NEW RESULTS ON DISCRETE-TIME

TIME-VARYING LINEAR SYSTEMS

A THESIS

Presented to

The Faculty of the Division of Graduate

Studies and Research

by

Khaled M. Hafez

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Information and Computer Science

Georgia Institute of Technology

March, 1975

NEW RESULTS ON DISCRETE TIME

TIME-VARYING LINEAR SYSTEMS

Approved:

Edward Kamen, Chairman

Lucio Chiaraviglio

Robert Kruse

Catherine Aust

John Gwynn

Date approved by Chairman: April 29, 1975

ACKNOWLEDGMENTS

The author wishes to gratefully thank his thesis advisor, Professor E. W. Kamen, who suggested the topic of this research and provided valuable counsel during the course of its completion.

The author also wishes to express deep thanks to his advisor Professor Lucio Chiaraviglio. His guidance, understanding, encouragement and continued assistance throughout the author's graduate program at Georgia Tech were without equal.

The members of my reading committee, Professors Robert Kruse of Emory University, and Catherine Aust of the Math Department, deserve my thanks for their careful reading of my thesis draft. I would also like to thank Professors Yves Rouchaleau of the University of Florida, and Allan Willisky of M.I.T. for reading and commenting on this thesis.

The author is also grateful to Professors James Gough, Jr., Miroslav Valach, James Sweeney, Robert Cooper, Philip J. Siegman, William Grosky, David Rogers, T.C.Ting, and Albert Badre of the faculty of ICS for their help and encouragement throughout the author's graduate program at Georgia Tech.

The author is particularly grateful to Professor John Gwynn, Jr. for his valuable assistance and help.

A note of thanks goes to Professor T. G. Windeknecht for initiating the author to system theory, introducing and recommending him to Professor Kamen.

Special thanks are given to Professor D. Paris, Director, Electrical Engineering Department, for his assistance, patience, and encouragement.

To my father, Mohammad Hafez, and my mother Zakia Hafez, whose forebearance and patience were without measure during the long periods of uncertainty and anxiety the author experienced as a student in the School of Information and Computer Science at Georgia Tech, I will never be able to express the extent of my gratitude. This thesis is dedicated to them.

This research has been supported by Grant No. DA-ARO-D-31-124-73-G171, from the U. S. Army Research Office (Durham). This support is gratefully acknowledged.

TABLE OF CONTENTS

SUMMARY

An investigation is made of an algebraic approach toward the study of discrete-time time-varying linear systems. The approach is based on a module structure over skew (noncommutative) polynomial rings. The thesis first adopts a global-in-time representation consisting of viewing the system coefficient matrices as elements of an arbitrary difference ring of time functions. The concepts of semilinear transformations and skew polynomial rings are then introduced and a module structure emerges as a consequence of incorporating the theory of noncommutative difference polynomials into the state variable description. This algebraic structure is used to study various properties of discrete-time time-varying linear systems and is applied to control design problems. In particular, via the construction of a characteristic polynomial in the time-varying case, new results on state-variable feedback and stabilization are given in the single-input case. A new global-in-time duality theory, termed T-duality, is introduced and is used in the construction of asymptotic state estimators and regulators in the single-variable case. Finally, a generalization of the above constructions and results to the multivariable case is given.

CHAPTER I

INTRODUCTION

1.1 Brief Historical Sketch

During the sixties an algebraic theory for linear constant discrete-time systems was established by Kalman [11] using a k[z]-module description where k[z] is the ring of polynomials in the indeterminate z over an arbitrary commutative field K. Two important avenues were since followed in connection with this approach. The first has relaxed the requirement that the system's elements be chosen from the field k; this resulted in the study of linear dynamical systems over various rings of scalars. In particular, Rouchaleau, Wyman, and Kalman [25,26,27] have studied the realization problem for linear systems over Noetherian domains, integrally closed and unique factorization domains, and principal ideal domains. Recently, Johnston [9] has given a general formulation for discrete-time linear time-invariant systems over an arbitrary ring with identity, extending some of Rouchaleau's results. The second avenue has studied the effects of replacing the operator ring k[z] by some other rings of operators; for infinite dimensional continuous-time constant systems Kamen [16] provided an algebraic representation in terms of modules over convolution algebras of Schwartz distributions; he also developed an algebraic theory for linear hereditary systems, including delay-differential systems [15], in terms of Noetherian operator rings (see also the papers by Morse [20] and Sontag [31]).

In the time-varying case, a new algebraic approach to discrete-time, linear, time-varying systems was developed very recently by Kamen [17] using skew (noncommutative) polynomial rings and a global-in-time representation specified in terms of a variable time-reference. Although noncommutative polynomials (mainly differential polynomials) appear in the engineering literature [21,36], it seems that the first serious attempt to apply this algebraic structure to the state space setting was made by Kamen [17].

1.2 The Problem: Goals and Objectives

Much of the work done in system theory has dealt with constant systems, as they are much easier to study in general than nonconstant systems. Most real systems are, however, time-varying and in recent years there has been an increasing interest in such systems as adaptive control systems, biological systems with time delays, integrated circuits with time-varying components, communication systems with time-varying channels, etc. At present, a complete theory for the study of linear time-varying systems is not available, although there exists a fairly well-developed state space theory based on the pointwise-in-time representation; that is, at every instant of time the system dynamics are given in terms of linear transformations between vector spaces over a field of scalars (usually the reals $\mathbf{R}$). In this setup, the conditions and the operations involved are given and carried out at every point in time. For example, many deep results in control [34] are based on uniform controllability which requires that the rank of the controllability matrix [29] be the same as the dimension of the given system at every point of a given interval. However, the "pointwise approach" is usually either tailored for restricted classes of time-varying systems such as those with slowly-varying coefficients [23,24] or is cumbersome from a computational standpoint since separate calculations must usually be performed at every instant of time [34,35].

On the other hand, one can also view systems globally-in-time, that is, the elements of the system coefficient matrices are elements of a certain ring of time functions. Criteria and operations can then be given in terms of the naturally associated module setup. For example, the uniform controllability condition then becomes a generation condition on the columns of the controllability matrix when viewed over the underlying ring of time functions. This viewpoint was adopted in [19] as a basis of study for certain classes of continuous-time, time-varying, linear systems.

The present work is devoted to the development of an algebraic theory for discrete-time, linear, time-varying systems that permits a global treatment of the system's structure and dynamical behavior. A major goal is to achieve more effective procedures for the study of system structure and properties, and to obtain a clearer or more complete picture of what the fundamental issues are, as opposed to the technical details. From an application viewpoint, the goal is to make effective use of (global-time) algebraic operations that are well suited for machine processing. For example, there is a good deal of motivation for constructing a system theory in terms of the ring of polynomials in time R[k], as operations within this structure are easily programmable.

1.3 Methodology

Our algebraic theory is based on a noncommutative structure consisting of a module framework whose ring of operators is noncommutative. More specifically, we shall incorporate the theory of noncommutative difference polynomials into a state variable description resulting in a new structure theory. The noncommutativity stems from the difference operator not commuting with the time functions in the polynomial ring structure.

1.4 Outline of Thesis

This thesis first places the proper class of systems under investigation into the proposed algebraic framework, and then proceeds by developing the structural properties of this setting.

Chapter II presents a global-in-time representation which will be the primary object of investigation and gives some specific examples of the rings of time functions which can be considered.

Chapter III develops a new algebraic framework which evolves naturally from the global-in-time representation presented in Chapter II. The contributions here include a new characterization of time-varying systems in terms of semilinear transformations, new results on difference equations developed via a skew polynomial structure, and a new adjoint construction which is central to the structure of "cyclic system."

Chapter IV exploits the algebraic properties of the framework. The contributions of this chapter consist of the introduction of the n-cyclicity concept and its relation to control canonical forms, the construction of a characteristic polynomial in the time-varying case and its system-theoretic interpretation, and results on state-variable feedback and stabilization, in the single-input case.

Chapter V introduces a new T-duality theory and investigates its relations to the existence of asymptotic state estimators and to the construction of regulators in the single-variable case. The main contributions here are the T-duality theory itself, the relation of this latter to the "T-adjoint" operation defined in Chapter III, and the constructions of asymptotic estimators and regulators.

Chapter VI generalizes the results of the preceding chapters to the multivariable case. The contribution here is a generalization of the usual technique used in the constant case to reduce the multivariable case to the single-variable one.

Finally, Chapter VII gives a brief discussion of the results and a summary of the thesis.

CHAPTER II

SYSTEM DESCRIPTION

This chapter reviews the conventional system definition and introduces a global-in-time representation which will form the primary object of investigation of the present work.

2.1 Pointwise-in-time Representation

Following [31], let

$\mathbf{Z}$ = ring of integers,

$\mathbf{R}$ = field of real numbers,

U = space of input values = $\mathbf{R}^m$ = space of m-element column vectors over $\mathbf{R}$,

X = state space = $\mathbf{R}^n$

Y = space of output values = $\mathbf{R}^p$.

Then,

Definition 2.1. An m-input p-output terminal n-dimensional discrete-time, linear, time-varying system over $\mathbf{R}$ (D.L.T.V . $\mathbf{R}$-system) is a collection of triples $\Sigma_k = (F(k),G(k),H(k))$ of linear maps

$$F(k): \quad X \longrightarrow X \ ,$$

$$G(k): \quad U \longrightarrow X \ ,$$

$$H(k): \quad X \longrightarrow Y \ ,$$

defining the equations

$$(2.1) \qquad x(k+1) = F(k)x(k) + G(k)u(k)$$

and

$$(2.2) \qquad y(k) = H(k)x(k)$$

where $k\epsilon\mathbf{Z}$, $x(k)\epsilon\mathbf{R}^n$ is the state at time k, $u(k)\epsilon\mathbf{R}^m$ is the input applied at time k, $y(k)\epsilon\mathbf{R}^p$ is the output at time k.

We shall usually not make a distinction between $F(k)$, $G(k)$, and $H(k)$ as linear maps or as matrices representing these maps with respect to the standard bases of $\mathbf{R}^n$, $\mathbf{R}^m$, and $\mathbf{R}^p$.

Let Φ: $\mathbf{Z}\times\mathbf{Z} \longrightarrow \mathbf{R}_{n\times n}$, where $\mathbf{R}_{n\times n}$ denotes the ring of $n\times n$ matrices over $\mathbf{R}$, be the $n\times n$ matrix function defined as follows

$$(2.3) \qquad \Phi(k,j) = F(k)F(k-1)\ldots F(j+1)F(j), \ j,k\epsilon\mathbf{Z}, \ k\geq j,$$

$$\Phi(k,k+1) = I_{n\times n} \quad \text{(the } n\times n \text{ identity matrix)},$$

$$\Phi(k,j) = \text{undefined for } j > k+1.$$

Then from equation (2.1), it is easy to see, by iteration, that the solution at the k^{th} instant starting from initial time k_o and state x_o is

$$(2.4) \qquad x(k) = \Phi(k-1,k_o)x_o + \sum_{j=k_o}^{k-1} \Phi(k-1,j+1)G(j)u(j) \; .$$

In definition 2.1, equations (2.1), (2.2) describe the dynamical behavior of a D.L.T.V. $\mathbf{R}$-system for every $k \epsilon \mathbf{Z}$, or alternatively, at every instant of time k the system dynamics are given in terms of the triple $\Sigma_k = (F(k), G(k), H(k))$ of linear transformations between vector spaces over the field of scalars $\mathbf{R}$. This fact will be referred to, in the sequel, as the pointwise-in-time representation.

2.2 Global-in-time Representation

Most existing theories are based primarily on the pointwise-in-time representation where the system is treated at every point of a given time interval. As a result, the time-variance of a given class of D.L.T.V. $\mathbf{R}$-systems defined over the interval is not clearly characterized in the algebraic framework provided by the underlying vector space structure. Further, the available pointwise-in-time methods of analysis and design are for the most part limited to certain classes of systems such as those with slowly-varying coefficients [23,24].

The main objective of the present research effort is the development of a general algebraic theory for large classes of D.L.T.V. $\mathbf{R}$-systems based on a global-in-time representation. Toward this latter we introduce the following

2.2.1 Notation

Let $\mathbf{R}^{\mathbf{Z}}$ be the commutative ring of real-valued functions defined on $\mathbf{Z}$ with pointwise addition and multiplication, and let $\sigma\colon \mathbf{R}^{\mathbf{Z}} \longrightarrow \mathbf{R}^{\mathbf{Z}}\colon f \longrightarrow \sigma f\colon k \longrightarrow f(k-1)$ be the right shift operator on $\mathbf{R}^{\mathbf{Z}}$. Let A_c denote the subring (field) of constant functions in $\mathbf{R}^{\mathbf{Z}}$ and consider the n-fold direct sum A_c^n. Then A_c^n is an n-dimensional (right) vector space over A_c, and $\mathbf{b}_n \equiv (b_i)_{1 \leq i \leq n}$ will denote the standard basis. Let A be a subring of $\mathbf{R}^{\mathbf{Z}}$ which contains A_c. Assume that A is an integral domain, and that the restriction of σ to A, denoted by the same letter, is an A-automorphism (1-1 and onto ring homomorphism). We shall let

$\sigma^{-1}: A \longrightarrow A: \alpha \longmapsto \sigma^{-1}\alpha: k \longmapsto \alpha(k+1)$ denote the left shift operator, the inverse of σ. Recall [5] that such a ring is referred to as a difference subring of $\mathbf{R}^{\mathbf{Z}}$. Let A^n be the free right A-module generated over A by $\mathbf{b}_n$. Throughout this work A_c will be identified with R and no distinction will be made between $M: A^m \longrightarrow A^n$ as an A-module morphism or as $m \times n$ matrix over A representing this morphism relative to the bases $\mathbf{b}_m$ and $\mathbf{b}_n$. The action of σ is extended to $n \times n$ matrices over A and to elements in A^n in the obvious way: if $M = (m_{ij})$, then $\sigma M = (\sigma m_{ij})$; $\sigma x = \Sigma b_i \sigma x_i$, where $x = \Sigma b_i x_i$ with $b_i \varepsilon \mathbf{b}_n$ and $x_i \varepsilon A$. Similar extensions are made for σ^{-1}.

<u>Definition 2.2</u>. A time-varying system over A (t-v A-system) is a triple $\Sigma = (F, G, H)$ of matrices over A defining the equations

$$\sigma^{-1}x = Fx + Gu \,, \tag{2.5}$$

and

$$y = Hx \,, \tag{2.6}$$

where F is $n \times n$, G is $n \times m$, H is $p \times n$, and $u \varepsilon (\mathbf{R}^{\mathbf{Z}})^m$.

From (2.4) and the fact that F, G, H are over A, it follows that if $u \varepsilon A^m$, then (2.5) has a solution x in A^n and consequently $y \varepsilon A^p$.

For convenience, we shall use the following version of (2.5)

$$x = D(\sigma x) + E(\sigma u), \text{ with } D = \sigma F,\ E = \sigma G \,. \tag{2.7}$$

Note that on applying σ (resp. σ^{-1}) to (2.5) (resp. 2.7) we get (2.7) (resp. 2.5). We shall be mainly concerned with t-v A-systems

$\Sigma = (D, E, H)$ defining the equations (2.6) and (2.7).

In contrast to the pointwise-in-time representation (2.1) and (2.2), the above definition sets up the appropriate class of systems without the need of any fixed time consideration. For this reason, we shall refer to it as the global-in-time representation (over A).

Observe that since A contains $\mathbf{R}$ (or A_c, the ring of constant functions), the theory of discrete-time, linear, time-invariant systems over $\mathbf{R}$ can be viewed as a subcase of that of t-v A-systems. On the other hand, from the standpoint of generality, one would like to take A as large as possible. We require however that A be an integral domain, because we then avoid difficulties due to the existence of zero divisors and we can use quotient field constructions.

Before concluding this section, we shall stop briefly to give some important examples of A together with their structural properties.

1. $\mathbf{R}[k]$, the ring of polynomials in time, is a principal ideal domain which, from a computational standpoint, is well adopted for machine processing.

2. The same is true of the ring $\mathbf{R}[e^{ak}]$, $a\epsilon\mathbf{R}$.

3. The subring of functions $\mathbf{R}^{\mathbf{Z}}$ which vanish only at a finite number of points is an integral domain

4. $\mathbf{R}[\sin ak, \cos bk]$, $a,b \epsilon \mathbf{R}$, is a Noetherian integral domain.

5. Let $A_{k_o} = \{\bar{f} | \bar{f} = f|_{I_{k_o}} = \text{the restriction of } f \text{ to } I_{k_o},\ f\epsilon A\}$, where $I_{k_o} = [k_o, \infty)$, $k_o \epsilon Z$, and A is a difference subring of $\mathbf{R}^{\mathbf{Z}}$. It is then clear that A_{k_o} with $\bar{\sigma}$ and $\bar{\sigma}^{-1}$ defined by

$$(\bar{\sigma}\bar{f})(k) = \begin{cases} (\sigma f)(k), & k = k_o+1, \ldots \\ 0, & k = k_o \end{cases}$$

where f is any element of A such that $f|_{I_{k_o}} = \bar{f}$, is a difference subring of $\mathbf{R}^{I_{k_o}}$.

Finally, a point to be emphasized here is that example 5 allows us to consider t-v A-systems defined on half intervals of the form $I_{k_o} = [k_o,\infty)$, $k_o \varepsilon \mathbf{Z}$.

2.2.2 Q(A)-Systems

Let A be a difference subring of $\mathbf{R}^{\mathbf{Z}}$, let Q(A) denote the field of quotients of A, and let M be a right A-module. Since A is a subring of Q(A), any right Q(A)-module is certainly a right A-module. In particular, Q(A) is a right A-module, and one can form the tensor product $M \otimes_A Q(A)$. Then this latter is an A-module and we have the following

Theorem 2.1. The multiplication defined by

$$M \otimes_A Q(A) \times Q(A) \longrightarrow M \otimes_A Q(A)$$

$$(m \otimes q, \lambda) \longrightarrow (m \otimes q)\lambda \overset{\Delta}{=} m \otimes q\lambda, \; m \varepsilon M, \; q, \lambda \varepsilon Q(A),$$

turns the A-module $M \otimes_A Q(A)$ into a Q(A)-module.

Proof : See [2] for a proof.

Moreover, if M is free on n-generators $\{m_1,\ldots,m_n\}$, then $M \otimes_A Q(A)$ is also a free Q(A)-module (an n-dimensional Q(A)-vector space) with basis $\{m_1 \otimes 1, \ldots, m_n \otimes 1\}$, and $u_M: M \longrightarrow M \otimes_A Q(A): m \longrightarrow m \otimes 1$ is an A-monomorphism termed the Q(A)-extension of M [2]. We shall use u_M to identify M with its image $u_M(M)$, an A-submodule of $M \otimes_A Q(A)$. Thus, every basis of M is also a basis of $M \otimes_A Q(A)$.

Next, A-homomorphisms can be extended as follows

Theorem 2.2. Let M, N be right A-modules, and let $E: M \longrightarrow N$ be an A-morphism. Then there exists a unique $Q(A)$-morphism, which will be denoted by the same letter, $E: M \otimes_A Q(A) \longrightarrow N \otimes_A Q(A)$ with $u_N E = E u_M$, where u_M (resp. u_N) is the $Q(A)$-extension of M (resp. of N).

Proof: See [2] for a proof.

Note that when M and N are finite and free, then any matrix E over A representing the morphism E relative to some fixed bases of M and N may also be regarded as a matrix over $Q(A)$ representing the extended morphism E relative to these same bases.

Recall that A is a difference subring of $\mathbf{R}^{\mathbf{Z}}$, or that the pair (A,σ) is a difference ring. Since A is an integral domain, σ can be extended to a $Q(A)$-automorphism [5] given by $\sigma: Q(A) \longrightarrow Q(A): p/q \longrightarrow \sigma p/\sigma q$. The pair $(Q(A),\sigma)$, where we denoted by the same letter the extension of σ(i.e., the $Q(A)$-automorphism whose restriction to A is the same as the A-automorphism σ), is called the quotient difference field of A.

With the constructional aids just sketched, given a t-v A-system $\Sigma = (D, E, H)$ let

$$U_{Q(A)} = A^m \otimes_A Q(A) ,$$

$$X_{Q(A)} = A^n \otimes_A Q(A) ,$$

$$Y_{Q(A)} = A^p \otimes_A Q(A).$$

As before a triple of matrices $\Sigma_{Q(A)} = (D_{Q(A)}, E_{Q(A)}, H_{Q(A)})$, together with the equations

$$x = D_{Q(A)}(\sigma x) + E_{Q(A)}(\sigma u), \tag{2.8}$$

$$y = H_{Q(A)}x, \tag{2.9}$$

where $x \varepsilon X_{Q(A)}$, $u \varepsilon U_{Q(A)}$, $y \varepsilon Y_{Q(A)}$, $D_{Q(A)}$ is an nxn matrix over Q(A), $E_{Q(A)}$ is an nxm matrix over Q(A) and $H_{Q(A)}$ is an nxp matrix over Q(A), defines a t-v Q(A)-system.

Then given a t-v A-system $\Sigma = (D, E, H)$, we can always embed it in a Q(A)-system by viewing the matrices D, E, and H as matrices over Q(A) with respect to the same bases, or alternatively, by extending the A-morphisms D, E, and H. However, although t-v. Q(A)-systems are well defined from a global-in-time point of view, they, unfortunately, may not be definable at every point in time. More precisely, if $K_o \varepsilon \mathbf{Z}$ is a zero of $g \varepsilon A$ (i.e., $g(k_o) = 0$), and $q = f/g$ is an element of a Q(A)-system, then this element will "blow up" at time k_o.

What lies between A and Q(A) is of great interest. In particular, we can broaden the class of t-v A-systems by taking instead of the ring A, the largest subring $\overline{A}$ of Q(A) contained in $\mathbf{R}^{\mathbf{Z}}$. More precisely, given the ring A, let

$$N = \{g \varepsilon A \mid g(k) \neq 0, \text{ for all } k \varepsilon \mathbf{Z}\} .$$

N is clearly a multiplicative subset of A; we can thus construct

$$A_N = \{f/g \mid f \varepsilon A,\ g \varepsilon N\} , \tag{2.10}$$

the quotient ring of A with respect to N. Note that if A is a principal ideal domain (resp. Noetherian domain), then so is A_N. Let $\overline{A} = A_N$; it is then clear that $\overline{A}$ is the largest subring of Q(A) whose elements belong to $\mathbf{R}^Z$. Further, the restriction of the Q(A)-automorphism σ to $\bar{A}$ is an $\bar{A}$-automorphism, i.e., $\bar{A}$ is a difference subring of $\mathbf{R}^Z$. Hence, we can consider t-v $\bar{A}$-systems and shall do so throughout the rest of the work.

2.3 $\bar{A}$-Equivalence

As it will be seen later, the module $\bar{A}^n$ plays a role similar to the state space of a D.L.T.V $\mathbf{R}$-system, and our line of attack will, in most cases, aim at fully exploring the properties of some more simplified forms (e.g., canonical forms) of t-v $\bar{A}$-systems obtainable from the given ones by a change of coordinates. The precise formulation is given by

<u>Definition 2.3</u>. Two t-v $\bar{A}$-systems $\Sigma = (D, E, H)$ and $\hat{\Sigma} = (\hat{D}, \hat{E}, \hat{H})$ are said to be $\bar{A}$-equivalent if there is an $n \times n$ matrix (over $\bar{A}$) P, invertible over $\bar{A}$, such that

$$\hat{D} = P^{-1}D(\sigma P), \tag{2.11}$$

$$\hat{E} = P^{-1}E \ ,$$

$$\hat{H} = HP \ .$$

Note that the invertible matrix P takes us from the standard basis $\mathbf{b}_n$ of $\bar{A}^n$ to another basis $\hat{b} = (\hat{b}_i)_{1 \leq i \leq n}$. Hence, if $\hat{x}$ denotes the column vector whose elements are the coordinates of x relative to the new basis $\hat{b}$, then $P\hat{x} = x$, and referring back to equations (2.6) and (2.7) we have

$$P\hat{x} = D(\sigma P)(\sigma\hat{x}) + E(\sigma u), \quad \text{since } \sigma x = (\sigma P)(\sigma\hat{x}),$$

and

$$y = HPx \ .$$

Therefore

$$\hat{x} = \hat{D}(\sigma\hat{x}) + \hat{E}(\sigma u),$$

$$y = \hat{H}\hat{x} \quad ,$$

where $\hat{D}$, $\hat{E}$, and $\hat{H}$ are given by equations (2.11), and the point to be emphasized here is given in

Theorem 2.3. $\overline{A}$-equivalence corresponds to a coordinate change applied to the dynamical equations (2.6-7) of a t-v $\overline{A}$-system.

2.4 Summary

This chapter has given a rigorous formulation of a global-in-time representation for classes of discrete-time, linear, time-varying systems, and has introduced the important notion of $\overline{A}$-equivalence. The next chapter will set up a new algebraic framework for the study of t-v $\overline{A}$-systems.

CHAPTER III

ALGEBRAIC THEORY OF t-v $\bar{A}$-SYSTEMS

This chapter lays the foundations of a new algebraic theory toward the analysis and synthesis of time-varying systems. The algebraic framework we propose here is based on a module structure over a non-commutative (skew) polynomial ring. This type of setup evolves naturally from the global-in-time representation discussed in the previous chapter. In the next two sections we develop some mathematical background for our considerations. What we require is the notion of semilinear transformations and the concept of skew polynomial rings. The discussions represent a generalization of these concepts defined originally over various fields, to the case of commutative rings.

3.1 Semilinear Transformations

Semilinear transformations and their generalizations, pseudolinear transformations, were introduced and studied by Jacobson in [8]. The domain of these operators were vector spaces over arbitrary fields. Since t-v $\bar{A}$-systems are of prime interest, we are lead to consider modules rather than vector spaces. We thus extend the notion of a semilinear transformation as follows.

Let X be a finitely generated free right R-module with basis $b=(b_i)_{1\leq i\leq n}$ where R is a commutative difference ring with 1, $\tau:R \longrightarrow R$ is its R-automorphism, and τ is extended to n-vector and n×n matrices as in Chapter II, section 2.2.1.

<u>Definition 3.1</u>: A mapping $T: X \longrightarrow X$ is said to be a semilinear transformation (s.ℓ.t) (relative to τ) if

$$(3.1) \qquad (x+y)T = xT + yT\,,$$

$$(3.2) \qquad (x\alpha)T = xT(\tau\alpha)\,, \quad x,y \in X,\ \alpha \in R\,.$$

A homothety $h_\alpha: X \longrightarrow X: x \longrightarrow x\alpha$, where α is an invertible element of R, is an s.ℓ.t. relative to the inner automorphism $\zeta \longrightarrow \alpha^{-1}\zeta\alpha$. Another example is provided by taking X to be a vector space over the field of complex numbers C and the automorphism τ to be conjugation. If we take τ = identity map on C, then T is a linear transformation of X.

A matrix representation of an s.ℓ.t can be obtained as follows. T sends b_i into $b_iT = \Sigma b_j t_{ji}$, where the t_{ji} are elements of R. The $n\times n$ matrix (t_{ji}) is by definition the matrix of T relative to b, denoted hereafter by $M_b(T)$. Thus, $M_b(T)$ is defined by the relation

$$(3.3) \qquad (b_1T,\ldots,b_nT) = (b_1,\ldots,b_n)M_b(T)\,.$$

If $x = \Sigma b_i\zeta_i \equiv b\zeta$, where $\zeta = (\zeta_1,\ldots,\zeta_n)^t$ and t stands for transpose, then it readily follows that

$$(3.4) \qquad xT = M_b(T)(\tau\zeta)\,.$$

Hence, T is completely determined by τ and its matrix representation relative to a basis of X, and conversely; every R-automorphism $\tau: R \longrightarrow R$ and every $n\times n$ matrix over R defines an s.ℓ.t on X by (3.4).

Let P be an n×n invertible R-matrix defining a change of bases in X, say $x = b\zeta = b'\zeta'$, where $x\varepsilon X$ and $b' = bP$, or equivalently $P\zeta' = \zeta$. If we let $M_b(T) = B$ and $M_b'(T) = C$, then a simple computation shows that

$$C = P^{-1}B(\tau P) \ . \tag{3.5}$$

Two matrices B and C satisfying (3.5), where P is invertible, are said to be similar and will be denoted by $B \simeq C$.

Definition 3.2: Two s.ℓ.t's T_1 and T_2 are said to be similar (denoted $T_1 \simeq T_2$) if there exists a bijective linear transformation P in X such that

$$T_2 = P.T_1 \ . \ P^{-1} \tag{3.6}$$

where . denotes composition and τ_x: $X \longrightarrow X$: $\Sigma b_i\zeta_i \longrightarrow \Sigma b_i\tau(\zeta_i)$.

For later use, we record the following

Lemma 3.1: The matrices corresponding to two similar s.ℓ.t's with respect to the same basis are similar, and conversely.

Proof: Clear.

Having developed the concept of an s.ℓ.t, we are now in a position to set forth what is to be a main tool of our investigation, namely, the concept of an s.ℓ.t of a t-v $\overline{A}$-system.

Let $\Sigma = (D, E, H)$ be a t-v $\overline{A}$-system. The matrix D defines an s.ℓ.t on $\overline{A}^n$ relative to σ, the right shift operator; it is this s.ℓ.t which will be termed the s.ℓ.t of Σ. More precisely, Definition 3.3: Given a t-v $\overline{A}$-system $\Sigma = (D, E, H)$, the s.ℓ.t $T:\overline{A}^n \longrightarrow \overline{A}^n$: $x \longrightarrow xT = D(\sigma x)$ is called the s.ℓ.t of Σ.

The matrix representation of T relative to the standard basis $\mathbf{b}_n$ is thus equal to D, and the dynamical equations (2.6) and (2.7) take the following form when $u \varepsilon \overline{A}^n$

$$x = xT + E(\sigma u); \tag{3.7}$$

$$y = Hx . \tag{3.8}$$

Observe that the triple D, E, H of $\overline{A}$-linear maps defining the t-v $\overline{A}$-system does not include directly the effect of the shift operator σ. However, the triple T, E, H does as seen from the following

Theorem 3.2. A necessary condition for two t-v $\overline{A}$-system $\Sigma = (D,E,H)$ and $\hat{\Sigma} = (\hat{D},\hat{E},\hat{H})$ to be $\overline{A}$-equivalent is that $T = \hat{T}$, where T is the s.ℓ.t of Σ and $\hat{T}$ that of $\hat{\Sigma}$.

Proof: follows immediately from Lemma (3.1).

The above remarks suggest denoting the t-v $\overline{A}$-system $\Sigma = (D, E, H)$ by $\Sigma = (T, E, H)$. We shall do so throughout this work and note that similar considerations apply to t-v $Q(A)$-systems.

Finally, it is important to emphasize that the novel feature of associating an s.ℓ.t T with a t-v $\overline{A}$-system Σ defined in terms of D, E, H is that with respect to the $\overline{A}$ structure Σ is completely characterized by the triple T, E, H.

3.2 Skew Polynomial Rings

The above basic concept plays a central role in our setting. Of equally important significance is the concept of a skew polynomial ring. Again, we shall sketch only the basic prerequisites using Ore [22] as a basis.

Let (R,τ) be a difference ring where R is a commutative integral domain with 1, and let $R[z]$ denote the set of formal polynomials

$$\pi(z) = z^n a_n + z^{n-1} a_{n-1} + \ldots + a_o , \tag{3.9}$$

in the indeterminate z with coefficients taken on the right in R. With the usual addition, $R[z]$ is an additive group with R as a domain of multipliers. If in (3.9) $a_n \neq 0$, the integer n is called the degree of $\pi(z)$ and will be denoted by $\deg(\pi)$; $\pi(z)$ is said to be monic when $a_n = 1$. Let us define multiplication in $R[z]$ as follows,

$$z^i \cdot z^j = z^{i+j}, \; i \, , \; j \in \mathbf{Z} , \tag{3.10}$$

$$az = z(\tau a), \quad a \in R . \tag{3.11}$$

It readily follows that this is a noncommutative multiplication which turns $R[z]$ into an integral domain termed a skew polynomial ring and denoted hereafter by $R_\tau[z]$.

For example, $C_-[z]$ consisting of polynomials with coefficients in the field of complex numbers C and with multiplication

$$az = z\bar{a}, \quad \text{where } \bar{a} \text{ is the complex conjugate of } a,$$

is known as the complex-skew polynomial ring.

Since R is an integral domain, by (3.11-12) $\deg(\pi) = \mathrm{Max}\{i \,|\, a_i \neq 0\}$ is a degree function on $R_\tau[z]$ (see [4]).

Let $(Q(R),\tau)$ be the quotient difference field of R and construct the skew polynomial ring $Q(R)_\tau[z]$. If $\pi(z)\epsilon Q(R)_\tau[z]$ is such that $\pi(z) = \pi_1(z)\ \pi_2(z)$, then $\pi_2(z)$ $(\pi_1(z))$ is called a right-hand (left-hand) divisor of $\pi(z)$. A right and left Eucledian algorithm exist for any two polynomials with the consequence that $Q(R)_\tau[z]$ is a left and right principal ideal domain. If $\pi_1(z)$, $\pi_2(z)$ are arbitrary polynomials, then the monic polynomial of lowest degree which is right-hand (left-hand) divisible by both $\pi_1(z)$ and $\pi_2(z)$ will be called the least common left(right) multiple of $\pi_1(z)$ and $\pi_2(z)$ and will be denoted by $[\pi_1(z),\ \pi_2(z)]_\ell([\pi_1(z),\ \pi_2(z)]_r)$. A highest common left(right) factor $(\pi_1(z),\pi_2(z))_\ell$ $((\pi_1(z),\pi_2(z))_r)$ and a least-common left(right multiple $[\pi_1(z),\ \pi_2(z)]_\ell([\pi_1(z),\pi_2(z)]_r)$ exist for any two polynomials $\pi_1(z)$ and $\pi_2(z)$ in $Q(R)_\tau[z]$. Two polynomials $\pi_1(z)$ and $\pi_2(z)$ are <u>similar</u>, denoted $\pi_1(z) \simeq \pi_2(z)$, if $\theta_1(z)\pi_1(z) = \pi_2(z)\theta_2(z)$, for some polynomials $\theta_1(z)$ and $\theta_2(z)$ such that $\theta_1(z)$ and $\pi_2(z)$ are relatively left prime and $\pi_1(z)$, $\theta_2(z)$ are relatively right-prime. Equivalently, $\pi_1(z) \simeq \pi_2(z)$ iff $Q(R)_\tau[z]/\pi_1(z)Q(R)_\tau[z]$ is isomorphic to $Q(R)_\tau[z]/\pi_2(z)Q(R)_\tau[z]$ when both are viewed as right $Q(R)_\tau[z]$-modules. In this case, a similar isomorphism holds also between the quotients module the left ideals generated by $\pi_1(z)$ and $\pi_2(z)$.

Unfortunately, most of the above properties do not carry over to $R_\tau[z]$. For example, $R_\tau[z]$ is not necessarily a left or right principal ideal domain; further the highest common left or right factor and the least common left or right multiple of two polynomials in $R_\tau[z]$ may not exist. However, the notion of similarity of two polynomials, as will be seen later, is extendable to $R_\tau[z]$. We also have the following important result.

Lemma 3.3. Let $\pi_1(z)$, $\pi_2(z)\varepsilon R_\tau[z]$ be two polynomials of degrees ≥ 0. If the leading coefficient of $\pi_2(z)$ is a unit in R, then there exist unique polynomials $q(z), r(z)\varepsilon R_\tau[z]$ such that

$$\pi_1(z) = \pi_2(z)q(z) + r(z), \quad \text{with } \deg(r) < \deg(\pi_2) .$$

Proof: See [18] for a proof in the commutative case (i.e., when $R_\tau[z]$ is the usual ring of polynomials) which immediately generalizes to $R_\tau[z]$.

The requirement that τ be an R-automorphism leads quite naturally to another skew polynomial ring

$$R_{\tau^{-1}}[z] = \{\pi(z) = z^{n-1}b_{n-1}+\ldots+b_o \,|\, b_i \,\varepsilon\, R\} ,$$

where z is an indeterminate, and multiplication is defined by

$$z^i \cdot z^j = z^{(i+j)}, \quad i,j\varepsilon \mathbf{Z} , \tag{3.12}$$

$$bz = z(\tau^{-1}b), \quad b\,\varepsilon\,\mathbf{R} . \tag{3.13}$$

There exists a very useful relationship between $R_\tau[z]$ and $R_{\tau^{-1}}[z]$:

Theorem 3.4: The mapping $*$: $R_\tau[z] \longrightarrow R_{\tau^{-1}}[z]$: $\pi(z) \longrightarrow \pi^*(z)$, where $\pi^*(z) = (\tau^{-1}a_n)z^n+\ldots+(\tau^{-1}a_1)z + \tau^{-1}a_o$, $\pi(z) = z^na_n+\ldots+za_1 + a_o$, is a ring antiisomorphism of $R_\tau[z]$ onto $R_{\tau^{-1}}[z]$.

Proof: It is clear that $*$ is 1-1 and onto; in addition, it is easy to check that $(z)^* = z$ and $(\pi_1(z) + \pi_2(z))^* = \pi_1^*(z) + \pi_2^*(z)$.

Let $\pi(z) = z^n a_n + \ldots + a_o$, then

$$(z\ \pi(z))^* = (\tau^{-1}a_n)z^{n+1} + \ldots + (\tau^{-1}a_o)z$$

$$= ((\tau^{-1}a_n)z^n + \ldots + \tau a_o)z$$

$$= \pi^*(z)z^*.$$

By using this last equality and a double induction on the degrees of $\pi_1(z)$ and $\pi_2(z)$, we can prove that $(\pi_1(z)\pi_2(z))^* = \pi_2^*(z)\pi_1^*(z)$. For suppose that $\pi_1(z) = a\varepsilon R$, the proof is then trivial when the degree of $\pi_2(z) = 0$. We thus let $\pi_2(z)$ be any polynomial in $R_\tau(z)$ of degree >0, and write

$$\pi_2(z) = b + z\bar{\pi}_2(z) \ ,$$

where $b\varepsilon R$ and deg $(\bar{\pi}_2) <$ deg(π_2). Then

$$\pi_2^*(z) = b^* + \bar{\pi}_2^*(z)z^* \ .$$

Further, a first induction on degree of $\pi_2(z)$, yields

$$(a\pi_2(z))^* = (a[b + z\bar{\pi}_2(z)])^* \ ,$$

$$= (ab + az\bar{\pi}_2(z))^* \ ,$$

$$= (ab)^* + (z(\tau a)\bar{\pi}_2(z))^* \ ,$$

$$= (ab)^* + ((\tau a)\bar{\pi}_2(z))^* z^* \ ,$$

$$= b^*a^* + \bar{\pi}_2^*(z)(\tau a)^* z^* \quad \text{(induction hyp.)},$$

$$= b^*a^* + \bar{\pi}_2^*(z)z^*a^* \ ,$$

$$= (b^* + \bar{\pi}_2^*(z)z^*)a^* \ ,$$

$$= \pi_2^*(z)a^* \, .$$

Next, let $\pi_1(z)$ be any polynomial of degree >0, and write

$$\pi_1(z) = z\bar{\pi}_1(z) + a, \quad a\varepsilon A \text{ , and } \deg(\bar{\pi}_1) < \deg(\pi_1) \text{ .}$$

Using the second induction on $\deg(\pi_1)$, we have

$$\begin{aligned}
(\pi_1(z)\cdot\pi_2(z))^* &= ([z\bar{\pi}_1(z) + a]\pi_2(z))^*, \\
&= (z\bar{\pi}_1(z)\pi_2(z) + a\pi_2(z))^*, \\
&= (z\bar{\pi}_1(z)\pi_2(z))^* + (a\pi_2(z))^*, \\
&= (\bar{\pi}_1(z)\pi_2(z))^* z^* + \pi_2^*(z)a^*, \\
&= \pi_2^*(z)\bar{\pi}_1^*(z)z^* + \pi_2^*(z)a^* \quad \text{(induction hyp.)}, \\
&= \pi_2^*(z)[\bar{\pi}_1^*(z)z^* + a^*], \\
&= \pi_2^*(z)\pi_1^*(z) \text{ .}
\end{aligned}$$

Note that the inverse of the antiisomorphism $(*)$, denoted by $(\overset{*}{*})$ is defined as follows: if $\pi(z) = z^n b_n + \ldots + b_o \,\varepsilon\, R_{\tau^{-1}}[z]$, then $\pi^{\overset{*}{*}}(z) = (\tau b_n)z^n + \ldots + \tau b$.

The ring $R_\tau[z]$ will be called the ring of difference polynomials in τ; the ring of difference polynomials in τ^{-1}, $R_{\tau^{-1}}[z]$, is referred to as the T-adjoint of $R_\tau[z]$. This adjoint construction will be utilized later.

As developed by Amitsur [1], a similar antiisomorphism exists in the differential case, i.e., when we consider the ring of differential polynomials $F(t)$ in the indeterminate t, where F is a commutative field of characteristic zero with a derivation d (a mapping $d:F \longrightarrow F$: $a \longrightarrow a'$ such that $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ and where multiplication is defined by $at = ta + a'$. This antiisomorphism is, however, unique and is from $F[t]$ onto $F[t]$.

3.3 Difference Equations

This section introduces difference equations along lines similar to those used by Amitsur [1] for the differential case.

Let $Q(R)_\tau[z]$ be the ring of difference polynomials in τ defined over the quotient difference field $Q(R)$ of R, and let $\pi(z) = z^n a_n + \ldots + a_o$ be a polynomial in $Q(R)_\tau[z]$. Then

$$Q(R) \times Q(R)_\tau[z] \longrightarrow Q(R)$$

$$(3.14) \qquad (q,\ \pi(z)) \longrightarrow \pi(q) \overset{\Delta}{=} (\tau^n q)a_n + (\tau^{n-1}q)a_{n-1} + \ldots + qa_o,$$

defines a module structure on $Q(R)$.
If $a_o a_n \neq o$ then $\pi(q) = u$, $u \in Q(R)$, is an n^{th} order linear difference equation over $Q(R)$; when $u = 0$, $\pi(q) = 0$ is a homogeneous linear difference equation. If $\mathbf{C} = \{q \in Q(R) \mid \tau q = q\}$ is the constant subfield of $Q(R)$, then it is well known that the solutions in $Q(R)$ of $\pi(q) = 0$ form a $\mathbf{C}$-module of dim $\leq n$, $n = \deg(\pi)$. The dimension r of this space is called the nullity of $\pi(z)$. Polynomials whose degrees are the same as their nullities are termed completely solvable in $Q(R)$.

It is easy to see that $y \in Q(R)$ is a solution of $\pi(q) = u$ if and only if

$$(3.15) \qquad y\pi(z) = (z-1)\pi_1(z) + u, \text{ for some } \pi_1(z) \in Q(R)_\tau[z] \ .$$

In particular, y is a solution of $\pi(q) = 0$ if and only if

$$(3.16) \qquad y\pi(z) = (z-1)\pi_1(z)$$

If $y \neq 0$ in (3.16), then

$$\pi(z) = y^{-1}(z-1)\pi_1(z) = (z(\tau y)^{-1} - y^{-1})\pi_1(z)\ ,$$

$$\pi(z) = (z - (\tau y)y^{-1})((\tau y^{-1})\pi_1(z))$$

An element $a \in Q(R)$ is called a left(right) root of $\pi[z]$ if $\pi(z) = (z-a)\pi_1(z)(\pi(z) = \pi_1(z)(z-a))$.
Thus

<u>Lemma 3.5</u>. $0 \neq y \in Q(R)$ is a solution of $\pi(q) = 0$ iff $(\tau y)y^{-1}$ is a left root of $\pi(z)$.

<u>Lemma 3.6</u>. $z - a \simeq z - 1$, i.e., $z - a$ is similar to $z - 1$, if and only if $a = (\tau b)b^{-1}$, for some $b \in Q(R)$.

<u>Proof</u>: $z-a \approx z-1$ if and only if there exist $b,\ c \in Q(R)$ such that

$$b(z-a) = (z-1)c\ .$$

Hence $c = ba$, $a = (\tau b)b^{-1}$, and conversely.

Let $\pi(z)$ be a polynomial of degree n and nullity r and let $\overline{\pi}(z)$ be the least common right multiple of all left factors of $\pi(z)$ which are similar to $z-1$. We then have

<u>Lemma 3.7</u>. $\overline{\pi}(z)$ is of degree r and is a completely solvable polynomial in $Q(R)$ which is unique up to right multiplication by an element of $Q(R)$. Further $\overline{\pi}(q) = 0$ is the minimal order equation in $Q(R)$

satisfied by all solutions of $\pi(q) = 0$ in $Q(R)$.

Proof : In view of the above lemmas, the proof is similar to the one given by Amitsur [1], and is therefore omitted.

For later use, we record the following

Lemma 3.8. A necessary and sufficient condition for n elements $y_1,\ldots,y_n$ of $Q(R)$ to be C-independent is that the least common right multiple of the polynomials $z-(\tau y_i)y_i^{-1}$ be of degree n.

Proof : Again the proof is similar to the one given in [1], and is omitted.

Similar considerations apply to $Q(R)_{\tau^{-1}}[z]$, the T-adjoint of $Q(R)_{\tau}[z]$. Hence, if $\pi(z) = z^n b_n + \ldots + b_o$ is an n^{th} degree polynomial of $Q(R)_{\tau^{-1}}[z]$, with $b_o \neq 0$, then

$$(\tau^{-n}q)b_n + \ldots + q\,b_o = u\ , \quad u \in Q(R)\ ,$$

is an n^{th} order difference equation over $Q(R)$, and $0 \neq y \in Q(R)$ is a solution of $\pi(q) = 0$ if and only if $(\tau^{-1}y)y^{-1}$ is a left root of $\pi(z)$. For later use we note the following

Theorem 3.9. $y(\tau y^{-1})$ is a right root of $\pi(z)$ if and only if y is a nonzero solution of the T-adjoint equation $\pi^*(q) = 0$.

Proof : Readily follows from the fact that the antiisomorphism (*) implies that an element $y \in Q(R)$ is a right(left) root of $\pi(z)$ if and only if $\tau^{-1}y$ is a left(right) root of $\pi^*(z)$.

In like manner, from polynomials in $R_\tau[z]$ and in $R_{\tau^{-1}}[z]$, one can define difference equations over R. Unfortunately, they do not enjoy the same properties as those defined over Q(R); this is primarily due to the absence of the Eucledian algorithm and to the fact that not every element in R is a unit. For instance, although $y \in R$ is a solution of $\pi(q) = 0$ ($\pi(q) = u$, $u \in R$), where $\pi(z) \in R_\tau[z]$, if and only if (3.16) ((3.15)) holds, Lemma (3.5) is not true unless y is a unit in R. Note, however, that since $R \subset Q(R)$, then $R_\tau[z]$ ($R_{\tau^{-1}}[z]$) is a subring of $Q(R)_\tau[z]$ ($Q(R)_{\tau^{-1}}[z]$) and any difference equation over R can be viewed as a difference equation over Q(R).

3.4 Module Structure

One of the simplest yet most important tools of our investigation is the module structure naturally induced by an s.ℓ.t. This is a generalization of the usual module structure associated with a linear transformation. Toward the former we note the following

Consider a t-v $\overline{A}$-system $\Sigma = (T, E, H)$ and denote by $\overline{A}[T]$ the ring of transformation in $\overline{A}^n$ generated by T and the elements of $\overline{A}$. It readily follows that

$$\overline{A}[T] = \{\pi(T) \mid \pi(T) = T^n a_n + T^{n-1} a_{n-1} + \ldots + a_o, a_i \in \overline{A}\}$$

As noted by Jacobson [8], $\overline{A}[T]$ is isomorphic to $\overline{A}_\sigma[z]$. T thus induces a natural right $A_\sigma[z]$-module structure on $\overline{A}^n$ which proves to be very useful in studying t-v $\overline{A}$-systems. The first system theoretic application of this module structure was made by Kamen [17]. We sum up the above discussion in the following

Theorem 3.10. With pointwise addition and the following multiplication

$$\overline{A}^n \times \overline{A}_\sigma[z] \longrightarrow \overline{A}^n \; : x \longrightarrow x\pi(z) \stackrel{\Delta}{=} x\pi(T), \tag{3.17}$$

$\overline{A}^n$ is a right $\overline{A}_\sigma[z]$-module.

We thus propose to study the action of T on $\overline{A}^n$ in terms of this $\overline{A}_\sigma[z]$-module structure. As it will be seen later, this algebraic framework incorporates in a natural way, structural and dynamical aspects of t-v $\overline{A}$-systems.

3.5 Summary

In this chapter, we have laid the foundations for a global-time algebraic theory of time-varying linear systems. This algebraic approach is based on a module framework defined over a skew polynomial ring.

CHAPTER IV

n-CYCLICITY AND FEEDBACK

In this chapter we begin to exploit the mathematical and structural properties of the module framework formulated in the previous chapter.

4.1 n-Cyclicity

Referring for details to [18,4] we first recall several algebraic concepts.

Let S be an arbitrary integral domain (not necessarily commutative) with 1 and let X be a right S-module. Then X is said to be cyclic if it is generated by a single element g. The annihilator of g given by

$$\mathrm{Ann}(g) = \{\alpha \varepsilon S \mid g\alpha = 0\}$$

is a right ideal of S, and X is isomorphic to the right quotient module S/Ann(g).

Let R be a commutative difference ring with R-automorphism $\tau : R \to R$, and let $T: X \longrightarrow X$ be an s.ℓ.t. relative to τ where X is a free finite (right) R-module of dimension n. As noted before, the skew polynomial ring $R_\tau[z]$ is a noncommutative integral domain, and T induces a right $R_\tau[z]$-module structure on X.

Definition 4.1 : (X,T), or simply T, is said to be cyclic if X is a cyclic right $R_\tau[z]$-module.

Note that because of the noncommutativity of $R_\tau[z]$, it is not necessarily true that elements in Ann(g), where g is a generator of the cyclic module X, annihilate every element in X.

To say that T is cyclic with generator g is clearly equivalent to saying that the elements $\{g, gT, \ldots, gT^n, \ldots\}$ generate X over R. If R were a field K then $\{g, gT, \ldots, gT^{n-1}\}$ would form a basis for X as a K-vector space. It is not surprising that this very important property may fail to hold for R-modules even when R is a principal ideal domain. Nevertheless, we would like to avail ourselves of this possibility which will be referred to as n-cyclicity. More specifically,

Definition 4.2: A cyclic s.ℓ.t. T is said to be n-cyclic if there exists an element $g \varepsilon X$ such that $\{g, gT, \ldots, gT^{n-1}\}$ form a basis for X. In this case g will be referred to as an n-cyclic generator of T.

The n-cyclicity concept is closely related to the form of the annihilating ideal Ann(g) as revealed by the following

Theorem 4.1: The s.ℓ.t. T is n-cyclic with n-cyclic generator g if and only if $\mathrm{Ann}(g) = \Psi(z)R_\tau[z]$ where $\Psi(z)$ is a monic polynomial of degree n.

Proof: If T is n-cyclic and g is an n-cyclic generator, then $\{g, gT, \ldots, gT^{n-1}\}$ or, equivalently, $\{g, gz, \ldots, gz^{n-1}\}$ form a basis for X, and it readily follows that every nonzero polynomial annihilating g should be of degree $>n-1$. Further, the relation

$$gz^n = gz^{n-1}\beta_n + \ldots + \beta_1, \quad \beta_i \varepsilon R,$$

yields the fact that g is annihilated by the monic polynomial

$$(4.1) \qquad \Psi(z) = z^n - z^{n-1}\beta_n - \ldots - \beta_1 \ .$$

Now that $\mathrm{Ann}(g) = \Psi(z)R_\tau[z]$ follows immediately by Lemma 3.3.

Conversely, let $\mathrm{Ann}(g) = \Psi(z)R_\tau[z]$, where $g \in X$ and $\Psi(z)$ is given by (4.1). Then $\{g, gz, \ldots, gz^{n-1}\}$ are linearly independent over R. For if

$$g\alpha_1 + \ldots + gz^{n-1}\alpha_n = 0, \ \alpha_i \in R \ ,$$

then $\pi(z) = z^{n-1}\alpha_n + \ldots + \alpha_1 \in \mathrm{Ann}(g)$, which contradicts $\mathrm{Ann}(g) = \Psi(z)R_\tau[z]$ since $\deg(\pi) < \deg(\Psi)$. Finally, the relation $0 = g\Psi(z)$ coupled with the fact that g is a generator of X implies that $\{g, gz, \ldots, gz^{n-1}\}$ generates the R-module X, whence the conclusion.

The polynomial $\Psi(z)$ alluded to in the above theorem plays a major role in subsequent developments. We shall refer to it as the order of the n-cyclic generator g, or as a characteristic polynomial of the n-cyclic s.ℓ.t. T.

Let T be n-cyclic and let g, g' be two n-cyclic generators of T. The cyclic module X has then the presentations $X \cong R_\tau[z]/\Psi(z)R_\tau[z]$ and $X \cong R_\tau[z]/\Psi'(z)R_\tau[z]$, where $\Psi(z)$ and $\Psi'(z)$ are the orders of g and g' respectively, so that $R_\tau[z]/\Psi(z)R_\tau[z] \cong R_\tau[z]/\Psi'(z)R_\tau[z]$. We say that $\Psi(z)$ and $\Psi'(z)$ are similar elements of $R_\tau[z]$, i.e., similar polynomials. We recall [4] in this case that the isomorphism $\phi: R_\tau[z]/\Psi(z)R_\tau[z] \longrightarrow R_\tau[z]/\Psi'(z)R_\tau[z]$ is determined by an element $\pi(z) \in R_\tau[z]$ such that

$$(4.2) \qquad \pi(z)\Psi(z) = \Psi'(z)\pi'(z) ,$$

for some polynomial $\pi'(z) \in R_T[z]$. Note that $\pi(z)$ and $\pi'(z)$ have the same degree and the same leading coefficient.

From the above discussion, it is then clear that all characteristic polynomials of an n-cyclic s.ℓ.t. T are similar, and we shall refer to any one of them as "the" characteristic polynomial of T.

Lemma 4.2. The characteristic polynomial of an n-cyclic s.ℓ.t. is unique up to a similarity.

Proof : Clear.

Throughout the remaining part of this work we shall consider a fixed n-cyclic generator g for a given n-cyclic s.ℓ.t. T, and take the order of g as the characteristic polynomial of T. The pair (T,g) will be referred to as an n-cyclic pair.

The correspondence between similar n-cyclic s.ℓ.t.'s and similar characteristic polynomials is a one-one correspondence as revealed by the following.

Lemma 4.3. A necessary and sufficient condition for two n-cyclic s.ℓ.t.'s to be similar is that their characteristic polynomials be similar.

Proof : straightforward.

We conclude this section by noting the following alternative (matrix) formulation of n-cyclicity. Let T be n-cyclic and g an n-cyclic generator. Fix a basis in X and view $g, gT, \ldots, gT^{n-1}$ as column vectors, then

Lemma 4.4. (T,g) is n-cyclic if and only if the matrix $[g\ gT \dots gT^{n-1}]$ is invertible over R.

Proof : Clear.

4.2 Control Canonical Forms

As one of the first facts justifying the introduction of the n-cyclicity concept we prove the following important

Theorem 4.5. Consider the single-input t-v $\overline{A}$-system $\Sigma = (T,e,-)$ where E in this case consists of a single column e, and H is immaterial. Then, a necessary and sufficient condition for the pair (T,e) to be n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$ is that there exists a basis $c = (c_i)_{1\leq i\leq n}$ for $\overline{A}^n$ relative to which T and e have the matrix representations

$$(4.3) \qquad M_c(T) = \begin{bmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ \vdots & & & 0 \\ 0 & & & 1 \\ \beta_1 & \sigma^{-1}\beta_2 & \dots & \sigma^{n-1}\beta_n \end{bmatrix}, \quad e = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

Proof : Suppose that (T,e) is n-cyclic and define the elements $(c_i)_{1\leq i\leq n}$ as follows

$$(4.4) \qquad c_i = e\Psi^{(n-i+1)}(z),$$

where

$$\psi^{(1)}(z) = 1,$$

$$\Psi^{(2)}(z) = \Psi^{(1)}(z)z - \sigma^{-n+1}\beta_n \ ,$$
$$\vdots$$
$$\Psi^{(n)}(z) = \Psi^{(n-1)}(z)z - \sigma^{-1}\beta_2 .$$

It readily follows that $c \equiv (c_i)_{1 \leq i \leq n}$ is a basis for $\overline{A}^n$ since the c_i's are a triangular linear combination of $\{e, ez, \ldots, ez^{n-1}\}$ which form a basis by the n-cyclicity of the pair (T,e). Using the two equations

$$0 = e\Psi(z), \text{ and } c_n = e,$$

we can easily compute $M(T)$ as follows.

$$\begin{aligned}
c_1 T = c_1 z &= e\Psi^{(n)}(z)z, \\
&= e(\Psi^{(n-1)}(z)z - \sigma^{-1}\beta_2)z \ , \\
&= e\Psi^{(n-1)}(z)z^2 - ez\beta_2 \ , \\
&= e(\Psi^{(n-2)}(z)z - \sigma^{-2}\beta_3)z^2 - ez\beta_2 \ , \\
&= e\Psi^{(n-2)}(z)z^3 - ez^2\beta_3 - ez\beta_2 \ , \\
&\ \ \vdots \\
&= ez^n - ez^{n-1}\beta_n - \ldots - ez\beta_2 = c_n\beta_1 .
\end{aligned}$$

Similarly,

$$c_2 T = c_2 z = ez^{n-1} - ez^{n-2}(\sigma^{-1}\beta_n) - \ldots - ez(\sigma^{-1}\beta_3) = c_1 + c_n(\sigma^{-1}\beta_2),$$
$$\vdots$$
$$c_n T = c_n z = \quad \ldots \quad = c_{n-1} + c_n(\sigma^{-n+1}\beta_n).$$

Thus

$$M_c(T) = \begin{bmatrix} 0 & 1 & & & 0 \\ \vdots & & \ddots & & \\ \vdots & & & \ddots & 1 \\ \beta_1 & \sigma^{-1}\beta_2 & \ldots & & \sigma^{-n+1}\beta_n \end{bmatrix}, \quad e = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} .$$

The converse follows from the fact that the preceding proof can be carried also in the reverse direction.

Let $\hat{\Sigma} = (\hat{T}, \hat{e}, -)$ be the t-v $\overline{A}$-system whose matrices are given by (4.3) (i.e., $M_{b_n}(\hat{T}) = M_c(T)$ and $\hat{e} = [0\ldots01]^t$). Then

<u>Definition 4.3</u>. $\hat{\Sigma} = (\hat{T},\hat{e},-)$ is called the control canonical form of $\Sigma = (T,e,-)$.

In terms of a change of coordinates, we have the following picture

$$(4.5) \qquad \underset{b_n}{\overline{A}^n} \xrightarrow{\quad P \quad} \underset{(ez^{i-1})_{1\le i\le n}}{\overline{A}^n} \xrightarrow{\quad S \quad} \underset{c}{\overline{A}^n} ,$$

where $(ez^{i-1})_{1\le i\le n}$ is the basis resulting from the n-cyclicity of (T,e), c is the control canonical basis, and the nonsingular matrices P and S are given (and denoted hereafter) by

$$(4.6) \qquad P = [e \;\; ez \; \ldots \; ez^{n-1}], \quad S = \begin{bmatrix} -\sigma^{-1}\beta_2 & -\sigma^{-2}\beta_3 \ldots & -\sigma^{-n+1}\beta_n & 1 \\ \vdots & & \cdot^{\cdot^{1}} & \vdots \\ \vdots & -\sigma^{-2}\beta_n & & \vdots \\ -\sigma^{-1}\beta_n & 1 & & \\ 1 & 0 & \ldots & 0 \end{bmatrix}$$

Thus, we have the

Corollary 4.6. A t-v $\overline{A}$-system $\Sigma = (T,e,-)$ has a control canonical form if and only if it is $\overline{A}$-equivalent to a t-v $\overline{A}$-system $\hat{\Sigma} = (\hat{T},\hat{e},-)$ with $M_{\mathfrak{b}_n}(\hat{T})$ and $\hat{e}$ given by (4.3).

Proof : the nonsingular matrix giving the $\overline{A}$-equivalence is PS.

Now we are very close to giving a system-theoretic interpretation of the characteristic polynomial corresponding to an n-cyclic pair (T,g) of a t-v $\overline{A}$-system; we have still to make the following observations.

Let $\Sigma = (T,e,-)$ be a t-v $\overline{A}$-system where the pair (T,e) is n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$, and let

$$(4.7) \qquad \phi : \overline{A}^n \longrightarrow \overline{A}_\sigma[z]/\Psi(z)\overline{A}_\sigma[z].$$

be the $\overline{A}_\sigma[z]$-module isomorphism which sends e to 1 (mod $\Psi(z)$). Assume that $u \in \overline{A}$; the dynamical equation $x = xT + e(\sigma u)$ defining the system Σ can also be written as

$$(4.8) \qquad x(z-1) + e(\sigma u) = 0 .$$

We shall assume that $x \in \overline{A}^n$ is some fixed solution of (4.8). Then, applying ϕ, we have

$$\phi(x(z-1) + e(\sigma u)) = \pi_x(z)(z-1) + \sigma u = 0 \pmod{\Psi(z)},$$

where $\pi_x(z)$ ($= \phi(x)$) can be chosen of degree $<n$. It therefore follows that there exists $y_x \in \overline{A}$ such that

(4.9) $$\pi_x(z)(z-1) + \sigma u = \Psi(z)y_x .$$

By applying the antiisomorphism $(*)$: $\overline{A}_\sigma[z] \longrightarrow \overline{A}_{\sigma^{-1}}[z]$: $\pi(z) \longrightarrow \pi^*(z)$, to both sides of equation (4.9), we get

$$(\sigma^{-1}y_x)\Psi^*(z) = (z - 1)\pi_x^*(z) + u ,$$

where $\Psi^*(z) = z^n - \sum_{i=1}^{n} z^{i-1}(\sigma^{-1}\beta_i)$.

By (3.15), it follows that $\sigma^{-1}y_x$ is a solution of the difference equation

(4.10) $$\Psi^*(q) = \sigma^{-n}q - \sigma^{-n+1}q(\sigma^{-n}\beta_n) - \ldots - q(\sigma^{-1}\beta_1) = u, \quad q \in \overline{A}$$

i.e., $$\Psi^*(\sigma^{-1}y_x) = u .$$

We now come to the following picture. Since (T,e) is n-cyclic it then follows by Theorem 4.5 that there is a basis $c = (c_i)_{1\leq i\leq n}$ relative to which $\Sigma = (T,e,-)$ has a control canonical form (4.3). Let $\hat{x}_i$, $i=1,\ldots,n$, be the coordinates relative to c of any fixed solution x of $x = xT + e(\sigma u)$. Referring back to the definition of c (see (4.4)), we have

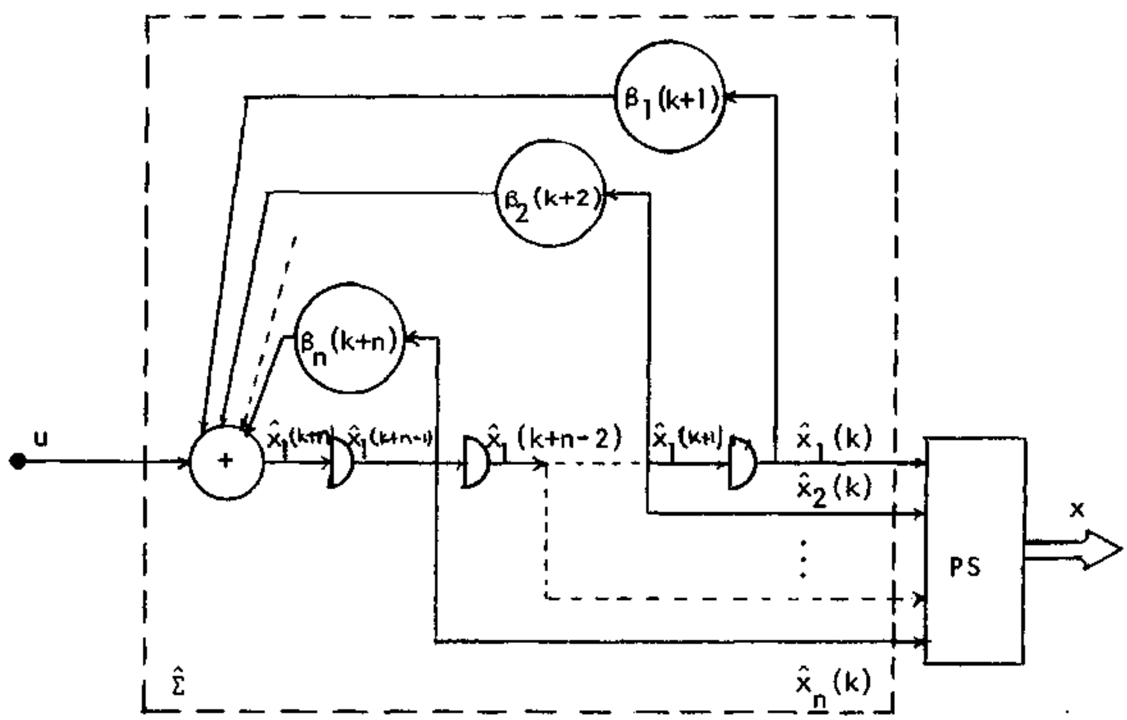
$$x = \sum_{i=1}^{n} c_i\hat{x}_i = \sum_{i=1}^{n} (e\Psi^{(n-i+1)}(z))\hat{x}_i ,$$

$$= e(\sum_{i=1}^{n} \Psi^{(n-i+1)}(z)\hat{x}_i) .$$

If we take $\pi_x(z) = \sum_{i=1}^{n} \Psi^{(n-i+1)}(z)\hat{x}_i$, then by equating coefficients, equation (4.9) yields

$$\text{(4.11)} \qquad \begin{aligned} \hat{x}_1 &= \sigma^{-1} y_x \, , \\ \hat{x}_2 &= \sigma^{-1} \hat{x}_1 \, , \\ \hat{x}_3 &= \sigma^{-1} \hat{x}_2 \, , \\ &\vdots \\ \hat{x}_n &= \sigma^{-1} \hat{x}_{n-1} \, . \end{aligned}$$

The first coordinate $\hat{x}_1$ is therefore a solution of $\Psi^*(q) = u$, i.e., $\Psi^*(\hat{x}_1) = u$, and the rest of the coordinates are obtainable from the first one by repetitive application of σ^{-1}. Hence we have the following simulation of the system $\Sigma = (T, e, -)$



Note that $\hat{x}$ passes through a device with gain PS which is memoryless in the sense that its output $x(k)$ at time k depends only on its input $\hat{x}(k)$ at the same time k. Since the components of $\hat{x}$ are shifts of the solution q of $\Psi^*(q) = u$, it is clear that $\Psi^*(z)$ plays a major role in Σ's dynamical behavior. In other words, the adjoint of the characteristic polynomial $\Psi(z)$ is central to dynamical behavior in the cyclic case. As will be seen later, this polynomial setting is particularly useful in the problem of achieving stability via state feedback.

4.3 Feedback

Let us focus our attention upon the control canonical form. Here a key problem is that of coefficient assignability of the characteristic polynomial for the closed-loop system. For discrete-time, linear, constant systems defined over an arbitrary field, complete reachability was found to be the necessary and sufficient condition for constructing an arbitrary characteristic polynomial of the closed-loop system [10]. For D.L.T.V. **R**-systems, we can consider characteristic polynomials defined for each point of the time interval. However "pointwise characteristic polynomials" are not very useful for solving certain problems and, in fact, can lead to incorrect conclusions. For a simple example illustrating this, let us consider the continuous-time constant system defined over **R** by

$$\frac{dx(t)}{dt} = Fx(t) , \tag{4.12}$$

where $t \in \mathbf{R}$, and F is an nxn matrix over **R**.

It is well-known that (4.12) is stable (see definition 4.5 below), if and only if, all the roots of the characteristic polynomial of F (i.e., the eigenvalues of F) have negative real parts. However, if the roots of the pointwise characteristic polynomials of the time-varying system $\frac{dx(t)}{dt} = F(t)x(t)$, have negative real parts, this does not necessarily imply that the system is stable [33].

Nevertheless, as we proceed now to show, our algebraic framework can be used to great advantage in the study of state variable feedback for the single-input case, and in particular it yields a simple constructive procedure for stabilization by feedback.

Let $\Sigma = (T,e,-)$ be a single-input t-v $\overline{A}$-system of dimension n, we then have

Theorem 4.7. (T,e) is n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$ if and only if, given any fixed n^{th} degree monic polynomial $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i$ in $\overline{A}_\sigma[z]$, there exists a row vector $w = (w_1,\ldots,w_n)$, $w_i \in \overline{A}$, such that the s.ℓ.t. of the closed-loop system $\Sigma_1 = (T_1,e,-)$, where $M_{\mathbf{b}_n}(T_1) = D - e(\sigma w)$, is n-cyclic with n-cyclic generator e and characteristic polynomial $\lambda(z)$.

Proof : if (T,e) is n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$, then $\Sigma = (T,e,-)$ is $\overline{A}$-equivalent to $\hat{\Sigma} = (\hat{T},\hat{e},-)$, where $\hat{D}$, the matrix of $\hat{T}$ relative to the standard basis $\mathbf{b}_n$, and $\hat{e}$ are in the control canonical form, i.e.,

$$\hat{D} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ \cdot & & & 1 \\ \beta_1 & \sigma^{-1}\beta_2 & \cdots & \sigma^{-n+1}\beta_n \end{bmatrix}, \hat{e} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Let $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i \in \overline{A}\sigma[z]$, and define relative to the control canonical basis c (see (4.4)), the row vector $\hat{w} = (\hat{w}_1,\ldots,\hat{w}_n)$, where $\hat{w}_i = \sigma^{-i}\beta_i - \sigma^{-i}\alpha_i$, $i = 1,2,\ldots,n$. It then follows that

$$\hat{D} - \hat{e}(\sigma\hat{w}) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ & & & 1 \\ \alpha_1 & \sigma^{-1}\alpha_2 & \cdots & \sigma^{-n+1}\alpha_n \end{bmatrix},$$

and by Theorem 4.5, $\Sigma_1 = (T_1, e_1, -)$, where $M_{\mathbf{b}_n}(T_1) = D - e(\sigma w)$, and $w = \hat{w}(PS)^{-1}$ is such that (T_1, e) is n-cyclic with characteristic polynomial $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i$.

Conversely, if for any $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i$ in $\overline{A}_\sigma[z]$, there exists an row vector w such that $\Sigma_1 = (T_1, e, -)$, where $M_{\mathbf{b}_n}(T_1) = D - e(\sigma w)$, is such that (T_1, e) is n-cyclic with characteristic polynomial $\lambda(z)$, then we have

$$\begin{aligned} eT_1 &= (D - e(\sigma w))\sigma e, \\ &= D\sigma e - e(\sigma(we)), \\ &= eT - ea, \quad a \in \overline{A}. \end{aligned}$$

$$eT_1^2 = (D - e(\sigma w))\sigma(eT - ea),$$
$$\vdots \quad = eT^2 - (eT)a_1 - ea_o, \quad a_o \text{ and } a_1 \varepsilon \overline{A},$$
$$eT_1^{n-1} = eT^{n-1} - (eT^{n-2})a_2 - \cdots - ea_o \ .$$

Hence, the determinant of $[e\ eT \ldots eT^{n-1}]$ is equal to that of $[e\ eT_1 \ldots eT_1^{n-1}]$, and the conclusion follows by lemma 4.4.

<u>Corollary 4.8</u>. If $\Sigma = (T,e,-)$ satisfies the hypothesis of above theorem, then there exists a bijection between n^{th} degree monic polynomials $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i$ and s.ℓ.t.'s $\tilde{T}$ (relative to σ) defined by $\tilde{T}: \overline{A}^n \longrightarrow \overline{A}^n: c_i \longrightarrow e(\sigma w_i), i=1,\ldots,n$, where $c = (c_i)_{1\leq i\leq n}$ is the control canonical basis and $w = (w_1,\ldots,w_n)$ is the row vector defined in the above theorem, (i.e., $M_c(\tilde{T}) = [e(\sigma w_1)\ldots e(\sigma w_n)]$), such that $T_1 = T - \tilde{T}$ is an n-cyclic s.ℓ.t. with n-cyclic generator e and characteristic polynomial $\lambda(z)$.

<u>Proof</u>: Since both $\tilde{T}$ and T are s.ℓ.t.'s of $\overline{A}^n$ relative to σ, it then follows that $T_1 = T - \tilde{T}$ is also such an s.ℓ.t. The corollary will thus follow from Theorem 4.5 if we can show that $d = (d_i)_{1\leq i\leq n}$, $d_i = e\lambda^{(n-i+1)}(T_1)$, is a basis for $\overline{A}^n$. For this, let $w = (w_1,\ldots,w_n)$, where $w_i = \sigma^{-i}\beta_i - \sigma^{-i}\alpha_i$, $i = 1,2,\ldots,n$ and use induction on i.
For $i = 1$, $\lambda^{(1)}(T_1) = \psi^{(1)}(T) = \psi^{(1)}(z) = 1$, by definition (see (4.4)).

Suppose that $$\lambda^{(i-1)}(T_1) = \psi^{(i-1)}(T) \ ,$$

then

$$\begin{aligned}
e\lambda^{(i)}(T_1) &= e[\lambda^{(i-1)}(T_1)T_1 - \sigma^{-n+i-1}\alpha_{n-i+2}] \\
&= e[\Psi^{(i-1)}(T)T_1 - \sigma^{-n+i-1}\alpha_{n-i+2}] \\
&= e[\Psi^{(i-1)}(T)(T-\tilde{T}) - \sigma^{-n+i-1}\alpha_{n-i+2}] \\
&= e[\Psi^{(i-1)}(T)T - \sigma w_{n-i+2} - \sigma^{-n+i-1}\alpha_{n-i+2}] \\
&= e[\Psi^{(i-1)}(T)T - (\sigma^{-n+i-1}\beta_{n-i+2} - \sigma^{-n+i-1}\alpha_{n-i+2}) - \sigma^{-n+i-1}\alpha_{n-i+2}] \\
&= e[\Psi^{(i-1)}(T)T - \sigma^{-n+i-1}\beta_{n-i+2}] \\
&= e\Psi^{(i)}(T) ,
\end{aligned}$$

$d = (d_i)_{1\le i\le n}$ therefore forms a basis for $\overline{A}^n$ since $c = (c_i)_{1\le i\le n}$ does, and the correspondence $\lambda \leftrightarrow \tilde{T}$ is obviously 1-1 and onto.

Definition 4.4. The pair $(\tilde{T}, \lambda(z))$ or, simply the s.ℓ.t. $\tilde{T}$, is called a control law for $\Sigma = (T,e,-)$ where (T,e) is n-cyclic.

With this definition, we can interpret the passage from T to T_1 via $\tilde{T}$ as the passage from the open-loop system $\Sigma = (T,e,-)$ to the closed-loop system $\Sigma_1 = (T_1,e,-)$ via the control law $\tilde{T}$. Note the remarkable similarity to the constant case [12].

4.4 Applications

4.4.1 Stability and Algebraic Equivalence

Of great importance in system theory is stabilization through the use of state variable feedback. We now proceed, in this section and the next one, to show how the preceding results lend themselves to yield a simple and effective solution to this problem.

For the purpose of this discussion, we let $\| \ \|$ denote the Euclidean norm in $\mathbf{R}^n$ and recall [13] the following definitions. Let

$$(4.13) \qquad x(k+1) = F(k)x(k), \ k \in \mathbf{Z}$$

be an n-dimensional free (i.e., zero input) D.L.T.V. $\mathbf{R}$-system.

Definition 4.5. The system (4.13) is said to be stable (uniformly in the sense of Lyapunov) if given $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\|x_o\| < \delta$ implies that $\|x(k)\| \leq \varepsilon$ for any k_o and for all $k \geq k_o$, where $x(k)$ is the solution of (4.13) at the k^{th} instant starting from the initial state $x_o = x(k_o)$ at time k_o.

Definition 4.6. The system (4.13) is said to be asymptotically stable (uniformly in the sense of Lyapunov) if it is stable and if every motion starting near the origin 0 converges to 0 a $k \to \infty$, i.e., there exists a $\gamma > 0$, and for any $\varepsilon > 0$ there corresponds a positive $\mu(\varepsilon, \gamma)$ such that $\|x_o\| \leq \gamma$ implies that $\|x(k)\| \leq \varepsilon$ for all $k \geq k_o + \mu$ and for any k_o, where $x(k)$ is the solution of (4.13) starting from the initial state $x_o = x(k_o)$ at time k_o.

If (4.12) is a constant system (i.e., $F(k) \equiv F \in \mathbf{R}_{n \times n}$), it is well known then that a necessary and sufficient condition for the system to be asymptotically stable is that all the eigenvalues of F lie in the unit circle of the complex plane.

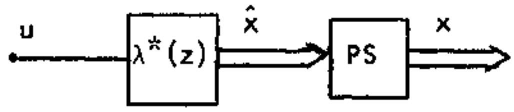
Next, let $\Sigma = (T,e,-)$ be a single-input t-v $\bar{A}$-system such that (T,e) is n-cyclic with characteristic polynomial $\Psi(z)$. We then know that the closed-loop system $\Sigma_1 = (T_1,e,-)$, $M_{\mathbf{b}_n}(T_1) = M_{\mathbf{b}_n}(T) - e(\sigma w)$, can have any desired characteristic polynomial $\lambda(z)$ in $\bar{A}_\sigma[z]$. Let us choose $\lambda(z) = z^n$. It then follows from the proof of Theorem 4.5 that relative to the control canonical basis $c = (c_i)_{1 \le i \le n}$ the closed-loop system $\Sigma_1 = (T_1,e,-)$ has the following dynamical equation

$$\hat{x} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ & & & 1 \\ 0 & \cdots\cdots & & 0 \end{bmatrix} \sigma \hat{x} + \begin{bmatrix} 0 \\ \vdots \\ \\ 1 \end{bmatrix} \sigma u \, . \tag{4.14}$$

In other words, the closed-loop system Σ_1 is $\bar{A}$-equivalent to the constant system (4.14), the matrix of the $\bar{A}$-equivalence being PS (see (4.5)), and this constant system (4.14) is obviously asymptotically stable since the eigenvalues of F, i.e., the roots of $\lambda(z) = z^n$, lie in the unit circle. Further, it is clear from equation (4.14) that

$$\hat{x}(k_o+n) = 0, \quad \text{for any } \hat{x}(k_o) \in \mathbf{R}^n \text{ and any initial time } k_o. \tag{4.15}$$

In light of the interpretation of the characteristic polynomial given in section 4.2, we can summarize the above situation in the following picture



Since (4.15) holds and since $x = (PS)\hat{x}$, it readily follows that the closed-loop system $\Sigma_1 = (T_1, e, -)$ is asymptotically stable regardless of the properties of the transformation PS. Moreover, from (4.15), $x(k) \to 0$ in n steps.

The interesting fact here is that we have been able to stabilize t-v $\bar{A}$-systems without resorting to any topological conditions such as uniform boundedness.

<u>Example 4.1</u>. Let us take $A = \mathbf{R}[k]$, the ring of polynomials in time and form $\bar{A} = \{p/q \mid p,q \in A, \text{ and } q(k) \neq 0, \forall k \in \mathbf{Z}\}$. Consider the single-input t-v $\bar{A}$-system

$$x(k) = \begin{bmatrix} k & 0 \\ 1/2 & 2k-1 \end{bmatrix} x(k-1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-1) . \tag{4.16}$$

Thus,

$$M_{\mathfrak{b}_2}(T) = \begin{bmatrix} k & 0 \\ 1/2 & 2k-1 \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 1 \end{bmatrix} .$$

It is easy to see that

$$P = [e \quad eT] = \begin{bmatrix} 1 & k \\ 1 & 2k-1/2 \end{bmatrix},$$

and that det $P = k-1/2$ is a unit in $\bar{A}$. (T,e) is therefore n-cyclic, and expressing eT^2 in terms of e and eT, we obtain the characteristic polynomial

$$\Psi(z) = z^2 - z\,\frac{6k - 11k + 4}{2k - 1} - \frac{-4k^3 + 8k^2 - 3k}{2k - 1}\,.$$

The control canonical form is therefore given by

$$\hat{x}(k) = \begin{bmatrix} 0 & 1 \\ \dfrac{-4k^3 + 8k^2 - 3k}{2k - 1} & \dfrac{6k^2 + k - 1}{2k + 1} \end{bmatrix} \hat{x}(k-1) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k-1).$$

If we pick $\lambda(z) = z^2$, then

$$\sigma\hat{w} = \begin{bmatrix} \dfrac{-4k^3 + 8k^2 - 3k}{2k - 1} & \dfrac{6k^2 + k - 1}{2k + 1} \end{bmatrix}.$$

Since

$$PS = \begin{bmatrix} \dfrac{-4k^2 + 1}{2k + 1} & 1 \\ \dfrac{-4k^2 + 1}{2(2k + 1)} & 1 \end{bmatrix}, \quad (PS)^{-1} = \frac{2}{-2k + 1} \begin{bmatrix} 1 & -1 \\ \dfrac{4k^2 - 1}{2(2k + 1)} & \dfrac{-4k^2 + 1}{2k + 1} \end{bmatrix},$$

then $\sigma w = (\sigma\hat{w})\sigma\,(PS)^{-1}$ is given by

$$w = \frac{2}{-2k + 3} \begin{bmatrix} \dfrac{8k^4 - 20k^3 + 10k^2 + 5k - 3}{2(2k+1)(2k-1)} & \dfrac{-16k^4 + 32k^3 - 8k^2 - 8k+3}{(2k+1)(2k-1)} \end{bmatrix}.$$

The closed-loop system has the form

$$(4.17)\quad x(k) = \frac{2}{-2k+3}\begin{bmatrix} \dfrac{-16k^4+32k^3-8k^2-8k+3}{2(2k+1)(2k-1)} & \dfrac{16k^4-32k^3+8k^2+8k-3}{(2k+1)(2k-1)} \\[2ex] \dfrac{-16k^4+32k^3-8k^2-8k+3}{4(2k+1)(2k-1)} & \dfrac{16k^4-32k^3+8k^2+8k-3}{2(2k+1)(2k-1)} \end{bmatrix} x(k-1)$$

$$+ \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k-1)$$

By direct computation, one can verify that (4.17) is asymptotically stable for any initial time k_o and initial state $x(k_o)$. In fact

$$(D - e(\sigma w))(j)(D - e(\sigma w))(j-1) = 0 \ , \ \forall j \in \mathbf{Z} \ ,$$

and therefore $x(k_o + 2) = 0, \ \forall k_o \in \mathbf{Z}$.

4.4.2 Stability and Topological Equivalence

It is well known [14,36] that algebraic equivalence between time-varying systems does not in general preserve stability properties. Lyapunov transformations, however, do preserve stability; toward this latter, we recall the following facts.

An nxn matrix $P = (p_{ij})$ over $\mathbf{R}^{\mathbf{Z}}$ is said to be uniformly bounded if there exists a $\mu > 0$ such that $|p_{ij}(k)| < \mu$, $\forall k \in \mathbf{Z}$ and where $|\cdot|$ is the absolute value.

<u>Definition 4.7</u>. An nxn matrix P over $\bar{A}$ is said to represent a Lyapunov transformation if it is uniformly bounded and if there exists a positive constant γ such that $0 < \gamma < |\det P(k)|, \forall k \in \mathbf{Z}$.

It is well known [36], that this definition is equivalent to requiring that both P and its inverse P^{-1} be uniformly bounded. Further, the Lyapunov transformations form a group and if $x = P\hat{x}$, where P is Lyapunov, then the t-v $\overline{A}$ systems

$$x = D(\sigma x) \ , \tag{4.18}$$

$$\hat{x} = \hat{D}(\sigma\hat{x}) \ , \tag{4.19}$$

have the same stability properties. It is because of this that we shall refer to the two systems (4.18) and (4.19) as being topologically equivalent.

If the Lyapunov transformation $x = P\hat{x}$ is such that the resulting system (4.19) is constant, i.e., $\hat{D}$ is a constant matrix, then the system (4.18) is said to be reducible. Again both systems have the same stability properties.

With the above preliminaries we can state and prove the following

<u>Theorem 4.9</u>. Let $\Sigma = (T,e,-)$ be a single-input t-v $\overline{A}$-system, where D and e are uniformly bounded, and such that (T,e) is n-cyclic with characteristic polynomial $\Psi(z) = z - \sum_{i=1}^{n} z^{i-1}\beta_i$. If there exists a γ such that $0 < \gamma < |\det p(k)|$ for all $k \in \mathbf{Z}$, where $P = [e \ eT \ldots eT^{n-1}]$, then there exists a control law $\tilde{T}$ whose matrix is uniformly bounded and such that the closed-loop system $\Sigma_1 = (T_1,e,-)$ is reducible to a constant one with arbitrary eigenvalues.

<u>Proof</u> : In view of Theorem 4.5, $\Sigma = (T,e,-)$ is $\overline{A}$-equivalent to $\hat{\Sigma} = (\hat{T}, \hat{e}, -)$ in control canonical form and the $\overline{A}$-equivalence is given by the transformation PS. Since $M_{\mathfrak{b}_n}(T) = D$ and e are uniformly bounded,

then so is P and the assumption that $0 < \gamma < |\det P(k)|, \forall k \in \mathbf{Z}$ implies therefore that P in Lyapunov. Recall from (4.5) that

$$M_{(ez^{i-1})_{1\le i\le n}}(T) = P^{-1} M_{\mathbf{b}_n}(T)(\sigma P) = \begin{bmatrix} 0 & \dots & & \beta_1 \\ 1 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \dots & 1 & \beta_n \end{bmatrix},$$

where the β_n's are the coefficients of the characteristic polynomial $\Psi(z)$. Hence, since $M_{\mathbf{b}_n}(T) = D$ is uniformly bounded and P is Lyapunov, the β_i's are uniformly bounded. It readily follows that S (see (4.6)) is uniformly bounded and is in fact Lyapunov since its determinant is equal to $(-1)^n$. The transformation PS is therefore Lyapunov and if we choose $\lambda(z) = z^n - \sum_{i=1}^{n} z^{i-1}\alpha_i$, $\alpha_i \in \mathbf{R}$, the characteristic polynomial of the closed-loop system, then this latter is $\overline{A}$-equivalent to a constant system. Clearly, the feedback control law $\hat{T}$ (see (Def.(4.4)) has a uniformly bounded matrix.

Note that if $\lambda(z)$ is a stable polynomial (i.e., with zeros in the unit circle), then the closed-loop system is also stable.

4.4.3 Specification of Fundamental Sets

Let $\pi(z) = z^n + \sum_{i=1}^{n} z^{i-1}\alpha_i$ be a monic polynomial of degree n in $\overline{A}_{\sigma^{-1}}[z]$ and consider the associated homogeneous linear difference equation

$$(4.20) \qquad \pi(q) = \sigma^{-n}q + (\sigma^{-n+1}q)\alpha_n + \dots + q\alpha_1 = 0, \quad q \in \overline{A}$$

If $\pi(z)$ is completely solvable in $\overline{A}$, then the set of solutions of (4.20) forms an $\mathbf{R}$-vector space of dimension n.

Let $y_1,\dots,y_n$ be n elements in $\bar{A}$, then they are said to be linearly **R**-dependent if there exist constants $c_1,\dots,c_n$, $c_i \in \mathbf{R}$, not all zero such that

$$y_1c_1+\dots+y_nc_n = 0 . \tag{4.21}$$

If whenever (4.21) holds, $c_i = 0$ for $i = 1,\dots,n$, then $y_1,\dots,y_n$ are said to be linearly **R**-independent. A fundamental set of solutions of a completely solvable polynomial of degree n is a set of n linearly **R**-independent solutions of (4.20); in this case, any solution of (4.20) is an **R**-linear combination of this set.

Next, let $y_1,\dots,y_n$ be n nonzero elements of $\bar{A}$ and let

$$\lambda(z) = [z-(\sigma^{-1}y_1)y_1^{-1},\dots,z-(\sigma^{-1}y_n)y_n^{-1}] \tag{4.22}$$

denote the monic least common right multiple of the polynomials $z - (\sigma^{-1}y_i)y_i^{-1}$, $i = 1,2,\dots,n$. As mentioned in section 3.2, the Euclidean algorithm in $Q(A)_{\sigma^{-1}}[z]$ implies the existence and uniqueness of $\lambda(z)$ in $Q(A)_{\sigma^{-1}}[z]$. Let

$$C = \begin{bmatrix} y_1 & \dots & y_n \\ \sigma^{-1}y_1 & & \sigma^{-1}y_n \\ \vdots & & \vdots \\ \sigma^{-n+1}y_1 & \dots & \sigma^{-n+1}y_n \end{bmatrix} \tag{4.23}$$

be the "Casorati matrix" of $y_1,\dots,y_n$.

It is well known [6] that if $(\det C)(k) \neq 0, \forall k \in \mathbf{Z}$, then there exists a unique homogeneous linear difference equation of the n^{th} order having $y_1,\ldots,y_n$ as a fundamental set of solutions and that the coefficients $\alpha_i \varepsilon \bar{A}$, $i = 1,\ldots,n$ of this difference equation are given by the system of linear equations

$$C^t(\alpha_1,\ldots,\alpha_n)^t = (-\sigma^{-n}y_1,\ldots,-\sigma^{-n}y_n)^t \tag{4.24}$$

As we proceed now to show, this linear difference equation obtained under the assumptions that $\det C(k) \neq 0$, $\forall k \varepsilon \mathbf{Z}$ (or equivalently, $\det C$ is a unit in $\bar{A}$), is nothing else but the difference equation

$$\lambda(q) = 0 \ , \tag{4.25}$$

associated with $\lambda(z)$, the monic least common right multiple of the polynomials $z - (\sigma^{-1}y_i)y_i^{-1}$, $i = 1,\ldots,n$. It is easy to see that if $y_1,\ldots,y_n$ are $\mathbf{R}$-dependent then $\det C = 0$. Since $\det C(k) \neq 0$, $\forall k \varepsilon \mathbf{Z}$, the elements, $y_1, y_2,\ldots,y_n$ are therefore $\mathbf{R}$-independent and by Lemma 3.8 it follows that $\lambda(z)$ is of degree n. Further, $\lambda(z)$ is a completely solvable monic polynomial, $\lambda(z) = z^n + \sum_{i=1}^{n} z^{i-1}\alpha'_i$, $\alpha'_i \varepsilon Q(A)$, and $y_1, y_2,\ldots,y_n$ is a fundamental set of solutions of the associated homogeneous equation (4.25). By the uniqueness of the difference equation having $y_1,\ldots,y_n$ as a fundamental set of solutions and corresponding to the n^{th} degree monic polynomial $\lambda(z)$, it follows that $\alpha'_i = \alpha_i$, $i=1,\ldots,n$, and $\lambda(z) \varepsilon \bar{A}_{\sigma^{-1}}[z]$.

<u>Example 4.2</u>. Let $A = \mathbf{R}[k]$ and let $\bar{A} = \{p/q \mid p,q \varepsilon A \text{ and } q(k) \neq 0, \forall k \varepsilon z\}$. Let $y_1 = 1$, $y_2 = k$, $y_3 = k^2$. It is then easy to check that $\det C = 2$ where C is the "Casorati matrix" of y_1, y_2, y_3 (see (4.23)). Referring

to [22] for details, we recall that the least common right multiple of two polynomials $\pi_1(z)$, $\pi_2(z)$ in $Q(A)_{\sigma^{-1}}[z]$ is given by

$$[\pi_1(z),\pi_2(z)] = \pi_1(z)\pi_3^{-1}(z)\pi_2(z)\pi_4^{-1}(z)\ \pi_3(z)\ldots\pi_{n-1}^{-1}(z)\pi_{n-2}(z)\pi_n^{-1}(z)\pi_{n-1}(z)a,$$

where the element $a \in Q(A)$ must be chosen so that the resulting polynomial is monic, and where $\pi_i(z)$, for $i = 3,\ldots,n$, are given by the Euclidean algorithm

$$\pi_1(z) = \pi_2(z)\theta_1(z) + \pi_3(z)\ ,$$

$$\pi_2(z) = \pi_3(z)\theta_2(z) + \pi_4(z)\ ,$$

$$\vdots$$

$$\pi_{n-2}(z) = \pi_{n-1}(z)\theta_{n-2}(z) + \pi_n(z),$$

$$\pi_{n-1}(z) = \pi_n(z)\theta_{n-1}(z)\ .$$

A simple computation gives

$$z - 1 = (z - \frac{k+1}{k})\cdot(1 + \frac{1}{k})\ ,$$

$$z - \frac{k+1}{k} = \frac{1}{k}(z(k+1) - (k+1))\ .$$

Hence

$$[z-1,\ z - \frac{k+1}{k}] = (z-1)\cdot k\cdot(z - \frac{k+1}{k})\cdot\frac{1}{k+1}\ ,$$

$$= (z-1)(z-1)\ ,$$

$$= z^2 - z2 + 1\ .$$

Similarly,

$$z^2 - z2 + 1 = (z - \frac{(k+1)^2}{k^2})(z + \frac{2k^2}{(k+1)^2}) + \frac{2}{k^2} ,$$

$$z - \frac{(k+1)^2}{k^2} = \frac{2}{k^2} (z \frac{(k+1)^2}{2} - \frac{(k+1)^2}{2}) ,$$

whence

$$[z^2 - z2 + 1, \ z - \frac{(k+1)^2}{k^2}] = (z^2 - z2+1)\cdot\frac{k^2}{2}\cdot (z - \frac{(k+1)^2}{k^2})\cdot\frac{2}{(k+1)^2} ,$$

$$= (z^2 - z2+1)(z-1) ,$$

i.e.,

$$\lambda(z) = [z^2 - z2 + 1, \ z - \frac{(k+1)^2}{k^2}] = z^3 - z^2 3 + z3 - 1 ,$$

and the homogeneous linear difference equation in q is given by

$$\sigma^{-3}q - (\sigma^{-2}q)3 + (\sigma^{-1}q)3 - q = 0 .$$

The above way of computing the difference equation having a given set of elements as a fundamental set of solutions is believed to be simpler in general than the usual way which consists of solving the system (4.24) of linear equations.

Finally, it is clear that if $\Sigma = (T,e,-)$ is a single-input t-v $\overline{A}$-system where (T,e) is n-cyclic with characteristic polynomial $\Psi(z)$, and if $y_1,\ldots,y_n$ are n nonzero elements of $\overline{A}$ with the property that the determinant of their "Casorati matrix" C is a unit in $\overline{A}$, then by feedback we can change $\Psi^*(z)$ to $\lambda^*(z)$, where $\lambda^*(z)$ is the completely solvable polynomial for which $\{y_1,\ldots,y_n\}$ is a fundamental set of solutions. Moreover, referring back to section 4.2, we see that C is a fundamental matrix solution of the zero-input system $\hat{x} = \hat{x}\hat{T}$, in the control canonical basis. Hence,

Theorem 4.10. Let $\Sigma = (T,e,-)$ be a t-v $\overline{A}$-system where (T,e) is n-cyclic with characteristic polynomial $\Psi(z)$ and let $y_1,\dots,y_n$, be n nonzero elements of $\overline{A}$. If det C, C the "Casorati matrix" of $y_1,\dots,y_n$, is a unit in $\overline{A}$, then by feedback the T-adjoint of the characteristic polynomial of the closed-loop system can be made completely solvable with $\{y_1,\dots,y_n\}$ as a fundamental set of solutions. Further, C is a fundamental matrix solution of the zero-input closed-loop system in the control canonical basis.

The above concludes our discussion of control canonical forms and related matter. In the final part of this chapter, we would like to make a little detour and establish some connections with the existing results.

A fundamental concept in the theory of linear time-varying systems is uniform controllability [29]. In the one-input case, Silverman [30] has shown that uniform controllability is necessary and sufficient for the existence of control canonical (phase-variable) forms for continuous-time systems.

In order to make a connection between our n-cyclicity concept and uniform controllability of t-v $\overline{A}$-systems in the single-input case we consider the system $\Sigma = (T,e,-)$ and recall that, when viewed pointwise, it is said to be uniformly controllable if

$$\mathrm{rank}[e(k),\ \phi(k,k)e(k-1),\dots,\phi(k,k-n+2)e(k-n+1)] = n,\ \forall k \in \mathbf{Z},$$

where $\phi(\cdot,\cdot)$ is as defined in (2.3).

The above condition turns out to be equivalent to

$$\text{rank } P(k) = \text{rank } [e \;\; eT \;\ldots\; eT^{n-1}](k) = n, \; \forall\, k \in \mathbf{Z},$$

and this latter is equivalent to saying that $\det P(k) \neq 0, \forall k \in \mathbf{Z}$, i.e., $\det P$ is a unit in $\bar{A}$. Hence,

(4.26) Uniform Controllability of $\Sigma = (T,e,-) \Leftrightarrow (T,e)$ is n-cyclic.

It is very interesting that our algebraic theory is quite similar to that given by Kalman for discrete-time, linear, constant systems. In particular, via the cyclic module structure, we can solve a "coefficients assignment problem" in the time-varying case, which as we have seen is very useful in studying the effect of state-variable feedback.

4.5 Summary

This chapter has investigated the important concept of n-cyclicity of an s.ℓ.t. with a series of results in stabilization and feedback as an outcome. The next chapter introduces a new concept of duality for t-v $\bar{A}$-systems and investigates its consequences.

CHAPTER V

T-DUALITY

Unlike the continuous-time case where linear time-varying systems have unique dual, or adjoint, systems, there are several possible dual systems one can associate with discrete-time time-varying linear systems [28]. It is the purpose of this chapter to introduce the concept of T-duality and to apply it to the construction of asymptotic observers.

5.1 Dual of an s.ℓ.t.

Let (X,T) be the pair consisting of an n-dimensional free right R-module X with basis $b = (b_i)_{1\leq i\leq n}$, together with an s.ℓ.t. $T:X \longrightarrow X$ relative to the R-automorphism τ of a commutative difference ring R with 1. Let X^* denote the dual module of X, that is, the set of all R-homomorphism $\xi:X \longrightarrow R:x \longrightarrow \xi x$ with the usual addition and the following multiplication

$$X^*\times R \longrightarrow X^*:\ (\xi,\alpha) \longrightarrow \xi\alpha : x \rightsquigarrow (\xi\alpha)(x) = (\xi x)\alpha = \xi(x\alpha)\,.$$

The module X^* is thus considered as a right R-module, and it is well known that the set of elements $b^* = (b_i^*)_{1\leq i\leq n}$ with $b_i^*(b_j) = \delta_{ij}$, where δ_{ij} denotes the Kronecker delta: $\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$, is a basis for X^* termed the dual basis of b.

Following [3] we introduce the

Definition 5.1. The dual of T is the mapping T^* defined by

$$T^* : X^* \longrightarrow X^* : \xi \longrightarrow \xi T^* : x \rightsquigarrow \tau^{-1}(\xi(xT)) \ .$$

Since ξT^* is clearly additive as a mapping $X \longrightarrow R$, and

$$\begin{aligned}(\xi T^*)(x\alpha) &= \tau^{-1}(\xi(x\alpha T)) \ , \\ &= \tau^{-1}(\xi(xT)\tau\alpha) \ , \\ &= \tau^{-1}(\xi(xT))\alpha \ , \\ &= \xi T^*(x)\alpha \ ,\end{aligned}$$

where $\alpha \in R$, it therefore follows that T^* is well-defined.

Lemma 5.1: The dual T^* of T is an s.ℓ.t. relative to τ^{-1}, and $M_{b^*}(T^*) = \tau^{-1}(M_b(T))^t$.

Proof : The second part of the lemma can be readily verified by computation. For the first part, let ξ_1, $\xi_2 \in X^*$ and consider

$$\begin{aligned}((\xi_1 + \xi_2)T^*)(x) &= \tau^{-1}((\xi_1 + \xi_2)(xT)) \ , \\ &= \tau^{-1}(\xi_1(xT) + \xi_2(xT)) , \\ &= (\xi_1 T^*)(x) + (\xi_2 T^*)(x) \ .\end{aligned}$$

If $\alpha \in R$, then

$$\begin{aligned}(\xi\alpha)T^*(x) &= \tau^{-1}(\xi\alpha(xT)) , \\ &= \tau^{-1}(\xi(xT)\alpha) , \\ &= \tau^{-1}(\xi(xT))\tau^{-1}\alpha \ , \\ &= (\xi T^*)(x)\tau^{-1}\alpha \ .\end{aligned}$$

Note that the requirement that $\tau : R \longrightarrow R$ be a ring automorphism is essential to define the dual of an s.ℓ.t. relative to τ. In like manner, the dual of an s.ℓ.t. relative to τ^{-1} is an s.ℓ.t. relative to τ.

Next, recall that the similarity of two s.ℓ.t.'s $T_i : X \longrightarrow X$, $i = 1,2$, was denoted by $T_1 \simeq T_2$.

<u>Lemma 5.2</u>. $T_1 \simeq T_2$ implies $T_1^* \simeq T_2^*$.

<u>Proof</u>: $T_1 \simeq T_2$ if and only if there exists an invertible n×n matrix P such that $M_b(T_2) = P^{-1}M_b(T_1)(\tau P)$. On applying τ^{-1} to and transposing the matrices of the last equation we get

$$\tau^{-1}(M_b\ (T_2))^t = p^t(\tau^{-1}(M_b(T_1)))^t\tau^{-1}(p^{-1})^t .$$

By Lemma 5.1, it follows that

$$M_{b^*}(T_2^*) = p^t\ M_{b^*}(T_1^*)\ \tau^{-1}(p^t)^{-1},$$

and the conclusion follows by considering the invertible matrix $p^* = (p^t)^{-1}$ and replacing it in the last equation.

As in section 3.8 the skew polynomial ring $R_{\tau^{-1}}[z]$ is associated with the s.ℓ.t. T^* which induces a right $R_{\tau^{-1}}[z]$-module structure on X^* as follows,

$$X^* \times R_{\tau^{-1}}[z] \longrightarrow X^*: \ (\xi,\ \pi(z)) \longrightarrow \xi\pi(z) \triangleq \xi\pi(T^*) \ .$$

Again we define n-cyclicity as before: $T^*: X^* \longrightarrow X^*$ is n-cyclic and ξ is an n-cyclic generator, or briefly (T^*,ξ) is n-cyclic if

$\{\xi, \xi T^*, \ldots, \xi T^{*n-1}\}$ form a basis for X^*. In this case, X^* has the representation $X^* \cong R_{\tau^{-1}}[z]/\chi(z)R_{\tau^{-1}}[z]$, where $\chi(z)$, the order of ξ, is an n^{th} degree monic polynomial in $R_{\tau^{-1}}[z]$.

The relation between the n-cyclicity of an s.ℓ.t. T and that of its dual T^* is given by

Theorem 5.3. T is n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$ if and only if T^* is n-cyclic with characteristic polynomial $\Psi^*(z) = z^n - \sum_{i=1}^{n} z^{i-1}(\sigma^{-i}\beta_i)$.

Proof : Let g be an n-cyclic generator whose order is $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$. We then know that there exists a basis $c = (c_i)_{1\leq i\leq n}$ (the control canonical basis (4.4)) such that

$$M_c(T) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & 1 \\ \beta_1 & \sigma^{-1}\beta_2 & \cdots & \sigma^{-n+1}\beta_n \end{bmatrix}$$

It readily follows that

$$M_{c^*}(T^*) = \begin{bmatrix} 0 & \cdots & & & \sigma^{-1}\beta_1 \\ 1 & & & & \vdots \\ 0 & & & & \vdots \\ \vdots & & & & \\ 0 & \cdots & 0 & 1 & \sigma^{-n}\beta_n \end{bmatrix}$$

Hence,
$$c_1^*T^* = c_2^* ,$$
$$c_2^*T^* = c_3^* ,$$
$$\vdots$$
$$c_{n-1}^*T^* = c_n^* ,$$
$$c_n^*T^* = c_1^*(\sigma^{-1}\beta_1)+\ldots+c_n^*(\sigma^{-n}\beta_n),$$

and $\{c_1^*, c_1^*T^*, \ldots, c_1^*T^{*n-1}\}$ forms a basis for X^*. The s.ℓ.t. T^* is therefore n-cyclic and $\Psi^*(z) = z^n - \sum_{i=1}^{n} z^{i-1}(\sigma^{-i}\beta_i)$ is the order of c_1^*.

The converse is proved by reversing the above steps.

It is because of this result that we termed $R_{\tau^{-1}}[z]$ (in section 2.2) as the T-adjoint of $R_\tau[z]$.

The definition of the dual of an s.ℓ.t. given above is also given by Bourbaki [3]. It seems, however, that the investigation of T^*'s properties in terms of the skew polynomial ring $R_{\tau^{-1}}[z]$ is new.

5.2 T-dual of a t-v $\bar{A}$-System

In this section we introduce a new type of duality, the T-duality, which evolves naturally from the global-in-time representation we have been dealing with.

Recall that each m×n matrix over $\bar{A}$ was considered to be the unique representation of a morphism $\bar{A}^n \longrightarrow \bar{A}^m$ in the standard bases $\mathbf{b}_n$ and $\mathbf{b}_m$ and that a matrix and its corresponding morphism were denoted by the same symbol (see section 2.2.1).

Consider the t-v $\bar{A}$-system $\Sigma = (T,E,H)$ and recall that $M_{\mathbf{b}_n}(T) = D$.

Definition 5.2. The T-dual of $\Sigma = (T,E,H)$ is the triple of $\overline{A}$-matrices $\Sigma^* = (\sigma^{-1}D^t, H^t, E^t)$ defining the dynamical equations

$$\xi = \sigma^{-1}D^t(\sigma^{-1}\xi) + H^t\eta \ , \tag{5.1}$$

$$\phi = E^t\xi \ , \tag{5.2}$$

where $\eta \in (\mathbf{R}^{\mathbf{Z}})^p$, and t stands for matrix transposition. Note that when $\eta \in (\overline{A})^p$, then $\xi \in \overline{A}^n$ and $\phi \in \overline{A}^m$.

Since $D = \sigma F$ (see (2.7)), it follows that $\sigma^{-1}D^t = F^t$ and equation (5.1) takes the more familiar form

$$\xi = F^t(\sigma^{-1}\xi) + H^t\eta \ . \tag{5.3}$$

A comparison between equation (5.3) and (2.7) shows that the T-dual system evolves in "reverse" time.

Let $T^*: (\overline{A}^n)^* \longrightarrow (\overline{A}^n)^*$ be the dual of T; then T^* is an s.ℓ.t. relative to σ^{-1} and $M_{\mathbf{b}_n}(T^*) = \sigma^{-1}D^t = F^t$. As before, we introduce the

Definition 5.3. The dual T^* of the s.ℓ.t. of a t-v $\overline{A}$-system Σ is called the s.ℓ.t. of the T-dual system Σ^*.

The s.ℓ.t. T^* is thus defined as follows

$$T^*: (\overline{A}^n)^* \longrightarrow (\overline{A}^n)^*: \xi \longrightarrow \xi T^* = F^t(\sigma^{-1}\xi).$$

Viewing $H^t(E^t)$ as the matrix of the $\overline{A}$-morphism $H^t: (\overline{A}^p)^* \longrightarrow (\overline{A}^n)^*$ ($E^t : (\overline{A}^n)^* \longrightarrow (\overline{A}^m)^*$) with respect to the bases $\mathbf{b}_p^*$, $\mathbf{b}_n^*$ ($\mathbf{b}_n^*$, $\mathbf{b}_m^*$), equations (5.2) and (5.3) could also be written as

$$\xi = \xi T^* + H^t\eta \ , \tag{5.4}$$

and,

$$\phi = E^t \xi . \tag{5.5}$$

Let P^* be an n×n invertible matrix over $\overline{A}$, defining the change of coordinates $P^*\hat{\xi} = \xi$ in $(\overline{A}^n)^*$. Equations (5.2) and (5.3) take then the form

$$\hat{\xi} = (P^{*-1}F^t\sigma^{-1}P^*)(\sigma^{-1}\hat{\xi}) + (P^{*-1}H^t)\eta , \tag{5.6}$$

$$\phi = (E^tP^*)\hat{\xi} . \tag{5.7}$$

If we let
$$P^{*-1}F^t(\sigma^{-1}P^*) = \hat{F}^t ,$$
$$P^{*-1}H^t = \hat{H}^t ,$$
$$E^tP^* = \hat{E}^t ,$$

then the triple $\hat{\Sigma}^* = (\hat{F}^t, \hat{H}^t, \hat{E}^t)$ defining the dynamical equations (5.6) and (5.7) is an $\overline{A}$-equivalent system to Σ^*in the sense of definition 2.3. Hence, $\overline{A}$-equivalence in the T-dual framework corresponds to a coordinate change applied to the equations (5.4) and (5.5). Clearly, the s.ℓ.t.'s T^* and $\hat{T}^*$ of Σ^* and $\hat{\Sigma}^*$ are similar. We shall therefore denote the T-dual of the t-v $\overline{A}$-system $\Sigma = (T,E,H)$ by the triple $\Sigma^* = (T^*, H^t, E^t)$.

Upon endowing the free n-dimensional right $\overline{A}$-module $(\overline{A}^n)^*$ with the right $\overline{A}_{\sigma^{-1}}[z]$-module structure induced by T^*, all the results obtained for t-v $\overline{A}$-systems can be duplicated for their T-duals when these latter are considered as t-v $\overline{A}$-systems in the T-dual framework.

For example, if the single-input t-v $\overline{A}$-system $\Sigma^* = (T^*, h^t, -)$ is such that (T^*, h^t) is n-cyclic, then there exists a basis $c^* = (c_i^*)_{1 \le i \le n}$ relative to which

$$(5.8) \qquad M_{c^*}(T^*) = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ \vdots & & & 1 \\ \beta_1 & \sigma\beta_2 & \cdots & \sigma^{n-1}\beta_n \end{bmatrix}, \quad h^t = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}$$

where $\chi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$ is the characteristic polynomial of T^* (the order of the n-cyclic generator h^t).

The above result has the following important consequence

Theorem 5.4. Let $\Sigma = (T, -, h)$ be a single-output t-v $\overline{A}$-system, where H in this case consists of a row vector h, and E is immaterial. If the single-input T-dual system $\Sigma^* = (T^*, h^t, -)$ is such that (T^*, h^t) is n-cyclic with characteristic polynomial $\chi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$, then Σ is $\overline{A}$-equivalent to $\Sigma_0 = (T_0, -, h_0)$ whose matrices have the form

$$(5.9) \qquad M(T_0) = \begin{bmatrix} 0 & \cdots & & \sigma\beta_1 \\ 1 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & \sigma^n\beta_n \end{bmatrix}, \quad h = [0 \ \ldots \ 1] ,$$

and conversely.

Proof : The fact that (T^*, h^*) is n-cyclic with characteristic polynomial $\chi(z) = z^n - \sum_{i=1}^{n} z^{n-1}\beta_i$ implies that Σ^* is $\overline{A}$-equivalent to $\hat{\Sigma}^* = (\hat{T}^*, \hat{h}^*)$ in the control canonical form (5.8). Hence $\Sigma = (T, -, h)$

is $\overline{A}$-equivalent to $\Sigma_0 = (T_0,-,h_0)$ whose matrices have the form (5.9).

<u>Definition 5.4</u>. The form (5.9) is called the observability canonical form of Σ.

In the next section the observability canonical form will be applied to the problem of state estimation.

Finally, we note the interesting fact that there is a connection between uniform observability, in the single-output case, and the n-cyclicity of T^* similar to the one which exists between uniform controllability and the n-cyclicity of T (see (4.26)). More specifically, let us recall [29] that the single-output t-v $\overline{A}$-system $\Sigma = (T,-,h)$, when viewed pointwise-in-time, is said to be uniformly observable if

$$\text{rank}[h^t(k),\ \phi^t(k,k)h^t(k+1),\ldots,\phi^t(k,k+n-2)h^t(k+n-1)] = n, \forall\, k \in \mathbf{Z},$$

where $\Phi(\cdot,\cdot)$ is defined by (2.3).

If we let $P^* = [h^t\ h^tT^* \ldots h^tT^{*n-1}]$, then the above rank condition is the same as the condition rank $P^*(k) = n$, $\forall\, k \in \mathbf{Z}$, and this latter is equivalent to saying that det $P^*(k) \neq 0, \forall\, k \in \mathbf{Z}$, i.e., det P^* is a unit in $\overline{A}$. Hence

(5.10) Uniform Observability of $\Sigma = (T,-,h) \Leftrightarrow (T^*,h^t)$ is n-cyclic.

<u>5.3 State Estimation</u>

In most cases, the states of a given system are not available, i.e., they can not be measured directly. Thus, to design control laws for time-varying systems, one first must design a system which estimates the state variables. In this section, sufficient conditions are given to insure the existence of such state estimators.

Let $\Sigma = (T,E,H)$ be a t-v $\bar{A}$-system defining the dynamical equations

(5.11) $$x = D(\sigma x) + E(\sigma u),$$

(5.12) $$y = Hx \ ,$$

<u>Definition 5.5</u>. Σ is said to have an asymptotic state estimator given by

(5.13) $$\underset{\sim}{x} = D(\sigma \underset{\sim}{x}) + L(\sigma y - \sigma H(\sigma \underset{\sim}{x})) + E(\sigma u) \ ,$$

$$\underset{\sim}{y} = \underset{\sim}{x} \ ,$$

If the $n \times p$ matrix L is over $\bar{A}$ and $\tilde{x}(k) = x(k) - \underset{\sim}{x}(k) \longrightarrow 0$ with $k \longrightarrow \infty$.

In this definition $\underset{\sim}{x}(k)$ is the state at time k estimated from past outputs and inputs. Note that equation (5.13) can be written as

(5.14) $$\underset{\sim}{x} = (D - L\sigma H)(\sigma \underset{\sim}{x}) + L(\sigma y) + E(\sigma u),$$

and by subtracting (5.11) from (5.14), we see that $\tilde{x}$, the error between the real state x and the estimated one $\underset{\sim}{x}$, satisfies the equation

(5.15) $$\tilde{x} = (D - L(\sigma H))(\sigma \tilde{x}) \ .$$

The following theorem tells us when such an estimator exists in the single-output case.

<u>Theorem 5.5</u>. Let $\Sigma = (T,E,h)$ be a single-output t-v $\bar{A}$-system. If (T^*,h^t) is n-cyclic, then there exists a state estimator with $\tilde{x}(k) \longrightarrow 0$ in n steps.

<u>Proof</u> : If (T^*,h^t) is n-cyclic, then by Theorem 5.4 it follows that $\Sigma = (T,E,h)$ is $\overline{A}$-equivalent to $\Sigma_0 = (T_0,E_0,h_0)$ in the observability canonical form

$$M(T_0) = \begin{bmatrix} 0 & \cdots & 0 & \sigma\beta_1 \\ 1 & & & \vdots \\ \vdots & & & \vdots \\ 0 & \cdots & 1 & \sigma^n\beta_n \end{bmatrix}, \quad h_0 = [0 \ \ldots \ 1] ,$$

where the β_i's are the coefficients of the characteristic polynomial $\chi(z)$ of $(\hat{T}^*,h^t)$. Let J be the n×n invertible matrix over $\overline{A}$ giving the $\overline{A}$-equivalence. If we let $\lambda(z) = z^n \in \overline{A}_\sigma[z]$ and define $\ell_0 = (\ell_{01},\ldots,\ell_{0n})$, where $\ell_{0i} = \sigma^i\beta_i$ for $i = 1,\ldots,n$, then

$$M(T_0) = \ell_0(\sigma h_0) = \begin{bmatrix} 0 & \cdots & & 0 \\ 1 & & & \vdots \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Let

$$\underset{\sim}{x} = (M(T_0) - \ell_0(\sigma h_0))(\sigma\underset{\sim}{x}_0) + \ell_0(\sigma y_0) + E_0(\sigma u) .$$

This is an asymptotic state estimator for the given system in the observable canonical form. For it is easy to see that if $\tilde{x}_0 = x_0 - \underset{\sim}{x}_0$, then

$$\tilde{x}_0 = \begin{bmatrix} 0 & \cdots & & 0 \\ 1 & & & \vdots \\ \vdots & & & \\ 0 & \cdots & 1 & 0 \end{bmatrix} (\sigma\tilde{x}_0) ,$$

and it is clear that $\tilde{x}_0(k_o+n-1) = 0$, for any k_o and any initial state $x(k_o-1)$.

Now the system

$$\underset{\sim}{x} = (D-\ell(\sigma h))(\sigma\underset{\sim}{x}) + \ell(\sigma y) + E(\sigma u),$$

where $\ell = \ell_0 J$, is an asymptotic state estimator for $\Sigma = (T,E,h)$ since $\tilde{x} = x-\underset{\sim}{x}$ is related to $\tilde{x}$ by J, i.e., $\tilde{x} = J\tilde{x}_0$, and $\tilde{x}_0(k) \longrightarrow 0$ ($\tilde{x}_0(k_o+n-1) = 0$) in n steps.

5.4 The Regulator Problem

By the usual definition, a regulator for a given system consists of a control law (or a controller) and of an asymptotic state estimator [10]. In previous sections, we have established conditions under which the characteristic polynomial of the closed-loop system can be assigned arbitrarily (i.e., can design an arbitrary control law, see section 4.3) and an asymptotic estimator can be designed. In this section we combine these results to construct regulators for single-input single-output t-v $\bar{A}$-systems.

Let $\Sigma = (T,e,h)$ be a single-input single-output t-v $\bar{A}$-system and assume that (T,e) is n-cyclic and that h^t is an n-cyclic generator of T^*. Then by Theorem 4.7, given any monic polynomial $\lambda(z)$ of degree n in $\bar{A}[z]$, there exists a row vector w such that (T_1,e) where $M_{\mathfrak{b}_n}(T_1) = M_{\mathfrak{b}_n}(T) - e(\sigma w)$, is n-cyclic with characteristic polynomial $\lambda(z)$, i.e., Σ has an arbitrary control law (see (Def. 4.4)). Let us choose a stable control law characterized by $\lambda(z) = z^n$. Then, the closed-loop system $x = (D-e(\sigma w))\sigma x$ is stable. On the other hand, by hypothesis (T^*,h^t) is n-cyclic, and, by Theorem 5.5, Σ has an asymptotic state estimator which gives an estimate $\underset{\sim}{x}$ of the real state x of Σ such that the error $\tilde{x} = x - \underset{\sim}{x}$ approaches zero as $k \longrightarrow \infty$. Now substitute $u = w\underset{\sim}{x}$ into equation (5.11). We then get

$$\begin{aligned} x &= D(\sigma x) + e(\sigma w)(\sigma \underset{\sim}{x}) , \\ &= D(\sigma x) + e(\sigma w)(\sigma x - \sigma\tilde{x}) , \\ &= (D - e(\sigma w))\sigma x - e(\sigma w)(\sigma\tilde{x}) , \end{aligned}$$

which is clearly stable since $\tilde{x}(k) \longrightarrow 0$ as $k \longrightarrow \infty$ and w was chosen to yield a stable closed-loop system.

We summarize the above discussion in the following.

Theorem 5.6. Let $\Sigma = (T,e,h)$ be a single-input single-output t-v $\overline{A}$-system such that (T,e) is n-cyclic and h^t is an n-cyclic generator of T^*, the dual of T. Then there exists a control law such that $\lambda(z) = z^n$ is the characteristic polynomial (of the s.ℓ.t.) of the closed-loop system and there exists an asymptotic state estimator such that the overall system is stable.

5.5 Summary

This chapter has introduced the important concept of T-duality. The construction of asymptotic state estimators was given in the single-output case. These estimators and the results of section 4.3 on feedback were used in constructing regulators for single-input single-output t-v $\overline{A}$-systems. The next chapter extends these results to the general case.

CHAPTER VI

MULTIVARIABLE t-v $\overline{A}$-SYSTEMS

The generalization of the previous results to the multivariable case, even for constant systems, is by no means a trivial matter, and the methods used are much less transparent. The difficulties could be attributed to the fact that inputs and outputs of multivariable systems are generally coupled in the sense that an input may control more than one output, and an output may be controlled by more than one input.

In this chapter, we investigate the multivariable case and make effective use of our previous results. More specifically, our line of attack will aim at reducing the multivariable case to the single-variable one and then applying the already established techniques. As it will be seen, this treatment of the multivariable case is new (in the time-varying case), simple, and constructive.

6.1 Stabilization by Feedback

Let us consider an m-input p-output t-v $\overline{A}$-system $\Sigma = (T,E,H)$ together with its dynamical equations

(6.1) $$x = D(\sigma x) + E(\sigma u) ,$$

(6.2) $$y = Hx ,$$

where $M_{b_n}(T) = D$.

Let e_i be the i^{th} column of E; that is, $E = [e_1\ e_2 \cdots e_m]$. If T were n-cyclic and e_i, for some $i = 1,2,\ldots,n$, were an n-cyclic generator, then the results of section 4.3 could (after minor modification) be applied here and the s.ℓ.t. of the closed-loop system could be assigned any characteristic polynomial in $\overline{A}_\sigma[z]$. Consequently, the system could be stabilized by appropriate feedback. In the following discussion, sufficient conditions under which Σ can be transformed, by feedback, to a t-v $\overline{A}$-system whose s.ℓ.t. is n-cyclic with any e_i, $1 \leq i \leq r$ (r to be specified), as an n-cyclic generator are derived.

An interesting point to be noted here is that t-v $\overline{A}$-systems appear in our framework as constant systems, and the proof of the just mentioned result is based on the one employed by Heymann [7] in solving the pole placement problem for time-invariant systems.

Let V denote the set of elements of $\overline{A}^n$ defined as follows

$$V = \{e_1, e_1 z, \ldots, e_1 z^{n-1};\ e_2, e_2 z, \ldots, e_2 z^{n-1}; \ldots;\ e_m, e_m z, \ldots, e_m z^{n-1}\}.$$

We shall select a special subset of V as follows. Start with e_1 and then proceed to $e_1 z$, $e_1 z^2, \ldots$ up to $e_1 z^{\nu_1 - 1}$ where the integer ν_1 is selected such that the element $e_1 z^{\nu_1}$ can be expressed as a Q(A)-linear combination of $\{e_1, e_1 z, \ldots, e_1 z^{\nu_1 - 1}\}$ (Q(A) is the quotient field of A). In other words, ν_1 is the smallest integer such that $e_1, e_1 z, \ldots, e_1 z^{\nu_1 - 1}$, $e_1 z^{\nu_1}$ are linearly dependent over Q(A). Then take e_2 and proceed as before until $e_2 z^{\nu_2}$ is a linear combination of $\{e_1, e_1 z_1, \ldots, e_1 z^{\nu_1 - 1}, e_2, e_2 z, \ldots, e_2 z^{\nu_2 - 1}\}$. Assume that there exists an integer ν_r (selected in the above fashion) such that $\nu_1 + \nu_2 + \ldots + \nu_r = n$ and view each $e_i z^{j_i - 1}$, $1 \le i \le r$, $1 \le j_i \le \nu_i$, as a column vector with respect to the standard basis $\mathbf{b}_n$ of $\overline{A}^n$. Let B denote the n×n matrix defined by

$$(6.3) \qquad B = [e_1 \; e_1 z \ldots e_1 z^{\nu_1 - 1} \; e_2 \ldots e_2 z^{\nu_2 - 1} \ldots \; e_r \ldots e_r z^{\nu_r - 1}].$$

<u>Theorem (6.1)</u>. If B is invertible over $\overline{A}$ (or equivalently det B is a unit in $\overline{A}$), then there exists an n×n matrix W_i over $\overline{A}$ such that the single-input closed loop system $\Sigma_i = (T_i, e_i, H)$, where $1 \le i \le r$ and $M_{\mathbf{b}_n}(T_i) = M_{\mathbf{b}_n}(T) - E(\sigma W_i)$, is such that (T_i, e_i) is n-cyclic.

<u>Proof</u> : Without loss of generality we prove the theorem for i=1. Let S denote the m×n matrix defined by

$$S = \begin{array}{c} [0 \ldots 0 \; s_2 \; 0 \ldots 0 \; s_3 \; 0 \ldots \; 0\,] \\ \qquad\quad \uparrow \qquad\qquad\quad \uparrow \qquad\qquad\quad \uparrow \\ \text{the } \nu_1\text{th column} \quad \text{the } (\nu_1+\nu_2)\text{th column} \quad \text{the } n\text{th column} \end{array}$$

where s_i is the i^{th} column of $I_{m\times m}$, the m×m unit matrix. Since B is invertible over $\overline{A}$, define the m×n matrix W_1, over $\overline{A}$, by

$$(6.4) \qquad W_1 = SB^{-1} \; ,$$

and let $T_1 : \overline{A}^n \longrightarrow \overline{A}^n$ be the s.ℓ.t. whose matrix is given by

$$M_{\mathbf{b}_n}(T_1) = M_{\mathbf{b}_n}(T) - E(\sigma W_1) \; .$$

Now we claim that (T_1, e_1) is n-cyclic; that is, $[e_1 \; e_1 T_1 \ldots e_1 T_1^{n-1}]$ is invertible over $\overline{A}$ (see Lemma 4.4), or equivalently, det $[e_1 \; e_1 T_1 \ldots e_1 T_1^{n-1}]$ is a unit in $\overline{A}$. In fact, we shall prove that det $[e_1 \; e_1 T_1 \ldots e_1 T_1^{n-1}] =$

ε det B, $\varepsilon = \pm 1$. From (6.3) we can write $W_1 B = S$ explicitly as

$$(6.5) \qquad W_1[e_1 \ldots e_1 z^{\nu_1 - 1} \; e_2 \ldots e_2 z^{\nu_2 - 1} \ldots e_r \ldots e_r z^{\nu_r - 1}] =$$

$$[0 \;\ldots\; s_2 \quad 0 \;\ldots\; s_3 \quad \ldots \quad 0\,]$$

and consider $e_1 T_1$; by definition and from (6.5), it is easy to verify that

$$\begin{aligned}
e_1 T_1 &= (D - E(\sigma W_1))\sigma e_1 \ , \\
&= D(\sigma e_1) - E\sigma(W_1 e_1) = e_1 T \ , \\
e_1 T_1^2 &= (D-E(\sigma W_1))\sigma(e_1 T), \\
&= D\sigma(e_1 T) - E\sigma(W_1(e_1 T)) = e_1 T^2 \ , \\
&\vdots \\
e_1 T_1^{\nu_1 - 1} &= \ \ldots\ldots \quad \ldots\ldots \quad = e_1 T^{\nu_1 - 1} , \\
e_1 T_1^{\nu_1} &= (D - E(\sigma W_1))\sigma(e_1 T^{\nu_1 - 1}) \ , \\
&= D\sigma(e_1 T^{\nu_1 - 1}) - E\sigma(W_1(e_1 T^{\nu_1 - 1})) \ , \\
&= e_1 T^{\nu_1} - E s_2 = e_1 T^{\nu_1} - e_2 \ , \\
&= e_2 - \ldots\ldots \ , \\
e_1 T_1^{\nu_1 + 1} &= (D - E(\sigma W_1))\sigma(e_2 - e_1 T^{\nu_1}) \ , \\
&= e_2 T - \ \ldots\ldots \ , \\
&\vdots \\
e_1 T_1^{n-1} &= (D - E(\sigma W_1))\sigma(e_r T^{\nu_r - 2} - \ldots) = e_r T^{\nu_r - 1} - \ldots ,
\end{aligned}$$

where, in the above equations, the ellipsis ... stand for linear combinations of the preceding vectors. It readily follows that $\det\ [e_1\ e_1T_1\ \dots\ e_1T_1^{n-1}] = \varepsilon \det B$. Hence, $[e_1\ \ e_1T_1 \dots\ e_1T_1^{n-1}]$ is invertible over $\overline{A}$ and (T_1,e_1) is n-cyclic.

Corollary (6.9). If det B is a unit in $\overline{A}$, then the closed-loop t-v $\overline{A}$-system $\Sigma_1 = (T_1,E,H)$ has an n-cyclic s.ℓ.t. T_1 whose characteristic polynomial $\lambda(z)$ can be arbitrarily chosen.

Proof: By the above theorem, the t-v $\overline{A}$-system $\Sigma' = (T',E,H)$, where $M_{\mathfrak{b}_n}(T') = M_{\mathfrak{b}_n}(T) - E(\sigma W_1')$, is such that (T',e_1) is n-cyclic with characteristic polynomial $\Psi(z) = z^n - \sum_{i=1}^{n} z^{i-1}\beta_i$ (order of e_1). The system $\hat{\Sigma} = (T',e_1,H)$ satisfies the conditions of theorem 4.7; it therefore follows, by appropriate choice of the row vector w, that the closed-loop system $\hat{\Sigma}_1 = (T_1,e_1,H)$, where $M_{\mathfrak{b}_n}(T_1) = M_{\mathfrak{b}_n}(T') - e_1(\sigma w)$, is such that (T_1,e_1) is n-cyclic with characteristic polynomial $\lambda(z)$. If we let W denote the nxn matrix

$$W = \begin{pmatrix} w \\ 0 \\ \\ \\ 0 \end{pmatrix} ,$$

it readily follows that $E(\sigma W) = e_1(\sigma w)$. Hence, if we define $W_1 = W_1' + W$, then $M_{\mathfrak{b}_n}(T_1) = M_{\mathfrak{b}_n}(T) - E(\sigma W_1)$, and the control law W_1 renders the s.ℓ.t. T_1 of the closed-loop system $\Sigma_1 = (T_1,E,H)$ n-cyclic with characteristic polynomial $\lambda(z)$.

Corollary (6.3). If det B is a unit in $\bar{A}$, then the closed-loop system $\Sigma_1 = (T_1, E, H)$ can be made asymptotically stable.

Proof: Readily follows by the results of section 4.4.1 and by the above corollary when we take $\lambda(z) = z^n$.

Finally, note that the feedback gain matrix (or control law) W_1 which renders the s.ℓ.t. T_1 of the closed-loop system $\Sigma_1 = (T_1, E, H)$ n-cyclic is obviously not unique, since any e_i, $2 \le i \le r$, would yield a control law W_i accomplishing the same result.

Illustrative Example 6.1. Let $A = R[k]$, let $\bar{A} = \{p/q \mid p, q \in A$, and $q(k) \neq 0, \forall\, k \in \mathbb{Z}\}$, and consider the t-v $\bar{A}$-system $\Sigma = (T, E, -)$ with

$$M_{\mathbf{b}_n}(T) = \begin{bmatrix} 0 & 0 & k^2+1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad E = \begin{bmatrix} 0 & k \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

It is easy to verify that

$$e_1 T = \begin{bmatrix} k^2+1 \\ 0 \\ 1 \end{bmatrix}, \quad e_1 T^2 = \begin{bmatrix} k^2+1 \\ 0 \\ k^2-2k+4 \end{bmatrix},$$

and that e_1, e_1T are linearly independent but (e_1, e_1T, e_1T^2) are not. Hence we consider e_2 and form the matrix

$$B = [e_1 \ e_1T \ e_2] = \begin{bmatrix} 0 & k^2+1 & k \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

which is invertible over $\overline{A}$ with inverse

$$B^{-1} = 1/k^2+1 \begin{pmatrix} -1 & k & k^2+1 \\ 1 & -k & 0 \\ 0 & k^2+1 & 0 \end{pmatrix} .$$

Define W_1 by

$$W_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} 1/k^2+1 \begin{pmatrix} -1 & k & k^2+1 \\ 1 & -k & 0 \\ 0 & k^2+1 & 0 \end{pmatrix} ,$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 1/k^2+1 & -k/k^2+1 & 0 \end{pmatrix} ,$$

and form

$$M_{\mathfrak{b}_n}(T_1) = M_{\mathfrak{b}_n}(T) - E(\sigma W_1) = \begin{pmatrix} 0 & 0 & k^2+1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & k \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1/k^2+1 & -k/k^2+1 & 0 \end{pmatrix},$$

$$= \begin{pmatrix} -k/(k-1)^2+1 & k(k-1)/(k-1)^2+1 & k^2+1 \\ -1/(k-1)^2+1 & 1+(k-1)/(k-1)^2+1 & 0 \\ 1 & 0 & 1 \end{pmatrix} .$$

Now

$$e_1T_1 = \begin{bmatrix} k^2+1 \\ 0 \\ 1 \end{bmatrix}, \quad e_1T_1^2 = \begin{bmatrix} k^2-k+1 \\ -1 \\ (k-1)^2+2 \end{bmatrix}$$

and $\det [e_1 \; e_1T_1 \; e_1T_1^2] = -(k^2+1) = -\det B$; that is, (T_1,e_1) is n-cyclic.

6.2 Asymptotic Estimators

Let $\Sigma = (T,E,H)$ be a t-v $\overline{A}$-system and consider its T-dual system $\Sigma^* = (T^*,H^t,E^t)$. If Σ^*, as a t-v $\overline{A}$-system, satisfies the conditions of Theorem 6.1 then we can find an asymptotic estimator (see definition 5.5) of Σ. More specifically, if the n×n matrix

$$(6.6) \qquad 0 = [h_1^t \; h_1^tT^* \ldots h_1^t T^{*\nu_1 - 1} \; h_2^t \; h_2^tT^* \ldots h_2^tT^{*\nu_2 - 1} \ldots h_r^t \; h_r^tT^* \ldots h_r^tT^{*\nu_r - 1}]$$

is invertible over $\overline{A}$, where $\nu_1+\ldots+\nu_r = n$ and ν_i, $i=1,\ldots,r$, is selected such that $h_i^tT^{*\nu_i}$ is a linear combination of the preceding vectors, then there exists an n×p matrix L^t over $\overline{A}$ such that (T_1^*,h^t) is n-cyclic where

$$(6.7) \qquad M_{\mathbf{b}_n^*}(T_1^*) = M_{\mathbf{b}_n^*}(T^*) - H^t(\sigma^{-1}L^t) .$$

By Theorem 5.5 the single-output t-v $\overline{A}$-system $\Sigma_1 = (T_1,E,h_1)$ has an asymptotic estimator. It follows, by a similar argument to the one used in the proof of Corollary 6.3, that $\Sigma_1 = (T_1,E,H)$ has an asymptotic state estimator.

6.3 The Regulator Problem

In light of the results of the preceding two sections, the construction of the regulator in the multivariable case is straightforward and will be omitted.

Finally, we note in passing that stabilization by feedback and the existence of exponential estimators were established, in the continuous-time case, for a class of time-varying systems termed index-invariant [19,34,35]. These latter can roughly be characterized as having their controllability properties invariant under time-variance. The comparison between our conditions and the conditions used in [34, 35] for index-invariant systems is somewhat difficult since the setups and the techniques of generating certain "canonical" bases are different. We believe, however, that our treatment is more transparent and is easy to implement.

6.4 Summary

This chapter has extended the results of Chapters IV and V to the multivariable case.

CHAPTER VII

CONCLUSIONS

7.1 General Remarks

This research effort has centered on a new algebraic approach toward the study of discrete-time time-varying linear systems. The methodology was based on a noncommutative algebraic framework consisting of a module structure over skew polynomial rings, and the resulting algebraic structure was used to study large classes of time-varying linear systems, termed t-v $\bar{A}$-system. The outcome was a network of interesting system theoretic results. The n-cyclicity concept, for example, was found to be equivalent to the existence of control canonical forms -- well known for their usefulness in state variable feedback. The n-cyclicity concept is a "time-varying version" of the usual cyclicity concept which has been successfully used in Kalman's algebraic theory for discrete time time-invariant linear systems. The equivalence between n-cyclicity and uniform controllability in the single-input case makes it possible to apply module theory to the study of dynamical properties in the time-varying case.

The T-duality theory which evolved as a consequence of adopting a global representation in terms of semilinear transformations made it possible, together with the module structure associated with the "T-adjoint" constructions, to treat T-dual systems as t-v $\bar{A}$-system. This dual framework allowed us to solve the estimation problem by dualizing the results on state-variable feedback.

As for the multivariable case, the simplicity of our approach results from that of Heymann's for the constant case. The process of reducing the multivariable case to the single input case allows us to apply the n-cyclicity concept with the resulting characteristic polynomial to the study of multivariable systems without having to use any general canonical forms.

On the other hand, the computational tasks the designers are faced with in control problems are usually of sizeable magnitude; this is especially true, if the computations must be carried out at every instant of time. Through the global representation and the incorporation of the theory of noncommutative difference polynomials into the state variable description, the computations involved in such problems are performed within the $\bar{A}$-structure. If $\bar{A}$ is such that operations in it are easily programmable, then the constructions presented in this work can be carried out on a computer. This gives, for example, control laws which are specified in terms of an arbitrary time-reference.

Finally, it is believed that for certain classes of time-varying systems, deep system theoretic results can be obtained under more relaxed conditions than those considered here. In fact Kamen is currently investigating this problem for t-v A-systems, where A is a semi-local ring of time functions.

7.2 Summary

Chapter II reviewed the basic system description. The usual pointwise-in-time definition was discussed and a global-in-time representation was given.

Chapter III layed the foundations of our algebraic approach. The concepts of semilinear transformations and skew polynomial rings were introduced and adapted to the system theoretical needs of our investigations. This resulted in a module structure whose rings of operators was noncommutative.

Chapter IV presented the n-cyclicity concept of an s.ℓ.t. and exploited its relation to control canonical forms and state variable feedback in the single-input case. The so-called characteristic polynomial of an s.ℓ.t. was defined and a system theoretical interpretation of it was also given. The "coefficient assignment" and stabilization problems were considered.

Chapter V introduced a new global-in-time duality theory which was termed T-duality. This T-duality provided the necessary elements for the construction of asymptotic state estimators in the single-output case, which then lead to the construction of regulators.

Chapter VI generalized the results of Chapters III, IV and V to the multivariable case.

Finally, some general remarks and a summary of this thesis were given in Chapter VII.

REFERENCES

1. Amitsur, A. C. Differential Polynomials and Division Algebras, Ann. Math. 59 (March 1954), 245-278.

2. Birkhoff, G. and Maclane, S. Algebra, Macmillan Company, New York (1968).

3. Bourbaki, N. Algebra I, Chapter 1-3, Elements of Mathematics, Ed. Herman.

4. Cohn, P. M. Free Rings and Their Relations, Academic Press, London (1971).

5. Cohn, R. H. Difference Algebra, Interscience Publ., a division of John Wiley, New York (1965).

6. Fort, T. Finite Differences, Oxford University Press, American House (1948).

7. Heymann, M. Comments on Pole Assignment in Multi-input Controllable Linear Systems, IEEE Trans. Automatic Control, Vol. AC-13 (1968), 748-749.

8. Jacobson, N. Pseudo-Linear Transformations, Ann. Math. 38 (May 1936), 484-507.

9. Johnston, R. D. Linear Systems over Various Rings, Ph. D. Dissertation, Electronic Systems Lab., Dept. of Elec. Eng., MIT, Cambridge, Mass.

10. Kalman, R. E., Falb, P. L., and Arbib, M. A. Topics in Mathematical Systems Theory, McGraw-Hill, New York (1969).

11. Kalman, R. E. Algebraic Theory of Linear Systems, Rep. from Proc. Third Allerton Conference on Circuit and System Theory (1965), 563-577.

12. __________ Lectures on Controllability and Observability, Centro Internazionale Mathematica Estivo, Bologna, Italy (1968).

13. Kalman, R. E. and Bertran, J. E. Control System Analysis and Design via the "Second Method" of Lyapunov. II Discrete-Time Systems Journal of Basic Engineering, Transactions of the ASME (June, 1960), 394-400.

14. Kalman, R. E. On the Stability of Time-Varying Linear Systems, Rep. from IRE Trans. of the Professional Group on Circuit Theory, CT-9, 4 (Dec. 1962), 419-422.

15. Kaman, E. W. On an Algebraic Theory of Systems Defined by Convolution Operators, Mathematical System Theory, Vol. 9, No. 1, to appear.

16. Module Structure of Infinite-Dimensional Systems with Applications to Controllability. to appear in SIAM J. Control.

17. __________, A New Algebraic Approach to Linear Time-Varying Systems, submitted for publication.

18. Lang, S. Algebra, Addison-Wesley (1965).

19. Morse, A. S. and Silverman, L. M. Structure of Index-Invariant Systems, SIAM J. Control, Vol. 11, 2 (May, 1973), 215-225.

20. Morse, A. S., "Ring Models for Delay-Differential Systems," Proc. IFAC Symp. on Multivariable Technological Systems, Manchester, U.K., 1974, pp. S6.1-S6.7.

21. Newcomb, R. A. Local Time-Variable Synthesis, Fourth Colloquim on Microwave Communications, Budapest, 1970.

22. Ore, O. Theory of Non-Commutative Polynomials, Ann. Math., 36 (1933), 480-508.

23. Ramnath, R. V. A Multiple Time Scales Approach to the Analysis of Linear Systems, Tech. Re. AFFDL-TR-68-60, Princeton University, Princeton, New Jersey (Oct.,1968).

24. ________ A New Approach to the Design of Time-Varying Control Systems with Application to the Space Shuttle Boost, Charles Stark Draper Lab, MIT, Cambridge, Mass. 02139 (May 1973).

25. Rouchaleau, Y. Linear, Discrete-Time, Finite Dimensional, Dynamical Systems over Some Classes of Commutative Rings, Ph.D. Dissertation, Stanford University (1972).

26. Rouchaleau, Y., and Wyman, B. F. Linear Dynamical Systems over Integral Domains, Journal of Computer and Systems Sciences, Vol. 9, No. 2 (Oct. 1974), 129-142.

27. Rouchaleau, Y., Wyman, B. F., and Kalman, R. E. Algebraic Structure of Linear Dynamical Systems, III, Realization Theory Over a Commutative Ring, Proc. Nat. Aca. Science 69 (Nov. 1972), 3404-3406.

28. Sarachik, P. E., and Kreindler, E. Concerning Adjoints of Discrete-Time Systems, IEEE Trans. Automatic Control, AC-10 (July, 1965), 350-352.

29. Silverman, L. M. and Meadows, H. E. Degrees of Controllability in Time-Variable Linear Systems, Proc. of the National Electronic Conference, Vol. 21 (1965), 689-693.

30. Silverman, L. M. Transformation of Time-Variable Systems to Canonical (phase-variable) Form, IEEE Trans. Automatic Control, Vol. AC-11 (1966), 300-303.

31. Sontag, E. D. On Feedback Control of Delay-Differential and other Finitary Systems, Center for Mathematical System Theory, University of Florida (to appear).

32. Weiss, L. Controllability, Realization, and Stability of Discrete-Time Systems, SIAM J. Control, Vol. 10 (1972), 230-251.

33. Wiberg, D. M. State Space and Linear Systems, Schawm's Outline Series, McGraw-Hill Book Company (1971).

34. Wolovich, W. A. On the Stabilization of Controllable Systems, IEEE Trans. Automatic Control, AC-13 (1968), 569-592.

35. Wolovich, W. A., On State Estimation of Observable Systems, Proc. 1968 JACC, Ann Arbor, Mich. (1968), 210-220.

36. Zadeh, L., and Desoer, E. Linear Systems Theory, McGraw-Hill, 1963.

VITA

Khaled Mohammad Hafez was born in Haifa, Palestine, on March 20, 1945. Mr. Hafez received the Licence es-Sciences in Mathematics from the C.E.M. (Beirut) in 1969. He graduated with a M.S. in Mathematics from the American University of Beirut in 1972.

The School of Information and Computer Science at Georgia Institute of Technology granted Mr. Hafez a Doctor of Philosophy degree in 1975. He was appointed to positions of graduate research and teaching assistantships during his graduate studies.

Mr. Hafez has co-authored papers in various aspects of systems theory and dynamical processes and was a Senator of the Graduate Student Government at Georgia Tech.