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Organization: Georgia Tech Research Corp
Submitted By:
Westdickenberg, Michael - Principal Investigator
Title:
Isentropic Euler Equations

Project Participants

Senior Personnel

Name: Westdickenberg, Michael
Worked for more than 160 Hours: Yes
Contribution to Project:

Post-doc

Graduate Student

Undergraduate Student

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts

Yann Brenier (University of Nice, France)
Wilfrid Gangbo (Georgia Institute of Technology, Atlanta)
Philippe G. LeFloch (Universite Paris VI & CNRS, Paris)
Giuseppe Savare (University of Pavia, Italy)
Jon A. Wilkening (University of California, Berkeley)

Activities and Findings

Research and Education Activities:

With Philippe G. LeFloch we wrote a research paper 'Finite Energy Solutions to the Isentropic Euler Equations with Geometric Effects' on global existence for the system of isentropic Euler equations. The paper was published in J. Math. Pures et Appl. We reported on the result on several occasion, including seminars, conferences, and a summer school. We started working on extending our result to the one-dimensional relativistic isentropic Euler equations.

With Wilfrid Gangbo we wrote a research paper 'Optimal Transport for the System of Isentropic Euler Equations', which introduces a new variational time discretization for the isentropic Euler equations that relies on ideas from optimal transport theory. The paper was published in Comm. PDE. We reported on our results at conferences and seminars.

Together with Jon Wilkening we implemented this time discretization and did numerical experiments. We wrote a research paper 'Variational Particles Schemes for the Porous Medium Equation and for the System of Isentropic Euler Equations' on the results. The paper was published in M2AN. We reported about these findings on several occasions, including seminars and conferences.

With Felix Otto and Gianluca Crippa we finished lecture notes on previous work of Felix Otto, Camillo De Lellis, and myself, on the structure of entropy solutions to scalar conservation laws. These lecture notes appeared in a book published by the Unione Matematica Italiana (Springer).

We wrote and submitted a research paper 'Projections onto the Cone of Optimal Transport Maps and Compressible Fluid Flows'. In the paper, we give a characterization of the polar cone to the cone of optimal transport maps and show convergence of the variational time discretization introduced in our paper with Wilfrid Gangbo (see above) towards a measure-valued solution of the isentropic Euler equations. The paper appeared in J. Hyp. Differ. Eqns. We presented the result at several workshops and seminars.

With Yann Brenier, Wilfrid Gangbo, and Giuseppe Savare we wrote a research paper 'Sticky particle dynamics with interactions', which is currently being finished and will be submitted soon.

We are currently writing a research paper about a variant of the variational time discretization first studied with Wilfrid Gangbo (see above). The new version considerably simplifies the construction and applies not only to the isentropic, but also to the full Euler equations. Again we show convergence towards a measure-valued solution.

We developed and implemented a new numerical method for the two-dimensional isentropic and the full Euler equations, based on our new time discretization. WE are currently writing up our results.

Findings:

With Philippe G. LeFloch we proved global existence of entropy solutions for the system of isentropic Euler equations with geometric effects, in particular spherically symmetric solutions in the whole space (with the origin included). This has been an open question for some time. Its proof required new a priori bounds on the density and the total energy. We are currently generalizing our result to the special relativistic case. We already proved that the crucial higher integrability estimate for the density also holds for the relativistic case. We are currently investigating convergence of a suitable approximation toward a measure-valued solution.

With Wilfrid Gangbo we introduced a new time discretization for the system of isentropic Euler equations. It is based on the intuition that admissible solutions to this hyperbolic conservation law should decrease its total energy at maximal rate, as suggested by Dafermos. We heavily use machinery from the the theory of optimal transport and of gradient flows on spaces of probability measures. One crucial step in this method involves the projection of a square integrable vector field onto the closed convex cone of optimal transport maps, which are gradients of convex functions. We give a characterization of the polar cone to this cone. More precisely, we show that the difference between the given vector field and its metric projection can be written as the sum of the divergence of a measure (which we call a stress tensor field), taking values in the nonnegative matrices, and a divergence-free term. We derived precise bounds on the size of the stress tensor. This characterization allowed us to prove that the sequence of approximate solutions generated by our variational time discretization converges to a (suitably defined) measure-valued solution of the isentropic Euler equations. We also clarified by a formal argument how the isentropic Euler equations (in the irrotational regime) can be understood as a second order ordinary differential equation on the space of probability measures.

With Jon Wilkening we implemented the new time discretization mentioned above for the system of isentropic Euler equations, in the special case of one space dimension. Our numerical results are very promising: The scheme captures very well shocks and discontinuities. Nonnegativity of the density and conservation of mass is built in. Without much additional work it is even possible to define higher-order schemes, using the same general framework.

With Yann Brenier, Giuseppe Savare, and Wilfrid Gangbo we considered compressible pressureless flows in Lagrangian coordinates in one space dimension. We assumed that the fluid self-interacts through a force field that generated by the fluid. It is quite natural to consider this system as a second-order differential inclusion defined on the space of optimal transport maps, which form a convex cone in a Hilbert space. Assuming a sticky particle dynamics (particles stick together after collisions) we showed, however, that the second order differential inclusion actually reduces to a first order differential inclusion, for which a well-developed theory exists. We proved the existence, semigroup property, and stability of solutions. Surprisingly, even the non-sticky case (when particles may separate after collisions) can be described in terms of a first order differential inclusion. We again obtained existence, semigroup property, and stability results.

The variational time discretization discussed above formally converges to solutions of the isentropic Euler equations in the irrotational regime, where the velocity is a gradient field. By modifying the construction, we got rid of this restriction and were able to construct measure-valued solutions of the isentropic Euler equations that are not necessarily irrotational. The new construction is considerable simpler than the old one and can be applied also to the full Euler equations (where formally the thermodynamical entropy is just transported along the flow). We established displacement convexity for the internal energy functional depending not only on the density, but also on the entropy. Moreover, we show that the family of weak entropies, which play a central role for the existence results for the one-dimensional isentropic Euler equations, are displacement convex as well. This observation will likely be important when showing that the variational time discretization converges to a weak (and not only a measure-valued) solution in one space dimension.

This modified approach also considerably simplifies the numerical implementation of the variational time discretization. We implemented the scheme in two space dimensions using a moving mesh with reconnection. The variational nature of the method takes care of the usual difficulties that arise in Lagrangian methods, such as cell degeneracies and mesh tangling, and proved to be quite accurate. It remains to compare the efficiency and robustness of the method to that of more standard approaches.

Training and Development:

The work on the project deepened our understanding of optimal transport theory and differential inclusions and their relation to partial differential equations, in particular hyperbolic conservation laws. We got the opportunity to teach about optimal transport to graduate students.

Outreach Activities:

I presented my work on several occasions, including seminars, conferences, and workshops. I gave a summer school on existence of finite energy solutions in Helsinki, Finland, in 2007 and another one in Trieste, Italy, in 2011. I gave two introductory lectures about my research at the Research Horizon Seminar in our department, in which faculty members describe their research area to beginning graduate students. Moreover, I gave a special topics course on optimal transportation for the graduate students in our department.

I served as a referee for several journals, and as an NSF panelist.

Journal Publications

Philippe G. LeFloch and Michael Westdickenberg, "Finite energy solutions to the isentropic Euler equations with geometric effects", J. Math. Pures et Appl., p. 389, vol. 88, (2007). Published, 10.1016/j.matpur.2007.07.004

Wilfrid Gangbo and Michael Westdickenberg, "Optimal Transport for the system of isentropic Euler equations", Comm. PDE, p. 1041, vol. 34, (2009). Published, 10.1080/03605300902892345

Jon Wilkening and Michael Westdickenberg, "Variational time discretization for the isentropic Euler equations", M2AN, p. 133, vol. 44, (2010). Published, 10.1051/m2an/2009043

Michael Westdickenberg, "Projections onto the Cone of Optimal Transport Maps and Compressible Fluid Flows", J. Hyperbolic Differ. Equ., p. , vol. 7, (2010). Published, 10.1142/S0219891610002244

Books or Other One-time Publications

Gianluca Crippa, Felix Otto, and Michael Westdickenberg, "Regularizing effect of nonlinearity in multidimensional scalar conservation laws", (2008). Book, Published
Bibliography: Lectures Notes of the Unione Matematica Italiana, Springer

Web/Internet Site

Other Specific Products

Contributions

Contributions within Discipline:

Global existence of spherically symmetric entropy solutions to the isentropic Euler equations in the whole space has been an open question for some time. Our result will help to understand what is happening at the origin, where truly multidimensional effects play a role. The compactness framework we developed is applicable also in other contexts.

Optimal transport theory offers a whole new toolbox that we hope to harness for the study of hyperbolic conservation laws. It touches on and combines ideas from the calculus of variations, convex analysis, and flows on spaces of probability measures, which will be very useful for studying the properties of the compressible gas dynamics equations.

The interpretation of conservation laws as steepest descents comes with a very natural time discretization, which can be used to develop a new class of numerical schemes. This methods are of Lagrangian type, but the usual ad hoc fixes needed to guarantee well-definedness of the method are not needed anymore since the variational nature already takes care of problems like mesh tangling.

Contributions to Other Disciplines:

It has been a long-standing conjecture in fluid dynamics that solutions to the Navier-Stokes-equations (which take into account the viscosity of the fluid, which is always present in the physical world) should converge to solutions of the inviscid Euler equations in the limit of small viscosity (thus high Reynold number). Our compactness framework for finite energy solutions to the isentropic Euler equations has been used to give the first rigorous mathematical proof that this convergence is indeed happening.

Contributions to Human Resource Development:

We gave introductory lectures about our research and a special topics course on optimal transport theory to graduate students.

Contributions to Resources for Research and Education:

Contributions Beyond Science and Engineering:

Conference Proceedings

Categories for which nothing is reported:

Organizational Partners

Any Web/Internet Site

Any Product

Contributions: To Any Resources for Research and Education

Contributions: To Any Beyond Science and Engineering

Any Conference