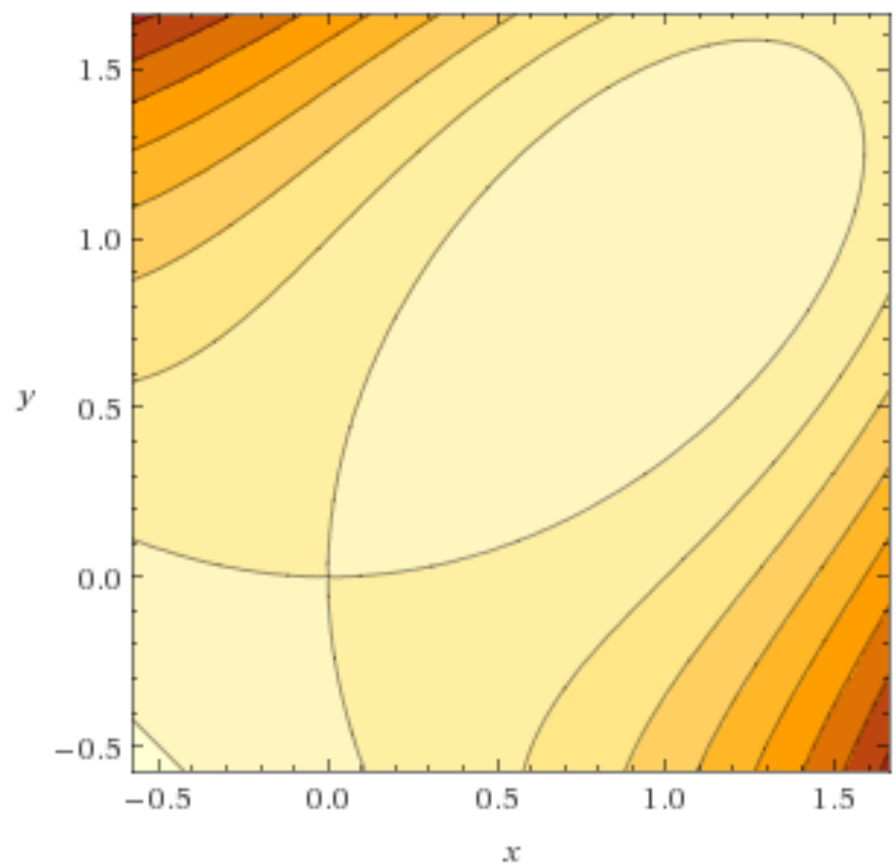
plot z(x,y) = 3xy –x^3 – y^3

Examples Random

Input interpretation:

plot $z(x, y) = 3\,x\,y - x^3 - y^3$

Contour plot:



Directional Derivative

Find the directional derivative of $f = z \ln(x/y)$ at (1,1,2) toward the point (2,2,1) and state what it represents.

Mixed Partial Derivatives

$f(x,y)$ is a function with continuous 1st and 2nd partial derivatives on D, and $f_{xy}(x,y) = 0$ everywhere on D.

a) What can we say about $f(x,y)$ on D?
b) Provide two functions that have this property.

Gravitation

Newton's Law of Gravitation in R^3 is

F =

a) Describe the shape of the level surfaces
b) Provide physical interpretations of the level surfaces and the gradient of F

Tangent Plane Intercepts

Show that, for all tangent planes to the given surface, the sum of their intercepts is the same.

$$\text{surface: } \sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$$

Having trouble with your audio?
- make sure speakers are not muted
- navigate to Meeting >> Audio Setup Wizard

Other issues?
- navigate to Help >>Troubleshooting
- see Quick Start Guide (PDF)

The strength of an electric field at a point due to an infinitely long wire along the *y*-axis is given by

$$E(x,y,z)=\frac{k}{\sqrt{x^2+z^2}}$$

Describe, in words, the level surfaces of *E*. What do they represent?

Circle the correct answer.

The set of all points whose distance from the z-axis is 4 is the:

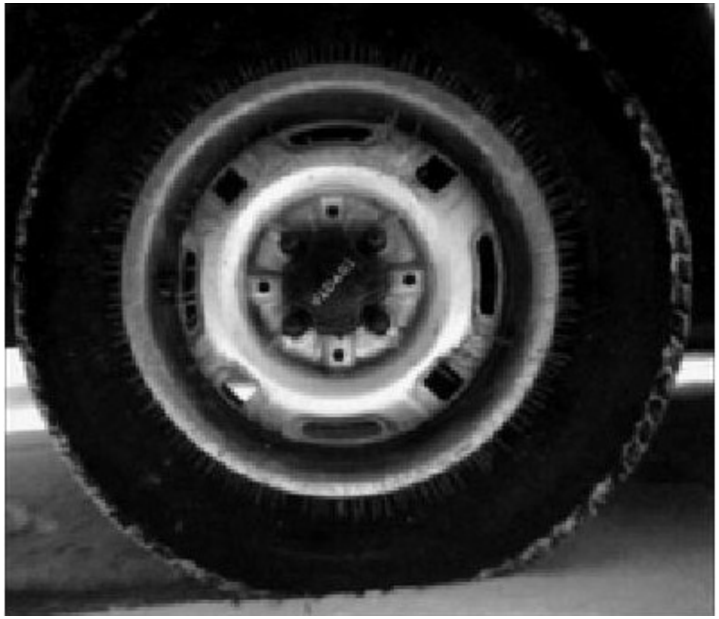
a) sphere of radius 4 centered on the z-axis
b) line parallel to the z-axis 4 units away from the origin
c) cylinder of radius 4 centered on the z-axis
d) plane $z = 4$

Announcements

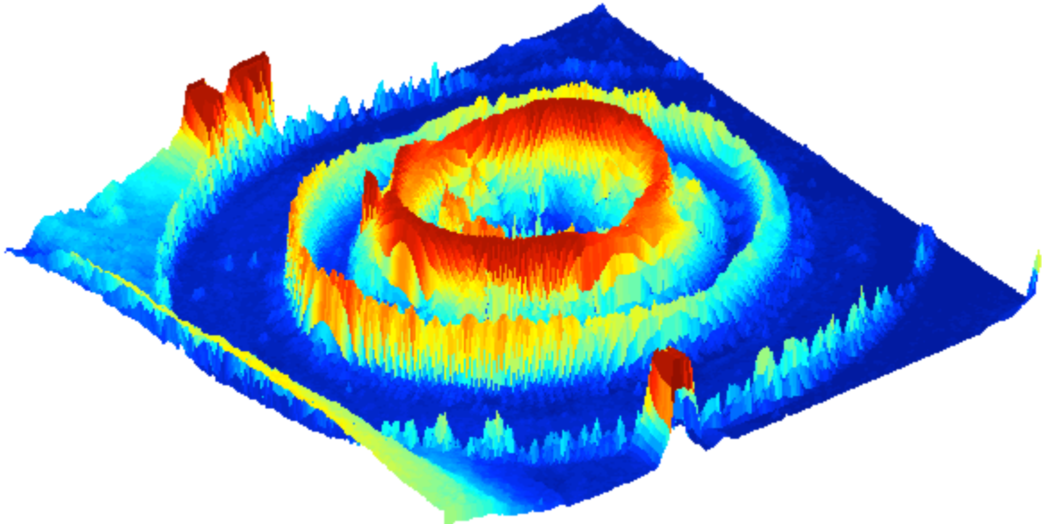
- HW4 due tomorrow
- Quiz 1 marked on Friday:

The Gradient and Local Maxima

A) Image $z = z(x,y)$



B) Surface Plot of $z(x,y)$

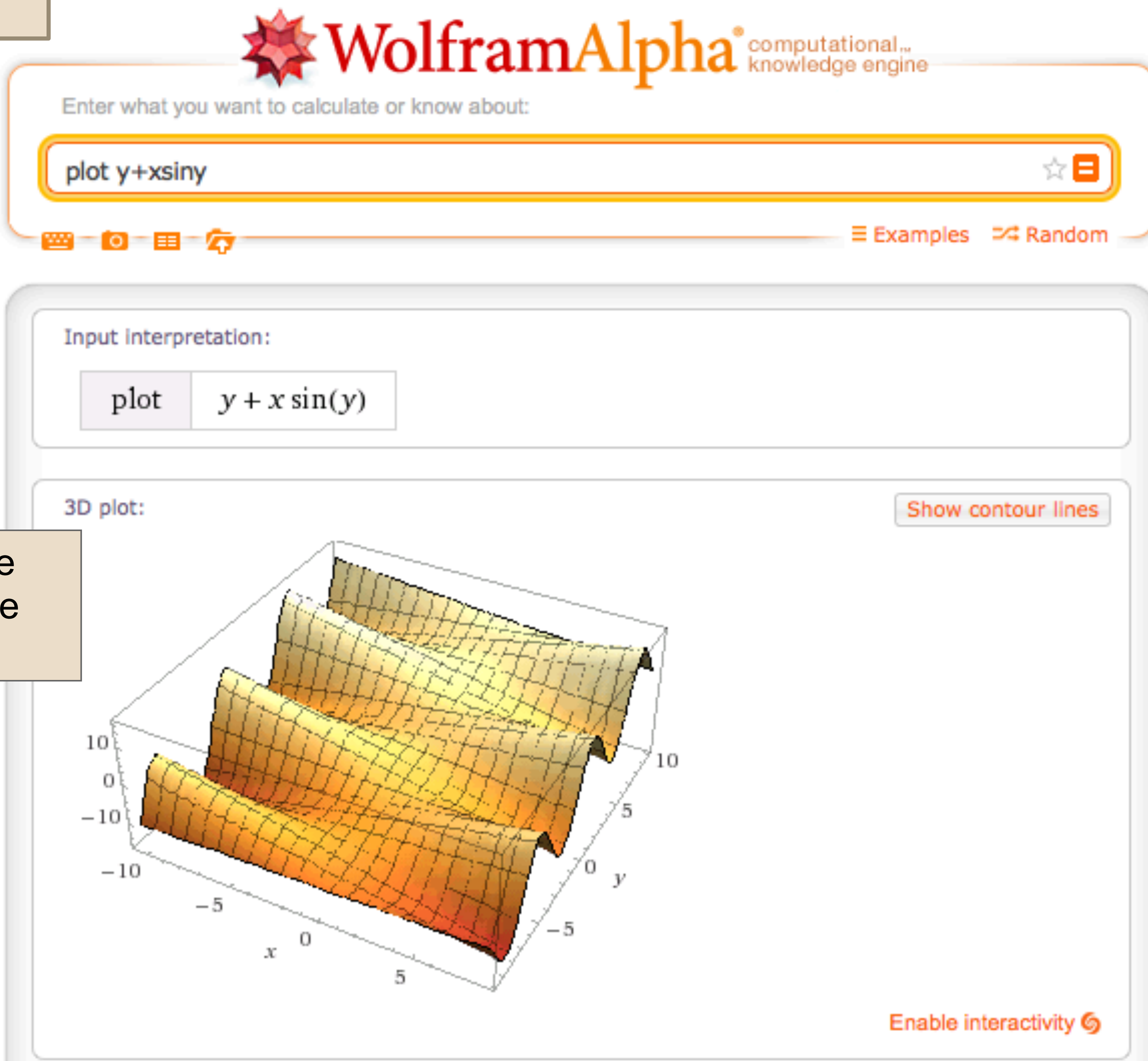


1) Place a dot on image (A) that could correspond to a local maximum.
2) What characteristics does the gradient vector have at local maxima?

Stationary Points

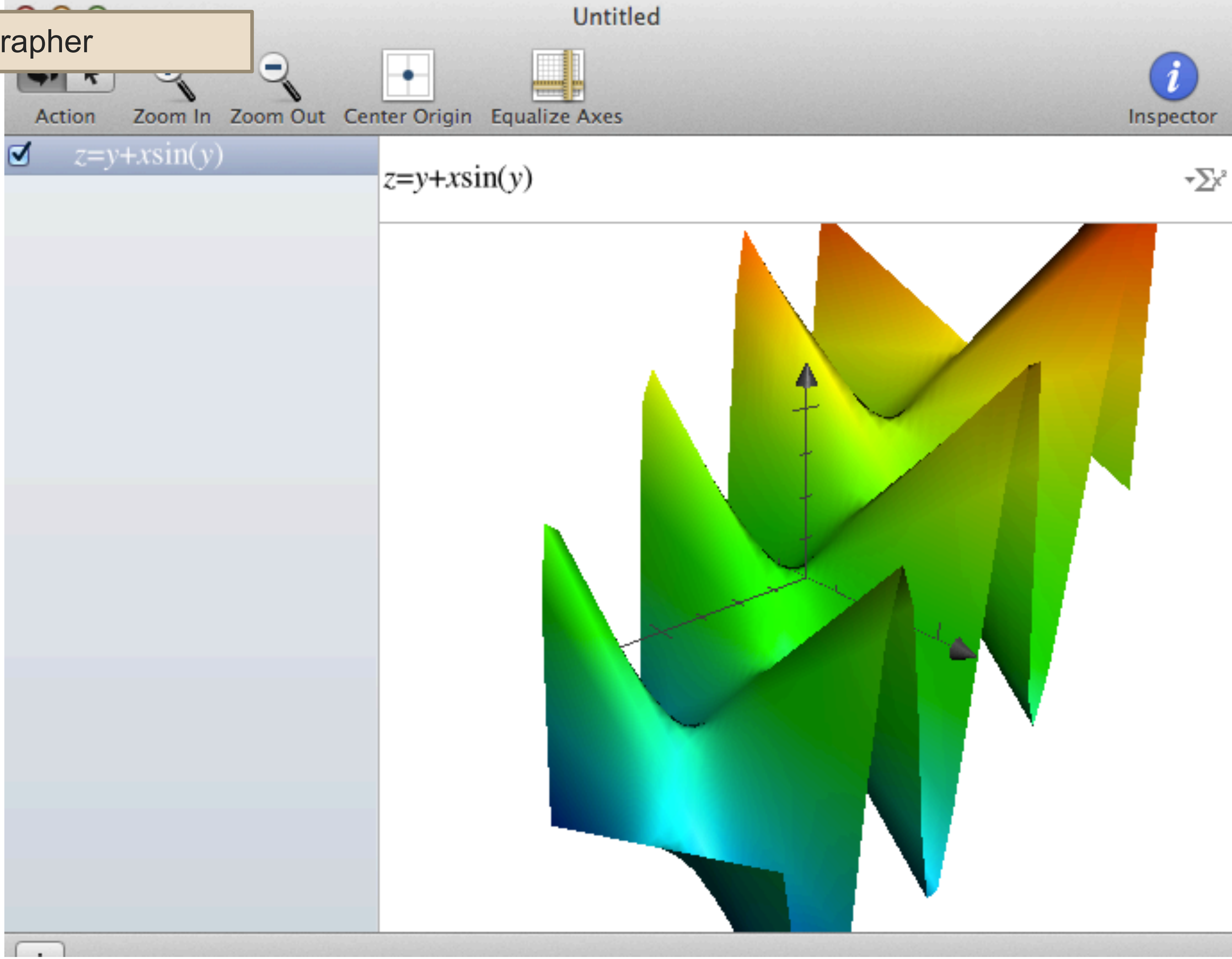
Find and describe the stationary points of $f(x,y) = y + x \sin(y)$.

WolframAlpha



Place dots on the 3D plot where the gradient is zero.

Grapher


Untitled
Action
Zoom In
Zoom Out
Center Origin
Equalize Axes
Inspector
z=y+xsin(y)
z=y+xsin(y)

Local Minima/Maxima

Let $f(x,y) = x^2 + kxy + y^2$.

a) Where are the stationary points?

b) For what values of k will f have a saddle at the origin?

c) For what values of k will f have a local min at the origin?

d) For what values of k is the second partials test inconclusive?

Part a)

WolframAlpha computational... knowledge engine

plot x^2 + y^2 + 2xy

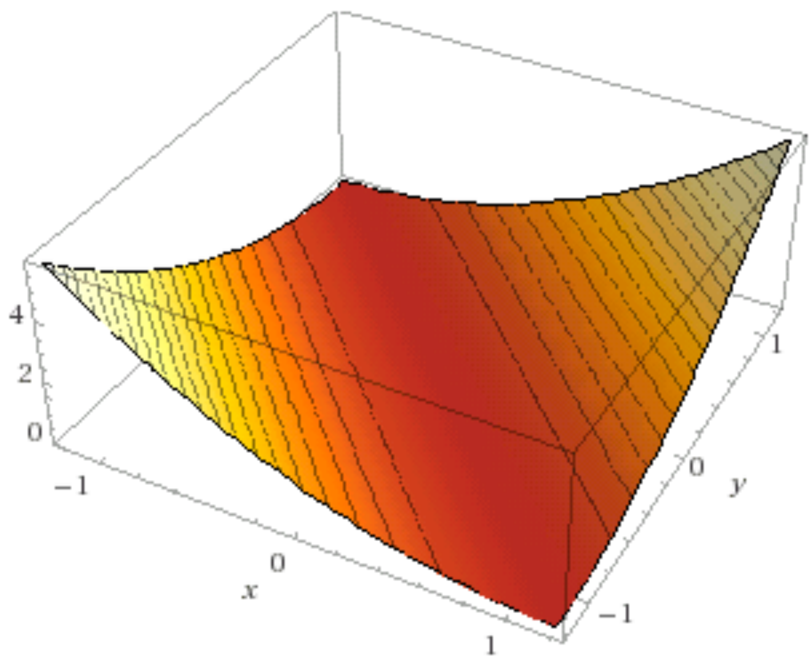
Examples Random

Input interpretation:

plot	$(x^2 + y^2) + 2\,x\,y$

3D plot:

Show mesh



Part b)

plot x^2 + y^2 - 8xy

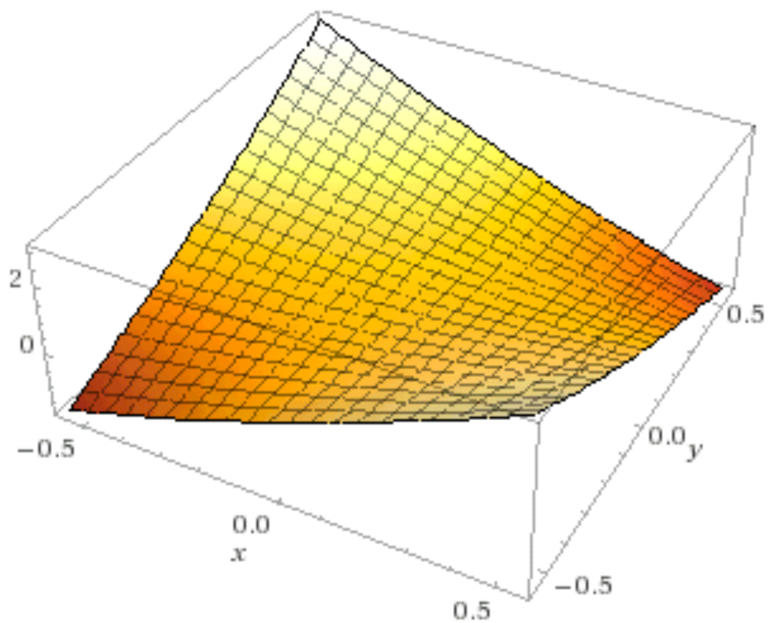
Examples Random

Input interpretation:

plot	$(x^2 + y^2) - 8\,x\,y$

3D plot:

Show contour lines



Enable interactivity

Temperature Optimization

The shape of a wire in R^3 can be modeled by $x^2 + y^2 \leq 1$. If the temperature of the wire is $T(x,y) = xy$, find the coldest and hottest regions of the wire.

Let's try group work in Adobe Connect

- You'll solve the question that we started at the end of Tuesday's recitation
- Three breakout rooms
- Everyone randomly assigned to a room
- Not graded
- You'll have 10 to 15 minutes
- I'll circulate between rooms

I suggest starting by discussing a solution strategy with the other people in your group using a mic and/or text chat.

Second Partials Test

Let $f(x,y) = x^2 + kxy + y^2$.

a) Where are the stationary points?
b) For what values of k will f have a saddle at the origin?
c) For what values of k will f have a local min at the origin?
d) For what values of k is the second partials test inconclusive?

Part a)

WolframAlpha computational... knowledge engine

plot x^2 + y^2 + 2xy

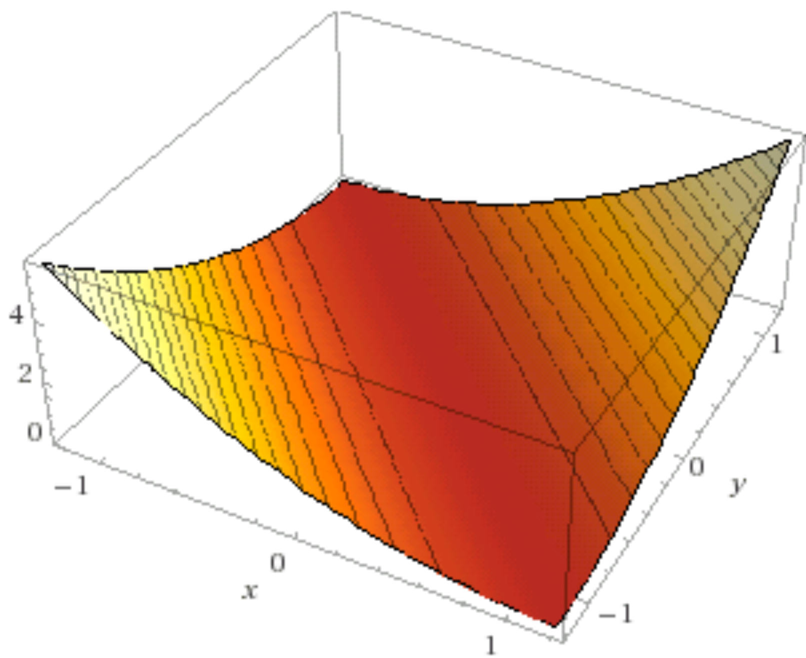
Examples Random

Input interpretation:

plot	$(x^2 + y^2) + 2\,x\,y$

3D plot:

Show mesh



Part b)

plot x^2 + y^2 - 8xy

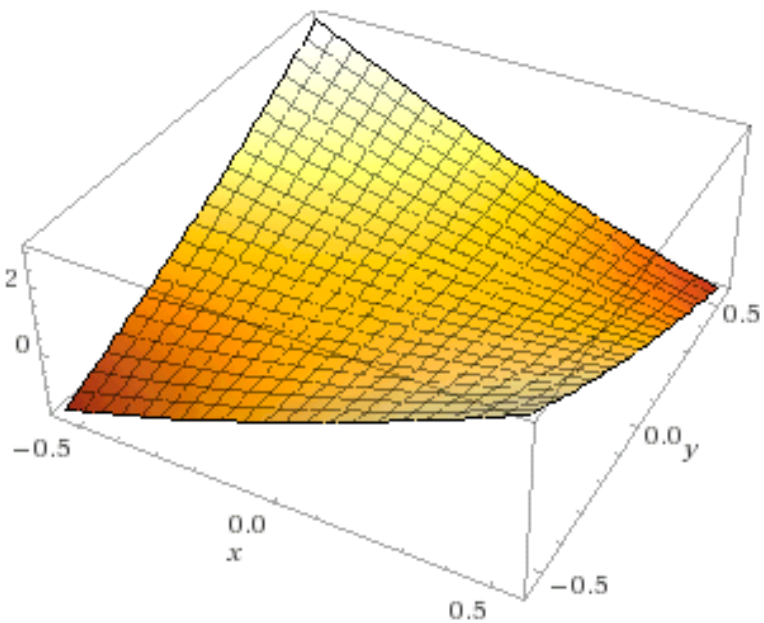
Examples Random

Input interpretation:

plot	$(x^2 + y^2) - 8\,x\,y$

3D plot:

Show contour lines



Enable interactivity

Lagrange Multipliers

The shape of a wire in three dimensions can be modeled by $x^2 + y^2 \leq 1$.
If the temperature of the wire is $T(x,y) = xy$, find the coldest and hottest regions of the wire.

- Pop quiz grading
  - Correct 5 points
  - Name on page 3 points
  - Did not take: 0 points.
- Time: 15 minutes
- To submit your work, either
  a) work on whiteboard in breakout room
     - press the **SAVE** button
  b) work on paper and give work to facilitator
     - leave 2 inch margin
     - write your name and QH6 at the top
     - facilitator can email quiz to cdlops@pe.gatech.edu
  c) work on paper and take a photo of your work
     - email your photo to me before 8:30
     - write in text chat that you are emailing your work to me

Pop Quiz

Find the cubic approximation of $f(x,y) = 4x \cos(y)$ near the origin.

Announcements

- **HW 5**
  - due tonight at 11:59 pm
  - seven questions on Taylor approximations from Section 14.9
- **HW 6**
  - fifteen questions on integration from Section 15.2 and 15.3
  - due Thursday at 11:59 pm
- **Quiz 2:** Tuesday March 4

How would you like to spend the rest of the recitaiton? Text your preference.

a) A Taylor approximation example and some integration
b) Integration examples

Do you have questions about the homeworks and/or the quiz?

Find and sketch the area of the triangular region with vertices (1,1), (4,1), (3,2).

Change the order of integration and integrate

$$\int_{1/2}^{1}\int_{x^3}^{x} dy\, dx$$

Change the order of integration and integrate

$$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx\,dy$$

Change the order of integration and integrate

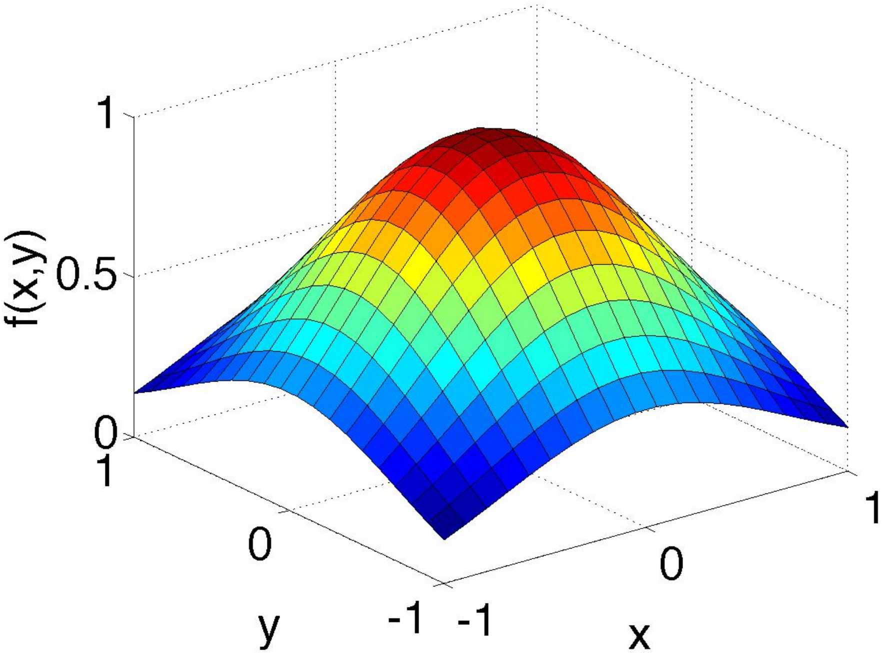
$$\int_1^3 \int_{-x}^{x^2} dy\,dx$$

Quadratic Approximation

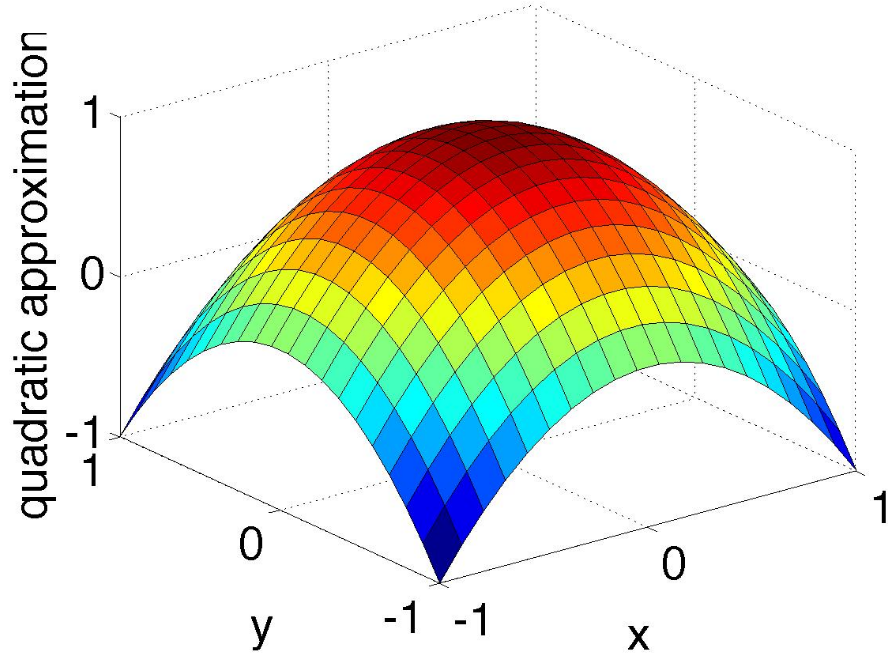
Find the quadratic approximation to $f(x,y) = \exp(-x^2 - y^2)$ near the origin.

Surface Plots

$f(x,y) = \exp(-x^2 - y^2)$

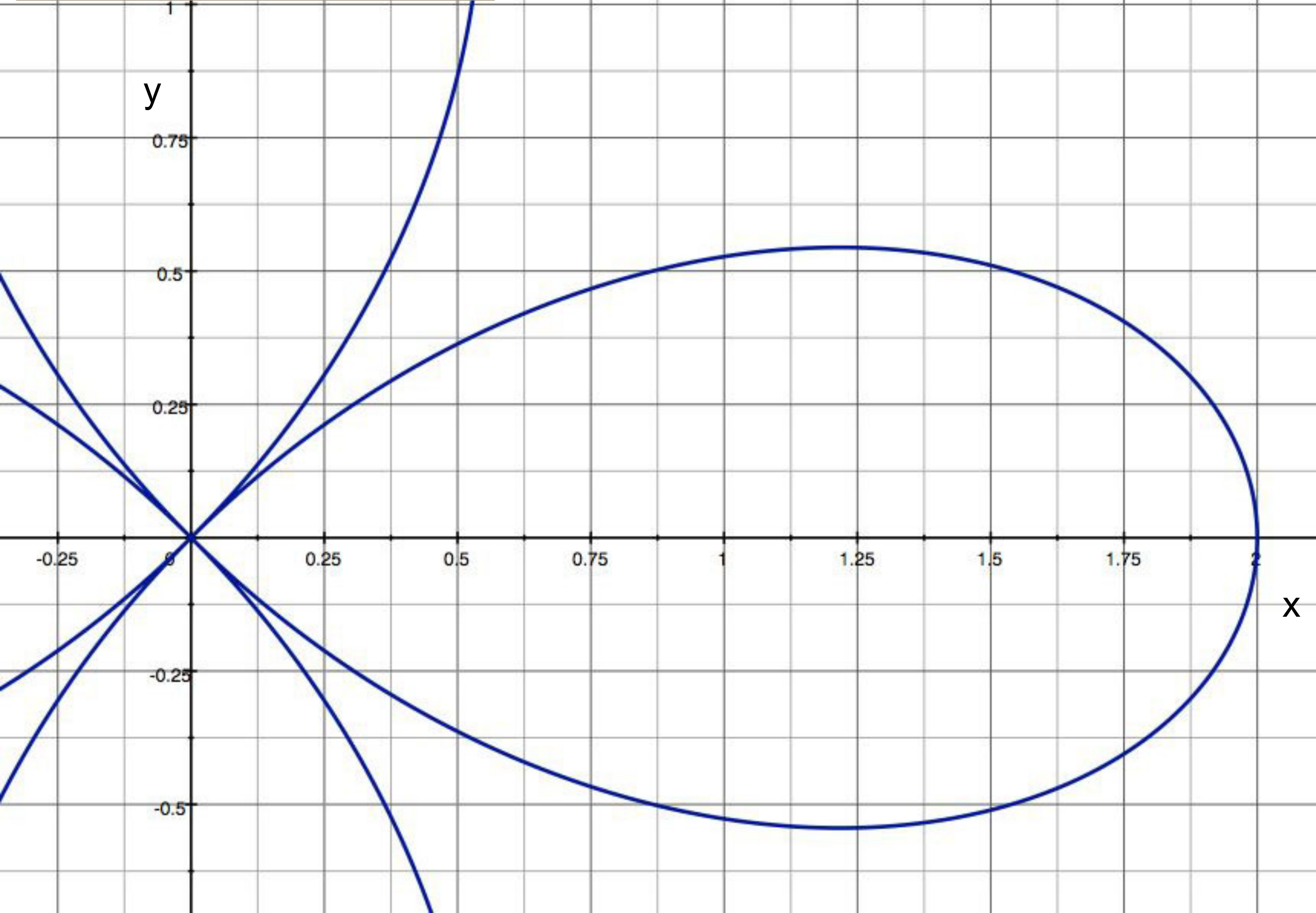


quadratic approximation
$1 - x^2 - y^2$

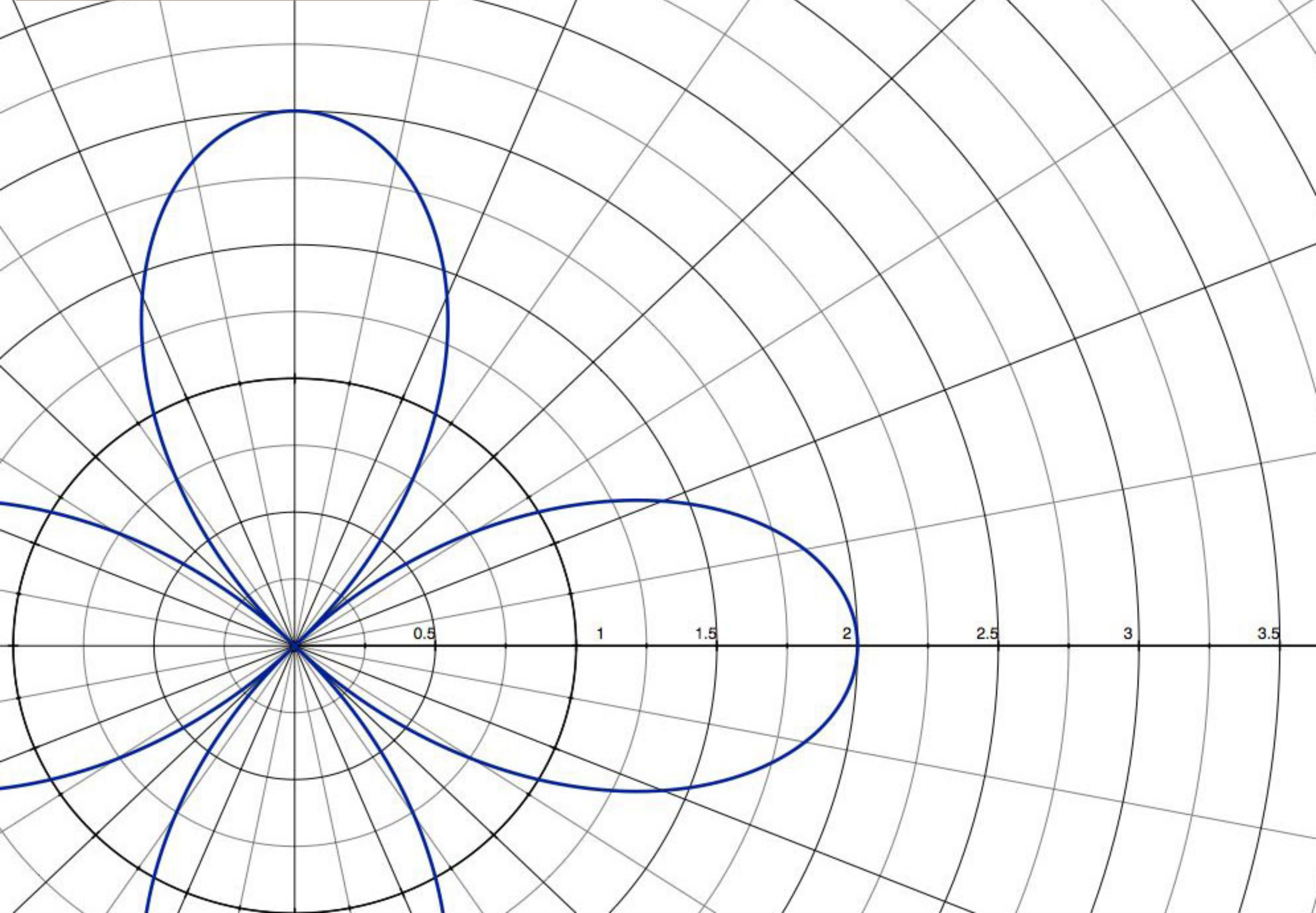


Sketch the petal curve $r = 2\cos(2\theta)$ and find the area of one petal.

Petal Area: Cartesian Grid



Petal Area: Polar Grid



1) Sketch and find the area of the region inside the curve $r = 5 + \cos(\theta)$ (from last year's quiz).

2) Sketch the region of integration, change the order of integration, and then integrate

$$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx\,dy$$

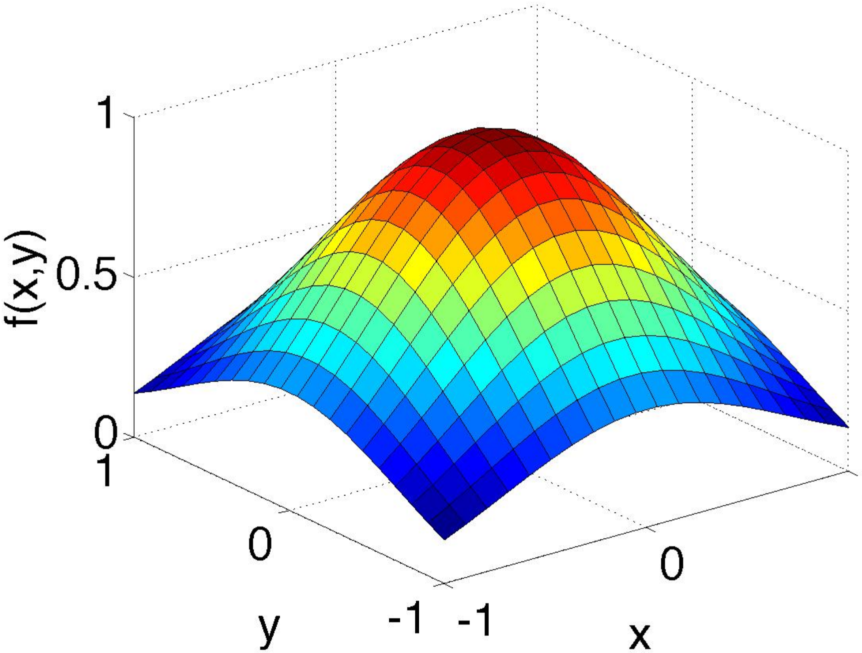
3) Sketch the region of integration, change the order of integration, and then integrate

$$\int_{1}^{3}\int_{-x}^{x^2} dy\,dx$$

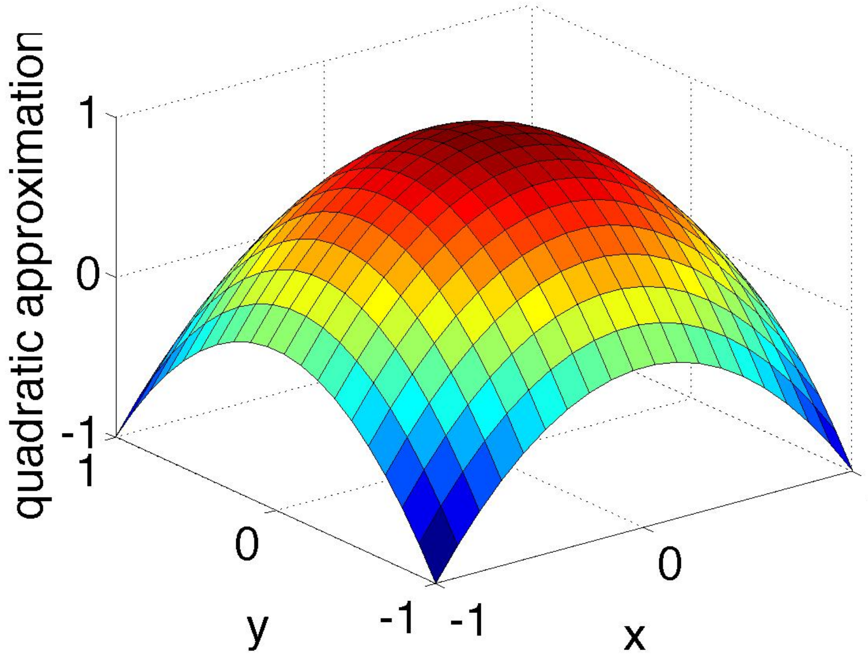
4) Find the quadratic approximation to $f(x,y) = \exp(-x^2 - y^2)$ near the origin.

Surface Plots

$f(x,y) = \exp(-x^2 - y^2)$



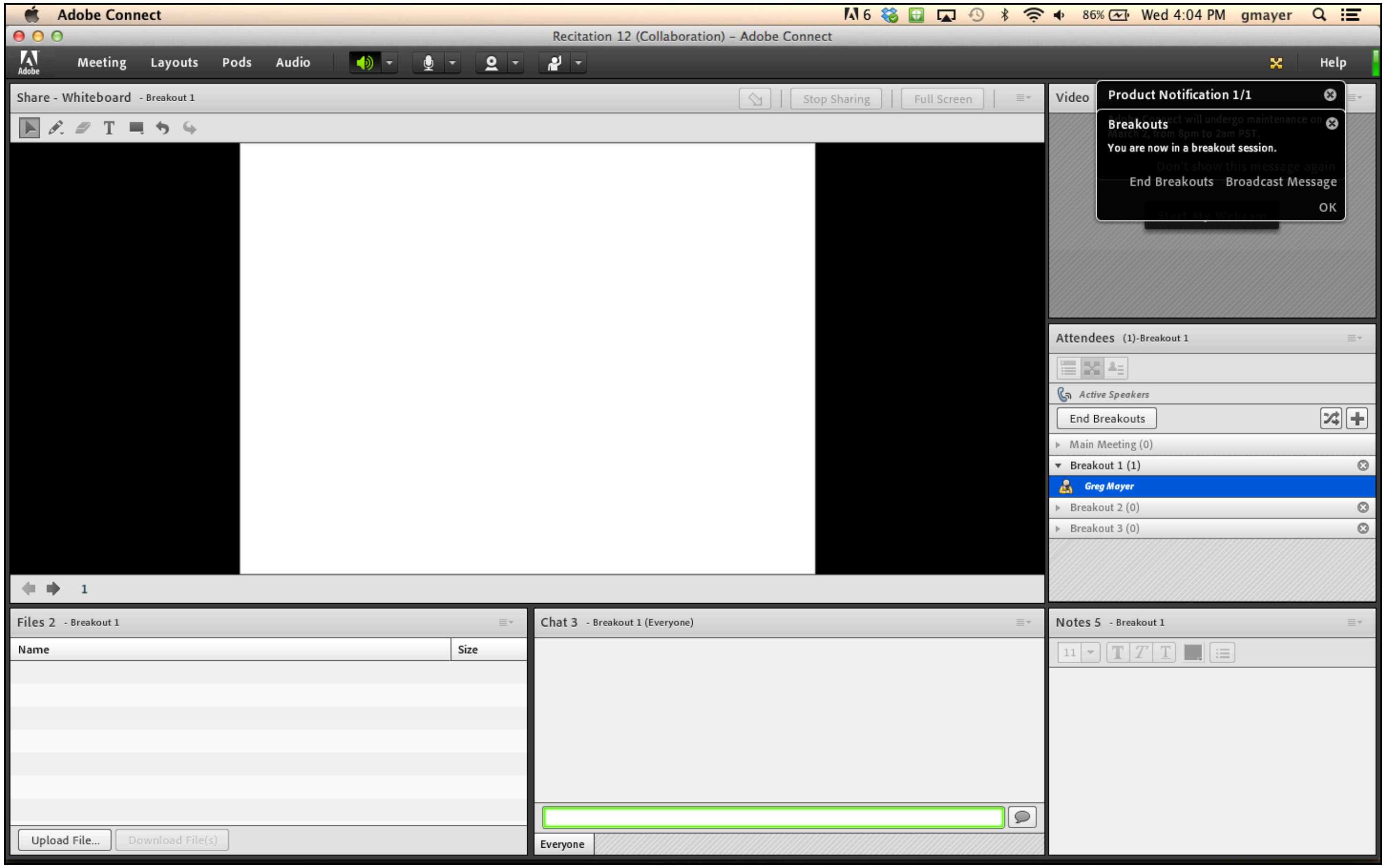
quadratic approximation
$1 - x^2 - y^2$



Set up a triple integral that represents the volume of the region bounded by $y^2 + z^2 = 1$ and the planes $y = x$, $x = 0$, $z = 0$.

Breakout Rooms in Connect

we need to first switch to the “Collaboration” layout



If 08 ≤ N ≤ 11 students: 2 rooms
If 12 ≤ N ≤ 15 students: 3 rooms
If 16 ≤ N ≤ 19 students: 4 rooms

Represent the volume of the region bounded by $z^2 = y$, $y + z = 2$, $x = 0$, $z = 0$, $x = 2$.
Set up the integral(s) in at least two different ways.

Set up a triple integral that represents the volume of the region bounded by
$x^2 + y^2 + z^2 = 2$, and by $x^2 + y^2 = 1$.

Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, bounded above by $x^2 + y^2 + 4z^2 = 36$, and bounded below by $z = 1$.

Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, bounded above by $x^2 + y^2 + 4z^2 = 36$, and bounded below by $z = 1$.

Next HW due Thursday Mar 13: any questions?

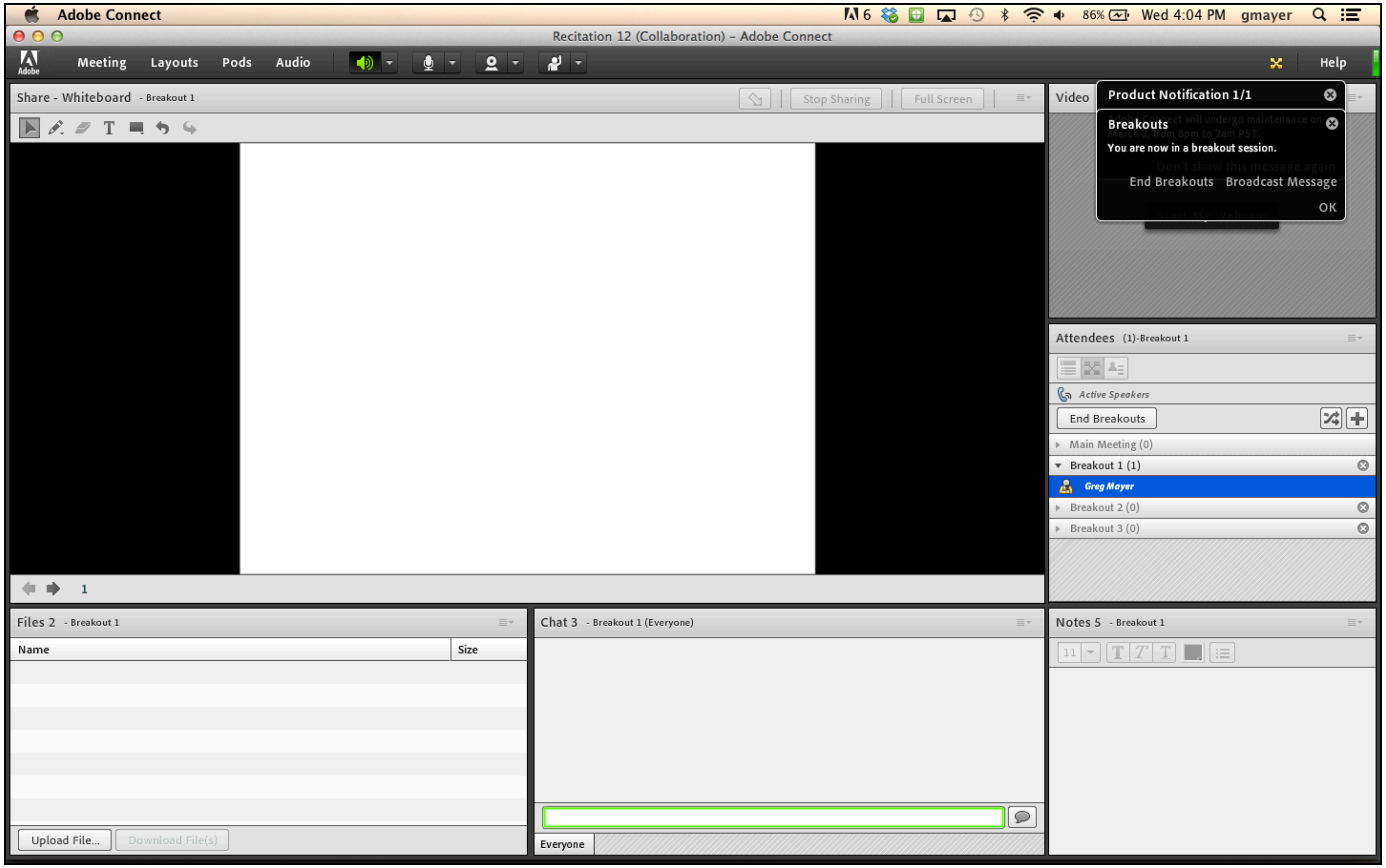
Quiz 3 on Thurs Mar 27

Graded group work on Thurs Mar 13:

- 3 to 5 students per group
- random grouping
- include a rough sketch that describes the region of integration
- technical issues? email me, we'll figure something out
- can't make it? email me, we'll figure something out

The question is from last year's Quiz 3

Let V be the volume between the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$ above the plane $z = 8$ and below the plane $z = 10$. Set up the volume as a triple integral . Do not Evaluate we need to first switch to the “Collaboration” layout



If 06 ≤ N ≤ 10 students: 2 rooms
If 11 ≤ N ≤ 15 students: 3 rooms
If 16 ≤ N ≤ 20 students: 4 rooms

Set-up an integral that represents the volume of the solid bounded above by $z = 1$, and below by $z^2 = x^2 + y^2$. Set this integral up in at least two different ways.

Change the order of integration.

$$V = \int_0^2 \int_0^{9-x^2} \int_0^{2-x} dz\ dy\ dx$$

Set up a triple integral that represents the volume of the region bounded by $x^2 + z^2 = 4$ and the planes $y + z = 6$, $x = 0$, $y = 0$, $z = 0$.

Set up an integral that represents the volume of solid bounded by $z = x^2 + y^2$, and $z = y$. Use cylindrical coordinates.

Structure:

- 3 to 5 students per group
- random grouping
- include a rough sketch that describes the region of integration
- technical issues? email me, we'll figure something out
- can't make it? email me, we'll figure something out

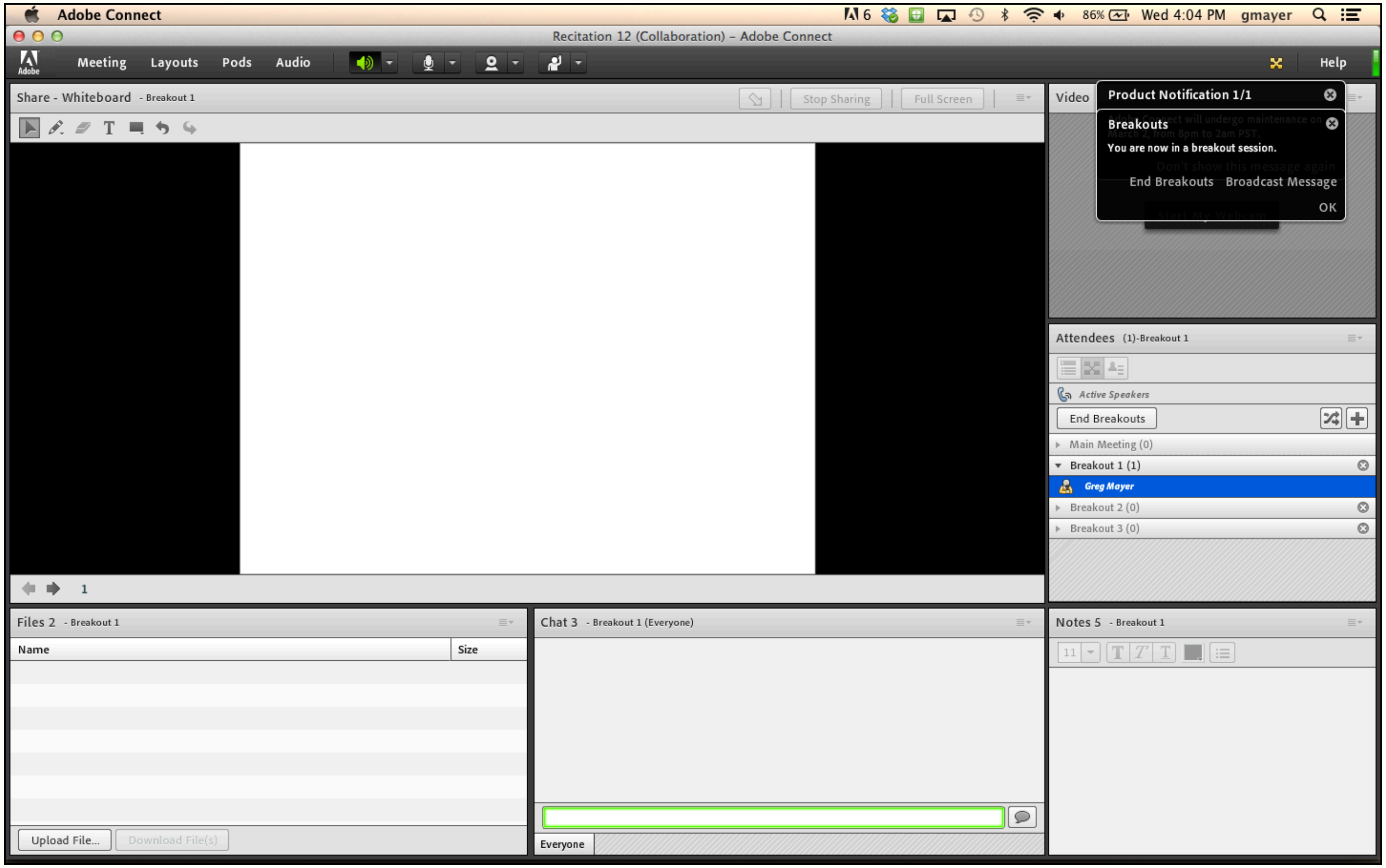
The question is from last year's Quiz 3.

> Let V be the volume between the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$ above the plane $z = 8$ and below the plane $z = 10$. Set up the volume as a triple integral. Do not Evaluate

You'll have about 15 minutes. Your group only needs one correct solution for full marks, but there are many ways to solve the problem. You can use Cartesian coordinates, you can use polar (cylindrical) coordinates

Breakout Rooms in Connect

we need to first switch to the “Collaboration” layout



If 06 ≤ N ≤ 10 students: 2 rooms
If 11 ≤ N ≤ 15 students: 3 rooms
If 16 ≤ N ≤ 20 students: 4 rooms

Let V be the volume between the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$ above the plane $z = 8$ and below the plane $z = 10$. Set up the volume as a triple integral . Do not Evaluate

Announcements

Next HW due tonight: any questions?

Quiz 3 on Thurs Mar 27: two weeks from now

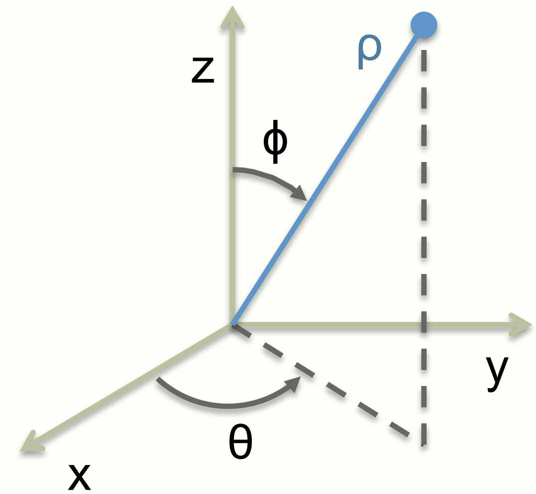
Set up an integral that represents the volume of the “ice cream cone” bounded by $x^2 + y^2 + z^2 = 1$, and $z^2 = 3(x^2 + y^2)$. Use cylindrical coordinates.

Fill in the blanks.

$x = \rho \cos\theta$ ________

$y = \rho \sin\theta$ ________

$z = \rho$ ____________



Provide a geometric interpretation of each expression.

a) $\rho \sin \phi = 1$

b) $\rho \cos \phi = 1$

Set-up an integral that represents the volume bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2(x^2 + y^2)^{1/2}$. Use spherical coordinates.

Quiz 3

For your quiz:

- 2 pages of 8 1/2 x 11 inch notes (both sides) allowed
- Calculators allowed.

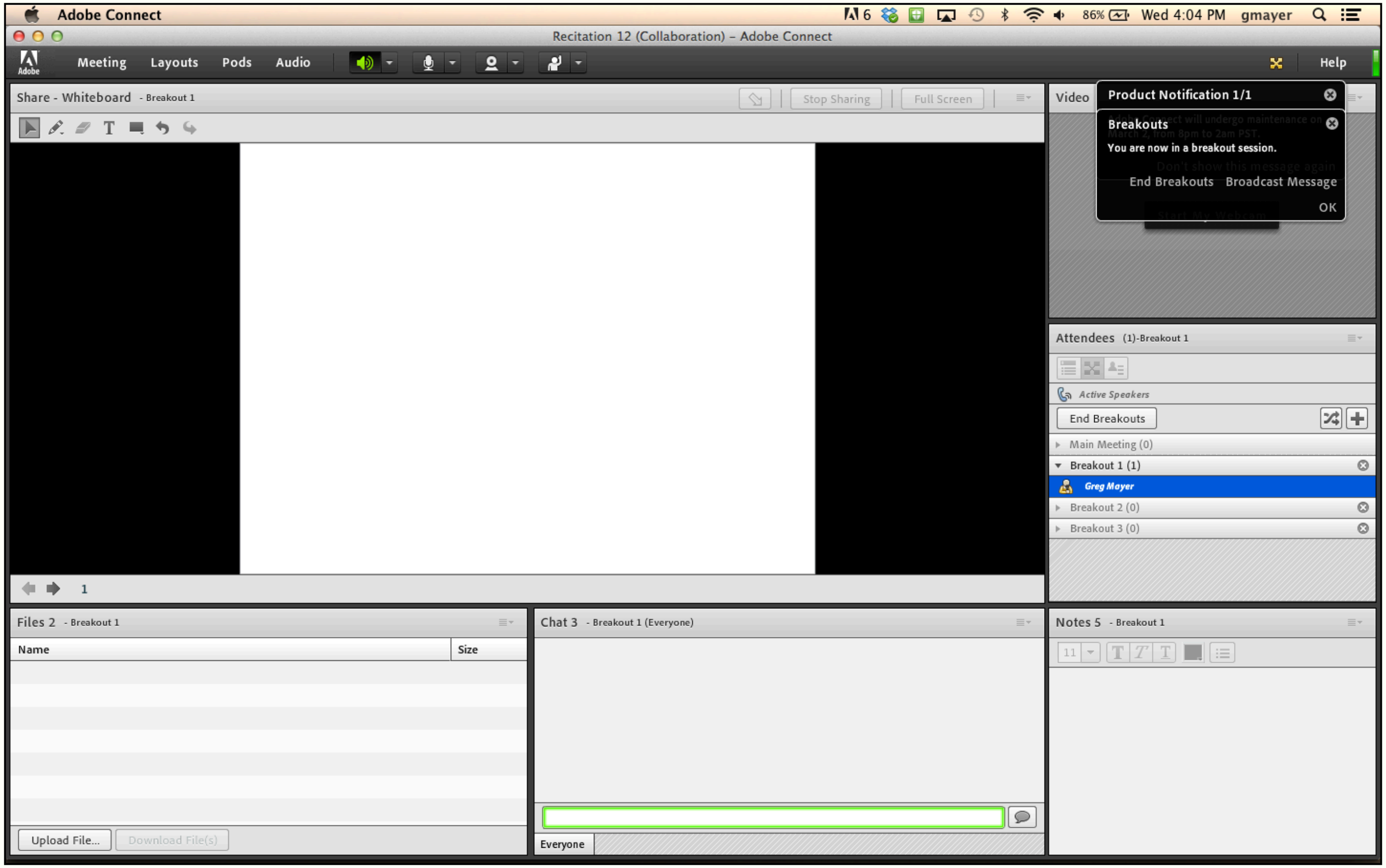
Also:

- Office hours: Wednesday, 7:30 pm to 9:30 pm
- If you can, during quiz connect to https://georgiatech.adobeconnect.com/distancecalculusquiz/
- What topics could be on the quiz?

    - **HW7:** triple integrals in Cartesian and cylindrical coordinates
    - **Extra problems for Quiz 3**: spherical coordinates

Breakout Rooms in Connect

we need to first switch to the "Collaboration" layout



If 06 ≤ N ≤ 10 students: 2 rooms
If 11 ≤ N ≤ 15 students: 3 rooms
If 16 ≤ N ≤ 20 students: 4 rooms

Use spherical coordinates to set-up an integral that represents the volume of the solid bounded by

$$0 \le x \le 1$$

$$0 \le y \le \sqrt{1 - x^2}$$

$$\sqrt{x^2 + y^2} \le z \le \sqrt{2 - (x^2 + y^2)}$$

Recitation 19	Line Integrals and Work

Fill in the Blanks: Work is the ________________ transferred to or from an object by

means of a ________________ acting on the _______________ .

Work Over a Straight Path

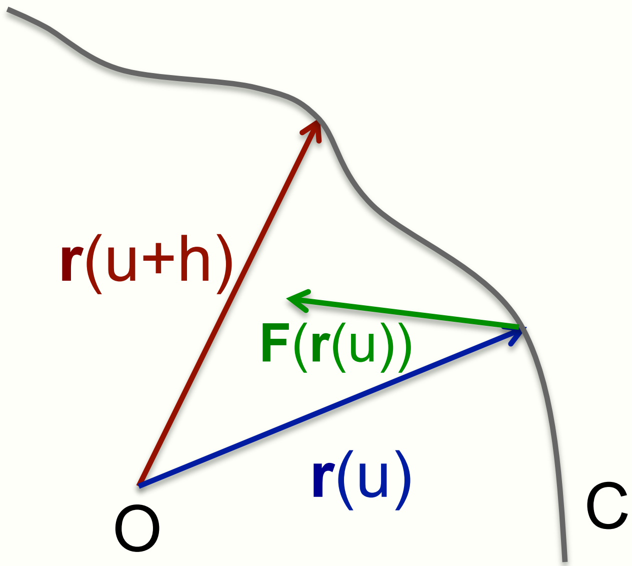
Force **F** is applied to an object as it moves from x = a to x = b along the x-axis.

	Applied Force	Work
Case 1	**F** = 4**i**	W =
Case 2	**F** = 4**i** – 2**j**	W =

we need to extend this concept to curved paths in R^3

Work Over a Curved Path

Force **F** applied to an object as it moves from **r**(u) to **r**(u + h) along curve C.



Work done by force **F** from **r**(u) to **r**(u+h) is W(u + h) – W(u).

	Applied Force	Work
Case 3	**F** = **F**(**r**(u))	W(u + h) – W(u) ≈

Calculating Work

Set up an integral that represents the total work.

a) $\mathbf{F} = (x + 2y)\mathbf{i} + (2x + y)\mathbf{j}$, path is $y = x^2$ from (0,0) to (2,4).

b) $\mathbf{F} = (x - y)\,\mathbf{i} - xy\,\mathbf{j}$, along the line from (2,3) to (1,2).

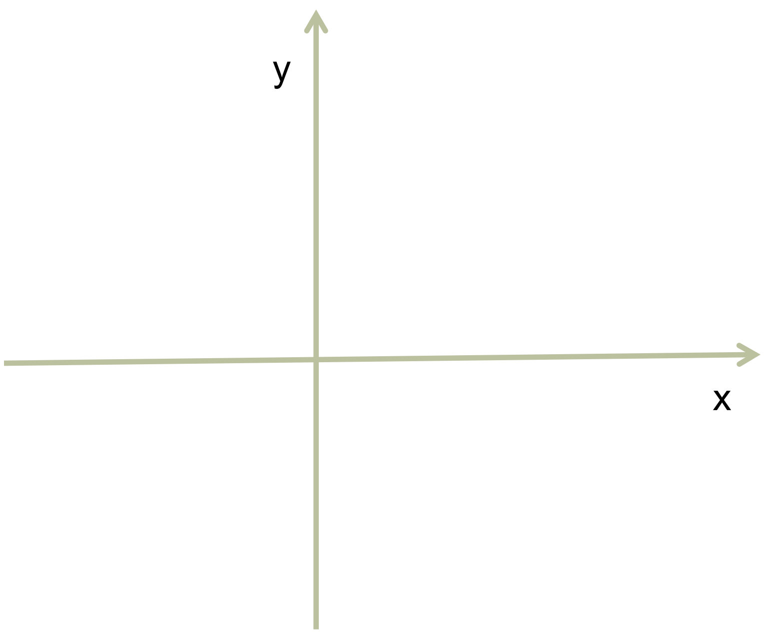
Announcements

Quiz 4: Tuesday April 15 (two weeks away)
Homework 8: due Friday at 11:59 pm
My prediction: one last pop quiz, this week or next

$$\text{circulation} = \Gamma = \int_C \vec{v}(\vec{r}) \cdot d\vec{r}$$

Sketch the velocity field for **v**, and calculate the circulation over curve C, where C is the circle of radius R.

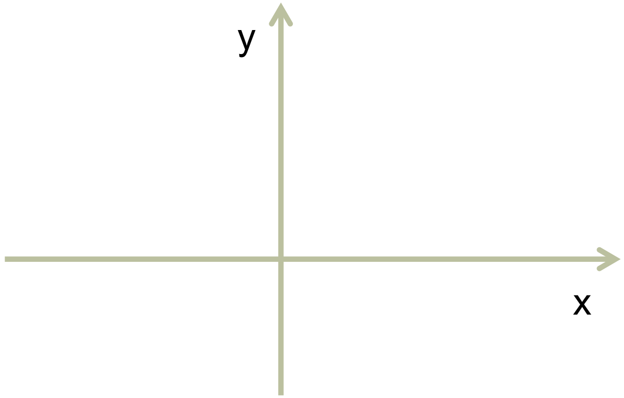
$$v = \begin{cases} 2\mathrm{i}, & R \le y \le R \\ 0, & \text{else} \end{cases}$$



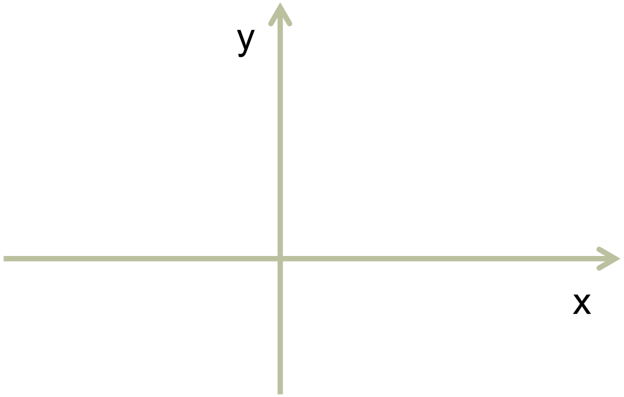
Circulation Examples

Sketch the velocity field for **v**, and calculate the circulation over curve C, where C is the circle of radius R.

b) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$



c) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$



Application of Circulation

The circulation of a vector field **V** around a directed closed curve is defined as

$$\text{circulation} = \Gamma = \int_C \vec{v}\left(\vec{r}\right)\cdot d\vec{r}$$

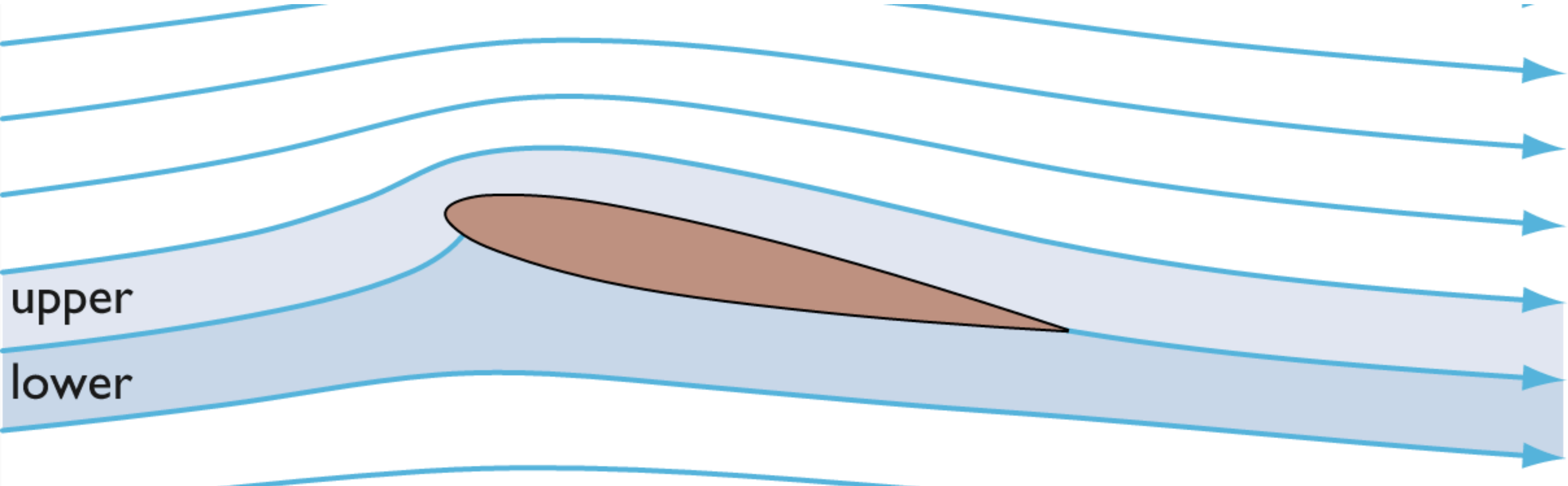
take C to be a closed path around the wing on its surface

- Upward lift force is proportional to circulation, Γ
- Note the cross-sectional profile of the wing in this photograph

Application of Circulation

$$\text{circulation} = \Gamma = \int_C \vec{v}\left(\vec{r}\right) \cdot d\vec{r}$$

take C to be a closed path around the wing on its surface



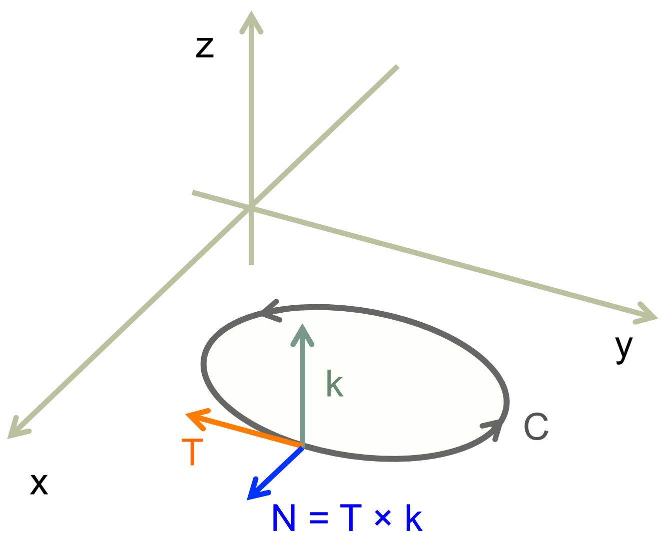
- Write Γ as $\Gamma = \Gamma_{upper} + \Gamma_{lower}$
- Γ_{upper} and Γ_{lower} have opposite signs
- the magnitude of **V** along the upper surface of the wing is greater than along the lower surface: net circulation is non-zero

Announcements

Quiz 4: Tuesday April 15
My prediction: one last pop quiz, next week?
Homework 8: due Friday Apr 4 at 11:59 pm. Questions?
Homework 9: due Friday Apr 11 at 11:59 pm. Questions?
Survey: please complete the brief technical issues survey, email sent yesterday

$$\text{circulation} = \int_C \vec{v} \cdot \vec{r}'\,dt = \int_C \vec{v} \cdot \vec{T}\,dt$$

$$\text{flux} = \int_C \vec{v} \cdot \vec{N}\,dt$$ N is the <u>outward</u> pointing, unit, normal vector of curve C



COUNTERCLOCKWISE MOTION
IN XY PLANE

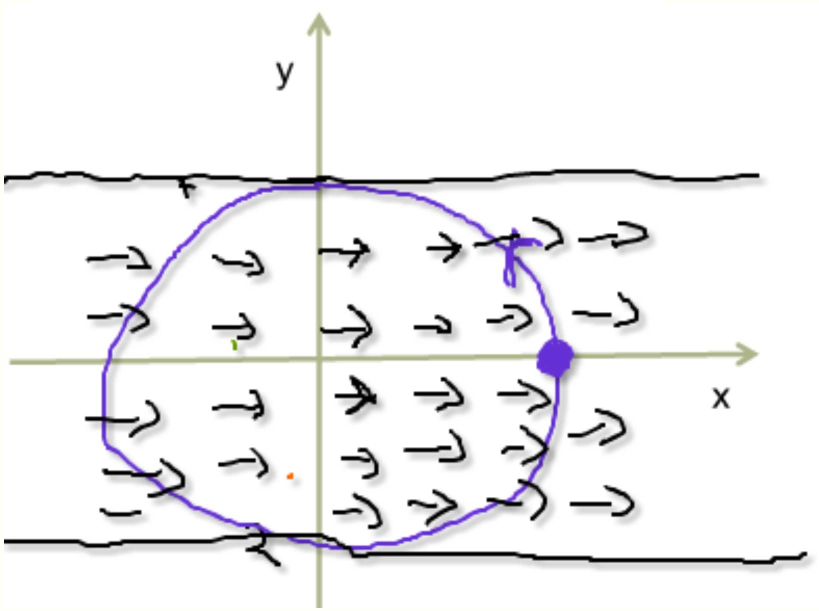
The textbook derives a computational formula for flux:

k is the unit vector parallel to the z-axis

$$\text{flux} = \int_C M\,dy - N\,dx$$

Calculate the flux over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\vec{\mathrm{i}}, -R \le y \le R \\ 0, \text{ else} \end{cases}$$



Announcements

Quiz 4: Tuesday April 15
My prediction: one last pop quiz, this week?
Homework 8: due Tues Apr 8 at 11:59 pm. Questions?
Homework 9: due Tues Apr 8 at 11:59 pm. Questions?
Survey: please complete the brief technical issues survey, email sent last Wed.

Graded group work activity on **Thursday**.

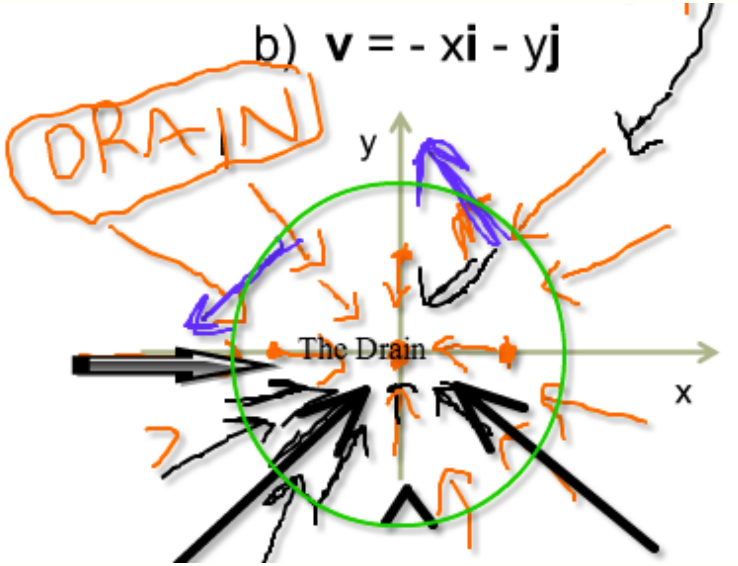
Problem 1 (10 points)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$, and C its boundary Consider the line integral

$\int_C xy\,dx - xy^2\,dy$. Use Green's theorem to convert to an double integral, and express this as a double integral in polar coordinates with limits.

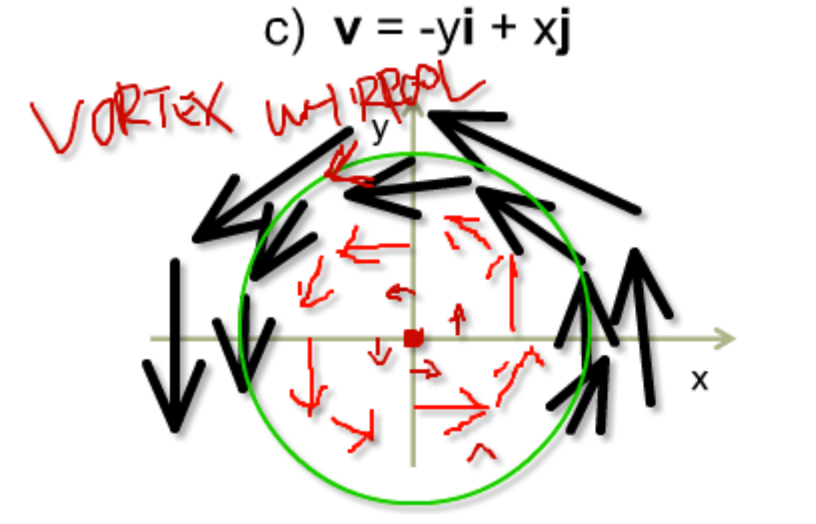
Circulation Examples

Calculate the flux over curve C, where C is the circle of radius R.



Circulation Examples

Calculate the flux over curve C, where C is the circle of radius R.



Summary

Fill in the blanks:

a) Circulation measures flow ________________ path C.

b) Flux measures the flow ____________________ of C.

velocity field equation	velocity field equation	circulation	flux	is v conservative?
pipe	$\mathbf{v} = 2\mathbf{i}$ for $-R \leq y \leq +R$, $\mathbf{v} = \mathbf{0}$ otherwise			
drain	$\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$			
vortex, whirlpool	$\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$			

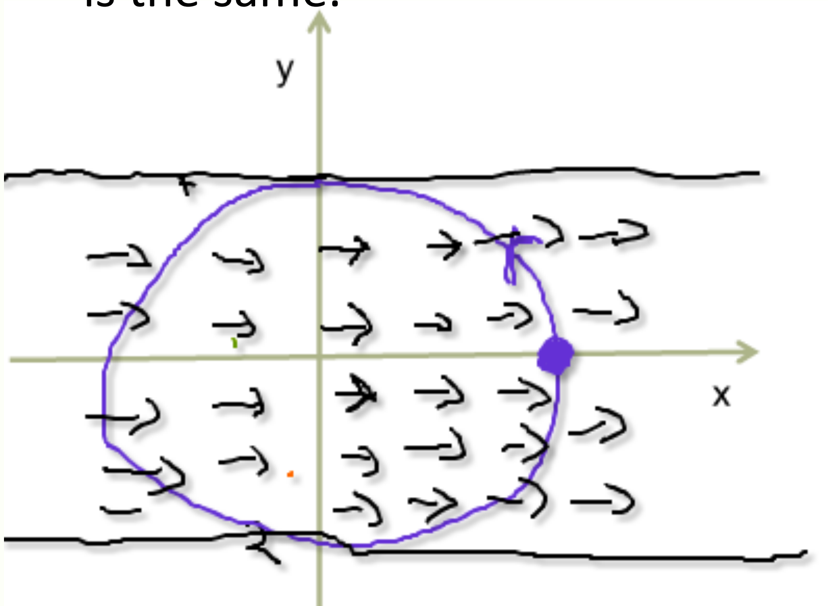
Conservative Vector Fields

Recall the Pipe example.

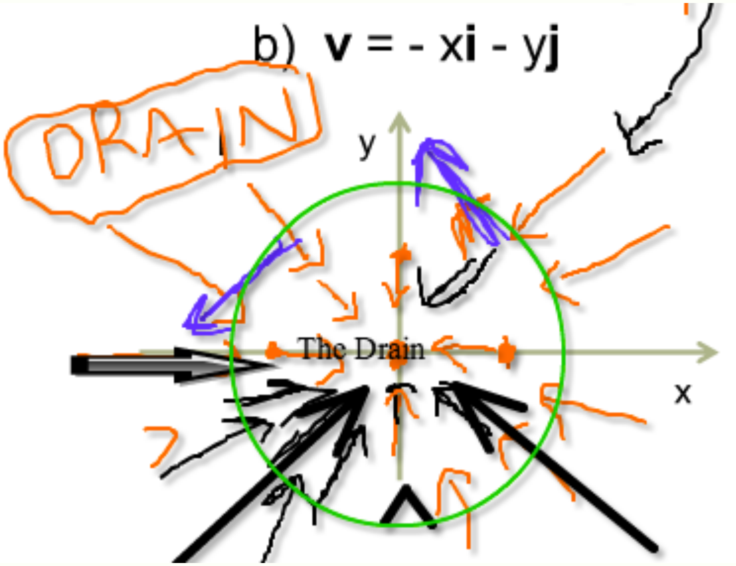
a) Why was the circulation zero?

b) For any path that starts and ends at point A, and stays inside "the pipe", the circulation is _____________ .

c) For all paths that starts at A and ends at point B, the integral ______________ is the same.



Is this vector field conservative?



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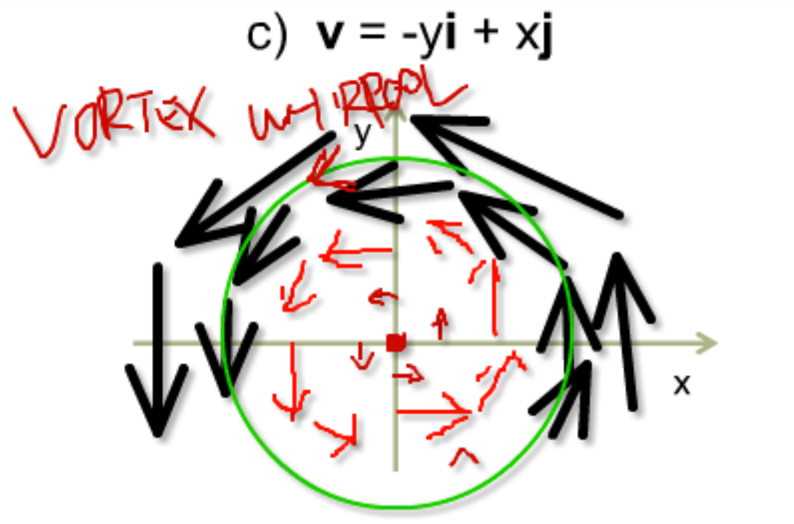


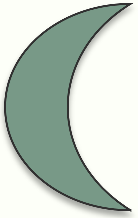
Green's Theorem

If D is a region that is ________________________________,
and P and Q are scalar fields that are differentiable on D,
and C is the boundary of D, then:

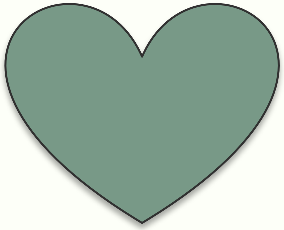
Below are five regions. For which regions can we apply Green's Theorem?

a)

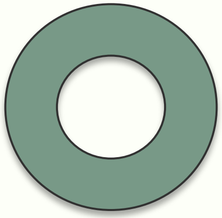
b)



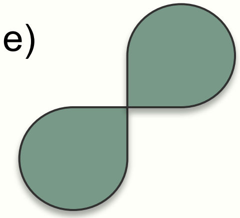
c)



d)



e)



a) Evaluate $\oint_C y^2\,dx + 2xy\,dy$, C is one loop of $r = 2\sin 2\theta$

b) Change the integral so that it represents the area of one loop.

a) Evaluate $\oint_C y^2\,dx + 2xy\,dy$, C is one loop of $r = 2\sin 2\theta$

b) Change the integral so that it represents the area of one loop.

Announcements

Quiz 4: Tuesday April 15
Homework 8: due Tues Apr 15 at 11:59 pm. Questions?
Homework 9: due Tues Apr 15 at 11:59 pm. Questions?
Questions for Quiz 4 (not graded)
Office Hours: Monday 7:30 to 9:30
Survey: please complete the brief technical issues survey, email sent last Wed.

Graded group work activity. Solve the question below in groups of 3 to 5 students, you have about 10 minutes. I'll circulate from room to room.

Problem 1 (10 points)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$,
and C its boundary Consider the line integral

$\int_C xy\,dx - xy^2\,dy$. Use Green's theorem to convert to an double integral,

and express this as a double integral in polar coordinates with limits.

Problem 1 (10 points)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$,
and C its boundary Consider the line integral

$\int_C xy\,dx - xy^2\,dy$. Use Green ' s theorem to convert to an double integral,
and express this as a double integral in polar coordinates with limits.

Fundamental Theorem of Line Integrals

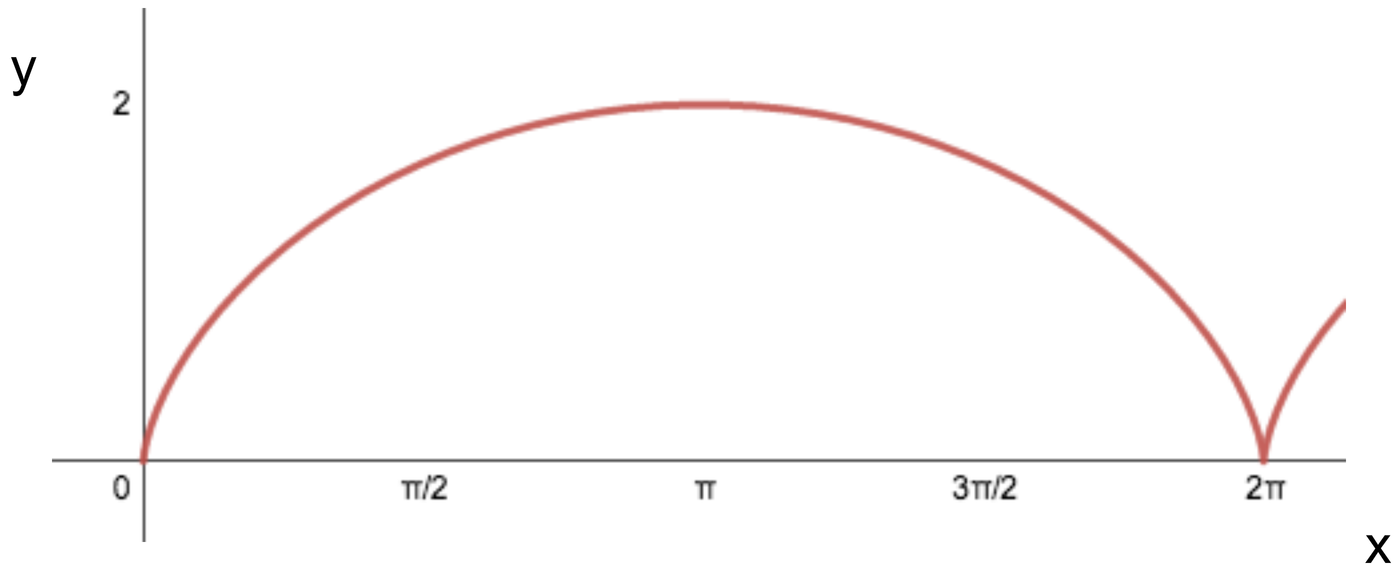
If **F** is a conservative field, then:

Example

Calculate line integral of **F** = $(x^2 - y)$**i** + $(y^2 - x)$**j**, over path

r = $a \cos(t)$**i** + $b \sin(t)$**j**, $0 \leq t \leq 2\pi$

The Cycloid

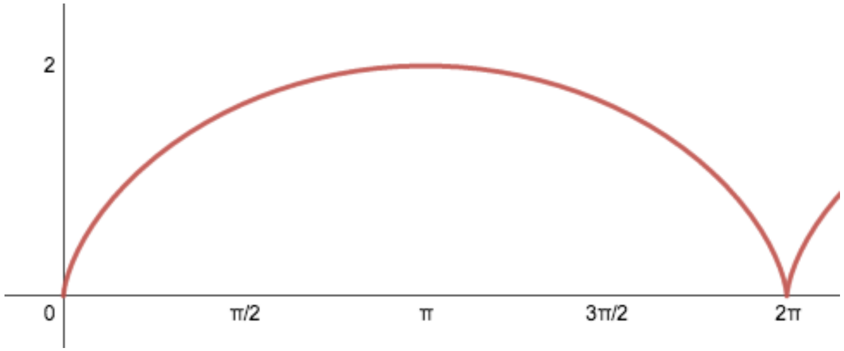


The curve traced by a point on a rolling wheel is

$$x(t) = t - \sin(t)$$
$$y(t) = 1 - \cos(t)$$

The Cycloid

Find the area under one arch of the cycloid:
$x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$



Recitation 24 | Today: Pop Quiz, Surface Integrals

- There's a pop quiz today! :D
- You have a few minutes to review your notes.
- Start time: 8:10
- Ends at: 8:30?
- Pop quiz grading
  - 5 points: on the right track
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, either
  a) **work on whiteboard in breakout room**
    - write in text chat that you'd like to work in breakout room,
    - submit work by letting me know when done, or email me a screen capture of your work
  b) **work on paper and give work to facilitator**
    - leave 2 inch margin
    - write your name and QH6 at the top
    - facilitator can email quiz to cdlops@pe.gatech.edu
  c) **work on paper and take a photo of your work**
    - email your photo to me before 8:40
    - write in text chat that you are emailing your work to me

Pop Quiz

Set up as a double integral, the surface integral of F dot n ds , where the surface is $z(x,y) = x^2 - y^2$, $F = x\mathbf{i} + z\mathbf{k}$, and $0 \le x \le 1$, $-1 \le y \le 1$.

Announcements

Quiz 4: Marked on Friday? Monday? I'm not sure yet.
Last HW : due Sun Apr 27
Technical issues during lecture yesterday: fiber cut?

Engagement Survey
<u>Please</u> complete the brief engagement survey, email sent last Tuesday.

Technical Survey
Follow-up question: I often let students write on the board at any time. In what ways, if any, did this help your learning in recitations?

Find an equation in x, y, z, for the surface whose parametric representation is

$$\mathbf{r} = Au\cos(v)\mathbf{i} + Bu\sin(v)\mathbf{j} + u^2\mathbf{k}, \quad u \geq 0, \quad 0 \leq v \leq \pi.$$

Describe and sketch the surface.

Find parametric representations for the following surfaces.

a) the upper half of $4x^2 + 9y^2 + z^2 = 36$

b) the part of the plane $z = x + 2$ inside the cylinder of $x^2 + y^2 = 1$

a) What properties does a parametric representation of a surface need to have?

b) Find a parametric representation for the part of the plane $z = x + 2$ in the first octant and inside the cylinder $x^2 + y^2 = 1$.

Announcements

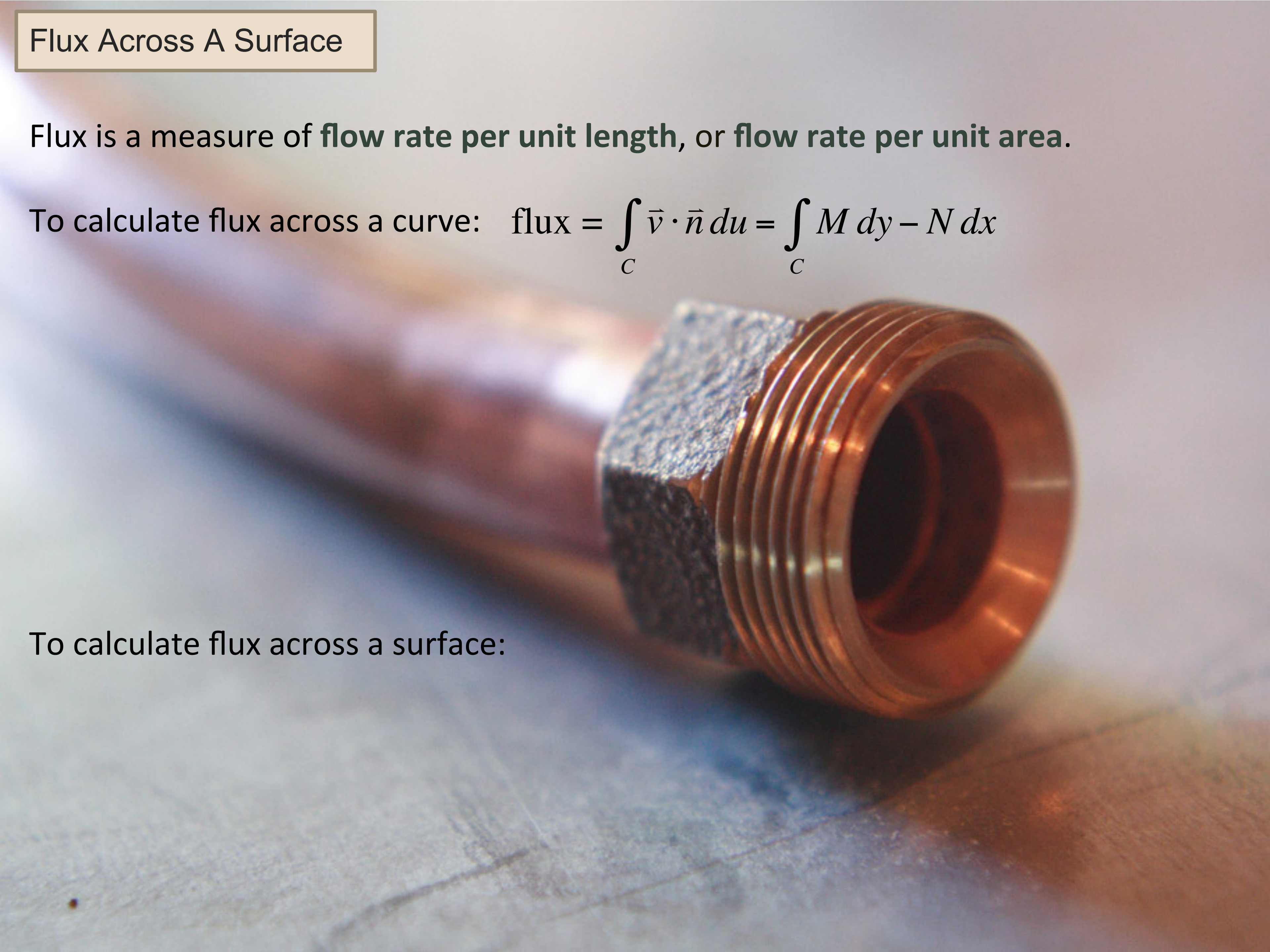
Quiz 4: marked yesterday, grades should be entered today.
HW grades: check in t-square that I entered grades correctly
Last HW : due Sun Apr 27
Cut-off for final exam: I don't know if there is one, or what cut-off would be

Engagement Survey
Please complete the brief engagement survey, reminder email sent yesterday.

Technical Survey
Follow-up question: most students didn't communicate with microphones very often. Why do you think this was the case?

Surface Area

Calculate the surface area of $z = y^2$, for $0 \le x \le a$, $0 \le y \le b$.



Flux Across A Surface

Flux is a measure of **flow rate per unit length**, or **flow rate per unit area**.

To calculate flux across a curve: $\text{flux} = \int_C \vec{v} \cdot \vec{n}\, du = \int_C M\, dy - N\, dx$

To calculate flux across a surface:

The Pop Quiz (from last rectation)

Set up as a double integral, the surface integral of F dot n ds , where the surface is $z(x,y) = x^2 - y^2$, $F = x\mathbf{i} + z\mathbf{k}$, and $0 \le x \le 1$, $-1 \le y \le 1$.

Divergence

In a two-dimensional, steady-state, incompressible fluid flow, the velocity, **v,** of the flow is **v** = f(x,y)**i**+ g(x,y)**j**, where f(x,y) and g(x,y) must satisfy $\nabla \cdot \mathbf{v} = 0$.

If f(x,y) = x/2, and v(0,0) = 0i + 0j, find g(x,y), and sketch **v**.

Archimedes Principle

Prove Archimedes Principle

Volume as a Surface Integral

Express the volume, V, of an object with a surface integral.

E = electric field. Then, Gauss's Law states that:

charge = ε_0 (flux of **E** through closed surface)

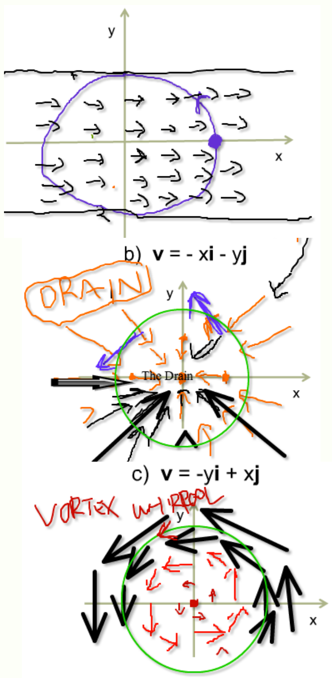
Find the charge conained in a solid hemisphere if **E** = x**i** + y**j** + z**k**.

Recitation 26 — Today: Divergence

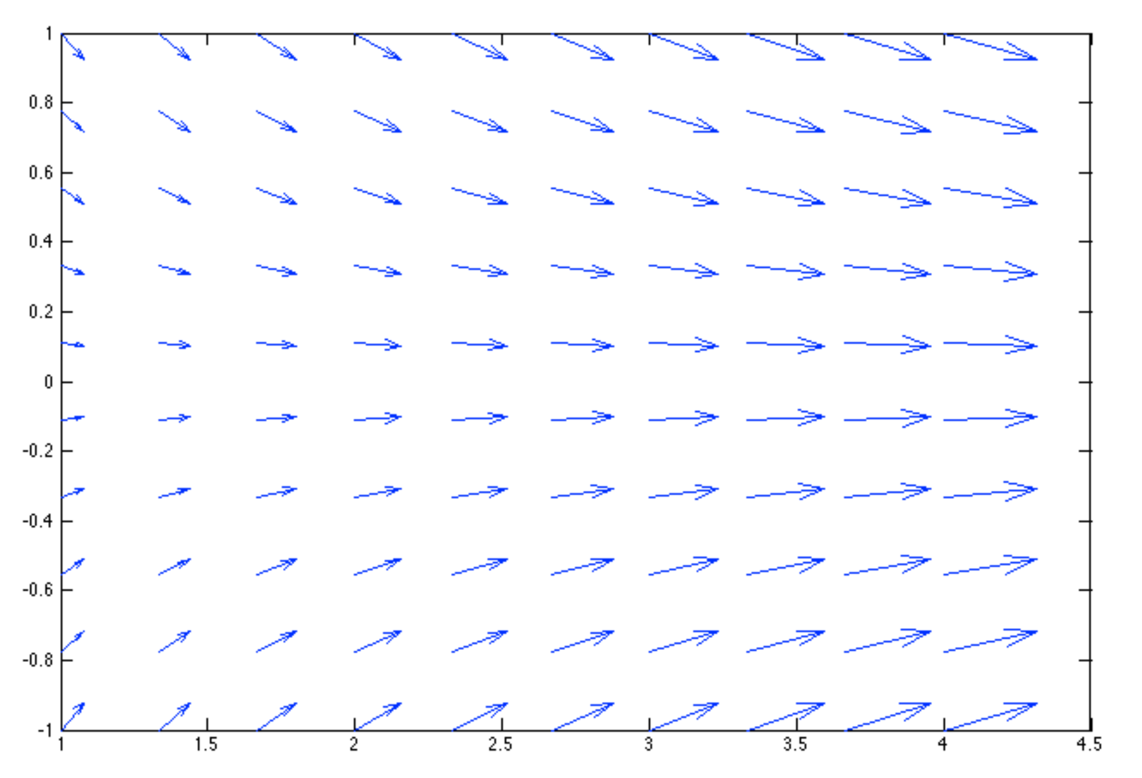
Divergence measures a flow's tendancy to ________________ .

If $\mathbf{v}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$, then $\operatorname{div}(\mathbf{v})$ = _______________.

velocity field equation	velocity field equation	divergence
pipe	$\mathbf{v} = 2\mathbf{i}$ for $-R \le y \le +R$, $\mathbf{v} = \mathbf{0}$ otherwise	
drain	$\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$	
vortex, whirlpool	$\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$	
nozzle	$\mathbf{v} = x\mathbf{i} - y\mathbf{j}$ $x \ge 1$	



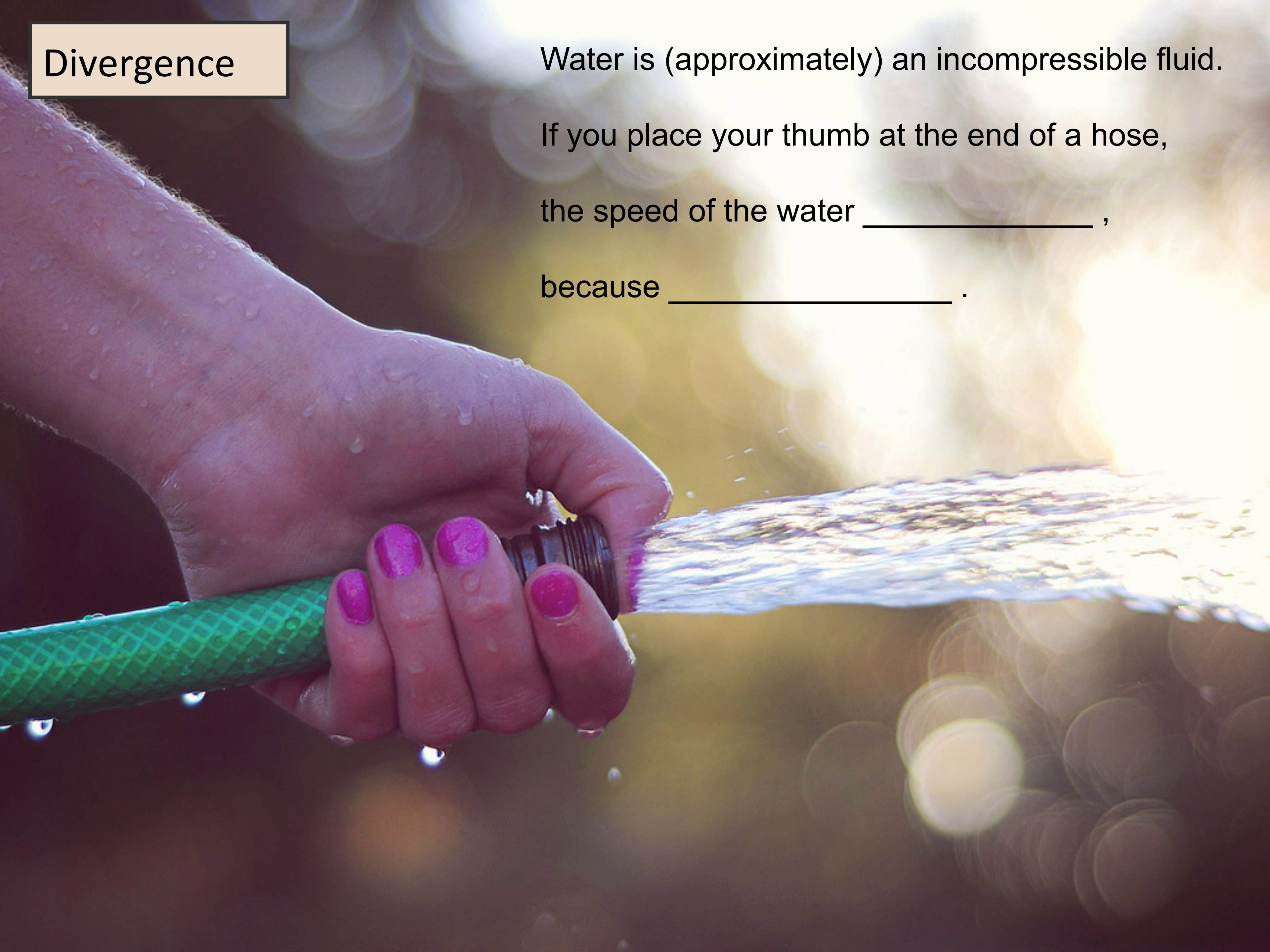
Incompressible Fluids



If a fluid is incompressible, then its divergence is _____ .

The field **v** = x**i** – y**j** could represent an incompressible flow.

As x increases, flow moves towards _____________, and its speed _________ .



Divergence

Water is (approximately) an incompressible fluid.

If you place your thumb at the end of a hose,

the speed of the water _____________ ,

because ________________ .

Announcements

Quiz 4: grades entered Tuesday.
HW grades: check in t-square that I entered grades correctly
Last HW: due Sun Apr 27
Cut-off for final exam:
Pop-quiz adjustments: made Wednesday
Graded activities: I'll apply adjustments today, only to those writing final
Tablets and mics: please return to facilitator
Final Exam
If attending Grady: May 2, on campus.
If not attending Grady: facilitator has instructions.

Engagement Survey
Follow-up question:
Is it important to get to know other students in recitation? Why/why not?

The divergence theorem states that:

Archimedes Principle

Upward buoyant force =

Prove Archimedes Principle

E = electric field. Then, Gauss's Law states that:

total charge = (ε_0)(flux of **E** through closed surface)

Find the total charge contained in a solid hemisphere if **E** = x**i** + y**j** + z**k**.

Office Hours for Quiz 2

- 2 pages of 8 1/2 x 11 inch notes (both sides) allowed
- Calculators allowed.
- Covers HW 4,5,6 and the “Questions for quiz 2” HW
- Office hours: Sunday and Monday, 7:30 pm to 9:30 pm
- If you can, during quiz connect to https://georgiatech.adobeconnect.com/distancecalculusquiz/
- What topics could be on the quiz?
    - **HW4:** surfaces and optimization, Lagrange multipliers
    - **HW5:** Taylor approximations and estimating their error (see last question)
    - **HW6:** setting up and evaluating double integrals
    - **Questions for Quiz 2:** polar integrals

Quadratic surface: a question from last year's quiz 2

Consider the surface

$$-6\,\mathbf{x} + \mathbf{x}^2 + 4\,\mathbf{y} + \mathbf{y}^2 + 8\,\mathbf{z} - \mathbf{z}^2 = 4$$

This is a qudratic surface. Find out the center, and what kind it is. Draw a picture, labeling the center and the axes.

Quadratic surface: a question from last year's quiz 2

WolframAlpha computational knowledge engine

x^2 - 6x +4y +y^2 +8z -z^2= 4

Examples Random

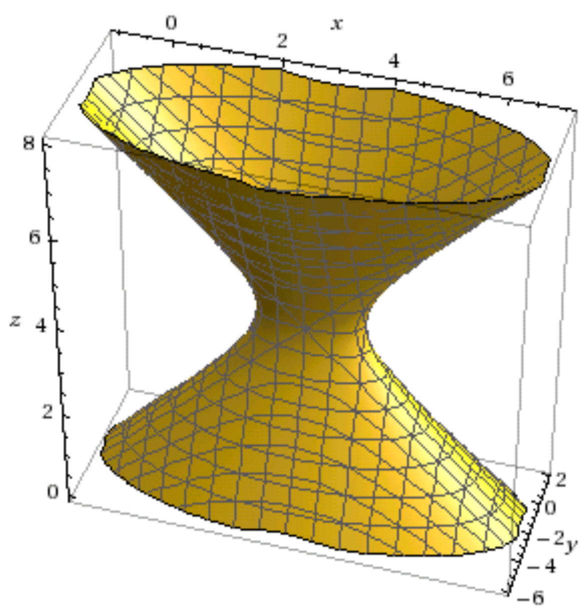
Input:

$$x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$$

Geometric figure:

one-sheeted hyperboloid

Surface plot:



Enable interactivity

An Optimization Problem

Find the minimum value of the function $f(x,y) = x^2 + (y - 2)^2$ subject to the constraint $x^2 - y^2 = 1$.

An Optimization Problem

Find the minimum of the function $f(x,y) = (x/a)^2 + (y/b)^2$ subject to the constraint $x + y = L$. The numbers a, b, and L are positive constants.

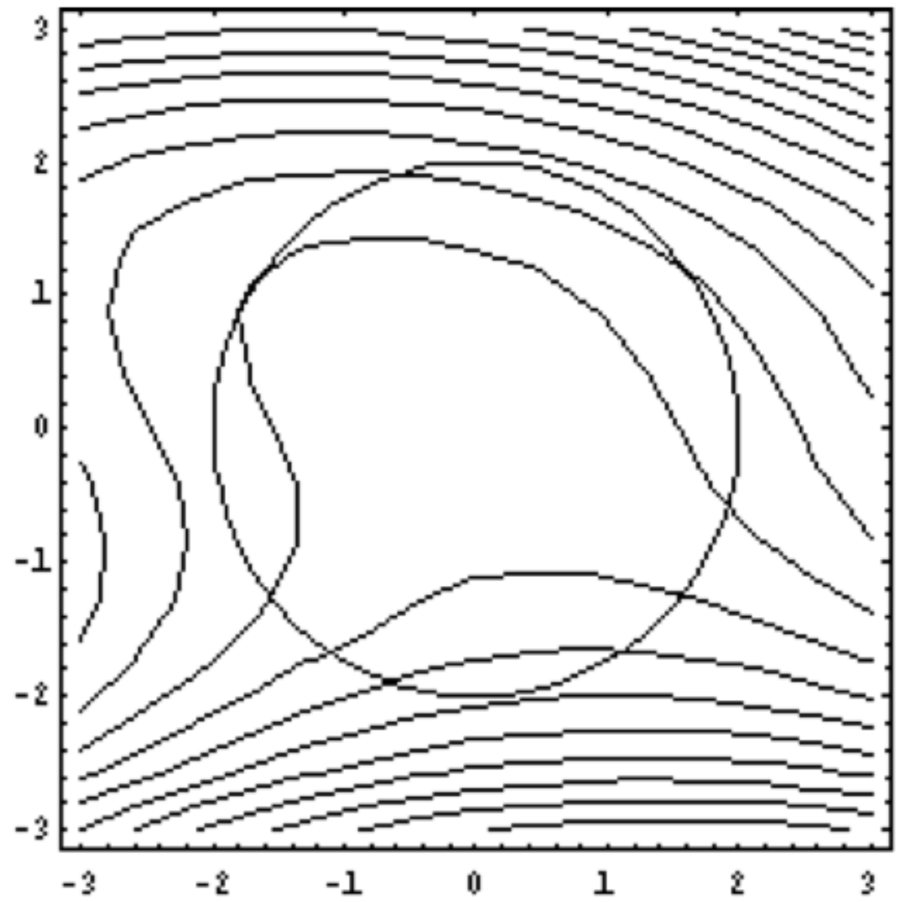
An Optimization Problem

A company produces widgets at N factories. The cost of producing x_i widgets at factory i is x_i^2/a_i, where $a_i > 0$. Minimize the total cost of producing L widgets.

A Conceptual Lagrange Multipliers Question

The diagram shows a contour plot of f(x,y), and the circle of radius 2 centered at (0,0). How many local maximums and mins does f(x,y) have on the perimeter of the circle?

Assume the origin is a global max of f(x,y).



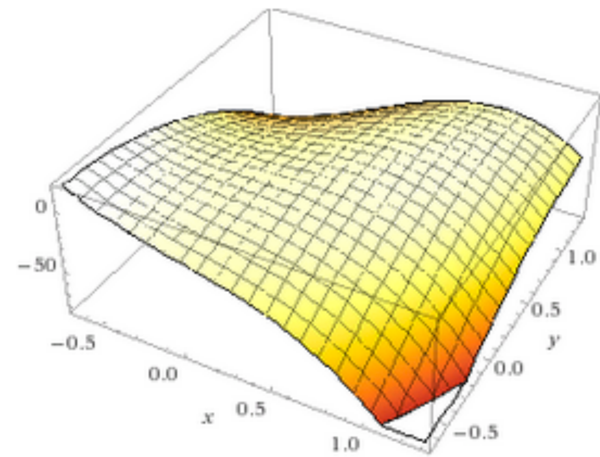
Max/Min Electrostatic Potential

The electrostatic potential in the region $0 \le x \le 1$, $0 \le y \le 1$, is given by $V = 48xy - 32x^3 - 24y^2$. Find the locations of the minimum and maximum values.

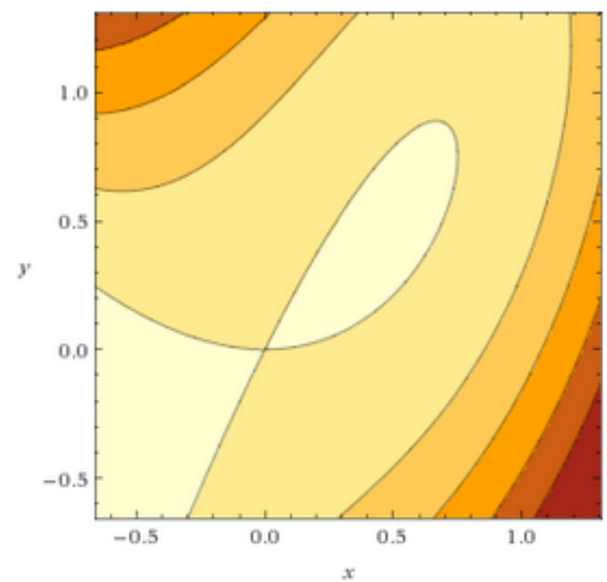
Max/Min Electrostatic Potential

plot $(48\,x\,y - 32\,x^3) - 24\,y^2$

3D plot:



Contour plot:



Setting Up a Polar Integral

Set up, but do not evaluate, an integral representing the area the region enclosed by $r = 2 - 2\cos\theta$. Sketch the region of integration.

Convert a Cartesian Integral to a Polar Integral

a) Sketch the region of integration
b) Express the integral in polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2 + y^2}\, dy\, dx$$

2) Sketch the region of integration, change the order of integration, and then integrate

$$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx\,dy$$

1) Sketch and find the area of the region inside the curve $r = 5 + \cos(\theta)$ (from last year's quiz).

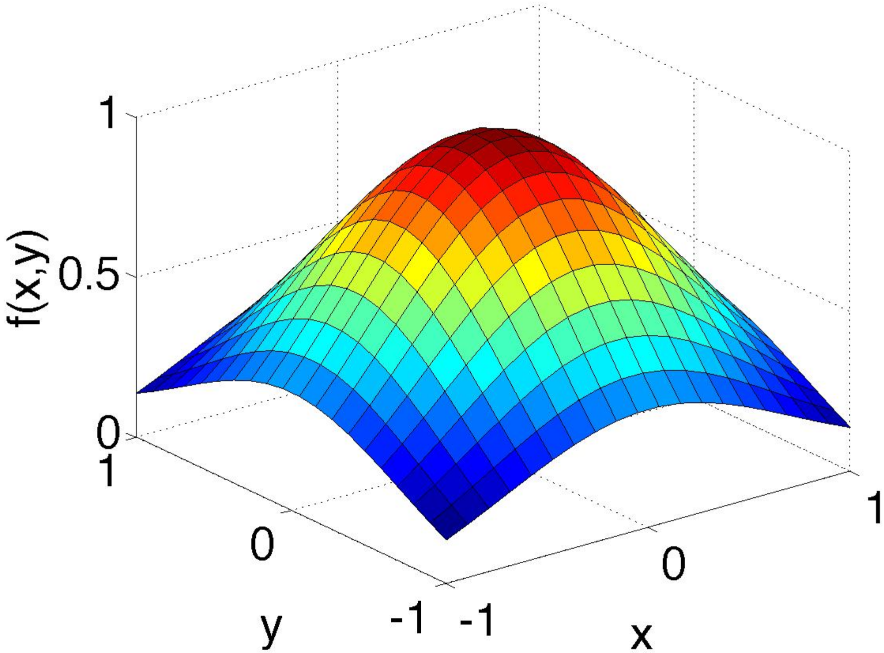
3) Sketch the region of integration, change the order of integration, and then integrate

$$\int_1^3 \int_{-x}^{x^2} dy\, dx$$

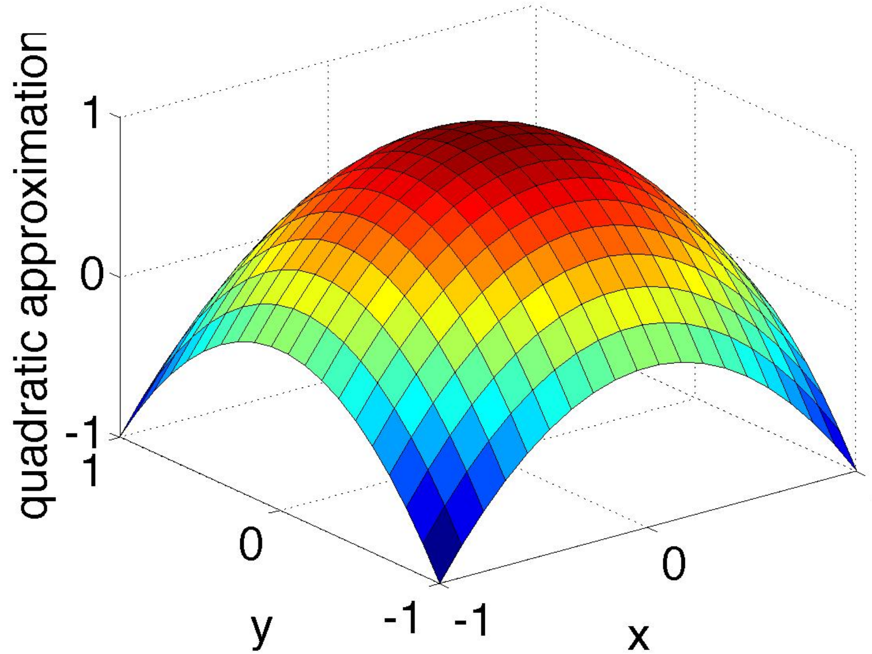
4) Find the quadratic approximation to $f(x,y) = \exp(-x^2 - y^2)$ near the origin.

Surface Plots

$f(x,y) = \exp(-x^2 - y^2)$



quadratic approximation
$1 - x^2 - y^2$



Quiz 3 Review

For your quiz:

- 2 pages of 8 1/2 x 11 inch notes (both sides) allowed
- Calculators allowed.

Also:

- Office hours: Wednesday, 7:30 pm to 9:30 pm
- If you can, during quiz connect to https://georgiatech.adobeconnect.com/distancecalculusquiz/

Set-up integrals that provide the centroid of the region bounded by $r = 1 + \cos\theta$. The mass density at any point in the region is proportional to its distance to the origin.

A region, with constant density D, is bounded by $x^2 + y^2 = a^2$, and $x^2 + z^2 = a^2$. Find the moment of inertia about the x-axis. Use Cartesian coordinates.

A region above the xy plane, with constant density D, is bounded above by $z^2 = x^2 + y^2$, and below by $z = x^2 + y^2$. Find the moment of inertia about the z-axis. Use cylindrical coordinates.

Set-up an integral that represents the volume of the solid bounded by $x^2 + y^2 + (z - R)^2 = R^2$. Use spherical coordinates.

Set up an integral that represents the volume of solid bounded by $2x = x^2 + y^2$, and $2z = 4 + x$. Use cylindrical coordinates.

Change the order of integration.

$$V = \int_0^2 \int_0^{9-x^2} \int_0^{2-x} dz \; dy \; dx$$

Set up a triple integral that represents the volume of the region bounded by $x^2 + z^2 = 4$ and the planes $y + z = 6$, $x = 0$, $y = 0$, $z = 0$.

Set-up an integral that represents the volume bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2(x^2 + y^2)^{1/2}$. Use spherical coordinates.

Welcome Back!

1. Announcements
2. Vector Derivatives (14.1)

Surveys

- **Teaching assistant survey**
  - it focused on evaluating your TA
  - closed in December
  - results sent by email

- **Engagement survey**
  - made available yesterday
  - closes next week Monday
  - we'll discuss its results next week
  - it focuses on how we can improve recitations
  - PLEASE complete this survey

Research Contact

If you have any questions about the research we are conducting, contact information has changed to:

Technologies: same as last semester

Recitations run in Wimba (yay!)

- Wimba technical problems, can contact technical support http://www.wimba.com/services/support

- Recordings of our recitations on Tegrity gatech.tegrity.com
- Tablets, mics: please bring and use them
- All homework on MyMathLab
- Piazza: link in t-square

Adobe Connect

- Made available to Georgia Tech
- I’m looking into what would be involved in switching from Wimba to Adobe Connect

Grading Weights: Same as Last Semester

	QH6 (%)	All other sections (%)
Homework	10	10
Final	25	25
Quizzes	60	65
Recitations	5	0
Total	100	100

+ random pop quizzes

Questions, Office Hours

Office Hours

Generally held on the night before quizzes (same as last semester)

Questions

email:

phone (office):

phone (cell):

Definition of Torque

Torque, τ, is defined as

$\tau =$

Angular Momentum

If the position of a particle with constant mass *m* is **r**(t), its angular momentum is **L**(t) = m**r**(t) × **r′**(t).

Show that **L′**(t) is equal to torque.

$$L' = \frac{d}{dt}\left(m\vec{r} \times \vec{r}'\right) = m\left(\vec{r}' \times \vec{r}' + \vec{r} \times \vec{r}''\right)$$

$$= m\,\vec{r} \times \vec{r}''$$

$$= \vec{r} \times \left(m\vec{r}''\right)$$

$$= \vec{r} \times \vec{F}$$

Zero Angular Momentum

Show that if the torque is a zero vector for all t, then the angular momentum of the particle is constant for all t.

$$\tau = \vec{0} = \vec{r} \times \vec{F}$$

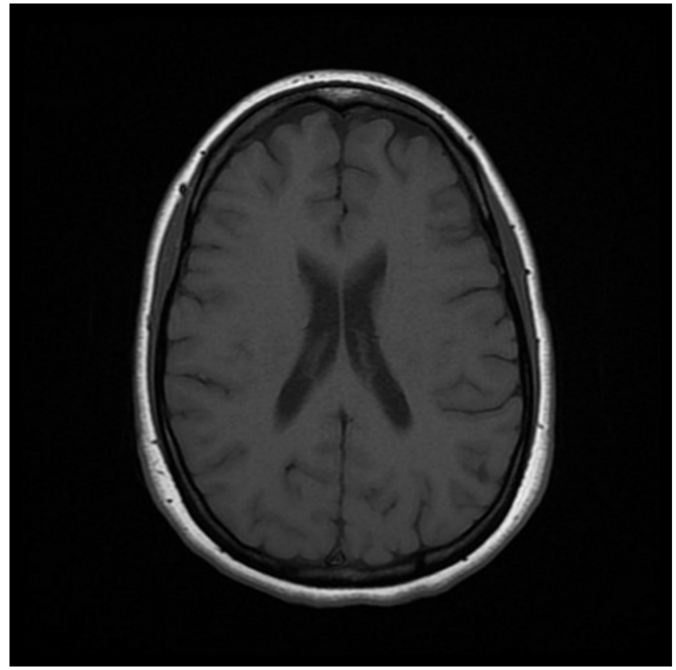
$$= \vec{r} \times (m\vec{r}\,'') = \vec{L}\,' \Rightarrow \vec{L} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

(TOO EASY?)

Magnetic Resonance Imaging (MRI)

MRI Scanner

MRI Image



$$\text{applied magnetic field, } \vec{B} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix}, \ \omega = \text{known constant}$$

The applied field creates a measurable signal, $\vec{M}(t)$.

The Bloch Equation

$$\frac{d\vec{M}}{dt} = \vec{M} \times \vec{B}, \quad \vec{M} = \begin{bmatrix} M_x(t) \\ M_y(t) \\ M_z(t) \end{bmatrix}$$

solve: $y'' = -k^2 y$

LDE CC : $y = e^{rt}$

$r^2 e^{rt} = -k e^{rt} \Rightarrow r^2 = -k^2$

$\Rightarrow r = ik$

Solve the differential equation, plot the solution.

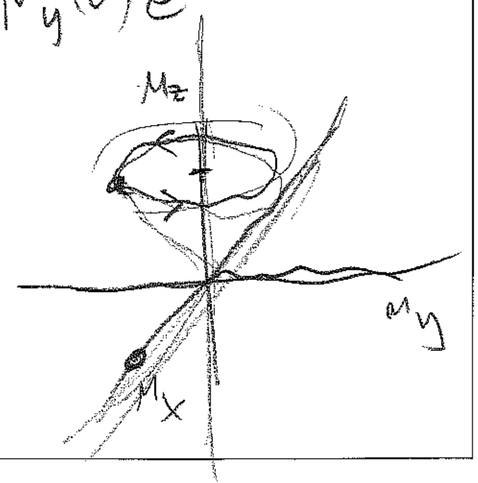
$$\begin{bmatrix} M_x' \\ M_y' \\ M_z' \end{bmatrix} = \begin{bmatrix} i & j & k \\ M_x & M_y & M_z \\ 0 & 0 & \omega \end{bmatrix} = \begin{bmatrix} \omega M_y \\ -\omega M_x \\ 0 \end{bmatrix}$$

$M_x' = \omega M_y \Rightarrow M_x'' = \omega M_y' = -\omega^2 M_x \Rightarrow M_x = e^{-i\omega t}$

$M_y' = -\omega M_x \qquad M_y'' = \omega M_x' = -\omega^2 M_y \Rightarrow M_y = M_y(0) e^{-i\omega t}$

$M_z' = 0 \Rightarrow M_z = C = M_z(0)$

$e^{i\omega t} = \cos\omega t + i \sin\omega t \qquad \begin{cases} M_x = M_x(0) \operatorname{cis} \omega t \\ M_y = M_y(0) \operatorname{cis} \omega t \end{cases}$



Recitation 02

Today: Vector Functions (13.1, 13.2)

While we're waiting to start: describe situations where the following is true for all t.

$$\vec{r}(t) \cdot \frac{d\vec{r}}{dt} = 0$$

THIS IS A NICE WARM-UP ACTIVITY.
LET STUDENTS COME UP WITH ANSWERS, SEE IF THEY UNDERSTAND,
EXAMPLES: (1) CIRCULAR MOTION, (2) STATIONARY OBJECT,

(1) CIRCULAR MOTION

CAN BE $\vec{r} = \cos t\,\hat{i} + \sin t\,\hat{j}$, $\vec{r}\,' = -\sin t\,\hat{i} + \cos t\,\hat{j} \Rightarrow \vec{r} \cdot \vec{r}\,' = 0 \quad \forall t$

(2) STATIONARY OBJECT

CAN BE: $\vec{r} = c_1\hat{i} + c_2\hat{j}$, $\vec{r}\,' = \vec{0} \Rightarrow \vec{r} \cdot \vec{r}\,' = 0, \quad \forall t$

HW 1

- HW1 is on MyMathLab
- due next Tuesday at 11:59 pm
- covers 13.1 and 13.2

Velocity, Acceleration

1) The position of particle is $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$.
   a) Show that position is perpendicular to its velocity
   b) For what values of *t* do the position and acceleration have the same direction?
2) Provide another example of a vector function, $\mathbf{s}(t)$ that satisfies $\mathbf{s}''(t)=\mathbf{s}(t)$ for all *t*.

1) a) SHOWN ON PREVIOUS SLIDE.

b) WE WANT TO FIND VALUES OF t THAT SATISFY

$$\vec{r}(t) = m\vec{r}''(t), \quad m = \text{constant}$$

THIS EQUATION IS NEVER SATISFIED.

2) SEE EARLIER SLIDE.

Integration

A moving object starts at point (1,0) and its velocity is given by the vector [2, 4t]. Its position is given by:

$$\vec{r}\,' = \vec{r}\,'(t) = \begin{bmatrix} 2 \\ 4t \end{bmatrix}$$

$$\vec{r} = \begin{bmatrix} 2t \\ 2t^2 \end{bmatrix}$$

ask students: what else do we need?

$$\vec{r} = \begin{bmatrix} 2t + c_1 \\ 2t^2 + c_2 \end{bmatrix}$$

use initial condition:

$$\vec{r} = \begin{bmatrix} 2t + 1 \\ 2t^2 + 0 \end{bmatrix}$$

Group Work

1. Group size is 2 or 3 students
2. Someone is in your group when they write their initials on board
3. Students can create breakout rooms
4. Have 10 minutes
5. Reflect on the problem for a minute before moving into groups

Integration

Consider the conjecture: $\int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt = \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt$

Provide an example to the other members of your group of an f(t) and a g(t) such that
1) the conjecture is **not** satisfied.
2) the conjecture **is** satisfied (for non-zero f and g).

1) For example: $f = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $g = \begin{bmatrix} t \\ 0 \end{bmatrix}$

LHS $= 0$

RHS $\neq 0$

2) $f = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

LHS $= 0$, RHS $= 0$

Recitation 03	Today: Vector Functions (13.1, 13.2)

Consider the conjecture: $\int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt = \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt$

Provide an example to the other members of your group of an f(t) and a g(t) such that
1) the conjecture is **not** satisfied.
2) the conjecture **is** satisfied (for non-zero f and g).

---

1) f(t) = $\begin{bmatrix} t \\ 0 \end{bmatrix}$ g(t) = $\begin{bmatrix} 0 \\ t \end{bmatrix}$ (any orthogonal functions will do)

$LHS: \int_a^b \vec{f}(t)\cdot\vec{g}(t)\,dt = 0$

$RHS: \int_a^b \vec{f}(t)\,dt \cdot \int_a^b \vec{g}(t)\,dt = 0$

Announcements

- Survey: reminder sent last night, only 5 people responded as of yesterday …
- HW2: due Tues Feb 21 at 11:59 pm, covers sections 13.1 to 13.5
- HW1: due tonight, any questions related to the HW?
- Thursday: Graded Group Work: Question 1 from last years Quiz 1, group size 2 or 3

At what point does the twisted cubic
$r_1(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$,
intersect the plane $x + 2y + 3z = 34$?

Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

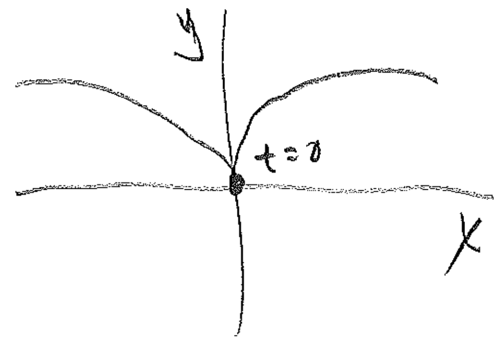
Sketching Vector Functions

Sketch $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j}$ $\Rightarrow$ Let $\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j} = t^3\,\hat{i} + t^2\,\hat{j}$

Ask students:

$x^3 = y^2$ implies $(t^3)^3 = (t^2)^2$, so $t = 0, \pm 1$.

Ask students: what is wrong? anything?



A better approach

let $x = t^3 \Rightarrow t = x^{1/3}$ (1)

let $y = t^2 \Rightarrow t = y^{1/2}$ (2)

equate (1) and (2) : $y^{1/2} = x^{1/3} \Rightarrow y = x^{2/3}$

DOMAIN / RANGE: can x be negative? can y be negative?

ANSWER: $x \in \mathbb{R}$, $y \in [0, \infty)$

ASK: WHAT IS THE TANGENT VECTOR AT $t=0$?

is $\vec{r}\,'(0) = 3t^2\,\hat{i} + 2t\,\hat{j}$?

No! $\vec{r}(0)$ is undefined.

Wolfram Alpha Syntax

WolframAlpha computational knowledge engine

plot x(t)=t^3, y(t) = t^2

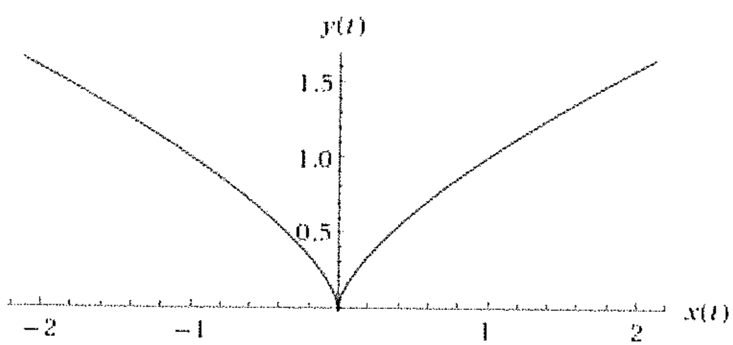
Examples Random

Input interpretation:

plot $x(t) = t^3$

$y(t) = t^2$

Parametric plot:



(t from −1.285 to 1.285)

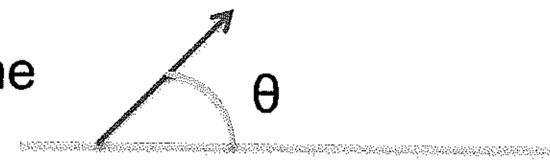
Enable interactivity

Projectile Motion (13.2)

A projectile is fired at angle θ with speed v_0.

a) derive its horizontal distance as a function of time

b) derive its maximum height



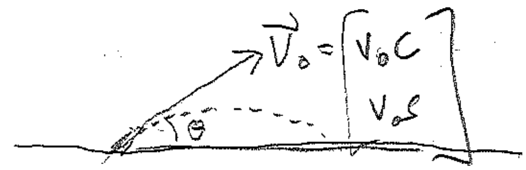
PROJECTILE PROBLEM | RECITATION 05 |

A projectile is fired at angle θ with speed v_0.

a) Find dist. travelled

b) Find ~~the~~ max. height.

(Assume: $\frac{d\vec{v}}{dt} = -g\hat{j}$.)



a)

Q What is $\vec{v}$?

A $\vec{v} = \underline{C_1}\hat{i} + \underline{(c_2 - gt)}\hat{j}$

but $\vec{v}(0) = \vec{v}_0$: $\vec{v} = \vec{v}_0 - gt\hat{j}$ (constant vector)

*I wonder if I need to give them $v'(t)$

If necessary: $\vec{v}_0 = \begin{bmatrix} v_x(0) \\ v_y(0) \end{bmatrix}$

THIS PROBLEM MIGHT BE TOO EASY: STUDENTS KNEW ANSWER FROM PHYSICS CLASS

Q what is $\vec{r}$?

A $\vec{r} = \vec{v}_0 t - \frac{gt^2}{2}\hat{j} + c_1\hat{i} + c_2\hat{j}$

$\vec{r}(0) = \vec{0}$ so $c_1 = c_2 = 0$

$\Rightarrow \boxed{\vec{r}(t) = \vec{V}_0 t - \frac{gt^2}{2}\hat{j}}$

θ = launch angle

R = "range"

$c = \cos\theta$

$s = \sin\theta$

V_0 = speed

Q what is $\vec{r}$ when object hits ground?

$\vec{r}_g = R\hat{i} = \boxed{V_0 ct\hat{i}} + (v_0 st - \frac{gt^2}{2})\hat{j}$

hits ground when $\hat{j}$ component is zero:

$\Rightarrow R = V_0 c t^*$, $V_0 s t^* = \frac{g t^{*2}}{2}$ or $t^* = 2v_0 gs$

long distance

$\Rightarrow t = \frac{2V_0 s}{g} \Rightarrow R = \frac{2V_0^2 cs}{g} = \frac{V_0^2 \sin 2\theta}{g}$

SHOULD ANSWER BE: $\frac{V_0^2 \sin 2\theta}{g}$?

b)

Q What is y-comp. of $\vec{v}$ at max height?

A 0

$\Rightarrow 0 = v_0 s - gt \Rightarrow t = \frac{v_0 \sin\theta}{g}$

$\Rightarrow$ use $\vec{r}(t)$ to get height:

y-component of $\vec{r} = v_0 st - \frac{gt^2}{2} = \frac{v_0^2 s^2}{g} - \frac{g\left(\frac{v_0 s}{g}\right)^2}{2}$

$= \frac{v_0^2 s^2}{2g}$

$s = \sin\theta$

Tangent Vectors

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

a) How is the unit tangent vector, $\mathbf{T}(t)$, defined mathematically?

b) Suppose $x = t^2$, $y = t^3$, $z = t^2$, and t is any real number. Then what is the unit tangent vector when $t = 0$?

a) $\vec{T} = \dfrac{d\vec{r}/dt}{\|\vec{r}\|}$

b) when $t \neq 0$, $\vec{T} = \dfrac{1}{\|\vec{r}\|}\begin{bmatrix} 2t \\ 3t^2 \\ 2t \end{bmatrix} = \dfrac{t}{\|r\|}\begin{bmatrix} 2 \\ 3t \\ 2 \end{bmatrix}$

$$= \frac{t}{\sqrt{8t^2+9t^4}}\begin{bmatrix} 2 \\ 3t \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{8+9t^2}}\begin{bmatrix} 2 \\ 3t \\ 2 \end{bmatrix}$$

$$\vec{T}(0) = \lim_{t \to 0} \vec{T}(t) = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

Position Perpendicular to Tangent

r(t) is the position of a moving particle. Show that ||**r**(t)|| = constant iff **r** ⊥ **r**'

if $\|\vec{r}\| = C$,

then $\|\vec{r}\|^2 = c^2$

$\Rightarrow \vec{r}\cdot\vec{r} = c^2$

$\Rightarrow 2\vec{r}\cdot\vec{r} = 0$ (took derivative)

$\Rightarrow \vec{r} \perp \vec{r}$

For converse, start with $\vec{r} \perp \vec{r}\,'$ and work backwards.

QH6 Recitation 04	Today: Tangents, Arc Length (13.3)

Let **r**(t) = x(t)**i** + y(t)**j** + z(t)**k**.

a) How is the unit tangent vector, **T**(t), defined mathematically?

b) Suppose x = t^2, y = t^3, z = t^2, and *t* is any real number. Then what is the unit tangent vector when t = 0?

a) $\vec{T} = \vec{r}\,'(t) / \|\vec{r}\,'(t)\|$

b) $\vec{r}\,'(t) = \begin{bmatrix} 2t \\ 3t^2 \\ 2t \end{bmatrix}$, $\|\vec{r}\,'(t)\| = \sqrt{8t^2 + 9t^4} = t\sqrt{8+9t^2}$

$\vec{T} = 0/0$, so simplify/factor:

l'Hospital on this is messy

$\vec{r}\,' = t \begin{bmatrix} 2 \\ 3t \\ 2 \end{bmatrix}$, $\vec{T} = \begin{bmatrix} 2 \\ 3t \\ 2 \end{bmatrix} \Big/ \sqrt{8+9t^2}$

$\vec{T}(0) = \frac{1}{\sqrt{8}} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

Announcements

- Quiz 1 is exactly 3 weeks away
- office hours, night before quiz
- HW2: Tue Feb 21 at 11:59 pm, sections 13.1-13.5 (hard?)
- Today: Graded Group Work: Question 1 from last years Quiz 1, group size 2 or 3

At what point does the twisted cubic

$r_1(t) = t\,i + t^2\,j + t^3\,k$,

intersect the plane $x + 2y + 3z = 34$?

Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

Group Work

1. Group size: 2 to 3 students
2. Someone is in your group when they write their initials on board
3. Students can create breakout rooms
4. Colors:
   a) Every student uses a different color
   b) Every student signs initials (or name) on board in their color
5. Only have 10 minutes
6. **Press SAVE button to submit your work**
7. What does the ERASE button do?

At what point does the twisted cubic

$r_1(t) = t\,i + t^2 j + t^3 k$,

intersect the plane $x + 2y + 3z = 34$?

Find their intersection and find the cosine of the angle between the tangent to the curve and the normal to this plane.

$\vec{r}'(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k}$

$\|\vec{r}'(2)\| = \sqrt{1+4^2+12^2} = \sqrt{161}$

intersect at: $t + 2t^2 + 3t^3 = 34$

plug-and-check: $t = 2$

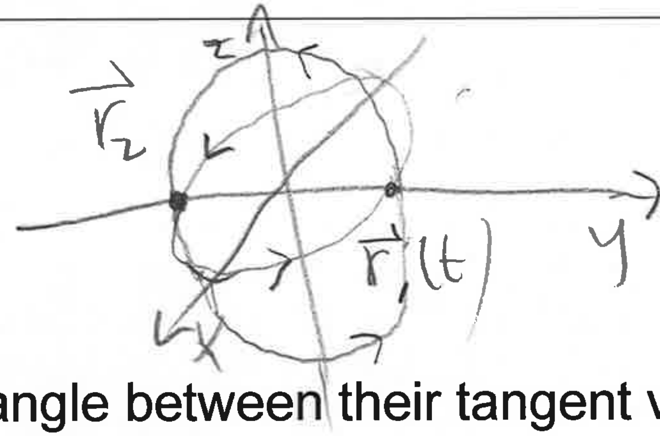
point is: $(2, 2^2, 2^3)$

angle is: $\theta = \arccos\left(\dfrac{\vec{r}'(2)}{\|\vec{r}'(2)\|} \cdot \dfrac{\vec{N}}{\|\vec{N}\|}\right)$

$\vec{r}'(2) \cdot \vec{N} = \begin{bmatrix}1\\4\\12\end{bmatrix} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = 1 + 8 + 36 = 45$

$\theta = \cos^{-1}\left(\dfrac{45}{\sqrt{161}\cdot\sqrt{14}}\right)$ ✓

Intersection Angle



$r_1(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$
$r_2(u) = \cos(u)\mathbf{j} + \sin(u)\mathbf{k}$

Find the point of intersection, and the angle between their tangent vectors at that point.

Intersect when $\vec{r}(t) = \vec{r}(u)$ for some value(s) of t, u.

By inspection, $t = \pi/2, \frac{3\pi}{2}$, so $\vec{r}_1 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$

$u = 0, \pi$, so $\vec{r}_2 = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$

$\Rightarrow$ POINT IS $(0, \pm 1, 0)$.

AT $(0, +1, 0)$

$\left.\begin{aligned} \vec{r}_1{}'\left(\tfrac{\pi}{2}\right) &= (-1)\hat{i} \\ \vec{r}_2{}'(0) &= \hat{k} \end{aligned}\right\} 90^\circ$

AT $(0, -1, 0)$

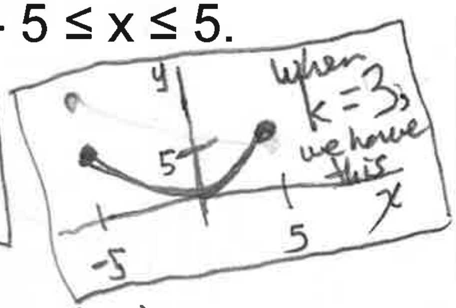
$\left.\begin{aligned} \vec{r}_1{}'(3\pi/2) &= -\hat{j} \\ \vec{r}_2{}'(\pi) &= \hat{k} \end{aligned}\right\} 90^\circ$

RECITATION 05

Arc Length

A cable is suspended between two poles that are 10 m apart. Find the length of the cable, if the cable's shape is y(x) = k [cosh(x/k) – 1], - 5 ≤ x ≤ 5.

$$y = k\left(\cosh\left(\tfrac{x}{k}\right) - 1\right) = k\left[\left(e^{x/k} + e^{-x/k}\right) - 1\right]$$



$$L = \int_{-5}^{+5} \sqrt{1 + (y')^2}\, dx, \quad y' = k \sinh\left(\tfrac{x}{k}\right)\frac{1}{k} - 0$$

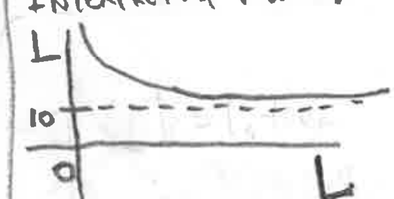
$$= \int_{-5}^{5} \sqrt{1 + \sinh^2 \tfrac{x}{k}}\, dx$$

$$= \int_{-5}^{5} \sqrt{\cosh^2 \tfrac{x}{k}}\, dx$$

$$= k \sinh\left(\tfrac{x}{k}\right)\Big|_{-5}^{+5} = \boxed{2k \sinh\left(\tfrac{5}{k}\right)}$$

$$= k\left[\sinh\left(\tfrac{5}{k}\right) - \sinh\left(\tfrac{-5}{k}\right)\right]$$

INTERPRETING RESULT:



HORIZ. ASYMPTOTE AT L=10, AS ONE WOULD EXPECT. k relates to the "stiffness" of cable.

R05

Arc Length (Quiz 1, Q2)

REC. 05

Find the arc length between 1 and t for the curve :

$$\mathbf{r}(s) = s\,\mathbf{i} + \left(2 - s^2\right)\mathbf{j} + \left(s^2 - 4\right)\mathbf{k}$$, $r' = i - 2sj + 2sk$

(Don't evaluate the integral)

$$L = \int_1^t \|r'\|\,ds$$

$$= \int_1^t \sqrt{1 + 4s^2 + 4s^2}\,ds$$

$$\boxed{L = \int_1^t \sqrt{1 + 8s^2}\,ds}$$ DONE.

If we wanted to go further: use trig subs, $\sin u = \sqrt{8}\,s$

ROS, QUIZ 1, QUESTION 3

$\vec{r} = \begin{bmatrix} 2t \\ t \\ 2t^2 \end{bmatrix}$, $\vec{V} = \begin{bmatrix} 2 \\ 1 \\ 4t \end{bmatrix}$, $|\vec{v}| = \sqrt{5+16t^2}$, $\frac{d}{dt}|v| = \frac{1}{2}\left(5+16t^2\right)^{-\frac{1}{2}} \cdot (32t)$

$= \frac{16t}{(5+16t^2)^{1/2}}$

$$\boxed{a_T = \frac{d}{dt}|v|} = \frac{16t}{(5+16t^2)^{-1/2}}$$

$$\boxed{a_N = K\left(|v|^2\right) = \frac{|\vec{V} \times \vec{a}|}{|v|}}$$

$\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $\vec{V} \times \vec{a} = \begin{vmatrix} i & j & k \\ 2 & 1 & 4t \\ 0 & 0 & 4 \end{vmatrix} = 4\hat{i} - 8\hat{j}$, $|v \times a| = \sqrt{80}$

$|\vec{v}| = \sqrt{5+16t^2}$

$$a_N = \frac{\sqrt{80}}{5+16t^2}$$

alternate formula

$a_N = \sqrt{|a|^2 - a_T^2}$

$= \sqrt{\sqrt{0^2+0^2+4^2}^2 - \frac{16^2t^2}{5+16t^2}}$

$= \sqrt{16 - \frac{16t^2}{5+16t^2}}$

$= \frac{1}{\sqrt{5+16t^2}}\sqrt{16(5+16t^2) - 16^2t^2} = \frac{\sqrt{80}}{}$

ROS

RO5

a) $r(-\frac{\pi}{2}) = [0, -1, -1]$

b) $T(-\frac{\pi}{2}) = \frac{v}{\|v\|}(-\frac{\pi}{2}) = [1, 0, 0]$

$V = \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}, \quad v(-\frac{\pi}{2}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |v| = 1$

$T = \begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}$

c) $N = \frac{dT/dt}{|dT/dt|} = \begin{bmatrix} -\cos t \\ -\sin t \\ 0 \end{bmatrix} / 1 = -\begin{bmatrix} c \\ s \\ 0 \end{bmatrix}$

$N(-\frac{\pi}{2}) = -\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

d) $B = T \times N = \begin{vmatrix} i & j & k \\ -s & c & 0 \\ -c & s & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ +s^2 + c^2 \end{bmatrix}$ (?)

$B(-\frac{\pi}{2}) = \begin{bmatrix} 0 \\ 0 \\ +1 \end{bmatrix}$

e) OSCULATING: $\vec{N} \times \vec{T} = \begin{vmatrix} -c & -s & 0 \\ -s & c & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c^2 - s^2 \end{bmatrix}$ ⇒ only has z-component → $0x + 0y + Cz = D$

⇒ $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$, $0x + 0y + (-1)z = D$

$r(-\frac{\pi}{2}) = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \Rightarrow D = 1$, plane is $z = -1$

NORMAL $\vec{B} \times \vec{N}$

RECTIFYING $\vec{B} \times \vec{T}$

RO5

Position Perpendicular to Tangent

r(t) is the position of a moving particle. Show that ||**r**(t)|| = constant for all t iff **r** ⊥ **r**' for all t.

if $\|\vec{r}\| = \text{constant}$

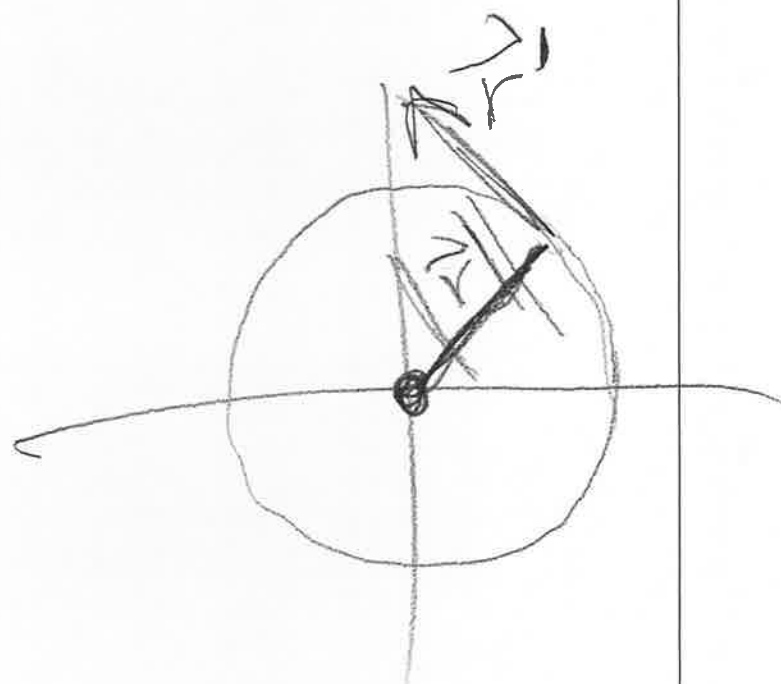
$\Rightarrow \|r\|^2 = c^2$

$\Rightarrow r \cdot r = c^2$

$\Rightarrow r' \cdot r + r \cdot r' = 0$

$\Rightarrow 2\vec{r} \cdot \vec{r}\,' = 0$

$\Rightarrow r \perp r'$



R05

$$a_T = \frac{d}{dt}|\mathbf{v}|$$

$$a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$

Write a in the form $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ at the given value of t without finding T and N.

$\mathbf{r}(t) = (e^t\sqrt{2})\mathbf{i} + (e^t\cos t)\mathbf{j} + (e^t\sin t)\mathbf{k}, \quad t = 0$

$c = \cos t$, $s = \sin t$

$$V = \sqrt{2}e^t\, i + \left[e^t(\cos t - \sin t)\right]\hat{j} + e^t(c+s)\hat{k},$$

$$a = \sqrt{2}e^t\, i + e^t\left((c-s)+(-s-c)\right)\hat{j} + e^t\left[(c+s)+(-s+c)\right]\hat{k}$$

$$= \sqrt{2}e^t\hat{i} + (-2s)e^t\hat{j} + 2ce^t\hat{k}$$

$$|v| = \left[2e^{2t} + e^{2t}(1 - \cancel{\sin 2t}) + e^{2t}(1 + \cancel{\sin 2t})\right]^{1/2}$$

$$\frac{d}{dt}|v| = \frac{1}{2}\left[\sim\sim\right]^{-1/2}\left(4e^{2t} + \cancel{4}e^{2t}\right)$$

$$\frac{d}{dt}|v|(0) = \frac{1}{2}\left[2+1+1\right]^{-1/2}(8) = 2$$

$|a|^2(0) = 2+4$, $a_N(0) = \sqrt{6-2^2}$ $\Rightarrow$ $\vec{a} = 2\vec{T} + \sqrt{2}\vec{N}$

Level Curves

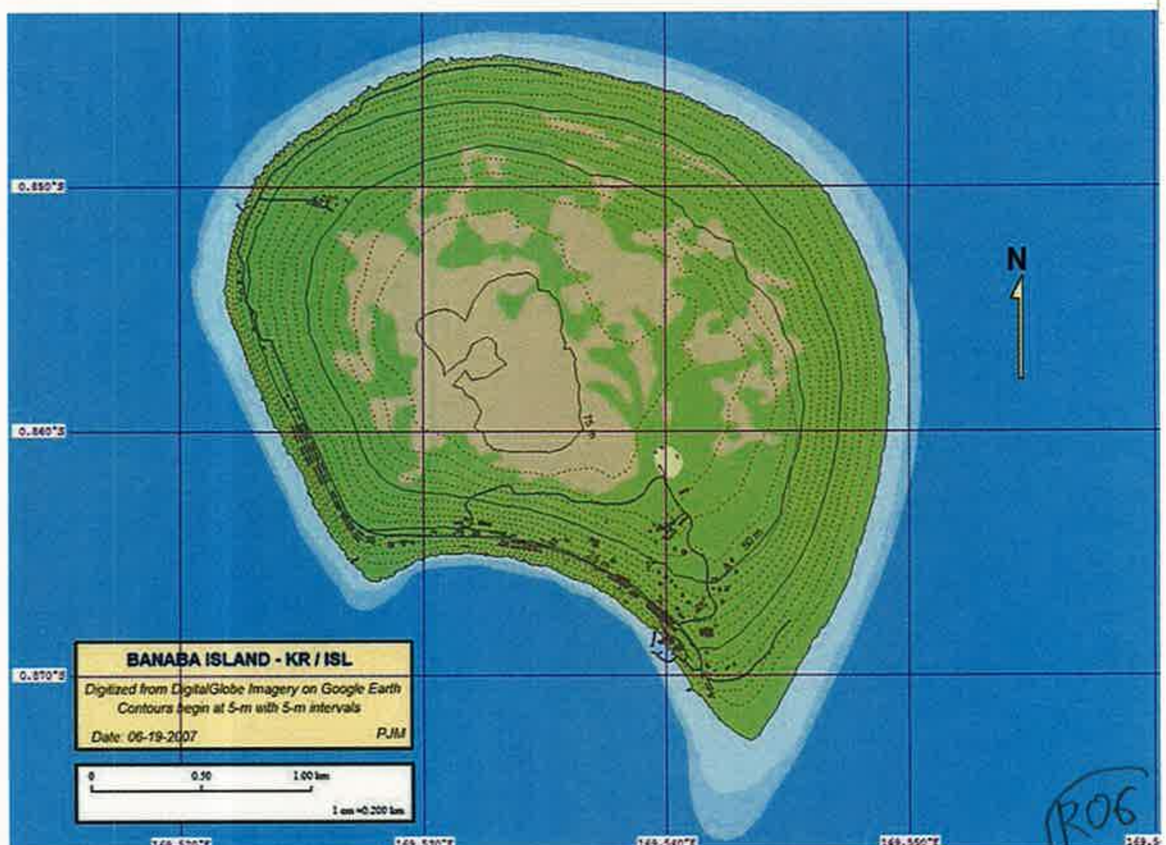
R06

1. The level curves of z = f(x,y) are the curves that satisfy the equation:

$z = f = \text{CONSTANT}$

2. The level curves in a topographic map describe:

CURVES OF CONSTANT ELEVATION

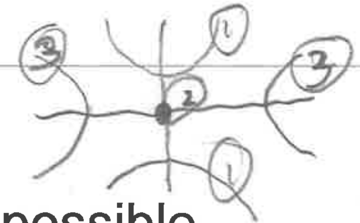


R06

Quadratic Surfaces

R07

examples: ① $-x^2+y^2=1$ ② $-x^2+y^2=0$ ③ $-x^2+y^2=-1$



Consider z = Ax² + By², A and B are constants. Describe all possible surfaces for the following cases.

i) A = B = 0

ii) AB > 0

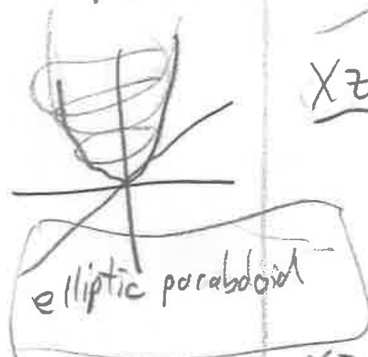
iii) AB < 0

i) $z=0$, xy-plane

ii) A, B have same sign

$Ax^2+By^2=C$, ellipse

A, B positive, $C>0$ ⟹

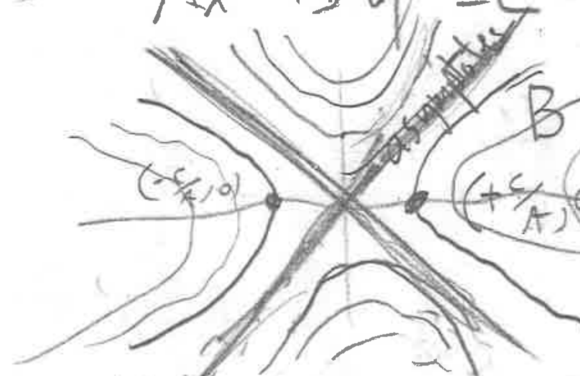


A, B negative, $C<0$

iii) A, B have different signs

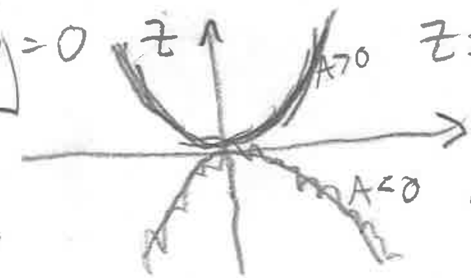
$Ax^2+By^2=C$ is a hyperbola

$B<0$, $A>0$, $C>0$



XZ PLANE

if $y=0$ $z=Ax^2$



YZ PLANE

$X=0$ $z=By^2$

hyperbolic paraboloid

R07

Case ii, AB > 0

WolframAlpha computational knowledge engine

plot z = -2*x^2 - 3*y^2

Examples Random

Input interpretation:

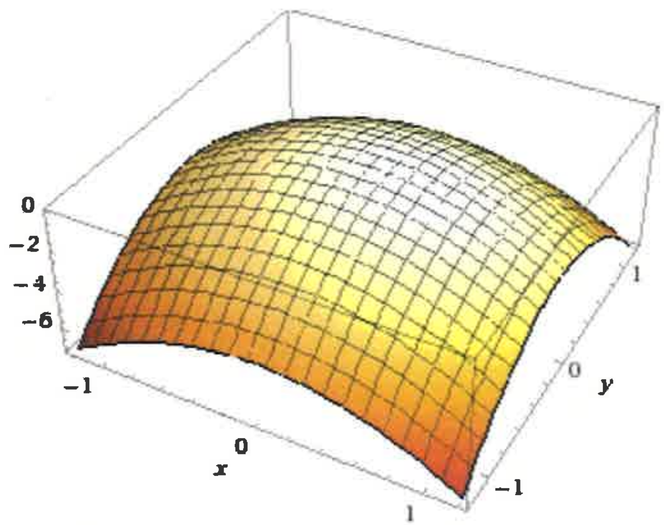
plot $z = -2x^2 - 3y^2$

Geometric figure:

elliptic paraboloid

3D plot:

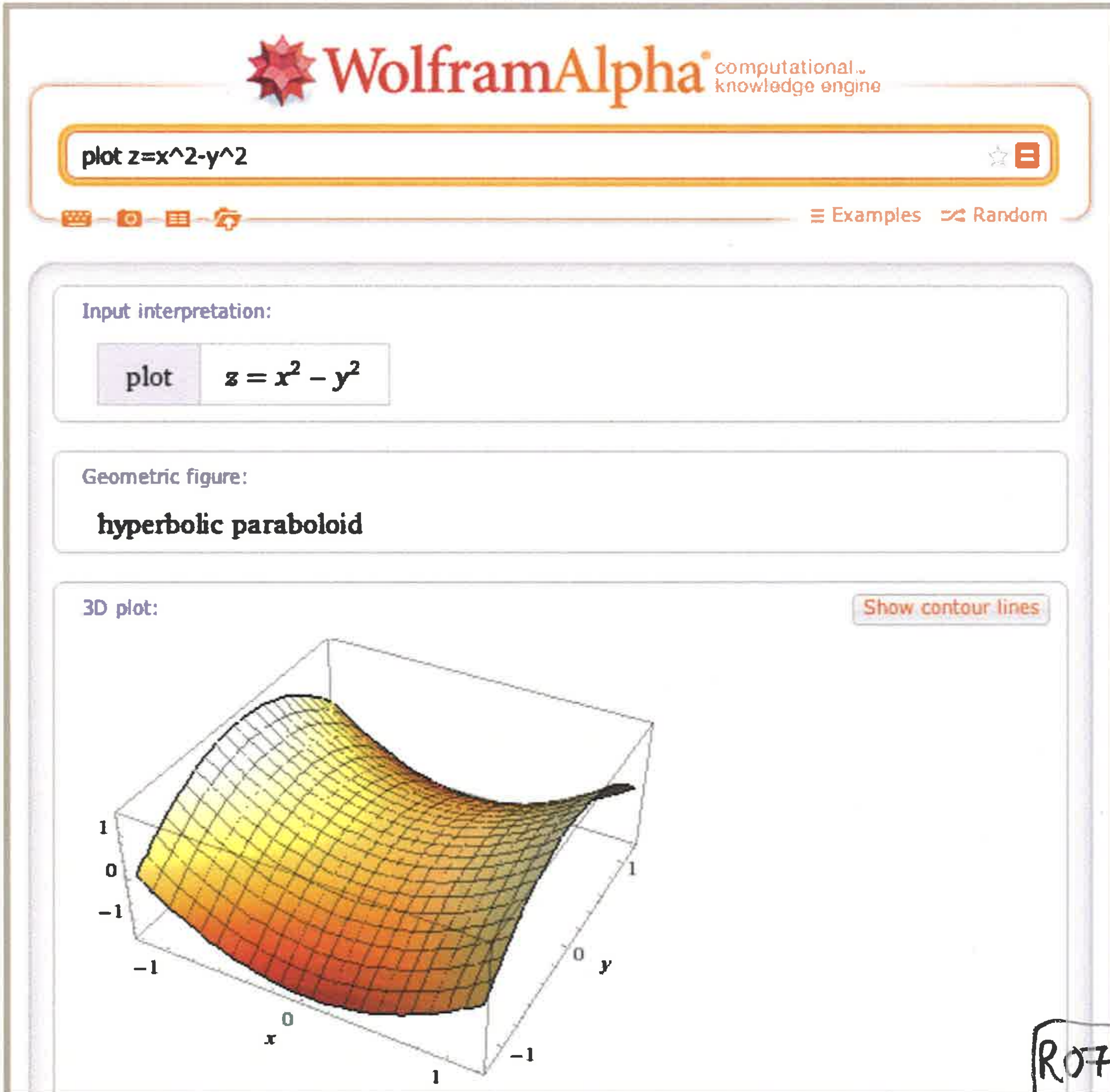
Show contour lines



Enable interactivity

R07

Case iii, AB < 0



R07

R07 5

SKETCH DOMAIN OF $g(x,y) = \frac{\sqrt{x+1}}{yx^2+xy^2}$

W a) Denom can't be zero.

Q what ~~equation~~ relation does that give us?

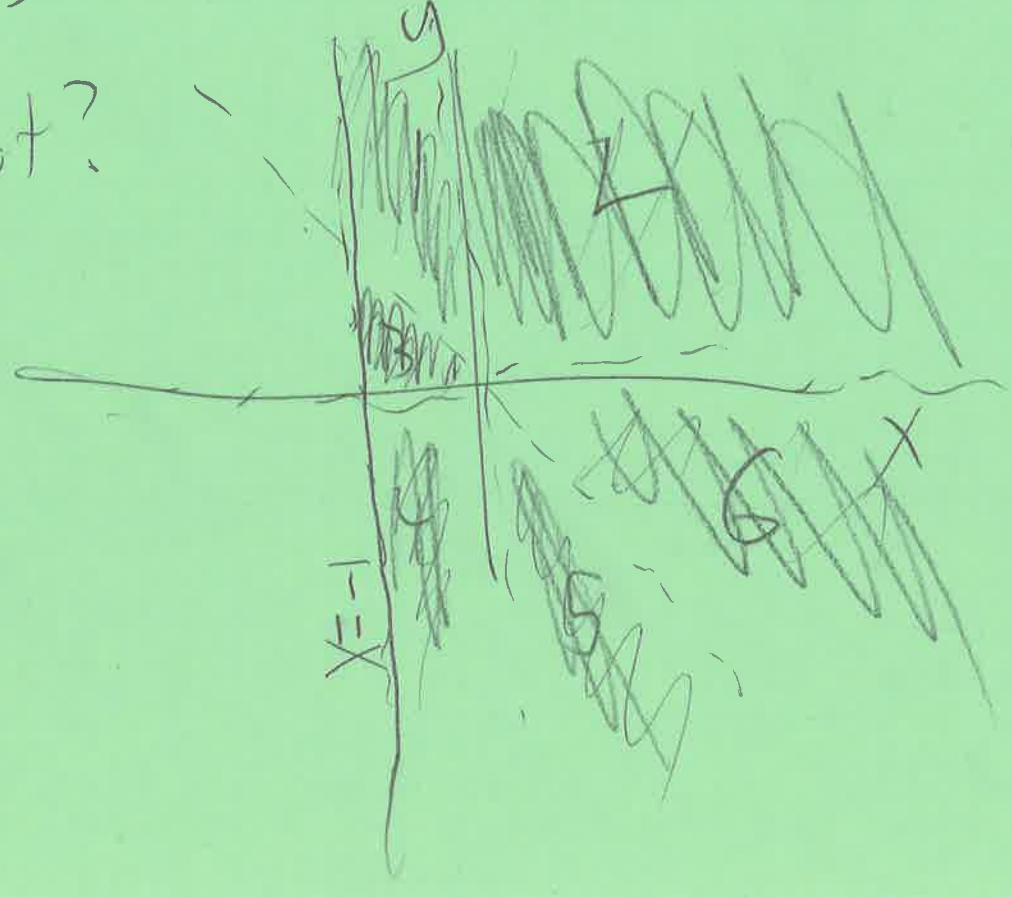
A $0 \neq yx(x+y) \Rightarrow$ $x \neq 0$, $y \neq 0$, $y \neq -x$

W) b) ~~When~~ Argument of SQRT must be positive.

Q what relation does that give us?

A $x \geq -1$

Q Plot?



R07

RECITATION 07

a) $$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Q what 3 tricks have we used?
- factoring
- l'Hospital
- Taylor

Q If we substitute limit point, we obtain?

A $0/0$, undefined

Q Can we apply l'Hopital's rule

A No - only in functions of one variable.

W let $y = m(x-1)$ (family of lines passing through limit point)

students tried $x = 1$ and found limit DNE

key

$$\lim_{x\to 1} \frac{x(x-1)^3 + m^2(x-1)^2}{4(x-1)^2 + 9m^3(x-1)^3}$$

$$= \lim_{x\to 1} \frac{x(x-1) + m^2}{4 + 9m^3(x-1)}$$

$$= \frac{m^2}{4}$$

$\Rightarrow$ DNE because value of limit depends on path of approach

b) find $F = \nabla f = (x + \sin y)\hat{i} + (x\cos y - 2y)\hat{j}$. Find f.

$f = \frac{x^2}{2} + x\sin y + C(y)$

$f_y = x\cos y + C'(y)$, $C' = -2y \Rightarrow C = -y^2$

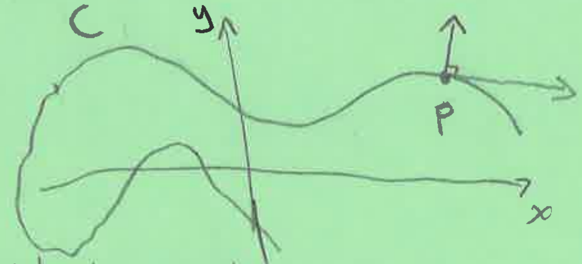
$\Rightarrow f = \frac{x^2}{2} + x\sin y - y^2$.

R07

~~R09~~ 3
R10, "O"

$g(x,y) = K$. Show $\nabla g \perp C$.

WRITE



$\vec{T} = \frac{\vec{r}'}{\|r'\|}$

Q What vector would ∇g be $\perp$ to at P?

A Tangent!

W Let $\vec{r} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ be a parameterization of curve C. Then, $\vec{T} = \begin{bmatrix} x' \\ y' \end{bmatrix} / \|r'\|$

$g = g(x(t), y(t)) = K$ (given)

$\frac{dg}{dt} = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = 0$ (chain rule (for function of two var's))

$= \left[\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right] \cdot \left[\frac{dx}{dt}, \frac{dy}{dt}\right]$

$= \nabla g \cdot \vec{r}\,'(t) = 0$

$\Rightarrow \boxed{\nabla g \perp \vec{T}}$

WRITE $g(x,y) = K$ is ~~the~~ ~~a~~ set of level curves!

R07

QH6 Recitation 08 | Quiz Review, Gradients

Having trouble with your audio?
- make sure speakers are not muted
- navigate to Meeting >> Audio Setup Wizard

Other issues?
- navigate to Help >>Troubleshooting
- see Quick Start Guide (PDF)

Let **F** = ∇f = (x+sin(y))**i** + (xcos(y)-2y)**j**. Find f(x,y).

We know that:

$$\frac{\partial f}{\partial x} = \_\_\_\_\_\_\_\_$$ ← (get students to tell you what this is)

Therefore, by integration,

$$f(x,y) = \frac{x^2}{2} + x\sin(y) + \_\_\_\_\_\_$$

To find h(y), diff w.r.t. y:

$$\frac{\partial f}{\partial y} = 0 + x\cos y + h'(y) \Rightarrow h'(y) = 2y$$

$$h = y^2$$

$$\Rightarrow f(x,y) = \underline{\frac{x^2}{2} + x\sin y + y^2}$$

ASK STUDENTS:
- could constant of integration be a function of y?
-

Quiz 1

As announced on Friday

- Covers HW1,2,3 + additional problems
- 2 sheet of 8 1/2 x 11 motes (both sides)
- Calculators allowed

Office Hours

- In Adobe Connect at https://georgiatech.adobeconnect.com/distancecalculusofficehours/
- Tuesday and Wednesday 8:00 pm to 9:30 pm

Prepare

- Solve HWs on MyMathLab
- Practice Quiz

During Quiz

- I'll be in Adobe Connect https://georgiatech.adobeconnect.com/distancecalculusquiz/
- Grady HS students: Klaus 2447

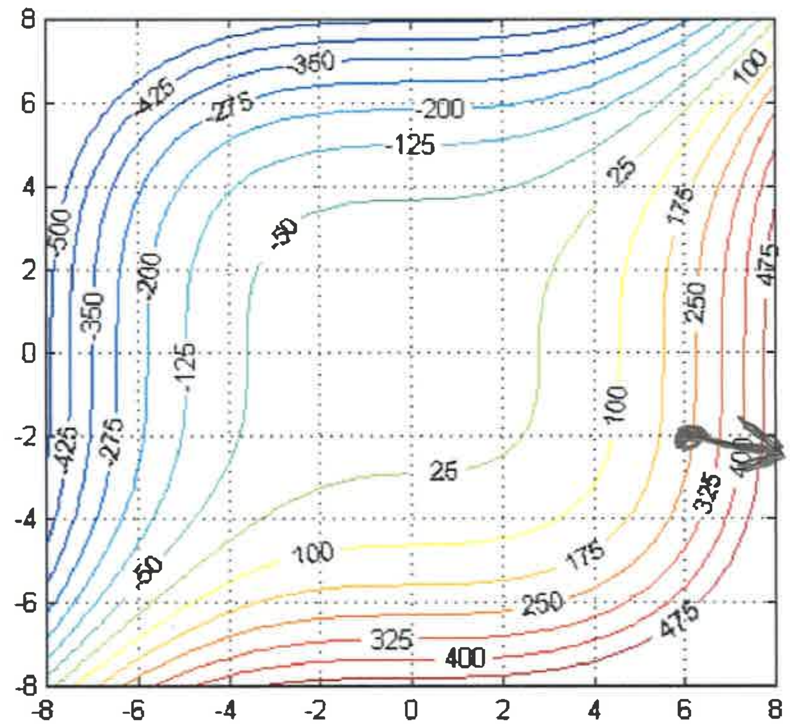
Do You Have Any Questions?

R08

Gradient and Level Curves

At which point will the gradient vector have the largest magnitude?

a) (0,2)
b) (-4,-4)
c) (0,0)
d) (6,-2)



Explain why, and sketch the gradient at that point.

Because $|\nabla f| = \sqrt{f_x^2 + f_y^2}$ and (d) is where lines are most "dense"

Vector is in direction of steepest ascent and is perpendicular to level curve

R08

Tangent Plane

"the surface" f is the set of all points that satisfy $3xy - x^3 - y^3 - z$

For $z = 3xy - x^3 - y^3$, find an equation for the tangent plane at $(1,1)$ and determine where the tangent plane is horizontal. What do those points represent?

Tangent plane given by (normal vector) $\cdot$ (vector in plane) $= 0$.

The vector normal to the surface is in $\mathbb{R}^3$.

$0 = 3xy - x^3 - y^3 - z$, LET $f(x,y,z) = 3xy - x^3 - y^3 - z = 0$

The normal vector is parallel to $\nabla f = f_x \hat{i} + f_y \hat{j} + f_z \hat{k}$.

$$\Rightarrow \nabla f = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \\ -1 \end{bmatrix}, \quad \nabla f(1,1,1) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

At $(1,1)$, $z = 1$, plane is $\nabla f(1,1,1) \cdot \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = 0 \Rightarrow z = 1$ is tangent plane.

Tangent plane is horizontal when $f_x = f_y = 0$:

$$\left.\begin{aligned} f_x &= 3y - 3x^2 = 0 \\ f_y &= 3x - 3y^2 = 0 \end{aligned}\right\} \begin{aligned} y &= x^2 \\ x &= y^2 \end{aligned}$$

, both satisfied at $(0,0)$, $(1,1)$

These point can indicate local max/min

R08

Wolfram Alpha

WolframAlpha computational knowledge engine

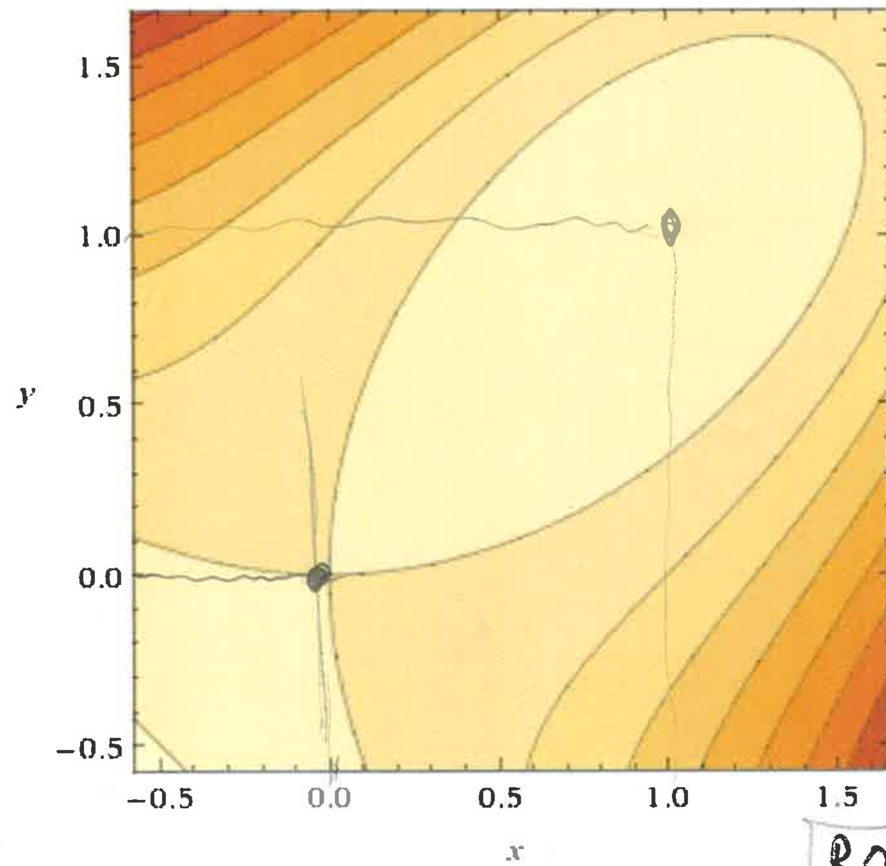
plot z(x,y) = 3xy −x^3 − y^3

Examples Random

Input interpretation:

plot $z(x, y) = 3xy - x^3 - y^3$

Contour plot:



points where $\nabla f = \vec{0}$ can indicate local minima/maxima!

ROB

Directional Derivative

Find the directional derivative of f = z ln(x/y) at (1,1,2) toward the point (2,2,1) and state what it represents.

$$\vec{\nabla} f(x,y,z) = z\left(\frac{1}{x/y}\right)\frac{d}{dx}\left(\frac{x}{y}\right)\hat{i} + z\left(\frac{1}{x/y}\right)\frac{d}{dy}\left(\frac{x}{y}\right)\hat{j} + \ln\left(\frac{x}{y}\right)\hat{k}$$

$$= \frac{z}{x}\hat{i} - \frac{z}{y}\hat{j} + \ln\left(\frac{x}{y}\right)\hat{k}$$

$$\nabla f(1,1,2) = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$\nabla f \cdot \begin{bmatrix} 1-2 \\ 1-2 \\ 2-1 \end{bmatrix} = 0 \Rightarrow \nabla f \text{ is } \perp \text{ to } \vec{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Rate at which function (surface) is increasing in that direction.

$\Rightarrow$ $\vec{u}$ is in tangent plane! Because ∇f is normal to surface f

R08

Mixed Partial Derivatives

f(x,y) is a function with continuous 1st and 2nd partial derivatives on D, and $f_{xy}(x,y) = 0$ everywhere on D.

a) What can we say about f(x,y) on D?
b) Provide two functions that have this property.

If $\frac{\partial^2 f}{\partial x \partial y} = 0$, then,

Int wrt y: $\frac{\partial f}{\partial x} = F(x) \leftarrow$ function of x only

Int wrt x: $f(x,y) = \tilde{F}(x) + \tilde{G}(y)$

a) functions of this form are expressed as a sum of univariate functions.

b) $f = Ax + By$, or $f = Ax^2 + By^2$

RaS

Gravitation

What is the formula that describes Newton's Law of Gravitation~~?~~ in $\mathbb{R}^3$

F = $GmM/r^2 = GmM/(x^2+y^2+z^2)$

DESCRIBE
a) ~~Sketch~~ the level surfaces
b) State, in words, what the surfaces describe

a) $x^2+y^2+z^2 = K$

SPHERES

b) regions of constant grav. force.

ROB

Tangent Plane

Show that, for all tangent planes to the given surface, the sum of their intercepts is the same.

$$\text{surface: } \sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$$

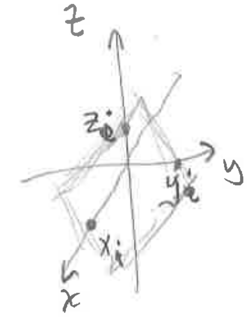
(SOLUTION ON NEXT PAGE)

R08

TANGENT PLANE

SHOW: Sum of intercepts of all tangent planes of surface is the same.

SURFACE: $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{a}$ (1)



Q What information regarding/related to this plane do we need?

A Intercepts.

Q How many intercepts are there?

A 3

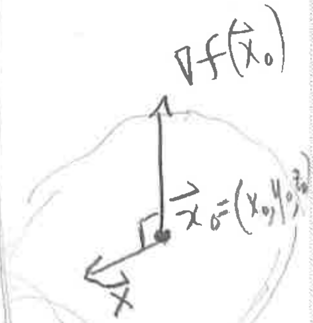
S Lets plot them & give them names: x_i, y_i, z_i

W NEED: $x_i + y_i + z_i =$ constant

S What equation do we need to find x_i, y_i, z_i?

A Equation of tangent plane:

$$\nabla f(\vec{x}_0) \cdot \underbrace{(\vec{x} - \vec{x}_0)}_{\text{vector in plane}} = 0$$



Q why is the dot product zero?

A $\perp$

$$\Rightarrow \underbrace{f_x}_{=1/\sqrt{x_0}}(x - x_0) + \underbrace{f_y}_{=1/\sqrt{y_0}}(y - y_0) + \underbrace{f_z}_{=1/\sqrt{z_0}}(z - z_0) = 0$$

$\vec{x}$ = vector in plane

Find x_i: $\frac{x_i - x_0}{\sqrt{x_0}} = \frac{y_0}{\sqrt{y_0}} + \frac{z_0}{\sqrt{z_0}}$ (at $x_i, y = z = 0$)

Solve for x_i: $x_i = \sqrt{x_0}(\sqrt{y_0} + \sqrt{z_0}) + x_0 = \sqrt{x_0}\sqrt{a}$

Similar process $y_i = \sqrt{y_0}\sqrt{a}$

$z_i = \sqrt{z_0}\sqrt{a}$

$\Rightarrow (\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0})\sqrt{a} = \sqrt{a}\sqrt{a}$ by (1)

R08

QH6 Recitation 09	Quadratic Surfaces, Extreme Values

Having trouble with your audio?
- make sure speakers are not muted
- navigate to Meeting >> Audio Setup Wizard

Other issues?
- navigate to Help >>Troubleshooting
- see Quick Start Guide (PDF)

The strength of an electric field at a point due to an infinitely long wire along the *y*-axis is given by

$$E(x,y,z)=\frac{k}{\sqrt{x^2+z^2}}$$

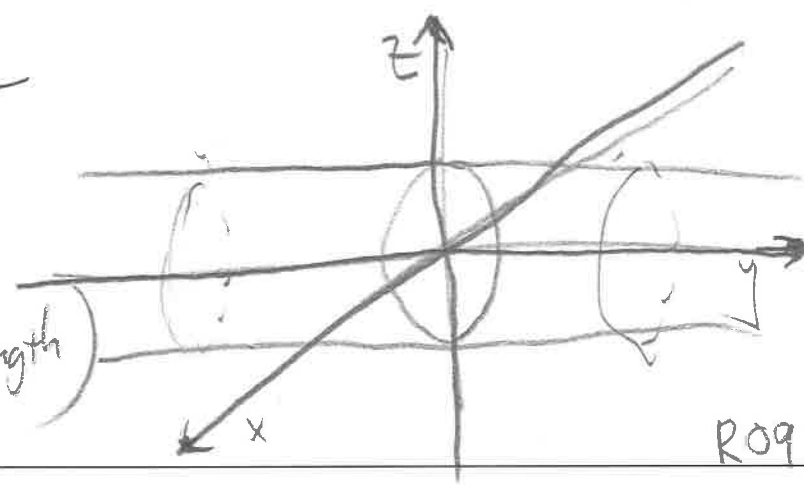
Describe, in words, the level surfaces of *E*. What do they represent?

Level surfaces given by $\frac{k}{\sqrt{x^2+z^2}} = \underline{\text{constant}}$

Rearrange: $\underline{x^2+z^2 = K_2^2}$

This is a <u>circular cylinder</u>

(Represent surfaces of constant field strength)



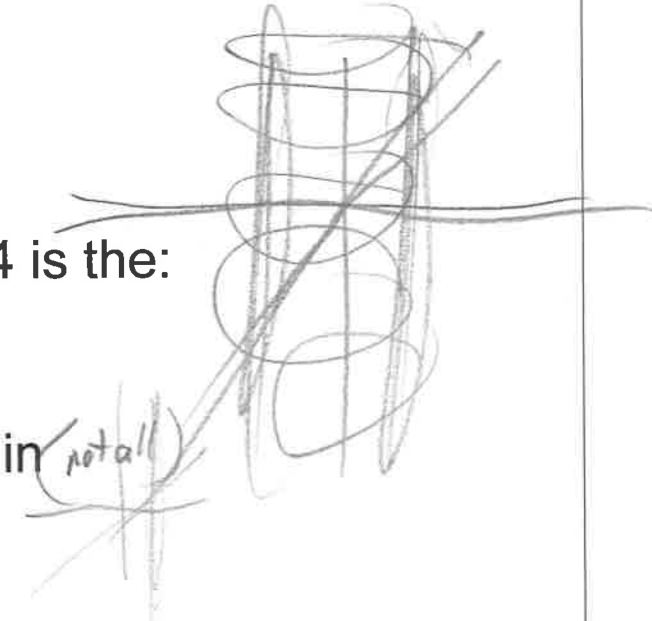
R09

Quadratic Surfaces

Circle the correct answer.

The set of all points whose distance from the z-axis is 4 is the:

a) sphere of radius 4 centered on the z-axis (origin)
b) line parallel to the z-axis 4 units away from the origin (not all)
c) cylinder of radius 4 centered on the z-axis
d) plane $z = 4$



C

Let $P(x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$.

The equation that represents the distance between P and the z-axis is

$$D = \|P - (0,0,z)\| = \sqrt{(x_0-0)^2 + (y_0-0)^2 + (z_0-z)^2}$$

But for the distance between P and the z-axis, $z = z_0$

$D = 4$, so $\sqrt{x_0^2 + y_0^2} = 4$, cylinder!

RDA

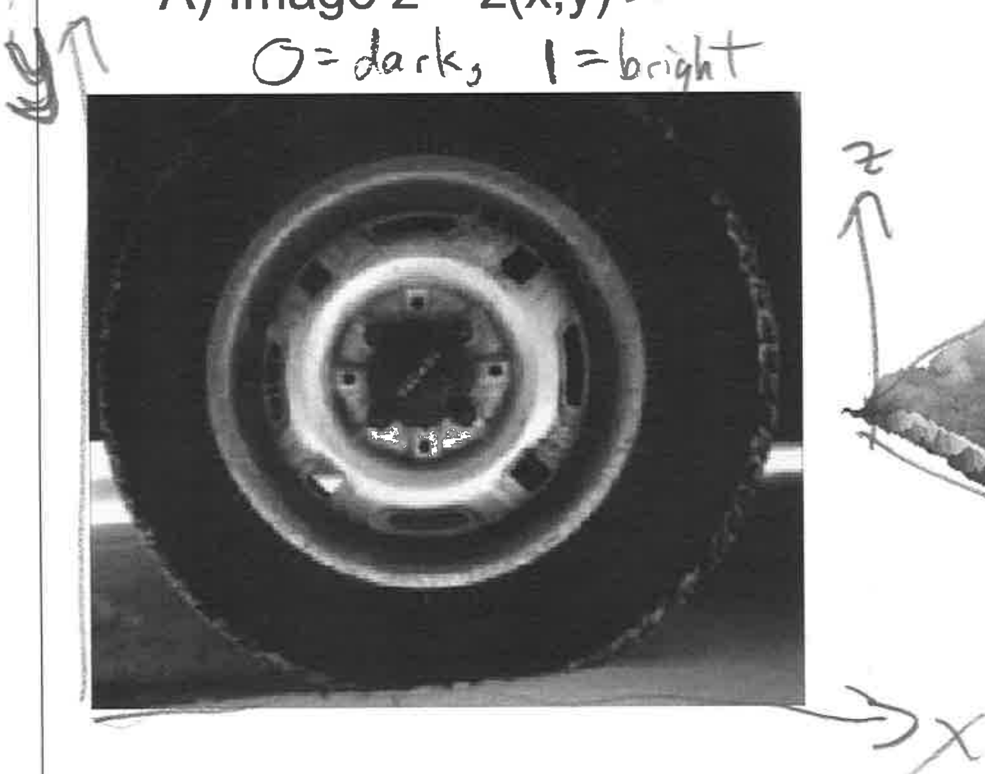
Announcements

- Next HW should be posted today
- Quiz 1 marked on Friday or next Tuesday
- What did you think of Quiz 1?

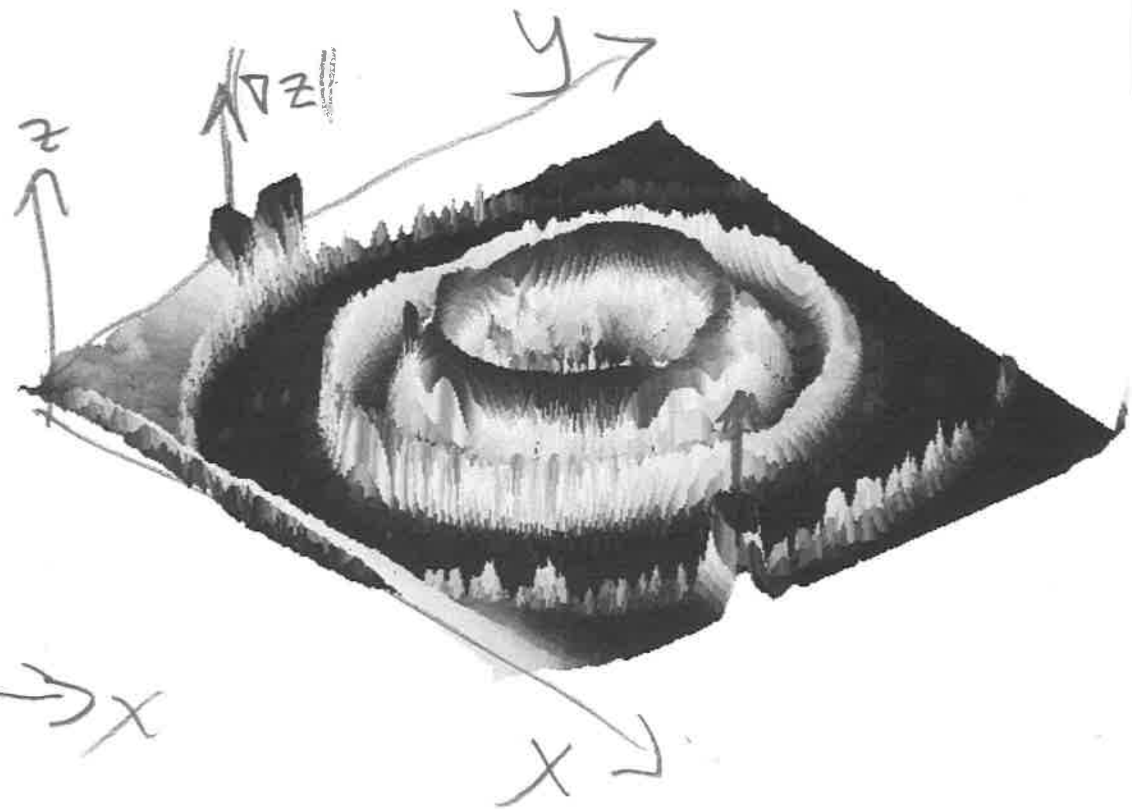
R09

The Gradient and Local Maxima

A) Image z = z(x,y) =
0 = dark, 1 = bright



B) Surface Plot of z(x,y)



1) Place a dot on image (A) that could correspond to a local maximum.
2) What characteristics does the gradient vector have at local maxima?

$f_x = f_y = 0$, Vector points up/down
x and y components of ∇z are zero.

R09

Stationary Points

Find and describe the stationary points of f(x,y) = y + x sin(y).

To find stationary points, first calculate ∇f, set to zero.

$$\nabla f(x,y) = \begin{bmatrix} \sin y \\ 1 + x\cos y \end{bmatrix} = \vec{0},$$

Solve: $\sin y = 0 \Rightarrow y = n\pi,\ n \in \mathbb{Z}$

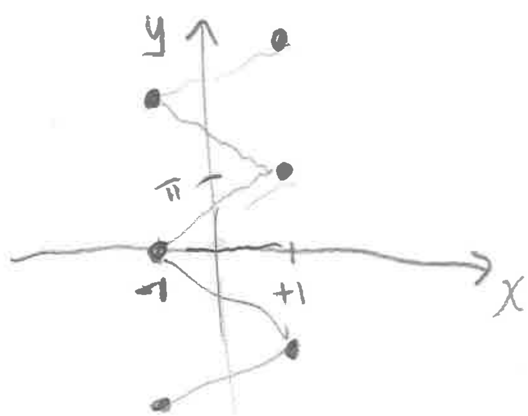
$1 + x\cos y = 0 \Rightarrow$

$1 + x\cos(n\pi) = 0$

$1 + x(-1)^n = 0$

$x = \pm 1$

n	x	y
0	-1	0
$+1$	$+1$	π
$+2$	-1	2π



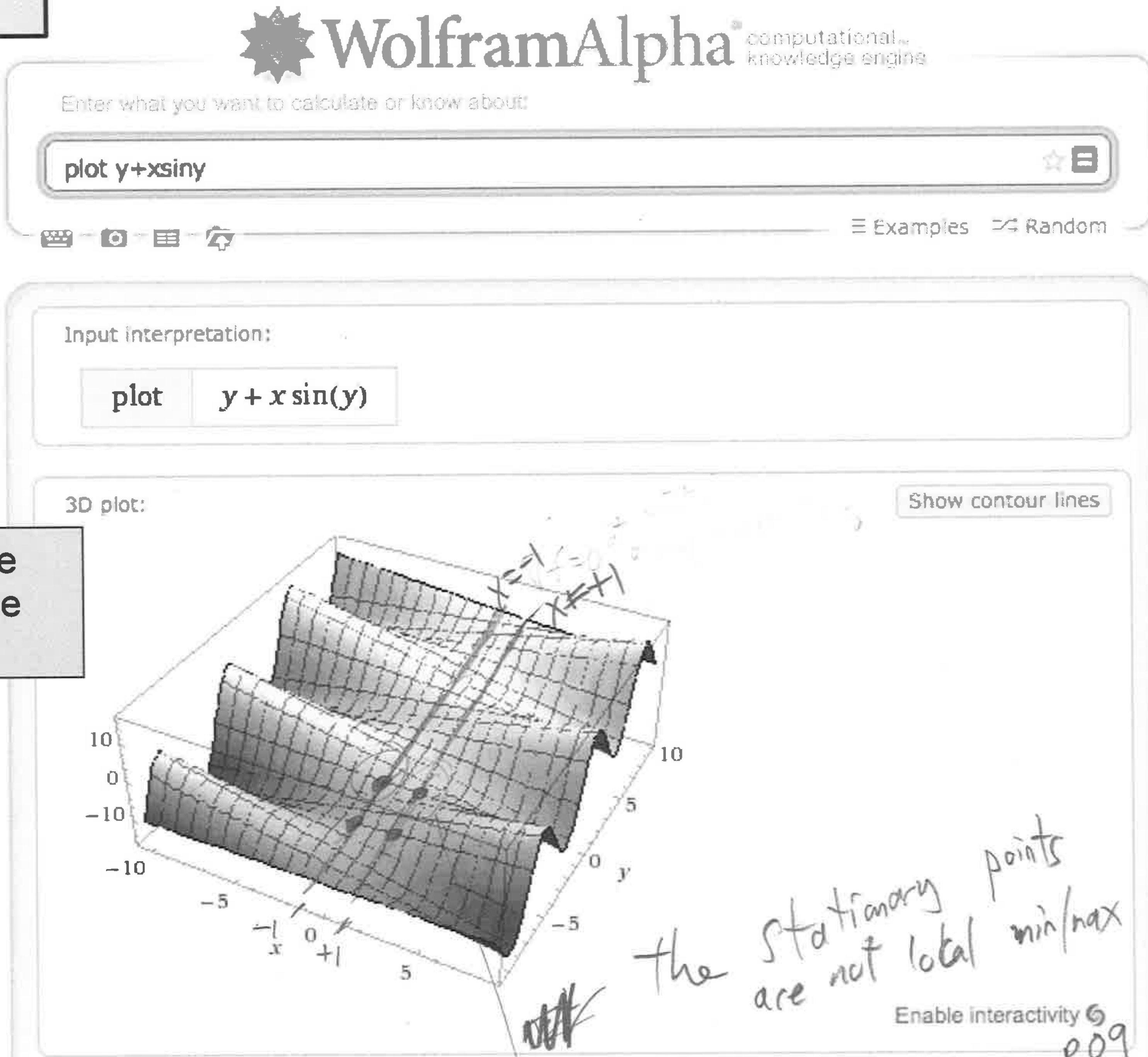
STATIONARY POINTS AT $(-1, 2n\pi)$, $(+1, 2(n+1)\pi)$

ARE THEY LOCAL MIN/MAX?

$$D = f_{xx}f_{yy} - (f_{xy})^2 = 0 \cdot 0 - (\cos y)^2 < 0 \Rightarrow \text{saddles!}$$

RQ

WolframAlpha



Place dots on the 3D plot where the gradient is zero.

the stationary points are not local min/max

R09

Second Partials Test

Let $f(x,y) = x^2 + kxy + y^2$, $\boxed{k \text{ is a constant}}$

a) Where are the stationary points?

b) For what values of k will f have a saddle at the origin?

c) For what values of k will f have a local min at the origin?

d) For what values of k is the second partials test inconclusive?

a) $\nabla f = \begin{bmatrix} 2x+ky \\ 2y+kx \end{bmatrix} = 0 \Rightarrow \begin{cases} ky = -2x, & x = -\frac{k}{2}y \\ kx = -2y, & k\left(\frac{-k}{2}y\right) = -2y \Rightarrow k = \pm 2 \end{cases}$

$\boxed{\text{if } x = y = 0, \text{ SP.}}$

$\boxed{\text{if } k = 2, \text{ SP along } y = -x.}$

$\Rightarrow$ stationary points along $y = \pm x$ when $k = \pm 2$.

b) Saddles when $D = f_{xx}f_{yy} - f_{xy}^2 < 0$

$f_{xx} = f_{yy} = 2$, $f_{xy} = k$, so $D = 4 - k^2$, so saddle at origin if $|k| > 2$

c) Local min if $D > 0$ and $f_{xx} > 0$, so $|k| < 2$

d) inconclusive if $D = 0$, or $k = \pm 2$

RCA

$T = xy$

RO9 27

ALTERNATIVE

$T = xy$
$= r\cos\theta \, r\sin\theta$
$T(\theta) = \cos\theta\sin\theta$
$T' = \cos(2t)$
etc

Q Draw contour of T:

circle: $g = x^2+y^2-1 = 0$

Q Which contour lines intersects circle at one point?

Q Y/N: is this the hottest point?

$\nabla T \parallel$ normal to circle

$\Rightarrow \nabla T = \lambda \nabla(x^2+y^2-1)$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$\Rightarrow \lambda = \frac{y}{2x} = \frac{x}{2y} \Rightarrow y^2 = x^2$

$\Rightarrow$ four points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$

test all four, coldest on line passing through $(-t, t)$, $(t, -t)$, $t = \frac{1}{\sqrt{2}}$

hottest region is lines passing $(+,+)$, $(-,-)$

Let's try group work in Adobe Connect

- You'll solve the question that we started at the end of Tuesday's recitation
- Three breakout rooms
- Everyone randomly assigned to a room
- Not graded
- You'll have 10 to 15 minutes
- I'll circulate between rooms

I suggest starting by discussing a solution strategy with the other people in your group using a mic and/or text chat.

Second Partials Test

Let $f(x,y) = x^2 + kxy + y^2$. k is a constant

a) Where are the stationary points?

b) For what values of k will f have a saddle at the origin?

c) For what values of k will f have a local min at the origin?

d) For what values of k is the second partials test inconclusive?

a) $\nabla f = \begin{bmatrix} 2x+ky \\ 2y+kx \end{bmatrix} = 0 \Rightarrow \begin{cases} ky=-2x, \; x=-\frac{k}{2}y \\ kx=-2y, \; k(\frac{-k}{2}y)=-2y \end{cases} \Rightarrow k=\pm 2$

$\boxed{\text{if } x=y=0, \text{ SP.}}$

$\boxed{\text{if } k=2, \text{ SP along } y=-x.}$

$\Rightarrow$ stationary points along $y=\pm x$ when $k=\pm 2$.

b) Saddles when $D = f_{xx}f_{yy} - f_{xy}^2 < 0$

$f_{xx}=f_{yy}=2$, $f_{xy}=k$, so $D=4-k^2$, so saddle at origin if $|k|>2$

c) Local min if $D>0$ and $f_{xx}>0$, so $|k|<2$

d) inconclusive if $D=0$, or $k=\pm 2$

R10

Part a)

WolframAlpha computational.. knowledge engine

plot x^2 + y^2 + 2xy

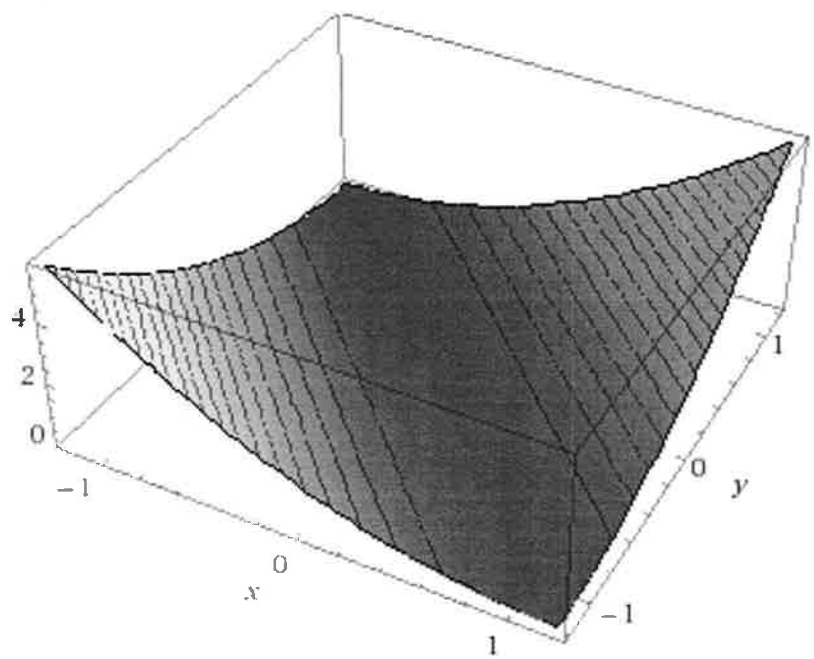
Examples Random

Input interpretation:

plot	$(x^2 + y^2) + 2\,x\,y$

3D plot:

Show mesh



Part b)

WolframAlpha computational... knowledge engine

plot x^2 + y^2 - 8xy

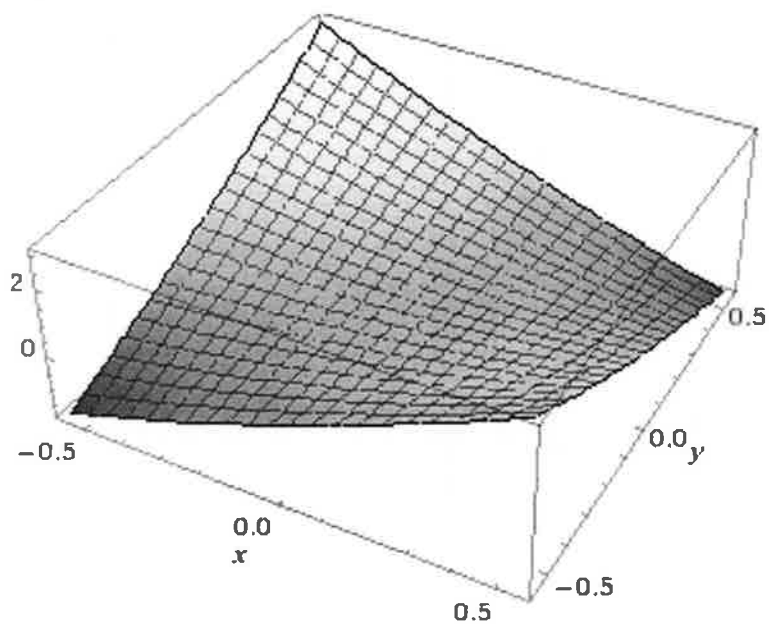
Examples Random

Input interpretation:

plot	$(x^2 + y^2) - 8\,x\,y$

3D plot:

Show contour lines



Enable interactivity

R10 2/2

$T = xy$

ALTERNATIVE

$T = xy$

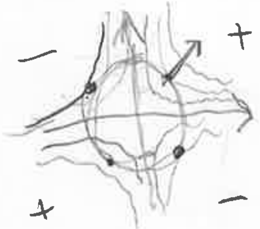
$= r\cos\theta \, r\sin\theta$

$T(\theta) = \cos\theta \sin\theta$

$T' = \cos(2t)$

etc

Q Draw contour of T:



circle: $g = x^2 + y^2 - 1 = 0$

Q which contour lines intersects circle at one point?

Q Y/N: is this the hottest point?

$\nabla T \parallel$ normal to circle

$\Rightarrow \nabla T = \lambda \nabla(x^2 + y^2 - 1)$

$\begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$

$\Rightarrow \lambda = \frac{y}{2x} = \frac{x}{2y} \Rightarrow y^2 = x^2$

$\Rightarrow$ four points $\left(\pm \frac{1}{\sqrt{2}}, \pm \frac{1}{\sqrt{2}}\right)$

test all four, coldest on line passing through $(-t, t), (t, -t)$, $t = \frac{1}{\sqrt{2}}$

hottest region is lines passing $(+,+)$, $(-,-)$

QH6 Recitation 11	Taylor Approximations, Integration

- Pop quiz grading
  - Correct 5 points
  - Name on page 3 points
  - Did not take: 0 points.
- Time: 15 minutes
- To submit your work, either
  a) work on whiteboard in breakout room
     - press the **SAVE** button
  b) work on paper and give work to facilitator
     - leave 2 inch margin
     - write your name and QH6 at the top
     - facilitator can email quiz to cdlops@pe.gatech.edu
  c) work on paper and take a photo of your work
     - email your photo to me before 8:30
     - write in text chat that you are emailing your work to me

R11

Pop Quiz

LET $c = \cos y$, $s = \sin y$

Find the cubic approximation of f(x,y) = 4x cos(y) near the origin.

DERIVATIVE	EVALUATE AT (0,0)
$f_x = 4c$	4
$f_y = -4xs$	0
$f_{xx} = 0$	0
$f_{xy} = -4s$	0
$f_{yy} = -4xc$	0
$f_{xxx} = 0$	0
$f_{xxy} = 0$	0
$f_{xyy} = -4c$	-4
$f_{yyy} = 4xs$	0

The cubic "term" has the form:

$\frac{1}{3!}\left(f_{xxx}(0,0) + 3f_{xxy}(0,0) + 3f_{xyy}(0,0) + f_{yyy}(0,0)\right)$

$$\Rightarrow f \approx 0 + \frac{1}{1!}(4x + 0y) + \frac{1}{2!}(0x^2 + 0xy + 0y^2) + \underbrace{\frac{1}{3!}\left(0x^3 + 3\cdot 0x^2y - 3\cdot 4xy^2 + 0y^3\right)}$$

$$= 4x - 2xy^2$$

RJ

Announcements

- **HW 5**
  - due tonight at 11:59 pm
  - seven questions on Taylor approximations from Section 14.9
- **HW 6**
  - fifteen questions on integration from Section 15.2 and 15.3
  - due Thursday at 11:59 pm
- **Quiz 2:** Tuesday March 4

How would you like to spend the rest of the recitaiton? Text your preference.

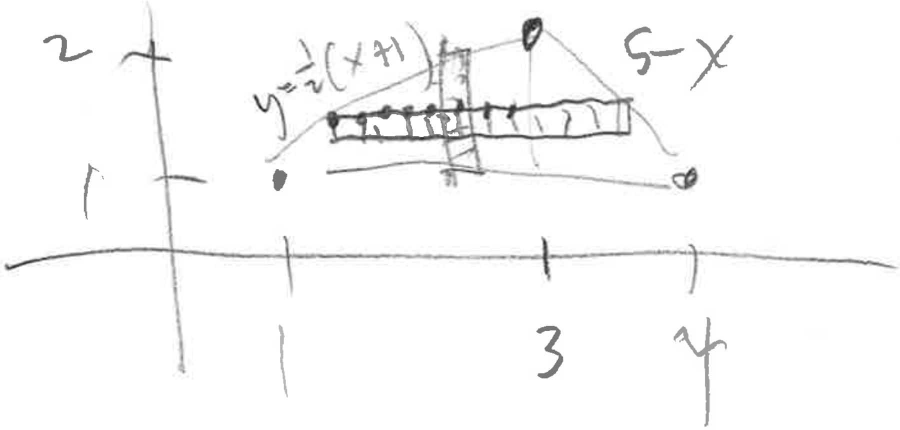
a) A Taylor approximation example and some integration

b) Integration examples

Do you have questions about the homeworks and/or the quiz?

R11

Set up an integral that represents the area of the triangular region with vertices (1,1), (4,1), (3,2).



$y = 5 - x \Rightarrow x = 5 - y$

$y = \frac{1}{2}(x+1) \Rightarrow x = 2y - 1$

INTERVALS

$y \in [1, 2]$

$x \in [2y-1, 5-y]$

$$\int_1^2 \int_{2y-1}^{5-y} dx\,dy$$

INTERVALS

R_1 $x \in [1, 3]$

$y \in [1, \frac{1}{2}(x+1)]$

R_2 $x \in [3, 4]$

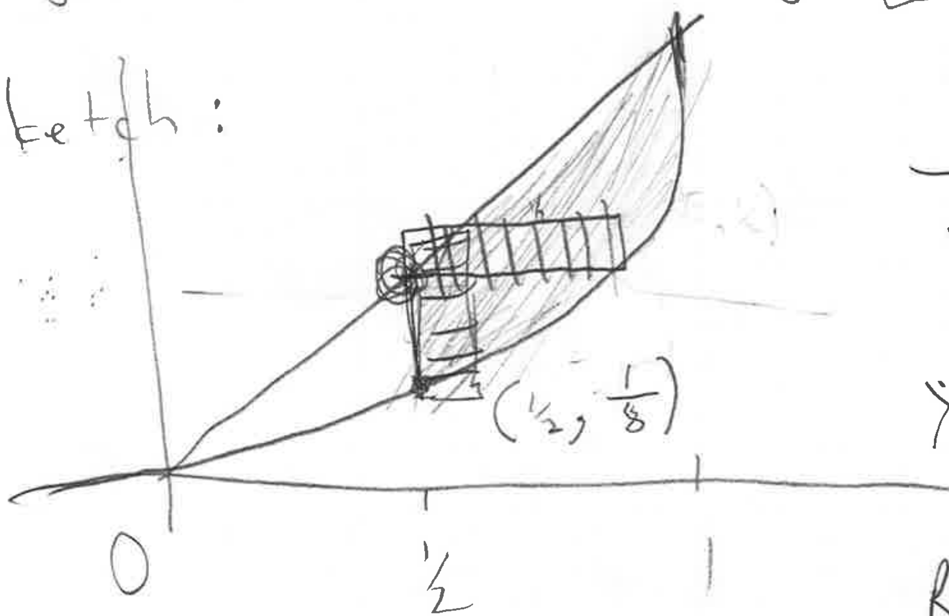
$y \in [1, 5-x]$

Change the order of integration and integrate

$$\int_{1/2}^{1}\int_{x^3}^{x} dy\,dx$$

Region of integration: $x \in [\tfrac{1}{2}, 1]$, $y \in [x^3, x]$

sketch:



REGION 2

$y \in [\tfrac{1}{2}, 1]$

$x \in [y, \sqrt[3]{y}]$

REGION 1

$y \in [\tfrac{1}{8}, \tfrac{1}{2}]$

$x \in [\tfrac{1}{2}, \sqrt[3]{y}]$

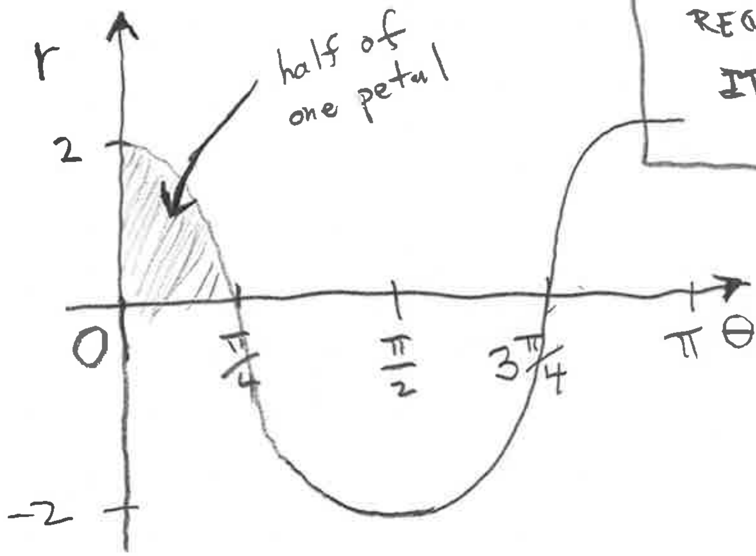
$$A = \int_{1/8}^{1/2}\int_{1/2}^{\sqrt[3]{y}} dx\,dy + \int_{1/2}^{1}\int_{y}^{\sqrt[3]{y}} dx\,dy$$

$$= \int_{1/8}^{1/2}\left(y^{1/3} - \tfrac{1}{2}\right)dy + \int_{1/2}^{1}\left(y^{1/3} - y\right)dy$$

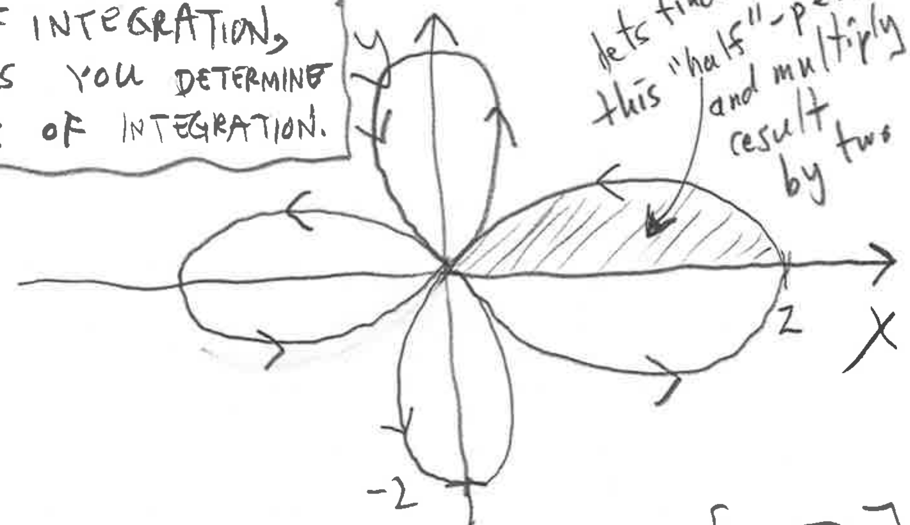
$$= \left(y^{4/3} - y/2\right)\Big|_{1/8}^{1/2} + y^{4/3} - y^2/2\,\Big|_{1/2}^{1} =$$

QH6 Recitation 12	Integration

Sketch the petal curve r = 2cos(2θ) and find the area of one petal.



ALWAYS SKETCH YOUR REGIONS OF INTEGRATION, IT HELPS YOU DETERMINE LIMITS OF INTEGRATION.



lets find area of this "half"-petal and multiply result by two

THE BOUNDS OF INTEGRATION FOR THE "HALF" PETAL ARE $\theta \in [0, \pi/4]$, $r \in [0, 2\cos(2\theta)]$

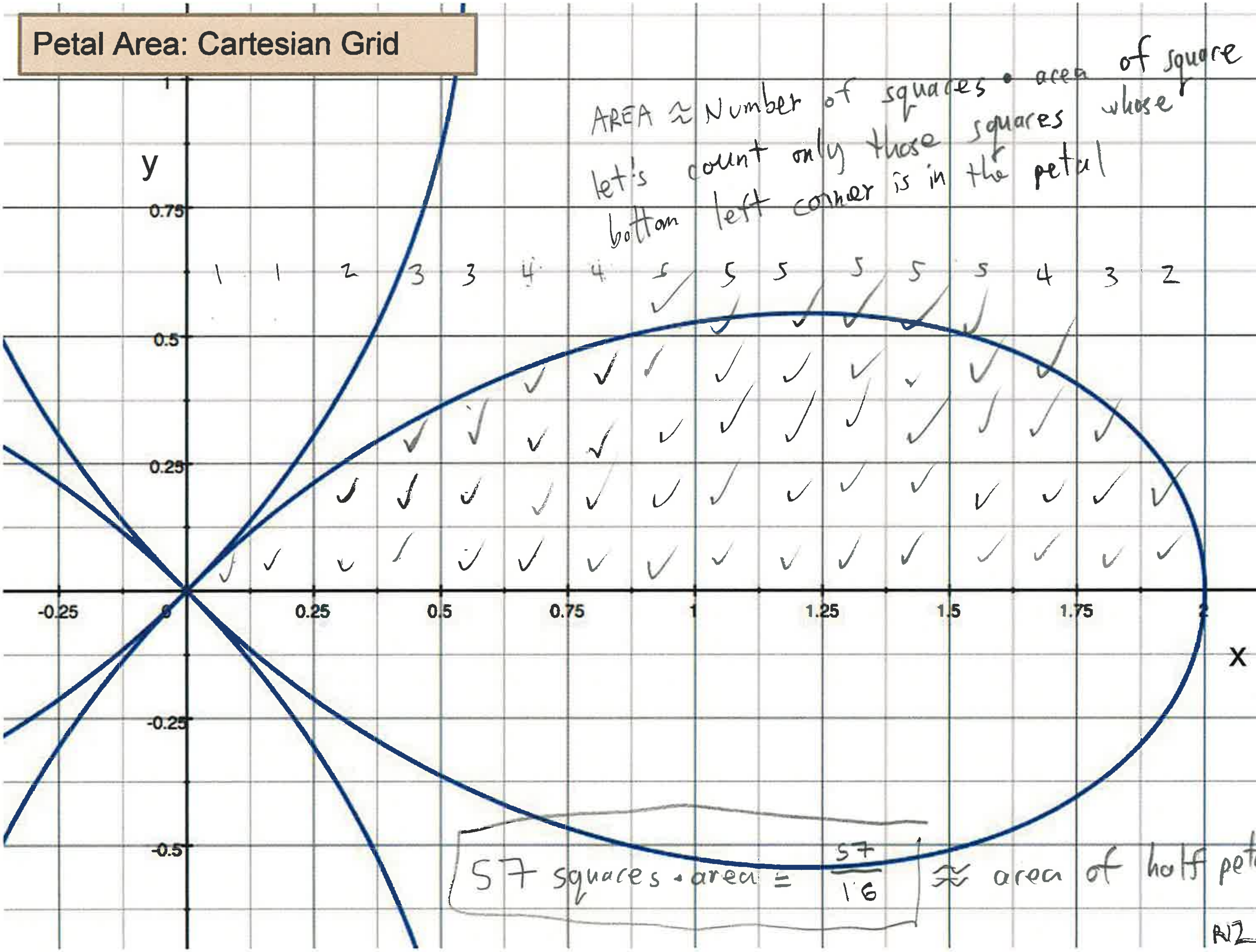
$$\Rightarrow \text{AREA} = 2\int_0^{\pi/4}\int_0^{2\cos(2\theta)} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[(2\cos 2\theta)^2/2\right]d\theta$$

$$= 4\int\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right)d\theta$$

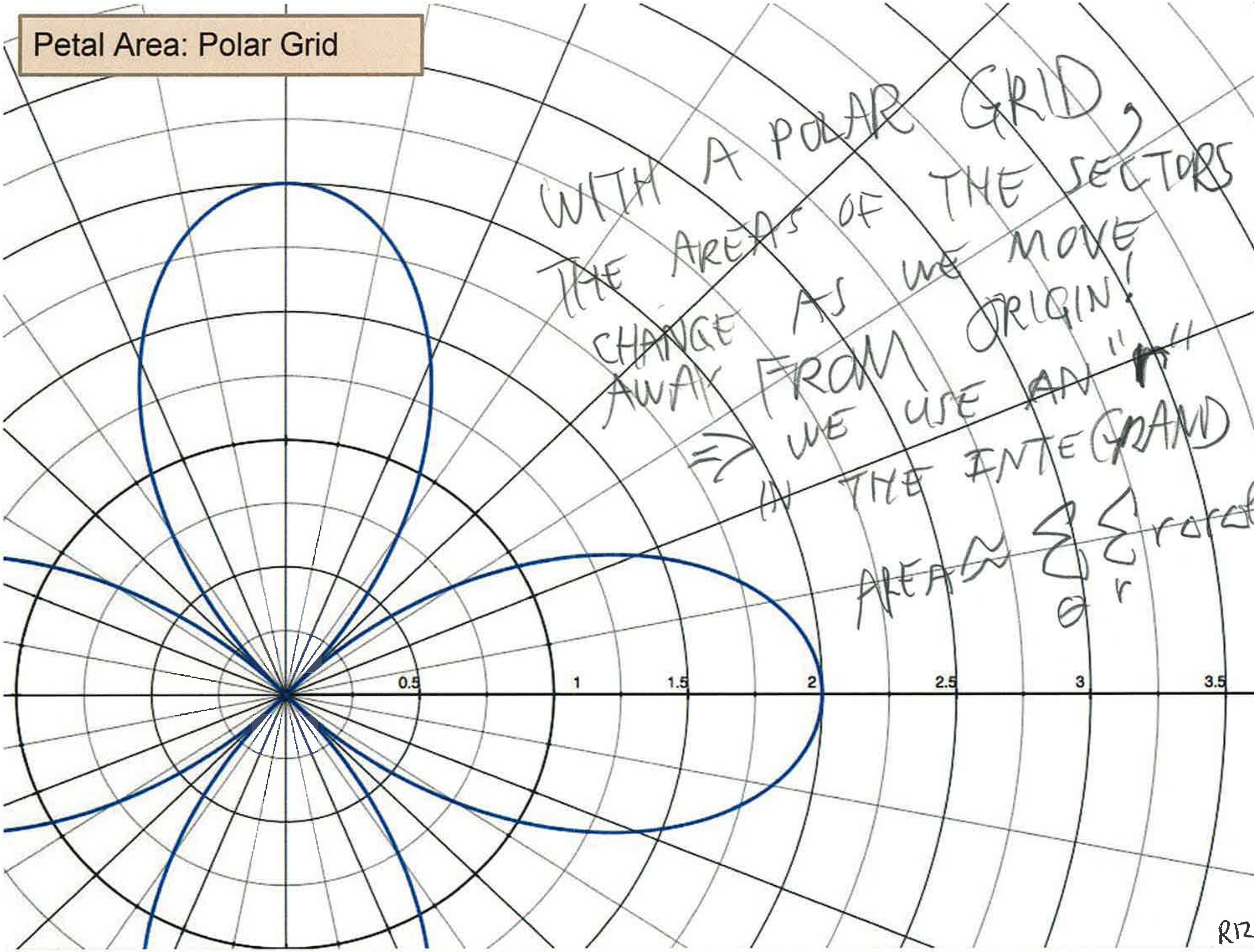
$$= 2\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4} = \pi/2$$

(why is there an r in the integrand?)

Petal Area: Cartesian Grid



Petal Area: Polar Grid



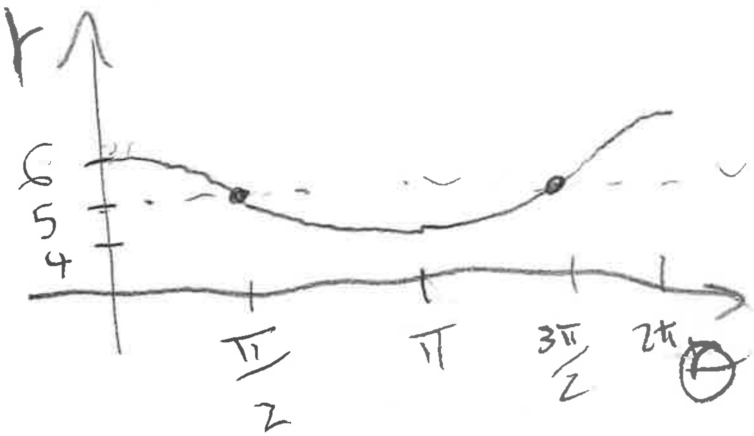
WITH A POLAR GRID, THE AREAS OF THE SECTORS CHANGE AS WE MOVE AWAY FROM ORIGIN!

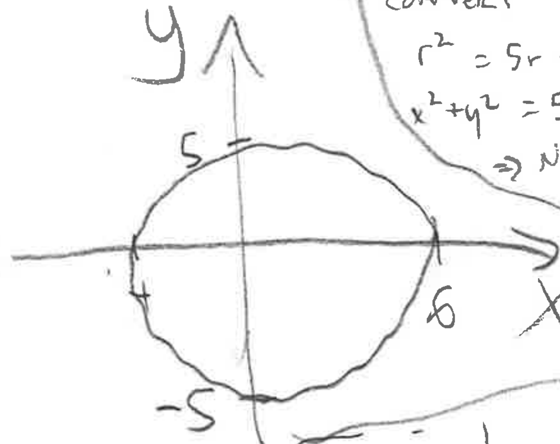
$\Rightarrow$ WE USE AN "r" IN THE INTEGRAND

AREA $\approx \sum_{\theta} \sum_{r} r \, dr \, d\theta$

R12

1) Sketch and find the area of the region inside the curve r = 5 + cos(θ) (from last year's quiz).





CONVERT TO CARTESIAN!

$r^2 = 5r + r\cos\theta$

$x^2+y^2 = 5\sqrt{x^2+y^2} + x$

$\Rightarrow$ NOT A CIRCLE RADIUS 5!

circle, radius 5? NO

area $\approx \pi r^2 = 25\pi$?

$$\text{Area} = \int_0^{\pi}\int_0^{5+\cos\theta} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi} (5+\cos\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \left(25 + 10\cos\theta + \cos^2\theta\right) d\theta$$

$$= \frac{1}{2}\left[25\theta + 10\sin\theta + \int\left(\frac{1}{2}+\frac{1}{2}\cos 2\theta\right)d\theta\right]$$

$$= \frac{25\pi}{2} + 0 + \frac{1}{2}\left[\frac{\theta}{2} + \frac{1}{4}\sin 2\theta\right]\Big|_0^{\pi}$$

$= \frac{25\pi}{2} + \frac{\pi}{4} = \frac{51\pi}{4} \approx 40$, not very close to 25π.

R12

2) Sketch the region of integration, change the order of integration, and then integrate

$$\int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx\,dy$$

THE "INNER" INTEGRAL USES $x \in [-\sqrt{y+1}, \sqrt{y+1}]$

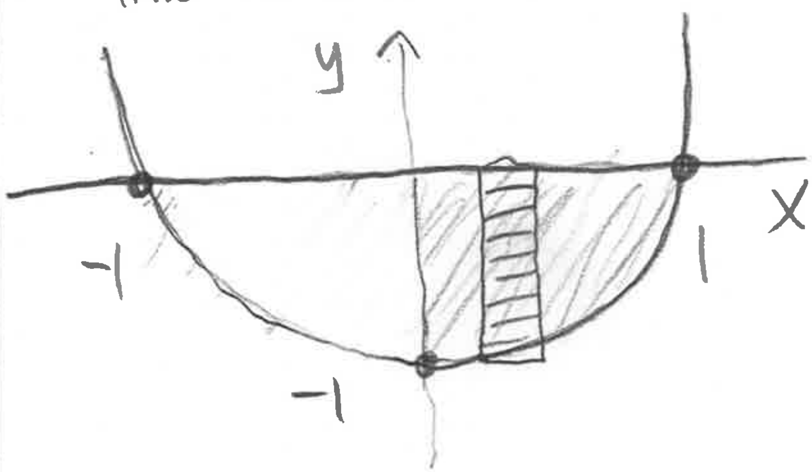
OR $-\sqrt{y+1} \leq x \leq \sqrt{y+1}$

OR: $x^2 \leq y+1$

OR: $y \geq x^2 - 1$

OUR "OUTER" INTEGRAL GIVES US $-1 \leq y \leq 0$. This

THIS YIELD THE REGION BELOW.



IN THIS REGION, $x \in [-1, +1]$, $y \in [x^2-1, 0]$.

$$\Rightarrow \text{AREA} = \int_0^1 \int_{x^2-1}^{0} dy\,dx$$

$$= \int_0^1 1 - x^2\,dx$$

$$= 2\int_0^1 1-x^2\,dx \quad \text{(even integrand, symmetric interval)}$$

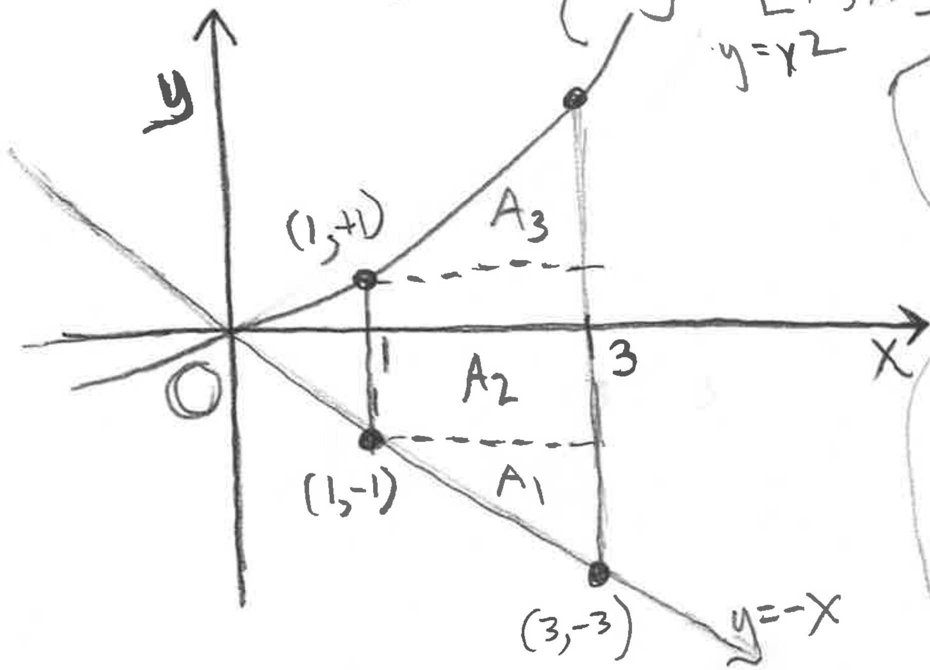
$$= 2\left(x - \frac{x^3}{3}\right)\Big|_0^1 = 2/3$$

RK

3) Sketch the region of integration, change the order of integration, and then integrate

$$\int_1^3 \int_{-x}^{x^2} dy\,dx$$

we are given $\begin{cases} x \in [1,3] \\ y \in [-x, x^2] \Rightarrow -x \leq y \leq x^2 \end{cases}$



$$A_1 = \int_{-3}^{+1}\int_{-y}^{3} dx\,dy = \int_{-3}^{1} x\Big|_{-y}^{3} dy = \int_{-3}^{1} 3+y\,dy = \left(3y+\frac{y^2}{2}\right)\Big|_{-3}^{1} = \left(3+\frac{1}{2}\right)-\left(-9+\frac{9}{2}\right) = 12-4 = 8$$

$$A_2 = \int_{-1}^{+1}\int_{1}^{3} dx\,dy = \int_{-1}^{1}(3-1)\,dy = 2\cdot(1+1) = 4$$

$$A_3 = \int_1^9\int_{\sqrt{y}}^{3} dx\,dy = \int_1^9 \left(3-\sqrt{y}\right)dy = \left(3y-\frac{2}{3}y^{3/2}\right)\Big|_1^9 = \dots = \frac{20}{3}$$

$$\Rightarrow \text{Area} = A_1 + A_2 + A_3 = 8 + 4 + \frac{20}{3} = \frac{56}{3}$$

R12

Quadratic Approximation

Find the quadratic approximation to f(x,y) = exp(–x² – y²) near the origin.

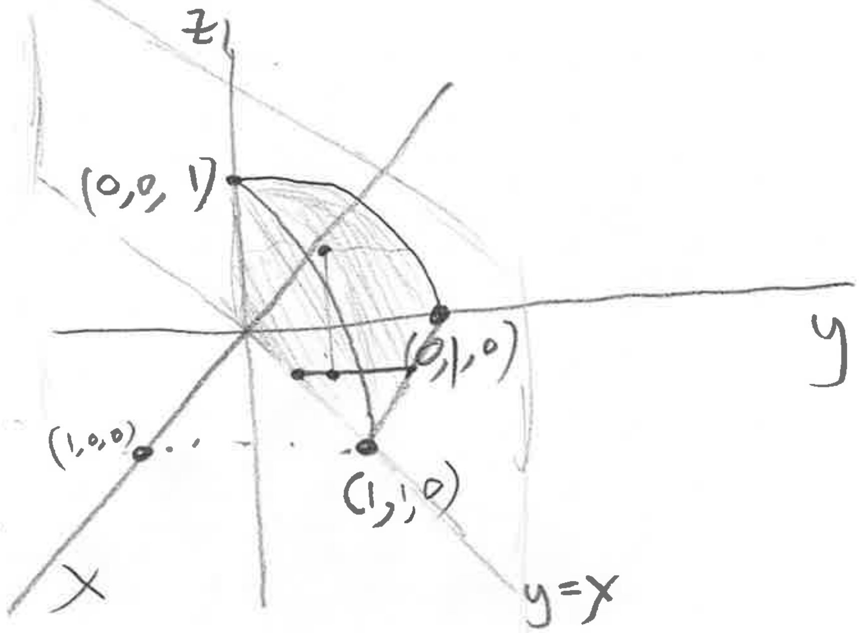
DERIVATIVE	AT (0,0)
$f_x = -2xf$	0
$f_y = -2yf$	0
$f_{xx} = -2f + 4x^2 f$	-2
$f_{xy} = 4xyf$	0
$f_{yy} = -2f + 4y^2 f$	-2

$$f \approx 1 + \frac{1}{2!}\left(-2x^2 - 2y^2\right) \quad \boxed{= 1 - x^2 - y^2}$$

R12

QH6 Recitation 14	Triple Integrals

Set up a triple integral that represents the volume of the region bounded by $y^2 + z^2 = 1$ and the planes $y = x$, $x = 0$, $z = 0$.



$x=0$ is the yz-plane

$z=0$ is the xy plane

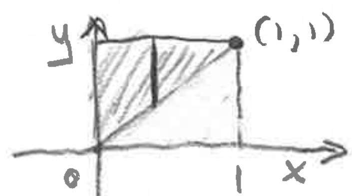
$y^2+z^2=1$ is a cylinder.

(1) ALWAYS TRY TO SKETCH REGION/SOLID.

(2) SELECT ORDER OF INTEGRATION, FIND LIMITS

$x \in [0,1]$ (position of strip)

$y \in [x,1]$ (where strips start/stop)



projection of solid onto xy plane.

$z \in [0,\sqrt{1-y^2}]$ (where columns start/stop)

$$\Rightarrow V = \int_0^1 \int_x^1 \int_0^{\sqrt{1-y^2}} dz\,dy\,dx$$

$$\int_0^1 \int_0^{\sqrt{1-z^2}} \int_0^y dx\,dy\,dz$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_x^{\sqrt{1-z^2}} dy\,dz\,dx$$

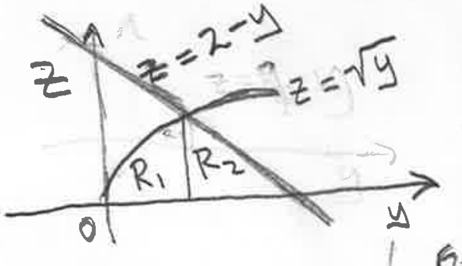
$$\int_0^1 \int_0^y \int_0^{\sqrt{1-y^2}} dz\,dx\,dy$$

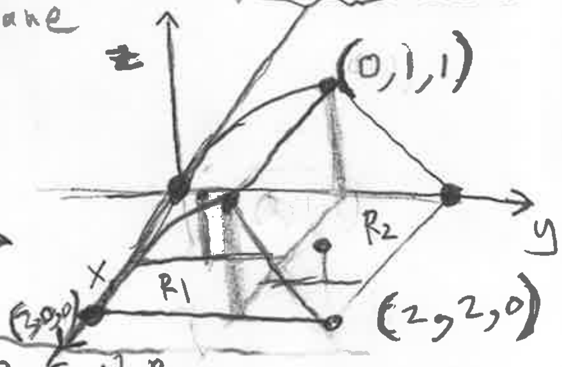
EASY TO SEE.

Set up a triple integral that represents the volume of the region bounded by
$z^2 = y$, $y + z = 2$, $x = 0$, $z = 0$, $x = 2$. Set up the integral in at least two different ways.

$x=0$ is the yz plane
$z=0$ is the xy plane

horizontal strips
vertical columns





REGION R_1 | REGION R_2

$x \in [0, 2]$ | $x \in [0, 2]$

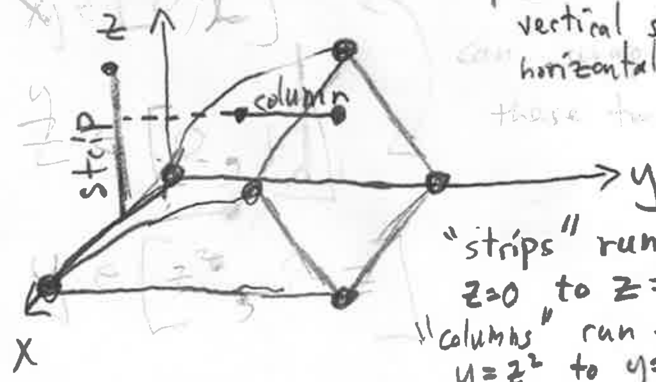
$y \in [0, 1]$ | $y \in [1, 2]$

$z \in [0, \sqrt{y}]$ | $z \in [0, 2-y]$

$$V = \int_0^2 \int_0^1 \int_0^{\sqrt{y}} dz\,dy\,dx + \int_0^2 \int_1^2 \int_0^{2-y} dz\,dy\,dx$$

ANOTHER APPROACH

vertical strip
horizontal column



"strips" run from $z=0$ to $z=1$
"columns" run from $y=z^2$ to $y=2-z$

$x \in [0, 2]$

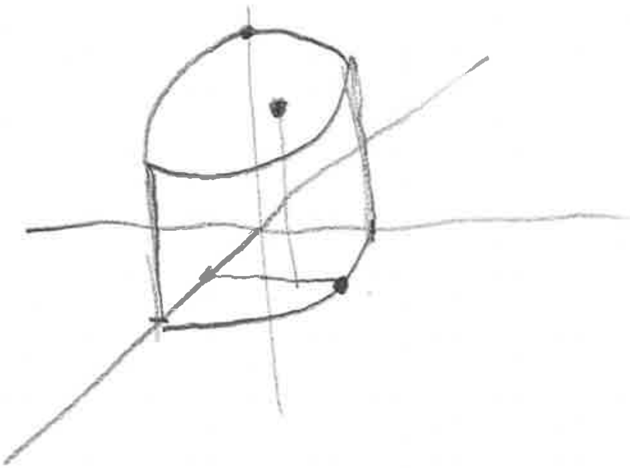
$z \in [0, 1]$

$y \in [z^2, 2-z]$

$$V = \int_0^2 \int_0^1 \int_{z^2}^{2-z} dy\,dz\,dx$$

R14

Set up a triple integral that represents the volume of the region bounded by $\underbrace{x^2 + y^2 + z^2 = 2}_{\text{sphere}}$, and by $\underbrace{x^2 + y^2 = 1}_{\text{cylinder}}$.



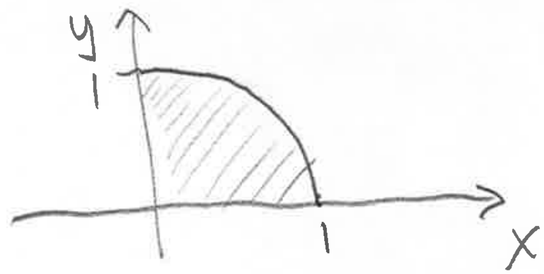
$x \in [0, 1]$

$y \in [0, \sqrt{1-x^2}]$

$z \in [0, \sqrt{2-x^2-y^2}]$

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{2-x^2-y^2}} dz\,dy\,dx$$

PROJECT solid ONTO X-Y PLANE



(every point "pushed" down into xy-plane)

$r \in [0, 1]$

$\theta \in [0, \pi/2]$

$z \in [0, \sqrt{2-r^2}]$

$$V = 8\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

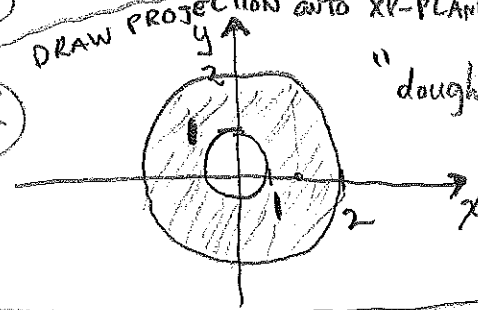
R14

QH6 Recitation 15 | Triple Integrals

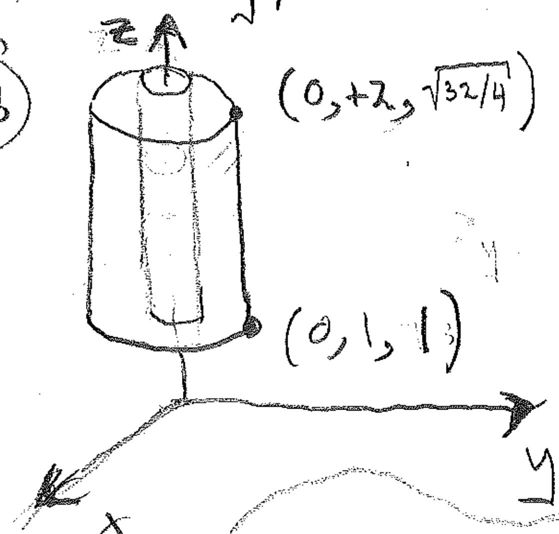
Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, above by $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

① How should we start? Sketch solid, sketch in x-y-plane.

② DRAW PROJECTION ONTO XY-PLANE



③



(sketch the solid before the coordinate axes)

④ Find points on solid. (Helps visualize)

⑤ Limits of integration

$x \in [0, 2]$

$y \in [0, \sqrt{4 - x^2}]$ "outer" (solid only)

$z \in [1, \sqrt{36/4 - \frac{1}{4}(x^2 + y^2)}]$

⑥ SET UP INTEGRALS AS OUTER MINUS INNER:

$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^{\sqrt{9 - \frac{1}{4}(x^2+y^2)}} dz\,dy\,dx - 4\int_0^2 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{9 - \frac{1}{4}(x^2+y^2)}} dz\,dy\,dx$$

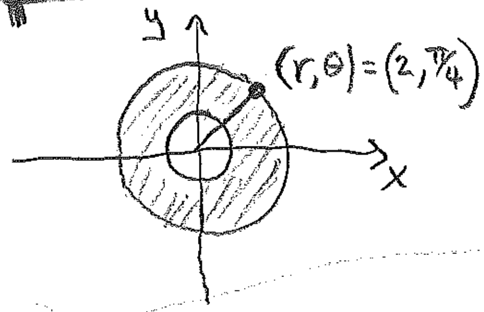
⑦ ANOTHER APPROACH:

$$V = 4\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_1^{\sqrt{\cdots}} dz\,dy\,dx + 4\int_1^2 \int_0^{\sqrt{4-x^2}} \int_1^{\sqrt{\cdots}} dz\,dy\,dx$$

QH6 Recitation 15 | Triple Integrals

Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, above by $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

ANOTHER APPROACH : POLAR / CYLINDRICAL.



$\theta \in [0, 2\pi]$

$r \in [1, 2]$

$z \in \left[1, \sqrt{36/4 - 1/4(x^2+y^2)}\right]$

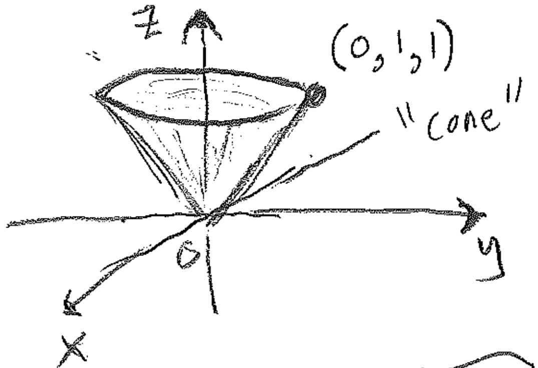
$$V = \int_0^{2\pi} \int_1^2 \int_1^{\sqrt{36/4 - 1/4(x^2+y^2)}} r\,dz\,dr\,d\theta$$

THINK ABOUT:

1/ Why do we need the "r" in the integrand?

2/ COULD WE USE $dr\,dz\,d\theta$?

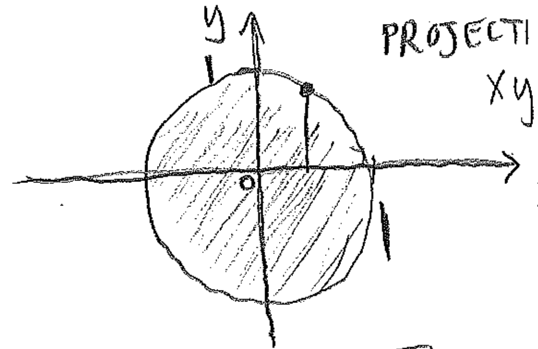
Set-up an integral that represents the volume of the solid bounded above by $z = 1$, and below by $z^2 = x^2 + y^2$. Set this integral up in at least two different ways.



$x \in [0, 1]$

$y \in [0, \sqrt{1-x^2}]$

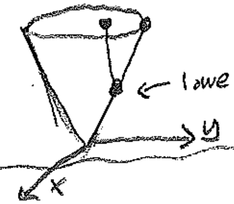
$z \in [\sqrt{x^2+y^2}, 1]$



PROJECTION ONTO xy-PLANE

$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\,dy\,dx, \quad \text{or} \quad V = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^{1} dz\,dx\,dy$$

BE CAREFUL WITH Z-COORDINATES: LOWER POINT ON SOLID IS THE CONE'S OUTER SURFACE!



← lower bound on integral is $z = \sqrt{x^2+y^2}$

POLAR/CYLINDRICAL

$\theta \in [0, 2\pi]$

$r \in [0, 1]$

$z \in [\sqrt{x^2+y^2}, 1]$

OR

$\theta \in [0, 2\pi]$

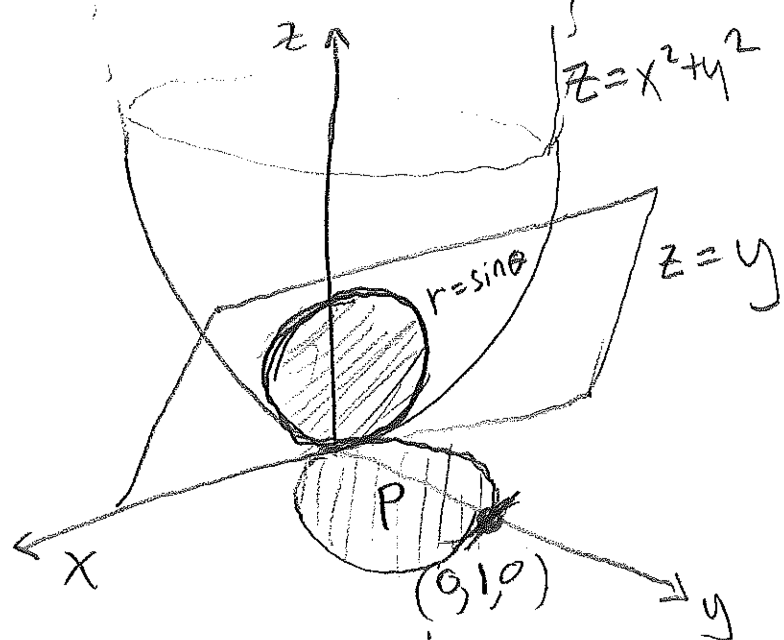
$z \in [0, 1]$

$r \in [0, z]$

$$V = \int_0^{2\pi} \int_0^1 \int_{\sqrt{x^2+y^2}}^{1} r\,dz\,dr\,d\theta, \quad \text{OR} \quad V = \int_0^{2\pi} \int_0^1 \int_0^z r\,dr\,dz\,d\theta$$

Recitation 16 — Triple Integrals, Cylindrical Coordinates

Set up an integral that represents the volume of solid bounded by $z = x^2 + y^2$, and $z = y$. Use cylindrical coordinates.



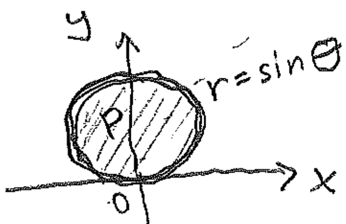
$z=x^2+y^2$ is an elliptic paraboloid, which intersects $z=y$ when:

$z=y=x^2+y^2$

In polar/cylindrical:

$r\sin\theta = r^2$

or: $r=\sin\theta$



P = projection of solid onto xy-plane. On a given ray, r runs from 0 to $\sin\theta$.

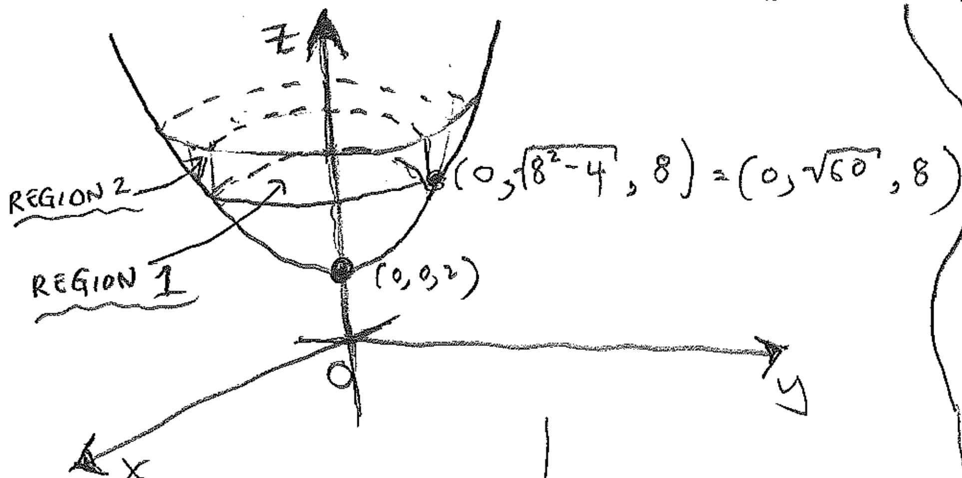
$\theta \in [0, \pi]$

$r \in [0, \sin\theta]$

$z \in [x^2+y^2, y]$, or $z \in [r^2, r\sin\theta]$

$$V = \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

Let V be the volume between the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$ above the plane $z = 8$ and below the plane $z = 10$. Set up the volume as a triple integral. Do not Evaluate



POLAR : $dr\,dz\,d\theta$

$\theta \in [0, 2\pi]$

$z \in [8, 10]$

For r, solve: $-r^2 + z^2 = 4$

$\Rightarrow r = \sqrt{z^2-4}$

$\Rightarrow r \in [0, \sqrt{z^2-4}]$

$$\Rightarrow V = \int_0^{2\pi} \int_8^{10} \int_0^{\sqrt{z^2-4}} r\,dr\,dz\,d\theta$$

POLAR : $dz\,dr\,d\theta$

REGION 1

$\theta \in [0, 2\pi]$

$r \in [0, 8]$

$z \in [8, 10]$

REGION 2

$\theta \in [0, 2\pi]$

$r \in [8, 10]$

$z \in [\sqrt{4+r^2}, 10]$

$$V = \int_0^{2\pi} \int_0^{8} \int_8^{10} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_8^{10} \int_{\sqrt{4+r^2}}^{10} r\,dz\,dr\,d\theta$$

CARTESIAN : $dz\,dy\,dx$

REGION 1

$x \in [0, 8]$

$y \in [0, \sqrt{4+x^2}]$

$z \in [8, 10]$

REGION 2

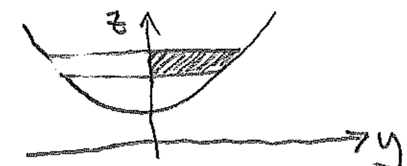
$x \in [8, 10]$

$y \in [0, \sqrt{4+x^2}]$

$z \in [\sqrt{4+x^2+y^2}, 10]$

$$V = 4\int_0^8 \int_0^{\sqrt{4+x^2}} \int_8^{10} dz\,dy\,dx + 4\int_8^{10} \int_0^{\sqrt{4+x^2}} \int_{\sqrt{\quad}}^{10} dz\,dy\,dx$$

CARTESIAN : $dx\,dy\,dz$



$z \in [8, 10]$

$y \in [0, \sqrt{z^2-4}]$

$x \in [0, \sqrt{z^2-y^2-4}]$

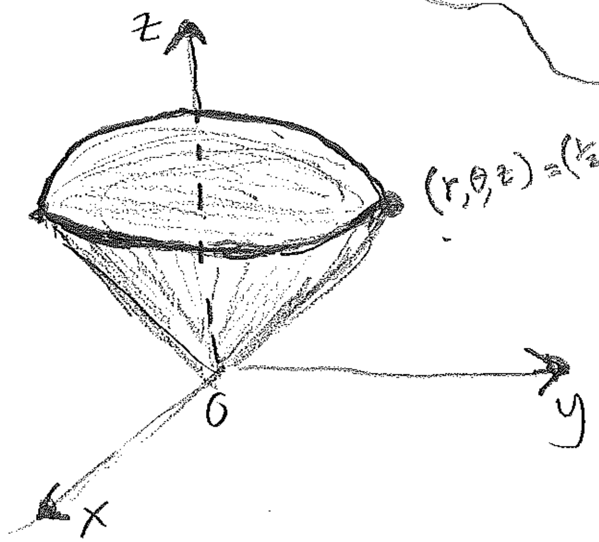
$$V = 4\int_8^{10} \int_0^{\sqrt{z^2-4}} \int_0^{\sqrt{z^2-y^2-4}} dx\,dy\,dz$$

Set up an integral that represents the volume of the "ice cream cone" bounded by $x^2 + y^2 + z^2 = 1$, and $z^2 = 3(x^2 + y^2)$. Use cylindrical coordinates.

The sphere and cone intersect on: $z^2 = 1 - x^2 - y^2 = 3(x^2+y^2)$

in polar: $1 - r^2 = 3r^2$

$r = 1/2$,



$\Rightarrow$ surfaces intersect when $r = 1/2$, and when $z^2 = 3((1/2)^2) = 3/4$, or when $z = \sqrt{3}/2$.

(we don't really need z-coordinate)

Use $dz\,dr\,d\theta$

$\theta \in [0, 2\pi]$

$r \in [0, 1/2]$

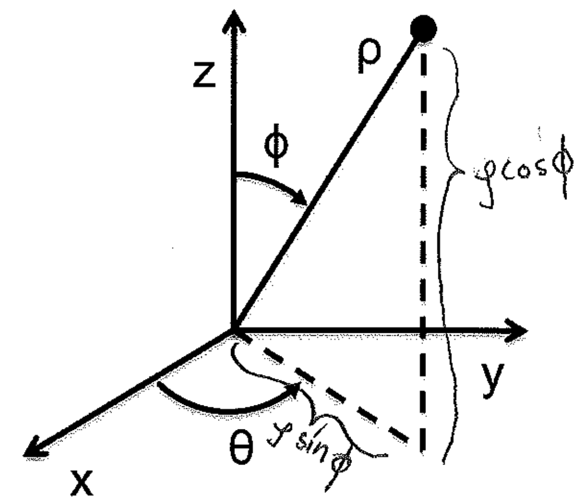
$z \in [\sqrt{3}r, \sqrt{1-r^2}]$

$$\boxed{V = \int_0^{2\pi}\int_0^{1/2}\int_{\sqrt{3}r}^{\sqrt{1-r^2}} r\,dr\,d\theta}$$

If interested, $V = \frac{\pi}{3}(2 - \sqrt{3})$

Recitation 17 | Triple Integrals in Spherical Coordinates

Fill in the blanks.



1. $x = \rho\cos\theta$ $\underline{\ \sin\phi\ }$
2. $y = \rho\sin\theta$ $\underline{\ \sin\phi\ }$
3. $z = \rho$ $\underline{\ \cos\phi\ }$

Cartesian in terms of spherical

There are equ's for expressing spherical in terms of Cartesian:

$\rho = \sqrt{x^2+y^2+z^2}$, $\tan\theta = y/x$, $\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Provide a geometric interpretation of each expression.

a) $\rho\sin\phi = 1$, $x^2+y^2 = \rho^2\sin^2\phi\,(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\phi$. But $\rho\sin\phi = 1$, so $x^2+y^2 = 1^2 \Rightarrow$ cylinder radius 1

b) $\rho\cos\phi = 1$ $\Rightarrow$ the plane $z=1$, from 3

The xy-plane in spherical coord. is: $\phi = \pi/2$, from 3

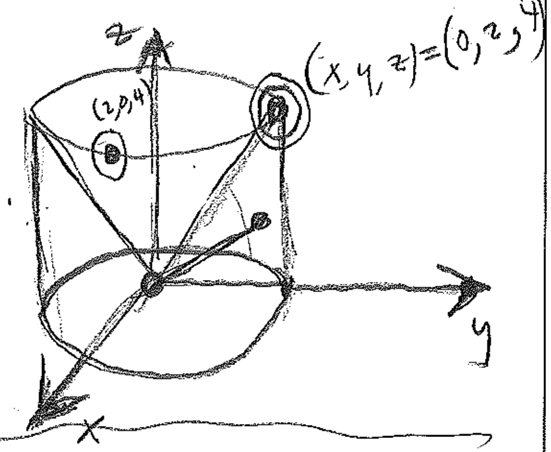
(because we need the value of ϕ that sets $z=0$)

Set-up an integral that represents the volume bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2(x^2 + y^2)^{1/2}$.

① Sketch solid: $x^2+y^2=4$ is a cylinder

$\frac{z}{2} = \sqrt{x^2+y^2}$ is a cone

bounded above by cone, below by plane.



② Integration limits

$\rho \in [0, 2\csc\phi]$, from (*)

$\phi \in \left[\arctan(\tfrac{1}{2}), \tfrac{\pi}{2}\right]$, the xy-plane is $\phi = \pi/2$, and the "top" of the surface: $\phi = \arctan(1/2)$

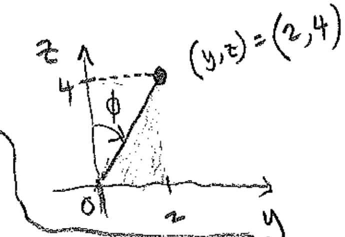
$\theta \in [0, 2\pi]$ (symmetric about z-axis)

Spherical Coordinates of ⊙:

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+16} = 2\sqrt{5}$

$\phi = \tan^{-1}\left(\frac{2}{4}\right)$

$\theta = 0$



③ WRITE INTEGRAL

$$V = \int_0^{2\pi}\int_{\operatorname{atan}\frac{1}{2}}^{\pi/2}\int_0^{2\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

(*) OUTER SURFACE OF CYLINDER:

$\rho^2\sin^2\phi = 4$, or $\rho\sin\phi = 2$

or $\rho = 2\csc\phi$.

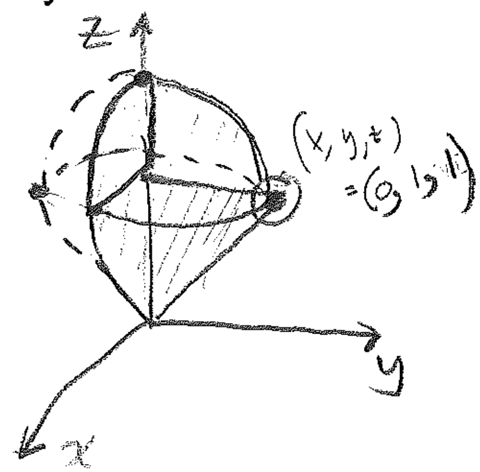
NOTE

we almost always use $d\rho\, d\phi\, d\theta$, or $d\rho\, d\theta\, d\phi$

Set-up an integral that represents the volume of the solid bounded by

$$\left.\begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le \sqrt{1 - x^2} \end{array}\right\} \text{cylinder } x^2 + y^2 \le 1$$

$$\underbrace{\sqrt{x^2 + y^2}}_{\text{cone}} \le z \le \underbrace{\sqrt{2 - (x^2 + y^2)}}_{\text{sphere}}$$

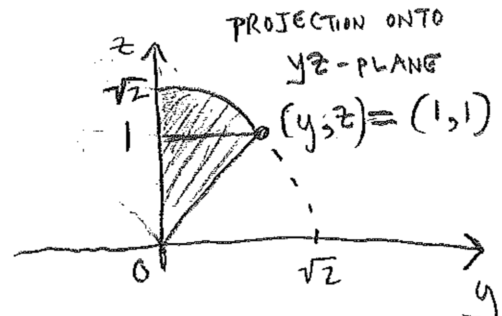


② INTEGRATION LIMITS

$\rho \in [0, \sqrt{2}]$

$\phi \in [0, \pi/4]$

$\theta \in [0, \pi/2]$

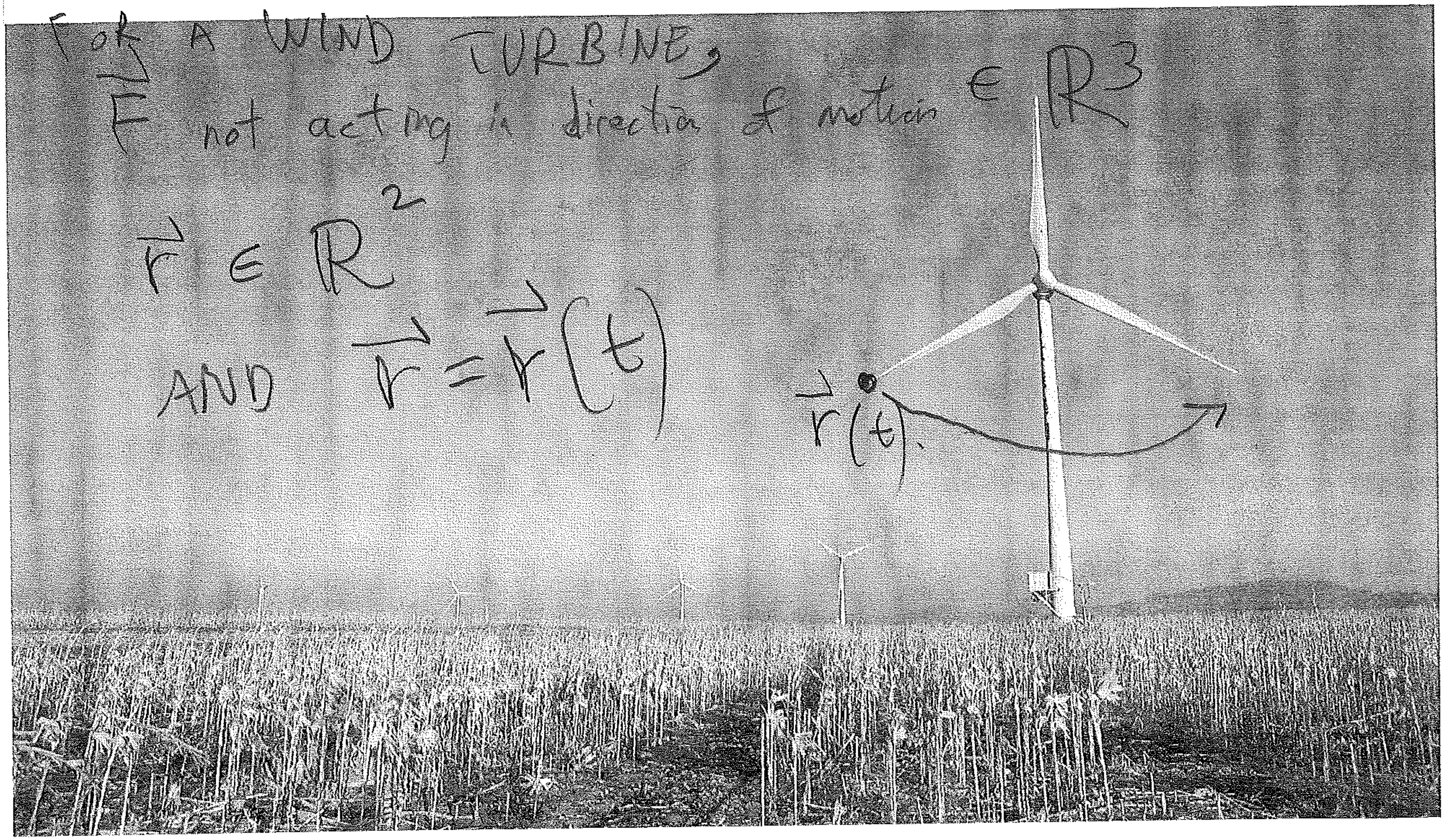


$$V = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\sqrt{2}}{6} \pi (2 - \sqrt{2})$$

(NOTE: solid is a "quarter" of an ice-cream cone)

Recitation 19	Line Integrals and Work

Fill in the Blanks: Work is the ENERGY transferred to or from an object by means of a FORCE acting on the OBJECT.



FOR A WIND TURBINE,

$\vec{F}$ not acting in direction of motion $\in \mathbb{R}^3$

$\vec{r} \in \mathbb{R}^2$

AND $\vec{r} = \vec{r}(t)$

$\vec{r}(t)$

Work Over a Straight Path

Force **F** is applied to an object as it moves from x = a to x = b along the x-axis.

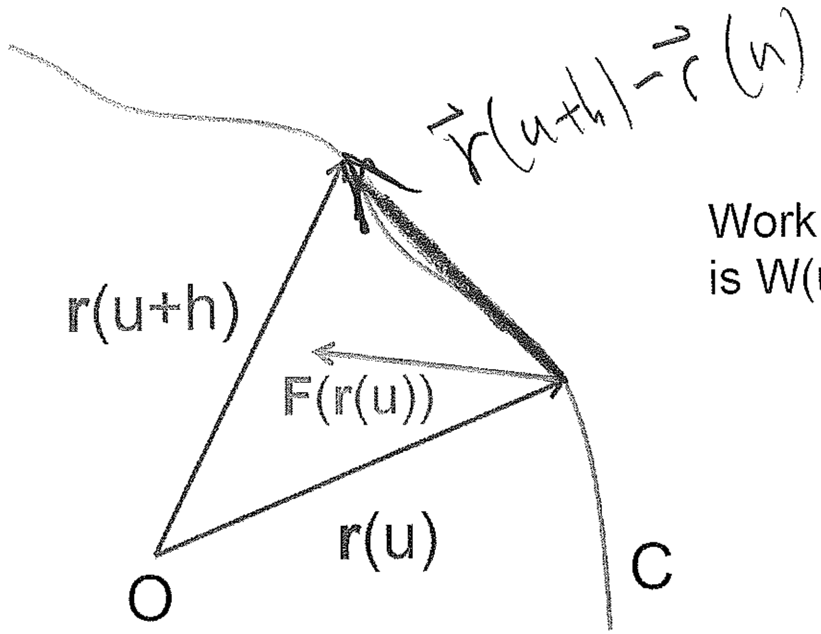
	Applied Force	Work
Case 1	F = 4i	W = $\vec{F}\cdot\vec{r} = 4i\cdot(b-a)i = 4(b-a)$
Case 2	F = 4i – 2j	W = $\vec{F}\cdot\vec{r} = 4i\cdot(b-a)i = 4(b-a)$

we need to extend this concept to curved paths in R^3

work is a <u>SCALAR</u>, calculated with a DOT PRODUCT.

Work Over a Curved Path

Force **F** applied to an object as it moves from **r**(u) to **r**(u + h) along curve C.



Work done by force **F** from **r**(u) to **r**(u+h) is W(u + h) – W(u).

	Applied Force	Work
Case 3	**F** = **F**(**r**(u))	$\dfrac{W(u+h) - W(u)}{h} \approx \dfrac{F(\vec{r}(u)) \cdot (\vec{r}(u+h) - \vec{r}(u))}{h}$

DIVIDE BOTH SIDES BY h;
TAKE LIMIT AS $h \to 0$,
INTEGRATE;

$$W' = F(\vec{r}) \cdot \vec{r}\,'$$

$$W = \int_a^b F(\vec{r}) \cdot \vec{r}\,' \, du$$

Examples

Set up an integral that represents the total work.

a) $\mathbf{F} = (x + 2y)\mathbf{i} + (2x + y)\mathbf{j}$, path is $y = x^2$ from (0,0) to (2,4).

b) ~~$\mathbf{F} = x\cos(y)\,\mathbf{i} - y\sin(x)\,\mathbf{j}$, along polygon connecting (0,0), (1,0), (1,1), (0,1), in the indicated order.~~ (Tedious)

a) Find $\vec{r}$: let $x = u$, $y = u^2$, parametric representation of $\vec{r}$. $u \in [0, 2]$

$$F = \begin{bmatrix} u + 2u^2 \\ 2u + u^2 \end{bmatrix}, \quad \vec{r} = \begin{bmatrix} u \\ u^2 \end{bmatrix}, \quad r' = \begin{bmatrix} 1 \\ 2u \end{bmatrix}$$

$$W = \int_0^2 \begin{bmatrix} u + 2u^2 \\ 2u + u^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2u \end{bmatrix} du = \int_0^2 (u + 2u^2) + (2u + u^2 + 4u^2 + 2u^3)\, du$$

b) $\vec{F} = (x - y)\hat{i} - xy\hat{j}$, from (3,3) to (1,2)

$$\vec{r} = \begin{bmatrix} 2 - u \\ 3 - u \end{bmatrix}, \quad \vec{r}' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \vec{F} = \begin{bmatrix} -1 \\ -6 - 5u + u^2 \end{bmatrix}$$

$$W = \int_0^1 \begin{bmatrix} -1 \\ +6 - 5u + u^2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} du = \int_0^1 1 - 6 + 5u - u^2\, du = -\frac{17}{6}$$

Announcements

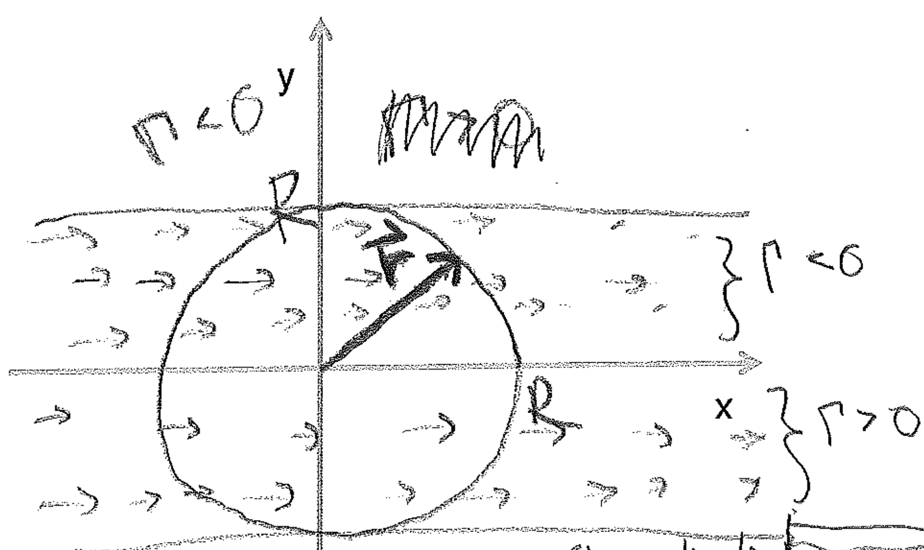
Quiz 4: Tuesday April 15 (two weeks away)
Homework 8: due Friday at 11:59 pm
My prediction: one last pop quiz, this week or next

Recitation 20 — Line Integrals: Circulation and Flux

$$\text{circulation} = \Gamma = \int_C \vec{v}(\vec{r}) \cdot d\vec{r}$$

Sketch the velocity field for **v**, and calculate the circulation over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\hat{i}, & R \le y \le R \\ 0, & \text{else} \end{cases}$$



$\vec{r} = R\cos t\,\hat{i} + R\sin t\,\hat{j}$

$\vec{r}\,' = -R\sin t\,\hat{i} + R\cos t\,\hat{j}$

$\vec{v} = 2\hat{i}$

$\vec{v} \cdot \vec{r}\,' = -2R\sin t$

$\Gamma = \int_0^{2\pi} -2R\sin t\,dt$

$= 0 = \int_0^{\pi} + \int_{\pi}^{2\pi}$

If $y>0$, $\Gamma<0$: moving against flow direction
If $y<0$, $\Gamma>0$: moving with flow direction

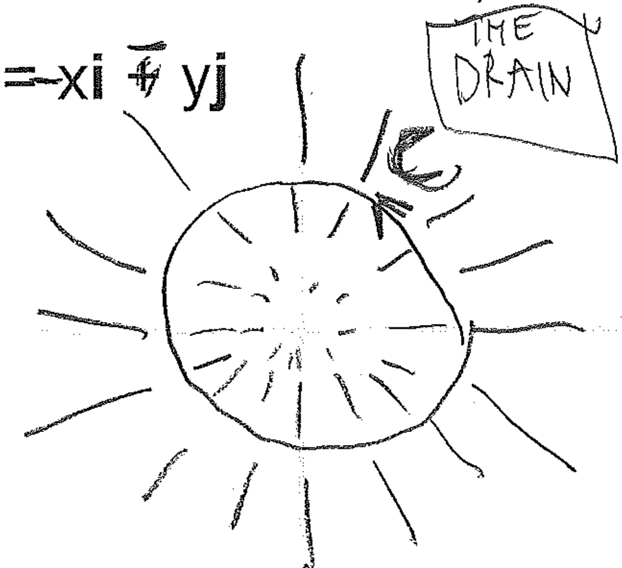
Why? Not cirulation cancellation.

Circulation Examples

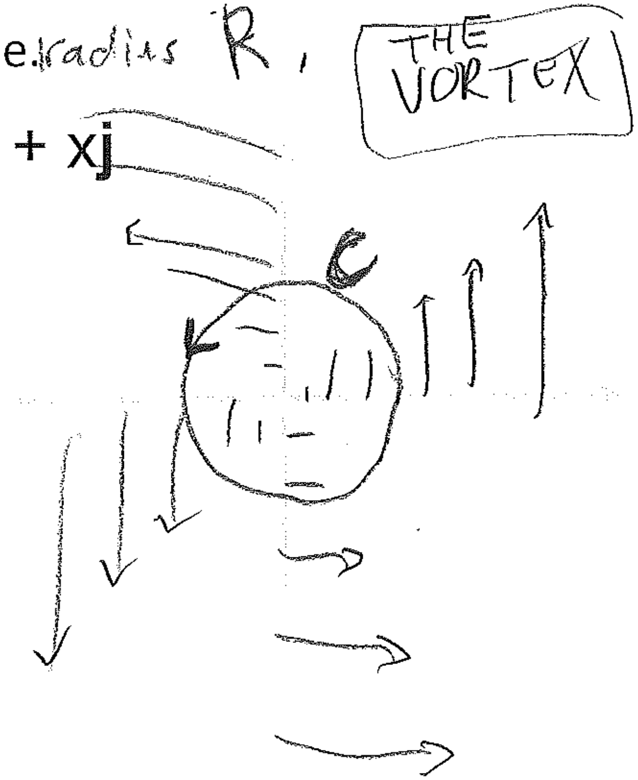
a) Draw the velocity field

b) Calculate the circulation, when C is the ~~unit~~ circle. radius R,

1) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$



2) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$



$\vec{r} = R\cos(t)\hat{i} + R\sin(t)\hat{j}$

$\vec{r}\,' = -RS\,\hat{i} + RC\,\hat{j}$, $x = R\cos t$, $y = R\sin t$

$$\Gamma = R^2\int_0^{2\pi} \begin{bmatrix}-C\\-S\end{bmatrix}\cdot\begin{bmatrix}-S\\C\end{bmatrix} dt$$

$= 0$ (why? flow is not "circulatory" & $\vec{v}\cdot\vec{r}\,'$ always zero)

$\vec{r}$ same.

$$\Gamma = \int_0^{2\pi} \begin{bmatrix}-R\sin\\R\cos\end{bmatrix}\cdot\begin{bmatrix}-RS\\RC\end{bmatrix} dt$$

$$= R^2\int_0^{2\pi} s^2 + c^2\, dt$$

$= 2\pi R^2 \Rightarrow$ bigger the circle, bigger the Γ

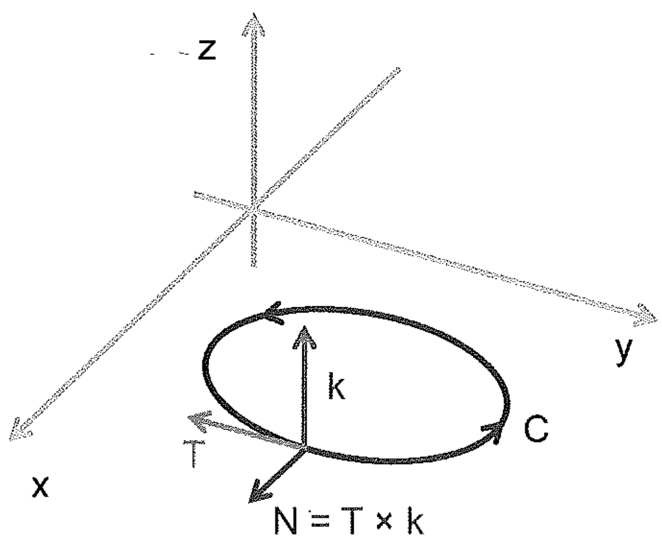
Announcements

Quiz 4: Tuesday April 15
My prediction: one last pop quiz, next week?
Homework 8: due Friday Apr 4 at 11:59 pm. Questions?
Homework 9: due Friday Apr 11 at 11:59 pm. Questions?
Survey: please complete the brief technical issues survey, email sent yesterday

Circulation and Flux

$$\text{circulation} = \int_C \vec{v}\cdot\vec{r}'\,dt = \int_C \vec{v}\cdot\vec{T}\,dt$$

$$\text{flux} = \int_C \vec{v}\cdot\vec{N}\,dt$$ N is the outward pointing, unit, normal vector of curve C



COUNTERCLOCKWISE MOTION
IN XY PLANE

The textbook derives a computational formula for flux:

minus?

$$\text{flux} = \oint_C M\,dy + N\,dx$$

$\oint_C$ means integral over curve C moving counter clockwise!

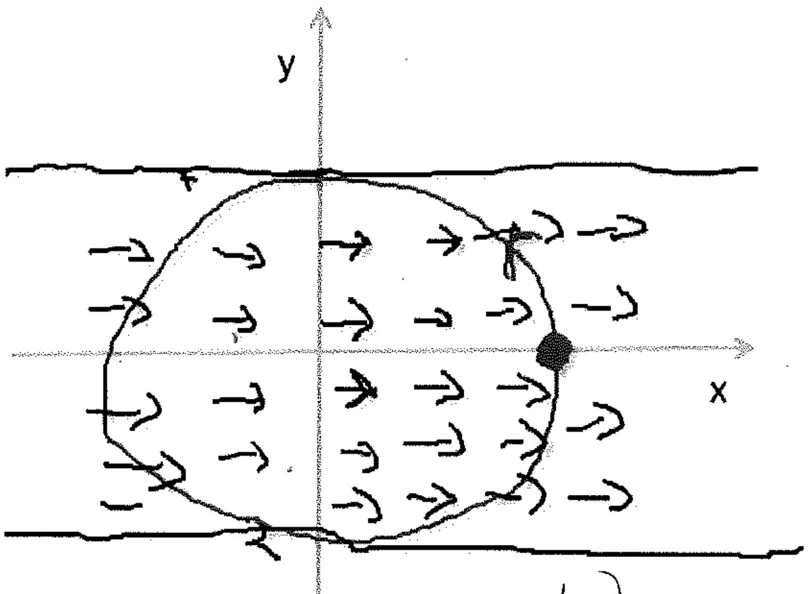
$$\vec{v} = M\hat{i} + N\hat{j} = M(x,y)\hat{i} + N(x,y)\hat{j}$$

Recitation 21 | Circulation, Flux, Conservative Fields

$$\text{flux} = \oint_C M\,dy - N\,dx$$

Calculate the flux over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\vec{i}, & -R \le y \le R \\ 0, & \text{else} \end{cases}$$



(Flow in = Flow out)

$\vec{r} = \begin{bmatrix} R\cos t \\ R\sin t \end{bmatrix}$, $t \in [0, 2\pi]$

flux $= \oint M\,dy + N\,dx$

Need dy, dx in terms of t:

$\left.\begin{array}{l} x = R\cos t \\ y = R\sin t \end{array}\right\} \Rightarrow \begin{array}{l} dx = -R\sin t\,dt \\ dy = +R\cos t\,dt \end{array}$

And: $M = 2$, $N = 0$.

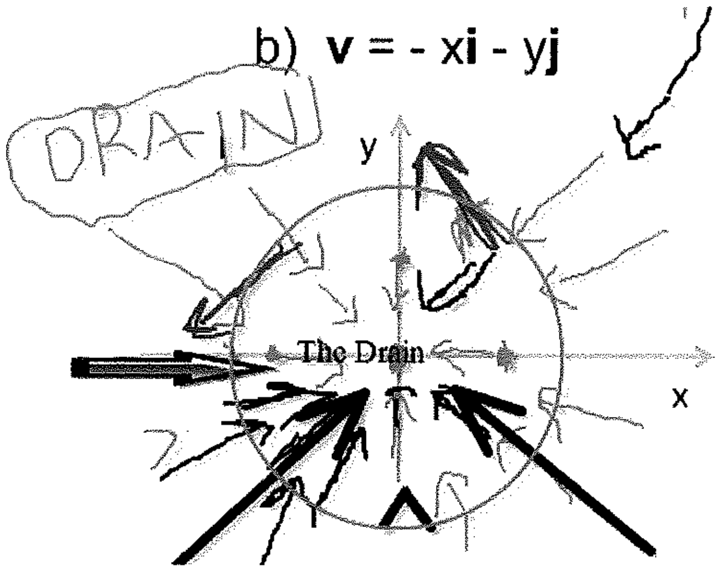
$\Rightarrow$ flux $= \int_0^{2\pi} 2R\cos t\,dt = 0$

Why? Flux is the net flow out of region enclosed by C.

Circulation Examples

Calculate the flux over curve C, where C is the circle of radius R.

b) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$



$$\text{flux} = \oint M\,dy - N\,dx, \quad \boxed{\begin{array}{l} M = -R\cos t \\ N = -R\sin t \end{array}}$$

$$= \int_C -R\cos t\,dy - (-R\sin t)\,dx$$

negative!

$$= \int_0^{2\pi} \underbrace{-R\cos t}_{M}\,\underbrace{R\cos t\,dt}_{dy} - \underbrace{(-R\sin t)}_{N}\,\underbrace{(-R\sin t\,dt)}_{dx}$$

$$= -\int_0^{2\pi} R^2(\cos^2 t + \sin^2 t)\,dt$$

$$= -2\pi R^2$$

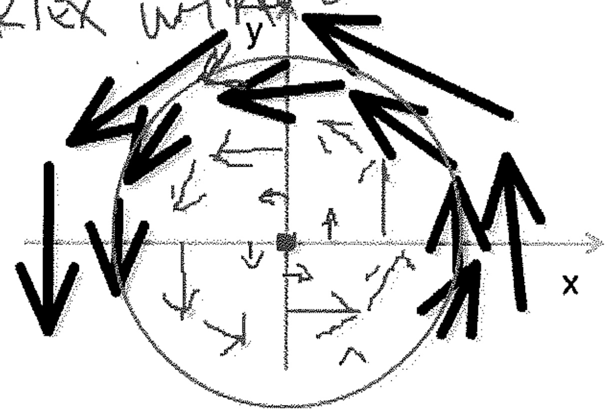
ASK: why is flux non-zero?
why is flux negative?

Circulation Examples

Calculate the flux over curve C, where C is the circle of radius R.

c) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

VORTEX WHIRPOOL



what will flux be?

$M = -R\sin t$, $N = R\cos t$

$$\text{flux} = \int_0^{2\pi} (-R\sin t)(R\cos t\,dt) - (R\cos t)(-R\sin t\,dt)$$

$$= \int_0^{2\pi} 0\,dt$$

$$= 0$$

Ask: why is ~~ф~~ flux = 0?

Answer: no flow in/out of C.

Conservative Vector Fields

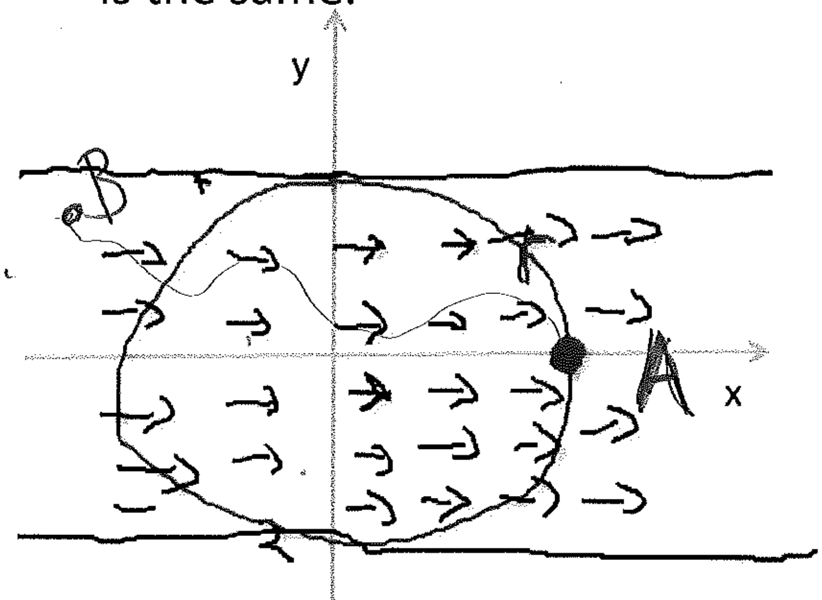
Recall the Pipe example.

a) Why was the circulation zero?

net cancellation

b) For any path that starts and ends at point A, and stays inside "the pipe", the circulation is $\underline{\quad 0 \quad}$.

c) For all paths that starts at A and ends at point B, the integral $\underline{\int_C \vec{V} \cdot d\vec{r}}$ is the same.



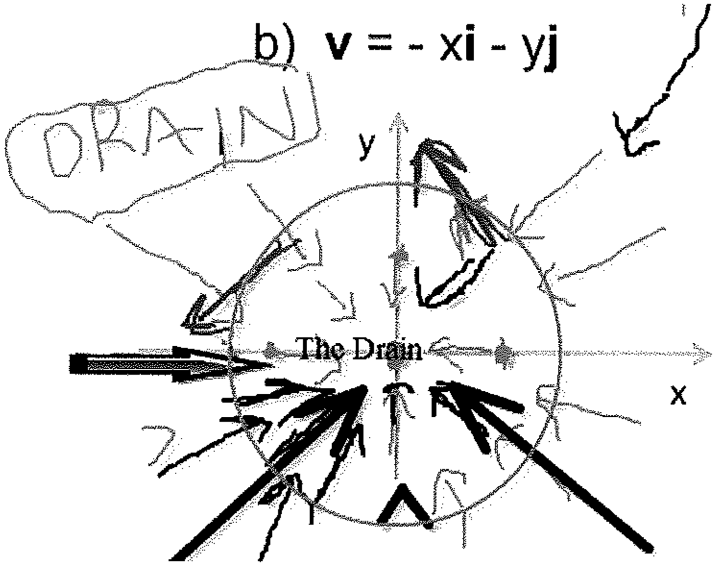
C = any path from A to B

$\vec{V}$ is conservative.

Conservative Vector Fields

Is this vector field conservative?

b) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$



If conservative, $\exists$ scalar field S s.t. $\nabla S = \vec{v}$.

Assume S exists. Then;

$$\frac{\partial S}{\partial x} = -x \Rightarrow S = \frac{-x^2}{2} + \underline{f(y)}$$

$$\frac{\partial S}{\partial y} = -y \Rightarrow S = \frac{-y^2}{2} + \underline{g(x)}$$

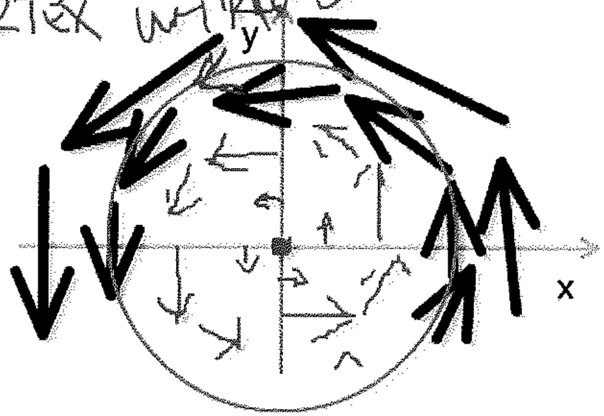
$f(y) = -y^2/2$, $g(x) = -x^2/2$, $\underbrace{S = -x^2/2 - y^2/2}_{\text{paraboloid opening downward, "potential field"}}$, and $\vec{v}$ is conservative.

Conservative Vector Fields

Is this vector field conservative?

c) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

VORTEX WHIRLPOOL



Assume exists. Then

$$\frac{\partial S}{\partial x} = -y \Rightarrow S = -xy + f(y)$$

$$\frac{\partial S}{\partial y} = x \Rightarrow S = +xy + g(x)$$

$\Rightarrow$ ~~∄ f(x)~~ No $f(y)$ and $g(x)$ exist to make those equal.

$\Rightarrow$ ~~∄~~ S DNE

$\Rightarrow \vec{v}$ not conservative.

Summary

Fill in the blanks:

a) Circulation measures flow (along) tangent to path C.

b) Flux measures the flow out of of C.

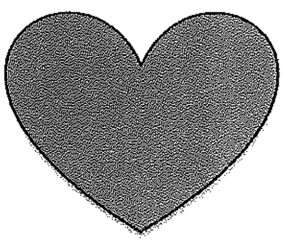
velocity field equation	velocity field equation	circulation	flux	is v conservative?
pipe	**v** = 2**i** for –R ≤ y ≤ +R, **v** = **0** otherwise	0	0	Y
drain	**v = -xi - yj**	0	$2\pi R^2$	Y
vortex, whirlpool	**v = -yi + xj**	$2\pi R^2$	0	N

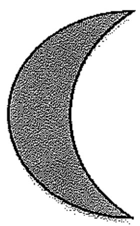
Green's Theorem (special case of Stoke's Thm)

If D is a region that is closed, simple,
and P and Q are scalar fields that are differentiable on D,
and C is the boundary of D, then:

$$\text{(OUTWARD) } \text{FLU}\cancel{X} = \oint \vec{F}\cdot\vec{n}\,ds = \oint_C M\,dy - N\,dx = \iint_D \underbrace{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}_{\vec{\nabla}\cdot\vec{F} = \text{divergence}} dx\,dy$$

Below are five regions. For which regions can we apply Green's Theorem?

a) 
OK

b) 
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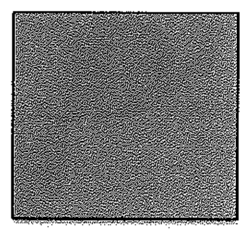
c) 
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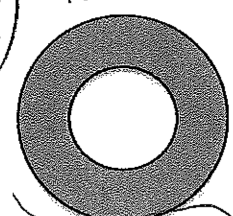
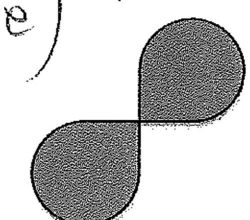
d) NOT SIMPLE 

e) NOT SIMPLE, 

need to do special things: write two integrals.

simple = no holes, boundary "non-self-intersecting". C divides plane into one interior and one exterior.

Recitation 22 | Green's Theorem

a) Evaluate $\oint_C y^2\,dx + 2xy\,dy$, C is one loop of $r = 2\sin 2\theta$

b) Change the integral so that it represents the area of one loop.

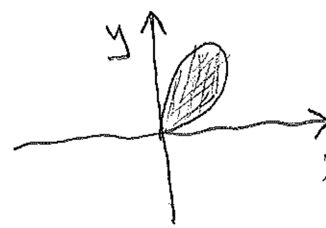
a) We need a formulation of Green's theorem. We can use

$$\text{flux} = \oint M\,dy - N\,dx = \iint_D \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy \left(= \iint_D \overbrace{\begin{bmatrix} M \\ N \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}}^{\vec{V}\cdot\vec{\nabla}} \right)$$

$= \cancel{\text{AREA}}$

(some textbooks use a slightly different formula)

$$\Rightarrow \left.\begin{aligned} M &= 2xy, & \partial M/\partial x &= 2y \\ N &= -y^2, & \partial N/\partial y &= -2y \end{aligned}\right\}$$ integrand is zero, so answer is zero.



b) For area, we need $\frac{\partial M}{\partial x} + \partial N/\partial y = 1$.

We can choose:

$$\left.\begin{aligned} M &= +3xy & \Rightarrow\ M_x &= 3y \\ N &= -y^2 & \Rightarrow\ N_y &= -2y \end{aligned}\right\} \quad \text{AREA} = \iint M_x + N_y\,dx\,dy = \int_0^{\frac{\pi}{2}} \int_0^{2\sin 2\theta} r\,dr\,d\theta$$

Announcements

Quiz 4: Tuesday April 15
Homework 8: due Tues Apr 15 at 11:59 pm. Questions?
Homework 9: due Tues Apr 15 at 11:59 pm. Questions?
Questions for Quiz 4 (not graded)
Office Hours: Monday 7:30 to 9:30
Survey: please complete the brief technical issues survey, email sent last Wed.

Graded group work activity. Solve the question below in groups of 3 to 5 students, you have about 10 minutes. I'll circulate from room to room.

Problem 1 (10 points)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$, and C its boundary Consider the line integral

$\int_C xy\,dx - xy^2\,dy$. Use Green's theorem to convert to an double integral,

and express this as a double integral in polar coordinates with limits.

Problem 1 (10 points)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$, and C its boundary Consider the line integral

diff wrt y   diff wrt x

$\int_C xy\,dx - xy^2\,dy$. Use Green's theorem to convert to an double integral, and express this as a double integral in polar coordinates with limits.

$$\left.\begin{aligned} M &= -xy^2, & \frac{\partial M}{\partial x} &= -y^2 \\ N &= -xy, & \frac{\partial N}{\partial y} &= -x \end{aligned}\right\} \quad \text{Area} = \iint \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$$

$$= \iint \left(-y^2 - x\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} \left(-r^2\sin^2\theta - r\cos\theta\right) r\,dr\,d\theta$$

Fundamental Theorem of Line Integrals

If **F** is a conservative field, then: $\int_C \vec{F}\cdot d\vec{r} = \int \nabla f \cdot d\vec{r} = f(\vec{b}) - f(\vec{a})$

$\vec{b} = \vec{r}(b),\ \vec{a} = \vec{r}(a)$

Example

Calculate line integral of $\mathbf{F} = (x^2-y)\mathbf{i} + (y^2 - x)\mathbf{j}$, over path

$\mathbf{r} = \cancel{a}^{c_1}\cos(t)\mathbf{i} + \cancel{b}^{c_2}\sin(t)\mathbf{j},\ \ 0 \le t \le 2\pi$

Is $\vec{F}$ conservative?

$\frac{\partial f}{\partial x} = x^2 - y \Rightarrow f = \frac{x^3}{3} - yx + \phi_1(y)$

$\partial f/\partial y = y^2 - x \Rightarrow f = \frac{y^3}{3} - xy + \phi_2(x)$

$\phi_1 = y^3/3,\quad \phi_2 = x^3/3,\quad f = \frac{1}{3}(x^3+y^3) - xy$

$\Rightarrow$ conservative.

EASIER WAY:

$\frac{\partial}{\partial y}\frac{\partial f}{\partial x}\ (x^2 - y)$

should equal

$\partial/\partial x\ \frac{\partial f}{\partial y} = \partial/\partial x(y^2 - x)$

it does.

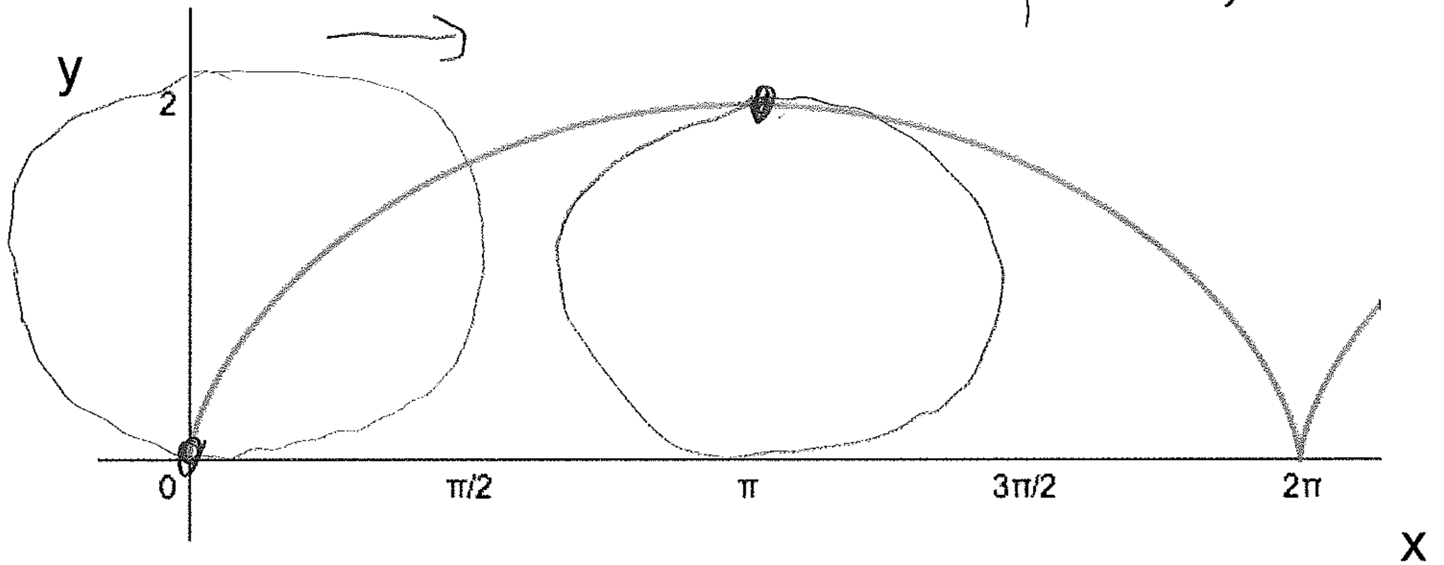
$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = f(2\pi) - f(0)$

$= 0$

Because $f(2\pi) = f(0)$

The Cycloid

sometimes we are given a path in terms of a paramter, t.



The curve traced by a point on a rolling wheel is

$$x(t) = t - \sin(t)$$
$$y(t) = 1 - \cos(t)$$

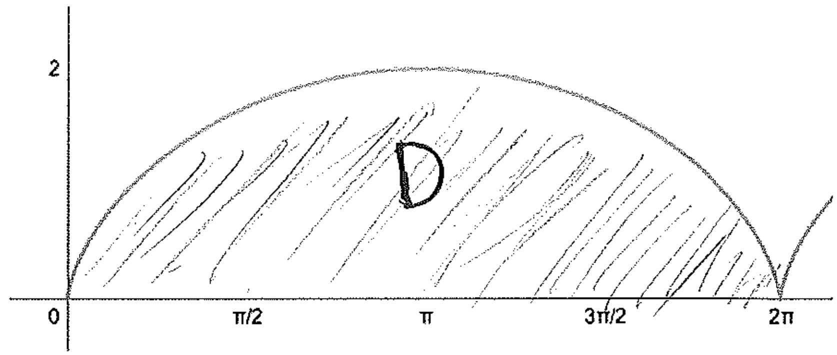
The Cycloid

Find the area under one arch of the cycloid:
$x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$

call the area D

$$A = \iint_D dx\,dy$$

We don't have $y = y(x)$ explicitly.
What can we do?



Introduce

$$\left.\begin{aligned} M &= x, \quad \frac{\partial M}{\partial x} = 1 \\ N &= 1, \quad \frac{\partial N}{\partial y} = 0 \end{aligned}\right\} A = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \oint_C x\,dy - 0\,dx$$

$$= \int_{C_1} x\,dy + \int_{C_2} x\,dy$$

$$= \int_0^{2\pi} t \cdot 0\,dt + \int_0^{2\pi} \sin^2 t - t\sin t\,dt$$

$$= \dots = 3\pi$$

On C_1: $x = t$, $y = 0$, $dy = 0\,dt$

On C_2: $x = t - \sin t$, $y = 1 - \cos t$, $dy = -\sin t$

Recitation 24	Today: Pop Quiz, Surface Integrals

- There's a pop quiz today! :D
- You have a few minutes to review your notes.
- Start time: 8:10
- Ends at: 8:30?
- Pop quiz grading
  - 5 points: on the right track
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, either
  - a) **work on whiteboard in breakout room**
    - write in text chat that you'd like to work in breakout room,
    - submit work by letting me know when done, or email me a screen capture of your work
  - b) **work on paper and give work to facilitator**
    - leave 2 inch margin
    - write your name and QH6 at the top
    - facilitator can email quiz to cdlops@pe.gatech.edu
  - c) **work on paper and take a photo of your work**
    - email your photo to me before 8:40
    - write in text chat that you are emailing your work to me

Pop Quiz

Set up as a double integral, the surface integral of F dot n ds , where the surface is $\cancel{z=}$ x^2 $\cancel{-y^2}$, and $0 \le x \le 1$, $-1 \le y \le 1$. $F = x\hat{i} + z\hat{k}$, surface is $z = x^2 - y^2$

Flux $= \iint F \cdot \vec{n}\, d\sigma$, need parameterization:

$\vec{r} = u\hat{i} + v\hat{j} + (u^2 - v^2)\hat{k}$,

$$\frac{\partial \vec{r}}{\partial u} = \begin{bmatrix} 1 \\ 0 \\ 2u \end{bmatrix}, \quad \frac{\partial \vec{r}}{\partial v} = \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix}$$

$$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \begin{bmatrix} -2u \\ +2v \\ 1 \end{bmatrix}$$ ✓, $\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^2 + 4v^2 + 1}$

$$\Rightarrow \vec{n} = \frac{1}{\sqrt{\sim}} \begin{bmatrix} -2u \\ 2v \\ 1 \end{bmatrix}$$

$$\vec{F}(x,y) = \begin{bmatrix} u \\ 0 \\ u^2 - v^2 \end{bmatrix}$$

$$\vec{F} \cdot \vec{n} = \frac{-2u^2 + u^2 - v^2}{\|\vec{n}\|} = -(u^2 + v^2) / \sqrt{\sim}$$

$$\vec{F} \cdot \vec{n}\, ds = \frac{-(u^2+v^2)}{\sqrt{\sim}} \sqrt{4u^2 + 4v^2 + 1}\, du\, dv$$

$$= (-u^2 - v^2)\, du\, dv$$

$$= -\int_0^1 \int_{-1}^{+1} u^2 + v^2\, dv\, du$$

Announcements

Quiz 4: Marked on Friday? Monday? I'm not sure yet.
Last HW : due Sun Apr 27
Technical issues during lecture yesterday: fiber cut?

Engagement Survey
Please complete the brief engagement survey, email sent last Tuesday.

Technical Survey
Follow-up question: I often let students write on the board at any time. In what ways, if any, did this help your learning in recitations?

Parametric Representations

Find an equation in x, y, z, for the surface whose parametric representation is

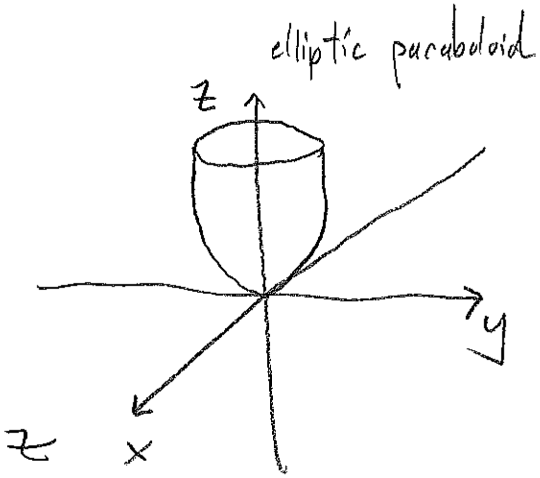
$$\mathbf{r} = Au\cos(v)\mathbf{i} + Bu\sin(v)\mathbf{j} + u^2\mathbf{k},\quad u \geq 0,\quad 0 \leq v \leq \pi.$$

Describe and sketch the surface.

We must identify functions $x(u,v)$, $y(u,v)$, $z(u,v)$.

We can use:

$$\left.\begin{aligned} x &= Au\cos v \\ y &= Bu\sin v \\ z &= u^2 \end{aligned}\right\} \text{ where } u \geq 0,\ v \in [0, \pi]$$



We need an equation in x, y, z:

Try: $\left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 = u^2\cos^2 v + u^2\sin^2 v = z$

Parametric Representations

Find parametric representations for the following surfaces.

a) the upper half of $4x^2 + 9y^2 + z^2 = 36$

b) the part of the plane $z = x + 2$ inside the cylinder of $x^2 + y^2 = 1$

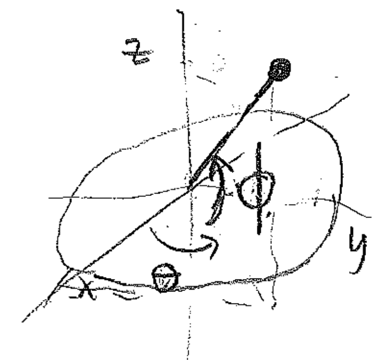
why? let $\bar{x} = x/3$, $\bar{y} = y/2$, $\bar{z} = \frac{z}{6}$
then $\bar{x} = \cos\theta \cos\phi$
etc

a) Divide by 36: $\frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{6^2} = 1$ (1)

Ellipsoid. Try a modification of spherical:

$$\left.\begin{aligned} \frac{x}{3} &= \cos\theta \cos\phi \\ y/2 &= \sin\theta \cos\phi \\ z/6 &= \sin\phi \end{aligned}\right\}$$

satisfies (1) for all $\theta \in [0, 2\pi]$, $\phi \in [0, \pi/2]$



NOTE: could also use $z = 6\cos\phi$, $x = 3\cos\theta\sin\phi$, etc

b) Try: $\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \\ z &= r c + 2 \end{aligned}\right\}$ $x^2 + y^2 \le 1$ or $r^2 \le 1$, $\theta \in 2\pi$

Recitation ~~24~~ 25	Today: Surface Integrals

a) What properties does a parametric representation of a surface need to have?

1) continuous
2) one-to-one
3) satisfy a given equation

b) Find a parametric representation for the part of the plane $z = x + 2$ in the first octant and inside the cylinder $x^2 + y^2 = 1$.

Parametric representation is:

$$\vec{r} = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

where

$$x = u\cos v$$

$$y = u\sin v$$

$$z = u\cos v + 2$$

which is 1, 2, 3, for $u \in [0,1]$, $v \in [0, \frac{\pi}{2}]$

first octant

Announcements

Quiz 4: marked yesterday, grades should be entered today.
HW grades: check in t-square that I entered grades correctly
Last HW : due Sun Apr 27
Cut-off for final exam: I don't know if there is one, or what cut-off would be

Engagement Survey
Please complete the brief engagement survey, reminder email sent yesterday.

Technical Survey
Follow-up question: most students didn't communicate with microphones very often. Why do you think this was the case?

Surface Area of z = f(x,y)

We want the surface area of $z = y^2$, over $0 \le x \le a$, $0 \le y \le b$.

a) Find a parametric representation for the surface.

b) Find equation of normal at an arbitrary point on the surface.

c) Set-up an integral that represents the surface area

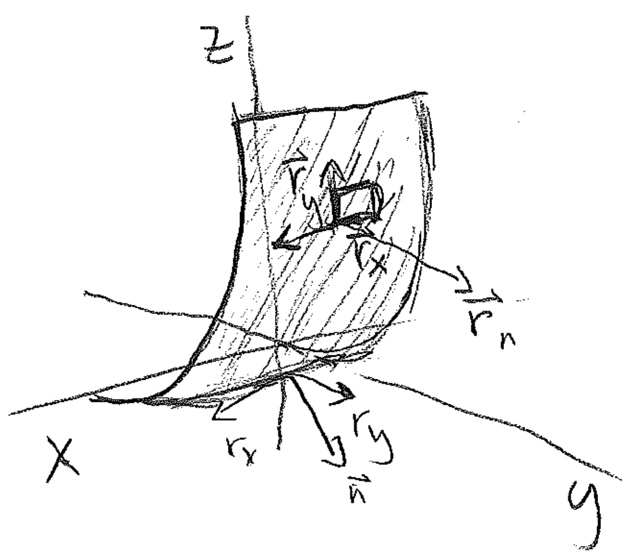
a) $x = u$, $u \in [0, a]$

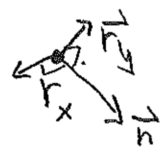
$y = v$, $v \in [0, b]$

$z = v^2$

$\Rightarrow \vec{r} = \begin{bmatrix} u \\ v \\ v^2 \end{bmatrix}$, $\vec{r}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{r}_v = \begin{bmatrix} 0 \\ 1 \\ 2v \end{bmatrix}$, $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = \begin{bmatrix} 0 \\ -2v \\ 1 \end{bmatrix}$

Jacobian $= \|\vec{r}_u \times \vec{r}_v\| = \sqrt{1+4v^2}$





$$\text{Area} = \int_0^a \int_0^b \sqrt{1+4v^2}\, dv\, du$$

$$= \iint dS$$

Flux Across A Surface

Flux is a measure of **flow rate per unit length**, or **flow rate per unit area**.

To calculate flux across a curve: $$\text{flux} = \int_C \vec{v}\cdot\vec{n}\,du = \int_C M\,dy - N\,dx$$

C = any curve (doesn't have to be closed)

$\vec{v}$ = velocity

$\vec{n}$ = unit normal

u = parameter

To calculate flux across a surface:

Jacobian

$$\text{FLUX} = \iint_S \vec{v}\cdot\vec{n}\,dS = \iint_S \vec{v}\cdot\vec{n}\,\|\vec{N}\|\,du\,dv$$

$\vec{N} = \det(\vec{r}_u \times \vec{r}_v)$

Pop Quiz

Set up as a double integral, the surface integral of F dot n ds , where the surface is $\cancel{z=}$ $x^2 \cancel{+ y^2}$, and $0 \le x \le 1$, $-1 \le y \le 1$. $F = x\hat{i} + z\hat{k}$, surface is $z = x^2 - y^2$

FLUX $= \iint F \cdot \vec{n}\, d\sigma$, need parameterization:

$\vec{r} = u\hat{i} + v\hat{j} + (u^2 - v^2)\hat{k}$,

$\frac{\partial \vec{r}}{\partial u} = \begin{bmatrix} 1 \\ 0 \\ 2u \end{bmatrix}$, $\frac{\partial \vec{r}}{\partial v} = \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix}$

$\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \begin{bmatrix} -2u \\ +2v \\ 1 \end{bmatrix}$ ✓, $\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4u^2 + 4v^2 + 1}$

$\Rightarrow \vec{n} = \frac{1}{\sqrt{\quad}} \begin{bmatrix} -2u \\ 2v \\ 1 \end{bmatrix}$

$\vec{F}(x,y) = \begin{bmatrix} u \\ 0 \\ u^2 - v^2 \end{bmatrix}$

$\vec{F} \cdot \vec{n} = \frac{-2u^2 + u^2 - v^2}{\|\vec{n}\|} = -(u^2 + v^2)/\sqrt{\quad}$

$\vec{F} \cdot \vec{n}\, dS = \frac{-(u^2 + v^2)}{\sqrt{\quad}} \sqrt{4u^2 + 4v^2 + 1}\; du\,dv$

$= (-u^2 - v^2)\, du\,dv$

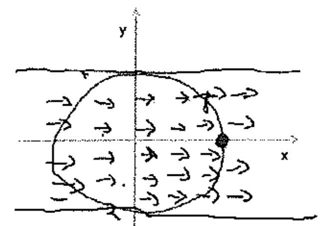
$= -\int_0^1 \int_{-1}^{+1} u^2 + v^2 \; dv\,du$

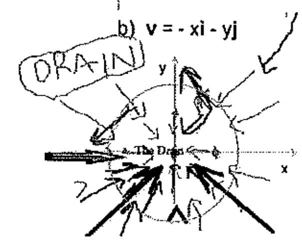
Recitation 26 | Today: Divergence

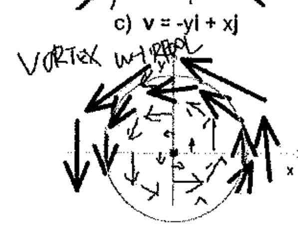
Divergence measures a flow's tendancy to expand / explode.

If $\mathbf{v}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j}$, then $\text{div}(\mathbf{v}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$.

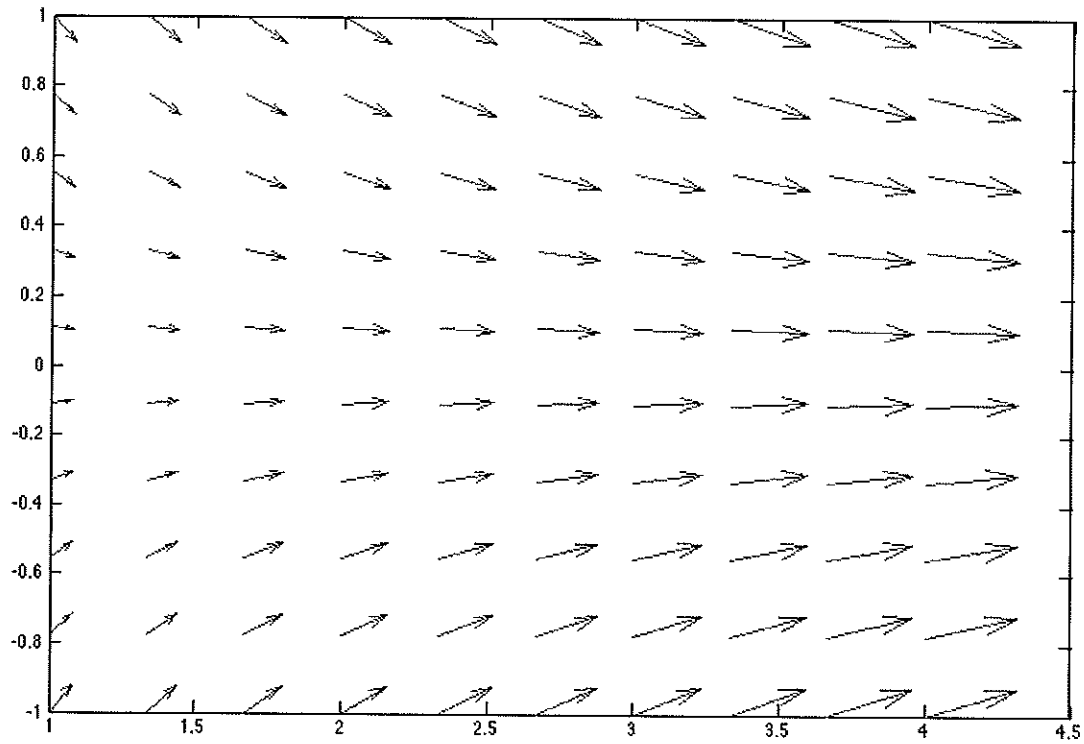
velocity field equation	velocity field equation	divergence
pipe	$\mathbf{v} = 2\mathbf{i}$ for $-R \le y \le +R$, $\mathbf{v} = \mathbf{0}$ otherwise	$\nabla \cdot \vec{v} = \frac{\partial}{\partial x}(2) + \partial/\partial y\,(0) = 0$ flow not expanding
drain	$\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$	-2 flow is "compressing"
vortex, whirlpool	$\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$	0
nozzle	$\mathbf{v} = x\mathbf{i} - y\mathbf{j}$ $x \ge 1$	0 as x increases flow is compressing, and its speed increasing







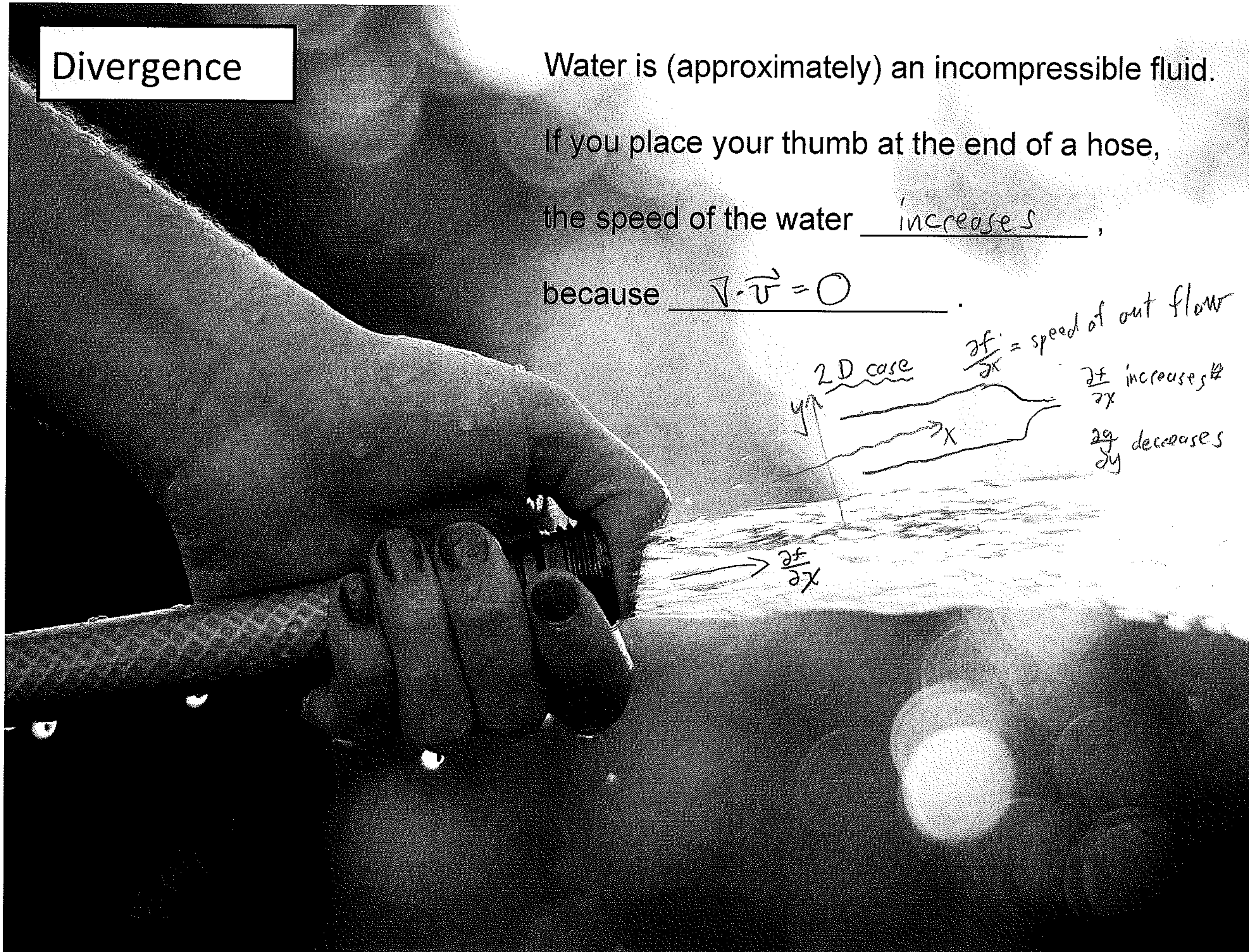
Incompressible Fluids



If a fluid is incompressible, then its divergence is 0 .

The field **v** = x**i** – y**j** could represent an incompressible flow

As x increases, flow moves towards x-axis, and its speed increases.



Divergence

Water is (approximately) an incompressible fluid.

If you place your thumb at the end of a hose,

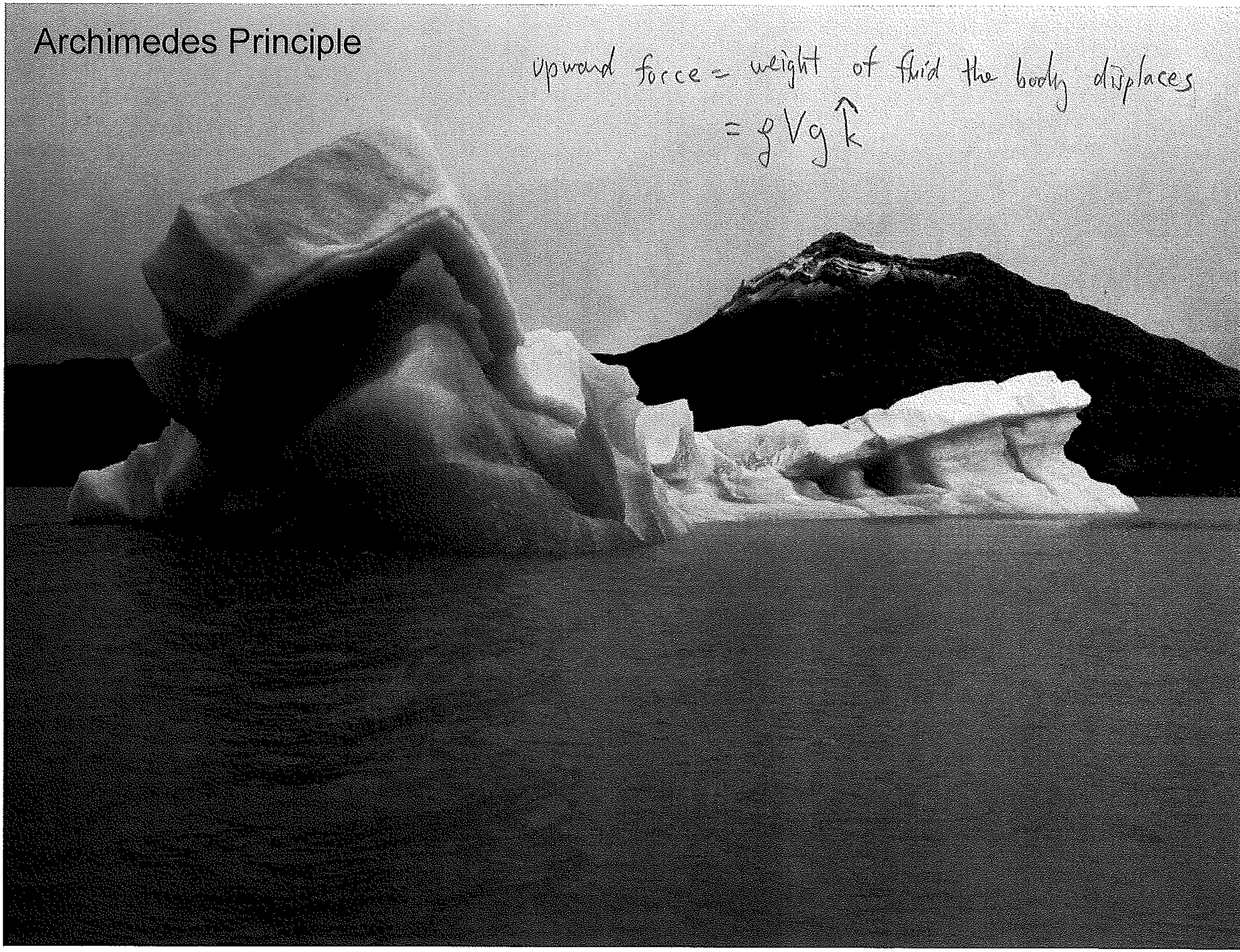
the speed of the water __increases__,

because __$\nabla \cdot \vec{v} = 0$__.

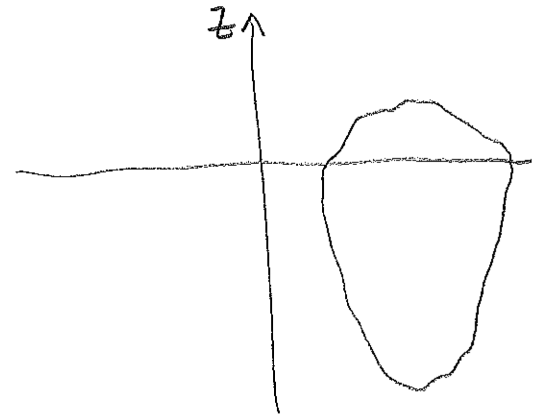
Archimedes Principle

upward force = weight of fluid the body displaces

$= \rho V g \hat{k}$



Prove Archimedes Principle



Let S be surface of underwater portion of object.

dS = area of small section.

$\vec{P}$ = pressure = $\vec{F}/dS$

$$\vec{F} = \vec{P}\, dS$$

$$= -|P|\,\vec{n}\, dS, \quad \vec{n} = \text{unit inward normal}$$

$$= -g\rho z\, \vec{n}\, dS, \quad \text{because } P = -g\rho z$$

$$\text{Total force} = \iint_S -g\rho z\, \vec{n}\cdot\vec{k}\, dS, \quad \text{because horizontal components cancel.}$$

$$= \iint_S \vec{v}\cdot(\vec{n})\, dS, \quad \text{let } \vec{v} = \begin{bmatrix} 0 \\ 0 \\ g\rho z \end{bmatrix}, \quad \vec{n} = \text{unit inward normal}$$

$$= \iiint_V \nabla\cdot\left(\begin{bmatrix} 0 \\ 0 \\ g\rho z \end{bmatrix}\right) dV, \quad \text{by div. theorem}$$

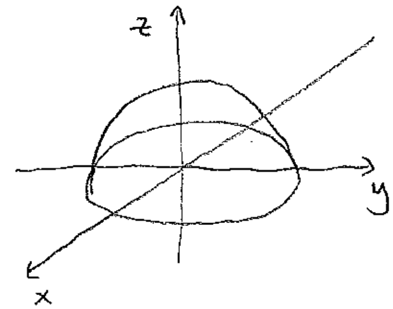
$$= g\rho V, \quad V = \text{volume of object that is submerged}$$

Electric Charge

E = electric field. Then, Gauss's Law states that:

total charge = (ε_0)(flux of **E** through closed surface)

Find the total charge contained in a solid hemisphere if **E** = x**i** + y**j** + z**k**.



EASY WAY

$$\text{charge} = \varepsilon_0 \oiint \vec{E}\cdot\vec{n}\, d\sigma, \quad \vec{n} \text{ unit outward}$$

$$= \varepsilon_0 \iiint \nabla\cdot\vec{E}\, dV, \quad \text{by div. thm.}$$

$$= \varepsilon_0 \iiint (1+1+1)\, dV$$

$$= 3\varepsilon_0 \cdot \tfrac{2}{3}\pi R^3, \quad R = \text{radius of hemisphere},$$

$$= 2\varepsilon_0 \pi R^3$$

HARDER WAY

parameterize top surface: $\vec{n} = \frac{1}{R}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

bottom surface: flux is zero.

$$\vec{E}\cdot\vec{n} = \frac{1}{R}(x^2+y^2+z^2) = \frac{R^2}{R} = R \quad \text{(on surface)}$$

$$d\sigma = \|\vec{n}\|\, du\, dv$$

$$\vec{r} = \begin{bmatrix} R\cos u \sin v \\ R\sin u \sin v \\ R\cos v \end{bmatrix}, \quad \left|\frac{\partial r}{\partial u}\times\frac{\partial r}{\partial v}\right| = R^2\sin v, \quad \text{charge} = \varepsilon_0 \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \overbrace{R}^{\vec{E}\cdot\vec{n}}\big(\overbrace{R^2\sin v}^{\text{Jacobian}}\big)\, dv\, du = 2\varepsilon_0\pi R^3$$

Quadratic surface: a question from last year's quiz 2

Consider the surface

$$-6x + x^2 + 4y + y^2 + 8z - z^2 = 4$$

This is a qudratic surface. Find out the center, and what kind it is. Draw a picture, labeling the center and the axes.

Express in a standard form by completing the square

$$(x^2 - 6x + 9 - 9) + (y^2 + 4y + 4 - 4) - (z^2 - 8 + 16 - 16) = +4$$

$$(x-3)^2 + (y+2)^2 - (z-4)^2 + 16 = 4 + 9 + 4$$

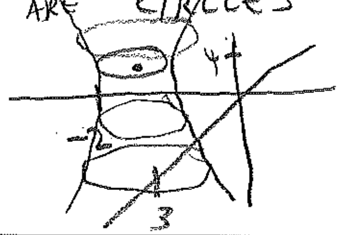
$$(x-3)^2 + (y+2)^2 - (z-4)^2 = 1$$

LEVEL CURVES CAN BE FOUND BY SETTING $z = \text{CONSTANT} = K$.

$$(x-3)^2 + (y+2)^2 = 1 + (K-4)^2$$

$\Rightarrow$ LEVEL CURVES ARE CIRCLES, RADIUS $1 + (K-4)^2$.

AT WHAT VALUE OF z ARE CIRCLES THE SMALLEST? $\boxed{\text{WHEN } K = 4 = z}$



HYPERBOLOID, ONE SHEET,
CENTER AT $(3, -2, +4)$

Quadratic surface: a question from last year's quiz 2

WolframAlpha computational knowledge engine

x^2 - 6x +4y +y^2 +8z -z^2= 4

Examples Random

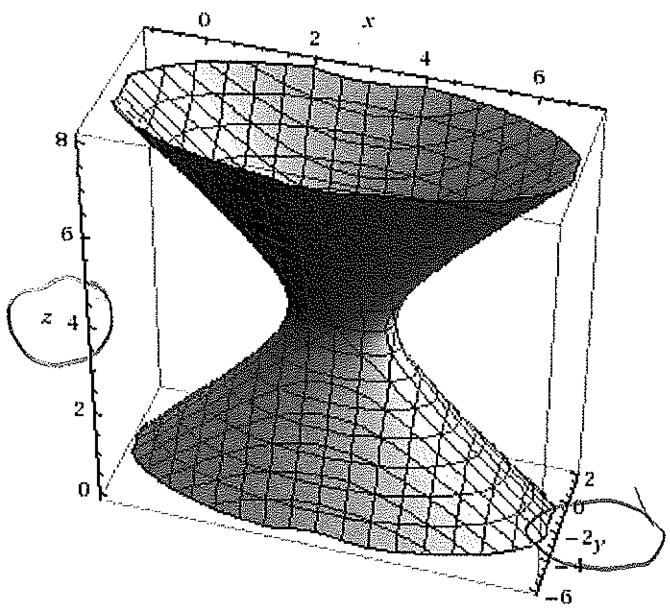
Input:

$$x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$$

Geometric figure:

one-sheeted hyperboloid

Surface plot:



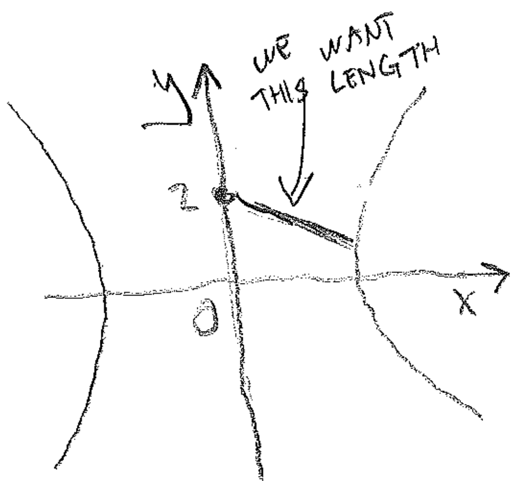
CENTER AT $(3, -2, +4)$

Enable interactivity

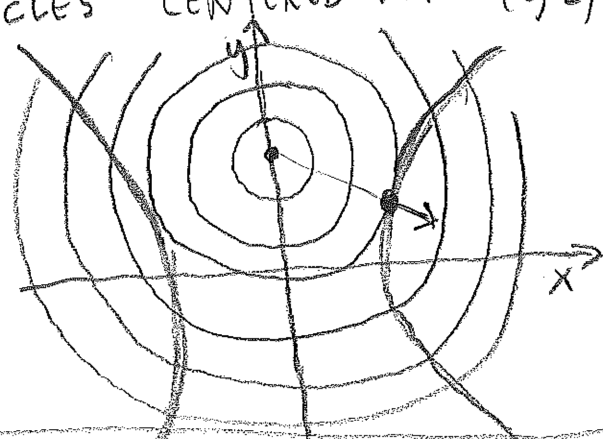
An Optimization Problem

Find the minimum value of the function $f(x,y) = x^2 + (y-2)^2$ subject to the constraint $x^2 - y^2 = 1$.

OBSERVE THAT $f(x,y)$ GIVES THE SQUARE OF THE DISTANCE BETWEEN THE POINT $(0,2)$ AND ANY OTHER POINT ON THE PLANE. OUR CONSTRAINT IS A HYPERBOLA, SO WE ARE LOOKING FOR THE DISTANCE BETWEEN $(0,2)$ AND THE HYPERBOLA.



ALSO NOTE THAT THE LEVEL CURVES OF f ARE CIRCLES CENTERED AT $(0,2)$:



AT THE MINIMUM VALUE OF $f(x,y)$, ∇f IS PARALLEL TO $\nabla(x^2-y^2)$

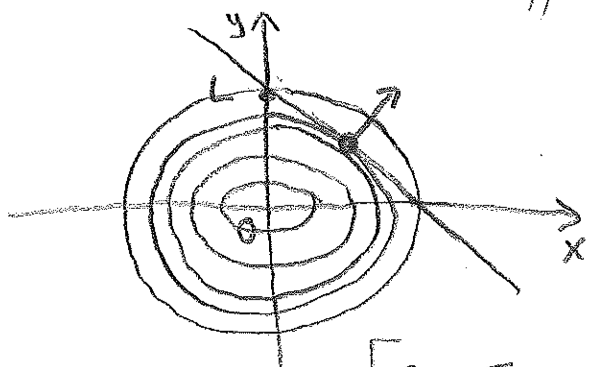
AT THE POINT WHERE $f(x,y)$ IS MINIMUM, ∇f IS PARALLEL TO $\nabla(x^2-y^2)$:

$\nabla f = \lambda \nabla(x^2-y^2)$. $\Rightarrow \begin{bmatrix} 2x \\ 2y-4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ -2y \end{bmatrix}$. SOLVING THIS SYSTEM, WITH $x^2-y^2=1$, YIELDS $\lambda = 1$, $y = 1$, $x = \pm\sqrt{2}$, $f = 3$.

An Optimization Problem

Find the minimum of the function $f(x,y) = (x/a)^2 + (y/b)^2$ subject to the constraint $x + y = L$. The numbers a, b, and L are positive constants.

LEVEL CURVES OF $f(x,y)$ ARE ELLIPSES; OUR CONSTRAINT IS A STRAIGHT LINE $y = L - x$.



THE CURVE THAT TOUCHES THE LINE ONLY ONCE DOES SO AT THE POINT WHERE $\nabla f = \lambda \nabla\left(\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2\right)$.

$$\nabla f = \begin{bmatrix} 2\frac{x}{a}\cdot\frac{1}{a} \\ 2\frac{y}{b}\cdot\frac{1}{b} \end{bmatrix} = \begin{bmatrix} \frac{2x}{a^2} \\ \frac{2y}{b^2} \end{bmatrix}, \quad \nabla(x+y) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

SO $\nabla f = \lambda \nabla(x+y)$ AND $x+y=L$ YIELDS;

$$\left.\begin{aligned} 2x &= \lambda a^2 \\ 2y &= \lambda b^2 \\ x+y &= L \end{aligned}\right\} \quad \left.\begin{aligned} x &= \lambda a^2/2 \\ y &= \lambda b^2/2 \\ \frac{\lambda a^2}{2} &+ \frac{\lambda b^2}{2} = L \\ \Rightarrow \lambda &= L/\left(\tfrac{a^2}{2} + \tfrac{b^2}{2}\right) \end{aligned}\right\} \quad \begin{aligned} x &= \tfrac{a^2}{2} L/\left(\tfrac{a^2}{2} + \tfrac{b^2}{2}\right) \\ y &= \tfrac{b^2}{2} L/\left(\tfrac{a^2}{2} + \tfrac{b^2}{2}\right) \\ f_{minimum} &= \frac{L^2}{4}\left(\frac{a^2}{2} + \frac{b^2}{2}\right)^2 + \frac{L^2}{4}\left(\frac{a^2}{2} + \frac{b^2}{2}\right)^2 \end{aligned}$$

An Optimization Problem

A company produces widgets at N factories. The cost of producing x_i widgets at factory i is x_i^2/a_i, where $a_i > 0$. Minimize the total cost of producing L widgets.

TOTAL COST IS $C = \sum_{i=1}^{N} x_i^2/a_i$, OUR CONSTRAINT IS $\sum_{i=1}^{N} x_i = L$

THIS IS AN N-DIMENSIONAL CASE OF THE PREVIOUS EXAMPLE, EXCEPT OUR ELLIPSES HAVE THE FORM

$$\frac{x_1^2}{a_1} + x_2^2/a_2 + \dots + \frac{x_N^2}{a_N}$$

$$\nabla C = \lambda \left(\sum x_i \right)$$

$$\nabla C = \begin{bmatrix} 2x_1/a_1 \\ 2x_2/a_2 \\ 2x_3/a_3 \\ \vdots \\ 2x_N/a_N \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$\Rightarrow$ the i'th equation is $2x_i/a_i = \lambda$, OR $x_i = \frac{a_i \lambda}{2}$. Subsitute into constraint:

$$L = \sum x_i = \sum \frac{a_i \lambda}{2} = \frac{\lambda}{2} \sum a_i, \text{ so } \lambda = 2L/\sum a_i.$$

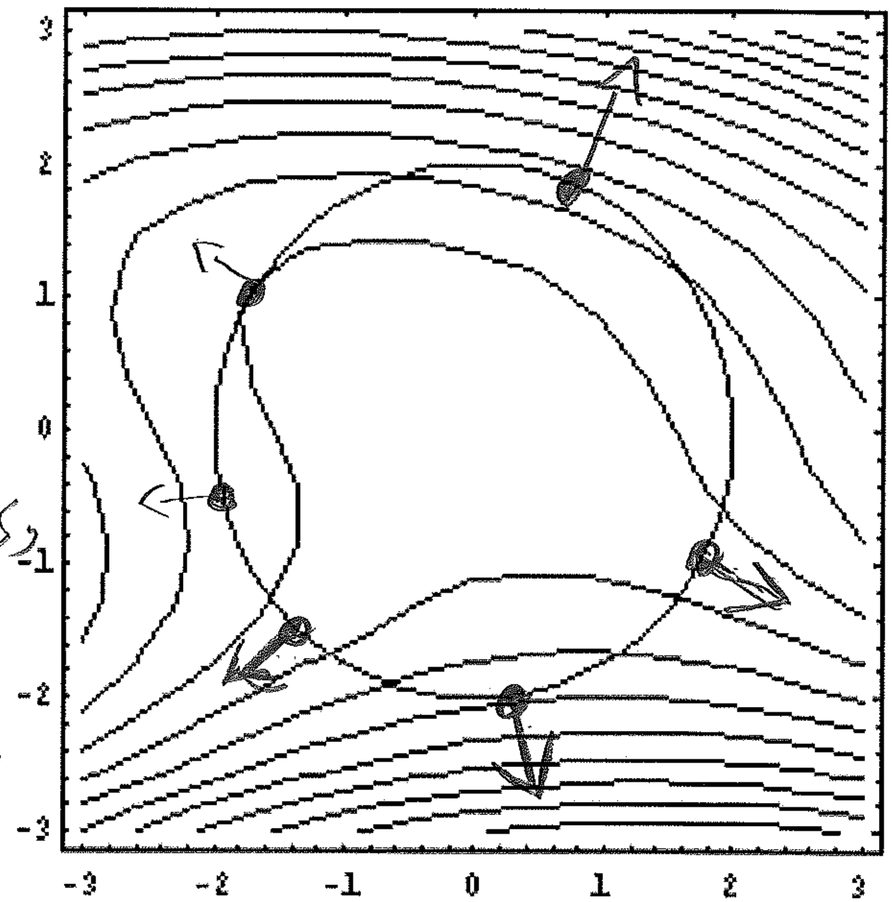
$$\Rightarrow x_i = \frac{a_i \lambda}{2} = \frac{a_i L}{\sum a_i},$$ and minimum cost is $C_{minimum} = \sum \frac{\left(\frac{a_i L}{\sum a_i}\right)^2}{a_i}$

$$= L^2 / \sum a_i$$

A Conceptual Lagrange Multipliers Question

The diagram shows a contour plot of f(x,y), and the circle of radius 2 centered at (0,0). How many local maximums and mins does f(x,y) have on the perimeter of the circle?

Assume the origin is a global max of f(x,y).



POSSIBLY 6 MINS AND MAX'S, WHICH ARE LOCATED WHERE ∇f IS PARARALLEL TO ∇(CONSTRAINT)

Max/Min Electrostatic Potential

The electrostatic potential in the region $0 \le x \le 1$, $0 \le y \le 1$, is given by $V = 48xy - 32x^3 - 24y^2$. Find the locations of the minimum and maximum values.

FIND CRITICAL POINTS:

$\partial V/\partial x = 0 = 48y - 96x^2$

$\partial V/\partial y = 0 = 48x - 48y$

SOLVING YIELDS $(0,0)$, $(\frac{1}{2}, \frac{1}{2})$.

AT THESE CRITICAL POINTS,

$V(0,0) = 0$

$V(\frac{1}{2}, \frac{1}{2}) = 2$.

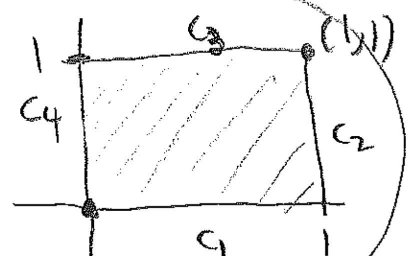
NOW CHECK BOUNDARIES.

ALONG C_1, $y = 0$, so

$V = -32x^3$, $x \in [0,1]$

$\Rightarrow$ minimum could be at $(1,0)$,

$V(1,0) = -32$



ALONG C_2, $V = 48y - 32 - 24y^2$, $y \in [0,1]$

$0 = dV/dy = 48 - 48y \Rightarrow y = 1$

$\Rightarrow$ at $(1,1)$, $V(1,1) = -8$

ALONG C_3: $V = 48x - 32x^3 - 24$, $x \in [0,1]$

$0 = \partial V/\partial x = 48 - 96x^2$, $x = \sqrt{2}/2$

$V(\frac{\sqrt{2}}{2}, 1) = 8(2\sqrt{2} - 3)$

ALONG C_4: $V = -24y^2$

$\Rightarrow$ one critical point at origin.

CHECK CORNERS:

$V(0,0) = 0$

$V(0,1) = -24$

$V(1,1) = -8$

$V(1,0) = -32$

COMPARING ALL VALUES WE FOUND,

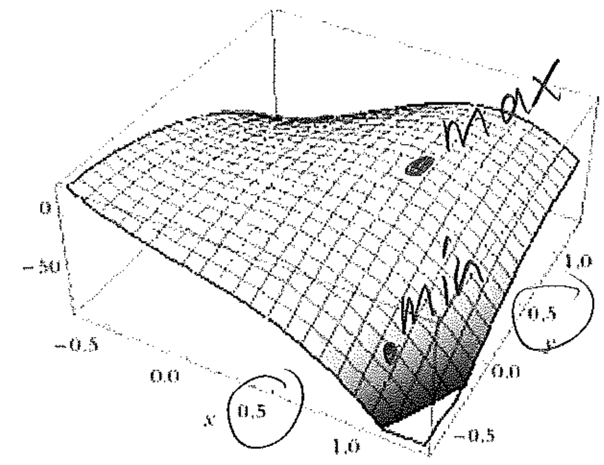
MAXIMUM AT $(\frac{1}{2}, \frac{1}{2})$

MINIMUM AT $(1,0)$

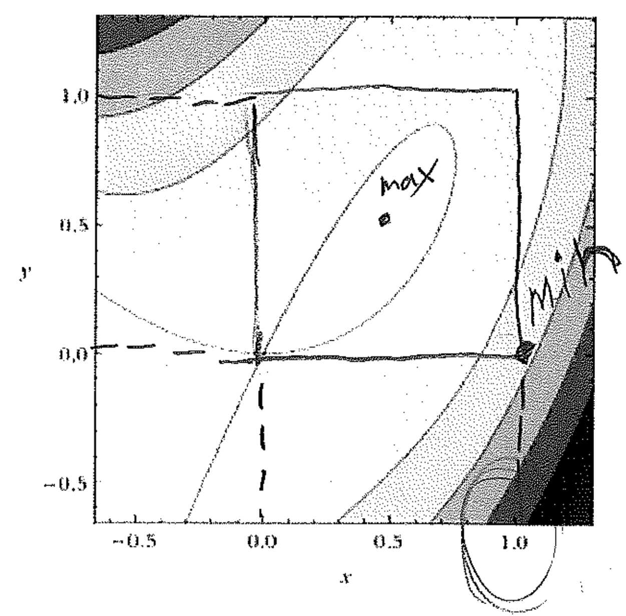
Max/Min Electrostatic Potential

plot $(48\,x\,y - 32\,x^3) - 24\,y^2$

3D plot:

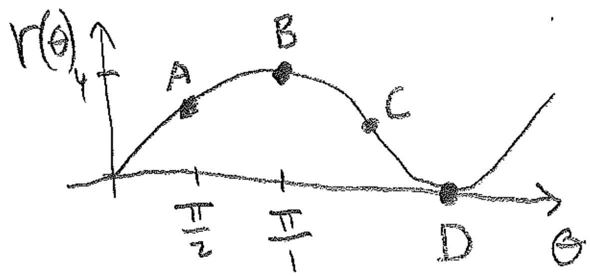


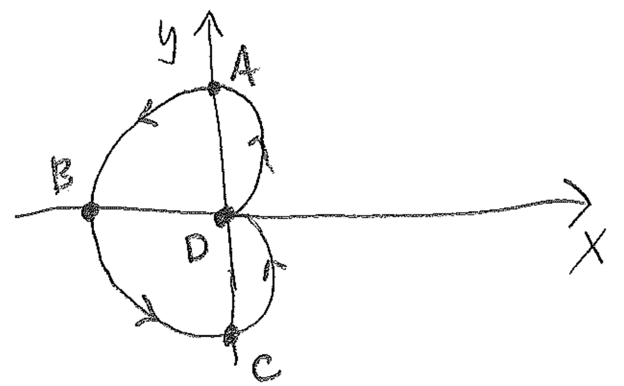
Contour plot:



Setting Up a Polar Integral

Set up, but do not evaluate, an integral representing the area the region enclosed by r = 2 – 2cosθ. Sketch the region of integration.





$$\text{AREA} = \int_0^{2\pi} \int_0^{2-2\cos\theta} r \, dr \, d\theta$$

Convert a Cartesian Integral to a Polar Integral

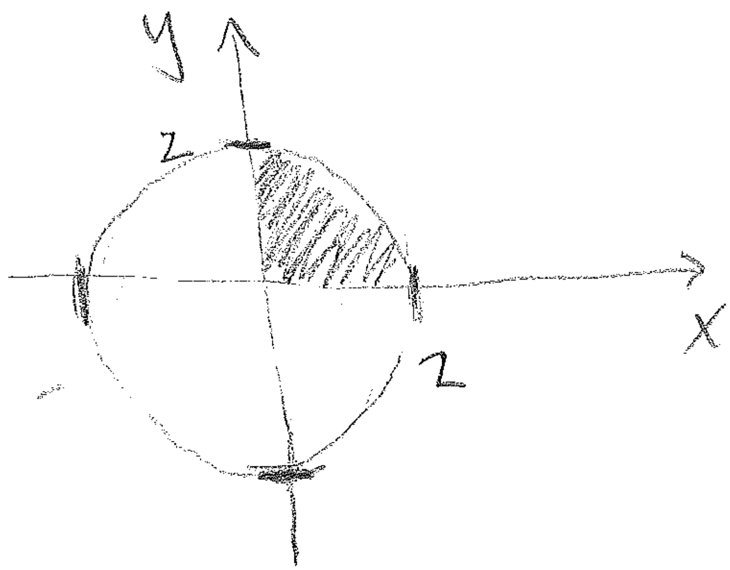
a) Sketch the region of integration
b) Express the integral in polar coordinates

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\, dy\, dx$$

$\sqrt{x^2+y^2} = r$,

$x \in [0, 2]$

$y \in [0, \sqrt{4-x^2}]$, or $y^2 \leq 4 - x^2$, or $x^2+y^2 \leq 4$



$r \in [0, 2]$

$\theta \in [0, \pi/2]$

AREA $= \int_0^{\pi/2} \int_0^2 r \cdot r \, dr\, d\theta$

Quiz 3 Review

For your quiz:

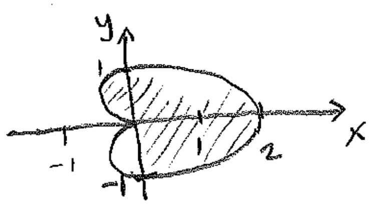
- 2 pages of 8 1/2 x 11 inch notes (both sides) allowed
- Calculators allowed.

Also:

- Office hours: Wednesday, 7:30 pm to 9:30 pm
- If you can, during quiz connect to https://georgiatech.adobeconnect.com/distancecalculusquiz/

Set-up integrals that provide the centroid of the region bounded by r = 1 + cosθ. The mass density at any point in the region is proportional to its distance to the origin.

① PLOT REGION



θ	r
0	2
$\pi/2$	1
π	0
$3\pi/2$	1

(NOTE: by symmetry, y_M should work out to be zero.)

② LIMITS OF INTEGRATION

$\theta \in [0, 2\pi]$

$r \in [0, 1+\cos\theta]$

③ WRITE INTEGRAL FOR MASS

$$\text{MASS} = M = \int_0^{2\pi}\int_0^{1+\cos\theta} \lambda\, r\, dr\, d\theta, \quad \lambda = kr \ (k = \text{constant})$$

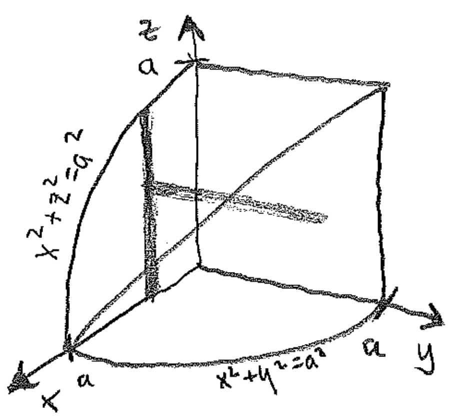
④ WRITE INTEGRALS FOR COORDS OF CENTROID

$$x_M = \frac{1}{M}\int_0^{2\pi}\int_0^{1+\cos\theta} \lambda (r\cos\theta)(r)\, dr\, d\theta = \frac{1}{M}\int_0^{2\pi}\int_0^{1+\cos\theta} k r^3 \cos\theta\, dr\, d\theta$$

$$y_M = \frac{1}{M}\int_0^{2\pi}\int_0^{1+\cos\theta} k r^3 \sin\theta\, dr\, d\theta, \quad \Rightarrow \text{CENTROID AT } (x_M, y_M)$$

A region, with constant density D, is bounded by $x^2 + y^2 = a^2$, and $x^2 + z^2 = a^2$. Find the moment of inertia about the x-axis. Use Cartesian coordinates.

① PLOT/DRAW REGION.



② LIMITS OF INTEGRATION

$y \in [0, \sqrt{a^2-x^2}]$

$z \in [0, \sqrt{a^2-x^2}]$

$x \in [0, a]$

(USING SYMMETRY; WE'LL CONSIDER 1ST OCTANT & MULTIPLY BY 8)

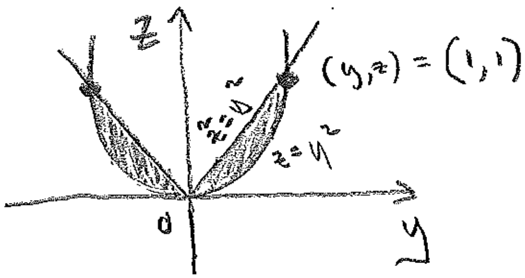
③ SET-UP INTEGRAL

$$I = 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2}} (y^2+z^2) D \, dy\,dz\,dx$$

A region with constant density D is bounded above by $z^2 = x^2 + y^2$, and below by $z = x^2 + y^2$. Find the moment of inertia about the z-axis. Use cylindrical coordinates.

① SKETCH REGION

SLICE THROUGH yz-PLANE



② LIMITS OF INTEGRATION

$z \in [r^2, r]$

$r \in [0, 1]$

$\theta \in [0, 2\pi]$

③ SET-UP INTEGRAL

$$I_z = \int_0^{2\pi} \int_0^1 \int_{r^2}^{r} D\, r \cdot r\, dz\, dr\, d\theta$$

$$= D \int_0^{2\pi} \int_0^1 \int_{r^2}^{r} r^2\, dz\, dr\, d\theta$$

Set-up an integral that represents the volume of the solid bounded by
$x^2 + y^2 + (z - R)^2 = R^2$. (*)

HOW CAN WE DESCRIBE SURFACE IN SPHERICAL COORDINATES?

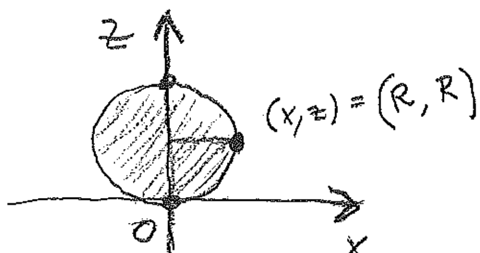
sub: $x = \rho \sin\phi \cos\theta$
$y = \rho \sin\phi \sin\theta$
$z = \rho \cos\phi$ into (*):

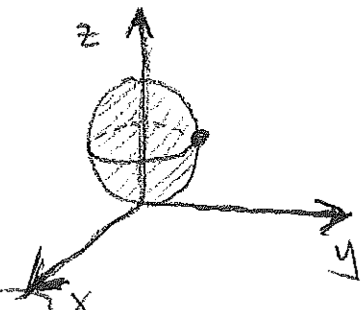
$$\rho^2 \sin^2\phi + (\rho\cos\phi - R)^2 = R^2$$

$$\rho^2 \sin^2\phi + \rho^2 \cos^2\phi + R^2 + 2\rho R\cos\phi = R^2$$

simplifies to: $\rho = 2R\cos\phi$

① SKETCH



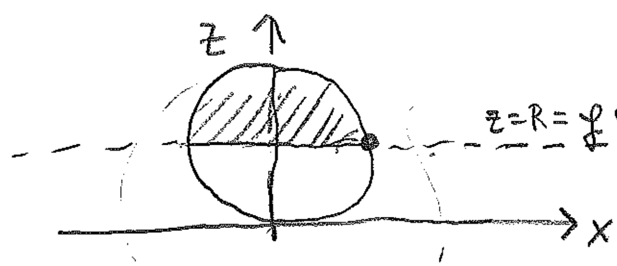


$\boxed{\rho \text{ extends from origin to point on sphere}}$.

$$\left.\begin{aligned} \rho &\in [0, 2R\cos\phi] \\ \phi &\in [0, \pi/2] \\ \theta &\in [0, 2\pi] \end{aligned}\right\} \text{FULL SOLID: } \boxed{V = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2R\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}$$

WHAT IF WE WERE ASKED FOR JUST THE VOLUME OF THE UPPER HALF OF THE SPHERE?

$\rho \in [R\sec\phi, 2R\cos\phi]$
$\phi \in [0, \pi/4]$
$\theta \in [0, 2\pi]$



$z = R = \rho\cos\phi$, or $\rho = R\sec\phi$

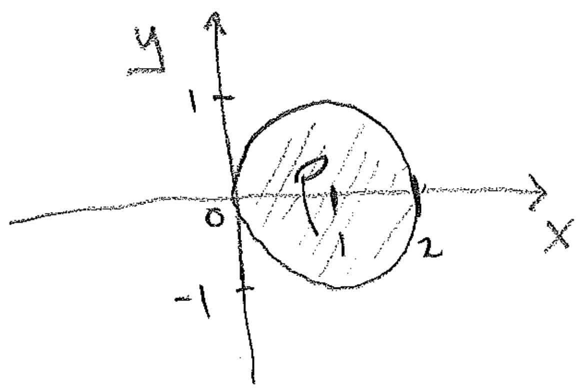
Set up an integral that represents the volume of solid bounded by $2x = x^2 + y^2$, and $z=0$
$2z = 4 + x$. Use cylindrical coordinates.

$$V = \int_{-\pi/2}^{+\pi/2} \int_0^{2\cos\theta} \int_0^{2+\frac{1}{2}r\cos\theta} r\,dz\,dr\,d\theta = \frac{5\pi}{2}$$

$$2x = x^2 + y^2$$

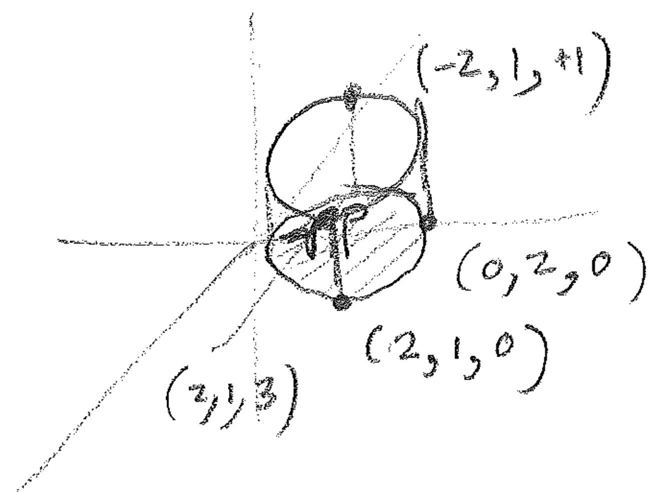
in polar: $2r\cos\theta = r^2$

$$\Rightarrow r = 2\cos\theta$$



$$2z = 4 + x$$

Becomes $z = 2 + \frac{1}{2}r\cos\theta$

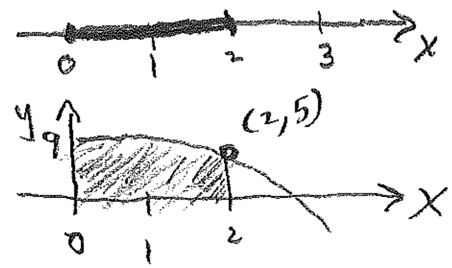


Change the order of integration.

$$V = \int_0^2 \int_0^{9-x^2} \int_0^{2-x} dz\ dy\ dx$$

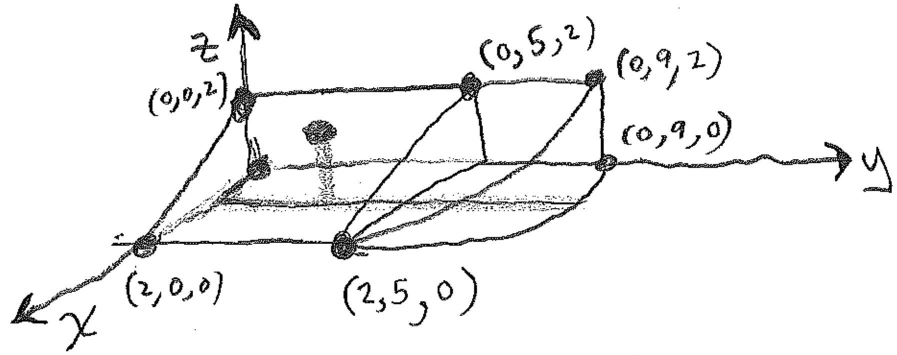
$x \in [0,2]$,

$y \in [0, 9-x^2]$,



PROJECTION ONTO XY PLANE

$z \in [0, 2-x]$



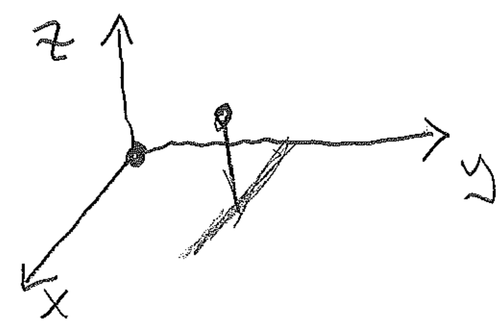
HORIZONTAL "STRIPS"
VERTICAL "COLUMNS"

<u>Try dydxdz:</u>

$y \in [0,5]$, $y \in [5,9]$

$x \in [0,2]$, $x \in [0, \sqrt{9-y}]$

$z \in [0, 2-x]$

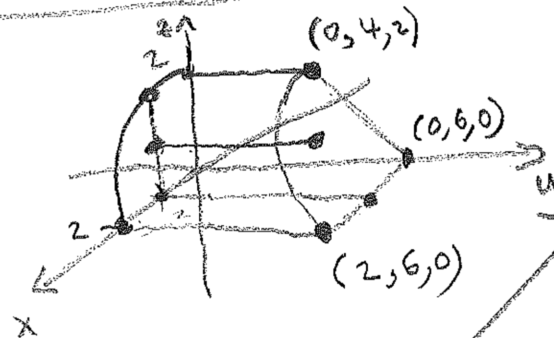


$$V = \int_0^5 \int_0^2 \int_0^{2-x} dz\,dx\,dy + \int_5^9 \int_0^{\sqrt{9-y}} \int_0^{2-x} dz\,dx\,dy$$

QH6 Recitation 14	Triple Integrals

Set up a triple integral that represents the volume of the region bounded by $x^2 + z^2 = 4$ and the planes $y + z = 6$, $\underbrace{x = 0}_{\text{yz plane}}$, $\underbrace{y = 0}_{\text{xz plane}}$, $\underbrace{z = 0}_{\text{xy plane}}$.

1/ FIRST SKETCH THE REGION



2/ CHOOSE ORDER OF INTEGRATION:

$x \in [0,2]$ (position of strips)

$z \in [0, \sqrt{4-x^2}]$ (strips stop at 6)

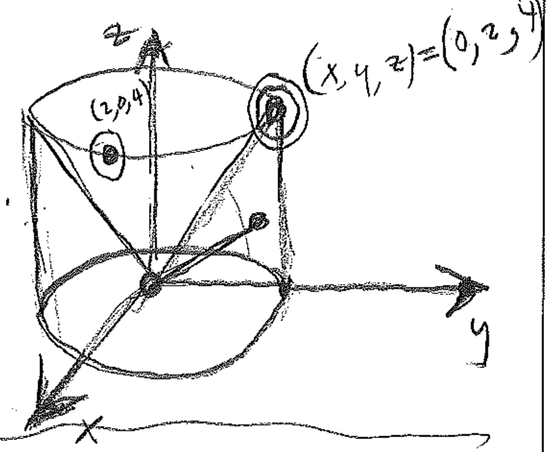
$y \in [0, 6-z]$

Set-up an integral that represents the volume bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2(x^2 + y^2)^{1/2}$.

① Sketch solid: $x^2+y^2=4$ is a cylinder

$\frac{z}{2} = \sqrt{x^2+y^2}$ is a cone

bounded above by cone, below by plane.



② Integration limits

$\rho \in [0, 2\csc\phi]$, from ✱

$\phi \in \left[\arctan(\tfrac{1}{2}), \tfrac{\pi}{2}\right]$, the xy-plane is $\phi = \pi/2$, and the "top" of the surface: $\phi = \arctan(1/2)$

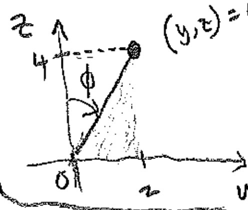
$\theta \in [0, 2\pi]$ (symmetric about z-axis)

Spherical Coordinates of ⊙:

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+16} = 2\sqrt{5}$

$\phi = \tan^{-1}\left(\frac{2}{4}\right)$

$\theta = 0$



③ WRITE INTEGRAL

$$V = \int_0^{2\pi}\int_{\operatorname{atan}\frac{1}{2}}^{\pi/2}\int_0^{2\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta,$$

✱ OUTER SURFACE OF CYLINDER:

$\rho^2 \sin^2\phi = 4$, or $\rho\sin\phi = 2$

or $\rho = 2\csc\phi$.

NOTE

we almost always use $d\rho\, d\phi\, d\theta$, or $d\rho\, d\theta\, d\phi$