HYBRID IS GOOD:
STOCHASTIC OPTIMIZATION AND APPLIED STATISTICS FOR OR

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HYBRID IS GOOD: STOCHASTIC OPTIMIZATION AND APPLIED STATISTICS FOR OR

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SUMMARY

In the first part of this thesis, we study revenue management in resource exchange alliances. An important way in which sellers collaborate is through the formation of alliances since seller alliances enable the sale of products combined from the resources of several sellers. For example, a vacation package may consist of airline tickets for 2 people, a hotel room for 4 nights, and car rental for 5 days. The resources used to offer the combined product are provided by 3 sellers: the airline, the hotel, and the car rental company. Another example of a widely used alliance is between carriers such as airlines and ocean carriers. The examples above illustrate that alliances are important in various industries and that alliances can be structured in many different ways. The detail rules of an alliance are clearly important for both the stability of the alliance, as well as the well-being of each member of the alliance.

In this study, we focus on resource exchange alliances. We propose an alliance design model that takes into account how the alliance members compete after the resource exchange by selling substitutable (and also complementary) products. It will be shown that a resource exchange alliance can increase both profits and consumer surplus at the same time that it increases horizontal competition. Currently, airline revenue management systems do not take into account the effect of alliances on the competition they are facing. For example, airline revenue management systems treat seats that they give to another airline in a resource exchange alliance as sales instead of as an increase in the resources available to the other airline for use in selling competing products.

We present a stochastic optimization model with equilibrium constraints (SMPEC) to determine the optimal amount of each resource to be exchanged. By "optimal" we mean the resource exchange that maximizes the sum of the alliance members' profits after the exchange, taking into account the resulting competition. The motivation for maximizing the sum of the profits is that, with the exchange of resources such as seat capacity or container capacity, alliance members can also exchange money as partial compensation for the exchanged resources, and the objective is to maximize the total amount of money available to the alliance. For each resource exchange, the competition among alliance members is modeled as a noncooperative game in which each alliance member chooses the prices for its own products, subject to its own capacity constraints (which depend on the resource exchange), to maximize its own profit. Assuming a linear model of demands for products as a function of both the prices chosen by the seller of the products as well as the prices chosen by the other alliance members for their products, necessary conditions are derived for an equilibrium, as well as sufficient conditions for uniqueness of an equilibrium. We show that the sufficient conditions hold under reasonable conditions, and that the equilibrium can be computed efficiently.

We develop a trust region algorithm to search for an optimal resource exchange. To compare the effect of a resource-exchange alliance on profits under different conditions, we also formulate a model of the no alliance case and a model of the perfect coordination case (where alliance members perfectly coordinate pricing decisions). We present a computational study in which we investigate the effects of: (i) the level of product differentiation between alliance members, and (ii) the difference in convenience level between an alliance and the setting with no alliance on the relative profits of the cases with and without an alliance, and the case with perfect coordination.

In the second part of this thesis, we study the estimation of risk measures in risk management. In the financial industry, sell-side analysts periodically publish recommendations of underlying securities with target prices. These recommendations reflect specific economic conditions and influence investors decisions and thus price movements. However, this type of analysis does not provide risk measures associated with underlying companies even though controlling risk is one of the most important aspects of investment decision making. Analysis and control of risk are important in many other industries in addition to financial services. For example, headlines of the past year have highlighted volatile oil prices impacting industries around the world, ranging from energy, to transportation, to technology. The impact of rising oil prices on total supply chain cost is substantial and one cannot make the right strategic decisions without understanding the risk associated with oil price movements in the business.

Now let us consider a situation where there exists information composed of economic and market variables which can be considered as a set of predictors for a variable of interest. In that case, one would be interested in the estimation of a risk measure conditional on observed values of predictors. In this study, we discuss linear regression approaches to the estimation of law invariant conditional risk measures. In particular, Value-at-Risk (VaR) and Average Value-at-Risk (AVaR) measures are discussed in detail. Two estimation procedures are considered and compared; one is based on residual analysis of the standard least squares method and the other is in the spirit of the M-estimation approach used in robust statistics.

First approach is to apply the standard least squares estimation procedure and then to make an adjustment of the estimate of the intercept parameter based on random error values. Since true values of the error term are unknown, it is natural idea to replace true error values by residual values. We refer to this estimation approach as the Least Squares Residuals (LSR) method. In fact, LSR approach can be easily applied to a considerably larger class of law invariant risk measures. An alternative approach is in the spirit of robust statistics; it is based on minimization of an appropriate error function. For the VaR risk measure, the error function is readily available and the corresponding robust regression approach is known as the quantile regression method. For general coherent risk measures, the situation is more delicate. Possibility of constructing the corresponding M-estimators is rather challenging, and such estimators certainly do not exist for the AVaR risk measure. Nevertheless, it is possible to construct the approximations and formulate the optimization problem as a linear programming. It could be remarked here that the approach based on mixing M-estimators is somewhat restrictive and constructing an appropriate error function for a particular risk measure could be quite involved.

Next, we derive large sample statistical inference of the estimators. We first show that both estimators are consistent. Furthermore, we investigate the efficiency of two estimation procedures by computing corresponding asymptotic variances. Finite sample properties of the proposed estimators are also investigated and compared with theoretical derivations in an extensive Monte Carlo study. Simulation results, under different distribution assumptions of the error term, indicate that the LSR estimators usually perform better than their (mixed) quantiles counterparts. Typically, the mean absolute error, asymptotic variance, and empirical variance of the LSR estimators are smaller than that of quantile based estimators.

Finally, empirical results on the real data (different financial asset classes including Credit Default Swap and US equities) are also provided to illustrate the performance of the estimators. Prediction performances on the real data example suggest similar conclusions: LSR estimators typically perform better.

CHAPTER I

REVENUE MANAGEMENT IN RESOURCE EXCHANGE ALLIANCES

1.1 Introduction

1.1.1 Optimal Alliance Design When Friends Become Competitors

An important way in which carriers such as airlines and ocean carriers collaborate is through the formation of alliances. For example, in an airline alliance each alliance member (marketing member) can sell tickets for flights operated by another alliance member (operating member) and the marketing member can put its own code on the flight. That enables airlines to sell tickets for itineraries that include flights operated by multiple airlines, thereby dramatically increasing the number of itinerary products that each airline can sell.

Another example of a widely used carrier alliance is the type of alliance that ocean container carriers enter into when they introduce new joint services. A "service" is a cycle (also called a "loop" or a "rotation") of voyages that repeat according to a regular schedule, typically with weekly departures at each port included in the cycle. Suppose the cycle is ports A,B,C,D,E,A. A set of ships is dedicated to the service, with each ship visiting the ports in the sequence A,B,C,D,E,A,B,.... To offer weekly departures at each port included in the cycle, the headway between successive ships traversing the cycle must be one week. Thus, if it takes a ship n weeks to complete one cycle, then n ships are needed to offer the service with weekly departures at each port in the cycle. For many services that visit ports in Asia and North America, and services that visit ports in Asia and Europe, it takes a ship approximately 6 weeks to complete one cycle, and thus 6 ships are needed to offer the service. Taking into account that a large container ship can cost several hundred million US dollars (and the trend is towards even larger container ships, because larger container ships tend to have significantly lower per unit operating costs), it becomes clear that for even the large carriers it would require an enormous investment to introduce a new service. A solution is for several carriers to enter into an alliance to offer a new service. Many services that visit ports in Asia and North America, and services that visit ports in Asia and Europe, are offered by alliances between two carriers. Each carrier in the alliance provides one or more ships to be used for the service. The capacity on each ship is then allocated to all the alliance members, often in proportion to the capacity that the alliance member contributed to the service. For example, if carrier 1 contributes 2 ships and carrier 2 contributes 4 ships to the service, and all the ships in the service have the same capacity, then carrier 1 can use 1/3 of each ship's capacity, and carrier 2 can use 2/3 of each ship's capacity. That way, each carrier in the alliance can offer weekly departures at each port in the service even though it did not have enough ships by itself to do so.

Vacation packages provide another example of seller alliances enabling the sale of products combined from the resources of several sellers. For example, a vacation package may consist of airline tickets for 2 people, a hotel room for 4 nights, and car rental for 5 days. The resources used to provide the combined product are provided by 3 sellers: the airline, the hotel, and the car rental company. Computers and peripherals provide another example of products combined from the resources of several sellers. There are many similar examples.

The examples above illustrate that alliances are or can be important in various industries, and that alliances can be structured in many different ways. The detail rules of an alliance are clearly important for both the stability of the alliance, as well as the well-being of each member of the alliance (e.g. see Boyd (1998) and Vinod (2005) for the discussion of the basic alliance types in the airline industry). The major distinguishing factors between different alliance structures involve the control of the inventory of the resources and the pricing of the products that alliance members offer for sale. For example, in a so-called "free-sell" airline alliance, the alliance members agree in advance of the selling season on the transfer prices at which operating members will sell capacity on flights to marketing members. However, under free-sell, during the selling season the operating members still control the availability of all the capacity on the flights operated by them, even if the flights are included in the code-share agreement. Both legal and operational reasons prevent airlines in alliances from merging their revenue management systems (Barla and Constantatos (2006)).

Another type of alliance structure is a so-called “resource exchange” or “hard block” alliance, in which the sellers exchange resources (for example, seat space on various flights or container capacity on various voyages, and possibly money). After the exchange, each seller can control the received resources as though they are the owner of the resources. Resource exchange alliances are more common among ocean carriers than airlines. An example of a resource exchange alliance between ocean carriers was given above. As an example of a resource exchange alliance between airlines, airline 1 may receive 15 seats on flight A operated by airline 2, and airline 2 may receive 10 seats on flight B operated by airline 1 as well as $2000. After the exchange, airline 1 controls the revenue management for the 15 seats on flight A that it received from airline 2, as well as for the remaining seats on the flights that it operates, and similarly, airline 2 controls the revenue management for the 10 seats on flight B that it received from airline 1, as well as for the remaining seats on the flights that it operates.

Since the control of transfer prices by free-sell alliances may cause suspicions of price collusion, resource exchange alliances have a potential benefit over free-sell alliances regarding competition and anti-trust regulation. However, we should mention that the structure of carrier alliances varies from alliance to alliance, and no carrier alliance is structured as simply as the stylistic cases of free-sell alliances or resource exchange alliances.

After formation of an alliance the alliance members compete to sell substitute products. In that way, alliances increase competition (more specifically, alliances increase horizontal competition). Currently, airline revenue management systems do not take into account the effect of alliances on the competition they are facing. For example, airline revenue management systems treat seats that they give to another airline in a resource exchange alliance as sales (Vinod (2005)), instead of as an increase in the resources available to the other airline for use in selling competing products.

In this study we focus on resource exchange alliances. We propose an alliance design model that takes into account how the alliance members compete after the resource exchange by selling substitutable (and also complementary) products. It will be shown that a resource exchange alliance can increase both profits and consumer surplus at the same time that it increases horizontal competition.

First we provide an economic motivation for interest in resource exchange alliances. Specifically, in Section 1.2 we consider a model with two sellers, each of whom sells one type of resource. Customers are interested in a product that requires both resource types. First we consider the case without an alliance, in which each seller sets the price for its resource, and customers buy resources from both sellers to obtain the desired product. Then we compare the equilibrium prices, quantities, profits, and consumer surpluses without an alliance with the prices, quantities, profits, and consumer surpluses that would result from perfect coordination. It is shown that the equilibrium prices without an alliance are higher than the prices under perfect coordination, and the equilibrium quantities without an alliance are lower than the quantities under perfect coordination. Intuitively this happens because without an alliance each seller is implicitly attempting to gather a larger share of the total revenue. This effect is especially pronounced if the capacity is large, and it results in both the total profit and the consumer surplus being smaller without an alliance than under perfect coordination.

Second we consider a resource exchange alliance. We show that both the total profit and the consumer surplus of a resource exchange alliance with exchange quantities chosen to maximize the total profit are always greater than the total profit and the consumer surplus respectively without an alliance (except if the capacity is small, in which case the equilibrium prices, quantities, profits, and consumer surpluses are the same for the settings with an alliance, without an alliance, and with perfect coordination). In addition, we show that the equilibrium prices, quantities, profits, and consumer surpluses are equal for a resource exchange alliance with exchange quantities chosen to maximize the total profit and for perfect coordination, except when the sellers' products are complementary (which would be unusual in a resource exchange alliance) and the capacity is large.

In Section 1.3, we consider models of no alliance, perfect coordination, and a resource exchange alliance for the case in which each seller has multiple resources. For resource exchange alliances we formulate an optimization model to determine the amount of each resource to be exchanged, taking into account the consequences of the exchange on the subsequent competition among the alliance members. If one assumes that after the resources have been exchanged, each alliance member chooses the prices of its products to maximize its own profit, and that this behavior of the alliance members leads to an equilibrium, then the problem can be formulated as a mathematical program with equilibrium constraints. An important question is whether, for each resource exchange, there exists an equilibrium and, if so, whether it is unique. In Section 1.3.6 we show how to determine whether a unique equilibrium exists, and how to compute it. A trust region algorithm is used to solve the mathematical program with equilibrium constraints. Illustrative numerical results are provided in Section 1.4, and we compare the results for the cases with no alliance, perfect coordination, and a resource exchange alliance.

1.1.2 Related Literature

There are broadly two streams of literature related to this study — literature that study the impact of alliances, such as the impact of airline alliances on pricing, competition, and public welfare; and literature that address the design of alliance agreements. The literature on alliance design is sparse relative to the literature on the impact of alliances. Also, most papers on alliances have addressed either ocean shipping alliances or airline alliances.

The literature on ocean shipping alliances have addressed questions such as network design under alliances, choice of resource exchange amounts, revenue sharing, or the stability of alliances. For example, Midoro and Pitto (2000) investigated factors which affect the stability of liner shipping alliances, and Slack et al. (2002) empirically examined the changes in services made by container shipping lines in response to the formation of alliances. Song and Panayides (2002) analyzed two examples using cooperative game theory to investigate the rationale behind and decision-making behavior in liner shipping alliances. Lu et al. (2010) studied a model of a resource exchange alliance between two carriers to determine the resource exchange or purchase amount to maximize the profit of an individual alliance member. Agarwal and Ergun (2010) considered a service network design problem in which ocean carriers share capacity on their ships. Their design problem does not take into account that carriers will compete when they share capacity on the same ships.

The literature on airline alliances have addressed questions such as the choice of flights to include in code-share agreements, the choice of transfer prices or proration rates in free-sell alliances, the effect of alliances on booking limits and the number of seats sold, and the effect of cargo alliances on the passenger market. For example, Brueckner (2001) considers a model with two airlines, with and without an alliance, and showed that for most parameter values, the alliance decreases the amount sold of the common interhub product, and increases the amounts sold of all the other products, especially the shared interline products. Sivakumar (2003) presented Code Share Optimizer, a tool built by United Airlines that considers the interaction between proration agreements, demand, fares, and market shares. O'Neal et al. (2007) built a code-share flight profitability tool to automate the code-share flight selection process at Delta airlines. Abdelghany et al. (2009) also presented a model for airlines to determine a set of flights for a code-share agreement. Zhang et al. (2004) examined the effect of an air cargo alliance between two passenger airlines on the passenger market. Netessine and Shumsky (2005) consider a model with multiple airlines, in which each airline has two fare classes for each flight, and each airline chooses a booking limit for each flight. The horizontal competition setting involves two airlines with one flight each, in which demand that is not accommodated on the first choice airline overflows to the other airline. In the vertical competition setting connecting passengers travel on flights of more than one airline. The equilibrium booking limits are compared with the booking limits under perfect coordination. The question of transfer prices that achieves perfect coordination is also investigated. These transfer prices are functions of the booking limits of both airlines, and also depends on the expectations of functions of random demand. Thus these coordinating transfer prices are not numbers determined before the airlines make their booking limit decisions. Wen and Hsu (2006) proposed a multi-objective optimization model to determine flight frequencies on airline code-share alliance networks. Barla and Constantatos (2006) consider a market with three competitors, two of which decide to cooperate where demand is uncertain. Under a "strategic alliance (SA)", the partners (a) jointly choose capacity in order to maximize their total expected profit, (b) share this capacity among themselves based on the Nash bargaining outcome, and (c) market their capacity shares independently after demand is revealed. They show that the profits of the cooperating firms is greater under SA than under a full merger (in their model, a merger does not include maintaining different brands), and thus SA is not necessarily a second best solution that is justified by regulations restricting airline mergers. Houghtalen et al. (2010) used the model in Agarwal and Ergun (2010) to choose capacity exchange prices for air cargo carriers. Their model also does not take into account that air cargo carriers (and freight forwarders) will compete when they exchange capacity.

Wright et al. (2010) formulate a Markov-game model of two airlines under a free-sell alliance. They first describe centralized booking control which gives an upper-bound on the total revenue for the alliance, and they find that no Markovian transfer-pricing scheme with decentralized booking control can guarantee the same revenues as centralized booking control. They examine static and dynamic transfer-pricing schemes, and show that the performance of static transfer-pricing schemes depends on the homogeneity and stability of the relative values that each airline places on the inventory used in interline itineraries. They also conclude that there is no one best dynamic proration scheme.

Hu et al. (2011) also study a model of a free-sell airline alliance. Similar to our model, their model is a two-stage model with the alliance design decision in the first stage and operational selling decisions of individual airlines in the second stage, formulated as a Nash equilibrium problem. Their alliance design decisions are static proration rates, whereas our alliance design decisions are static resource exchange amounts. In their model the prices and proration rates are the same irrespective of which airline sells the interline itinerary, whereas our model makes provision for different prices and demands for the same interline itinerary sold by different marketing airlines. Their second-stage decisions are static booking limits, whereas our second-stage decisions are static product prices. The booking limits in their model are capacity allocations to different itineraries, and not nested booking limits on the flight legs. The demand in both models may be random. However, in their model the demand for different itineraries (and fare classes) are assumed to be independent, and also independent of the second-stage decisions (booking limits), whereas in our model the demand for different itineraries are allowed to be dependent, and to depend on the second-stage decisions (prices). In both models existence and uniqueness of a Nash equilibrium in the second stage is somewhat problematic — for their model, a Nash equilibrium always exists, but is not unique, whereas for our model existence and uniqueness of a Nash equilibrium can be guaranteed in special cases (for example, when the demands for products are independent of the prices of other products), but not in general. For our model, existence and uniqueness of a Nash equilibrium can be verified numerically for a given demand model. In both papers, total profits under alliances are compared with total profits under a centralized solution, and it is investigated when the profits are equal. In our study we compare the consumer surplus in addition to total seller profits.

1.2 *Two-Resource Model*

Consider 2 sellers, indexed by -1 and 1. Each seller produces one resource. Seller i produces resource i, and a maximum quantity b_i of resource i can be consumed. Seller i has a constant marginal cost of c_i per unit of resource i consumed, and seller i chooses the price $\tilde{y}_i + c_i$ per unit of resource i, that is, $\tilde{y}_i$ denotes the price in excess of the marginal cost c_i per unit of resource i. Customers want to consume a product that requires one unit of each resource. (In this section, there is no demand for a product that consists of only one resource.) Thus customers buy units of a product consisting of one unit of each resource and pay $c_{-1}+\tilde{y}_{-1}+c_1+\tilde{y}_1$ per unit of product. The demand d for products depends on the prices as follows:

$$d \quad = \quad \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}, \tag{1.2.1}$$

where $\tilde{\alpha}$ and $\tilde{\beta}$ are positive constants known to each seller. Assume that $\tilde{\alpha} > 0$, that is, demand is positive if each seller charges only its marginal cost. The detailed calculations for this section are given in Appendix A.

1.2.1 No Alliance

First consider the case with no alliance, which is modeled as a non-cooperative game. Let $b_{\min} := \min\{b_{-1}, b_1\}$. Thus, the number of products sold is given by $\min\{b_{\min},\ \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}$, and the profit of seller i is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) \quad := \quad \tilde{y}_i \min\{b_{\min},\ \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\}\}.$$

If $b_{\min} \geq \tilde{\alpha}/3$, then the equilibrium prices are given by

$$\tilde{y}_i^* \quad = \quad \frac{\tilde{\alpha}}{3\tilde{\beta}}, \tag{1.2.2}$$

the equilibrium demand is equal to

$$\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1}^* + \tilde{y}_1^*) \quad = \quad \frac{\tilde{\alpha}}{3} \quad > \quad 0, \tag{1.2.3}$$

the resulting profit of seller i is equal to

$$\tilde{y}_i^* \min\{b_{\min},\ \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i}^* + \tilde{y}_i^*)\}\} \quad = \quad \frac{\tilde{\alpha}^2}{9\tilde{\beta}}, \tag{1.2.4}$$

and thus the total profit of both sellers together is equal to

$$\tilde{y}_{-1}^* \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1}^* + \tilde{y}_1^*)\right] + \tilde{y}_1^* \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1}^* + \tilde{y}_1^*)\right] \quad = \quad \frac{2\tilde{\alpha}^2}{9\tilde{\beta}}, \tag{1.2.5}$$

and the consumer surplus is equal to

$$\frac{1}{2}\left(\frac{\tilde{\alpha}}{\tilde{\beta}} - \frac{2\tilde{\alpha}}{3\tilde{\beta}}\right)\frac{\tilde{\alpha}}{3} \quad = \quad \frac{\tilde{\alpha}^2}{18\tilde{\beta}}. \tag{1.2.6}$$

If $b_{\min} \leq \tilde{\alpha}/3$, then all pairs of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\ [\tilde{\alpha} - 2b_{\min}]/\tilde{\beta})$ and $([\tilde{\alpha} - 2b_{\min}]/\tilde{\beta},\ b_{\min}/\tilde{\beta})$ are equilibria. For all of these equilibrium prices the total price is equal to $(\tilde{\alpha} - b_{\min})/\tilde{\beta}$, the demand is equal to $b_{\min}$, the resulting profit of seller i is equal to $\tilde{y}_i b_{\min}$, and thus the total profit of both sellers together is equal to

$$\tilde{y}_{-1} b_{\min} + \tilde{y}_1 b_{\min} \quad = \quad \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} b_{\min}, \tag{1.2.7}$$

and the consumer surplus is equal to

$$\frac{1}{2}\left(\frac{\tilde{\alpha}}{\tilde{\beta}} - \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}}\right) b_{\min} \quad = \quad \frac{b_{\min}^2}{2\tilde{\beta}}. \tag{1.2.8}$$

1.2.2 Perfect Coordination

In this section we determine the maximum achievable total profit of the two sellers together, that is, the total profit if the sellers would perfectly coordinate pricing.

The total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \quad := \quad [\tilde{y}_{-1} + \tilde{y}_1] \min\{b_{\min},\ \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}.$$

If $b_{\min} \geq \tilde{\alpha}/2$, then the optimal total price is equal to

$$\bar{y}_{-1} + \bar{y}_1 \quad = \quad \frac{\tilde{\alpha}}{2\tilde{\beta}}. \tag{1.2.9}$$

Note that (1.2.2) and (1.2.9) show that $\tilde{y}^*_{-1} + \tilde{y}^*_1 > \bar{y}_{-1} + \bar{y}_1$, that is, the total of the equilibrium prices is greater than the optimal total price. (These results are reminiscent of the comparison of the cases with and without vertical integration by Spengler (1950); however, the setting here is different because one seller does not buy a product from another seller and add a mark-up before reselling it.) The corresponding demand is equal to

$$\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1) \quad = \quad \frac{\tilde{\alpha}}{2} \quad > \quad \frac{\tilde{\alpha}}{3} \quad = \quad \tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1), \tag{1.2.10}$$

the total profit of both sellers together is equal to

$$[\bar{y}_{-1} + \bar{y}_1]\left[\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1)\right] \quad = \quad \frac{\tilde{\alpha}^2}{4\tilde{\beta}}, \tag{1.2.11}$$

and the consumer surplus is equal to

$$\frac{1}{2}\left(\frac{\tilde{\alpha}}{\tilde{\beta}} - \frac{\tilde{\alpha}}{2\tilde{\beta}}\right)\frac{\tilde{\alpha}}{2} \quad = \quad \frac{\tilde{\alpha}^2}{8\tilde{\beta}}. \tag{1.2.12}$$

If $b_{\min} \le \tilde{\alpha}/2$, then the optimal total price is given by $\bar{y}_{-1} + \bar{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, with corresponding demand equal to $b_{\min}$. The total profit of both sellers together is equal to $(\bar{y}_{-1} + \bar{y}_1)\, b_{\min} = (\tilde{\alpha} - b_{\min})\, b_{\min}/\tilde{\beta}$, and the consumer surplus is equal to $\left[\tilde{\alpha}/\tilde{\beta} - (\tilde{\alpha} - b_{\min})/\tilde{\beta}\right] b_{\min}/2 = b_{\min}^2/(2\tilde{\beta})$.

Note that when capacity is small, $b_{\min} \le \tilde{\alpha}/3$, the total profit of the setting with no alliance cannot be increased by coordination, and the consumer surplus is also the same for the two settings. When capacity is large, $b_{\min} \ge \tilde{\alpha}/2$, the relative amount by which the total profit can be increased is given by

$$\frac{\frac{\tilde{\alpha}^2}{4\tilde{\beta}} - \frac{2\tilde{\alpha}^2}{9\tilde{\beta}}}{\frac{2\tilde{\alpha}^2}{9\tilde{\beta}}} \quad = \quad \frac{1}{8},$$

and the relative amount by which the consumer surplus can be increased is given by

$$\frac{\frac{\tilde{\alpha}^2}{8\tilde{\beta}} - \frac{\tilde{\alpha}^2}{18\tilde{\beta}}}{\frac{\tilde{\alpha}^2}{18\tilde{\beta}}} \quad = \quad \frac{5}{4}.$$

When capacity is intermediate, $\tilde{\alpha}/3 \le b_{\min} \le \tilde{\alpha}/2$, then the relative amount by which the total profit can be increased is bounded by

$$0 \quad \le \quad \frac{\frac{\tilde{\alpha}-b_{\min}}{\tilde{\beta}} b_{\min} - \frac{2\tilde{\alpha}^2}{9\tilde{\beta}}}{\frac{2\tilde{\alpha}^2}{9\tilde{\beta}}} \quad \le \quad \frac{1}{8},$$

and the relative amount by which the consumer surplus can be increased is bounded by

$$0 \quad \le \quad \frac{\frac{b_{\min}^2}{2\tilde{\beta}} - \frac{\tilde{\alpha}^2}{18\tilde{\beta}}}{\frac{\tilde{\alpha}^2}{18\tilde{\beta}}} \quad \le \quad \frac{5}{4}.$$

This potential increase in profit is the major economic motivation for sellers' interest in alliances. The extent to which this increase can be attained by an alliance depends on the capacity and the customer choice behavior, including the extent to which the sellers can differentiate their products. In the next section we consider a resource exchange alliance and investigate the effect of both capacity and product differentiation on the total profit and the consumer surplus with and without an alliance.

1.2.3 Resource Exchange Alliance

Consider a resource exchange alliance involving the two sellers. Let $x_i \in [0, b_i]$ denote the amount of resource i that seller i makes available to seller $-i$, and let $x := (x_{-1}, x_1)$. Then the number of units of the two-resource product that seller i can sell is $q_i(x) := \min\{b_i - x_i,\, x_{-i}\}$. Assume that seller i pays seller $-i$ an amount c_{-i} for each unit of resource $-i$ that seller i consumes, so that each seller has marginal cost equal to $c_{-1} + c_1$ for the two-resource product.

Specifically, a resource exchange alliance with zero exchange of resources ($x = 0$) may be chosen, in which case the sellers sell only the separate resources as in the case without an alliance. Thus, in general, the total profit of an optimally designed resource exchange alliance is no less than the total profit without an alliance. We consider the setting in which each alliance member sells only the two-resource product, and products consisting of a single resource are not sold separately. Let y_i denote the difference between the price of seller i and the marginal cost $c_{-1} + c_1$ for the two-resource product.

The demand $d_i(y_i, y_{-i})$ for the product sold by seller i depends on the prices as follows:

$$d_i(y_i, y_{-i}) \quad = \quad \max\{0,\ \alpha - \beta y_i + \gamma y_{-i})\}, \tag{1.2.13}$$

where α and β are positive constants, and $\gamma \in (-\beta, \beta)$. Here provision is made for brand distinction between the products sold by the sellers. The constants are known to each seller. To keep the number of parameters in this example small, the constants α, β, and γ are the same for both sellers.

Thus, the number of units of product sold by seller i is given by

$$\min\left\{q_i(x),\, \max\{0,\ \alpha - \beta y_i + \gamma y_{-i})\}\right\},$$

and the profit of seller i is given by

$$g_i(x, y_i, y_{-i}) \quad := \quad y_i \min\left\{q_i(x),\, \max\{0,\ \alpha - \beta y_i + \gamma y_{-i}\}\right\}.$$

Next we establish a relation between $\tilde{\alpha}$ and $\tilde{\beta}$, and α, β and γ, to facilitate comparison among the settings with no alliance, with perfect coordination, and with an alliance. Consider prices $(\tilde{y}_{-1}, \tilde{y}_1)$ in the no-alliance setting, such that $\tilde{y}_{-1} + \tilde{y}_1 < \tilde{\alpha}/\tilde{\beta}$. Suppose that the two alliance members charge the same price $y_{-1} = y_1 = \tilde{y}_{-1} + \tilde{y}_1$ for the two-resource products. Then the total demand in the no-alliance setting given by (1.2.1) is equal to $\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > 0$, and the total demand in the alliance setting given by (1.2.13) is equal to $2(\alpha - \beta y_1 + \gamma y_1) = 2\alpha - 2(\beta - \gamma)(\tilde{y}_{-1} + \tilde{y}_1)$. Thus the total demand in the two settings is the same if $\tilde{\alpha} = 2\alpha$ and $\tilde{\beta} = 2(\beta - \gamma)$. It is also shown in Appendix **??**app:two-resource perfect coordination with product differentiation]A.4 that a model of perfect coordination with demand given by (1.2.13) leads to the same optimal prices, demands, profits, and consumer surplus as the model in Section 1.2.2

with demand given by (1.2.1) if $\tilde{\alpha} = 2\alpha$ and $\tilde{\beta} = 2(\beta - \gamma)$. Hence the results for the settings with no alliance, with perfect coordination, and with an alliance will be compared using $\tilde{\alpha} = 2\alpha$ and $\tilde{\beta} = 2(\beta - \gamma)$.

For the setting with an alliance, for any given resource exchange x, let $(y^*_{-1}(x), y^*_1(x))$ denote the equilibrium prices of the two sellers for the two-resource product (existence and uniqueness of the equilibrium are addressed in the detail calculations in Appendix A.0.3. The resulting profit of seller i is given by $g_i(x, y^*_i(x), y^*_{-i}(x))$. The alliance design problem is to choose $x \in [0, b_{-1}] \times [0, b_1]$ to maximize

$$f(x) \quad := \quad g_{-1}(x, y^*_{-1}(x), y^*_1(x)) + g_1(x, y^*_1(x), y^*_{-1}(x)).$$

Let x^* denote an optimal resource exchange.

A natural question is how the total profit $f(x^*)$ should be partitioned among the alliance members. First, note that if money can be exchanged together with the other resources, then any partition of the total profit can be achieved. In that case the Nash bargaining solution is easy: each alliance member receives its profit in the setting without an alliance plus half the difference between the maximum total profit $f(x^*)$ of the alliance and the total profit without an alliance.

Table 1 and Figure 1 summarize the results for the settings with no alliance, with perfect coordination, and with an alliance. The calculations are given in Appendix A.

Here we just mention that there are three cases regarding capacity: (1) Capacity $b_{\min}$ is large enough so that both sellers can be provided with sufficient product capacity $q_i(x)$ to make capacity not constraining in equilibrium ($b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$), (2) Capacity $b_{\min}$ is so small that the product capacity $q_i(x)$ of both sellers must be constraining in equilibrium ($b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), and (3) Capacity $b_{\min}$ is small enough that the product capacity $q_i(x)$ of at least one seller must be constraining in equilibrium, but large enough so that one seller can be provided with sufficient product capacity $q_i(x)$ to make capacity not constraining in equilibrium ($\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$). In addition, there are two cases

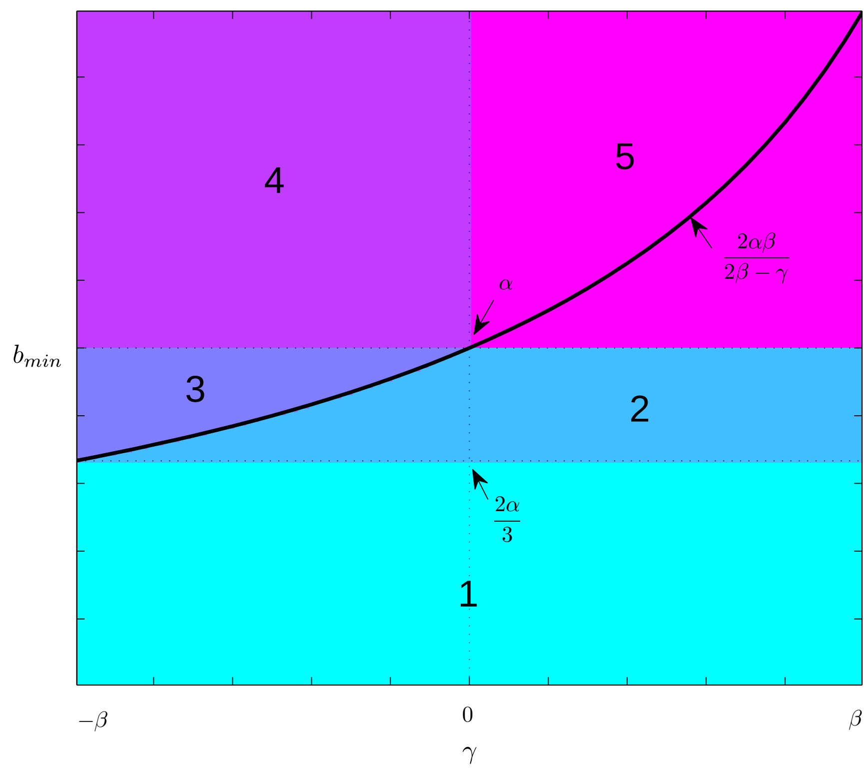


Figure 1: The regions distinguished in Table 1

regarding the degree of product differentiation: (1) $\gamma \geq 0$, and (2) $\gamma \leq 0$. Figure 2 shows a plot of the relative increase in total profit with an alliance over no alliance, that is, $(f(x^*) - [\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})])/[\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})]$, as a function of $b_{\min}/\alpha$ and γ/β. The figure shows that the relative increase is largest when the capacity is large ($b_{\min} \geq \alpha$) and the products of the sellers are substitutes ($\gamma \geq 0$). Figure 3 shows a plot of the relative gap in total profit between perfect coordination and an alliance, that is, $(\tilde{g}(\bar{y}_{-1}, \bar{y}_1) - f(x^*))/\tilde{g}(\bar{y}_{-1}, \bar{y}_1)$, as a function of $b_{\min}/\alpha$ and γ/β. The figure shows that the total profit under an alliance equals the total profit under perfect coordination, except when the capacity is large ($b_{\min} \geq 2\alpha/3$) and the products of the sellers are complements ($\gamma \leq 0$). Figure 4 shows a plot of the relative increase in consumer surplus with an alliance over no alliance, as a function of $b_{\min}/\alpha$ and γ/β. The figure shows that, similar to total profit, the relative increase is largest when the capacity is large ($b_{\min} \geq \alpha$) and the products of the sellers are substitutes ($\gamma \geq 0$).

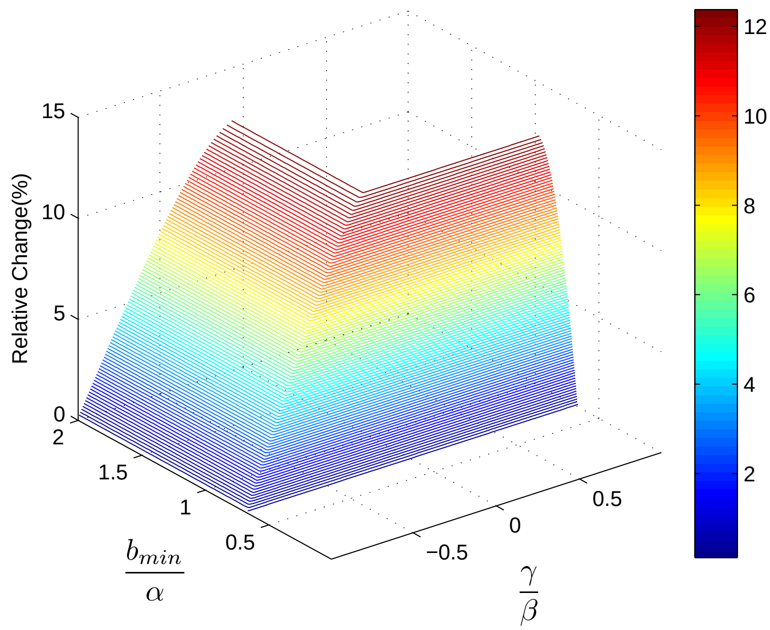


Figure 2: Plot of the relative increase in total profit with an alliance over no alliance, that is, $(f(x^*) - [\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})])/[\tilde{g}_{-1}(\tilde{y}^*_{-1}, \tilde{y}^*_1) + \tilde{g}_1(\tilde{y}^*_1, \tilde{y}^*_{-1})]$, as a function of $b_{\min}/\alpha$ and γ/β.

Table 1: Comparison of no alliance, perfect coordination, and a resource exchange alliance, in terms of price, demand, total profit, and consumer surplus, for a single product with two resources.

Region	Capacity	Cross-Price Coefficient	Quantity	No-Alliance	Perfect Coordination	Alliance
1	$0 \leq b_{\min} \leq \frac{2\alpha}{3}$	$\gamma \in (-\beta, \beta)$	Total Price	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$
			Total Demand	$b_{\min}$	$b_{\min}$	$b_{\min}$
			Total Profit	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$
			Consumer Surplus	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$
2	$\frac{2\alpha}{3} \leq b_{\min} \leq \min\left\{\alpha,\ \frac{2\alpha\beta}{2\beta-\gamma}\right\}$	$\gamma \in (-\beta, \beta)$	Total Price	$\frac{2\alpha}{3(\beta-\gamma)}$	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$
			Total Demand	$\frac{2\alpha}{3}$	$b_{\min}$	$b_{\min}$
			Total Profit	$\frac{4\alpha^2}{9(\beta-\gamma)}$	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$
			Consumer Surplus	$\frac{\alpha^2}{9(\beta-\gamma)}$	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$
3	$\frac{2\alpha\beta}{2\beta-\gamma} \leq b_{\min} \leq \alpha$	$\gamma \in (-\beta, 0]$	Total Price	$\frac{2\alpha}{3(\beta-\gamma)}$	$\frac{2\alpha - b_{\min}}{2(\beta-\gamma)}$	$\frac{\alpha}{2\beta-\gamma}$
			Total Demand	$\frac{2\alpha}{3}$	$b_{\min}$	$\frac{2\alpha\beta}{2\beta-\gamma}$
			Total Profit	$\frac{4\alpha^2}{9(\beta-\gamma)}$	$\frac{(2\alpha - b_{\min})b_{\min}}{2(\beta-\gamma)}$	$\frac{2\alpha^2\beta}{(2\beta-\gamma)^2}$
			Consumer Surplus	$\frac{\alpha^2}{9(\beta-\gamma)}$	$\frac{{b_{\min}}^2}{4(\beta-\gamma)}$	$\frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2}$
4	$\alpha \leq b_{\min}$	$\gamma \in (-\beta, 0]$	Total Price	$\frac{2\alpha}{3(\beta-\gamma)}$	$\frac{\alpha}{2(\beta-\gamma)}$	$\frac{\alpha}{2\beta-\gamma}$
			Total Demand	$\frac{2\alpha}{3}$	α	$\frac{2\alpha\beta}{2\beta-\gamma}$
			Total Profit	$\frac{4\alpha^2}{9(\beta-\gamma)}$	$\frac{\alpha^2}{2(\beta-\gamma)}$	$\frac{2\alpha^2\beta}{(2\beta-\gamma)^2}$
			Consumer Surplus	$\frac{\alpha^2}{9(\beta-\gamma)}$	$\frac{\alpha^2}{4(\beta-\gamma)}$	$\frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2}$
5	$\alpha \leq b_{\min}$	$\gamma \in [0, \beta)$	Total Price	$\frac{2\alpha}{3(\beta-\gamma)}$	$\frac{\alpha}{2(\beta-\gamma)}$	$\frac{\alpha}{2(\beta-\gamma)}$
			Total Demand	$\frac{2\alpha}{3}$	α	α
			Total Profit	$\frac{4\alpha^2}{9(\beta-\gamma)}$	$\frac{\alpha^2}{2(\beta-\gamma)}$	$\frac{\alpha^2}{2(\beta-\gamma)}$
			Consumer Surplus	$\frac{\alpha^2}{9(\beta-\gamma)}$	$\frac{\alpha^2}{4(\beta-\gamma)}$	$\frac{\alpha^2}{4(\beta-\gamma)}$

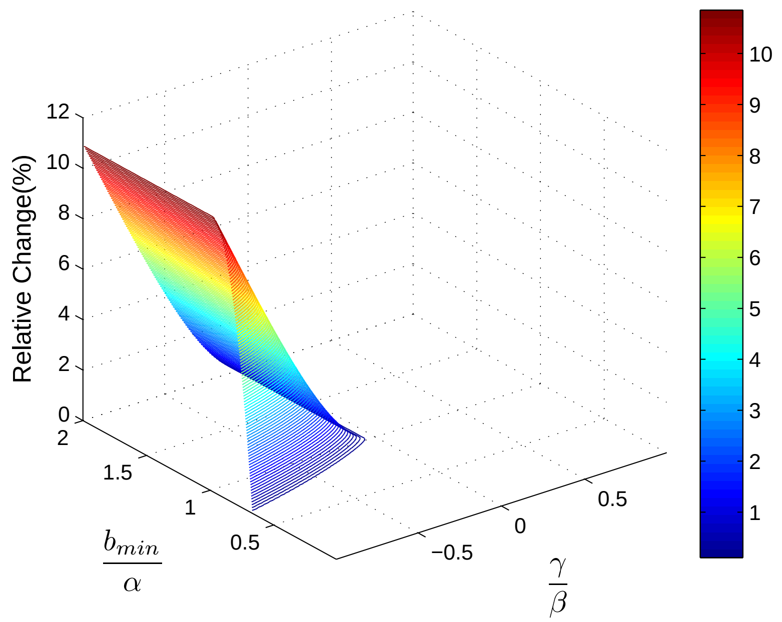


Figure 3: Plot of the relative gap in total profit between perfect coordination and an alliance, that is, $(\tilde{g}(\bar{y}_{-1}, \bar{y}_1) - f(x^*))/\tilde{g}(\bar{y}_{-1}, \bar{y}_1)$, as a function of $b_{\min}/\alpha$ and γ/β.

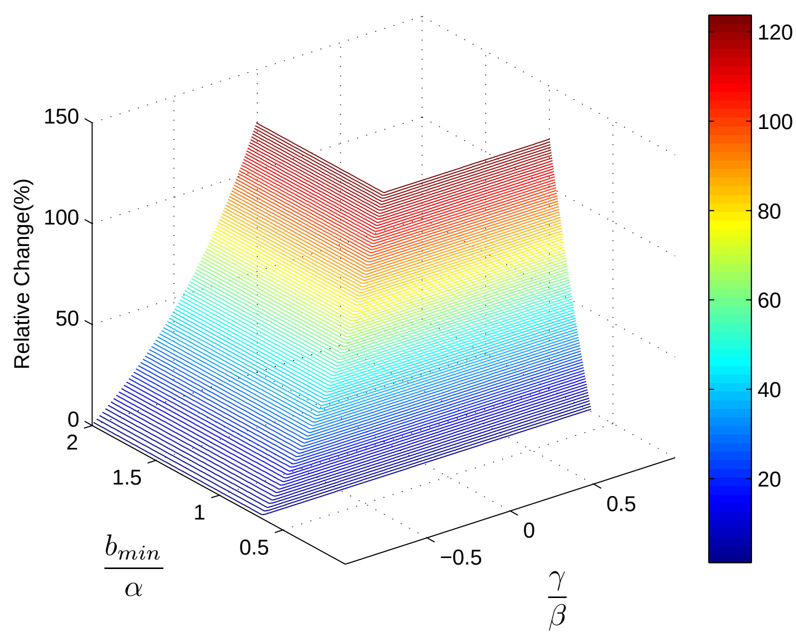


Figure 4: Plot of the relative increase in consumer surplus with an alliance over no alliance, as a function of $b_{\min}/\alpha$ and γ/β.

1.3 Multiple-Resource Model

In this Section we present a model for a resource exchange alliance with multiple resources. In addition to the alliance model, we also present models for the settings with no alliance and with perfect coordination to facilitate comparisons.

Consider 2 sellers, indexed by $i = \pm 1$. (It can be seen from the results in Section 1.3.3 that this analysis can be extended to a setting with more than 2 sellers, at the cost of more complicated notation.) Seller i produces k_i resource types indexed by $j = 1, \ldots, k_i$. For example, resource j may denote the flight of airline i scheduled to depart from Atlanta to New York every Monday at 8am. Initially, before any resource exchange, seller i has quantity $b_{i,j}$ of resource j, and a constant marginal cost of $c_{i,j}$ per unit of resource j consumed.

1.3.1 Multiple-Resource Network Example

In this section we provide an example with multiple resources to illustrate the models that will be formulated in later sections. An airline flight network is shown in Figure 5, and some flight data are given in Table 2.

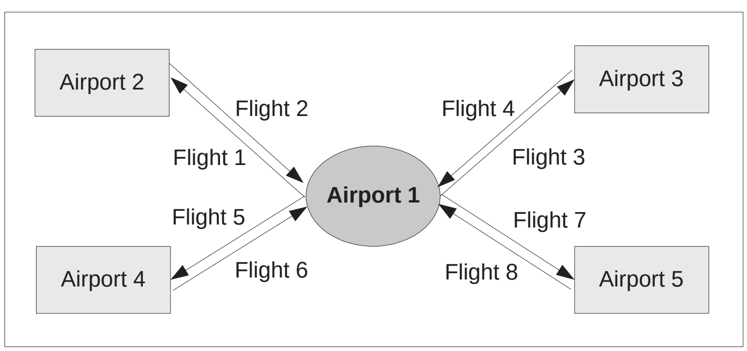


Figure 5: Multiple-resource network example

In this network, airport 1 is a connection hub for both airlines. Each airline operates 4 flights. For example, flight 5, taking place from airport 1 to airport 4, is operated by airline 1, and has a capacity of 300 seats. The set of products that can

Table 2: Flight information for the network example

Flight number	Airline	Departure	Arrival	Capacity
1	-1	1	2	300
2	-1	2	1	300
3	-1	1	3	300
4	-1	3	1	300
5	1	1	4	300
6	1	4	1	300
7	1	1	5	300
8	1	5	1	300

be sold by each airline is different in the case with no alliance and the case with an alliance.

Table 3 shows the products and the corresponding itineraries (here simply specified by the origin-destination pair) which could be offered by the two airlines. The column labeled "Airline" specifies which airlines can sell each product in the case with no alliance and the case with an alliance. For example, in the case with no alliance, product 7 can be sold by airline 1 only, and in the case with an alliance, product 7 can be sold by both airlines (A denotes both airlines under alliance). Product 17, involving travel from airport 3 to airport 4 via airport 1, can only be sold in the case with an alliance, and in that case it can be sold by both airlines. However, note that there is demand for travel from airport 3 to airport 4 both in the case with no alliance and in the case with an alliance. In the case with no alliance, all demand for travel from airport 3 to airport 4 is satisfied by buying two separate tickets; a ticket from airline -1 for travel from airport 3 to airport 1 and a ticket from airline 1 for travel from airport 1 to airport 4. In the case with an alliance, demand for travel from airport 3 to airport 4 can be satisfied in four different ways: (1) by buying a ticket from airline -1 for travel from airport 3 to airport 1 and a ticket from airline 1 for travel from airport 1 to airport 4, or (2) by buying a ticket from airline 1 for travel from airport 3 to airport 1 and a ticket from airline -1 for travel from airport 1 to airport 4, or (3) by buying a ticket for travel from airport 3 to airport 4 via airport 1 from airline -1, or (4) by buying a ticket for travel from airport 3 to airport 4 via airport 1 from airline 1. In the case with an alliance, the choices exercised by the buyers, and thus the resulting aggregate demand, depend on the prices of the airlines for the different products. In this paper we consider linear models of aggregate demand, as specified in more detail later.

Table 3: Product information for network example.

Product	Airline	Origin	Destination	Product	Airline	Origin	Destination
1	-1 or A	1	2	11	1 or A	4	5
2	-1 or A	2	1	12	1 or A	5	4
3	-1 or A	1	3	13	A only	2	4
4	-1 or A	3	1	14	A only	4	2
5	-1 or A	2	3	15	A only	2	5
6	-1 or A	3	2	16	A only	5	2
7	1 or A	1	4	17	A only	3	4
8	1 or A	4	1	18	A only	4	3
9	1 or A	1	5	19	A only	3	5
10	1 or A	5	1	20	A only	5	3

1.3.2 Resource Exchange Alliance Model

In this section we introduce a model of a resource exchange alliance involving multiple resources. After resource exchange, seller i may have some of each resource supplied by seller $-i$, as well as some of each resource supplied by itself. Index the union of the resources by $j = 1, \dots, k$, where $k = k_{-1} + k_1$. Let $b_i = (b_{i,1}, \dots, b_{i,k})$ denote the initial endowment of seller i of each resource ($b_{i,j} = 0$ if resource j is supplied by seller $-i$). Let x_j denote the amount of resource j that seller 1 makes available to seller -1. For example, $x = (-110, -120, -100, -150, 140, 170, 130, 160)$ for the network in Section 1.3.1 means that airline -1 gives 110 seats on flight 1 to airline 1, airline 1 gives 140 seats on flight 5 to airline -1, etc.

After resource exchange, seller i can sell m_i products, indexed by $\ell = 1, \ldots, m_i$. In the example in Table 3, $m_i = 20$ for $i = \pm 1$. Let $y_{i,\ell}$ denote the price of seller i for product ℓ in excess of the marginal cost of the product, and $d_{i,\ell}$ denote the demand for product ℓ of seller i. Consider the following linear demand model:

$$d_{i,\ell} \quad = \quad -\sum_{\ell'=1}^{m_i} E_{i,\ell,\ell'} y_{i,\ell'} + \sum_{\ell'=1}^{m_{-i}} B_{-i,\ell,\ell'} y_{-i,\ell'} + C_{i,\ell}, \tag{1.3.14}$$

where $E_{i,\ell,\ell'}$ denotes the rate of change of the demand for product ℓ of seller i with respect to the price of product ℓ' of the same seller i, and $B_{-i,\ell,\ell'}$ denotes the rate of change of the demand for product ℓ of seller i with respect to the price of product ℓ' of the other seller $-i$. Using matrix notation, $d_i = -E_i y_i + B_{-i} y_{-i} + C_i$, where $d_i, y_i, C_i \in \mathbb{R}^{m_i}$, $E_i \in \mathbb{R}^{m_i \times m_i}$, $B_i \in \mathbb{R}^{m_{-i} \times m_i}$, and attention is restricted to values of (y_{-1}, y_1) such that $d_i \geq 0$ for $i = \pm 1$. Let $A_i \in \mathbb{R}^{k \times m_i}$ be the "network matrix", i.e., $A_{i,j,\ell}$ denotes the amount of resource j consumed by each unit of product ℓ sold by seller i.

Next we introduce the two-stage alliance design problem. Given a first stage resource exchange decision $x \in \mathbb{R}^k$, at the second stage each seller i wants to solve the following optimization problem:

$$\begin{array}{lll} \max\limits_{y_i, d_i \in \mathbb{R}_+^{m_i}} & y_i^\mathsf{T} d_i & \\ \text{s.t.} & A_i d_i \;\; \leq \;\; b_i - ix & \\ & d_i \;\; = \;\; -E_i y_i + B_{-i} y_{-i} + C_i \;\; \geq \;\; 0. & \end{array} \tag{1.3.15}$$

We are interested in the Nash equilibrium defined by the two optimization problems (1.3.15) for $i = \pm 1$.

A stochastic version of the alliance design problem is as follows. At the first stage, when x is chosen, elements of matrices E_i and B_i, and vectors C_i, are random. However, the network matrices A_i are deterministic. Let $\xi := (E_{-1}, E_1, B_{-1}, B_1, C_{-1}, C_1)$

denote the random data vector. In the first stage the expected value with respect to the distribution of ξ of an objective (specified below) is optimized. Also, note that the Nash equilibrium associated with the second stage depends on the realization of ξ.

Let $Q_i := E_i + E_i^{\mathsf{T}} \in \mathbb{R}^{m_i \times m_i}$ denote the symmetric version of E_i. We assume that matrices E_i, and hence Q_i, are positive definite. Let I_m denote the $m \times m$ identity matrix, 0_m denotes the zero vector in $\mathbb{R}^m$, and $0_{m,n}$ denotes the zero matrix in $\mathbb{R}^{m \times n}$. Then the optimization problem (1.3.15) can be written as follows:

$$\begin{aligned} \min_{y_i \in \mathbb{R}_+^{m_i}} \quad & \tfrac{1}{2} y_i^{\mathsf{T}} Q_i y_i - y_i^{\mathsf{T}} B_{-i} y_{-i} - C_i^{\mathsf{T}} y_i \\ \text{s.t.} \quad & W_i \left(E_i y_i - B_{-i} y_{-i} \right) \;\geq\; \eta_i + i M_i x, \end{aligned} \tag{1.3.16}$$

where

$$W_i \;:=\; \begin{bmatrix} A_i \\ -I_{m_i} \end{bmatrix}, \qquad \eta_i \;:=\; W_i C_i + \begin{bmatrix} -b_i \\ 0_{m_i} \end{bmatrix}, \qquad M_i \;:=\; \begin{bmatrix} I_k \\ 0_{m_i,k} \end{bmatrix}.$$

A point $(y_{-1}^*(x), y_1^*(x))$ is a solution of the equilibrium problem if $y_1^*(x)$ is an optimal solution of problem (1.3.16) for $i = 1$ when $y_{-1} = y_{-1}^*(x)$, and also $y_{-1}^*(x)$ is an optimal solution of problem (1.3.16) for $i = -1$ when $y_1 = y_1^*(x)$. Note that $(y_{-1}^*(x), y_1^*(x))$ also depends on ξ, but the dependence is not shown in the notation. (The above problem is called a *generalized* Nash equilibrium problem since the feasible set of problem (1.3.16) depends on y_{-i}.) Let $V_i(x, \xi)$, $i = \pm 1$, denote the optimal objective values of problem (1.3.16) at the equilibrium point given data ξ, i.e.,

$$V_i(x, \xi) \;:=\; \tfrac{1}{2} y_i^*(x)^{\mathsf{T}} Q_i y_i^*(x) - y_i^*(x)^{\mathsf{T}} B_{-i} y_{-i}^*(x) - C_i^{\mathsf{T}} y_i^*(x). \tag{1.3.17}$$

Note that these functions are well defined only if the equilibrium point $(y_{-1}^*(x), y_1^*(x))$ exists and is unique. We will discuss existence and uniqueness of the equilibrium point in Section 1.3.3.

At the first stage, we consider designs of the resource exchange alliance that aim to maximize the total profit of the sellers. Let $b = b_1 - b_{-1} \in \mathbb{R}^k$. Note that $b_j > 0$ if resource j is supplied by seller 1 and $b_j < 0$ if resource j is supplied by seller -1. Let l_j and u_j be lower and upper bounds, respectively, such that $b_j l_j \geq 0$ and $b_j u_j \geq 0$, that is, l_j, u_j, and b_j have the same sign, and $|l_j| \leq |u_j| \leq |b_j|$. Then the first stage problem is as follows:

$$\begin{aligned}
\max_{x \in \mathbb{R}^k} \quad & \big\{ f(x) \ := \ \mathbb{E}\big[V_{-1}(x,\xi) + V_1(x,\xi)\big] \big\} \\
\text{s.t.} \quad & b_j x_j \ \geq \ 0 \qquad\qquad \forall\, j = 1, \dots, k \\
& |l_j| \ \leq \ |x_j| \ \leq \ |u_j| \qquad\qquad \forall\, j = 1, \dots, k.
\end{aligned} \tag{1.3.18}$$

As mentioned, the expectation in (1.3.18) is with respect to a specified probability distribution of the data vector ξ. In particular, if a single value for ξ is considered in the first stage, then problem (1.3.18) is deterministic and the expectation operator can be removed.

1.3.3 Existence and Uniqueness of Nash Equilibrium

Recall that the matrices Q_i are positive definite, and hence problem (1.3.16) is a convex quadratic programming problem. The first order (KKT) necessary and sufficient optimality conditions for problem (1.3.16) are

$$\begin{aligned}
Q_i y_i - B_{-i} y_{-i} - C_i - E_i^{\mathsf{T}} W_i^{\mathsf{T}} \lambda_i &= 0 \\
W_i \left(E_i y_i - B_{-i} y_{-i} \right) - \eta_i - i M_i x &\geq 0 \\
\lambda_i &\geq 0 \\
\lambda_i^{\mathsf{T}} \left[W_i \left(E_i y_i - B_{-i} y_{-i} \right) - \eta_i - i M_i x \right] &= 0,
\end{aligned} \tag{1.3.19}$$

where λ_i denotes the vector of Lagrange multipliers associated with the inequality constraints in (1.3.16). The optimality conditions (1.3.19) can be written as a variational inequality. For a closed convex set $C \subset \mathbb{R}^m$ and a point $z \in C$, we denote by $\mathcal{N}_C(z)$ the normal cone to C at $z \in C$,

$$\mathcal{N}_C(z) := \{ v \in \mathbb{R}^m : \ v^{\mathsf{T}}(z' - z) \leq 0, \ \forall z' \in C \}.$$

By definition, $\mathcal{N}_C(z) := \emptyset$ if $z \notin C$. Note that if C is a convex cone and $z \in C$, then $\mathcal{N}_C(z) = \{ v \in C^* : v^{\mathsf{T}} z = 0 \}$, where $C^* := \{ v \in \mathbb{R}^m : v^{\mathsf{T}} z \leq 0, \ \forall z \in$

$C\}$ is the polar cone of C. In particular, if $C = \mathbb{R}^m_+$ and $z \in C$, then $\mathcal{N}_C(z) = \{v \leq 0 : v_i = 0,\ i \in \mathcal{I}(z)\}$, where $\mathcal{I}(z) := \{i \in \{1, \ldots, m\} : z_i > 0\}$ Let us denote

$$z := (y_{-1}, \lambda_{-1}, y_1, \lambda_1) \in \mathbb{R}^{m_{-1}} \times \mathbb{R}^{k_{-1}+m_{-1}} \times \mathbb{R}^{m_1} \times \mathbb{R}^{k_1+m_1}.$$

Since the first equation of (1.3.19) can be written as

$$Q_i y_i - B_{-i} y_{-i} - C_i - E_i^{\mathsf{T}} W_i^{\mathsf{T}} \lambda_i \in \mathcal{N}_{\mathbb{R}^{m_i}}(y_i),$$

and the remaining two inequality conditions and the last equation of (1.3.19) as

$$-W_i \left(E_i y_i - B_{-i} y_{-i}\right) + \eta_i + iMx \in \mathcal{N}_{\mathbb{R}_+^{k_i+m_i}}(\lambda_i),$$

we can write the optimality conditions (1.3.19) as the following variational inequality

$$\mathcal{A}z + h \quad \in \quad \mathcal{N}_K(z), \tag{1.3.20}$$

where $K := \mathbb{R}^{m_{-1}} \times \mathbb{R}_+^{k_{-1}+m_{-1}} \times \mathbb{R}^{m_1} \times \mathbb{R}_+^{k_1+m_1}$ and

$$\mathcal{A} := \begin{bmatrix} Q_{-1} & E_{-1}^{\mathsf{T}} W_{-1}^{\mathsf{T}} & -B_1 & 0 \\ -W_{-1} E_{-1} & 0 & B_1 & 0 \\ -B_{-1} & 0 & Q_1 & -E_1^{\mathsf{T}} W_1^{\mathsf{T}} \\ B_{-1} & 0 & -W_1 E_1 & 0 \end{bmatrix}, \quad h := \begin{bmatrix} -C_{-1} \\ \eta_{-1} - Mx \\ -C_1 \\ \eta_1 + Mx \end{bmatrix}.$$

A widely used approach to establish existence and uniqueness of a solution to the optimality conditions, and thus existence and uniqueness of a Nash equilibrium, is to exploit monotonicity of the variational inequality. Note that matrix $\mathcal{A}$ is singular since the $m_1 \times (k_1 + m_1)$ matrix $E_1^{\mathsf{T}} W_1^{\mathsf{T}}$ has more columns than rows (and similarly for the matrix $E_{-1}^{\mathsf{T}} W_{-1}^{\mathsf{T}}$). Moreover, $\mathcal{A}$ has zero diagonal elements and hence cannot be positive semidefinite. Therefore, the operator $z \mapsto \mathcal{A}z + h$ cannot be monotone. This poses a certain problem for verification of existence and uniqueness of solutions of the variational inequality (1.3.20), and thus a different approach is required.

Consider the optimization problem

$$\begin{array}{lll} \min\limits_{y_{-1},y_1,\lambda_{-1},\lambda_1} & \sum\limits_{i=\pm1} \lambda_i^\mathsf{T} \left[W_i\left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - iM_i x\right] & \\ \text{s.t.} & Q_i y_i - B_{-i} y_{-i} - C_i - E_i^\mathsf{T} W_i^\mathsf{T} \lambda_i \;=\; 0, \quad i = \pm 1 & \\ & W_i\left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - iM_i x \;\geq\; 0, \quad i = \pm 1 & \\ & \lambda_i \;\geq\; 0, \quad i = \pm 1. & \end{array} \tag{1.3.21}$$

Note that the objective value of problem (1.3.21) is nonnegative at all feasible points, and $(y_{-1}^*, y_1^*, \lambda_{-1}^*, \lambda_1^*)$ is a solution of the optimality conditions (1.3.19) if and only if the optimal value of problem (1.3.21) is zero, in which case it is an optimal solution of problem (1.3.21). It follows from the first equation of (1.3.19) that

$$\lambda_i^\mathsf{T} W_i \;=\; y_i^\mathsf{T} Q_i E_i^{-1} - y_{-i}^\mathsf{T} B_{-i}^\mathsf{T} E_i^{-1} - C_i^\mathsf{T} E_i^{-1}.$$

After substitution of this into the objective, problem (1.3.21) becomes

$$\begin{array}{ll} \min\limits_{y_{-1},y_1,\lambda_{-1},\lambda_1} & \sum\limits_{i=\pm1} \left[\left(y_i^\mathsf{T} Q_i E_i^{-1} - y_{-i}^\mathsf{T} B_{-i}^\mathsf{T} E_i^{-1} - C_i^\mathsf{T} E_i^{-1}\right)\left(E_i y_i - B_{-i} y_{-i}\right) - \lambda_i^\mathsf{T}\left(\eta_i + iM_i x\right)\right] \\ \text{s.t.} & Q_i y_i - B_{-i} y_{-i} - C_i - E_i^\mathsf{T} W_i^\mathsf{T} \lambda_i \;=\; 0, \quad i = \pm 1 \\ & W_i\left(E_i y_i - B_{-i} y_{-i}\right) - \eta_i - iM_i x \;\geq\; 0, \quad i = \pm 1 \\ & \lambda_i \;\geq\; 0, \quad i = \pm 1. \end{array} \tag{1.3.22}$$

Note that the objective function of problem (1.3.22) is quadratic with its quadratic term $(y_{-1}^\mathsf{T}, y_1^\mathsf{T})\,\Psi\,(y_{-1}^\mathsf{T}, y_1^\mathsf{T})^\mathsf{T}$, where

$$\Psi \;:=\; \begin{bmatrix} Q_{-1} + B_{-1}^\mathsf{T} E_1^{-1} B_{-1} & -B_{-1} - Q_{-1} E_{-1}^{-1} B_1 \\ -B_1 - Q_1 E_1^{-1} B_{-1} & Q_1 + B_1^\mathsf{T} E_{-1}^{-1} B_1 \end{bmatrix}. \tag{1.3.23}$$

Note that problem (1.3.22) is a convex quadratic program if and only if the matrix Ψ, or equivalently the symmetric matrix $\Psi + \Psi^\mathsf{T}$, is positive semidefinite.

Theorem 1 *Suppose that the problem* (1.3.22) *is feasible and that the matrix* Ψ*, defined in* (1.3.23)*, is positive definite. Then problem* (1.3.22) *has an optimal solution* $(y_{-1}^*, y_1^*, \lambda_{-1}^*, \lambda_1^*)$ *with* (y_{-1}^*, y_1^*) *being unique. Moreover, if the optimal objective value of problem* (1.3.22) *is zero, then* (y_{-1}^*, y_1^*) *is the unique Nash equilibrium.*

Proof The objective value of problem (1.3.22) is bounded from below by zero. It is known that a quadratic program with a bounded from below objective value has an optimal solution. To establish uniqueness, consider the problem

$$\min_{(x,y)\in\mathcal{X}} \left\{f(x,y) := x^{\mathsf{T}}Qx + a^{\mathsf{T}}x + b^{\mathsf{T}}y\right\}, \tag{1.3.24}$$

where $\mathcal{X} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ is a convex set and Q is an $n_1 \times n_1$ positive definite matrix. Let (x_1^*, y_1^*) and (x_2^*, y_2^*) be two optimal solutions of (1.3.24). Consider the function $\phi(t) := f(tx_1^* + (1-t)x_2^*, ty_1^* + (1-t)y_2^*)$. Note that ϕ is a quadratic function, $\phi(t) = \alpha t^2 + \beta t + \gamma$, where $\alpha = (x_1^* - x_2^*)^{\mathsf{T}} Q (x_1^* - x_2^*)$. Note that $\alpha \geq 0$ since Q is positive definite, and thus ϕ is convex. Convexity of $\mathcal{X}$ and optimality of (x_1^*, y_1^*) and (x_2^*, y_2^*) implies that $\phi(t) \geq \phi(0) = \phi(1)$ for all $t \in [0,1]$. Moreover, convexity of ϕ implies that $\phi(t) \leq \phi(0) = \phi(1)$ for all $t \in [0,1]$. Hence $\phi(t) = \phi(0) = \phi(1)$ for all $t \in [0,1]$, and thus $\alpha = 0$. Since Q is positive definite it follows that $x_1^* = x_2^*$. Finally, if the optimal objective value of problem (1.3.22), and hence of problem (1.3.21), is zero, then $(y_{-1}^*, y_1^*, \lambda_{-1}^*, \lambda_1^*)$ satisfies the necessary and sufficient optimality conditions (1.3.19), and thus (y_{-1}^*, y_1^*) is the Nash equilibrium. ∎

Note that a similar approach can be used if there are more than two sellers. In such a case more than two sets of optimality conditions of the form (1.3.19) will be involved, and in the quadratic program (1.3.22) the index i will take on more than two values.

Hence, the question of existence and uniqueness of the Nash equilibrium can be answered with the following steps: (1) verification that the matrix Ψ (or the symmetric matrix $\Psi + \Psi^{\mathsf{T}}$) is positive definite, (2) solving the quadratic program (1.3.22) if Ψ is positive definite, and (3) verification that the optimal objective value is zero. Note that if Ψ is positive definite, then the quadratic program (1.3.22) can be solved efficiently and hence existence and uniqueness of the equilibrium point can be verified numerically. Some simple necessary conditions and sufficient conditions for Ψ to be positive definite can be identified, but it seems difficult to give simple conditions that are both necessary and sufficient for Ψ to be positive definite. A necessary condition for Ψ to be positive definite is that its block diagonal matrices $Q_{-1} + B_{-1}^{\mathsf{T}} E_1^{-1} B_{-1}$ and $Q_1 + B_1^{\mathsf{T}} E_{-1}^{-1} B_1$ are positive definite. Note that these matrices are indeed positive definite because E_{-1} and E_1 are positive definite. Also, note that if B_{-1} and B_1 are null matrices, then matrix Ψ is the block diagonal matrix $\mathrm{diag}(Q_{-1}, Q_1)$, and hence Ψ is positive definite because Q_{-1} and Q_1 are positive definite. More general, if matrices E_i are "significantly bigger" than B_i, then one may expect matrix Ψ to be positive definite. Intuitively, if the demand for a seller's product depends more strongly on the prices of that seller (and especially the price of that product) than the prices of the other seller, then one may expect matrix Ψ to be positive definite. Another instructive example is the following.

Example Suppose that the products of the two sellers are direct substitutes for each other, that is, for each product of seller i there is a product of seller $-i$ that is a close substitute. This allows the possibility that seller $-i$ may not be able to sell the substitute product because it does not have the resources to do so. It seems that in the applications of interest, the set of products can always be chosen so that this property holds. Hence, the matrices B_i are squared, i.e., $m_{-1} = m_1$. Suppose that the matrices E_i and B_i, $i = \pm 1$, are diagonal. Then $Q_i = E_i$ and

$$\Psi \quad = \quad \begin{bmatrix} E_{-1} + B_{-1}^2 E_1^{-1} & -B_{-1} - B_1 \\ -B_{-1} - B_1 & E_1 + B_1^2 E_{-1}^{-1} \end{bmatrix}.$$

Since matrices E_i are positive definite it follows that $E_1 + B_1^2 E_{-1}^{-1}$ is positive definite, and thus it follows by the Schur complement condition for positive definiteness that Ψ is positive definite if and only if the matrix $E_{-1} + B_{-1}^2 E_1^{-1} - (B_{-1} + B_1)^2 (E_1 + B_1^2 E_{-1}^{-1})^{-1}$ is positive definite. Since matrices E_i and B_i are diagonal, this matrix is positive definite if and only if the matrix

$$(E_{-1}+B_{-1}^2E_1^{-1})(E_1+B_1^2E_{-1}^{-1})-(B_{-1}+B_1)^2 \quad = \quad E_{-1}E_1+B_{-1}^2B_1^2E_{-1}^{-1}E_1^{-1}-2B_{-1}B_1$$

is positive definite. In turn this matrix is positive definite if and only if the matrix

$$E_{-1}^2E_1^2+B_{-1}^2B_1^2-2E_{-1}E_1B_{-1}B_1 \quad = \quad (E_{-1}E_1-B_{-1}B_1)^2$$

is positive definite. Note that the last matrix is always positive semidefinite and is positive definite if and only if matrix $E_{-1}E_1-B_{-1}B_1$ does not have any zero diagonal elements.

1.3.4 No Alliance Model

In this section, we present a model for the setting with no alliance. This model will be used to compare the profit under no alliance with the profit under an alliance and the profit under perfect coordination. First we describe the demand model for the setting with no alliance.

Under an alliance, there are a total of m distinct products. Some of the products may be offered by only one seller, and some of the products may be offered by both sellers. In the example in Table 3, $m=20$ and each of the 20 products is offered by both sellers in an alliance. These m products can be partitioned into three subsets: sets L_i, for $i=\pm 1$, of products which can be offered by seller i with and without an alliance, and set L_0 of products which could be offered only under an alliance. For the example in Table 3, L_{-1} contains products 1 to 6, L_1 contains products 7 to 12, and L_0 contains products 13 to 20.

As before, let $\tilde{y}_{i,\ell}$ denote the price of seller i for product $\ell \in L_i$. Suppose that the resulting demand for product $\ell \in L_i$ is given by

$$\tilde{d}_{i,\ell} \quad = \quad -\sum_{\ell' \in L_i} \tilde{E}_{i,\ell,\ell'}\tilde{y}_{i,\ell'} + \sum_{\ell' \in L_{-i}} \tilde{B}_{-i,\ell,\ell'}\tilde{y}_{-i,\ell'} + \tilde{C}_{i,\ell}. \tag{1.3.25}$$

Using matrix notation, $\tilde{d}_i = -\tilde{E}_i\tilde{y}_i + \tilde{B}_{-i}\tilde{y}_{-i} + \tilde{C}_i$, where $\tilde{d}_i, \tilde{y}_i, \tilde{C}_i \in \mathbb{R}^{|L_i|}$, $\tilde{E}_i \in \mathbb{R}^{|L_i|\times|L_i|}$, $\tilde{B}_i \in \mathbb{R}^{|L_{-i}|\times|L_i|}$, and attention is restricted to values of $(\tilde{y}_{-1}, \tilde{y}_1)$ such that

$\tilde{d}_i \geq 0$ for $i = \pm 1$. Let $\tilde{A}_{i,j,\ell}$ denote the amount of resource j consumed by each unit of product $\ell \in L_i$, and let $\tilde{A}_i \in \mathbb{R}^{k_i \times |L_i|}$ denote the network matrix.

Similar to the example with two resources in Section 1.2, the parameters E, B, C in demand model (1.3.14) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (1.3.25) should be related in a particular way to facilitate a fair comparison of the prices, demands, total profit, and consumer surplus between the settings with and without an alliance. The derivation of the relation is given in Appendix B.

The setting with no alliance is formulated as a non-cooperative game in which each seller i wants to solve the optimization problem

$$\begin{array}{rlcl} \max\limits_{\tilde{y}_i, \tilde{d}_i \in \mathbb{R}_+^{|L_i|}} & \tilde{y}_i^\mathsf{T} \tilde{d}_i & & \\ \text{s.t.} & \tilde{A}_i \tilde{d}_i & \leq & b_i \\ & \tilde{d}_i & = & -\tilde{E}_i \tilde{y}_i + \tilde{B}_{-i} \tilde{y}_{-i} + \tilde{C}_i \quad \geq \quad 0. \end{array} \tag{1.3.26}$$

The no alliance outcome is the Nash equilibrium defined by the two optimization problems (1.3.26) for $i = \pm 1$, as long as it exists and is unique. The Nash equilibrium is computed using the same approach described in Section 1.3.3.

1.3.5 Perfect Coordination Model

The models with and without an alliance presented above are compared with a perfect coordination model, given in this section. The perfect coordination model considers a setting in which the sellers coordinate pricing to maximize the sum of the sellers' profits, as given by the following optimization problem:

$$\begin{array}{rlcl} \max\limits_{(y_{-1}, y_1) \in \mathbb{R}^{m_{-1}} \times \mathbb{R}^{m_1}} & \sum\limits_{i=\pm 1} y_i^\mathsf{T} \left(-E_i y_i + B_{-i} y_{-i} + C_i\right) & & \\ \text{s.t.} & \sum\limits_{i=\pm 1} A_i \left(-E_i y_i + B_{-i} y_{-i} + C_i\right) & \leq & b_{-1} + b_1 \\ & -E_i y_i + B_{-i} y_{-i} + C_i & \geq & 0\,, \quad i = \pm 1. \end{array} \tag{1.3.27}$$

1.3.6 Solution Approach

In this section, we present a solution method for the multiple-resource model described in Section 1.3. Recall that in order to solve the problem (1.3.22), we have to solve the second-stage Nash equilibrium problem, and that problem (1.3.22) can be solved efficiently if the matrix Ψ defined in (1.3.23) is positive definite. Next consider the first stage problem (1.3.18). Recall that the expectation in (1.3.18) is taken with respect to the probability distribution of the random data vector ξ. We assume that we can sample from that distribution by using Monte Carlo sampling techniques and hence generate an independent and identically distributed sample $\xi^1, \ldots, \xi^N$. Next we approximate the expectation with the sample average and construct the following Sample Average Approximation (SAA) problem:

$$
\begin{array}{lllllll}
\max\limits_{x \in \mathbb{R}^k} & \left\{ \hat{f}_N(x) \right. & := & \left. \sum_{n=1}^N \left[V_{-1}(x, \xi^n) + V_1(x, \xi^n) \right] \right\} & & & \\
\text{s.t.} & b_j x_j & \geq & 0 & \forall\, j = 1, \ldots, k & & \\
 & |l_j| & \leq & |x_j| & \leq & |u_j| & \forall\, j = 1, \ldots, k.
\end{array} \tag{1.3.28}
$$

Theoretical properties of the SAA approach have been studied extensively (e.g., Shapiro et al. (2009)). Under mild conditions, the optimal objective value and optimal solution of the SAA problem (1.3.28) converge to the optimal objective value and optimal solution of the problem (1.3.18) (cf., Shapiro and Xu (2008)). The first-stage problem may not be convex, and thus it may be hard to solve problem (1.3.28) to optimality. For that reason, we may only ensure convergence to a stationary point of the problem (1.3.18). Nevertheless, in our numerical experiments, typically solutions seem to be stable and insensitive to the choice of starting point.

In order to solve the SAA problem (1.3.28) numerically, we need to compute derivatives $\nabla_x V_i(x, \xi^n)$ of the first-stage objective functions V_i at a feasible point x and sample point ξ^n. Consider a feasible point x, and assume that Ψ is positive definite and that the second-stage problem has an equilibrium point $(y^*_{-1}(x), y^*_1(x))$ (the equilibrium depends on ξ^n as well, but the dependence is not shown in the notation). Let $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$ be a solution of the system (1.3.19) of first order optimality conditions (and thus $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$ is also a solution of the quadratic programming problem (1.3.21)). Note that, since Ψ is positive definite, it holds that $(y^*_{-1}(x), y^*_1(x))$ is unique and is a continuous function of x (e.g., Bonnans and Shapiro (2000)).

Recall that Lagrange multipliers corresponding to inactive constraints are zeros. Let

$$\mathcal{I}_i(y_i, y_{-i}, x) \quad := \quad \left\{ j \in \{1, \ldots, k + m_i\} \ : \ [W_i\,(E_i y_i - B_{-i} y_{-i}) - \eta_i - iM_i x]_j = 0 \right\}$$

denote the index set of active constraints of the problem (1.3.16). Assume that the Lagrange multiplier vector is unique. It is said that the strict complementarity condition holds at an equilibrium point $(y^*_{-1}(x), y^*_1(x))$ if among the corresponding Lagrange multiplier vectors λ_i, there exists at least one such that $[\lambda_i]_j > 0$ for all $j \in \mathcal{I}_i(y^*_i(x), y^*_{-i}(x), x)$, for $i = \pm 1$, i.e., there are Lagrange multipliers corresponding to the active constraints that are positive.

Now, suppose that the strict complementarity condition holds at $(y^*_{-1}(x), y^*_1(x))$, with $[\lambda^*_i(x)]_j > 0$ for all $j \in \mathcal{I}_i(y^*_i(x), y^*_{-i}(x), x)$, for $i = \pm 1$. Then for small perturbations dx of x, the active constraints remain active and the inactive constraints remain inactive. Therefore, by linearizing the optimality conditions (1.3.19) at $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$, the following system of $m_{-1} + m_1 + 2k$ linear equations in $m_{-1} + m_1 + 2k$ unknowns $(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1)$ is obtained:

$$\begin{array}{rcll}
Q_i dy_i - B_{-i} dy_{-i} - E_i^{\mathsf{T}} W_i^{\mathsf{T}} d\lambda_i & = & 0, & i = \pm 1 \\
\left[W_i\,(E_i dy_i - B_{-i} dy_{-i}) - iM_i dx\right]_j & = & 0, \quad j \in \mathcal{I}_i(y^*_i(x), y^*_{-i}(x), x), & i = \pm 1 \\
\left[d\lambda_i\right]_j & = & 0, \quad j \notin \mathcal{I}_i(y^*_i(x), y^*_{-i}(x), x), & i = \pm 1.
\end{array} \tag{1.3.29}$$

Suppose that the linear system (1.3.29) is nonsingular. Then for any dx sufficiently small, the system (1.3.29) has a unique solution, and by the Implicit Function Theorem, the solution of (1.3.29) gives the differential of $(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x))$ at

x. More specifically, the system (1.3.29) can be written in the form

$$S(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1) = T\,dx,$$

where $S \in \mathbb{R}^{(m_{-1}+m_1+2k)\times(m_{-1}+m_1+2k)}$ and $T \in \mathbb{R}^{(m_{-1}+m_1+2k)\times k}$. If the matrix S is nonsingular, then $(dy_{-1}, dy_1, d\lambda_{-1}, d\lambda_1) = S^{-1}T\,dx$, and thus

$$\nabla(y^*_{-1}(x), y^*_1(x), \lambda^*_{-1}(x), \lambda^*_1(x)) = S^{-1}T.$$

It follows from (1.3.17) that

$$\begin{aligned} \nabla_x V_i(x,\xi) &= \nabla y^*_i(x)^\mathsf{T} Q_i y^*_i(x) - \nabla y^*_i(x)^\mathsf{T} B_{-i} y^*_{-i}(x) && (1.3.30)\\ &\quad -\nabla y^*_{-i}(x)^\mathsf{T} B^\mathsf{T}_{-i} y^*_i(x) - \nabla y^*_i(x)^\mathsf{T} C_i && (1.3.31)\\ \nabla^2_{xx} V_i(x,\xi) &= \nabla y^*_i(x)^\mathsf{T} Q_i \nabla y^*_i(x) - \nabla y^*_i(x)^\mathsf{T} B_{-i} \nabla y^*_{-i}(x) && (1.3.32)\\ &\quad -\nabla y^*_{-i}(x)^\mathsf{T} B^\mathsf{T}_{-i} \nabla y^*_i(x) && (1.3.33) \end{aligned}$$

can be calculated easily.

The analysis above shows that sufficient conditions for differentiability of V_i with respect to x at (x,ξ) are the strict complementarity condition and nondegeneracy of the system (1.3.29). These conditions are not necessary — for example, if $M_i = 0$ for $i = \pm 1$, then $V_i(x,\xi)$ is constant and hence differentiable with respect to x. Also, the expectation operator often smooths nondifferentiable functions. For example, if $\nabla_x V_i(x,\xi)$ exists for almost every ξ and a mild boundedness condition holds, then $\mathbb{E}[V_i(x,\xi)]$ is differentiable at x and $\nabla_x \mathbb{E}[V_i(x,\xi)] = \mathbb{E}[\nabla_x V_i(x,\xi)]$ (e.g., Shapiro et al. (2009) Theorem 7.44).

The derivatives in (1.3.30) and (1.3.33) are used to solve SAA problems (1.3.28) with a trust-region method. Numerical results are given in Section 1.4.

1.4 Numerical Examples

In this Section, we present numerical results to compare profits in settings with an alliance, no alliance, and perfect coordination, for the multiple-resource models described in Chapter 1.3. We present results for the network example given in Section 1.3.1. We first present the results for the deterministic case with known demand functions in Section 1.4.1, and then present results for the stochastic case with random demand functions in Section 1.4.2.

1.4.1 Deterministic Examples

We first describe how the input data E_i, B_i, and C_i for the numerical examples were chosen. For the example network, $m_{-1} = m_1 = 20$, and thus $E_i, B_i \in \mathbb{R}^{20\times 20}$ and $C_i \in \mathbb{R}^{20}$ for $i = \pm 1$. For each instance, a specific ratio $r_1 \in [0, 1)$ is chosen such that $|B_{-i,\ell,\ell'}| = r_1|E_{i,\ell,\ell'}|$. Thus, r_1 is similar to the ratio γ/β of the two-resource example in Section 1.2.3, and represents the level of differentiation between the sellers' products. For all instances, it was verified that the resulting matrix Ψ defined in (1.3.23) was positive definite.

For the no alliance setting, we used the transformations in Appendix B to obtain $\tilde{E}_i$, $\tilde{B}_i$, and $\tilde{C}_i$. In addition, we investigated the effect of a difference in product attractiveness between the settings with and without an alliance. As mentioned, in a setting without an alliance, a buyer may have to buy products from multiple sellers and combine them to obtain the product desired by the buyer. Under an alliance a seller may offer the combined product to the buyer, making it more convenient for the buyer to obtain the product ("one-stop shopping"). There may be additional ways in which an alliance increases demand. For example, with an airline alliance, the coordination of connecting flight schedules to reduce lay-over time or missed connections, rebooking in case of missed connections, and coordination of baggage handling, may further enhance the combined product under an alliance. This might increase the potential demand level under an alliance compared to that under no alliance. Motivated by these observations, we solved some instances in which the demands under no alliance is obtained using the transformations in Appendix B, but with a reduction in the demand for products assembled from more than one seller by a factor of $r_2 \in (0, 1]$ (in the notation of that section, the part of the demand for products in L_i derived from the demand for products in $L_{0,-1} \cup L_{0,1}$ was reduced by a factor of r_2).

The two-stage alliance design problem (1.3.18) was solved using a trust region algorithm. At each iteration, given the current value of the resource exchange vector x, the convex quadratic program (1.3.21) was solved. It was verified that the optimal objective value of (1.3.21) was zero, that is, the solution of (1.3.21) gave a solution of the second stage equilibrium problem (1.3.15) for $i = \pm 1$. It was also verified that the strict complimentary condition held and that the system (1.3.29) was nonsingular. Next the derivatives of the objective function of (1.3.18) with respect to x could be computed, and the trust region algorithm could execute the next iteration.

As mentioned, the objective function of (1.3.18) may not be convex. To address the concern of potential multiple local optima, for each instance we used 50 different starting points x_0 for the first iteration. For each instance, all 50 starting points lead to similar final solutions and final objective values.

For the no alliance model, the second-stage equilibrium problem had to be solved only once for each instance. For the perfect coordination model, the convex quadratic optimization problem (1.3.27) also had to be solved only once for each instance.

Table 4 presents the total profits under different levels of product differentiation represented by different values of r_1 for $r_2 = 1$ and with diagonal matrices E_i and B_i. The largest increase in profits relative to the no alliance setting was obtained under high levels of product differentiation. For example, when $r_1 = 0.2$, an alliance increases the profit of the no alliance setting by 7.92%, and perfect coordination increases the profit by 7.98%. Even under a low level of product differentiation ($r_1 = 0.8$), an alliance still increases the profit by 2.88%, and perfect coordination increases the profit by 4.99%. Similar results were obtained with non-diagonal matrices.

We also compared profits for different values of r_2. Table 5 compares the total profits under different levels of convenience represented by different values of r_2 for $r_1 = 0.5$ and with diagonal matrices E_i and B_i. As expected, the relative increase in profit is larger for smaller values of r_2.

1.4.2 Stochastic Examples

In this section, we present results for the stochastic model (that is, the first stage problem (1.3.18) with expectation in the objective) presented in Section 1.3. The random data E_i, B_i, and C_i followed a multivariate normal distribution with means as described in Section 1.4.1, standard deviations proportional to the means, and correlation coefficients of 0.6.

We generated and solved SAA problems with different sample sizes $N = 20, 40, \ldots, 500$. At each iteration of the first-stage problem, the second-stage problem was solved for each of the N sample points ξ^n. Then, for each of the N sample points ξ^n, the derivatives of $V_i(x, \xi^n)$ were computed as given in (1.3.30) and (1.3.33). The averages of these derivatives over the N sample points then gave the derivatives of the first-stage objective of the SAA problem (1.3.28).

Finally, after a resource exchange x was chosen by solving a SAA problem, we compared the total profits in the alliance, no alliance, and perfect coordination settings with an independent and identically distributed sample of 1000 sample points, independent of the samples used in the SAA problem. Table 6 reports the number of iterations of the trust region algorithm until termination, the resource exchange solution x_{opt} at termination, the objective value (obj_{opt}) of the SAA problem at x_{opt}, and the gradient norm ($\|g\|$) of the SAA objective function at x_{opt}, for different sample

Table 4: Comparison of total profit for a resource exchange alliance, no alliance, and perfect coordination, for different levels of product differentiation.

Deterministic Model ($r_2 = 1$)	$r_1 = 0.2$		$r_1 = 0.5$		$r_1 = 0.8$	
	Total Revenue	Relative increase (%)	Total Revenue	Relative increase (%)	Total Revenue	Relative increase (%)
No alliance	318060.00		322790.00		326980.00	
Perfect Coordination	343430.00	7.98	343340.00	6.37	343300.00	4.99
Alliance	343235.54	7.92	341615.26	5.83	336386.89	2.88

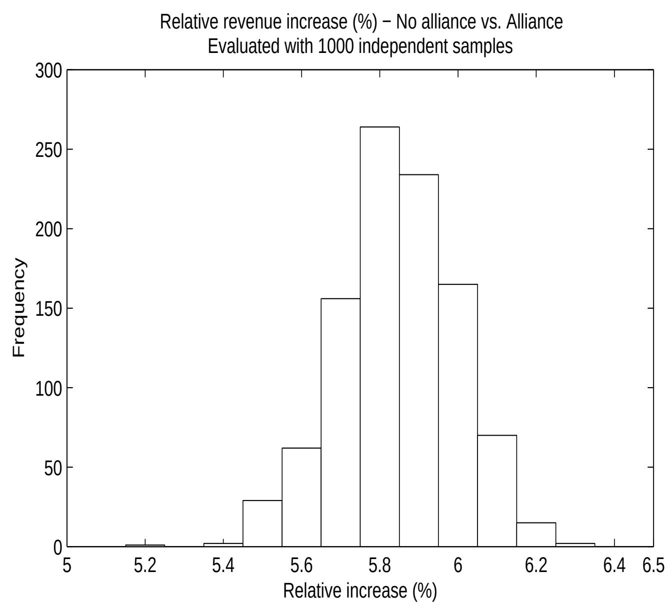
Table 5: Comparison of maximum achievable total revenue under different convenience level

Deterministic Model ($r_1 = 0.5$)	$r_2 = 0.2$ (High)		$r_2 = 0.6$		$r_2 = 1$ (No Difference)	
	Total Revenue	Relative increase (%)	Total Revenue	Relative increase (%)	Total Revenue	Relative increase (%)
No alliance	311590.00		318450.00		322790.00	
Perfect Coordination	343340.00	10.19	343340.00	7.82	343340.00	6.37
Alliance	341615.26	9.64	318450.00	7.27	341615.26	5.83

sizes N, for the network example in Section 1.3.1. As far as we know, these are the first stochastic mathematical programs with equilibrium constraints motivated by an application that have been solved.

Figure 6 presents a histogram of the pairwise difference in total profit between an alliance and no alliance, again using a sample of 1000 sample points, independent of the samples used in the SAA problem. The total profit under an alliance was larger for *all* 1000 sample points, with the percentage increase varying from 5.24% to 6.31%.

Figure 6: Histogram of the pairwise difference in total profit between an alliance and no alliance, using a sample of 1000 sample points.



1.4.3 Robustness With Respect to Resource Exchange

So far, we have compared the total profit under an alliance with the total profit under no alliance after computing the optimal exchange. An important question is how robust the improvement in total profit is with respect to choice of resource exchange. In this section we present a simple example to cast some light on the question.

Table 6: Optimal solution under different sample sizes for the stochastic case

n	$iter$	obj_{opt}	$\|g\|$	x_{opt}							
20	41	-340950.08	1.08E-04	144.41	154.96	139.45	148.01	-150.07	-158.56	-139.32	-152.32
60	36	-340983.90	1.83E-04	144.78	155.30	139.70	148.30	-149.84	-158.17	-138.93	-152.09
100	39	-340886.90	3.53E-05	144.35	154.93	139.36	147.87	-150.27	-158.53	-139.27	-152.48
140	37	-340889.10	1.51E-04	144.53	155.11	139.51	148.06	-150.16	-158.36	-139.05	-152.33
180	38	-340843.67	9.08E-06	144.60	155.22	139.63	148.16	-150.06	-158.25	-138.92	-152.25
220	39	-340856.47	1.46E-05	144.65	155.31	139.74	148.22	-149.98	-158.17	-138.85	-152.16
260	38	-340937.31	4.20E-05	144.61	155.28	139.70	148.19	-150.00	-158.22	-138.90	-152.19
300	43	-340933.57	3.25E-05	144.67	155.34	139.76	148.27	-149.94	-158.16	-138.82	-152.14
340	40	-341218.94	1.00E-05	144.61	155.27	139.69	148.21	-149.99	-158.22	-138.87	-152.19
380	42	-341170.71	2.46E-05	144.61	155.29	139.71	148.21	-149.99	-158.21	-138.86	-152.19
420	41	-341118.56	1.04E-04	144.57	155.25	139.65	148.17	-150.02	-158.26	-138.91	-152.25
460	40	-341222.77	4.41E-05	144.58	155.28	139.69	148.19	-149.99	-158.25	-138.91	-152.22
500	41	-341329.49	8.62E-05	144.61	155.32	139.73	148.23	-149.95	-158.20	-138.86	-152.18

[a] n: sample size

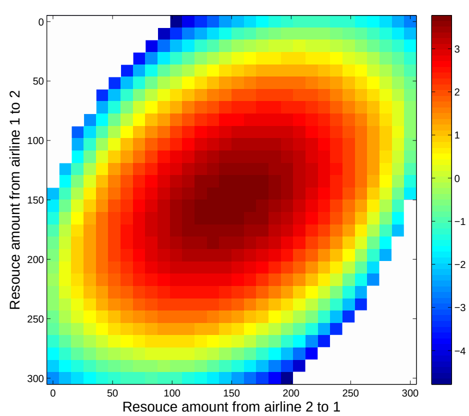
[b] $iter$: number of iterations when algorithm stopped

[c] obj_{opt}: objective function value at the optimal solution

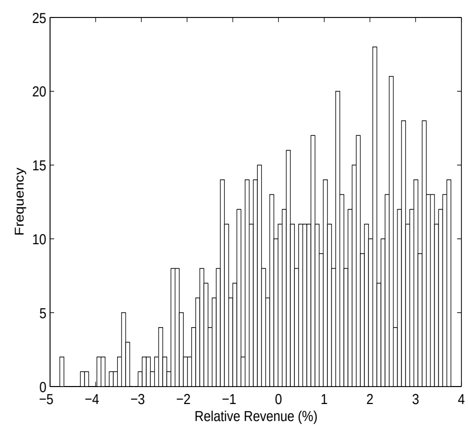
[d] $\|g\|$: gradient norm at the optimal solution

[e] x_{opt}: optimal solution

Suppose that airline -1 operates a flight with capacity 300 from A to B, and airline 1 operates a flight with capacity 300 from B to C. After resource exchange, each airline can offer three products: itineraries from A to B, from B to C, and from A via B to C. Figure 7(a) shows the percentage increase in total profit of the alliance relative to no alliance, as a function of the number of seats that airline 1 (airline -1) makes available to airline -1 (airline 1) shown on the x-axis (y-axis). Figure 7(b) shows a histogram of the percentage increase in total profit of the alliance relative to no alliance for 770 different resource exchanges. As shown, the percentage increase ranges from -4.78% to 3.77%, the alliance profit is larger than the no alliance profit for 68% of the exchanges, and the average percentage increase is 0.75%. Thus, an alliance with an exchange that is not carefully chosen could be worse than no alliance, but the improvement of an alliance over no alliance seems quite robust with respect to deviations from the optimal exchange.



(a) Percentage increase in total profit of the alliance relative to no alliance, as a function of the resource exchange.



(b) Histogram of percentage increase in total profit of the alliance relative to no alliance for 770 different resource exchanges.

Figure 7: Robustness of increase in total profit of the alliance relative to no alliance with respect to resource exchange.

1.5 Conclusion

In this study we presented an economic motivation for interest in alliances, by showing that without an alliance sellers will tend to price their products too high and sell too little, thereby foregoing potential profit, especially if the capacity is large. We showed that under a resource exchange alliance, some of the foregone profit can be captured. In fact, in the two-resource example, the alliance attained the same total profit as perfect coordination, except when capacity is large and the products of the sellers are complements.

We formulated the problem of determining the optimal amounts of resources to exchange as a mathematical program with equilibrium constraints, taking the competition into account that results from alliance members selling similar products. In general, mathematical programs with equilibrium constraints are hard to solve, especially in the stochastic case with random problem parameters. We used a trust region algorithm to search for an optimal exchange, and used it to solve example problems.

Many research questions regarding alliances remain. In this study we consider one type of alliance, namely resource exchange alliances. Such alliances are attractive because they do not require complicated coordination after the resource exchange has taken place, and because such alliances should not have anti-trust problems, since they enhance competition instead of reducing competition. However, there are many other potential alliance structures of interest that remain to be analyzed and compared in greater detail.

The problem of optimal revenue management under an alliance is very challenging, and has not received much attention in the literature. This study does not address operational level revenue management under an alliance — the purpose of this paper is to obtain insight into conditions under which a resource exchange alliance can provide greater profit than the setting without an alliance, and to propose a model and a method to compute good resource exchange amounts. Thus the problem of optimal revenue management under an alliance remains to be addressed.

CHAPTER II

CONDITIONAL VALUE-AT-RISK AND AVERAGE VALUE-AT-RISK: ESTIMATION AND ASYMPTOTICS

2.1 Introduction

In the financial industry, sell-side analysts periodically publish recommendations of underlying securities with target prices (e.g., the Goldman Sachs Conviction Buy List). These recommendations reflect specific economic conditions and influence investors' decisions and thus price movements. However, this type of analysis does not provide risk measures associated with underlying companies. We see similar phenomena in buy-side analysis as well. Each analyst or team covers different sectors (e.g., the airline industry vs. semi-conductor industry) and typically makes separate recommendations for the portfolio managers without associated risk measures. However, the risk measure of the companies that are covered are one of the most important factors for investment decision making. In this study, we consider ways to estimate risk measures for a single asset at given market conditions. This information could be useful for investors and portfolio managers to compare prospective securities and to pick the best ones. For example, when portfolio managers expect crude oil price to spike (due to inflation or geo-political conflicts), they could select securities less sensitive to oil price movements in the airline industry.

In order to formalize our discussion, let us introduce the following setting. Let $(\Omega, \mathcal{F})$ be a measurable space equipped with probability measure P. A measurable function $Y : \Omega \to \mathbb{R}$ is called a random variable. With random variable Y, we associate a number $\rho(Y)$ which we refer to as a *risk measure*. We assume that "smaller is better," i.e., between two possible realizations of random data, we prefer the one with a smaller value of $\rho(\cdot)$. The term "risk measure" is somewhat unfortunate since it can be confused with the term probability measure. Moreover, in applications, one often tries to reach a compromise between minimizing the expectation (i.e., minimizing on average) and controlling the associated risk. Thus, some authors use the term "mean-risk measure," or "acceptability functional" (e.g., Pflug and Römisch (2007)).

For historical reasons, we use here the "risk measure" terminology. Formally, risk measure is a function $\rho : \mathcal{Y} \to \mathbb{R}$ defined on an appropriate space $\mathcal{Y}$ of random variables. For example, in some applications, it is natural to use the space $\mathcal{Y} = L_p(\Omega, \mathcal{F}, P)$, with $p \in [1, \infty)$, of random variables having finite p-th-order moments.

It was suggested in Artzner et al. (1999) that a "good" risk measure should have the following properties (axioms), and such risk measures were called *coherent*.

(A1) *M*onotonicity: If $Y, Y' \in \mathcal{Y}$ and $Y \succeq Y'$, then $\rho(Y) \geq \rho(Y')$.

(A2) *C*onvexity:

$$\rho(tY + (1-t)Y') \leq t\rho(Y) + (1-t)\rho(Y')$$

for all $Y, Y' \in \mathcal{Y}$ and all $t \in [0, 1]$.

(A3) *T*ranslation Equivariance: If $a \in \mathbb{R}$ and $Y \in \mathcal{Y}$, then $\rho(Y + a) = \rho(Y) + a$.

(A4) *P*ositive Homogeneity: If $t \geq 0$ and $Y \in \mathcal{Y}$, then $\rho(tY) = t\rho(Y)$.

The notation $Y \succeq Y'$ means that $Y(\omega) \geq Y'(\omega)$ for a.e. $\omega \in \Omega$. We may refer, e.g., to Detlefsen and Scandolo (2005), Weber (2006), Föllmer and Schied (2011) for a further discussion of mathematical properties of risk measures.

An important example of risk measures is the *Value-at-Risk* measure

$$\mathsf{V@R}_\alpha(Y) = \inf\{t : F_Y(t) \geq \alpha\}, \tag{2.1.34}$$

where $\alpha \in (0, 1)$ and $F_Y(t) = \mathsf{Pr}(Y \leq t)$ is the cumulative distribution function (cdf) of Y, i.e., $\mathsf{V@R}_\alpha(Y) = F_Y^{-1}(\alpha)$ is the left side α-quantile of the distribution of Y. This risk measure satisfies axioms (A1),(A3) and (A4), but not (A2), and hence is not coherent. Another important example is the so-called *Average Value-at-Risk* measure, which can be defined as

$$\mathsf{AV@R}_\alpha(Y) = \inf_{t \in \mathbb{R}} \left\{ t + (1-\alpha)^{-1} \mathbb{E}[Y - t]_+ \right\} \tag{2.1.35}$$

(cf., Rockafellar and Uryasev (2002)), or equivalently

$$\mathsf{AV@R}_\alpha(Y) = \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_\tau(Y)d\tau. \tag{2.1.36}$$

Note that $\mathsf{AV@R}_\alpha(Y)$ is finite iff $\mathbb{E}[Y]_+ < \infty$. Therefore, it is natural to use the space $\mathcal{Y} = L_1(\Omega, \mathcal{F}, P)$ of random variables having finite first order moment for the $\mathsf{AV@R}_\alpha$ risk measure. The Average Value-at-Risk measure is also called the Conditional Value-at-Risk or Expected Shortfall measure. (Since we discuss here "conditional" variants of risk measures, we use the Average Value-at-Risk rather than Conditional Value-at-Risk terminology.)

The Value-at-Risk and Average Value-at-Risk measures are widely used to measure and manage risk in the financial industry (see, e.g., Jorion (2003), Duffie and Singleton (2003), Gaglianone et al. (2011) for the financial background and various applications). Note that in the above two examples, risk measures are functions of the distribution of Y. Such risk measures are called *law invariant*. Law invariant risk measures have been studied extensively in the financial risk management literature (e.g., Acerbi (2002), Frey and McNeil (2002), Scaillet (2004), Fermanian and Scaillet (2005), Chen and Tang (2005), Zhu and Fukushima (2009), Jackson and Perraudin (2000), Berkowitz et al. (2002), Bluhm et al. (2002), and reference therein). Sometimes, we write a law invariant risk measure as a function $\rho(F)$ of cdf F.

Now let us consider a situation where there exists information composed of economic and market variables $X_1, ..., X_k$ which can be considered as a set of predictors for a variable of interest Y. In that case, one can be interested in estimation of a risk measure of Y conditional on observed values of predictors $X_1, ..., X_k$. For example, suppose we want to measure (predict) the risk of a single asset given specific economic conditions represented by market index and interest rates. Then, for a random vector $\boldsymbol{X} = (X_1, ..., X_k)^\mathsf{T}$ of relevant predictors, the conditional version of a law invariant risk measure ρ, denoted $\rho(Y|\boldsymbol{X})$ or $\rho_{|\boldsymbol{X}}(Y)$, is obtained by applying ρ to the conditional distribution of Y given $\boldsymbol{X}$. In particular, $\mathsf{V@R}_\alpha(Y|\boldsymbol{X})$ is the α-quantile of the conditional distribution of Y given $\boldsymbol{X}$, and

$$\mathsf{AV@R}_\alpha(Y|\boldsymbol{X}) = \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_\tau(Y|\boldsymbol{X})d\tau. \tag{2.1.37}$$

Recently, several researchers have paid attention to estimation of the conditional risk measures. For the conditional Value-at-Risk, Chernozhukov and Umantsev (2001) used a polynomial type regression quantile model and Engle and Manganelli (2004) proposed the model which specifies the evolution of the quantile over time using a special type of autoregressive processes. In both models, unknown parameters were estimated by minimizing the regression quantile loss function. For conditional Average Value-at-Risk, Scaillet (2005) and Cai and Wang (2008) utilized Nadaraya-Watson (NW) type nonparametric double kernel estimation while Peracchi and Tanase (2008) and Leorato et al. (2010) used the semiparametric method.

In this study, we discuss procedures for estimation of conditional risk measures. Especially, we will pay attention to estimation of conditional Value-at-Risk and Average Value-at-Risk measures. We assume the following linear model (linear regression)

$$Y = \beta_0 + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X} + \varepsilon, \tag{2.1.38}$$

where β_0 and $\boldsymbol{\beta} = (\beta_1, ..., \beta_k)^\mathsf{T}$ are (unknown) parameters of the model and the error (noise) random variable ε is assumed to be independent of random vector $\boldsymbol{X}$. Meaning of the model (2.1.38) is that there is a true (population) value $\beta_0^*, \boldsymbol{\beta}^*$ of the respective parameters for which (2.1.38) holds. Sometimes, we will write this explicitly and sometimes suppress this in the notation.

Let $\rho(\cdot)$ be a law invariant risk measure satisfying axiom (A3) (Translation Equivariance), and $\rho_{|\boldsymbol{X}}(\cdot)$ be its conditional analogue. Note that because of the independence of ε and $\boldsymbol{X}$, it follows that $\rho_{|\boldsymbol{X}}(\varepsilon) = \rho(\varepsilon)$. Together with axiom (A3), this implies

$$\rho_{|\boldsymbol{X}}(Y) = \rho_{|\boldsymbol{X}}(\beta_0 + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X} + \varepsilon) = \beta_0 + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X} + \rho_{|\boldsymbol{X}}(\varepsilon) = \beta_0 + \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X} + \rho(\varepsilon). \tag{2.1.39}$$

Since $\beta_0 + \rho(\varepsilon) = \rho(\varepsilon + \beta_0)$, we can set $\rho(\varepsilon) = 0$ by adding a constant to the error term. In that case, for the true values of the parameters, we have $\rho_{|\boldsymbol{X}}(Y) = \beta_0^* + \boldsymbol{\beta}^{*\mathsf{T}}\boldsymbol{X}$. Hence, the question is how to estimate these (true) values $\beta_0^*, \boldsymbol{\beta}^*$ of the respective parameters.

This study is organized as follows. In Section 2.2, we describe two different estimation procedures for the conditional risk measures; one is based on residuals of the least squares estimation procedure and the other is based on the M-estimation approach. Asymptotic properties of both estimators are provided in Section 2.3. In Section 2.4, we investigate the finite sample and asymptotic properties of the considered estimators. We present Monte Carlo simulation results under different distribution assumptions of the error term. Later, we illustrate the performance of different methods on the real data (different financial asset classes) in Section 2.5. Finally, Section 2.6 gives some conclusion remarks and suggestions for future research directions.

2.2 *Basic Estimation Procedures*

Suppose that we have N observations (data points) $(Y_i, \boldsymbol{X}_i)$, $i = 1, ..., N$, which satisfy the linear regression model (2.1.38), i.e.,

$$Y_i = \beta_0 + \boldsymbol{\beta}^{\mathsf{T}} \boldsymbol{X}_i + \varepsilon_i, \quad i = 1, ..., N. \tag{2.2.40}$$

We assume that: (i) $\boldsymbol{X}_i$, $i = 1, ..., N$, are iid (independent identically distributed) random vectors, and write $\boldsymbol{X}$ for random vector having the same distribution as $\boldsymbol{X}_i$, (ii) the errors $\varepsilon_1, .., \varepsilon_N$ are iid with finite second order moments and independent of $\boldsymbol{X}_i$. We denote by $\sigma^2 = \mathrm{Var}[\varepsilon_i]$ the common variance of the error terms.

There are two basic approaches to estimation of the true values of β_0 and $\boldsymbol{\beta}$. One approach is to apply the standard Least Squares (LS) estimation procedure and then to make an adjustment of the estimate of the intercept parameter β_0. That is, let $\tilde{\beta}_0$ and $\tilde{\boldsymbol{\beta}}$ be the least squares estimators of the respective parameters of the linear model (2.2.40) and

$$e_i := Y_i - \tilde{\beta}_0 - \tilde{\boldsymbol{\beta}}^{\mathsf{T}} \boldsymbol{X}_i, \quad i = 1, ..., N, \tag{2.2.41}$$

be the corresponding residuals. By the standard theory of the LS method, we have that $\tilde{\beta}_0$ and $\tilde{\boldsymbol{\beta}}$ are unbiased estimators of the respective parameters of the linear model (2.1.38) provided $\mathbb{E}[\varepsilon] = 0$. Therefore, we need to make the correction $\tilde{\beta}_0 + \rho(\varepsilon)$ of the intercept estimator. If we knew the true values $\varepsilon_1, ..., \varepsilon_N$ of the error term, we could estimate $\rho(\varepsilon)$ by replacing the cdf F_ε of ε by its empirical estimate $\hat{F}_{\varepsilon,N}$ associated with $\varepsilon_1, ..., \varepsilon_N$, i.e., to estimate $\rho(F_\varepsilon)$ by $\rho(\hat{F}_{\varepsilon,N})$. Since true values of the error term are unknown, it is a natural idea to replace $\varepsilon_1, ..., \varepsilon_N$ by the residual values $e_1, ..., e_N$. Hence, we use the estimator $\tilde{\beta}_0 + \rho(\hat{F}_{e,N})$, where $\hat{F}_{e,N}$ is the empirical cdf of the residual values, i.e., $\hat{F}_{e,N}$ is the cdf of the probability distribution assigning mass $1/N$ to each point e_i, $i = 1, ..., N$ (see section 2.3.1 for further discussion). We refer to this estimation approach as the *Least Squares Residuals* (LSR) method.

An alternative approach is based on the following idea. Suppose that we can construct a function $h(y, \theta)$ of $y \in \mathbb{R}$ and $\theta \in \mathbb{R}$, convex in θ, such that the minimizer of $\mathbb{E}_F[h(Y, \theta)]$ will be equal to $\rho(F)$, i.e., $\rho(F) = \arg\min_\theta \mathbb{E}_F[h(Y, \theta)]$. Since $\rho(Y+a) = \rho(Y) + a$ for any $a \in \mathbb{R}$, it follows that the function $h(y, \theta)$ should be of the form $h(y, \theta) = \psi(y - \theta)$ for some convex function $\psi : \mathbb{R} \to \mathbb{R}$. We refer to $\psi(\cdot)$ as the *error function*. Therefore, we need to construct an error function such that

$$\rho(F) = \arg\min_\theta \mathbb{E}_F[\psi(Y - \theta)]. \tag{2.2.42}$$

This is equivalent to solving the equation

$$\mathbb{E}_F[\phi(Y - \theta)] = 0, \tag{2.2.43}$$

where $\phi(t) := \psi'(t)$. Note that the error function $\psi(\cdot)$ could be nondifferentiable, in which case the corresponding derivative function $\phi(\cdot)$ is discontinuous. That is, the function $\phi(\cdot)$ is monotonically nondecreasing.

The corresponding estimators $\hat\beta_0$ and $\hat{\boldsymbol\beta}$ are taken as solutions of the optimization problem

$$\operatorname*{Min}_{\beta_0, \boldsymbol\beta} \sum_{i=1}^N \psi\left(Y_i - \beta_0 - \boldsymbol\beta^\mathsf{T} \boldsymbol X_i\right). \tag{2.2.44}$$

In the statistics literature, such estimators are called M-estimators (the terminology which we will follow) and for an appropriate choice of the error function, this is the approach of robust regression (Huber (1981)). For the $\mathsf{V@R}_\alpha$ risk measure, the error function is readily available (recall that $[t]_+ = \max\{0, t\}$):

$$\psi(t) := \alpha[t]_+ + (1 - \alpha)[-t]_+. \tag{2.2.45}$$

The corresponding robust regression approach is known as the *quantile regression* method (cf. Koenker (2005)).

For coherent risk measures, the situation is more delicate. Let us make the following observations. Suppose that the representation (2.2.42) holds. Let F_1 and F_2 be two cdf such that $\rho(F_1) = \rho(F_2) = \theta$. Then it follows by (2.2.42) (by (2.2.43)) that

$\rho(tF_1 + (1-t)F_2) = \theta$ for any $t \in [0,1]$. This is quite a strong necessary condition for existence of a representation of the form (2.2.42). It certainly doesn't hold for the $\mathsf{AV@R}_\alpha$, $\alpha \in (0,1)$, risk measure (See proof of theorem 11, p.760, in Gneiting (2011)).

This shows that for general coherent risk measures, possibility of constructing the corresponding M-estimators is rather exceptional, and such estimators certainly do not exist for the $\mathsf{AV@R}_\alpha$ risk measure. Nevertheless, it is possible to construct the following approximations (this construction is essentially due to Rockafellar et al. (2008)).

Proposition 1 *Let $\psi_j : \mathbb{R} \to \mathbb{R}$, $j = 1,...,r$, be convex functions, $\lambda_j \in \mathbb{R}$ be such that $\sum_{j=1}^r \lambda_j = 1$ and*

$$\mathcal{E}(Y) := \inf_{\boldsymbol{\tau} \in \mathbb{R}^r} \left\{ \mathbb{E}\left[\sum_{j=1}^r \psi_j(Y - \tau_j) \right] : \sum_{j=1}^r \lambda_j \tau_j = 0 \right\}. \tag{2.2.46}$$

Moreover, let $S_j(Y)$ be a minimizer of $\mathbb{E}[\psi_j(Y-\theta)]$ over $\theta \in \mathbb{R}$. Then $S(Y) := \sum_{j=1}^r \lambda_j S_j(Y)$ is a minimizer of $\mathcal{E}(Y-\theta)$ over $\theta \in \mathbb{R}$.

Proof Proof Consider the problem

$$\operatorname*{Min}_{\theta, \boldsymbol{\tau}} \mathbb{E}\left[\sum_{j=1}^r \psi_j(Y - \theta - \tau_j) \right] \ \text{ s.t. } \sum_{j=1}^r \lambda_j \tau_j = 0. \tag{2.2.47}$$

By making change of variables $\eta_j = \theta + \tau_j$, $j = 1,...,r$, we can write this problem in the form

$$\operatorname*{Min}_{\theta, \boldsymbol{\eta}} \mathbb{E}\left[\sum_{j=1}^r \psi_j(Y - \eta_j) \right] \ \text{ s.t. } \sum_{j=1}^r \lambda_j \eta_j = \theta. \tag{2.2.48}$$

Since $S_j(Y)$ is a minimizer of $\mathbb{E}\left[\psi_j(Y-\eta_j)\right]$, it follows that $\eta_j = S_j(Y)$, $i = 1,..,r$, $\theta = S(Y)$, is an optimal solution of problem (2.2.48). This completes the prof. ∎

In particular, we can consider functions $\psi_j(\cdot)$ of the form (2.2.45), i.e.,

$$\psi_j(t) := \alpha_j[t]_+ + (1-\alpha_j)[-t]_+, \tag{2.2.49}$$

for some $\alpha_j \in (0,1)$, $j = 1, ..., r$. Then $S_j(Y) = \mathsf{V@R}_{\alpha_j}(Y)$ and hence the risk measure $\sum_{j=1}^r \lambda_j \mathsf{V@R}_{\alpha_j}(Y)$ is a minimizer of $\mathcal{E}(Y - \theta)$. We can view $\sum_{j=1}^r \lambda_j \mathsf{V@R}_{\alpha_j}(Y)$ as a discretization of the integral $\frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_\tau(Y)d\tau$ if we set $\Delta := (1-\alpha)/r$ and take

$$\lambda_j := (1-\alpha)^{-1}\Delta,\ \alpha_j := \alpha + (j - 0.5)\Delta,\ j = 1, ..., r. \tag{2.2.50}$$

For this choice of λ_j, α_j, and by formula (2.1.36), we have that

$$\mathsf{AV@R}_\alpha(Y) = \frac{1}{1-\alpha}\int_\alpha^1 \mathsf{V@R}_\tau(Y)d\tau \approx \sum_{j=1}^r \lambda_j \mathsf{V@R}_{\alpha_j}(Y) = S(Y). \tag{2.2.51}$$

Consider now the problem

$$\operatorname*{Min}_{\beta_0, \boldsymbol{\beta}} \mathcal{E}(Y - \beta_0 - \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X}). \tag{2.2.52}$$

By the definition (2.2.46) of $\mathcal{E}(\cdot)$, we can write this problem in the following equivalent form

$$\operatorname*{Min}_{\boldsymbol{\tau}, \beta_0, \boldsymbol{\beta}} \mathbb{E}\left[\textstyle\sum_{j=1}^r \psi_j(Y - \beta_0 - \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X} - \tau_j)\right] \ \text{ s.t. } \textstyle\sum_{j=1}^r \lambda_j \tau_j = 0. \tag{2.2.53}$$

The so-called Sample Average Approximation (SAA) of this problem is

$$\operatorname*{Min}_{\boldsymbol{\tau}, \beta_0, \boldsymbol{\beta}} \frac{1}{N}\sum_{i=1}^N \sum_{j=1}^r \psi_j(Y_i - \beta_0 - \boldsymbol{\beta}^\mathsf{T}\boldsymbol{X}_i - \tau_j) \ \text{ s.t. } \sum_{j=1}^r \lambda_j \tau_j = 0. \tag{2.2.54}$$

The above problem (2.2.54) can be formulated as a linear programming problem. Following Rockafellar et al. (2008), we consider the following estimators.

Mixed quantile estimator for $\mathsf{AV@R}_\alpha(\mathbf{Y}|\boldsymbol{x})$

We refer to $\check{\beta}_0 + \check{\boldsymbol{\beta}}^\mathsf{T}\boldsymbol{x}$ as the *mixed quantile estimator* of $\mathsf{AV@R}_\alpha(Y|\boldsymbol{x})$, where $(\check{\boldsymbol{\tau}}, \check{\beta}_0, \check{\boldsymbol{\beta}})$ is an optimal solution of problem (2.2.54).

This idea can be extended to a larger class of law invariant risk measures. For example, consider a risk measure

$$\rho(Y) := c\mathbb{E}[Y] + (1-c)\mathsf{AV@R}_\alpha(Y) \tag{2.2.55}$$

for some constants $c \in [0,1]$ and $\alpha \in (0,1)$. Recall that the minimizer of $\mathbb{E}[(Y - t)^2]$ is $t^* = \mathbb{E}[Y]$. Therefore, by taking function $\psi_0(t) := t^2$, functions $\psi_j(t)$ of the form (2.2.49), λ_j, and α_j given in (2.2.50), we can construct the corresponding error function

$$\mathcal{E}(Y) := \inf_{\boldsymbol{\tau} \in \mathbb{R}^{r+1}} \left\{ \mathbb{E}\left[\psi_0(Y - \tau_0) + \sum_{j=1}^{r} \psi_j(Y - \tau_j) \right] : c\tau_0 + \sum_{j=1}^{r} (1-c)\lambda_j \tau_j = 0 \right\}. \tag{2.2.56}$$

As another example, consider risk measures of the form

$$\rho(Y) := \int_0^1 \mathsf{AV@R}_\alpha(Y) d\mu(\alpha), \tag{2.2.57}$$

where μ is a probability measure on the interval $[0,1)$. By a result due to Kusuoka (2001), this measures form a class of the comonote law invariant coherent risk measures. By (2.1.36), we can write such risk measure as

$$\rho(Y) = \int_0^1 \int_\alpha^1 (1-\alpha)^{-1} \mathsf{V@R}_\tau(Y) d\tau d\mu(\alpha) = \int_0^1 w(\tau) \mathsf{V@R}_\tau(Y) d\tau, \tag{2.2.58}$$

where $w(\tau) := \int_0^\tau (1-\alpha)^{-1} d\mu(\alpha)$. Such risk measures are also called spectral risk measures (Acerbi (2002)). By making a discretization of the above integral (2.2.58), we can proceed as above.

It could be remarked here that while the LSR approach is quite general, the approach based on mixing M-estimators is somewhat restrictive. Constructing an appropriate error function for a particular risk measure could be quite involved.

2.3 Large Sample Statistical Inference

In the previous section, we formulated two approaches, the LSR estimators and mixed M-estimators, to estimation of the true (population) values of parameters $\beta_0^*, \boldsymbol{\beta}^*$ of the linear model (2.1.38) such that $\rho(\varepsilon) = 0$. For the $\mathsf{V@R}_\alpha$ risk measure, the corresponding M-estimators $\hat{\beta}_0$ and $\hat{\boldsymbol{\beta}}$ are taken as solutions of the optimization problem

(2.2.44), with the error function (2.2.45), and referred to as the quantile regression estimators. For the $\mathsf{AV@R}_\alpha$ risk measure and more generally comonotone risk measures of the form (2.2.58), we constructed the corresponding mixed quantile estimators $\check{\boldsymbol{\tau}}, \check{\beta}_0, \check{\boldsymbol{\beta}}$. In this section, we discuss statistical properties of these estimators. In particular, we address the question of which of these two estimation procedures is more efficient by computing corresponding asymptotic variances.

2.3.1 Statistical Inference of Least Squares Residual Estimators

The linear model (2.2.40) can be written as

$$\boldsymbol{Y} = \mathbb{X}[\beta_0; \boldsymbol{\beta}] + \boldsymbol{\epsilon}, \tag{2.3.59}$$

where $\boldsymbol{Y} = (Y_1, ..., Y_N)^\mathsf{T}$ is $N \times 1$ vector of responses, $\mathbb{X}$ is $N \times (k+1)$ data matrix of predictor variables with rows $(1, \boldsymbol{X}_i^\mathsf{T})$, $i = 1, ..., N$, (i.e., first column of $\mathbb{X}$ is column of ones), $\boldsymbol{\beta} = (\beta_1, ..., \beta_k)^\mathsf{T}$ vector of parameters and $\boldsymbol{\epsilon} = (\varepsilon_1, .., \varepsilon_N)^\mathsf{T}$ is $N \times 1$ vector of errors. By $[\beta_0; \boldsymbol{\beta}]$, we denote $(k+1) \times 1$ vector $(\beta_0, \boldsymbol{\beta}^\mathsf{T})^\mathsf{T}$. We assume that the conditions (i) and (ii), specified at the beginning of section 2.2, hold. It is also possible to view data points $\boldsymbol{X}_i$ as deterministic. In that case, we assume that $\mathbb{X}$ has full column rank $k+1$.

Let $\tilde{\beta}_0$ and $\tilde{\boldsymbol{\beta}}$ be the least squares estimators of the respective parameters of the linear model (2.2.40). Recall that these estimators are given by $[\tilde{\beta}_0; \tilde{\boldsymbol{\beta}}] = (\mathbb{X}^\mathsf{T}\mathbb{X})^{-1}\mathbb{X}^\mathsf{T}\boldsymbol{Y}$, vector of residuals $\boldsymbol{e} := \boldsymbol{Y} - \mathbb{X}[\tilde{\beta}_0, \tilde{\boldsymbol{\beta}}]$ is given by

$$\boldsymbol{e} = (\boldsymbol{I}_N - \boldsymbol{H})\boldsymbol{Y} = (\boldsymbol{I}_N - \boldsymbol{H})\boldsymbol{\epsilon},$$

where $\boldsymbol{I}_N$ is the $N \times N$ identity matrix, and $\boldsymbol{H} = \mathbb{X}(\mathbb{X}^\mathsf{T}\mathbb{X})^{-1}\mathbb{X}^\mathsf{T}$ is the so-called hat matrix. Note that $\text{trace}(\boldsymbol{H}) = k+1$ and we have that

$$\varepsilon_i - e_i = [1; \boldsymbol{X}_i^\mathsf{T}](\mathbb{X}^\mathsf{T}\mathbb{X})^{-1}\mathbb{X}^\mathsf{T}\boldsymbol{\epsilon}, \quad i = 1, ..., N. \tag{2.3.60}$$

If we knew errors $\varepsilon_1, .., \varepsilon_N$, we could estimate $\rho(\varepsilon)$ by the corresponding sample estimate based on the empirical cdf

$$\hat{F}_{\varepsilon,N}(\cdot) = N^{-1} \sum_{i=1}^{N} \mathbb{I}_{[\varepsilon_i,\infty)}(\cdot), \tag{2.3.61}$$

where $\mathbb{I}_A(\cdot)$ denotes the indicator function of set A. However, the true values of the errors are unknown. Therefore, in the LSR approach we replace them by the residuals computed by the least squares method and hence estimate $\rho(\varepsilon)$ by employing the respective empirical cdf $\hat{F}_{e,N}(\cdot)$ instead of $\hat{F}_{\varepsilon,N}(\cdot)$.

The first natural question is whether the LSR estimators are consistent, i.e., converge w.p.1 to their true values as the sample size N tends to infinity. It is well known that, under the specified assumptions, the LS estimators $\tilde{\beta}_0$ and $\tilde{\boldsymbol{\beta}}$ are consistent, with $\tilde{\beta}_0$ being consistent under the condition $\mathbb{E}[\varepsilon] = 0$. The question of consistency of empirical estimates of law invariant coherent risk measures was studied in Wozabal and Wozabal (2009). It was shown that, under mild regularity conditions, such estimators are consistent. In particular, the consistency holds for the comonotone risk measures of the form (2.2.58), i.e., $\rho(\hat{F}_{\varepsilon,N})$ converges w.p.1 to $\rho(F_\varepsilon)$ as $N \to \infty$. It is also possible to show that the difference $\rho(\hat{F}_{\varepsilon,N}) - \rho(\hat{F}_{e,N})$ tends w.p.1 to zero and hence $\rho(\hat{F}_{e,N})$ converges w.p.1 to $\rho(F_\varepsilon)$ as well. A rigorous proof of this could be quite technical and will be beyond the scope of this study.

We have that the LS estimator $\left[\tilde{\beta}_0; \tilde{\boldsymbol{\beta}}\right]$ asymptotically has normal distribution with the asymptotic covariance matrix $N^{-1}\sigma^2\boldsymbol{\Omega}^{-1}$, where $\boldsymbol{\mu} := \mathbb{E}[\boldsymbol{X}]$, $\boldsymbol{\Sigma} := \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\right]$ and $\boldsymbol{\Omega} := \begin{bmatrix} 1 & \boldsymbol{\mu}^\mathsf{T} \\ \boldsymbol{\mu} & \boldsymbol{\Sigma} \end{bmatrix}$. Consequently, for a given $\boldsymbol{x}$, the estimate $\tilde{\beta}_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}$ asymptotically has normal distribution with the asymptotic variance $N^{-1}\sigma^2[1;\boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{x}^\mathsf{T}]^\mathsf{T}$.

We also have that random vectors $(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}})$ and $\boldsymbol{e}$ are uncorrelated. Therefore, if errors ε_i have normal distribution, then vectors $(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}})$ and $\boldsymbol{e}$ jointly have a multivariate normal distribution and these vectors are independent. Consequently, $\tilde{\beta}_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}$ and $\rho(\hat{F}_{e,N})$ are independent. For nonnormal distribution, this independence holds asymptotically and thus asymptotically $\tilde{\beta}_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}$ and $\rho(\hat{F}_{e,N})$ are uncorrelated.

Asymptotics of empirical estimators of law invariant coherent risk measures were studied in Pflug and Wozabal (2010) and Shapiro et al. (2009). Derivation of the asymptotic variance of $\rho(\hat{F}_{\varepsilon,N})$, for a general law invariant risk measure, could be quite involved. Let us consider two important cases of the $\mathsf{V@R}_\alpha$ and $\mathsf{AV@R}_\alpha$ risk measures. We give below a summary of basic results, for a more technical discussion we refer to the Appendix.

In case of $\rho := \mathsf{V@R}_\alpha$, the LSR estimate of $\mathsf{V@R}_\alpha(\varepsilon)$ becomes

$$\widehat{\mathsf{V@R}}_\alpha(e) := \hat{F}_{e,N}^{-1}(\alpha) = e_{(\lceil N\alpha \rceil)}, \tag{2.3.62}$$

where $e_{(1)} \leq ... \leq e_{(N)}$ are order statistics (i.e., numbers $e_1, ..., e_N$ arranged in the increasing order), and $\lceil a \rceil$ denotes the smallest integer $\geq a$. Suppose that the cdf $F_\varepsilon(\cdot)$ has nonzero density $f_\varepsilon(\cdot) = F'_\varepsilon(\cdot)$ at $F_\varepsilon^{-1}(\alpha)$ and let

$$\omega^2 := \frac{\alpha(1-\alpha)}{\left[f_\varepsilon\left(F_\varepsilon^{-1}(\alpha)\right)\right]^2}. \tag{2.3.63}$$

LSR estimator of $\mathsf{V@R}_\alpha(\mathbf{Y}|\boldsymbol{x})$

Consider the LSR estimator $\tilde{\beta}_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}} + \widehat{\mathsf{V@R}}_\alpha(e)$ of $\mathsf{V@R}_\alpha(\boldsymbol{Y}|\boldsymbol{x})$. Suppose that the set of population α-quantiles is a singleton. Then the LSR estimator is a consistent estimator of $\mathsf{V@R}_\alpha(\boldsymbol{Y}|\boldsymbol{x})$, and the asymptotic variance of this estimator can be approximated by

$$N^{-1}\left(\omega^2 + \sigma^2[1;\boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{x}^\mathsf{T}]^\mathsf{T}\right). \tag{2.3.64}$$

Detailed derivation of above asymptotics is discussed in Appendix C.

For the $\rho := \mathsf{AV@R}_\alpha$ risk measure, the LSR estimate of $\mathsf{AV@R}_\alpha(\varepsilon)$ is given by

$$\begin{aligned} \widehat{\mathsf{AV@R}}_\alpha(e) &= \inf_{t\in\mathbb{R}}\left\{t + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=1}^N [e_i - t]_+\right\} \\ &= \widehat{\mathsf{V@R}}_\alpha(e) + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=1}^N \left[e_i - \widehat{\mathsf{V@R}}_\alpha(e)\right]_+ \\ &= e_{(\lceil N\alpha\rceil)} + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=\lceil N\alpha\rceil+1}^N \left(e_{(i)} - e_{(\lceil N\alpha\rceil)}\right). \end{aligned} \tag{2.3.65}$$

LSR estimator of $\mathsf{AV@R}_\alpha(\boldsymbol{Y}|\boldsymbol{x})$

Consider the LSR estimator $\tilde{\beta}_0+\boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}+\widehat{\mathsf{AV@R}}_\alpha(e)$ of $\mathsf{AV@R}_\alpha(\boldsymbol{Y}|\boldsymbol{x})$. This estimator is consistent and its asymptotic variance is given by

$$N^{-1}\left(\gamma^2+\sigma^2[1;\boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{x}^\mathsf{T}]^\mathsf{T}\right), \tag{2.3.66}$$

where $\gamma^2 := (1-\alpha)^{-2}\mathrm{Var}\big([\varepsilon-\mathsf{V@R}_\alpha(\varepsilon)]_+\big)$, $\boldsymbol{\Omega} := \begin{bmatrix} 1 & \boldsymbol{\mu}^\mathsf{T} \\ \boldsymbol{\mu} & \boldsymbol{\Sigma} \end{bmatrix}$, $\boldsymbol{\mu} := \mathbb{E}[\boldsymbol{X}]$ and $\boldsymbol{\Sigma} := \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^\mathsf{T}\right]$.

The above asymptotics are discussed in Appendix D.

Remark It should be remembered that the above approximate variances are asymptotic results. Suppose for the moment that $N<(1-\alpha)^{-1}$. Then $\lceil N\alpha\rceil = N$ and hence $\widehat{\mathsf{V@R}}_\alpha(\varepsilon) = \max\{\varepsilon_1,...,\varepsilon_N\}$. Consequently $\left[\varepsilon_i-\widehat{\mathsf{V@R}}_\alpha(\varepsilon)\right]_+ = 0$ for all $i=1,...,N$, and hence

$$\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) = \widehat{\mathsf{V@R}}_\alpha(\varepsilon) = \max\{\varepsilon_1,...,\varepsilon_N\}.$$

In that case the above asymptotics are inappropriate. In order for these asymptotics to be reasonable, N should be significantly bigger than $(1-\alpha)^{-1}$.

LSR approach can be easily applied to a considerably larger class of law invariant risk measures. For example, let us consider the entropic risk measure $\rho(Y) := \alpha^{-1}\log\mathbb{E}[e^{\alpha Y}]$, where $\alpha>0$ is a positive constant. This risk measure satisfies axioms (A1)–(A3), but it is not positively homogeneous (see Giesecke and Weber (2008) for the general discussion of utility-based shortfall risk including entropic risk measure). The empirical estimate of $\rho(\varepsilon)$ is

$$\rho(\hat{F}_{\varepsilon,N}) = \alpha^{-1}\log\left(N^{-1}\textstyle\sum_{i=1}^N e^{\alpha\varepsilon_i}\right). \tag{2.3.67}$$

Of course, as it was discussed above, the errors ε_i should be replaced by the respective residuals e_i in the construction of the corresponding LSR estimators. By using linearizations $e^{\alpha\varepsilon} = 1 + \alpha\varepsilon + o(\alpha\varepsilon)$ and $\log(1 + x) = x + o(x)$, we obtain that $N^{1/2}[\rho(\hat{F}_{\varepsilon,N}) - \rho(\varepsilon)]$ converges in distribution to normal with zero mean and variance σ^2 (by the Delta Theorem).

2.3.2 Statistical Inference of Quantile and Mixed Quantile Estimators

As it was discussed in section 2.2, the quantile regression is a particular case of the M-estimation method with the error function $\psi(\cdot)$ of the form (2.2.45). By the Law of Large Numbers (LLN), we have that N^{-1} times the objective function in (2.2.44) converges (pointwise) w.p.1 to the function $\Psi(\beta_0, \boldsymbol{\beta}) := \mathbb{E}\left[\psi(Y - \beta_0 - \boldsymbol{\beta}^\top \boldsymbol{X})\right]$. We also have

$$\begin{aligned} \Psi(\beta_0, \boldsymbol{\beta}) &= \mathbb{E}\left[\psi\left(\beta_0^* + \boldsymbol{\beta}^{*\top}\boldsymbol{X} + \varepsilon - \beta_0 - \boldsymbol{\beta}^\top \boldsymbol{X}\right)\right] \\ &= \mathbb{E}\left[\psi\left(\varepsilon - (\beta_0 - \beta_0^*) - (\boldsymbol{\beta} - \boldsymbol{\beta}^*)^\top \boldsymbol{X}\right)\right]. \end{aligned} \tag{2.3.68}$$

Under mild regularity conditions, derivatives of $\Psi(\beta_0, \boldsymbol{\beta})$ can be taken inside the integral (expectation) and hence

$$\begin{aligned} \nabla_{\beta_0}\Psi(\beta_0, \boldsymbol{\beta}) &= \mathbb{E}\left[\nabla_{\beta_0}\psi\left(\varepsilon - (\beta_0 - \beta_0^*) - (\boldsymbol{\beta} - \boldsymbol{\beta}^*)^\top \boldsymbol{X}\right)\right] \\ &= -\mathbb{E}\left[\psi'\left(\varepsilon - (\beta_0 - \beta_0^*) - (\boldsymbol{\beta} - \boldsymbol{\beta}^*)^\top \boldsymbol{X}\right)\right], \end{aligned} \tag{2.3.69}$$

$$\begin{aligned} \nabla_{\boldsymbol{\beta}}\Psi(\beta_0, \boldsymbol{\beta}) &= \mathbb{E}\left[\nabla_{\boldsymbol{\beta}}\psi\left(\varepsilon - (\beta_0 - \beta_0^*) - (\boldsymbol{\beta} - \boldsymbol{\beta}^*)^\top \boldsymbol{X}\right)\right] \\ &= -\mathbb{E}\left[\psi'\left(\varepsilon - (\beta_0 - \beta_0^*) - (\boldsymbol{\beta} - \boldsymbol{\beta}^*)^\top \boldsymbol{X}\right)\boldsymbol{X}\right]. \end{aligned} \tag{2.3.70}$$

Since ε and $\boldsymbol{X}$ are independent, we obtain that derivatives of $\Psi(\beta_0, \boldsymbol{\beta})$ are zeros at $(\beta_0^*, \boldsymbol{\beta}^*)$ if the following condition holds

$$\mathbb{E}[\psi'(\varepsilon)] = 0. \tag{2.3.71}$$

Since function $\Psi(\cdot,\cdot)$ is convex, it follows that if condition (2.3.71) holds, then $\Psi(\cdot,\cdot)$ attains its minimum at $(\beta_0^*, \boldsymbol{\beta}^*)$. If the minimizer $(\beta_0^*, \boldsymbol{\beta}^*)$ is unique, then the estimator $(\hat{\beta}_0, \hat{\boldsymbol{\beta}})$ converges w.p.1 to the population value $(\beta_0^*, \boldsymbol{\beta}^*)$ as $N \to \infty$, i.e., $(\hat{\beta}_0, \hat{\boldsymbol{\beta}})$ is a *consistent* estimator of $(\beta_0^*, \boldsymbol{\beta}^*)$ (cf. Huber (1981)). That is, (2.3.71) is the basic condition for consistency of $(\hat{\beta}_0, \hat{\boldsymbol{\beta}})$.

For the error function (2.2.45) of the quantile regression, we have

$$\psi'(t) = \begin{cases} \alpha - 1 & \text{if } t < 0, \\ \alpha & \text{if } t > 0. \end{cases} \tag{2.3.72}$$

(Note that here the error function $\psi(t)$ is not differentiable at $t = 0$ and its derivative $\psi'(t)$ is discontinuous at $t = 0$. Nevertheless, all arguments can go through provided that the error term has a continuous distribution.) Consequently,

$$\mathbb{E}[\psi'(\varepsilon)] = (\alpha - 1)F_\varepsilon(0) + \alpha(1 - F_\varepsilon(0)) = \alpha - F_\varepsilon(0), \tag{2.3.73}$$

and hence condition (2.3.71) holds iff $F_\varepsilon(0) = \alpha$, or equivalently $F_\varepsilon^{-1}(\alpha) = 0$ provided this quantile is unique. In that case, the estimator $(\hat{\beta}_0, \hat{\boldsymbol{\beta}})$ is consistent if the population value β_0^* is normalized such that $\mathsf{V@R}_\alpha(\varepsilon) = 0$. That is, for this error function, $\hat{\beta}_0 + \hat{\boldsymbol{\beta}}^\mathsf{T}\boldsymbol{x}$ is a consistent estimator of the conditional Value-at-Risk $\mathsf{V@R}_\alpha(Y|\boldsymbol{x})$ of Y given $\boldsymbol{X} = \boldsymbol{x}$.

It is also possible to derive asymptotics of the estimator $(\hat{\beta}_0, \hat{\boldsymbol{\beta}})$. That is, suppose that the cdf $F_\varepsilon(\cdot)$ has nonzero density $f_\varepsilon(\cdot) = F_\varepsilon'(\cdot)$ at $F_\varepsilon^{-1}(\alpha)$ and consider ω^2 defined in (2.3.63). Then $N^{1/2}\left[\hat{\beta}_0 - \beta_0^*; \hat{\boldsymbol{\beta}} - \boldsymbol{\beta}^*\right]$ converges in distribution to normal with zero mean vector and covariance matrix (cf., Koenker (2005))

$$\omega^2[1; \boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1; \boldsymbol{x}^\mathsf{T}]^\mathsf{T}, \tag{2.3.74}$$

i.e., N^{-1} times the matrix given in (2.3.74) is the asymptotic covariance matrix of $\left[\hat{\beta}_0; \hat{\boldsymbol{\beta}}\right]$.

Remark Note that by LLN, we have that $N^{-1}\sum_{i=1}^N \boldsymbol{X}_i$ and $N^{-1}\sum_{i=1}^N \boldsymbol{X}_i\boldsymbol{X}_i^\mathsf{T}$ converge w.p.1 as $N \to \infty$ to the vector $\boldsymbol{\mu}$ and matrix $\boldsymbol{\Sigma}$ respectively and that $\boldsymbol{\Sigma} - \boldsymbol{\mu}\boldsymbol{\mu}^\mathsf{T}$ is the covariance matrix of $\boldsymbol{X}$. In case of deterministic $\boldsymbol{X}_i$, we simply define vector $\boldsymbol{\mu}$ and matrix $\boldsymbol{\Sigma}$ as the respective limits of $N^{-1}\sum_{i=1}^N \boldsymbol{X}_i$ and $N^{-1}\sum_{i=1}^N \boldsymbol{X}_i\boldsymbol{X}_i^\mathsf{T}$, assuming that such limits exist. It follows then that $N^{-1}\mathbb{X}^\mathsf{T}\mathbb{X} \to \boldsymbol{\Omega}$ as $N \to \infty$.

The mixed quantile estimator $\check{\beta}_0+\check{\boldsymbol{\beta}}^{\mathsf{T}}\boldsymbol{x}$ can be justified by the following arguments. We have that an optimal solution $(\check{\boldsymbol{\tau}}, \check{\beta}_0, \check{\boldsymbol{\beta}})$ of problem (2.2.54) converges w.p.1 as $N \to \infty$ to the optimal solution $(\boldsymbol{\tau}^\star, \beta_0^\star, \boldsymbol{\beta}^\star)$ of problem (2.2.53), provided (2.2.53) has unique optimal solution. Because of the linear model (2.1.38), we can write problem (2.2.53) as

$$\operatorname*{Min}_{\boldsymbol{\tau},\beta_0,\boldsymbol{\beta}} \mathbb{E}\left[\textstyle\sum_{j=1}^r \psi_{\alpha_j}(\varepsilon + \beta_0^* - \beta_0 + (\boldsymbol{\beta}^* - \boldsymbol{\beta})^{\mathsf{T}}\boldsymbol{X} - \tau_j)\right] \ \text{ s.t. } \textstyle\sum_{j=1}^r \lambda_j \tau_j = 0, \quad (2.3.75)$$

where β_0^* and $\boldsymbol{\beta}^*$ are population values of the parameters. Similar to the proof of Proposition 1, by making change of variables $\eta_j = \beta_0 + \tau_j$, $j = 1, ..., r$, we can write problem (2.3.75) in the following equivalent form

$$\operatorname*{Min}_{\boldsymbol{\eta},\beta_0,\boldsymbol{\beta}} \mathbb{E}\left[\textstyle\sum_{j=1}^r \psi_{\alpha_j}(\varepsilon + \beta_0^* - \eta_j + (\boldsymbol{\beta}^* - \boldsymbol{\beta})^{\mathsf{T}}\boldsymbol{X})\right] \ \text{ s.t. } \textstyle\sum_{j=1}^r \lambda_j \eta_j = \beta_0. \quad (2.3.76)$$

It follows that if

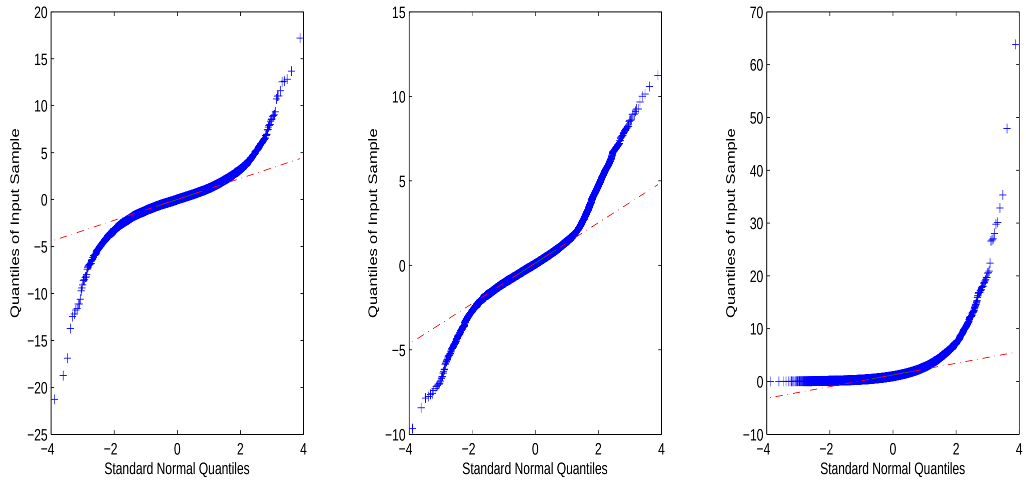
$$\sum_{j=1}^r \lambda_j \mathsf{V@R}_{\alpha_j}(\varepsilon) = 0, \quad (2.3.77)$$

then $(\beta_0^\star, \boldsymbol{\beta}^\star) = (\beta_0^*, \boldsymbol{\beta}^*)$. That is, $\check{\beta}_0+\check{\boldsymbol{\beta}}^{\mathsf{T}}\boldsymbol{x}$ is a consistent estimator of $\sum_{j=1}^r \lambda_j \mathsf{V@R}_{\alpha_j}(Y|\boldsymbol{x})$. Consequently for λ_j and α_j given in (2.2.50), we can use $\check{\beta}_0+\check{\boldsymbol{\beta}}^{\mathsf{T}}\boldsymbol{x}$ as an approximation of $\mathsf{AV@R}_\alpha(Y|\boldsymbol{x})$.

Asymptotics of the mixed quantile estimators are more involved. These asymptotics are discussed in Appendix E.

2.4 Simulation Study

To illustrate the performance of the considered estimators, we perform the Monte Carlo simulations where errors (innovations) in linear model (2.2.40) are generated from following different distributions; (1) Standard Normal (denoted as $N(0, 1)$), (2) Student's t distribution with 3 degrees of freedom (denoted as $t(3)$), (3) Skewed Contaminated Normal where standard normal is contaminated with 20% $N(1, 9)$ errors (denoted as $CN(1, 9)$), (4) Log-Normal with parameter 0 and 1 (denoted as $LN(0, 1)$).

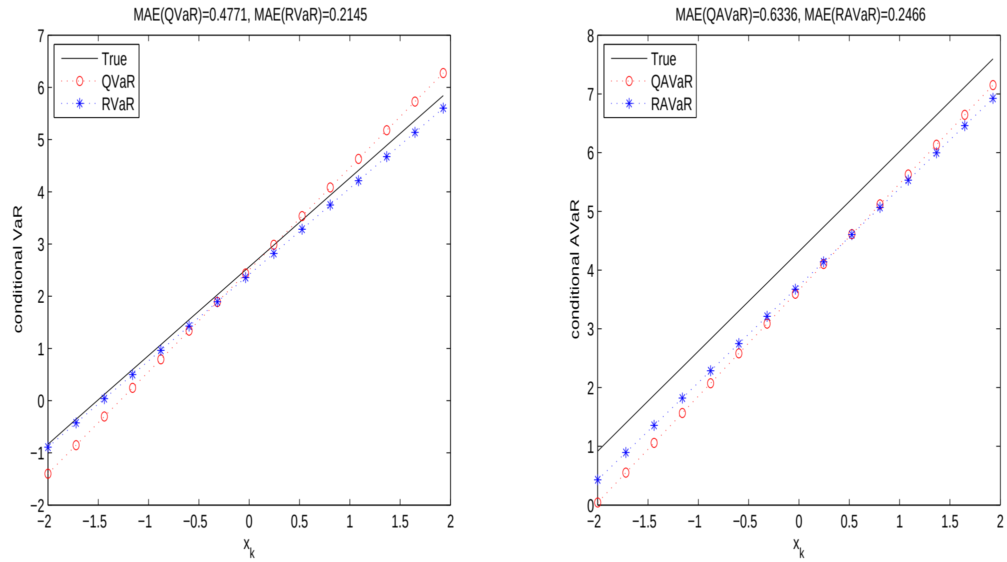


(a) N(0,1) vs. t(3) (b) N(0,1) vs. CN(1,9) (c) N(0,1) vs. LN(0,1)

Figure 8: Normal Q-Q plot for different error distributions

Note that error distributions (2)-(4) are heavy-tailed in contrast to the normal errors as shown in Figure 8. In fact, financial innovations often follow heavy-tailed distributions. We consider $\alpha = 0.9, 0.95, 0.99$, sample size $N = 500, 1000, 2000$ and $R = 500$ replications for each sample size. Conditional Value-at-Risk (VaR) and Average Value-at-Risk (AVaR) are estimated and compared with true (theoretical) values at given 500 test points x_k ($k = 1, 2, \ldots, 500$), which are equally spaced between -2 and 2 for each replication. Estimators obtained from different methods are computed; quantile based estimator (referred to as "QVaR") and LSR estimator (referred to as "RVaR") for the conditional VaR, mixed quantile estimator (referred to as "QAVaR") and LSR estimator (referred to as "RAVaR") for the conditional AVaR (as described in Section 2.2).

Figure 9 displays an example of estimation results where solid line is true (theoretical) VaR (AVaR), dash-circle line is QVaR (QAVaR), and dash-cross line is RVaR (RAVaR) given test points x_k. In this example, errors follow $CN(1, 9)$, $\alpha = 0.95$ and



(a) Conditional VaR (b) Conditional AVaR

Figure 9: Conditional VaR and AVaR: True vs. Estimated (Errors$\sim CN(1, 9)$, $\alpha = 0.95$, $N = 1000$)

$N = 1000$. In Figure 9 RVaR estimates are closer to true VaR values as Mean Absolute Error (MAE) confirms (MAE(QVaR)=0.4771 vs. MAE(RVaR)=0.2145). Performance of both estimators are worse for AVaR, yet RAVaR estimates are still closer to true AVaR values than QAVaR (MAE(QAVaR)=0.6336 vs. MAE(RAVaR)=0.2466) as shown in Figure 9.

To compare estimators under different error distributions, MAE (averaged over all test points) and variance of MAE (in parenthesis) across 500 replications are obtained as shown in Table 7. Regardless of the error distributions, RVaR (RAVaR) works better than QVaR (QAVaR); MAE and the variance of MAE are smaller. As we can expect, both estimators perform better for the conditional VaR than AVaR.

Figure 10 presents box-plots for both estimators (QAVaR and RAVaR) given $x = 1.006$ across 500 replications. Findings are similar to the one from Table 7; there are some evidence to suggest that RAVaR has smaller MAE than QAVaR. Also, RAVaR is more stable than QAVaR (MAE of QAVaR is more spread). Note that both estimators work better for normal distributions than other heavy-tailed distributions. We could

Table 7: MAE for different error distributions $\alpha = 0.95, N = 1000$ (averaged over all test points)

Error	QVaR	RVaR	QAVaR	RAVaR
$N(0,1)$	0.0762	0.0575	0.0990	0.0674
	(0.0037)	(0.0020)	(0.0058)	(0.0026)
$t(3)$	0.1758	0.1290	0.4255	0.3232
	(0.0188)	(0.0095)	(0.0808)	(0.0623)
$CN(1,9)$	0.3006	0.1955	0.3844	0.2311
	(0.0563)	(0.0225)	(0.0882)	(0.0316)
$LN(0,1)$	0.3905	0.2670	0.8957	0.6432
	(0.0959)	(0.0430)	(0.3896)	(0.2481)

observe the similar pattern for conditional VaR.

Table 8 illustrates sample size effect on MAE of estimators. As expected, both estimators perform better as sample size increases. MAE of RVaR (RAVaR) is still smaller than that of QVaR (QAVaR) across all sample sizes.

Next, we obtain asymptotic variances (derived in Section 2.3) and compare that with empirical (finite sample) variances of both estimators. Figure 11 reports asymptotic and finite sample efficiencies of both estimators for the conditional VaR where $R = 500$, and error follows $N(0,1)$ (results are similar for other error distributions). In Figure 11-(a), we see that asymptotic variance of RVaR (dash-dot line) is smaller than that of QVaR (solid line) except at x_k near 0. In fact, asymptotic variance is affected by how far x_k is away from 0 (which is the mean of explanatory variable in the simulation); when x_k is further from the mean, the difference between asymptotic variances of both estimators is bigger. Figure 11-(b) provides empirical variance of both estimators across 500 replications. Empirical variance of RVaR is (equal or) smaller than that of QVaR at all x_k. Figure 11-(c) and Figure 11-(d) compare asymptotic variances to empirical variances of both estimators. It is clear that

Table 8: MAE for different sample size N with $\alpha = 0.95$ (averaged over all test points)

Error	Estimator	$N = 500$	$N = 1000$	$N = 2000$
$N(0,1)$	QVaR	0.1129	0.0762	0.0569
	RVaR	0.0849	0.0575	0.0418
	QAVaR	0.1390	0.0990	0.0737
	RAVaR	0.0992	0.0674	0.0498
$t(3)$	QVaR	0.2420	0.1758	0.1277
	RVaR	0.1785	0.1290	0.0942
	QAVaR	0.5385	0.4255	0.3207
	RAVaR	0.4517	0.3232	0.2085
$CN(1,9)$	QVaR	0.4322	0.3006	0.2180
	RVaR	0.2928	0.1955	0.1447
	QAVaR	0.5471	0.3844	0.2658
	RAVaR	0.3373	0.2311	0.1636
$LN(0,1)$	QVaR	0.5814	0.3905	0.2959
	RVaR	0.4095	0.2670	0.1975
	QAVaR	1.1986	0.8957	0.7275
	RAVaR	0.9503	0.6432	0.4754

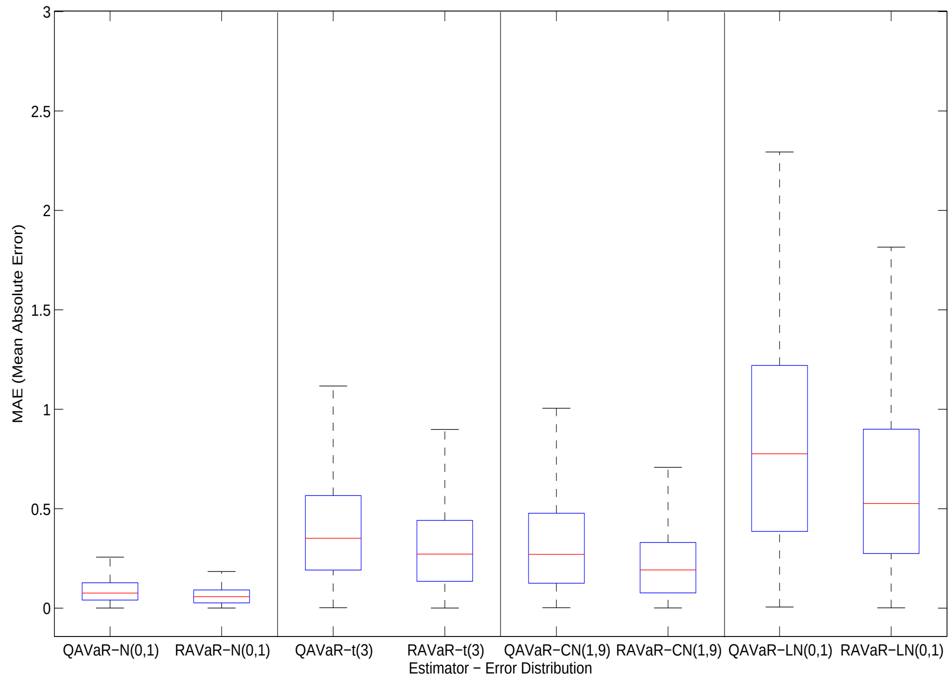
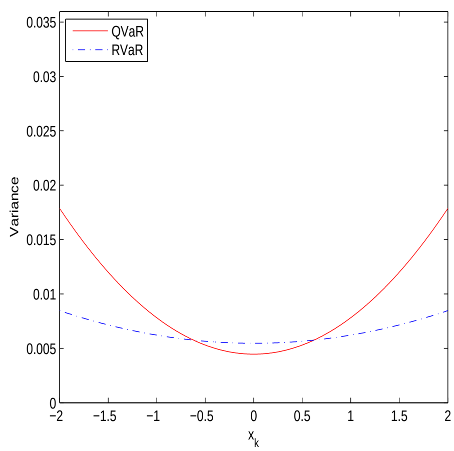


Figure 10: MAE for conditional AVaR given $x = 1.006$ under different error distributions ($\alpha = 0.95$, $N = 1000$)

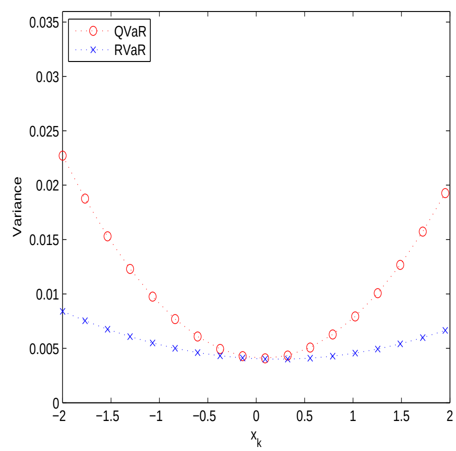
asymptotic variances are to provide a good approximation to the empirical ones for both estimators.

Figure 12 illustrates asymptotic and empirical variances of both estimators for AVaR. Insights obtained from the results are similar to the VaR case. However, Figure 12-(c) indicates that empirical variances of QAVaR are larger than asymptotic variances, especially when x_k is far from the mean. For this case, asymptotic efficiency of QAVaR may not very informative on its behavior in finite sample. Results are similar for other error distributions except $t(3)$. When the error follows $t(3)$, asymptotic (empirical) variances of QAVaR are smaller than that of RAVaR except when x_k is close to the boundary (as shown in Figure 13).

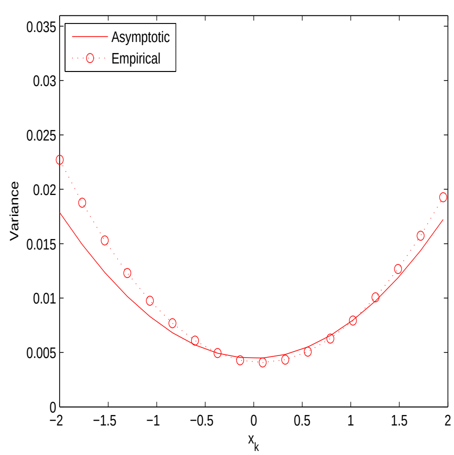
To further investigate the finite sample efficiencies and robustness of both estimators compared to the asymptotic ones, we provide empirical coverage probabilities (CP) of a two-sided 95% (nominal) confidence interval (CI) in Table 9 (difference



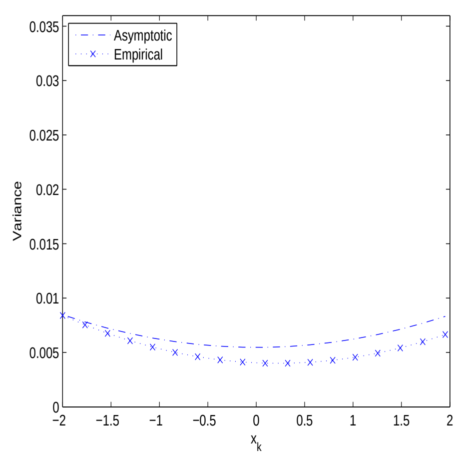
(a) Asymptotic variance



(b) Empirical variance

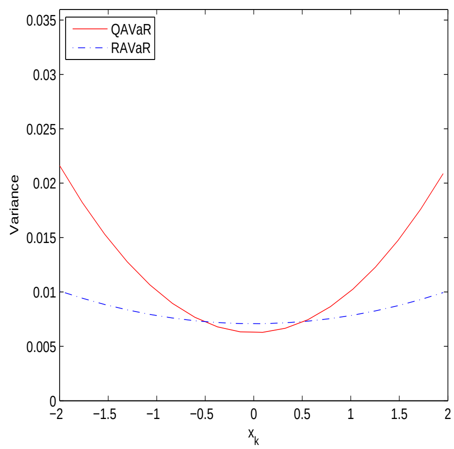


(c) QVaR: Asymptotic vs. Empirical

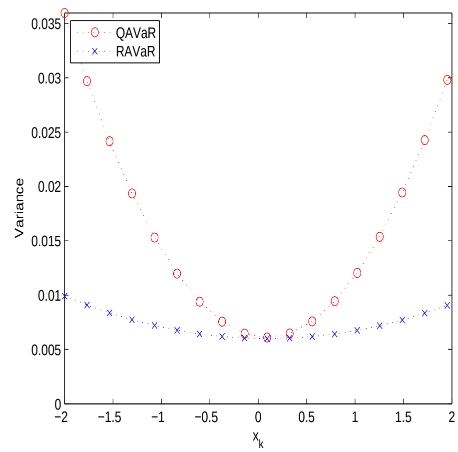


(d) RVaR: Asymptotic vs. Empirical

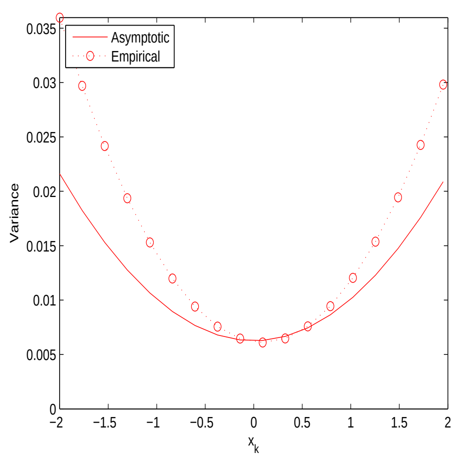
Figure 11: Conditional VaR: asymptotic and empirical variance (Error$\sim N(0,1)$, $\alpha = 0.95$, $N = 1000$, $R = 500$)



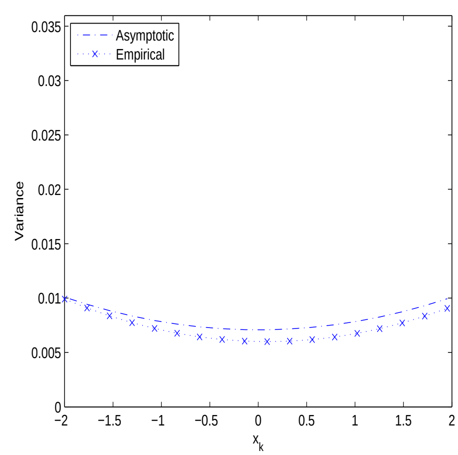
(a) Asymptotic variance



(b) Empirical variance

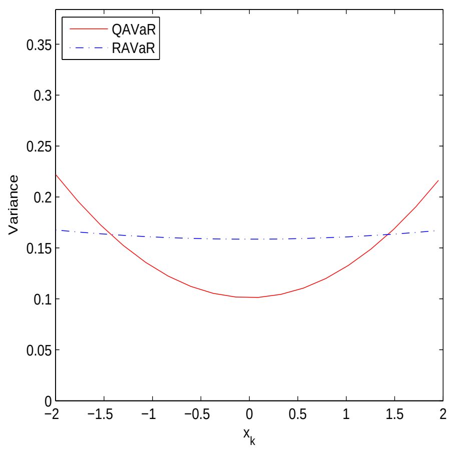


(c) QAVaR: Asymptotic vs. Empirical

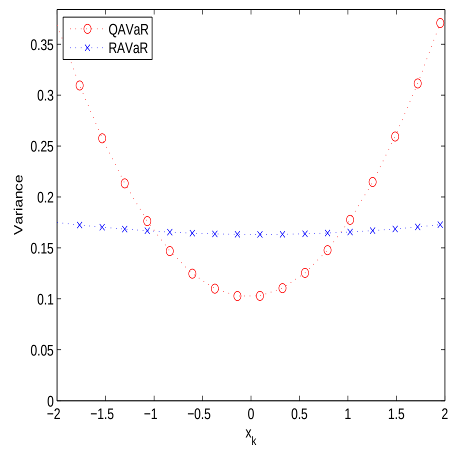


(d) RAVaR: Asymptotic vs. Empirical

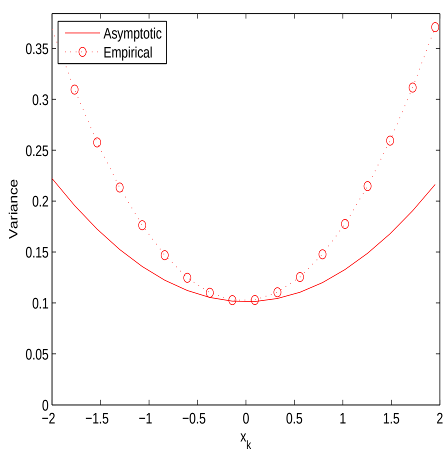
Figure 12: Conditional AVaR: asymptotic and empirical variance (Error$\sim N(0,1)$, $\alpha = 0.95$, $N = 1000$, $R = 500$)



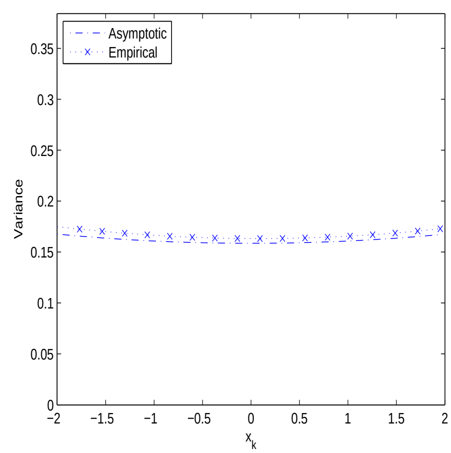
(a) Asymptotic variance



(b) Empirical variance



(c) QAVaR: Asymptotic vs. Empirical



(d) RAVaR: Asymptotic vs. Empirical

Figure 13: Conditional AVaR: asymptotic and empirical variance (Error$\sim t(3)$, $\alpha = 0.95$, $N = 1000$, $R = 500$)

Table 9: Coverage probability with $\alpha = 0.95, N = 1000$ (averaged over all test points)

Error	QVaR	RVaR	QAVaR	RAVaR
$N(0,1)$	0.9167	0.9551	0.8442	0.9552
	(0.0333)	(-0.0051)	(0.1058)	(-0.0052)
$t(3)$	0.9044	0.9269	0.7088	0.9080
	(0.0456)	(0.0231)	(0.2412)	(0.0420)
$CN(1,9)$	0.9262	0.9428	0.8824	0.9548
	(0.0238)	(0.0072)	(0.0676)	(-0.0048)
$LN(0,1)$	0.9185	0.9276	0.6930	0.9185
	(0.0315)	(0.0224)	(0.2570)	(0.0315)

between CP and 0.95 is given in parentheses). For each replication, the empirical confidence interval is calculated from the sample version of asymptotic variance (when applied to the values of an observed sample of a given size). Then, for given x_k, the proportion of the 500 replications where the obtained confidence interval contains the true (theoretical) value is calculated, and these proportions are averaged across all test points. For $N(0,1)$ and $CN(1,9)$ error distributions, the resulting CP of RVaR (RAVaR) is very close to 0.95 while empirical CI for QVaR (QAVaR) under-covers (resulting CP is smaller than 0.95). For $t(3)$ and $LN(0,1)$ error distributions, CP of RVaR (RAVaR) drops, yet maintains somewhat adequate CP which is a lot better than CP of QVaR (QAVaR). CI of QAVaR under-covers seriously (resulting CP is about 0.7) and this indicates QAVaR procedure may be very unstable and needs rather wider CI than other estimators to overcome its sensitivity. Note that RVaR (RAVaR) is more conservative than QVaR (QAVaR) regardless of the error distributions.

We could draw similar conclusions for other sample sizes and α values. That is, RVaR (RAVaR) performs better and provides stable results than QVaR (QAVaR) under different error distributions.

In addition, we estimate another law invariant risk measure given in (2.2.55) with $c = 0.7$ using different procedures (mixed quantile based and residual based methods). Quantile based estimator is referred to as "QRM" and LSR estimator is referred to as "RRM" for this risk measure. As before, we compare these estimators under different error distributions. Table 10 presents MAE (averaged over all test points) and variance of MAE (in parenthesis) across 500 replications of estimates. Similar to the cases of value-at-risk and average value-at-risk measures, RRM works better than QRM. That is, MAE and the variance of MAE computed for RRM are smaller. These results indicate that LSR estimators perform better than their mixed quantile counterparts for different risk measures.

Table 10: MAE for different error distributions $\alpha = 0.95, N = 1000$ of the risk measure (2.2.55)

Error	QRM	RRM
$N(0,1)$	0.0661	0.0404
	(0.0029)	(0.0009)
$t(3)$	0.2045	0.1240
	(0.0246)	(0.0106)
$CN(1,9)$	0.2158	0.1439
	(0.0298)	(0.0096)
$LN(0,1)$	0.9971	1.1442
	(0.1832)	(0.0944)

2.5 *Illustrative Empirical Examples*

In this section, we demonstrate considered methods to estimate conditional VaR and AVaR with real data; different financial asset classes. Let us first present an example of Credit Default Swap (CDS). CDS is the most popular credit derivative in the rapidly growing credit markets (see FitchRatings (2006) for a detailed survey of the credit derivatives market). CDS contract provides insurance against a default by a particular company, a pool of companies, or sovereign entity. The rate of payments made per year by the buyer is known as the CDS spread (in basis points). We focus on the risk of CDS trading (long or short position) rather than on the use of a CDS to hedge credit risk. The CDS dataset obtained from Bloomberg consists of 1006 daily observations from January 2007 to January 2011. Let the dependent variable Y be daily percent change, $(Y(t+1) - Y(t))/Y(t) * 100$, of Bank of America Corp (NYSE:BAC) 5-year CDS spread, explanatory variables X_1 be daily return of BAC stock price, and X_2 be daily percent change of generic 5-year investment grade CDX spread (CDX.IG). We use the term "percent change" rather than return because the return of CDS contract is not same as the return of CDS spread (e.g., see O'Kane and Turnbull (2003) for an overview of CDS valuation models). Residuals obtained from this dataset are heavy-tailed distributed (similar to Figure 8-(b)).

Figure 14 shows estimated conditional VaR (RVaR) of BAC CDS spread percent change (result of QVaR is similar). Since one can take either short or long position, we present both tail risk with all values of α which ranges from 0.01 to 0.99; $\alpha < 0.5$ corresponds to the left tail (short position) and right tail (long position), otherwise. It is clear that RVaR of certain dates are much higher (lower) than normal level due to the different daily economic conditions reflected by BAC stock price and CDX spread. This indicates the specific (daily) economic conditions should be taken account for the accurate estimation of risk, and therefore emphasize the importance of conditional risk measures. Note that given a specific date, estimated RVaR curve along the different α values is asymmetric since the distribution of CDS spread percent change is not symmetric.

To compare the prediction performance of both estimators, we forecast 603 one-day-ahead (tomorrow's) VaR (AVaR) given the current (today's) value of explanatory

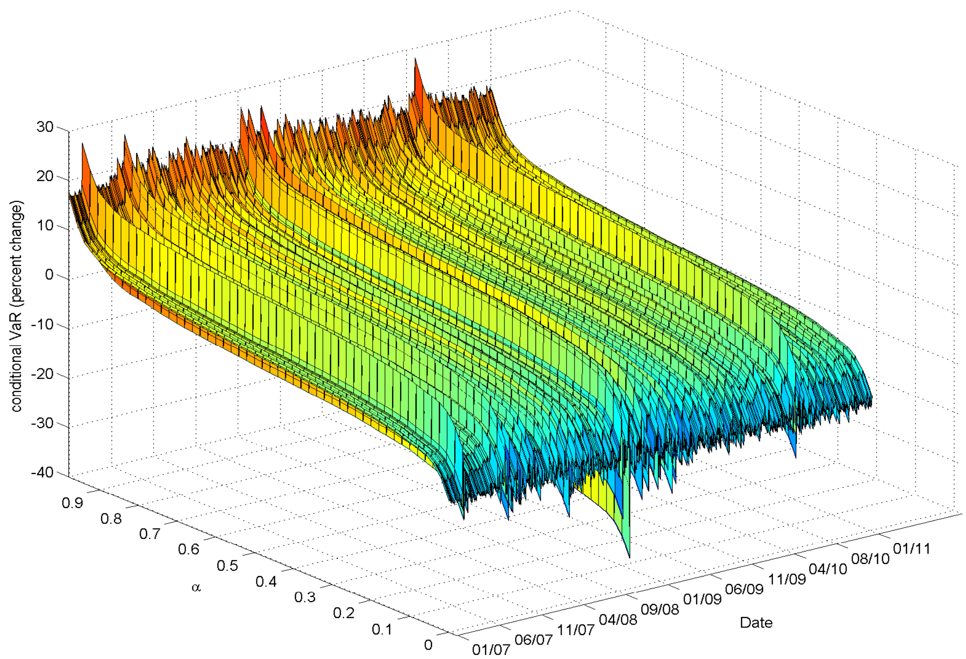


Figure 14: Estimated conditional VaR (RVaR) for BAC CDS spread percent change for $\alpha = 0.01, \ldots, 0.99$

variables using a rolling window of the previous 403 days. Figure 15 presents forecasting results of QVaR and RVaR with $\alpha = 0.05$ on 603 out-of-sample. Both estimators show similar behaviors, but RVaR seems little more stable. Following ideas in McNeil and Frey (2000) and Leorato et al. (2010), "violation event" is said to occur whenever observed CDS spread percent change falls below the predicted VaR (we can find a few violation events from Figure 15). Also, the forecast error of AVaR is defined as the difference between the observed CDS spread percent change and the predicted AVaR under the violation event. By definition, the violation event probability should be close to α and the forecast error should be close to zero. Table 11 presents the prediction performance (violation event probability for VaR, mean of forecast error for AVaR in parenthesis) of both estimators for $\alpha = 0.01$ and 0.05. In-sample statistics show that both estimators fit the data well; the violation event probabilities are very

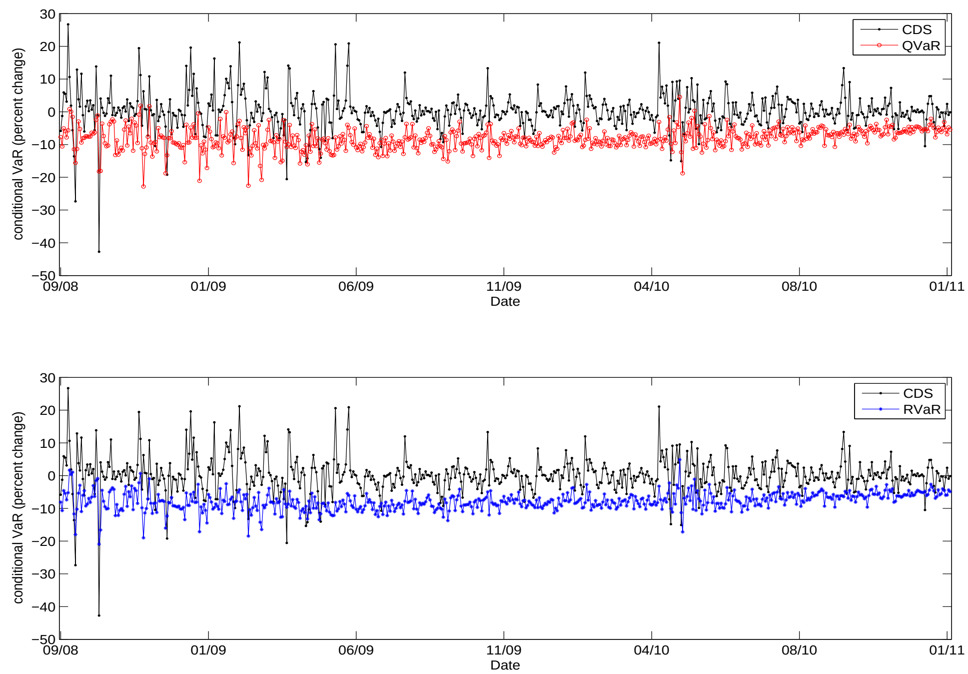


Figure 15: Risk prediction of BAC CDS: QVaR and RVaR (α=0.05)

close to α and forecast errors are very small. Out-of-sample performances of both estimators are very similar for $\alpha = 0.01$, even though the forecast errors increase a little compared to in-sample cases. For $\alpha = 0.05$, RVaR (RAVaR) seems perform better; event probabilities are closer to 0.05 and forecast errors are smaller.

Next, we apply considered methods to one of the US equities; International Business Machines Corp (NYSE). The dataset contains 1722 daily observation from December 2005 to December 2010. Let the dependent variable Y be the daily log return, 100*log(Y(t+1)/Y(t)), of IBM stock price, explanatory variables X_1 be the log return of S&P 500 index, and X_2 be the lagged log return. Similar to CDS example, we forecast 638 one-day-ahead (tomorrow's) VaR (AVaR) given the current (today's) value of explanatory variables using a rolling window of the previous 639 days. Residuals obtained from this dataset are heavy-tailed distributed. Table 12 compares the

Table 11: Risk prediction performance of BAC CDS

In-sample	α	Event(%)	Mean
QVaR(QAVaR)	0.01	0.9950	(0.1965)
RVaR(RAVaR)	0.01	0.9950	(-0.8630)
QVaR(QAVaR)	0.05	4.9751	(0.2287)
RVaR(RAVaR)	0.05	4.9751	(-0.0269)
Out-of-sample	α	Event(%)	Mean
QVaR(QAVaR)	0.01	0.8292	(1.4546)
RVaR(RAVaR)	0.01	0.8292	(1.1052)
QVaR(QAVaR)	0.05	3.6484	(1.3740)
RVaR(RAVaR)	0.05	4.4776	(-0.3722)

risk prediction performance of IBM stock return. Both estimators perform well for in-sample prediction. For out-of-sample prediction, both estimators behave similarly for $\alpha = 0.05$, but violation event probability is larger than 0.05. For $\alpha = 0.01$, RVaR (RAVaR) seems a bit better, but event probability exceeds 0.01. We provide the additional information of estimated regression coefficients and confidence intervals (upper and lower) for the empirical examples in Appendix F.

Finally, we illustrate how crude oil price had impacted the US airlines' risk as we mentioned in Section 2.1. Crude oil prices had continued to rise since May 2007 and peaked all time high in July 2008, right before the brink of the US financial system collapse. We compare the movement of estimated VaR for three airline stocks given crude oil price change; Delta Airlines, Inc (NYSE:DAL), American Airlines, Inc (NYSE:AMR), and Southwest Airlines Co (NYSE:LUV). Figure 16 depicts RVaR movement with $\alpha = 0.05$ from May 2007 to July 2008 (QVaR shows similar patterns). For easy comparison, we standardize all units relative to the starting date. As we can see, crude oil price had jumped 150% during this time span. On the other hand, RVaR of LUV increased about 15% while that of AMR increased 120% and that of DAL increased 90% (in magnitude). In fact, different airlines have different strategies

Table 12: Risk prediction performance of IBM stock

In-sample	α	Event(%)	Mean
QVaR(QAVaR)	0.01	1.0180	(-0.1305)
RVaR(RAVaR)	0.01	0.9397	(-0.3481)
QVaR(QAVaR)	0.05	5.0117	(0.0468)
RVaR(RAVaR)	0.05	4.9334	(-0.0225)
Out-of-sample	α	**Event(%)**	**Mean**
QVaR(QAVaR)	0.01	2.3511	(0.6171)
RVaR(RAVaR)	0.01	1.8809	(0.5023)
QVaR(QAVaR)	0.05	6.7398	(0.4787)
RVaR(RAVaR)	0.05	6.1129	(0.4778)

to hedge the risk on oil price fluctuations and this in turn affects the risk of airlines' stock movement. For example, Southwest Airlines is well known for hedging crude oil prices aggressively. On the other hand, Delta Airlines does little hedge against crude oil price, but operates a lot of international flights. American Airlines does not have strong hedging against crude oil price either, and operates less international flights than Delta Airlines. Our estimation results confirm the firm specific risk regarding crude oil price fluctuations.

2.6 Conclusions

Value-at-Risk and Average Value-at-Risk are widely used measures of financial risk. In order to accurately estimate risk measures, taking into account the specific economic conditions, we considered two estimation procedures for conditional risk measures; one is based on residual analysis of the standard least squares method (LSR estimator) and the other is based on mixed M-estimators (mixed quantile estimator). Large sample statistical inferences of both estimators are derived and compared. In addition, finite sample properties of both estimators are investigated and compared as well. Monte Carlo simulation results, under different error distributions, indicate

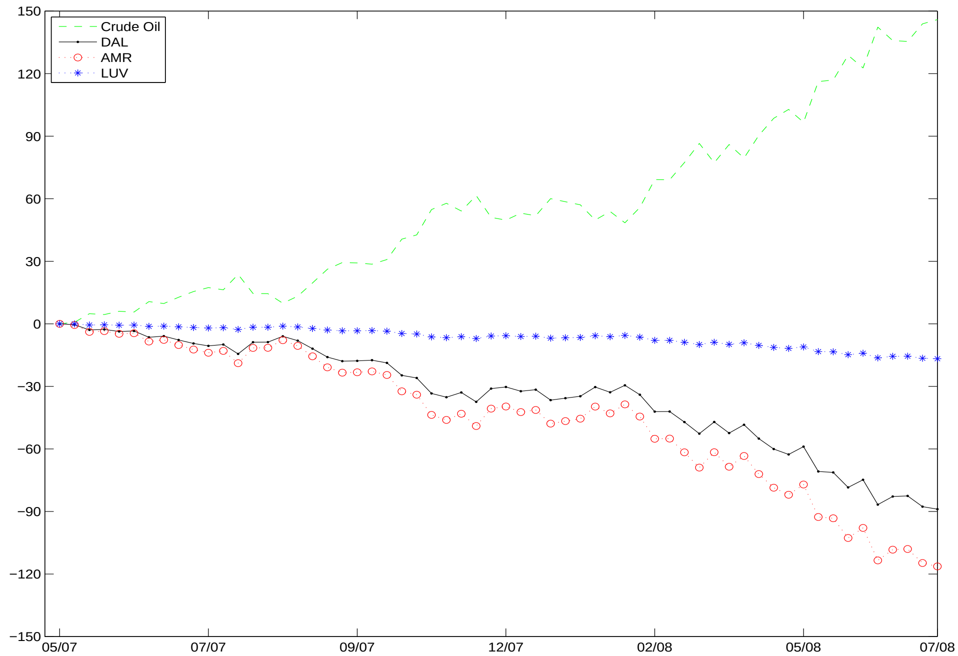


Figure 16: Airline equities: RVaR conditional on crude oil price (α=0.05)

that the LSR estimators perform better than their (mixed) quantiles counterparts. In general, MAE and asymptotic/empirical variance of the LSR estimators are smaller than that of quantile based estimators. We also observe that asymptotic variance of estimators approximates the finite sample efficiencies well for reasonable sample sizes used in practice. However, we may need more samples to guarantee an acceptable efficiency of the quantile based estimator for Average Value-at-risk compared to other estimators. Prediction performances on the real data example suggest similar conclusions. In fact, residual based estimators can be calculated easily and therefore the LSR method could be implemented efficiently in practice. Moreover, LSR method can be easily applied to the general class of law invariant risk measures. In this study, we assume a static model with independent error distributions. Extension of considered estimation procedures incorporating different aspects of (dynamic) time series models could be an interesting topic for the further study.

APPENDIX A

DERIVATION OF RESULTS FOR TWO-RESOURCE MODEL

A.0.1 No Alliance

First consider the case in which $b_{\min} \geq \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > 0$ (it is shown later for which input parameter values this condition holds). In this case the profit function of seller i is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) \quad = \quad \tilde{y}_i \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\right]$$

Then the best response function of seller i is given by

$$B_i(\tilde{y}_{-i}) \quad = \quad \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_{-i}}{2\tilde{\beta}}$$

Solving the system

$$\tilde{y}_i \quad = \quad \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_{-i}}{2\tilde{\beta}}$$

for $i = \pm 1$, the equilibrium $(\tilde{y}_{-1}^*, \tilde{y}_1^*)$ is obtained, where

$$\tilde{y}_i^* \quad = \quad \frac{\tilde{\alpha}}{3\tilde{\beta}} \quad > \quad 0$$

The demand at the equilibrium prices $(\tilde{y}_{-1}^*, \tilde{y}_1^*)$ is equal to

$$\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1}^* + \tilde{y}_1^*) \quad = \quad \frac{\tilde{\alpha}}{3} \quad > \quad 0 \tag{A.0.78}$$

Therefore, if $b_{\min} \geq \tilde{\alpha}/3$, then the equilibrium prices are given by (1.2.2), the equilibrium demand is given by (1.2.3), the resulting profit of seller i is given by (1.2.4), and thus the total profit of both sellers together is given by (1.2.5) and the consumer surplus is given by (1.2.6).

Next, consider the case in which $b_{\min} \leq \tilde{\alpha}/3$. Note that in this case $\tilde{\alpha} \geq 3b_{\min} > b_{\min}$.

Case (1): First, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$. In Figure 17, this corresponds to (a). Then $\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > b_{\min} > 0$, and thus the profit of seller i is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) \quad = \quad \tilde{y}_i b_{\min}$$

Thus, if $\tilde{y}_{-1} + \tilde{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, then the profit of seller i is increasing in $\tilde{y}_i$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 \geq \tilde{\alpha}/\tilde{\beta}$. In Figure 17, this corresponds to (b). Then the demand and profit of each seller is zero.

Case (3.1): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$. In Figure 17, this corresponds to (c). Then $0 < \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) < b_{\min}$, and thus the profit of seller i is given by

$$\tilde{g}_i(\tilde{y}_i, \tilde{y}_{-i}) \quad = \quad \tilde{y}_i \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-i} + \tilde{y}_i)\right]$$

Note that

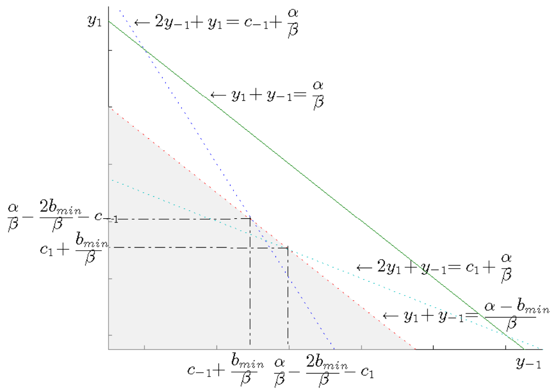
$$\partial \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})/\partial \tilde{y}_1 \quad = \quad \tilde{\alpha} - \tilde{\beta}\tilde{y}_{-1} - 2\tilde{\beta}\tilde{y}_1 \quad < \quad 0$$

Thus, if $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$, then the profit of seller 1 is decreasing in $\tilde{y}_1$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

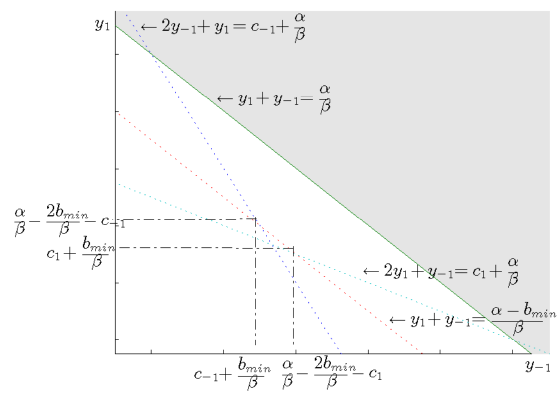
Case (3.2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{\alpha}/\tilde{\beta} > \tilde{y}_{-1} + \tilde{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $2\tilde{y}_{-1} + \tilde{y}_1 > \tilde{\alpha}/\tilde{\beta}$. In Figure 17, this corresponds to (d). It follows similarly to Case 3.1 that the profit of seller -1 is decreasing in $\tilde{y}_{-1}$, and hence such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (4.1): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1} + \tilde{y}_1 = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $0 \leq \tilde{y}_{-1} < b_{\min}/\tilde{\beta}$. Note that $\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) = b_{\min}$, and thus the corresponding profit of seller -1 is given by

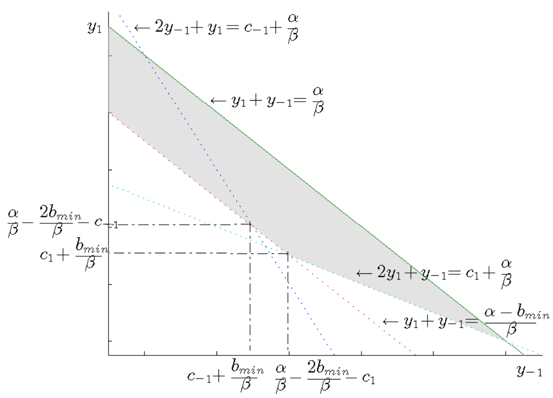
$$\tilde{g}_{-1}(\tilde{y}_{-1}, \tilde{y}_1) \quad = \quad \tilde{y}_{-1} b_{\min}$$



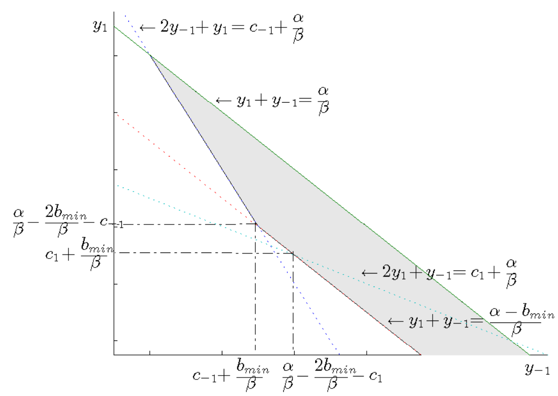
(a) Case 1: $\tilde{y}_{-1}+\tilde{y}_1<(\tilde{\alpha}-b_{\min})/\tilde{\beta}$.



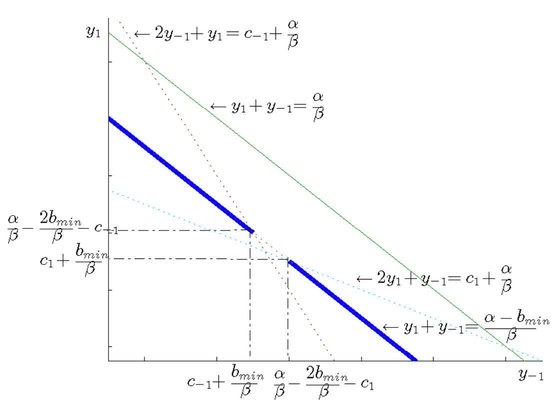
(b) Case 2: $\tilde{y}_{-1}+\tilde{y}_1\geq\tilde{\alpha}/\tilde{\beta}$.



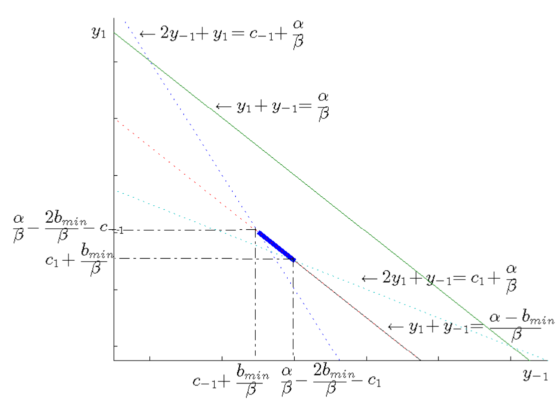
(c) Case 3.1: $\tilde{\alpha}/\tilde{\beta}>\tilde{y}_{-1}+\tilde{y}_1>(\tilde{\alpha}-b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1}+2\tilde{y}_1>\tilde{\alpha}/\tilde{\beta}$.



(d) Case 3.2: $\tilde{\alpha}/\tilde{\beta}>\tilde{y}_{-1}+\tilde{y}_1>(\tilde{\alpha}-b_{\min})/\tilde{\beta}$ and $2\tilde{y}_{-1}+\tilde{y}_1>\tilde{\alpha}/\tilde{\beta}$.



(e) Case 4: $\tilde{y}_{-1}+\tilde{y}_1=(\tilde{\alpha}-b_{\min})/\tilde{\beta}$ and ($\tilde{y}_{-1}<b_{\min}/\tilde{\beta}$ or $\tilde{y}_1<b_{\min}/\tilde{\beta}$).



(f) Case 5: The line segment between ($b_{\min}/\tilde{\beta}$, $\tilde{\alpha}/\tilde{\beta}-2b_{\min}/\tilde{\beta}$) and ($\tilde{\alpha}/\tilde{\beta}-2b_{\min}/\tilde{\beta}$, $b_{\min}/\tilde{\beta}$).

Figure 17: Different regions of the pair of prices $(\tilde{y}_{-1},\tilde{y}_1)$ corresponding to different cases.

Next, consider $\hat{y}_{-1} := \left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1\right)/2$. First, note that

$$\begin{aligned}
\tilde{y}_1 \quad \leq \quad \tilde{y}_{-1} + \tilde{y}_1 \quad = \quad \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} \quad &< \quad \frac{\tilde{\alpha}}{\tilde{\beta}} \\
\Rightarrow \quad \frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_1}{2} \quad &> \quad 0 \\
\Leftrightarrow \quad \tilde{\alpha} - \tilde{\beta}\left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2} + \tilde{y}_1\right) \quad &> \quad 0 \\
\Leftrightarrow \quad \tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right) \quad &> \quad 0
\end{aligned}$$

Also, note that

$$\begin{aligned}
\tilde{y}_{-1} \quad &< \quad b_{\min}/\tilde{\beta} \\
\Leftrightarrow \quad \tilde{y}_{-1} + \left(\tilde{\alpha} - b_{\min}\right)/\tilde{\beta} \quad &< \quad \tilde{\alpha}/\tilde{\beta} \\
\Leftrightarrow \quad 2\tilde{y}_{-1} + \tilde{y}_1 \quad &< \quad \tilde{\alpha}/\tilde{\beta} \\
\Leftrightarrow \quad \tilde{y}_{-1} \quad &< \quad \frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2} \quad = \quad \hat{y}_{-1}
\end{aligned}$$

and thus $\tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right) < \tilde{\alpha} - \tilde{\beta}\left(\tilde{y}_{-1} + \tilde{y}_1\right) = b_{\min}$. Thus the corresponding profit of seller -1 is given by

$$\tilde{g}_{-1}(\hat{y}_{-1}, \tilde{y}_1) \quad = \quad \hat{y}_{-1}\left[\tilde{\alpha} - \tilde{\beta}\left(\hat{y}_{-1} + \tilde{y}_1\right)\right]$$

Next, note that

$$
\begin{aligned}
& \tilde{y}_{-1} \quad < \quad b_{\min}/\tilde{\beta} \\
\Rightarrow \quad & \left(b_{\min} - \tilde{\beta}\tilde{y}_{-1}\right)^2 \quad > \quad 0 \\
\Leftrightarrow \quad & b_{\min}^2 + 2b_{\min}\tilde{\beta}\tilde{y}_{-1} + \tilde{\beta}^2\tilde{y}_{-1}^2 \quad > \quad 4b_{\min}\tilde{\beta}\tilde{y}_{-1} \\
\Leftrightarrow \quad & \left(b_{\min} + \tilde{\beta}\tilde{y}_{-1}\right)^2 \quad > \quad 4\tilde{\beta}\tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \left(\frac{b_{\min}/\tilde{\beta} + \tilde{y}_{-1}}{2}\right)\left(\frac{b_{\min} + \tilde{\beta}\tilde{y}_{-1}}{2}\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \left(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta} - \tilde{y}_{-1}\right)}{2}\right)\left(\frac{\tilde{\alpha} - \tilde{\beta}\left(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta} - \tilde{y}_{-1}\right)}{2}\right) \\
& \quad > \quad \tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2}\right)\left(\frac{\tilde{\alpha} - \tilde{\beta}\tilde{y}_1}{2}\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \left(\frac{\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1}{2}\right)\left(\tilde{\alpha} - \frac{\tilde{\beta}\left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_1\right)}{2} - \tilde{\beta}\tilde{y}_1\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \hat{y}_{-1}\left(\tilde{\alpha} - \tilde{\beta}\hat{y}_{-1} - \tilde{\beta}\tilde{y}_1\right) \quad > \quad \tilde{y}_{-1}b_{\min} \\
\Leftrightarrow \quad & \tilde{g}_{-1}(\hat{y}_{-1}, \tilde{y}_1) \quad > \quad \tilde{g}_{-1}(\tilde{y}_{-1}, \tilde{y}_1)
\end{aligned}
$$

Thus such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium.

Case (4.2): Next, consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ such that $\tilde{y}_{-1}+\tilde{y}_1 = (\tilde{\alpha}-b_{\min})/\tilde{\beta}$ and $0 \leq \tilde{y}_1 < b_{\min}/\tilde{\beta}$. Consider $\hat{y}_1 := \left(\tilde{\alpha}/\tilde{\beta} - \tilde{y}_{-1}\right)/2$. It follows similarly to Case 4.1 that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) > \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$ and thus such a pair of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ cannot be an equilibrium. In Figure 17, Case (4.1) and Case (4.2) correspond to (e).

Case (5): The only remaining pairs of prices to check are pairs $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\ \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta})$ and $(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\ b_{\min}/\tilde{\beta})$. In Figure 17, this corresponds to the line segment on (f). Consider any pair of prices $(\tilde{y}_{-1}, \tilde{y}_1) = (1-\gamma)(b_{\min}/\tilde{\beta},\ \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta}) + \gamma(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\ b_{\min}/\tilde{\beta})$ for $\gamma \in [0,1]$. It follows from $b_{\min} \leq \tilde{\alpha}/3$ that $0 < b_{\min}/\tilde{\beta} \leq \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta}$, and thus $\tilde{y}_i > 0$. Note that $\tilde{y}_{-1} + \tilde{y}_1 = (1-\gamma)(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta}) + \gamma(\tilde{\alpha}/\tilde{\beta} - b_{\min}/\tilde{\beta}) = (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, that $\tilde{y}_{-1} + 2\tilde{y}_1 =$

$(1-\gamma)(2\tilde{\alpha}/\tilde{\beta} - 3b_{\min}/\tilde{\beta}) + \gamma\tilde{\alpha}/\tilde{\beta} \geq \tilde{\alpha}/\tilde{\beta}$, where the inequality follows from $b_{\min} \leq \tilde{\alpha}/3$, and similarly $2\tilde{y}_{-1} + \tilde{y}_1 \geq \tilde{\alpha}/\tilde{\beta}$. Then, for any $\hat{y}_1 < \tilde{y}_1$, it holds that $\tilde{y}_{-1} + \hat{y}_1 < (\tilde{\alpha} - b_{\min})/\tilde{\beta}$, and thus it follows from Case (a) that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) < \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$. Also, for any $\hat{y}_1 > \tilde{y}_1$, it holds that $\tilde{y}_{-1} + \hat{y}_1 > (\tilde{\alpha} - b_{\min})/\tilde{\beta}$ and $\tilde{y}_{-1} + 2\hat{y}_1 > \tilde{\alpha}/\tilde{\beta}$, and thus it follows from Case (c) that $\tilde{g}_1(\hat{y}_1, \tilde{y}_{-1}) < \tilde{g}_1(\tilde{y}_1, \tilde{y}_{-1})$. Hence, given $\tilde{y}_{-1}$, $\tilde{y}_1$ is the best response for seller 1. Similarly, given $\tilde{y}_1$, $\tilde{y}_{-1}$ is the best response for seller -1.

Therefore, if $b_{\min} \leq \tilde{\alpha}/3$, then all pairs of prices $(\tilde{y}_{-1}, \tilde{y}_1)$ on the line segment between $(b_{\min}/\tilde{\beta},\ \tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta})$ and $(\tilde{\alpha}/\tilde{\beta} - 2b_{\min}/\tilde{\beta},\ b_{\min}/\tilde{\beta})$ are equilibria. For all of these equilibrium prices total price is equal to $(\tilde{\alpha} - b_{\min})/\tilde{\beta}$, the demand is equal to $b_{\min}$, the resulting profit of seller i is equal to $\tilde{y}_i b_{\min}$, and thus the total profit of both sellers together is given by (1.2.7) and the consumer surplus is given by (1.2.8).

A.0.2 Perfect Coordination

In this section we determine the maximum achievable total profit of the two sellers together, that is, the total profit if the sellers would perfectly coordinate pricing.

The total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \quad := \quad (\tilde{y}_{-1} + \tilde{y}_1) \min\{b_{\min},\ \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}\}$$

First consider the case in which $b_{\min} \geq \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1) > 0$. In this case the total profit of the two sellers is given by

$$\tilde{g}(\tilde{y}_{-1}, \tilde{y}_1) \quad := \quad (\tilde{y}_{-1} + \tilde{y}_1) \left[\tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\right]$$

The optimal total price $\bar{y}_{-1} + \bar{y}_1$ that maximizes the total profit is given by

$$\bar{y}_{-1} + \bar{y}_1 \quad = \quad \frac{\tilde{\alpha}}{2\tilde{\beta}} \quad > \quad 0$$

The demand at the optimal total price $\bar{y}_{-1} + \bar{y}_1$ is equal to

$$\tilde{\alpha} - \tilde{\beta}(\bar{y}_{-1} + \bar{y}_1) \quad = \quad \frac{\tilde{\alpha}}{2} \quad > \quad \frac{\tilde{\alpha}}{3} \quad = \quad \tilde{\alpha} - \tilde{\beta}(\tilde{y}^*_{-1} + \tilde{y}^*_1) \qquad \text{(A.0.79)}$$

Therefore, if $b_{\min} \geq \tilde{\alpha}/2$, then the optimal total price is given by (1.2.9), the corresponding demand is given by (1.2.10), the total profit of both sellers together is given by (1.2.11), and the consumer surplus is given by (1.2.12).

Next, consider the case in which $b_{\min} \leq \tilde{\alpha}/2$. In this case the optimal total price is given by

$$\tilde{y}_{-1} + \tilde{y}_1 \quad = \quad \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} \quad > \quad 0$$

with corresponding demand equal to $b_{\min}$. The total profit of both sellers together is equal to

$$(\tilde{y}_{-1} + \tilde{y}_1)\, b_{\min} \quad = \quad \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}} b_{\min}$$

and the consumer surplus is equal to

$$\frac{1}{2}\left[\frac{\tilde{\alpha}}{\tilde{\beta}} - \frac{\tilde{\alpha} - b_{\min}}{\tilde{\beta}}\right] b_{\min} \quad = \quad \frac{b_{\min}^2}{2\tilde{\beta}}$$

A.0.3 Resource Exchange Alliance

For given values of b_{-1} and b_1, the feasible set S_1 of two-resource products that can be sold by the two sellers is given by $S_1 := \{(q_{-1}(x), q_1(x)) \,:\, x_i \in [0, b_i],\, i = \pm 1\}$. Next we show that this set S_1 is equal to $S_2 := \{(q_{-1}, q_1) \in [0, b_{\min}]^2 \,:\, q_{-1} + q_1 \leq b_{\min}\}$. First, consider any $(q_{-1}(x), q_1(x)) \in S_1$ with corresponding $(x_{-1}, x_1) \in [0, b_{-1}] \times [0, b_1]$. Without loss of generality, suppose that $b_{-1} = b_{\min}$. Then $q_{-1}(x) + q_1(x) = \min\{b_{-1} - x_{-1},\, x_1\} + \min\{b_1 - x_1,\, x_{-1}\} \leq b_{-1} - x_{-1} + x_{-1} = b_{-1} = b_{\min}$, and thus $(q_{-1}(x), q_1(x)) \in S_2$. Next, consider any $(q_{-1}, q_1) \in S_2$. Choose $x_i = q_{-i}$ for $i = \pm 1$. Note that $x_i \in [0, b_i]$ since $q_{-i} \in [0, b_{\min}]$. Also, $x_i = q_{-i} \leq b_{\min} - q_i = b_{\min} - x_{-i} \leq b_{-i} - x_{-i}$, and thus $q_{-i}(x) = \min\{b_{-i} - x_{-i},\, x_i\} = x_i = q_{-i}$. Thus $(q_{-1}, q_1) \in S_1$, and hence $S_1 = S_2$. Hence, the first-stage decision variables may be considered to be the resource exchange quantities $x = (x_{-1}, x_1) \in [0, b_{-1}] \times [0, b_1]$, or equivalently the capacities $q = (q_{-1}, q_1) \in S_2$ of two-resource products after exchange.

Case 1. First consider the case in which $q_i > \alpha - \beta y_i + \gamma y_{-i} > 0$ for $i = \pm 1$ (it is considered later for which input parameter values and values of q and y this condition holds). In this case the profit function of each seller i is given by

$$g_i(y_i, y_{-i}) \quad = \quad y_i \left[\alpha - \beta y_i + \gamma y_{-i}\right]$$

Then the best response function of each seller i is given by

$$B_i(y_{-i}) \quad = \quad \frac{\alpha + \gamma y_{-i}}{2\beta}$$

Solving the system

$$y_i \quad = \quad \frac{\alpha + \gamma y_{-i}}{2\beta}$$

for $i = \pm 1$, the equilibrium (y_{-1}^*, y_1^*) is obtained, where

$$y_i^* \quad = \quad \frac{\alpha}{2\beta - \gamma} \quad > \quad 0 \tag{A.0.80}$$

Note that the equilibrium prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. The demand at the equilibrium prices (y_{-1}^*, y_1^*) is equal to

$$\alpha - \beta y_i^* + \gamma y_{-i}^* \quad = \quad \frac{\alpha\beta}{2\beta - \gamma} \quad > \quad 0 \tag{A.0.81}$$

The resulting profit of each seller is equal to

$$y_i^* \min\{q_i,\ \max\{0,\ \alpha - \beta y_i^* + \gamma y_{-i}^*\}\} \quad = \quad \frac{\alpha^2\beta}{(2\beta - \gamma)^2} \tag{A.0.82}$$

and thus the total profit of both sellers together is equal to

$$2\frac{\alpha^2\beta}{(2\beta - \gamma)^2} \tag{A.0.83}$$

Therefore, if $q_i \geq \alpha\beta/(2\beta - \gamma)$ for $i = \pm 1$, then the equilibrium prices are given by (A.0.80), the equilibrium demand is given by (A.0.81), the resulting profit of each seller is given by (A.0.82), and thus the total profit of both sellers together is given by (A.0.83). Note that $q_i \geq \alpha\beta/(2\beta - \gamma)$ for $i = \pm 1$ requires that $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$.

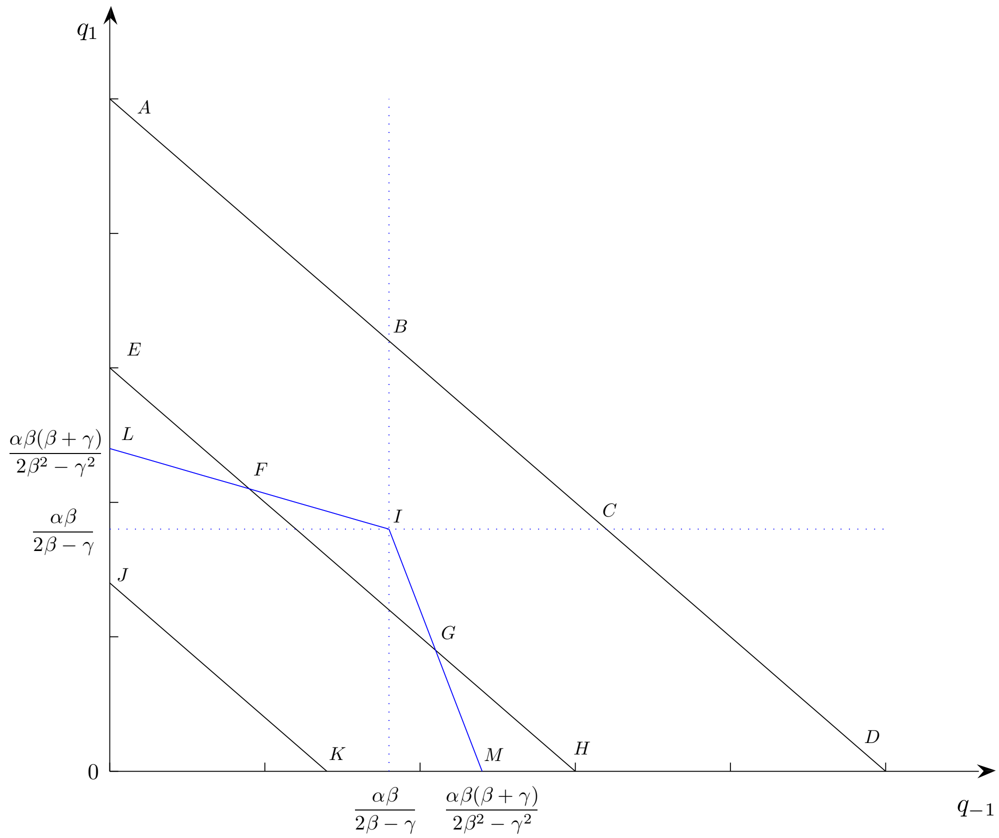


Figure 18: Different cases of capacity $b_{\min}$ for a resource exchange alliance.

Thus the results above hold if $b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$ and the resource exchange x is chosen such that $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i = \pm 1$. In Figure 18, the line $ABCD$ shows pairs (q_{-1}, q_1) such that $q_{-1} + q_1 = b_{\min} > 2\alpha\beta/(2\beta-\gamma)$, obtained with resource exchange $x = (x_{-1}, x_1)$ such that $x_i = q_{-i} = b_{\min} - q_i = b_{\min} - x_{-i} \leq b_{-i} - x_{-i}$. Thus, for the given value of $b_{\min} > 2\alpha\beta/(2\beta-\gamma)$, the set of points (q_{-1}, q_1) such that $q_i \geq \alpha\beta/(2\beta-\gamma)$ for $i = \pm 1$ and $q_{-1} + q_1 \leq b_{\min}$ corresponds to triangle BCI. All corresponding resource exchanges x lead to sales of two-resource products of $\alpha\beta/(2\beta-\gamma)$ by each seller, corresponding to point I, and provide total profit of $2\alpha^2\beta/(2\beta-\gamma)^2$.

Case 2. Next, consider the case in which $0 \leq q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i$ and $q_i > \alpha - \beta y_i + \gamma y_{-i} > 0$ (as before, it is considered later for which input parameter values and values of q and y this condition holds). In this case the profit function of seller $-i$ is given by

$$g_{-i}(y_{-i}, y_i) \;=\; y_{-i} q_{-i}$$

and the profit function of seller i is given by

$$g_i(y_i, y_{-i}) \;=\; y_i \left[\alpha - \beta y_i + \gamma y_{-i}\right]$$

Then the best response function of seller $-i$ is given by

$$B_{-i}(y_i) \;=\; \max\{y_{-i} \;:\; q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i\} \;=\; \frac{\alpha + \gamma y_i - q_{-i}}{\beta}$$

and the best response function of seller i is given by

$$B_i(y_{-i}) \;=\; \frac{\alpha + \gamma y_{-i}}{2\beta}$$

Solving the system

$$\begin{aligned} y_{-i} &= \frac{\alpha + \gamma y_i - q_{-i}}{\beta} \\ y_i &= \frac{\alpha + \gamma y_{-i}}{2\beta} \end{aligned}$$

the solution (y^*_{-1}, y^*_1) is obtained, where

$$\begin{aligned} y^*_{-i} &= \frac{2\alpha\beta + \alpha\gamma - 2\beta q_{-i}}{2\beta^2 - \gamma^2} \\ y^*_i &= \frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2} \end{aligned} \tag{A.0.84}$$

(It is checked later under what conditions $y^*_{-i}, y^*_i > 0$ and (y^*_{-i}, y^*_i) is the unique equilibrium.) The demands at the prices (y^*_{-i}, y^*_i) are equal to

$$d_{-i}(y^*_{-i}, y^*_i) \;=\; \alpha - \beta y^*_{-i} + \gamma y^*_i \;=\; q_{-i} \tag{A.0.85}$$

$$d_i(y^*_i, y^*_{-i}) \;=\; \alpha - \beta y^*_i + \gamma y^*_{-i} \;=\; \frac{\alpha\beta(\beta + \gamma) - \beta\gamma q_{-i}}{2\beta^2 - \gamma^2} \tag{A.0.86}$$

Recall that we are considering the case in which $q_{-i} \leq \alpha - \beta y_{-i} + \gamma y_i$ and $q_i > \alpha - \beta y_i + \gamma y_{-i}$. Note that $q_{-i} = \alpha - \beta y^*_{-i} + \gamma y^*_i$. Also note that $q_i > \alpha - \beta y^*_i + \gamma y^*_{-i}$ if and only if $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$. Examples of the line $q_i = \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ are given in Figure 18 by line LFI for $i = 1$ and by line MGI for $i = -1$. It can be verified that the intercept satisfies $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \in (0, 2\alpha\beta/(2\beta-\gamma))$. The slope of the lines are negative if $\gamma > 0$ and positive if $\gamma < 0$. Note that if $q_{-i} = \alpha\beta/(2\beta-\gamma)$, then $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2) = \alpha\beta/(2\beta-\gamma)$, and thus in all cases the lines go through $I = (\alpha\beta/(2\beta-\gamma),\ \alpha\beta/(2\beta-\gamma))$. In Figure 18, if $b_{\min} > 2\alpha\beta/(2\beta-\gamma)$, such as in the case in which the line $ABCD$ shows pairs (q_{-1}, q_1) such that $q_{-1}+q_1 = b_{\min}$, then the set of points (q_{-1}, q_1) such that $0 \leq q_{-1} \leq \alpha - \beta y^*_{-1} + \gamma y^*_1$, $q_1 > \alpha - \beta y^*_1 + \gamma y^*_{-1}$, and $q_{-1} + q_1 \leq b_{\min}$, corresponds to quadrilateral $ABIL$. (Note that $q_{-1} \leq \alpha\beta/(2\beta-\gamma)$, since it has already been shown that $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$ in triangle BCI.) Similarly, the set of points (q_{-1}, q_1) such that $0 \leq q_1 \leq \alpha - \beta y^*_1 + \gamma y^*_{-1}$, $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$, and $q_{-1} + q_1 \leq b_{\min}$, corresponds to quadrilateral $DCIM$ (note that $q_1 \leq \alpha\beta/(2\beta-\gamma)$). If $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) < b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$, such as in the case in which the line $EFGH$ shows pairs (q_{-1}, q_1) such that $q_{-1}+q_1 = b_{\min}$, then the set of points (q_{-1}, q_1) such that $0 \leq q_{-1} \leq \alpha - \beta y^*_{-1} + \gamma y^*_1$, $q_1 > \alpha - \beta y^*_1 + \gamma y^*_{-1}$, and $q_{-1} + q_1 \leq b_{\min}$, corresponds to triangle EFL, and the set of points (q_{-1}, q_1) such that $0 \leq q_1 \leq \alpha - \beta y^*_1 + \gamma y^*_{-1}$, $q_{-1} > \alpha - \beta y^*_{-1} + \gamma y^*_1$, and $q_{-1}+q_1 \leq b_{\min}$, corresponds to triangle HGM. It is verified in Case 3 that, if $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$, then $q_i \leq \alpha - \beta y^*_i + \gamma y^*_{-i}$ for $i = \pm 1$.

Next we verify that, if $q_{-i} \leq \alpha\beta/(2\beta-\gamma)$, then the prices y^*_{-i}, y^*_i given in (A.0.84) satisfy $y^*_{-i}, y^*_i > 0$, that is, the prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. First note that the denominator in the expressions for y^*_{-i}

and y_i^* is positive. Next consider the numerator in the expression for y_{-i}^*. Note that

$$\begin{aligned}
2\beta^2 &< 4\beta^2 - \gamma^2 = (2\beta+\gamma)(2\beta-\gamma) \\
\Leftrightarrow \quad \frac{\alpha\beta}{2\beta-\gamma} &< \frac{2\alpha\beta+\alpha\gamma}{2\beta}
\end{aligned}$$

Thus, if $q_{-i} \leq \alpha\beta/(2\beta-\gamma)$, then

$$\begin{aligned}
q_{-i} &< \frac{2\alpha\beta+\alpha\gamma}{2\beta} \\
\Leftrightarrow \quad 0 &< 2\alpha\beta+\alpha\gamma-2\beta q_{-i} \\
\Leftrightarrow \quad 0 &< \frac{2\alpha\beta+\alpha\gamma-2\beta q_{-i}}{2\beta^2-\gamma^2} = y_{-i}^*
\end{aligned}$$

Next consider the numerator in the expression for y_i^*. If $\gamma \leq 0$, then $\alpha(\beta+\gamma)-\gamma q_{-i} > 0$ (recall that $\gamma \in (-\beta, \beta)$), and thus

$$y_i^* = \frac{\alpha\beta+\alpha\gamma-\gamma q_{-i}}{2\beta^2-\gamma^2} > 0$$

Next, consider the case with $\gamma > 0$. Note that

$$\frac{\alpha\beta}{2\beta-\gamma} < \frac{\alpha\beta}{\gamma} < \frac{\alpha\beta+\alpha\gamma}{\gamma}$$

Thus, if $q_{-i} \leq \alpha\beta/(2\beta-\gamma)$, then

$$\begin{aligned}
q_{-i} &< \frac{\alpha\beta+\alpha\gamma}{\gamma} \\
\Leftrightarrow \quad 0 &< \alpha\beta+\alpha\gamma-\gamma q_{-i} \\
\Leftrightarrow \quad 0 &< \frac{\alpha\beta+\alpha\gamma-\gamma q_{-i}}{2\beta^2-\gamma^2} = y_i^*
\end{aligned}$$

Next we verify that, if $q_{-i} \leq \alpha\beta/(2\beta-\gamma)$ and $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$, then (y_{-i}^*, y_i^*) given in (A.0.84) is the unique equilibrium. First, recall that $B_i(y_{-i}) = (\alpha+\gamma y_{-i})/(2\beta)$ is the unique best response for seller i if the capacity q_i of seller i is not constraining. Note that if seller $-i$ chooses price y_{-i}^* and $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$, then the capacity q_i of seller i is not constraining, and thus y_i^* given in (A.0.84) is the unique best response for seller i to

y^*_{-i}. Next we verify that y^*_{-i} given in (A.0.84) is the unique best response for seller $-i$ to y^*_i. Given y^*_i, the profit of seller $-i$ is given by

$$
\begin{aligned}
g_{-i}(y_{-i}, y^*_i) &= y_{-i} \min\{q_{-i},\ \max\{0,\ \alpha - \beta y_{-i} + \gamma y^*_i\}\} \\
&= \begin{cases} y_{-i}q_{-i} & \text{if} \quad y_{-i} \le \frac{\alpha+\gamma y^*_i - q_{-i}}{\beta} \\ y_{-i}\left(\alpha - \beta y_{-i} + \gamma y^*_i\right) & \text{if} \quad \frac{\alpha+\gamma y^*_i - q_{-i}}{\beta} \le y_{-i} \le \frac{\alpha+\gamma y^*_i}{\beta} \\ 0 & \text{if} \quad y_{-i} \ge \frac{\alpha+\gamma y^*_i}{\beta} \end{cases}
\end{aligned}
$$

Thus $g_{-i}(y_{-i}, y^*_i)$ is a nondecreasing linear function of y_{-i} if $y_{-i} \le (\alpha + \gamma y^*_i - q_{-i})/\beta$. If $(\alpha + \gamma y^*_i - q_{-i})/\beta < y_{-i} < (\alpha + \gamma y^*_i)/\beta$, then $g_{-i}(y_{-i}, y^*_i)$ is a concave quadratic function of y_{-i}, with

$$
\begin{aligned}
g'_{-i}(y_{-i}, y^*_i) &= -2\beta y_{-i} + \alpha + \gamma y^*_i \\
&< -2\left(\alpha + \gamma y^*_i - q_{-i}\right) + \alpha + \gamma y^*_i \\
&= -\alpha - \gamma y^*_i + 2q_{-i} \\
&= -\alpha - \gamma \frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2} + 2q_{-i} \\
&= \frac{-2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i}}{2\beta^2 - \gamma^2}
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{-2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i}}{2\beta^2 - \gamma^2} &\le 0 \\
\Leftrightarrow \quad -2\alpha\beta^2 - \alpha\beta\gamma + (4\beta^2 - \gamma^2)q_{-i} &\le 0 \\
\Leftrightarrow \quad -\alpha\beta(2\beta + \gamma) + (2\beta - \gamma)(2\beta + \gamma)q_{-i} &\le 0 \\
\Leftrightarrow \quad -\alpha\beta + (2\beta - \gamma)q_{-i} &\le 0 \\
\Leftrightarrow \quad q_{-i} &\le \frac{\alpha\beta}{2\beta - \gamma}
\end{aligned}
$$

Hence, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$, then $g'_{-i}(y_{-i}, y^*_i) < 0$ for all $y_{-i} \in ((\alpha + \gamma y^*_i - q_{-i})/\beta,\ (\alpha + \gamma y^*_i)/\beta)$. Hence, the unique best response for seller $-i$ to y^*_i is $B_{-i}(y^*_i) = (\alpha + \gamma y^*_i - q_{-i})/\beta$. Therefore, if $q_{-i} \le \alpha\beta/(2\beta - \gamma)$ and $q_i > \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$, then (y^*_{-i}, y^*_i) given in (A.0.84) is the unique equilibrium.

The resulting profit of each seller is equal to

$$\begin{aligned}
g_{-i}(y_{-i}^*, y_i^*) &= y_{-i}^* q_{-i} \\
&= \frac{\alpha (2\beta + \gamma) q_{-i} - 2\beta q_{-i}^2}{2\beta^2 - \gamma^2} \\
g_i(y_i^*, y_{-i}^*) &= y_i^* \left(\alpha - \beta y_i^* + \gamma y_{-i}^*\right) \\
&= \left(\frac{\alpha\beta + \alpha\gamma - \gamma q_{-i}}{2\beta^2 - \gamma^2}\right)\left(\frac{\alpha\beta (\beta + \gamma) - \beta\gamma q_{-i}}{2\beta^2 - \gamma^2}\right) \\
&= \frac{\alpha^2 \beta (\beta + \gamma)^2 - 2\alpha\beta\gamma (\beta + \gamma) q_{-i} + \beta\gamma^2 q_{-i}^2}{(2\beta^2 - \gamma^2)^2} \qquad \text{(A.0.87)}
\end{aligned}$$

and thus the total profit of both sellers together is equal to

$$\begin{aligned}
G(q_{-i}) &= \frac{\alpha (2\beta + \gamma) q_{-i} - 2\beta q_{-i}^2}{2\beta^2 - \gamma^2} + \frac{\alpha^2 \beta (\beta + \gamma)^2 - 2\alpha\beta\gamma (\beta + \gamma) q_{-i} + \beta\gamma^2 q_{-i}^2}{(2\beta^2 - \gamma^2)^2} \\
&= \frac{\alpha (2\beta + \gamma)(2\beta^2 - \gamma^2) q_{-i} - 2\beta (2\beta^2 - \gamma^2) q_{-i}^2 + \alpha^2 \beta (\beta + \gamma)^2}{(2\beta^2 - \gamma^2)^2} \\
&\quad + \frac{\beta\gamma^2 q_{-i}^2 - 2\alpha\beta\gamma (\beta + \gamma) q_{-i}}{(2\beta^2 - \gamma^2)^2} \\
&= \frac{\alpha^2 \beta (\beta + \gamma)^2 + \alpha (4\beta^3 - 4\beta\gamma^2 - \gamma^3) q_{-i} - \beta (4\beta^2 - 3\gamma^2) q_{-i}^2}{(2\beta^2 - \gamma^2)^2} \qquad \text{(A.0.88)}
\end{aligned}$$

Therefore, if $q_{-i} \leq \alpha\beta/(2\beta - \gamma)$ and $q_i > \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$, then the equilibrium prices are given by (A.0.84), the equilibrium demand is given by (A.0.86), the resulting profit of each seller is given by (A.0.87), and thus the total profit of both sellers together is given by (A.0.88).

Case 3. Next consider the case in which $0 \leq q_i \leq \alpha - \beta y_i + \gamma y_{-i}$ for $i = \pm 1$. (It will be shown that this case holds if and only if $0 \leq q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$. In Figure 18 this case corresponds to two-resource product capacities (q_{-1}, q_1) in region $0LIM$. Thus the entire region $\{(q_{-1}, q_1) : q_i \geq 0,\ i = \pm 1\}$ is covered by Cases 1–3.) In this case the profit function of each seller i is given by

$$g_i(y_i, y_{-i}) = y_i q_i$$

Then the best response function of each seller i is given by

$$B_i(y_{-i}) \quad = \quad \max\{y_i \;:\; q_i \leq \alpha - \beta y_i + \gamma y_{-i}\} \quad = \quad \frac{\alpha + \gamma y_{-i} - q_i}{\beta}$$

Solving the system

$$y_i \quad = \quad \frac{\alpha + \gamma y_{-i} - q_i}{\beta}$$

for $i = \pm 1$, the equilibrium (y_{-1}^*, y_1^*) is obtained, where

$$y_i^* \quad = \quad \frac{\alpha(\beta + \gamma) - \beta q_i - \gamma q_{-i}}{\beta^2 - \gamma^2} \tag{A.0.89}$$

(It is checked later under what conditions $y_i^* > 0$ and (y_{-1}^*, y_1^*) is the unique equilibrium.) The demand of seller i at the prices (y_{-1}^*, y_1^*) is equal to

$$\alpha - \beta y_i^* + \gamma y_{-i}^* \quad = \quad q_i \quad > \quad 0 \tag{A.0.90}$$

Next we verify that, if $q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, then the prices y_i^* given in (A.0.89) satisfy $y_i^* > 0$ for $i = \pm 1$, that is, the prices are greater than the marginal cost $c_{-1} + c_1$ of the two-resource product. Note that $q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$ implies that $q_{-1} + q_1 \leq 2\alpha\beta/(2\beta - \gamma)$. For a given pair (q_{-1}, q_1) such that $q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, consider the line with slope -1 through the point (q_{-1}, q_1). For example, in Figure 18, $EFGH$ is such a line, with points (q_{-1}, q_1) on line segment FG satisfying $q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$; and JK is also such a line, with all points (q_{-1}, q_1) on line segment JK satisfying $q_i \leq \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$. We show that the prices y_i^* given by (A.0.89) corresponding to all points (q_{-1}, q_1) on line segment FG satisfy $y_i^* > 0$. It follows that the prices y_i^* given by (A.0.89) corresponding to all points (q_{-1}, q_1) on line segment JK also satisfy $y_i^* > 0$. The coordinates of point F are $([(2\beta^2 - \gamma^2)(q_{-1} + q_1) - \alpha\beta(\beta + \gamma)]/(2\beta^2 - \beta\gamma - \gamma^2),\ [\alpha\beta(\beta + \gamma) - \beta\gamma(q_{-1} + q_1)]/(2\beta^2 - \beta\gamma - \gamma^2))$ and the coordinates of point G are $([\alpha\beta(\beta + \gamma) - \beta\gamma(q_{-1} + q_1)]/(2\beta^2 - \beta\gamma -$

$\gamma^2)$, $[(2\beta^2-\gamma^2)(q_{-1}+q_1)-\alpha\beta(\beta+\gamma)]/(2\beta^2-\beta\gamma-\gamma^2))$. Consider the prices y_i^* given in (A.0.89). Note that

$$\begin{aligned}
y_i^* &= \frac{\alpha(\beta+\gamma)-\beta q_i-\gamma q_{-i}}{\beta^2-\gamma^2} > 0 \\
\Leftrightarrow\quad \alpha(\beta+\gamma)-\beta q_i-\gamma q_{-i} &> 0 \\
\Leftrightarrow\quad \beta q_i+\gamma(q_{-i}+q_i-q_i) &< \alpha(\beta+\gamma) \\
\Leftrightarrow\quad (\beta-\gamma)q_i+\gamma(q_{-i}+q_i) &< \alpha(\beta+\gamma) \qquad \text{(A.0.91)}
\end{aligned}$$

If (q_{-1}, q_1) is on line segment FG, then

$$\begin{aligned}
q_i &\leq \frac{\alpha\beta(\beta+\gamma)-\beta\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2} \\
\Leftrightarrow\quad (\beta-\gamma)q_i+\gamma(q_{-i}+q_i) &\leq (\beta-\gamma)\frac{\alpha\beta(\beta+\gamma)-\beta\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2}+\gamma(q_{-i}+q_i) \\
&= \frac{\alpha\beta^3-\alpha\beta\gamma^2+\beta^2\gamma(q_{-1}+q_1)-\gamma^3(q_{-i}+q_i)}{2\beta^2-\beta\gamma-\gamma^2} \\
&= \frac{\alpha\beta(\beta^2-\gamma^2)+(\beta^2-\gamma^2)\gamma(q_{-1}+q_1)}{2\beta^2-\beta\gamma-\gamma^2} \\
&= \frac{(\beta-\gamma)(\beta+\gamma)[\alpha\beta+\gamma(q_{-1}+q_1)]}{(\beta-\gamma)(2\beta+\gamma)} \\
&= \frac{(\beta+\gamma)[\alpha\beta+\gamma(q_{-1}+q_1)]}{2\beta+\gamma} \qquad \text{(A.0.92)}
\end{aligned}$$

Next, by separately considering the cases $\gamma \leq 0$ and $\gamma \geq 0$, we show that $[\alpha\beta+\gamma(q_{-1}+q_1)]/(2\beta+\gamma) < \alpha$, then it follows from (A.0.92) that $(\beta-\gamma)q_i+\gamma(q_{-i}+q_i) < \alpha(\beta+\gamma)$, and hence it follows from (A.0.91) that $y_i^* > 0$.

First, suppose that $\gamma \leq 0$. Note that

$$\begin{aligned}
-\gamma &< \beta \\
\Leftrightarrow\quad \beta &< 2\beta+\gamma \\
\Leftrightarrow\quad \frac{\alpha\beta}{2\beta+\gamma} &< \alpha \\
\Rightarrow\quad \frac{\alpha\beta+\gamma(q_{-1}+q_1)}{2\beta+\gamma} &< \alpha \qquad \text{(A.0.93)}
\end{aligned}$$

The last step follows since $\gamma \leq 0$ and $q_{-1}+q_1 \geq 0$. It follows from (A.0.91), (A.0.92) and (A.0.93) that, if $\gamma \leq 0$, then $y_i^* > 0$.

Next, suppose that $\gamma \geq 0$. Note that

$$\begin{aligned}
\gamma &< \beta \\
\Leftrightarrow \quad \beta &< 2\beta - \gamma \\
\Leftrightarrow \quad \frac{\alpha\beta(2\beta - \gamma + 2\gamma)}{(2\beta-\gamma)(2\beta+\gamma)} &< \alpha \\
\Leftrightarrow \quad \frac{\alpha\beta + \frac{2\alpha\beta\gamma}{2\beta-\gamma}}{2\beta+\gamma} &< \alpha \\
\Rightarrow \quad \frac{\alpha\beta + \gamma(q_{-1}+q_1)}{2\beta+\gamma} &< \alpha \qquad \text{(A.0.94)}
\end{aligned}$$

The last step follows since $\gamma \geq 0$ and $q_{-1}+q_1 \leq 2\alpha\beta/(2\beta-\gamma)$. It follows from (A.0.91), (A.0.92) and (A.0.94) that, if $\gamma \geq 0$, then $y_i^* > 0$.

Next we verify that, if $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, then (y_{-1}^*, y_1^*) given in (A.0.89) is the unique equilibrium. We verify that y_i^* given in (A.0.89) is the unique best response for seller i to y_{-i}^*. Given y_{-i}^*, the profit of seller i is given by

$$\begin{aligned}
g_i(y_i, y_{-i}^*) &= y_i \min\left\{q_i,\ \max\{0,\ \alpha - \beta y_i + \gamma y_{-i}^*\}\right\} \\
&= \begin{cases} y_i q_i & \text{if} \quad y_i \leq \frac{\alpha+\gamma y_{-i}^* - q_i}{\beta} \\ y_i\left(\alpha - \beta y_i + \gamma y_{-i}^*\right) & \text{if} \quad \frac{\alpha+\gamma y_{-i}^*-q_i}{\beta} \leq y_i \leq \frac{\alpha+\gamma y_{-i}^*}{\beta} \\ 0 & \text{if} \quad y_i \geq \frac{\alpha+\gamma y_{-i}^*}{\beta} \end{cases}
\end{aligned}$$

Thus $g_i(y_i, y_{-i}^*)$ is a nondecreasing linear function of y_i if $y_i \leq (\alpha + \gamma y_{-i}^* - q_i)/\beta$. If $(\alpha+\gamma y_{-i}^* - q_i)/\beta < y_i < (\alpha+\gamma y_{-i}^*)/\beta$, then $g_i(y_i, y_{-i}^*)$ is a concave quadratic function of y_i, with

$$\begin{aligned}
g_i'(y_i, y_{-i}^*) &= -2\beta y_i + \alpha + \gamma y_{-i}^* \\
&< -2\left(\alpha + \gamma y_{-i}^* - q_i\right) + \alpha + \gamma y_{-i}^* \\
&= -\alpha - \gamma y_{-i}^* + 2q_i \\
&= -\alpha - \gamma\frac{\alpha(\beta+\gamma) - \beta q_{-i} - \gamma q_i}{\beta^2-\gamma^2} + 2q_i \\
&= \frac{-\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2-\gamma^2)q_i}{\beta^2-\gamma^2}
\end{aligned}$$

If (q_{-1}, q_1) is on line segment FG, then

$$\begin{aligned}
q_i &\leq \frac{\alpha\beta(\beta+\gamma) - \beta\gamma(q_{-i}+q_i)}{2\beta^2 - \beta\gamma - \gamma^2} \\
\Leftrightarrow \quad 0 &\geq -\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma(q_{-i}+q_i) + (2\beta^2 - \beta\gamma - \gamma^2)q_i \\
&= -\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2 - \gamma^2)q_i \\
\Leftrightarrow \quad 0 &\geq \frac{-\alpha\beta^2 - \alpha\beta\gamma + \beta\gamma q_{-i} + (2\beta^2 - \gamma^2)q_i}{\beta^2 - \gamma^2} \\
\Leftrightarrow \quad g_i'(y_i, y_{-i}^*) &< 0
\end{aligned}$$

Hence, if (q_{-1}, q_1) is on line segment FG, then $g_i'(y_i, y_{-i}^*) < 0$ for all $y_i \in ((\alpha + \gamma y_{-i}^* - q_i)/\beta,\ (\alpha + \gamma y_{-i}^*)/\beta)$. Hence, the unique best response for seller i to y_{-i}^* is $B_i(y_{-i}^*) = (\alpha + \gamma y_{-i}^* - q_i)/\beta$. It follows in the same way that if (q_{-1}, q_1) is on line segment JK, then the unique best response for seller i to y_{-i}^* is $B_i(y_{-i}^*) = (\alpha + \gamma y_{-i}^* - q_i)/\beta$. Therefore, if $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, then (y_{-1}^*, y_1^*) given in (A.0.89) is the unique equilibrium.

The resulting profit of each seller i is equal to

$$y_i^* \min\{q_i,\ \max\{0,\ \alpha - \beta y_i^* + \gamma y_{-i}^*\}\} \quad = \quad \frac{\alpha(\beta+\gamma)q_i - \beta q_i^2 - \gamma q_{-i} q_i}{\beta^2 - \gamma^2} \tag{A.0.95}$$

and thus the total profit of both sellers together is equal to

$$\frac{\alpha(\beta+\gamma)(q_{-1}+q_1) - \beta(q_{-1}^2 + q_1^2) - 2\gamma q_{-1} q_1}{\beta^2 - \gamma^2} \tag{A.0.96}$$

Therefore, if $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-i}/(2\beta^2 - \gamma^2)$ for $i = \pm 1$, then the equilibrium prices are given by (A.0.89), the equilibrium demand is given by (A.0.90), the resulting profit of each seller is given by (A.0.95), and thus the total profit of both sellers together is given by (A.0.96).

Next we determine the value of (q_{-1}, q_1) that maximizes the total profit of both sellers together under Case 3. First we fix the value of $q_{-1} + q_1$ at some value $q \leq b_{\min}$, and choose q_1 to maximize the total profit subject to $q_{-1} + q_1 = q$. Thereafter we choose q to maximize the total profit subject to $q \leq b_{\min}$. It follows from (A.0.96) that the total profit is equal to

$$\begin{aligned}
& \frac{\alpha(\beta+\gamma)(q_{-1}+q_1)-\beta(q_{-1}^2+q_1^2)-2\gamma q_{-1}q_1}{\beta^2-\gamma^2} \\
= & \frac{\alpha(\beta+\gamma)(q_{-1}+q_1)-\beta(q_{-1}^2+2q_{-1}q_1+q_1^2)+2\beta q_{-1}q_1-2\gamma q_{-1}q_1}{\beta^2-\gamma^2} \\
= & \frac{\alpha(\beta+\gamma)q-\beta q^2+2(\beta-\gamma)(q-q_1)q_1}{\beta^2-\gamma^2} \\
= & \frac{\alpha(\beta+\gamma)q-\beta q^2+2(\beta-\gamma)qq_1-2(\beta-\gamma)q_1^2}{\beta^2-\gamma^2}
\end{aligned}$$

Let

$$H_1(q_1) \quad := \quad \frac{\alpha(\beta+\gamma)q-\beta q^2+2(\beta-\gamma)qq_1-2(\beta-\gamma)q_1^2}{\beta^2-\gamma^2}$$

Note that H_1 is a concave quadratic function that is maximized at $q_1^* = q/2$, and the corresponding value of q_{-1} is also $q_{-1}^* = q/2$. Recall that (A.0.96) applies if $q_i \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$. Note that

$$\begin{aligned}
& q_i^* \leq \frac{\alpha\beta(\beta+\gamma)}{2\beta^2-\gamma^2} - \frac{\beta\gamma}{2\beta^2-\gamma^2}q_{-i}^* \quad \text{for } i = \pm 1 \\
\Leftrightarrow \quad & \frac{q}{2} \leq \frac{\alpha\beta(\beta+\gamma)}{2\beta^2-\gamma^2} - \frac{\beta\gamma}{2\beta^2-\gamma^2}\frac{q}{2} \\
\Leftrightarrow \quad & q \leq \frac{2\alpha\beta}{2\beta-\gamma}
\end{aligned}$$

Next we choose q to maximize the total profit subject to $q \leq b_{\min}$ and $q \leq 2\alpha\beta/(2\beta-\gamma)$. Let

$$\begin{aligned}
H_2(q) \quad := \quad & H_1(q/2) \\
= \quad & \frac{\alpha(\beta+\gamma)q-\beta q^2+2(\beta-\gamma)q^2/2-2(\beta-\gamma)q^2/4}{\beta^2-\gamma^2} \\
= \quad & \frac{2\alpha(\beta+\gamma)q-(\beta+\gamma)q^2}{2(\beta-\gamma)(\beta+\gamma)} \\
= \quad & \frac{2\alpha q-q^2}{2(\beta-\gamma)}
\end{aligned}$$

Note that H_2 is a concave quadratic function and $H_2'(q^*) = 0 \Leftrightarrow q^* = \alpha$. Also note that $q^* = \alpha \leq 2\alpha\beta/(2\beta-\gamma)$ if and only if $\gamma \geq 0$. Let $a_{\min} := \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$. Then the value of (q_{-1}, q_1) that maximizes the total profit and that satisfies

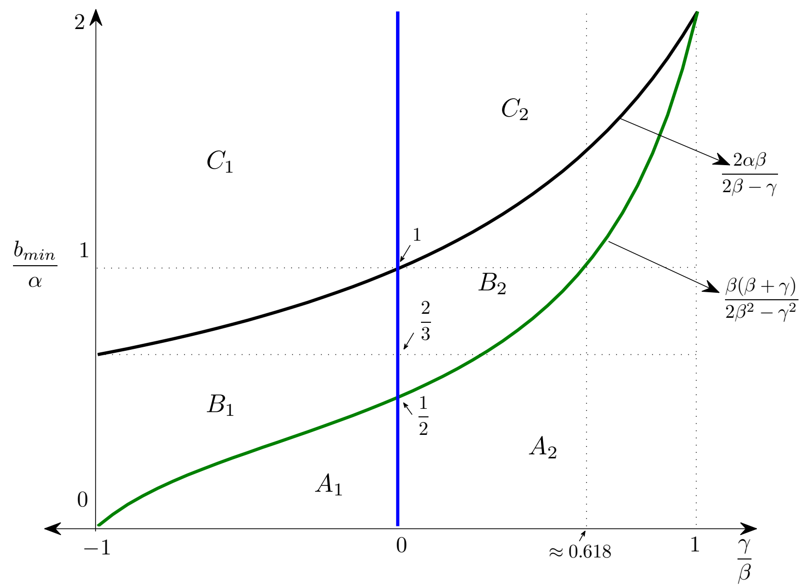


Figure 19: Different cases of the capacity ratio $b_{\min}/\alpha$ and the price coefficient ratio γ/β.

$q_i \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-i}/(2\beta^2-\gamma^2)$ for $i = \pm 1$, is $q^*_{-1} = q^*_1 = a_{\min}/2$. The corresponding total profit is $H_2(a_{\min}) = (2\alpha - a_{\min})a_{\min}/[2(\beta-\gamma)]$. This concludes Case 3.

Optimal exchange. Next, we compare the profits under Cases 1, 2, and 3, and determine the value of (q_{-1}, q_1), that is, the value of the exchange $x = (x_{-1}, x_1)$, that maximizes the total profit of both sellers together. Different cases hold, depending on the capacity ratio $b_{\min}/\alpha$ and the price coefficient ratio γ/β (recall that $\gamma/\beta \in (-1, 1)$). The different cases are depicted in Figure 19.

Case A (small capacity). $b_{\min}/\alpha \le [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $b_{\min} \le \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

In Figure 18, line JK shows an example of pairs (q_{-1}, q_1) such that $q_{-1}+q_1 = b_{\min}$ for a given value of $b_{\min} < \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$, and triangle $0JK$ shows pairs $(q_{-1}, q_1) \ge 0$ such that $q_{-1}+q_1 \le b_{\min}$. In this case, the capacity $b_{\min}$ is so small that all feasible values of (q_{-1}, q_1) correspond to Case 3. Recall that $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \in (0, 2\alpha\beta/(2\beta-\gamma))$.

Case A1. $\gamma/\beta \leq 0$ and $b_{\min}/\alpha \leq [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $\gamma \leq 0$ and $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

Recall that $2\alpha\beta/(2\beta-\gamma) \leq \alpha$ if and only if $\gamma \leq 0$. Since $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) < 2\alpha\beta/(2\beta-\gamma) \leq \alpha$, it follows that $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of (q_{-1}, q_1) that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (A.0.89), is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (A.0.90), is equal to $q_i^* = b_{\min}/2$.

Case A2. $\gamma/\beta \geq 0$ and $b_{\min}/\alpha \leq [1+\gamma/\beta]/[2-(\gamma/\beta)^2]$, that is, $\gamma \geq 0$ and $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$:

In this case, $b_{\min} \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) < 2\alpha\beta/(2\beta-\gamma)$ and $\alpha \leq 2\alpha\beta/(2\beta-\gamma)$. If $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq \alpha$, then $b_{\min} \leq \alpha$ and thus $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, the value of (q_{-1}, q_1) that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (A.0.89), is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (A.0.90), is equal to $q_i^* = b_{\min}/2$. Note that $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq \alpha$ if and only if $\gamma/\beta \leq (\sqrt{5}-1)/2 = 1/\varphi = \varphi - 1 \approx 0.618$, where φ denotes the golden ratio. If $\gamma/\beta > (\sqrt{5}-1)/2$ (and thus $\alpha < \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then there are two possibilities. If $b_{\min} \leq \alpha$, then as before, $q_{-1}^* = q_1^* = b_{\min}/2$, the equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to $q_i^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. Otherwise, if $\alpha < b_{\min}$, then $q_{-1}^* = q_1^* = \alpha/2$, the resulting equilibrium price of each seller, given by (A.0.89), is $y_i^* = \alpha/[2(\beta-\gamma)]$, the resulting equilibrium demand of each seller, given by (A.0.90), is equal to $q_i^* = \alpha/2$, and the maximum total profit is $(2\alpha-\alpha)\alpha/[2(\beta-\gamma)] = \alpha^2/[2(\beta-\gamma)]$. Note that in this case the optimal resource exchange x^* is such that $q^*_{-1}+q^*_1 = \alpha < b_{\min}$, that is, some capacity is not used.

Case B (intermediate capacity). $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \le b_{\min}/\alpha \le 2/(2-\gamma/\beta)$, that is, $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \le b_{\min} \le 2\alpha\beta/(2\beta-\gamma)$:

In Figure 18, line $EFGH$ shows an example of pairs (q_{-1}, q_1) such that $q_{-1} + q_1 = b_{\min}$ for a given value of $b_{\min} \in (\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2),\ 2\alpha\beta/(2\beta-\gamma))$, and triangle $0EH$ shows pairs $(q_{-1}, q_1) \ge 0$ such that $q_{-1} + q_1 \le b_{\min}$. In this case with intermediate capacity $b_{\min}$, there are feasible values of (q_{-1}, q_1) corresponding to Case 3, for example in pentagon $0LFGM$ in Figure 18, and there are feasible values of (q_{-1}, q_1) corresponding to Case 2, for example in triangles EFL and GHM in Figure 18.

Consider any two pairs (q_{-1}, q_1) and (q'_{-1}, q'_1) in triangle EFL such that $q_{-1} = q'_{-1}$. It follows from (A.0.84), (A.0.86), (A.0.87), and (A.0.88) that the equilibrium prices, the equilibrium demand, the profit of each seller, and thus the total profit of both sellers together are the same for (q_{-1}, q_1) and (q'_{-1}, q'_1). Therefore, for any point (q_{-1}, q_1) in triangle EFL, there is a point $(q_{-1},\ \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ on the boundary LF between triangle EFL and pentagon $0LFGM$ with the same total profit as at point (q_{-1}, q_1). Next, we show that the total profit as a function of (q_{-1}, q_1) is continuous on the boundary between triangle EFL and pentagon $0LFGM$. Recall from (A.0.96) that the total profit at a point (q_{-1}, q_1) in pentagon $0LFGM$ is equal to

$$\frac{\alpha(\beta+\gamma)\left(q_{-1}+q_1\right) - \beta\left(q_{-1}^2+q_1^2\right) - 2\gamma q_{-1}q_1}{\beta^2-\gamma^2}$$

Specifically, at the boundary point $(q_{-1},\ \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) - \beta\gamma q_{-1}/(2\beta^2-\gamma^2))$

the total profit is equal to

$$\frac{\left\{\begin{array}{c}\alpha(\beta+\gamma)\left(q_{-1}+\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}\right)\\ -\beta\left(q_{-1}^2+\left[\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}\right]^2\right)-2\gamma q_{-1}\frac{\alpha\beta(\beta+\gamma)-\beta\gamma q_{-1}}{2\beta^2-\gamma^2}\end{array}\right\}}{\beta^2-\gamma^2}$$

$$= \frac{\left\{\begin{array}{c}\left[\alpha^2\beta(\beta+\gamma)^2\left(2\beta^2-\gamma^2\right)-\alpha^2\beta^3(\beta+\gamma)^2\right]\\ +\left[\alpha(\beta+\gamma)\left(2\beta^2-\gamma^2\right)^2-\alpha\beta\gamma(\beta+\gamma)\left(2\beta^2-\gamma^2\right)+2\alpha\beta^3\gamma(\beta+\gamma)\right]q_{-1}\\ -\left[2\alpha\beta\gamma(\beta+\gamma)\left(2\beta^2-\gamma^2\right)\right]q_{-1}\\ +\left[-\beta\left(2\beta^2-\gamma^2\right)^2-\beta^3\gamma^2+2\beta\gamma^2\left(2\beta^2-\gamma^2\right)\right]q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)}$$

$$= \frac{\left\{\begin{array}{c}\alpha^2\beta\left(2\beta^2-\gamma^2-\beta^2\right)(\beta+\gamma)^2\\ +\alpha\left(4\beta^4-4\beta^2\gamma^2+\gamma^4-2\beta^3\gamma+\beta\gamma^3+2\beta^3\gamma-4\beta^3\gamma+2\beta\gamma^3\right)(\beta+\gamma)q_{-1}\\ -\beta\left(4\beta^4-4\beta^2\gamma^2+\gamma^4+\beta^2\gamma^2-4\beta^2\gamma^2+2\gamma^4\right)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)}$$

$$= \frac{\left\{\begin{array}{c}\alpha^2\beta\left(\beta^2-\gamma^2\right)(\beta+\gamma)^2\\ +\alpha\left(4\beta^4-4\beta^3\gamma-4\beta^2\gamma^2+3\beta\gamma^3+\gamma^4\right)(\beta+\gamma)q_{-1}\\ -\beta\left(4\beta^4-7\beta^2\gamma^2+3\gamma^4\right)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2\left(\beta^2-\gamma^2\right)}$$

$$= \frac{\left\{\begin{array}{c}\alpha^2\beta\left(\beta-\gamma\right)(\beta+\gamma)^3\\ +\alpha\left(4\beta^3-4\beta\gamma^2-\gamma^3\right)(\beta-\gamma)(\beta+\gamma)q_{-1}\\ -\beta\left(4\beta^2-3\gamma^2\right)(\beta-\gamma)(\beta+\gamma)q_{-1}^2\end{array}\right\}}{\left(2\beta^2-\gamma^2\right)^2(\beta-\gamma)(\beta+\gamma)}$$

$$= \frac{\alpha^2\beta(\beta+\gamma)^2+\alpha\left(4\beta^3-4\beta\gamma^2-\gamma^3\right)q_{-1}-\beta\left(4\beta^2-3\gamma^2\right)q_{-1}^2}{\left(2\beta^2-\gamma^2\right)^2}$$

which is the same as the total profit given by (A.0.88) for point $(q_{-1},\ \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)-\beta\gamma q_{-1}/(2\beta^2-\gamma^2))$ in triangle EFL. Thus the total profit as a function of (q_{-1},q_1) is continuous on the boundary between triangle EFL and pentagon $0LFGM$. The same observation applies to the total profit as a function of (q_{-1},q_1) in triangle GHM. Hence, in Case B with intermediate capacity, it is sufficient to optimize (q_{-1}, q_1) over pentagon $0LFGM$ only, that is, it is sufficient to restrict attention to feasible values of (q_{-1}, q_1) corresponding to Case 3. The rest of Case B follows in the same way as for Case A with small capacity.

Case B1. $\gamma/\beta \leq 0$ and $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \leq b_{\min}/\alpha \leq 2/(2-\gamma/\beta)$, that is, $\gamma \leq 0$ and $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$:

Consider the optimal value of (q_{-1}, q_1) in pentagon $0LFGM$. Since $b_{\min} \leq 2\alpha\beta/(2\beta-\gamma) \leq \alpha$, it follows that $b_{\min} = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, and thus the value of (q_{-1}, q_1) in pentagon $0LFGM$ that maximizes the total profit is $q_{-1}^* = q_1^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller is equal to $q_i^* = b_{\min}/2$.

Case B2. $\gamma/\beta \geq 0$ and $[1+\gamma/\beta]/[2-(\gamma/\beta)^2] \leq b_{\min}/\alpha \leq 2/(2-\gamma/\beta)$, that is, $\gamma \geq 0$ and $\alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2) \leq b_{\min} \leq 2\alpha\beta/(2\beta-\gamma)$:

If $\gamma/\beta \geq (\sqrt{5}-1)/2$ (and thus $\alpha \leq \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then $\alpha = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta-\gamma)\}$, the value of (q_{-1}, q_1) that maximizes the total profit is $q_{-1}^* = q_1^* = \alpha/2$, and the maximum total profit is $(2\alpha-\alpha)\alpha/[2(\beta-\gamma)] = \alpha^2/[2(\beta-\gamma)]$. The resulting equilibrium price of each seller, given by (A.0.89), is $y_i^* = \alpha/[2(\beta-\gamma)]$, and the resulting equilibrium demand of each seller, given by (A.0.90), is equal to $q_i^* = \alpha/2$. In this case the optimal resource exchange x^* is such that $q_{-1}^* + q_1^* = \alpha \leq b_{\min}$, that is, some capacity is not used. If $\gamma/\beta < (\sqrt{5}-1)/2$ (and thus $\alpha > \alpha\beta(\beta+\gamma)/(2\beta^2-\gamma^2)$), then there are two possibilities. If $\alpha \leq b_{\min}$, then as before, $q_{-1}^* = q_1^* = \alpha/2$, the equilibrium price of each seller is $y_i^* = \alpha/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to $q_i^* = \alpha/2$, and the maximum total profit is $\alpha^2/[2(\beta-\gamma)]$. Otherwise, if $b_{\min} \leq \alpha$, then $q_{-1}^* = q_1^* = b_{\min}/2$, the equilibrium price of each seller is $y_i^* = (2\alpha - b_{\min})/[2(\beta-\gamma)]$, the equilibrium demand of each seller is equal to

$q_i^* = b_{\min}/2$, and the maximum total profit is $(2\alpha - b_{\min})b_{\min}/[2(\beta - \gamma)]$.

Case C (large capacity). $b_{\min}/\alpha \geq 2/(2 - \gamma/\beta)$, that is, $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$:

In Figure 18, line $ABCD$ shows an example of pairs (q_{-1}, q_1) such that $q_{-1} + q_1 = b_{\min}$ for a given value of $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$, and triangle $0AD$ shows pairs $(q_{-1}, q_1) \geq 0$ such that $q_{-1} + q_1 \leq b_{\min}$. In this case with large capacity $b_{\min}$, there are feasible values of (q_{-1}, q_1) in quadrilateral $0LIM$ in Figure 18 corresponding to Case 3, there are feasible values of (q_{-1}, q_1) corresponding to Case 2, for example in quadrilaterals $ABIL$ and $DCIM$ in Figure 18, and there are feasible values of (q_{-1}, q_1) corresponding to Case 1, for example in triangle BCI in Figure 18.

For any point (q_{-1}, q_1) in $ABIL$, there is a point $(q_{-1}, \alpha\beta(\beta + \gamma)/(2\beta^2 - \gamma^2) - \beta\gamma q_{-1}/(2\beta^2 - \gamma^2))$ on the boundary IL between $ABIL$ and $0LIM$ with the same total profit as at point (q_{-1}, q_1). It was shown under Case B that the total profit as a function of (q_{-1}, q_1) is continuous on the boundary. The same observation applies to the total profit as a function of (q_{-1}, q_1) in $DCIM$. Hence, in Case C with large capacity, it is sufficient to optimize (q_{-1}, q_1) over quadrilateral $0LIM$ and triangle BCI only, that is, it is sufficient to restrict attention to feasible values of (q_{-1}, q_1) corresponding to Case 3 and Case 1.

Case C1. $\gamma/\beta \leq 0$ and $b_{\min}/\alpha \geq 2/(2 - \gamma/\beta)$, that is, $\gamma \leq 0$ and $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$:

Since $2\alpha\beta/(2\beta - \gamma) \leq \alpha$ and $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$, it follows that $2\alpha\beta/(2\beta - \gamma) = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta - \gamma)\}$, and thus the value of (q_{-1}, q_1) that maximizes the total profit over $0LIM$ is given by $q_{-1}^* = q_1^* = \alpha\beta/(2\beta - \gamma)$ represented by point I, and the corresponding total profit is $(2\alpha - 2\alpha\beta/(2\beta - \gamma))2\alpha\beta/(2\beta - \gamma)/[2(\beta - \gamma)] = 2\alpha^2\beta/(2\beta - \gamma)^2$. Also, as shown in Case 1, all values of (q_{-1}, q_1) in triangle BCI have the same total profit of $2\alpha^2\beta/(2\beta - \gamma)^2$. Thus, any point (q_{-1}, q_1) in triangle BCI represents an optimal resource exchange for Case C1. For all such optimal resource exchanges, the resulting equilibrium price of each seller, given by both (A.0.80) and (A.0.89), is $y_i^* = \alpha/(2\beta - \gamma)$, and the resulting equilibrium demand of each seller, given by both (A.0.81) and (A.0.90), is equal to $\alpha\beta/(2\beta - \gamma)$.

Case C2. $\gamma/\beta \geq 0$ and $b_{\min}/\alpha \geq 2/(2-\gamma/\beta)$, that is, $\gamma \geq 0$ and $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma)$:

Since $b_{\min} \geq 2\alpha\beta/(2\beta - \gamma) \geq \alpha$, it follows that $\alpha = \min\{\alpha, b_{\min}, 2\alpha\beta/(2\beta - \gamma)\}$, and thus the value of (q_{-1}, q_1) that maximizes the total profit over $0LIM$ is $q_{-1}^* = q_1^* = \alpha/2$, and the corresponding total profit is $(2\alpha - \alpha)\alpha/[2(\beta - \gamma)] = \alpha^2/[2(\beta - \gamma)]$. Also, all values of (q_{-1}, q_1) in triangle BCI have the same total profit of $2\alpha^2\beta/(2\beta - \gamma)^2$. Note that

$$\begin{aligned} 4\beta^2 - 4\beta\gamma + \gamma^2 &\geq 4\beta^2 - 4\beta\gamma \\ \Rightarrow \quad (2\beta - \gamma)^2 &\geq 4\beta(\beta - \gamma) \\ \Rightarrow \quad \frac{\alpha^2}{2(\beta - \gamma)} &\geq \frac{2\alpha^2\beta}{(2\beta - \gamma)^2} \end{aligned}$$

Thus the optimal point for Case C2 is $q_{-1}^* = q_1^* = \alpha/2$, and the maximum total profit is $\alpha^2/[2(\beta - \gamma)]$. The resulting equilibrium price of each seller, given by (A.0.89), is $y_i^* = \alpha/[2(\beta - \gamma)]$, and the resulting equilibrium demand of each seller, given by (A.0.90), is equal to $q_i^* = \alpha/2$.

Inspection of the results above for the settings with no alliance, perfect coordination, and a resource exchange alliance reveal that the results can be summarized by 5 cases, as in Table 1.

Consumer surplus. To calculate the consumer surplus associated with demand model (1.2.13), it is instructive to start with a utility model that leads to demand model (1.2.13). Consider a representative consumer who consumes z_{-1} units of the product sold by seller -1 and z_1 units of the product sold by seller 1. Suppose that the resulting utility is given by $U(z_{-1}, z_1) := a_{-1}z_{-1} + a_1 z_1 - b_{-1}z_{-1}^2/2 - b_1 z_1^2/2 -$

$cz_{-1}z_1$ with $b_{-1}, b_1, b_{-1}b_1 - c^2 > 0$. Given a price p_i for the product sold by each seller i, the consumer chooses quantities (z_{-1}, z_1) to maximize the consumer surplus $U(z_{-1}, z_1) - p_{-1}z_{-1} - p_1 z_1$. It follows that the chosen quantities satisfy

$$z_i \quad = \quad \frac{a_i b_{-i} - a_{-i}c}{b_{-1}b_1 - c^2} - \frac{b_{-i}}{b_{-1}b_1 - c^2}p_i + \frac{c}{b_{-1}b_1 - c^2}p_{-i}$$

This utility model leads to the demand model (1.2.13) if $\alpha = (a_i b_{-i} - a_{-i}c)/(b_{-1}b_1 - c^2)$, $\beta = b_i/(b_{-1}b_1 - c^2)$, and $\gamma = c/(b_{-1}b_1 - c^2)$ for $i = \pm 1$, that is, if $a_i = \alpha/(\beta - \gamma)$, $b_i = \beta/(\beta^2 - \gamma^2)$, and $c = \gamma/(\beta^2 - \gamma^2)$ for $i = \pm 1$.

In regions 1 and 2 in Table 1, the resulting consumer surplus is given by

$$U(b_{\min}/2, b_{\min}/2) - \frac{2\alpha - b_{\min}}{2(\beta - \gamma)}\frac{b_{\min}}{2} - \frac{2\alpha - b_{\min}}{2(\beta - \gamma)}\frac{b_{\min}}{2} \quad = \quad \frac{b_{\min}^2}{4(\beta - \gamma)}$$

In regions 3 and 4, the resulting consumer surplus is given by

$$\begin{aligned} & U(\alpha\beta/(2\beta - \gamma), \alpha\beta/(2\beta - \gamma)) - \frac{\alpha}{2\beta - \gamma}\frac{\alpha\beta}{2\beta - \gamma} - \frac{\alpha}{2\beta - \gamma}\frac{\alpha\beta}{2\beta - \gamma} \\ &= \frac{\alpha^2\beta^2}{(\beta - \gamma)(2\beta - \gamma)^2} \end{aligned}$$

In region 5, the resulting consumer surplus is given by

$$U(\alpha/2, \alpha/2) - \frac{\alpha}{2(\beta - \gamma)}\frac{\alpha}{2} - \frac{\alpha}{2(\beta - \gamma)}\frac{\alpha}{2} \quad = \quad \frac{\alpha^2}{4(\beta - \gamma)}$$

Thus, in region 1 all three settings have the same consumer surplus. In region 2, the consumer surplus under perfect coordination and under the alliance are the same, and as shown in Section 1.2.2, both are larger than the consumer surplus under no alliance. To compare the consumer surplus under the alliance and under no alliance in regions 3 and 4, note that

$$\begin{aligned} \frac{\alpha^2}{9(\beta - \gamma)} &\leq \frac{\alpha^2\beta^2}{(\beta - \gamma)(2\beta - \gamma)^2} \\ \Leftrightarrow \quad -4\beta\gamma + \gamma^2 &\leq 5\beta^2 \end{aligned}$$

which holds since $\gamma \in (-\beta, \beta)$, and thus in regions 3 and 4 the consumer surplus under the alliance is greater than the consumer surplus under no alliance. To compare the consumer surplus under the alliance and under perfect coordination in region 3, note that

$$\begin{aligned} \frac{b_{\min}^2}{4(\beta-\gamma)} &\geq \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2} \\ \Leftrightarrow \quad b_{\min} &\geq \frac{2\alpha\beta}{2\beta-\gamma} \end{aligned}$$

and thus in region 3 the consumer surplus under perfect coordination is greater than the consumer surplus under the alliance. To compare the consumer surplus under the alliance and under perfect coordination in region 4, note that

$$\begin{aligned} \frac{\alpha^2}{4(\beta-\gamma)} &\geq \frac{\alpha^2\beta^2}{(\beta-\gamma)(2\beta-\gamma)^2} \\ \Leftrightarrow \quad (2\beta-\gamma)^2 &\geq 4\beta^2 \end{aligned}$$

which holds since $\gamma \leq 0$ in region 4, and thus in region 4 the consumer surplus under perfect coordination is greater than the consumer surplus under the alliance. Finally, in region 5 the consumer surplus under perfect coordination and under the alliance are the same, and both are larger than the consumer surplus under no alliance by a factor of 9/4. Note that, similar to total profit, the consumer surplus under perfect coordination and under the alliance are the same except when capacity is large ($b_{\min} \geq 2\alpha\beta/(2\beta-\gamma)$) and the sellers' products are complements ($\gamma \leq 0$).

A.0.4 Perfect Coordination with Product Differentiation

The model of perfect coordination introduced in Section 1.2.2 (with details given in Section A.0.2) was based on a model of demand d for the two-resource product given by $d = \max\{0,\ \tilde{\alpha} - \tilde{\beta}(\tilde{y}_{-1} + \tilde{y}_1)\}$, and the model of an alliance introduced in Section 1.2.3 (with details given in Section A.0.3) was based on a model of demand $d_i(y_i, y_{-i})$ for the two-resource product of seller i given by $d_i(y_i, y_{-i}) = \max\{0,\ \alpha - \beta y_i + \gamma y_{-i}\}$, where $\tilde{\alpha} = 2\alpha + 2(\beta-\gamma)(c_{-1}+c_1)$ and $\tilde{\beta} = 2(\beta-\gamma)$. Thus, the model of perfect coordination in Section 1.2.2 does not make provision for different brands of the two-resource product, but the model of an alliance in Section 1.2.3

makes provision for different brands of the two-resource product. In this section we consider a model of perfect coordination that makes provision for different brands of the two-resource product.

The demand $d_i(y_i, y_{-i})$ for the brand i product sold is given as follows:

$$d_i(y_i, y_{-i}) \quad = \quad \alpha - \beta y_i + \gamma y_{-i}$$

where as before y_i denotes the excess of the price of the brand i product over the marginal cost $c_{-1} + c_1$, and we consider only values of (y_{-1}, y_1) such that $\alpha - \beta y_i + \gamma y_{-i} \geq 0$ for $i = \pm 1$.

First consider the case in which the capacity is not constraining (it is determined later what amount of capacity is sufficient for this condition to hold). In this case, the total profit is given by

$$g(y_{-1}, y_1) \quad := \quad y_{-1} d_{-1}(y_{-1}, y_1) + y_1 d_1(y_1, y_{-1}) \quad = \quad \alpha(y_{-1} + y_1) - \beta(y_{-1}^2 + y_1^2) + 2\gamma y_{-1} y_1$$

Note that

$$\nabla g(y_{-1}, y_1) \quad = \quad \begin{bmatrix} \alpha - 2\beta y_{-1} + 2\gamma y_1 \\ \alpha - 2\beta y_1 + 2\gamma y_{-1} \end{bmatrix}$$

$$\nabla^2 g(y_{-1}, y_1) \quad = \quad \begin{bmatrix} -2\beta & 2\gamma \\ 2\gamma & -2\beta \end{bmatrix}$$

and thus $\nabla^2 g(y_{-1}, y_1)$ is negative definite ($\beta > 0$, $\beta^2 - \gamma^2 > 0$), and hence g is a concave quadratic function. Therefore, the prices that maximize the total profit are given by

$$y_{-1}^* \quad = \quad y_1^* \quad = \quad \frac{\alpha}{2(\beta - \gamma)}, \tag{A.0.97}$$

and the corresponding total demand at the optimal prices is equal to α. Thus, if $b_{\min} \geq \alpha$, then the total profit of the two sellers under perfect coordination is given by $\frac{\alpha^2}{2(\beta-\gamma)}$. Note that the optimal prices, demand, profit, and consumer surplus are the same as for perfect coordination in Section 1.2.2 when $b_{\min} \geq \alpha$.

Next consider the case in which $b_{\min} < \alpha$. First we consider price points (y_{-1}, y_1) such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \le b_{\min}$, and then we consider price points (y_{-1}, y_1) such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \ge b_{\min}$. It follows from the results above for g that the point $(\check{y}_{-1}, \check{y}_1)$ that maximizes g subject to the constraint $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \le b_{\min}$ satisfies $d_{-1}(\check{y}_{-1}, \check{y}_1) + d_1(\check{y}_1, \check{y}_{-1}) = b_{\min}$, that is, $2\alpha - (\beta - \gamma)(\check{y}_{-1} + \check{y}_1) = b_{\min}$. Let

$$\begin{aligned} g_1(y_1) &:= g\left([2\alpha - b_{\min}]/[\beta - \gamma] - y_1, y_1\right) \\ &= \alpha \frac{2\alpha - b_{\min}}{\beta - \gamma} - \beta \frac{(2\alpha - b_{\min})^2}{(\beta - \gamma)^2} + 2(\beta + \gamma) \left(\frac{2\alpha - b_{\min}}{\beta - \gamma} - y_1 \right) y_1 \end{aligned}$$

Note that g_1 is a concave quadratic function with maximum at $\check{y}_1 = (2\alpha - b_{\min})/[2(\beta - \gamma)]$ (and thus $\check{y}_{-1} = \check{y}_1 = (2\alpha - b_{\min})/[2(\beta - \gamma)]$).

Next consider price points (y_{-1}, y_1) such that $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) \ge b_{\min}$, that is, $2\alpha - (\beta - \gamma)(y_{-1} + y_1) \ge b_{\min}$. The model should specify how capacity $b_{\min}$ is to be allocated between the two brands if $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) > b_{\min}$. There are various ways to allocate constrained capacity. Here we present one such way, the equal rationing rule, in detail, and then we point out other ways that lead to the same results. Under the equal rationing rule, if $d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1}) > b_{\min}$, then the same fraction λ of the demands $d_i(y_i, y_{-i})$ for the different brands is satisfied, where

$$\lambda = \frac{b_{\min}}{d_{-1}(y_{-1}, y_1) + d_1(y_1, y_{-1})} = \frac{b_{\min}}{2\alpha - (\beta - \gamma)(y_{-1} + y_1)}$$

Then, the total profit is given by

$$\begin{aligned} g_2(y_{-1}, y_1) &= \lambda y_{-1}(\alpha - \beta y_{-1} + \gamma y_1) + \lambda y_1(\alpha - \beta y_1 + \gamma y_{-1}) \\ &= b_{\min} \frac{\alpha(y_{-1} + y_1) - \beta(y_{-1} + y_1)^2 + 2(\beta + \gamma) y_{-1} y_1}{2\alpha - (\beta - \gamma)(y_{-1} + y_1)} \end{aligned}$$

Let $y := y_{-1} + y_1$, and let

$$\begin{aligned} g_3(y, y_1) &:= g_2(y - y_1, y_1) \\ &= b_{\min} \frac{\alpha y - \beta y^2 + 2(\beta + \gamma) y y_1 - 2(\beta + \gamma) y_1^2}{2\alpha - (\beta - \gamma) y} \end{aligned}$$

Recall that, in this case, $2\alpha-(\beta-\gamma)(y_{-1}+y_1) \geq b_{\min}$, and thus $y \leq (2\alpha-b_{\min})/(\beta-\gamma)$. First, consider any fixed value of $y \in [0,(2\alpha-b_{\min})/(\beta-\gamma)]$, and maximize $g_3(y,\cdot)$ with respect to y_1. Note that $g_3(y,\cdot)$ is a concave quadratic function with maximum at $\hat{y}_1 = y/2$ (and thus $\hat{y}_{-1} = \hat{y}_1 = y/2$). Next, let

$$\begin{aligned}
g_4(y) &:= g_2(y/2, y/2) \\
&= \frac{b_{\min}}{2}\frac{2\alpha y + \gamma y^2 - \beta y^2}{2\alpha - (\beta-\gamma)y} \\
&= \frac{b_{\min}}{2} y
\end{aligned}$$

Note that the maximum of g_4 over $y \in [0,(2\alpha-b_{\min})/(\beta-\gamma)]$ is attained at $y = (2\alpha-b_{\min})/(\beta-\gamma)$, and thus $\hat{y}_{-1} = \hat{y}_1 = (2\alpha-b_{\min})/[2(\beta-\gamma)]$. Therefore, if $b_{\min} < \alpha$, then the optimal prices are

$$y^*_{-1} = y^*_1 = \check{y}_{-1} = \check{y}_1 = \hat{y}_{-1} = \hat{y}_1 = \frac{2\alpha - b_{\min}}{2(\beta-\gamma)} \qquad \text{(A.0.98)}$$

with corresponding total demand equal to $b_{\min}$. Thus, the total profit under perfect coordination is equal to $(2\alpha - b_{\min})b_{\min}/[2(\beta-\gamma)]$. Note that the optimal prices, demand, profit and consumer surplus are also the same as for perfect coordination in Section 1.2.2 when $b_{\min} \leq \alpha$.

Other rationing rules also lead to the same results. For example, suppose that the demand for brand -1 is satisfied first and then the remaining capacity, if any, is used for brand 1. In this case, the total profit is given by

$$\begin{aligned}
g_5(y_{-1}, y_1) = \; & y_{-1}\min\{b_{\min}, \alpha - \beta y_{-1} + \gamma y_1\} \\
& + y_1 \min\{\max\{0, b_{\min} - (\alpha - \beta y_{-1} + \gamma y_1)\}, \alpha - \beta y_1 + \gamma y_{-1}\}
\end{aligned}$$

For this rationing rule the optimal prices are same as in (A.0.98).

APPENDIX B

DETAILS OF DEMAND TRANSFORMATION FOR NO ALLIANCE MODEL

The parameters E, B, C in demand model (1.3.14) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (1.3.25) should be related in a particular way to facilitate a fair comparison of the prices, demands, total profit, and consumer surplus between the settings with and without an alliance. In this section we derive the relation.

The relation between the demand models with and without an alliance is based on the assumption that the overall demand level for each product is the same with and without an alliance. Recall that L_i denotes the set of products which can be offered by seller i with and without an alliance, for $i = \pm 1$, and L_0 denotes the set of products which could be offered only under an alliance. In addition, let $L_{0,i} \subset L_0$ denote the set of products in L_0 that can be offered by seller i under an alliance, and let $L_{i,-i} \subset L_i$ denote the set of products in L_i that can be offered by seller $-i$ under an alliance, but not without an alliance. Thus, for the setting with an alliance the number of demand equations (and prices) for each seller i is $m_i = |L_i| + |L_{0,i}| + |L_{-i,i}|$, and for the setting without an alliance the number of demand equations (and prices) for each seller i is only $|L_i|$.

The following example is used to explain the derivation of the relation between the demand models. Seller -1 produces resource A, and seller 1 produces resources B and C. With an alliance, the following products are offered by each seller: Product A using 1 unit of resource A each, product B using 1 unit of resource B each, product C using 1 unit of resource C each, product BC using 1 unit of resource B and 1 unit of resource C each, and product A^2BC using 2 units of resource A, 1 unit of resource B, and 1 unit of resource C each. Without an alliance, product A is offered by seller -1 only and seller -1 captures all the demand for product A, and products B, C, and BC are offered by seller 1 only and seller 1 captures all the demand for products B, C, and BC. Product A^2BC is not offered by either seller, but there still is the same demand for product A^2BC; buyers buy each unit of product A^2BC by buying 2 units of product A from seller -1, and 1 unit of product BC from seller 1. As shown later, the demands for products A and BC derived from the demand for product A^2BC is added to the respective demands for products A and BC by themselves. Note that this derivation assumes that buyers buy each unit of product A^2BC by buying 1 unit of product BC from seller 1 instead of buying 1 unit of product B and 1 unit of product C separately from the same seller. This assumption may be questionable if the price of buying products B and C separately is less than the price of product BC. In the numerical work, we verified that the prices of multiple resource products offered by a seller were less than the sum of the prices of any products that could be bought separately to make up the multiple resource product. Thus, in this example, $L_{-1} = \{A\}$, $L_1 = \{B, C, BC\}$, $L_{0,-1} = \{A^2BC\}$, $L_{0,1} = \{A^2BC\}$, $L_{-1,1} = \{A\}$, and $L_{1,-1} = \{B, C, BC\}$. With an alliance, the demand for each product is given by (1.3.14):

$$
\begin{aligned}
d_{i,A} \quad &= \quad -E_{i,A,A}y_{i,A} - E_{i,A,B}y_{i,B} - E_{i,A,C}y_{i,C} - E_{i,A,BC}y_{i,BC} - E_{i,A,A^2BC}y_{i,A^2BC} \\
&\qquad + B_{-i,A,A}y_{-i,A} + B_{-i,A,B}y_{-i,B} + B_{-i,A,C}y_{-i,C} + B_{-i,A,BC}y_{-i,BC} \\
&\qquad + B_{-i,A,A^2BC}y_{-i,A^2BC} + C_{i,A} \\
d_{i,B} \quad &= \quad -E_{i,B,A}y_{i,A} - E_{i,B,B}y_{i,B} - E_{i,B,C}y_{i,C} - E_{i,B,BC}y_{i,BC} - E_{i,B,A^2BC}y_{i,A^2BC} \\
&\qquad + B_{-i,B,A}y_{-i,A} + B_{-i,B,B}y_{-i,B} + B_{-i,B,C}y_{-i,C} + B_{-i,B,BC}y_{-i,BC} \\
&\qquad + B_{-i,B,A^2BC}y_{-i,A^2BC} + C_{i,B}
\end{aligned}
$$

$$
\begin{aligned}
d_{i,C} &= -E_{i,C,A}y_{i,A} - E_{i,C,B}y_{i,B} - E_{i,C,C}y_{i,C} - E_{i,C,BC}y_{i,BC} \\
&\quad -E_{i,C,A^2BC}y_{i,A^2BC} + B_{-i,C,A}y_{-i,A} + B_{-i,C,B}y_{-i,B} + B_{-i,C,C}y_{-i,C} \\
&\quad +B_{-i,C,BC}y_{-i,BC} + B_{-i,C,A^2BC}y_{-i,A^2BC} + C_{i,C} \\
d_{i,BC} &= -E_{i,BC,A}y_{i,A} - E_{i,BC,B}y_{i,B} - E_{i,BC,C}y_{i,C} - E_{i,BC,BC}y_{i,BC} \\
&\quad -E_{i,BC,A^2BC}y_{i,A^2BC} + B_{-i,BC,A}y_{-i,A} + B_{-i,BC,B}y_{-i,B} + B_{-i,BC,C}y_{-i,C} \\
&\quad +B_{-i,BC,BC}y_{-i,BC} + B_{-i,BC,A^2BC}y_{-i,A^2BC} + C_{i,BC} \\
d_{i,A^2BC} &= -E_{i,A^2BC,A}y_{i,A} - E_{i,A^2BC,B}y_{i,B} - E_{i,A^2BC,C}y_{i,C} - E_{i,A^2BC,BC}y_{i,BC} \\
&\quad -E_{i,A^2BC,A^2BC}y_{i,A^2BC} + B_{-i,A^2BC,A}y_{-i,A} + B_{-i,A^2BC,B}y_{-i,B} \\
&\quad +B_{-i,A^2BC,C}y_{-i,C} + B_{-i,A^2BC,BC}y_{-i,BC} + B_{-i,A^2BC,A^2BC}y_{-i,A^2BC} + C_{i,A^2BC}
\end{aligned}
$$

To use these observations and the demand functions given by (1.3.14) for the alliance setting to derive the demand functions for the products with no alliance, first note that the demands in (1.3.14) depend on $|L_{0,-1}| + |L_{0,1}| + |L_{-1}| + |L_1| + |L_{-1,1}| + |L_{1,-1}|$ prices $y_{i,\ell}$, but the demands in (1.3.25) depend on only $|L_{-1}| + |L_1|$ prices. Thus, to derive the demands of the products with no alliance (as a function of the $|L_{-1}| + |L_1|$ prices $\tilde{y}$ with no alliance), it remains to determine appropriate values to substitute into (1.3.14) for the $|L_{0,-1}| + |L_{0,1}| + |L_{-1}| + |L_1| + |L_{-1,1}| + |L_{1,-1}|$ prices y given the prices $\tilde{y}$. First, consider the easy case: if a product ℓ is offered by the same seller i in both the setting with an alliance and the setting without an alliance, that is, $\ell \in L_i$, then simply substitute price $\tilde{y}_{i,\ell}$ for $y_{i,\ell}$ in the demand model (1.3.14). Thus, in the example above, $\tilde{y}_{-1,A}$, $\tilde{y}_{1,B}$, $\tilde{y}_{1,C}$, and $\tilde{y}_{1,BC}$ are substituted for $y_{-1,A}$, $y_{1,B}$, $y_{1,C}$, and $y_{1,BC}$ respectively. Next, if a product ℓ offered by a seller i in the alliance setting is not offered by any seller in the no alliance setting, that is, $\ell \in L_{0,i}$, but it can be assembled in the no alliance setting by buying a_{-1} units of product ℓ_{-1} from seller -1 and a_1 units of product ℓ_1 from seller 1, then substitute price $a_{-1}\tilde{y}_{-1,\ell_{-1}} + a_1\tilde{y}_{1,\ell_1}$ for $y_{i,\ell}$ in the demand model (1.3.14). Thus, in the example above, $2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}$ is substituted for y_{-1,A^2BC} and y_{1,A^2BC}. Next, if a product ℓ offered by a seller i in the alliance setting is not offered by seller i in the no alliance setting, but it is offered by seller $-i$ in the no alliance setting, that is, $\ell \in L_{-i,i}$), then we choose the price $y_{i,\ell}$ in the demand model (1.3.14) so that together with the other prices $y_{i',\ell'}$, $i' = \pm 1$, $\ell' \in L_{i'} \cup L_{0,i'}$, already determined as described above, will equate $d_{i,\ell}$ to zero. Note that if there are n such products, then n linear equations are obtained by equating the n linear expressions for $d_{i,\ell}$ to zero, and under reasonable conditions these equations can be solved for the n desired values of $y_{i,\ell}$. Thus, for the example above, the system of equations

$$\begin{aligned}
& -E_{1,A,A}y_{1,A} - E_{1,A,B}\tilde{y}_{1,B} - E_{1,A,C}\tilde{y}_{1,C} - E_{1,A,BC}\tilde{y}_{1,BC} \\
& -E_{1,A,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{-1,A,A}\tilde{y}_{-1,A} + B_{-1,A,B}y_{-1,B} \\
& +B_{-1,A,C}y_{-1,C} + B_{-1,A,BC}y_{-1,BC} + B_{-1,A,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{1,A} \\
& = 0 \\
& -E_{-1,B,A}\tilde{y}_{-1,A} - E_{-1,B,B}y_{-1,B} - E_{-1,B,C}y_{-1,C} - E_{-1,B,BC}y_{-1,BC} \\
& -E_{-1,B,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{1,B,A}y_{1,A} + B_{1,B,B}\tilde{y}_{1,B} \\
& +B_{1,B,C}\tilde{y}_{1,C} + B_{1,B,BC}\tilde{y}_{1,BC} + B_{1,B,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,B} \\
& = 0 \\
& -E_{-1,C,A}\tilde{y}_{-1,A} - E_{-1,C,B}y_{-1,B} - E_{-1,C,C}y_{-1,C} - E_{-1,C,BC}y_{-1,BC} \\
& -E_{-1,C,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{1,C,A}y_{1,A} + B_{1,C,B}\tilde{y}_{1,B} \\
& +B_{1,C,C}\tilde{y}_{1,C} + B_{1,C,BC}\tilde{y}_{1,BC} + B_{1,C,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,C} \\
& = 0 \\
& -E_{-1,BC,A}\tilde{y}_{-1,A} - E_{-1,BC,B}y_{-1,B} - E_{-1,BC,C}y_{-1,C} - E_{-1,BC,BC}y_{-1,BC} \\
& -E_{-1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{1,BC,A}y_{1,A} + B_{1,BC,B}\tilde{y}_{1,B} \\
& +B_{1,BC,C}\tilde{y}_{1,C} + B_{1,BC,BC}\tilde{y}_{1,BC} + B_{1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,BC} \\
& = 0
\end{aligned}$$

is solved for $y_{1,A}$, $y_{-1,B}$, $y_{-1,C}$, and $y_{-1,BC}$ as linear functions of $\tilde{y}_{-1,A}$, $\tilde{y}_{1,B}$, $\tilde{y}_{1,C}$, and $\tilde{y}_{1,BC}$. Suppose the solution is

$$\begin{aligned}
y_{1,A} &= b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0} \\
y_{-1,B} &= b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} + b_{-1,B,1,BC}\tilde{y}_{1,BC} + b_{-1,B,0} \\
y_{-1,C} &= b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC} + b_{-1,C,0} \\
y_{-1,BC} &= b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} + b_{-1,BC,1,BC}\tilde{y}_{1,BC} \\
&\quad + b_{-1,BC,0}
\end{aligned}$$

Now we are ready to use the observations above and the demand functions given by (1.3.14) for the alliance setting to derive the demand functions for the products with no alliance. For the example above, we obtain the following demand functions:

$$\begin{aligned}
\tilde{d}_{-1,A} &= -E_{-1,A,A}\tilde{y}_{-1,A} - E_{-1,A,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} \\
&\quad + b_{-1,B,1,BC}\tilde{y}_{1,BC} + b_{-1,B,0}) - E_{-1,A,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} \\
&\quad + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC} + b_{-1,C,0}) - E_{-1,A,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} \\
&\quad + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} + b_{-1,BC,1,BC}\tilde{y}_{1,BC} + b_{-1,BC,0}) \\
&\quad - E_{-1,A,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{1,A,A}(b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} \\
&\quad + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0}) \\
&\quad + B_{1,A,B}\tilde{y}_{1,B} + B_{1,A,C}\tilde{y}_{1,C} + B_{1,A,BC}\tilde{y}_{1,BC} \\
&\quad + B_{1,A,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,A} + 2\,[-E_{-1,A^2BC,A}\tilde{y}_{-1,A} \\
&\quad - E_{-1,A^2BC,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} \\
&\quad + b_{-1,B,1,BC}\tilde{y}_{1,BC} + b_{-1,B,0}) \\
&\quad - E_{-1,A^2BC,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC} \\
&\quad + b_{-1,C,0}) - E_{-1,A^2BC,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} \\
&\quad + b_{-1,BC,1,BC}\tilde{y}_{1,BC} + b_{-1,BC,0}) - E_{-1,A^2BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC})
\end{aligned}$$

$$
\begin{aligned}
&+B_{1,A^2BC,A}(b_{1,A,-1,A}\tilde{y}_{-1,A}+b_{1,A,1,B}\tilde{y}_{1,B}+b_{1,A,1,C}\tilde{y}_{1,C}+b_{1,A,1,BC}\tilde{y}_{1,BC}\\
&+b_{1,A,0})+B_{1,A^2BC,B}\tilde{y}_{1,B}+B_{1,A^2BC,C}\tilde{y}_{1,C}+B_{1,A^2BC,BC}\tilde{y}_{1,BC}\\
&+B_{1,A^2BC,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{-1,A^2BC}\\
&-E_{1,A^2BC,A}(b_{1,A,-1,A}\tilde{y}_{-1,A}+b_{1,A,1,B}\tilde{y}_{1,B}+b_{1,A,1,C}\tilde{y}_{1,C}+b_{1,A,1,BC}\tilde{y}_{1,BC}+b_{1,A,0})\\
&-E_{1,A^2BC,B}\tilde{y}_{1,B}-E_{1,A^2BC,C}\tilde{y}_{1,C}-E_{1,A^2BC,BC}\tilde{y}_{1,BC}\\
&-E_{1,A^2BC,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})\\
&+B_{-1,A^2BC,A}\tilde{y}_{-1,A}\\
&+B_{-1,A^2BC,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A}+b_{-1,B,1,B}\tilde{y}_{1,B}+b_{-1,B,1,C}\tilde{y}_{1,C}\\
&+b_{-1,B,1,BC}\tilde{y}_{1,BC}+b_{-1,B,0})\\
&+B_{-1,A^2BC,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A}+b_{-1,C,1,B}\tilde{y}_{1,B}+b_{-1,C,1,C}\tilde{y}_{1,C}\\
&+b_{-1,C,1,BC}\tilde{y}_{1,BC}+b_{-1,C,0})\\
&+B_{-1,A^2BC,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A}+b_{-1,BC,1,B}\tilde{y}_{1,B}+b_{-1,BC,1,C}\tilde{y}_{1,C}\\
&+b_{-1,BC,1,BC}\tilde{y}_{1,BC}+b_{-1,BC,0})\\
&+B_{-1,A^2BC,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{1,A^2BC}]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{d}_{1,B} \;=\; &-E_{1,B,A}(b_{1,A,-1,A}\tilde{y}_{-1,A}+b_{1,A,1,B}\tilde{y}_{1,B}+b_{1,A,1,C}\tilde{y}_{1,C}+b_{1,A,1,BC}\tilde{y}_{1,BC}+b_{1,A,0})\\
&-E_{1,B,B}\tilde{y}_{1,B}-E_{1,B,C}\tilde{y}_{1,C}-E_{1,B,BC}\tilde{y}_{1,BC}-E_{1,B,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})\\
&+B_{-1,B,A}\tilde{y}_{-1,A}+B_{-1,B,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A}+b_{-1,B,1,B}\tilde{y}_{1,B}+b_{-1,B,1,C}\tilde{y}_{1,C}\\
&+b_{-1,B,1,BC}\tilde{y}_{1,BC}+b_{-1,B,0})\\
&+B_{-1,B,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A}+b_{-1,C,1,B}\tilde{y}_{1,B}+b_{-1,C,1,C}\tilde{y}_{1,C}+b_{-1,C,1,BC}\tilde{y}_{1,BC}\\
&+b_{-1,C,0})\\
&+B_{-1,B,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A}+b_{-1,BC,1,B}\tilde{y}_{1,B}+b_{-1,BC,1,C}\tilde{y}_{1,C}\\
&+b_{-1,BC,1,BC}\tilde{y}_{1,BC}+b_{-1,BC,0})\\
&+B_{-1,B,A^2BC}(2\tilde{y}_{-1,A}+\tilde{y}_{1,BC})+C_{1,B}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{d}_{1,C} &= -E_{1,C,A}(b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0}) \\
&\quad -E_{1,C,B}\tilde{y}_{1,B} - E_{1,C,C}\tilde{y}_{1,C} - E_{1,C,BC}\tilde{y}_{1,BC} \\
&\quad -E_{1,C,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{-1,C,A}\tilde{y}_{-1,A} \\
&\quad +B_{-1,C,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} + b_{-1,B,1,BC}\tilde{y}_{1,BC} \\
&\quad +b_{-1,B,0}) \\
&\quad +B_{-1,C,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC} \\
&\quad +b_{-1,C,0}) \\
&\quad +B_{-1,C,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} \\
&\quad +b_{-1,BC,1,BC}\tilde{y}_{1,BC} + b_{-1,BC,0}) \\
&\quad +B_{-1,C,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{1,C} \\
\tilde{d}_{1,BC} &= -E_{1,BC,A}(b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0}) \\
&\quad -E_{1,BC,B}\tilde{y}_{1,B} - E_{1,BC,C}\tilde{y}_{1,C} + B_{1,BC,BC}\tilde{y}_{1,BC} \\
&\quad -E_{1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{-1,BC,A}\tilde{y}_{-1,A} \\
&\quad +B_{-1,BC,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} + b_{-1,B,1,BC}\tilde{y}_{1,BC} \\
&\quad +b_{-1,B,0}) \\
&\quad +B_{-1,BC,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC} \\
&\quad +b_{-1,C,0}) \\
&\quad +B_{-1,BC,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} \\
&\quad +b_{-1,BC,1,BC}\tilde{y}_{1,BC} + b_{-1,BC,0}) \\
&\quad +B_{-1,BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{1,BC} - E_{-1,A^2BC,A}\tilde{y}_{-1,A} \\
&\quad -E_{-1,A^2BC,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} + b_{-1,B,1,BC}\tilde{y}_{1,BC} \\
&\quad +b_{-1,B,0}) \\
&\quad -E_{-1,A^2BC,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC}
\end{aligned}
$$

$+b_{-1,C,0})$

$-E_{-1,A^2BC,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C}$

$+b_{-1,BC,1,BC}\tilde{y}_{1,BC} + b_{-1,BC,0})$

$-E_{-1,A^2BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC})$

$+B_{1,A^2BC,A}(b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0})$

$+B_{1,A^2BC,B}\tilde{y}_{1,B} + B_{1,A^2BC,C}\tilde{y}_{1,C} + B_{1,A^2BC,BC}\tilde{y}_{1,BC}$

$+B_{1,A^2BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{-1,A^2BC}$

$-E_{1,A^2BC,A}(b_{1,A,-1,A}\tilde{y}_{-1,A} + b_{1,A,1,B}\tilde{y}_{1,B} + b_{1,A,1,C}\tilde{y}_{1,C} + b_{1,A,1,BC}\tilde{y}_{1,BC} + b_{1,A,0})$

$-E_{1,A^2BC,B}\tilde{y}_{1,B} - E_{1,A^2BC,C}\tilde{y}_{1,C} - E_{1,A^2BC,BC}\tilde{y}_{1,BC}$

$-E_{1,A^2BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + B_{-1,A^2BC,A}\tilde{y}_{-1,A}$

$+B_{-1,A^2BC,B}(b_{-1,B,-1,A}\tilde{y}_{-1,A} + b_{-1,B,1,B}\tilde{y}_{1,B} + b_{-1,B,1,C}\tilde{y}_{1,C} + b_{-1,B,1,BC}\tilde{y}_{1,BC}$

$+b_{-1,B,0})$

$+B_{-1,A^2BC,C}(b_{-1,C,-1,A}\tilde{y}_{-1,A} + b_{-1,C,1,B}\tilde{y}_{1,B} + b_{-1,C,1,C}\tilde{y}_{1,C} + b_{-1,C,1,BC}\tilde{y}_{1,BC}$

$+b_{-1,C,0})$

$+B_{-1,A^2BC,BC}(b_{-1,BC,-1,A}\tilde{y}_{-1,A} + b_{-1,BC,1,B}\tilde{y}_{1,B} + b_{-1,BC,1,C}\tilde{y}_{1,C} + b_{-1,BC,1,BC}\tilde{y}_{1,BC}$

$+b_{-1,BC,0})$

$+B_{-1,A^2BC,A^2BC}(2\tilde{y}_{-1,A} + \tilde{y}_{1,BC}) + C_{1,A^2BC}$

Thus, the demand model given by (1.3.25) is obtained for the setting with no alliance.

For the example above, the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ are given by E, B, C as follows:

$$\begin{aligned}
\tilde{E}_{-1,A,A} &= E_{-1,A,A} + E_{-1,A,B}b_{-1,B,-1,A} + E_{-1,A,C}b_{-1,C,-1,A} + E_{-1,A,BC}b_{-1,BC,-1,A} \\
&\quad +2E_{-1,A,A^2BC} - B_{1,A,A}b_{1,A,-1,A} - 2B_{1,A,A^2BC} \\
&\quad +2(E_{-1,A^2BC,A} + E_{-1,A^2BC,B}b_{-1,B,-1,A} + E_{-1,A^2BC,C}b_{-1,C,-1,A} \\
&\quad +E_{-1,A^2BC,BC}b_{-1,BC,-1,A} + 2E_{-1,A^2BC,A^2BC} - B_{1,A^2BC,A}b_{1,A,-1,A} \\
&\quad -2B_{1,A^2BC,A^2BC} + E_{1,A^2BC,A}b_{1,A,-1,A} + 2E_{1,A^2BC,A^2BC} \\
&\quad -B_{-1,A^2BC,A} - B_{-1,A^2BC,B}b_{-1,B,-1,A} - B_{-1,A^2BC,C}b_{-1,C,-1,A} \\
&\quad -B_{-1,A^2BC,BC}b_{-1,BC,-1,A} - 2B_{-1,A^2BC,A^2BC}) \\
\tilde{E}_{1,B,B} &= E_{1,B,A}b_{1,A,1,B} + E_{1,B,B} - B_{-1,B,B}b_{-1,B,1,B} - B_{-1,B,C}b_{-1,C,1,B} \\
&\quad -B_{-1,B,BC}b_{-1,BC,1,B} \\
\tilde{E}_{1,B,C} &= E_{1,B,A}b_{1,A,1,C} + E_{1,B,C} - B_{-1,B,B}b_{-1,B,1,C} - B_{-1,B,C}b_{-1,C,1,C} \\
&\quad -B_{-1,B,BC}b_{-1,BC,1,C} \\
\tilde{E}_{1,B,BC} &= E_{1,B,A}b_{1,A,1,BC} + E_{1,B,BC} - B_{-1,B,B}b_{-1,B,1,BC} - B_{-1,B,C}b_{-1,C,1,BC} \\
&\quad -B_{-1,B,BC}b_{-1,BC,1,BC} \\
\tilde{E}_{1,C,B} &= E_{1,C,A}b_{1,A,1,B} + E_{1,C,B} - B_{-1,C,B}b_{-1,B,1,B} - B_{-1,C,C}b_{-1,C,1,B} \\
&\quad -B_{-1,C,BC}b_{-1,BC,1,B} \\
\tilde{E}_{1,C,C} &= E_{1,C,A}b_{1,A,1,C} + E_{1,C,C} - B_{-1,C,B}b_{-1,C,1,C} - B_{-1,C,C}b_{-1,C,1,C} \\
&\quad -B_{-1,C,BC}b_{-1,BC,1,C} \\
\tilde{E}_{1,C,BC} &= E_{1,C,A}b_{1,A,1,BC} + E_{1,C,BC} - B_{-1,C,B}b_{-1,B,1,BC} - B_{-1,C,C}b_{-1,C,1,BC} \\
&\quad -B_{-1,C,BC}b_{-1,BC,1,BC} \\
\tilde{E}_{1,BC,B} &= E_{1,BC,A}b_{1,A,1,B} + E_{1,BC,B} - B_{-1,BC,B}b_{-1,B,1,B} - B_{-1,BC,C}b_{-1,BC,1,B} \\
&\quad -B_{-1,BC,BC}b_{-1,BC,1,B} + E_{-1,A^2BC,B}b_{-1,B,1,B} + E_{-1,A^2BC,C}b_{-1,C,1,B} \\
&\quad +E_{-1,A^2BC,BC}b_{-1,BC,1,B} - B_{1,A^2BC,A}b_{1,A,1,B} - B_{1,A^2BC,B} \\
&\quad +E_{1,A^2BC,A}b_{1,A,1,B} + E_{1,A^2BC,B} - B_{-1,A^2BC,B}b_{-1,B,1,B} \\
&\quad -B_{-1,A^2BC,C}b_{-1,C,1,B} - B_{-1,A^2BC,BC}b_{-1,BC,1,B}
\end{aligned}$$

$$
\begin{aligned}
\tilde{E}_{1,BC,C} &= E_{1,BC,A}b_{1,A,1,C} + E_{1,BC,C} - B_{-1,BC,B}b_{-1,B,1,C} - B_{-1,BC,C}b_{-1,BC,1,C} \\
&\quad - B_{-1,BC,BC}b_{-1,BC,1,C} + E_{-1,A^2BC,B}b_{-1,B,1,C} + E_{-1,A^2BC,C}b_{-1,C,1,C} \\
&\quad + E_{-1,A^2BC,BC}b_{-1,BC,1,C} - B_{1,A^2BC,A}b_{1,A,1,C} - B_{1,A^2BC,C} \\
&\quad + E_{1,A^2BC,A}b_{1,A,1,C} + E_{1,A^2BC,C} - B_{-1,A^2BC,B}b_{-1,B,1,C} \\
&\quad - B_{-1,A^2BC,C}b_{-1,C,1,C} - B_{-1,A^2BC,BC}b_{-1,BC,1,C} \\
\tilde{E}_{1,BC,BC} &= E_{1,BC,A}b_{1,A,1,BC} - B_{1,BC,BC} + E_{1,BC,A^2BC} \\
&\quad - B_{-1,BC,B}b_{-1,B,1,BC} - B_{-1,BC,C}b_{-1,BC,1,BC} - B_{-1,BC,BC}b_{-1,BC,1,BC} \\
&\quad - B_{-1,BC,A^2BC} + E_{-1,A^2BC,B}b_{-1,B,1,BC} + E_{-1,A^2BC,C}b_{-1,C,1,BC} \\
&\quad + E_{-1,A^2BC,BC}b_{-1,BC,1,BC} + E_{-1,A^2BC,A^2BC} - B_{1,A^2BC,A}b_{1,A,1,BC} \\
&\quad - B_{1,A^2BC,BC} - B_{1,A^2BC,A^2BC} + E_{1,A^2BC,A}b_{1,A,1,BC} \\
&\quad + E_{1,A^2BC,BC} + E_{1,A^2BC,A^2BC} - B_{-1,A^2BC,B}b_{-1,B,1,BC} \\
&\quad - B_{-1,A^2BC,C}b_{-1,C,1,BC} - B_{-1,A^2BC,BC}b_{-1,BC,1,BC} - B_{-1,A^2BC,A^2BC} \\
\tilde{B}_{-1,B,A} &= -E_{1,B,A}b_{1,A,-1,A} - 2E_{1,B,A^2BC} + B_{-1,B,A} \\
&\quad + B_{-1,B,B}b_{-1,B,-1,A} + B_{-1,B,C}b_{-1,C,-1,A} + B_{-1,B,BC}b_{-1,BC,-1,A} \\
&\quad + 2B_{-1,B,A^2BC} \\
\tilde{B}_{-1,C,A} &= -E_{1,C,A}b_{1,A,-1,A} - 2E_{1,C,A^2BC} + B_{-1,C,A} - B_{-1,C,B}b_{-1,B,-1,A} \\
&\quad + B_{-1,C,C}b_{-1,C,-1,A} + B_{-1,C,BC}b_{-1,BC,-1,A} + 2B_{-1,C,A^2BC} \\
\tilde{B}_{-1,BC,A} &= -E_{1,BC,A}b_{1,A,-1,A} - 2E_{1,BC,A^2BC} + B_{-1,BC,B}b_{-1,B,-1,A} \\
&\quad + B_{-1,BC,C}b_{-1,C,-1,A} + B_{-1,BC,BC}b_{-1,BC,-1,A} + 2B_{-1,BC,A^2BC} \\
&\quad - E_{-1,A^2BC,A} - E_{-1,A^2BC,B}b_{-1,B,-1,A} - E_{-1,A^2BC,C}b_{-1,C,-1,A} \\
&\quad - E_{-1,A^2BC,BC}b_{-1,BC,-1,A} - 2E_{-1,A^2BC,A^2BC} + B_{1,A^2BC,A}b_{1,A,-1,A} \\
&\quad + 2B_{1,A^2BC,A^2BC} - E_{1,A^2BC,A}b_{1,A,-1,A} - 2E_{1,A^2BC,A^2BC} \\
&\quad + B_{-1,A^2BC,B}b_{-1,B,-1,A} + B_{-1,A^2BC,C}b_{-1,C,-1,A} + B_{-1,A^2BC,BC}b_{-1,BC,-1,A} \\
&\quad + 2B_{-1,A^2BC,A^2BC}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{B}_{1,A,B} &= -E_{1,A,B}b_{-1,B,1,B} - E_{-1,A,C}b_{-1,C,1,B} - E_{-1,A,BC}b_{-1,BC,1,B} \\
&\quad +B_{1,A,A}b_{1,A,1,B} + B_{1,A,B} - 2(E_{-1,A^2BC,B}b_{-1,B,1,B} \\
&\quad -E_{-1,A^2BC,C}b_{-1,C,1,B} - E_{-1,A^2BC,BC}b_{-1,BC,1,B} \\
&\quad +B_{1,A^2BC,A}b_{1,A,1,B} + B_{1,A^2BC,B} - E_{1,A^2BC,A}b_{1,A,1,B} - E_{1,A^2BC,B} \\
&\quad +B_{-1,A^2BC,B}b_{-1,B,1,B} + B_{-1,A^2BC,C}b_{-1,C,1,B} + B_{-1,A^2BC,BC}b_{-1,BC,1,B}) \\
\tilde{B}_{1,A,C} &= -E_{1,A,B}b_{-1,B,1,C} - E_{-1,A,C}b_{-1,C,1,C} - E_{-1,A,BC}b_{-1,BC,1,C} \\
&\quad +B_{1,A,A}b_{1,A,1,C} + B_{1,A,C} - 2(E_{-1,A^2BC,B}b_{-1,B,1,C} \\
&\quad -E_{-1,A^2BC,C}b_{-1,C,1,C} - E_{-1,A^2BC,BC}b_{-1,BC,1,C} \\
&\quad +B_{1,A^2BC,A}b_{1,A,1,C} + B_{1,A^2BC,C} - E_{1,A^2BC,A}b_{1,A,1,C} - E_{1,A^2BC,C} \\
&\quad +B_{-1,A^2BC,B}b_{-1,B,1,C} + B_{-1,A^2BC,C}b_{-1,C,1,C} + B_{-1,A^2BC,BC}b_{-1,BC,1,C}) \\
\tilde{B}_{1,A,BC} &= -E_{1,A,B}b_{-1,B,1,BC} - E_{-1,A,C}b_{-1,C,1,BC} - E_{-1,A,BC}b_{-1,BC,1,BC} \\
&\quad -E_{-1,A,A^2BC} + B_{1,A,A}b_{1,A,1,BC} + B_{1,A,BC} + B_{1,A,A^2BC} \\
&\quad -2(E_{-1,A^2BC,B}b_{-1,B,1,BC} - E_{-1,A^2BC,C}b_{-1,C,1,BC} - E_{-1,A^2BC,BC}b_{-1,BC,1,BC} \\
&\quad -E_{-1,A^2BC,A^2BC} + B_{1,A^2BC,A}b_{1,A,1,BC} + B_{1,A^2BC,BC} \\
&\quad +B_{1,A^2BC,A^2BC} - E_{1,A^2BC,A}b_{1,A,1,BC} - E_{1,A^2BC,BC} - E_{1,A^2BC,A^2BC} \\
&\quad +B_{-1,A^2BC,B}b_{-1,B,1,BC} + B_{-1,A^2BC,C}b_{-1,C,1,BC} + B_{-1,A^2BC,BC}b_{-1,BC,1,BC} \\
&\quad +B_{-1,A^2BC,A^2BC}) \\
\tilde{C}_{-1,A} &= C_{-1,A} + 2(C_{-1,A^2BC} + C_{1,A^2BC}) \\
\tilde{C}_{1,B} &= C_{1,B} \\
\tilde{C}_{1,C} &= C_{1,C} \\
\tilde{C}_{1,BC} &= C_{1,BC} + C_{-1,A^2BC} + C_{1,A^2BC}
\end{aligned}
$$

To state the relation between parameters E, B, C in demand model (1.3.14) and the parameters $\tilde{E}, \tilde{B}, \tilde{C}$ in demand model (1.3.25) in general, we first develop the notation needed for a concise representation. Let the rows and columns of matrix E_i be grouped so that the first group of rows and columns correspond to products in L_i, the second group of rows and columns correspond to products in $L_{0,i}$, and the third group of rows and columns correspond to products in $L_{-i,i}$. Hence E_i can be partitioned into submatrices as follows:

$$E_i = \begin{array}{c} \begin{array}{ccc} L_i & L_{0,i} & L_{-i,i} \end{array} \\ \left[\begin{array}{ccc} E_{i,i} & E_{i,0,i} & E_{i,-i,i} \\ E_{0,i,i} & E_{0,i,0,i} & E_{0,i,-i,i} \\ E_{-i,i,i} & E_{-i,i,0,i} & E_{-i,i,-i,i} \end{array} \right] \end{array} \begin{array}{c} \\ L_i \\ L_{0,i} \\ L_{-i,i} \end{array}$$

This grouping of the rows and columns of E_i implies that the rows and columns of d_i, y_i, B_i, and C_i are similarly grouped:

$$B_{-i} = \begin{array}{c} \begin{array}{ccc} L_{-i} & L_{0,-i} & L_{i,-i} \end{array} \\ \left[\begin{array}{ccc} B_{i,-i} & B_{i,0,-i} & B_{i,i,-i} \\ B_{0,i,-i} & B_{0,i,0,-i} & B_{0,i,i,-i} \\ B_{-i,i,-i} & B_{-i,i,0,-i} & B_{-i,i,i,-i} \end{array} \right] \end{array} \begin{array}{c} \\ L_i \\ L_{0,i} \\ L_{-i,i} \end{array} , \quad y_i = \left[\begin{array}{c} y_{i,i} \\ y_{i,0,i} \\ y_{i,-i,i} \end{array} \right]$$

$$C_i = \left[\begin{array}{c} C_{i,i} \\ C_{i,0,i} \\ C_{i,-i,i} \end{array} \right] , \quad d_i = \left[\begin{array}{c} d_{i,i} \\ d_{i,0,i} \\ d_{i,-i,i} \end{array} \right]$$

Note that given the prices $\tilde{y}$ in the no alliance setting, the prices for the same products in the alliance setting are $y_{i,i} = \tilde{y}_i \in \mathbb{R}^{|L_i|}$. Let $R_{i,i',\ell,\ell'}$ denote the number of units of product $\ell' \in L_{i'}$ used to assemble one unit of product $\ell \in L_{0,i}$. Then, given the prices $\tilde{y}$ in the no alliance setting, the price paid to assemble one unit of product $\ell \in L_{0,i}$ in the no alliance setting is

$$\sum_{i'=\pm 1} \sum_{\ell' \in L_{i'}} R_{i,i',\ell,\ell'} \tilde{y}_{i',\ell'}$$

Let $R_{i,i'} \in \mathbb{R}^{|L_{0,i}|\times|L_{i'}|}$ denote the matrix with entry $R_{i,i',\ell,\ell'}$ in the row corresponding to $\ell \in L_{0,i}$ and the column corresponding to $\ell' \in L_{i'}$. Then, given the prices $\tilde{y}$ in the no alliance setting, the prices paid to assemble each unit of product in $L_{0,i}$ is given by

$$y_{i,0,i} = \sum_{i'=\pm 1} R_{i,i'}\tilde{y}_{i'}$$

Next, consider the demand for products in $L_{-i,i}$.

$$\begin{aligned} d_{i,-i,i} &= -E_{-i,i,i}y_{i,i} - E_{-i,i,0,i}y_{i,0,i} - E_{-i,i,-i,i}y_{i,-i,i} + B_{-i,i,-i}y_{-i,-i} + B_{-i,i,0,-i}y_{-i,0,-i} \\ &\quad + B_{-i,i,i,-i}y_{-i,i,-i} + C_{i,-i,i} \\ &= -E_{-i,i,i}\tilde{y}_i - E_{-i,i,0,i}\sum_{i'=\pm 1} R_{i,i'}\tilde{y}_{i'} - E_{-i,i,-i,i}y_{i,-i,i} \\ &\quad + B_{-i,i,-i}\tilde{y}_{-i} + B_{-i,i,0,-i}\sum_{i'=\pm 1} R_{-i,i'}\tilde{y}_{i'} + B_{-i,i,i,-i}y_{-i,i,-i} + C_{i,-i,i} \end{aligned}$$

Then, given the prices $\tilde{y}$ in the no alliance setting, the value of $(y_{-1,1,-1}, y_{1,-1,1})$ is chosen to set $(d_{-1,1,-1}, d_{1,-1,1}) = 0$. The system of equations $(d_{-1,1,-1}, d_{1,-1,1}) = 0$ can be written as $-Dy_- + F\tilde{y} + C_- = 0$, where

$$y_- := \begin{bmatrix} y_{-1,1,-1} \\ y_{1,-1,1} \end{bmatrix}, \quad \tilde{y} := \begin{bmatrix} \tilde{y}_{-1} \\ \tilde{y}_1 \end{bmatrix}, \quad C_- := \begin{bmatrix} C_{-1,1,-1} \\ C_{1,-1,1} \end{bmatrix}$$

$$D := \begin{bmatrix} E_{1,-1,1,-1} & -B_{1,-1,-1,1} \\ -B_{-1,1,1,-1} & E_{-1,1,-1,1} \end{bmatrix}$$

$$F := \begin{bmatrix} -E_{1,-1,-1} - E_{1,-1,0,-1}R_{-1,-1} + B_{1,-1,0,1}R_{1,-1} & -E_{1,-1,0,-1}R_{-1,1} + B_{1,-1,1} + B_{1,-1,0,1}R_{1,1} \\ -E_{-1,1,0,1}R_{1,-1} + B_{-1,1,-1} + B_{-1,1,0,-1}R_{-1,-1} & -E_{-1,1,1} - E_{-1,1,0,1}R_{1,1} + B_{-1,1,0,-1}R_{-1,1} \end{bmatrix}$$

Under reasonable conditions D is nonsingular (more specifically, positive definite), and then the unique solution is $y_- = D^{-1}F\tilde{y} + D^{-1}C_-$. Let

$$
\begin{array}{cccc}
 & L_{1,-1} & L_{-1,1} & \\
D^{-1} = & \left[\begin{array}{cc} D^{-1}_{-1,-1} & D^{-1}_{-1,1} \\ D^{-1}_{1,-1} & D^{-1}_{1,1} \end{array}\right] & & \begin{array}{c} L_{1,-1} \\ L_{-1,1} \end{array}
\end{array}
, \qquad
\begin{array}{cccc}
 & L_{-1} & L_{1} & \\
F = & \left[\begin{array}{cc} F_{-1,-1} & F_{-1,1} \\ F_{1,-1} & F_{1,1} \end{array}\right] & & \begin{array}{c} L_{1,-1} \\ L_{-1,1} \end{array}
\end{array}
$$

Then

$$
\begin{aligned}
y_{i,-i,i} &= (D^{-1}_{i,-i}F_{-i,i} + D^{-1}_{i,i}F_{i,i})\tilde{y}_i + (D^{-1}_{i,-i}F_{-i,-i} + D^{-1}_{i,i}F_{i,-i})\tilde{y}_{-i} \\
&\quad + (D^{-1}_{i,-i}C_{-i,i,-i} + D^{-1}_{i,i}C_{i,-i,i}) \\
&= \sum_{i'=\pm 1}\left(\sum_{i''=\pm 1} D^{-1}_{i,i''}F_{i'',i'}\tilde{y}_{i'} + D^{-1}_{i,i'}C_{i',-i',i'}\right)
\end{aligned}
$$

Next, the demand model (1.3.14) is used to derive the demand for each product $\ell \in L_i$ that is offered in the no alliance setting:

$$
\begin{aligned}
d_{i,\ell} = &\Bigg[-\sum_{\ell'\in L_i} E_{i,\ell,\ell'}y_{i,i,\ell'} - \sum_{\ell'\in L_{0,i}} E_{i,\ell,\ell'}y_{i,0,i,\ell'} - \sum_{\ell'\in L_{-i,i}} E_{i,\ell,\ell'}y_{i,-i,i,\ell'} \\
&+ \sum_{\ell'\in L_{-i}} B_{-i,\ell,\ell'}y_{-i,-i,\ell'} + \sum_{\ell'\in L_{0,-i}} B_{-i,\ell,\ell'}y_{-i,0,-i,\ell'} + \sum_{\ell'\in L_{i,-i}} B_{-i,\ell,\ell'}y_{-i,i,-i,\ell'} + C_{i,\ell}\Bigg] \\
&+ \sum_{i'=\pm 1}\Bigg[\sum_{\ell'\in L_{0,i'}} R_{i',i,\ell',\ell}\Bigg(-\sum_{\ell''\in L_{i'}} E_{i',\ell',\ell''}y_{i',i',\ell''} - \sum_{\ell''\in L_{0,i'}} E_{i',\ell',\ell''}y_{i',0,i',\ell''} \\
&- \sum_{\ell''\in L_{-i',i'}} E_{i',\ell',\ell''}y_{i',-i',i',\ell''} + \sum_{\ell''\in L_{-i'}} B_{-i',\ell',\ell''}y_{-i',-i',\ell''} \\
&+ \sum_{\ell''\in L_{0,-i'}} B_{-i',\ell',\ell''}y_{-i',0,-i',\ell''} + \sum_{\ell''\in L_{i',-i'}} B_{-i',\ell',\ell''}y_{-i',i',-i',\ell''} + C_{i',\ell'}\Bigg)\Bigg]
\end{aligned}
$$

The first term in brackets above corresponds to the demand for product $\ell \in L_i$ by itself, and the second term in brackets corresponds to the demand for product ℓ to assemble products $\ell' \in L_{0,i'}$, $i' = \pm 1$. In terms of matrix notation, the demands for the products in L_i that are offered in the no alliance setting is given by

$$\begin{aligned} d_{i,i} &= [-E_{i,i}y_{i,i} - E_{i,0,i}y_{i,0,i} - E_{i,-i,i}y_{i,-i,i} + B_{i,-i}y_{-i,-i} + B_{i,0,-i}y_{-i,0,-i} \\ &\quad +B_{i,i,-i}y_{-i,i,-i} + C_{i,i}] \\ &\quad + \sum_{i'=\pm 1} \left[R^{\mathsf{T}}_{i',i}\,(-E_{0,i',i'}y_{i',i'} - E_{0,i',0,i'}y_{i',0,i'} - E_{0,i',-i',i'}y_{i',-i',i'} \right. \\ &\qquad\qquad \left. +B_{0,i',-i'}y_{-i',-i'} + B_{0,i',0,-i'}y_{-i',0,-i'} + B_{0,i',i',-i'}y_{-i',i',-i'} + C_{i',0,i'})\right] \end{aligned}$$

Next, replace $y_{i,i}$, $y_{i,0,i}$, and $y_{i,-i,i}$ with the expressions in terms of $\tilde{y}$ derived above. Then the demands $\tilde{d}_i$ for the products in L_i in the no alliance setting as a function of the prices $\tilde{y}$ in the no alliance setting are obtained, as follows:

$$\begin{aligned} \tilde{d}_i &= \Bigg[-E_{i,i}\tilde{y}_i - E_{i,0,i}\sum_{i'=\pm 1} R_{i,i'}\tilde{y}_{i'} - E_{i,-i,i}\sum_{i'=\pm 1}\left(\sum_{i''=\pm 1} D^{-1}_{i,i''}F_{i'',i'}\tilde{y}_{i'} + D^{-1}_{i,i'}C_{i',-i',i'}\right) \\ &\quad +B_{i,-i}\tilde{y}_{-i} + B_{i,0,-i}\sum_{i'=\pm 1} R_{-i,i'}\tilde{y}_{i'} \\ &\quad +B_{i,i,-i}\sum_{i'=\pm 1}\left(\sum_{i''=\pm 1} D^{-1}_{-i,i''}F_{i'',i'}\tilde{y}_{i'} + D^{-1}_{-i,i'}C_{i',-i',i'}\right) + C_{i,i}\Bigg] \\ &\quad + \sum_{i'=\pm 1}\Big[R^{\mathsf{T}}_{i',i}\,(-E_{0,i',i'}\tilde{y}_{i'} \\ &\quad -E_{0,i',0,i'}\sum_{i''=\pm 1} R_{i',i''}\tilde{y}_{i''} - E_{0,i',-i',i'}\sum_{i''=\pm 1}\left(\sum_{i'''=\pm 1} D^{-1}_{i',i'''}F_{i''',i''}\tilde{y}_{i''} + D^{-1}_{i',i''}C_{i'',-i'',i''}\right) \\ &\qquad +B_{0,i',-i'}\tilde{y}_{-i'} + B_{0,i',0,-i'}\sum_{i''=\pm 1} R_{-i',i''}\tilde{y}_{i''} \\ &\qquad +B_{0,i',i',-i'}\sum_{i''=\pm 1}\left(\sum_{i'''=\pm 1} D^{-1}_{-i',i'''}F_{i''',i''}\tilde{y}_{i''} + D^{-1}_{-i',i''}C_{i'',-i'',i''}\right) \\ &\quad +C_{i',0,i'})\Big] \end{aligned}$$

Note that the demands $\tilde{d}_i$ above are consistent with the demand model (1.3.25), for the following parameter values:

$$
\begin{aligned}
\tilde{E}_i &= E_{i,i} + E_{i,0,i}R_{i,i} + E_{i,-i,i}\sum_{i'=\pm 1} D^{-1}_{i,i'}F_{i',i} \\
&\quad -B_{i,0,-i}R_{-i,i} - B_{i,i,-i}\sum_{i'=\pm 1} D^{-1}_{-i,i'}F_{i',i} + R^{\mathsf{T}}_{i,i}E_{0,i,i} - R^{\mathsf{T}}_{-i,i}B_{0,-i,i} \\
&\quad + \sum_{i'=\pm 1} R^{\mathsf{T}}_{i',i}\Bigg(E_{0,i',0,i'}R_{i',i} + E_{0,i',-i',i'}\sum_{i''=\pm 1} D^{-1}_{i',i''}F_{i'',i} \\
&\quad -B_{0,i',0,-i'}R_{-i',i} - B_{0,i',i',-i'}\sum_{i''=\pm 1} D^{-1}_{-i',i''}F_{i'',i}\Bigg) \\
\tilde{B}_{-i} &= -E_{i,0,i}R_{i,-i} - E_{i,-i,i}\sum_{i'=\pm 1} D^{-1}_{i,i'}F_{i',-i} + B_{i,-i} + B_{i,0,-i}R_{-i,-i} \\
&\quad +B_{i,i,-i}\sum_{i'=\pm 1} D^{-1}_{-i,i'}F_{i',-i} - R^{\mathsf{T}}_{-i,i}E_{0,-i,-i} + R^{\mathsf{T}}_{i,i}B_{0,i,-i} \\
&\quad + \sum_{i'=\pm 1} R^{\mathsf{T}}_{i',i}\Bigg(-E_{0,i',0,i'}R_{i',-i} - E_{0,i',-i',i'}\sum_{i''=\pm 1} D^{-1}_{i',i''}F_{i'',-i} \\
&\quad +B_{0,i',0,-i'}R_{-i',-i} + B_{0,i',i',-i'}\sum_{i''=\pm 1} D^{-1}_{-i',i''}F_{i'',-i}\Bigg) \\
\tilde{C}_i &= -E_{i,-i,i}\sum_{i'=\pm 1} D^{-1}_{i,i'}C_{i',-i',i'} + B_{i,i,-i}\sum_{i'=\pm 1} D^{-1}_{-i,i'}C_{i',-i',i'} + C_{i,i} \\
&\quad + \sum_{i'=\pm 1} R^{\mathsf{T}}_{i',i}\Bigg(-E_{0,i',-i',i'}\sum_{i''=\pm 1} D^{-1}_{i',i''}C_{i'',-i'',i''} + B_{0,i',i',-i'}\sum_{i''=\pm 1} D^{-1}_{-i',i''}C_{i'',-i'',i''} \\
&\quad +C_{i',0,i'})
\end{aligned}
$$

APPENDIX C

ASYMPTOTICS FOR LSR ESTIMATOR OF VALUE-AT-RISK

Suppose, for the sake of simplicity, that support of the distribution of $\boldsymbol{X}_i$ is bounded, i.e., $\boldsymbol{X}_i$ is bounded w.p.1. Since $N^{-1}\mathbb{X}^{\mathsf{T}}\mathbb{X}$ converges w.p.1 to $\boldsymbol{\Omega}$ and by (2.3.60), we have that

$$|\varepsilon_i - e_i| \leq O_p(N^{-1}) \sum_{j=1}^{N} \varepsilon_j.$$

We can assume here that $\mathbb{E}[\varepsilon_i] = 0$, and hence $\sum_{j=1}^{N} \varepsilon_j = O_p(N^{1/2})$. It follows that

$$\left|\varepsilon_{(\lceil N\alpha \rceil)} - e_{(\lceil N\alpha \rceil)}\right| = O_p(N^{-1/2}). \tag{C.0.99}$$

Suppose now that the set of population α-quantiles is a singleton. Then $\hat{F}_{\varepsilon}^{-1}(\alpha)$ converges w.p.1 to the population quantile $F_{\varepsilon}^{-1}(\alpha) = \mathsf{V@R}_{\alpha}(\varepsilon)$, and hence by (C.0.99), we have that $e_{(\lceil N\alpha \rceil)}$ converges in probability to $F_{\varepsilon}^{-1}(\alpha)$. That is, $\widehat{\mathsf{V@R}}_{\alpha}(e)$ is a consistent estimator of $\mathsf{V@R}_{\alpha}(\varepsilon)$, and hence the estimator $\tilde{\beta}_0 + \boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{\beta}} + \widehat{\mathsf{V@R}}_{\alpha}(e)$ is a *consistent estimator* of $\mathsf{V@R}_{\alpha}(\boldsymbol{Y}|\boldsymbol{x})$.

Let us consider the asymptotic efficiency of the residual based $\mathsf{V@R}_{\alpha}$ estimator. It is known that $\tilde{\beta}_0 + \boldsymbol{x}^{\mathsf{T}}\tilde{\boldsymbol{\beta}}$ is an unbiased estimator of the true expected value $\beta_0 + \boldsymbol{x}^{\mathsf{T}}\boldsymbol{\beta}$ and $N^{1/2}\left[\tilde{\beta}_0 - \beta_0^* + \boldsymbol{x}^{\mathsf{T}}(\tilde{\boldsymbol{\beta}} - \boldsymbol{\beta}^*)\right]$ converges in distribution to normal with zero mean and variance

$$\sigma^2[1; \boldsymbol{x}^{\mathsf{T}}]\boldsymbol{\Omega}^{-1}[1; \boldsymbol{x}^{\mathsf{T}}]^{\mathsf{T}}. \tag{C.0.100}$$

Also, $N^{1/2}\left(\varepsilon_{(\lceil N\alpha \rceil)} - \mathsf{V@R}_{\alpha}(\varepsilon)\right)$ converges in distribution to normal with zero mean and variance

$$\omega^2 := \frac{\alpha(1-\alpha)}{\left[f_{\varepsilon}\left(F_{\varepsilon}^{-1}(\alpha)\right)\right]^2}, \tag{C.0.101}$$

provided that distribution of ε has nonzero density $f_\varepsilon(\cdot)$ at the quantile $F_\varepsilon^{-1}(\alpha)$.

Let us also estimate the asymptotic variance of the right hand side of (2.3.60). We have that N times variance of the second term in the right hand side of (2.3.60) can be approximated by

$$\sigma^2\mathbb{E}\left\{[1;\boldsymbol{X}_i^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{X}_i^\mathsf{T}]^\mathsf{T}\right\} = \sigma^2(k+1).$$

We also have that random vectors $(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}})$ and $\boldsymbol{e}$ are uncorrelated. Therefore, if errors ε_i have normal distribution, then vectors $(\tilde{\beta}_0, \tilde{\boldsymbol{\beta}})$ and $\boldsymbol{e}$ have jointly a multivariate normal distribution and these vectors are independent. Consequently, $\tilde{\beta}_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}$ and $\widehat{\mathsf{V@R}}_\alpha(e)$ are independent. For not necessarily normal distribution, this independence holds asymptotically and thus asymptotically $\tilde{\beta}_0+\boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol{\beta}}$ and $\widehat{\mathsf{V@R}}_\alpha(e)$ are uncorrelated.

Now, we can calculate the asymptotic covariance of the corresponding terms $\left(\varepsilon_{(\lceil N\alpha\rceil)} - \mathsf{V@R}_\alpha(\varepsilon)\right)$ and $\left(\varepsilon_{(\lceil N\alpha\rceil)} - e_{(\lceil N\alpha\rceil)}\right)$ as $\frac{-\sigma^2\left(k+1\right)}{2}$. Thus, asymptotic variance of the residual based $\mathsf{V@R}_\alpha$ estimator can be approximated as

$$N^{-1}\left(\omega^2 + \sigma^2[1;\boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{x}^\mathsf{T}]^\mathsf{T}\right). \tag{C.0.102}$$

APPENDIX D

ASYMPTOTICS FOR LSR ESTIMATOR OF AVERAGE VALUE-AT-RISK

The estimator $\widehat{\mathsf{AV@R}}_\alpha(e)$ can be compared with the corresponding random variable which is based on the errors instead of residuals

$$\begin{aligned} \widehat{\mathsf{AV@R}}_\alpha(\varepsilon) &:= \inf_{t\in\mathbb{R}}\left\{t + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=1}^N[\varepsilon_i - t]_+\right\} \\ &= \widehat{\mathsf{V@R}}_\alpha(\varepsilon) + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=1}^N\left[\varepsilon_i - \widehat{\mathsf{V@R}}_\alpha(\varepsilon)\right]_+ \\ &= \varepsilon_{(\lceil N\alpha\rceil)} + \tfrac{1}{(1-\alpha)N}\textstyle\sum_{i=\lceil N\alpha\rceil+1}^N\left[\varepsilon_{(i)} - \varepsilon_{(\lceil N\alpha\rceil)}\right]. \end{aligned} \tag{D.0.103}$$

Note that $\widehat{\mathsf{AV@R}}_\alpha(\varepsilon)$ is not an estimator since errors ε_i are unobservable.

By (C.0.99), we have that

$$\left|\widehat{\mathsf{V@R}}_\alpha(\varepsilon) - \widehat{\mathsf{V@R}}_\alpha(e)\right| = O_p(N^{-1/2}) \tag{D.0.104}$$

and it is known that $\widehat{\mathsf{AV@R}}_\alpha(\varepsilon)$ converges w.p.1 to $\mathsf{AV@R}_\alpha(\varepsilon)$ as $N\to\infty$, provided that ε has a finite first order moment. It follows that $\widehat{\mathsf{AV@R}}_\alpha(e)$ converges in probability to $\mathsf{AV@R}_\alpha(\varepsilon)$, and hence $\tilde\beta_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol\beta} + \widehat{\mathsf{AV@R}}_\alpha(e)$ is a consistent estimator of $\mathsf{AV@R}_\alpha(\boldsymbol{Y}|\boldsymbol{x})$.

Lets discuss asymptotic properties of the residual based $\mathsf{AV@R}_\alpha$ estimator. As it was pointed out in Appendix C, random vectors $(\tilde\beta_0, \tilde{\boldsymbol\beta})$ and $\boldsymbol{e}$ are uncorrelated, and hence asymptotically $\tilde\beta_0 + \boldsymbol{x}^\mathsf{T}\tilde{\boldsymbol\beta}$ and $\widehat{\mathsf{AV@R}}_\alpha(e)$ are independent and hence uncorrelated. Assuming that α-quantile of $F_\varepsilon(\cdot)$ is unique, we have by Delta theorem

$$\widehat{\mathsf{AV@R}}_\alpha(e) = \mathsf{V@R}_\alpha(\varepsilon) + (1-\alpha)^{-1}N^{-1}\sum_{i=1}^N[e_i - \mathsf{V@R}_\alpha(\varepsilon)]_+ + o_p(N^{-1/2}) \tag{D.0.105}$$

and

$$\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) = \mathsf{V@R}_\alpha(\varepsilon) + (1-\alpha)^{-1}N^{-1}\sum_{i=1}^N[\varepsilon_i - \mathsf{V@R}_\alpha(\varepsilon)]_+ + o_p(N^{-1/2}). \tag{D.0.106}$$

Equation (D.0.106) leads to the following asymptotic result (cf. Trindade et al. (2007), Shapiro et al. (2009), section 6.5.1)

$$N^{1/2}\big[\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) - \mathsf{AV@R}_\alpha(\varepsilon)\big] \xrightarrow{\mathcal{D}} \mathcal{N}(0,\gamma^2), \tag{D.0.107}$$

where $\gamma^2 = (1-\alpha)^{-2}\mathrm{Var}\big([\varepsilon - \mathsf{V@R}_\alpha(\varepsilon)]_+\big)$. Moreover, if distribution of ε has nonzero density $f_\varepsilon(\cdot)$ at $\mathsf{V@R}_\alpha(\varepsilon)$, then

$$\mathbb{E}\big[\widehat{\mathsf{AV@R}}_\alpha(\varepsilon)\big] - \mathsf{AV@R}_\alpha(\varepsilon) = -\frac{1-\alpha}{2Nf_\varepsilon(\mathsf{V@R}_\alpha(\varepsilon))} + o(N^{-1}). \tag{D.0.108}$$

From the equation (D.0.105) and (D.0.106), the asymptotic variance of $\big(\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) - \widehat{\mathsf{AV@R}}_\alpha(e)\big)$ can be bounded by $(1-\alpha)^{-1}N^{-2}\sigma^2\big(k+1\big)$ and we can approximate the asymptotic covariance of the corresponding terms, $\Big(\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) - \mathsf{AV@R}_\alpha(\varepsilon)\Big)$ and $\Big(\widehat{\mathsf{AV@R}}_\alpha(\varepsilon) - \widehat{\mathsf{AV@R}}_\alpha(e)\Big)$ as $\frac{-(1-\alpha)^{-1}N^{-2}\sigma^2\big(k+1\big)}{2}$. Thus, asymptotic variance of the residual based $\mathsf{AV@R}_\alpha$ estimator can be approximated as

$$N^{-1}\left(\gamma^2 + \sigma^2[1;\boldsymbol{x}^\mathsf{T}]\boldsymbol{\Omega}^{-1}[1;\boldsymbol{x}^\mathsf{T}]^\mathsf{T}\right). \tag{D.0.109}$$

APPENDIX E

ASYMPTOTICS FOR THE MIXED QUANTILE ESTIMATOR

It is possible to derive asymptotics of the mixed quantile estimator. For the sake of simplicity, let us start with a sample estimate of $S(X)$, with λ_j and α_j, $j = 1, ..., r$, given in (2.2.50). That is, let $X_1, ..., X_N$ be an iid sample (data) of the random variable X, and $X_{(1)} \leq ... \leq X_{(N)}$ be the corresponding order statistics. Then the corresponding sample estimate is obtained by replacing the true distribution F of X by its empirical estimate $\hat{F}$. Consequently, $(1-\alpha)^{-1} S(X)$ is estimated by

$$(1-\alpha)^{-1} \sum_{j=1}^{r} \lambda_j \hat{F}^{-1}(\alpha_j) = \frac{1}{r} \sum_{j=1}^{r} X_{(\lceil N\alpha_j \rceil)}. \tag{E.0.110}$$

This can be compared with the following estimator of $\mathsf{AV@R}_\alpha(X)$ based on sample version of (2.1.35):

$$\begin{array}{c} X_{(\lceil N\alpha \rceil)} + \frac{1}{(1-\alpha)N} \sum_{i=\lceil N\alpha \rceil+1}^{N} \left[X_{(i)} - X_{(\lceil N\alpha \rceil)} \right] = \\ \left(1 - \frac{N - \lceil N\alpha \rceil}{(1-\alpha)N}\right) X_{(\lceil N\alpha \rceil)} + \frac{1}{(1-\alpha)N} \sum_{i=\lceil N\alpha \rceil+1}^{N} X_{(i)}. \end{array} \tag{E.0.111}$$

Assuming that $N\alpha$ is an integer and taking $r := (1-\alpha)N$, we obtain that the right hand sides of (E.0.110) and (E.0.111) are the same.

Asymptotic variance of the mixed quantile estimator can be calculated as follows. Consider problem (2.3.76). The optimal solution of that problem is $\boldsymbol{\beta}^\star = \boldsymbol{\beta}^*$,

$$\eta_j^\star = \beta_0^* + \mathsf{V@R}_{\alpha_j}(\varepsilon) = \beta_0^* + F_\varepsilon^{-1}(\alpha_j),\ j = 1, ..., r,$$

and $\beta_0^\star = \sum_{j=1}^{r} \lambda_j \eta_j^\star = \beta_0^*$. Assume that ε has continuous distribution with cdf $F_\varepsilon(\cdot)$ and density function $f_\varepsilon(\cdot)$. Then conditional on $\boldsymbol{X}$, the asymptotic covariance matrix of the corresponding sample estimator $(\check{\boldsymbol{\beta}}, \check{\boldsymbol{\eta}})$ of $(\boldsymbol{\beta}^\star, \boldsymbol{\eta}^\star)$ is $N^{-1} \boldsymbol{H}^{-1} \boldsymbol{\Sigma} \boldsymbol{H}^{-1}$, where $\boldsymbol{H}$ is the Hessian matrix of second order partial derivatives of $\mathbb{E}\left[\sum_{j=1}^{r} \psi_{\alpha_j}(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X})\right]$ at the point $(\boldsymbol{\beta}^\star, \boldsymbol{\eta}^\star)$, and $\boldsymbol{\Sigma}$ is the covariance matrix of the random vector

$$\boldsymbol{Z} := \sum_{j=1}^{r} \nabla\psi_{\alpha_j}\left(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X}\right),$$

where the gradients are taken with respect to $(\boldsymbol{\beta}, \boldsymbol{\eta})$ at $(\boldsymbol{\beta}, \boldsymbol{\eta}) = (\boldsymbol{\beta}^\star, \boldsymbol{\eta}^\star)$ (e.g., Shapiro (1989)). We have

$$\sum_{j=1}^{r} \nabla_{\boldsymbol{\beta}}\psi_{\alpha_j}\left(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X}\right) = -\left(\sum_{j=1}^{r} \psi'_{\alpha_j}\left(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X}\right)\right)\boldsymbol{X},$$

$$\nabla_{\eta_j}\psi_{\alpha_j}\left(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X}\right) = -\psi'_{\alpha_j}\left(\varepsilon+\beta_0^*-\eta_j+(\boldsymbol{\beta}^*-\boldsymbol{\beta})^{\top}\boldsymbol{X}\right),$$

with $\psi'_{\alpha_j}(\cdot)$ is given in (2.3.72).

Note that $\mathbb{E}[\psi'_{\alpha_j}(\varepsilon - F_\varepsilon^{-1}(\alpha_j)] = 0$, $j = 1, ..., r$, (see (2.3.73)), and hence $\mathbb{E}[\boldsymbol{Z}] = 0$. Then $\boldsymbol{\Sigma} = \mathbb{E}\left[\boldsymbol{Z}\boldsymbol{Z}^{\top}\right]$ and we can compute $\boldsymbol{\Sigma} = \begin{bmatrix} \kappa\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^{\top}\right] & \boldsymbol{\Psi} \\ \boldsymbol{\Psi}^{\top} & \boldsymbol{\Delta} \end{bmatrix}$, where $\kappa = \mathbb{E}\left\{\left[\sum_{j=1}^{r} \psi'_{\alpha_j}\left(\varepsilon - F_\varepsilon^{-1}(\alpha_j)\right)\right]^2\right\}$, $\boldsymbol{\Psi} = [\boldsymbol{\Psi}_1, ..., \boldsymbol{\Psi}_r]$ with

$$\boldsymbol{\Psi}_j = \mathbb{E}\left[\left(\sum_{i=1}^{r} \psi'_{\alpha_i}\left(\varepsilon - F_\varepsilon^{-1}(\alpha_i)\right)\right)\psi'_{\alpha_j}\left(\varepsilon - F_\varepsilon^{-1}(\alpha_j)\right)\boldsymbol{X}\right],\ j = 1, ..., r,$$

and $\Delta_{ij} = \mathbb{E}\left[\psi'_{\alpha_i}\left(\varepsilon - F_\varepsilon^{-1}(\alpha_i)\right)\psi'_{\alpha_j}\left(\varepsilon - F_\varepsilon^{-1}(\alpha_j)\right)\right]$, $i, j = 1, ..., r$.

The Hessian matrix $\boldsymbol{H}$ can be computed as $\boldsymbol{H} = \begin{bmatrix} \gamma\mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^{\top}\right] & \boldsymbol{F} \\ \boldsymbol{F}^{\top} & \boldsymbol{D} \end{bmatrix}$, where $\gamma = \sum_{j=1}^{r} \gamma_j$ with

$$\begin{aligned}
\gamma_j &= \left.\frac{\partial\mathbb{E}\left[\psi'_{\alpha_j}(\varepsilon+\beta_0^*-\eta_j^\star+t)\right]}{\partial t}\right|_{t=0} \\
&= \left.\frac{\partial\left[\alpha_j(1-F_\varepsilon(F_\varepsilon^{-1}(\alpha_j)-t))+(\alpha_j-1)F_\varepsilon(F_\varepsilon^{-1}(\alpha_j)-t)\right]}{\partial t}\right|_{t=0} \\
&= \alpha_j f_\varepsilon(F_\varepsilon^{-1}(\alpha_j)) - (1-\alpha_j)f_\varepsilon(F_\varepsilon^{-1}(\alpha_j)) = f_\varepsilon(F_\varepsilon^{-1}(\alpha_j)),\ j = 1, ..., r,
\end{aligned}$$

$\boldsymbol{F} = [\boldsymbol{F}_1, ..., \boldsymbol{F}_r]$ with $\boldsymbol{F}_j = \gamma_j\mathbb{E}[\boldsymbol{X}]$, $j = 1, ..., r$, and $\boldsymbol{D} = \mathrm{diag}(\gamma_1, ..., \gamma_r)$.

Since $\check{\boldsymbol{\beta}}_0 = \boldsymbol{\lambda}^{\top}\check{\boldsymbol{\eta}}$, we have that $\check{\boldsymbol{\beta}}_0 + \check{\boldsymbol{\beta}}^{\top}\boldsymbol{x} = [\boldsymbol{x}^{\top}; \boldsymbol{\lambda}^{\top}][\check{\boldsymbol{\beta}}; \check{\boldsymbol{\eta}}]$, and hence the asymptotic variance of $\check{\boldsymbol{\beta}}_0 + \check{\boldsymbol{\beta}}_0{}^{\top}\boldsymbol{x}$ is given by $N^{-1}[\boldsymbol{x}^{\top}; \boldsymbol{\lambda}^{\top}]\boldsymbol{H}^{-1}\boldsymbol{\Sigma}\boldsymbol{H}^{-1}[\boldsymbol{x}; \boldsymbol{\lambda}]$.

APPENDIX F

ESTIMATED REGRESSION COEFFICIENTS FOR THE EMPIRICAL EXAMPLES

Table 13: Estimated coefficients, lower(LCI) and upper(UCI) confidence intervals for the empirical examples

BAC CDS spread example

Variables	Coefficients	LCI	UCI
Y (percent change of BAC CDS)			
Intercept	0.3956	0.0587	0.7325
X1 (return of BAC stock price)	-0.2164	-0.2849	-0.1479
X2 (percent change of generic 5-year CDX.IG)	0.4555	0.3574	0.5536

IBM stock example

Variables	Coefficients	LCI	UCI
Y (log retern of IBM stock)			
Intercept	0.036	-0.0479	0.12
X1 (daily log return of S&P 500 index)	-0.1733	-0.2539	-0.0927
X2 (lagged log return)	0.0009	0.0001	0.0017

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