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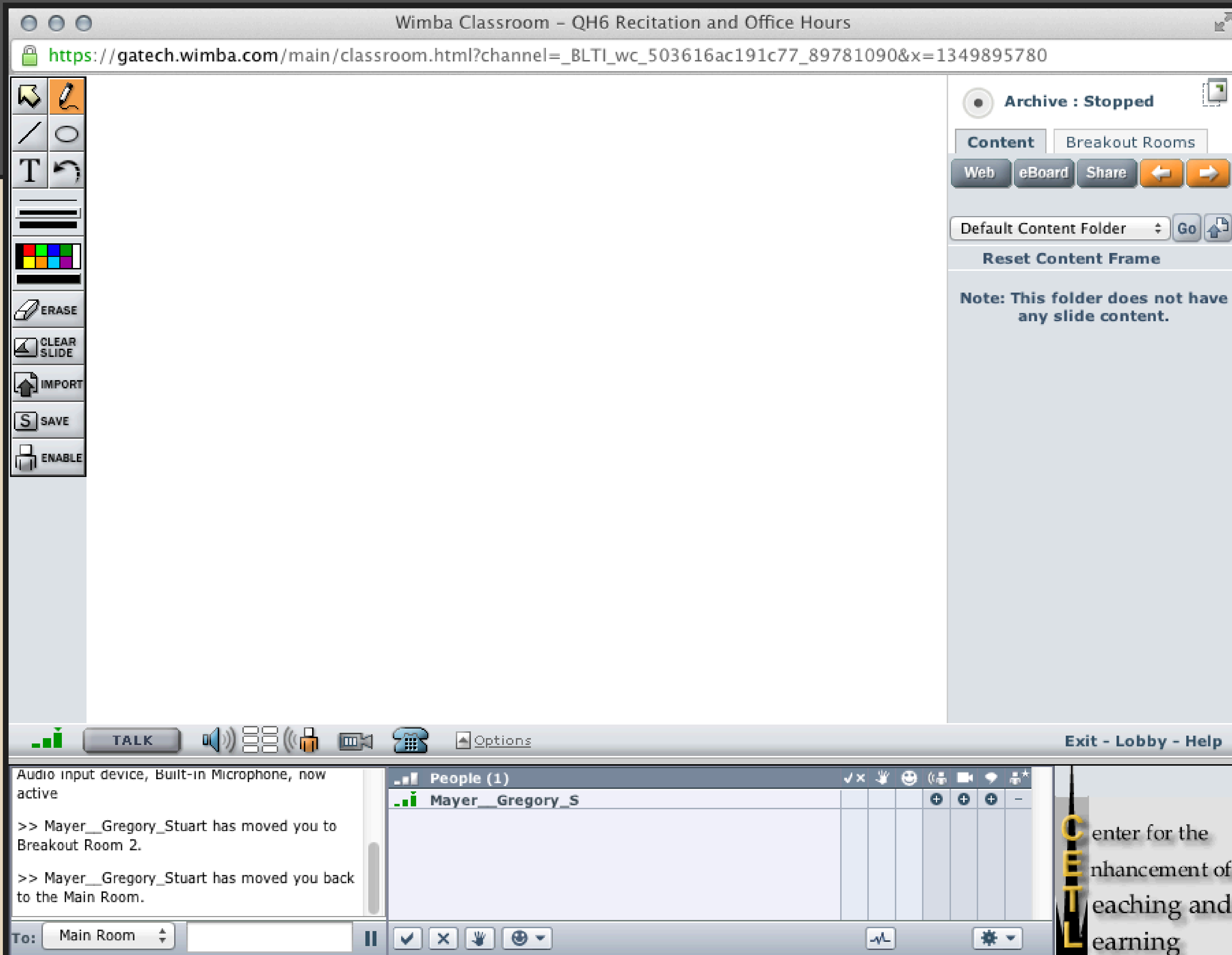
Contact Information
Questions regarding this document can be directed to Greg Mayer (gsmayer@gmail.com), who would be happy to hear your suggestions on how to improve this document.

Welcome to QH6 Recitation!

We'll get started at 8:05. See if you can:

- use the chat window (bottom left) to send a message
- use your mic to say hi: press and hold TALK button
- move yourself in and out of a breakout room (top right)
- draw a picture in the space below of something (drawings are always anonymous)


Wimba Classroom – QH6 Recitation and Office Hours
https://gatech.wimba.com/main/classroom.html?channel=_BLTI_wc_503616ac191c77_89781090&x=1349895780
ERASE
CLEAR SLIDE
IMPORT
SAVE
ENABLE
Archive : Stopped
Content
Breakout Rooms
Web
eBoard
Share
Default Content Folder
Go
Reset Content Frame
Note: This folder does not have any slide content.
TALK
Options
Exit - Lobby - Help
Audio input device, Built-in Microphone, now active
>> Mayer__Gregory_Stuart has moved you to Breakout Room 2.
>> Mayer__Gregory_Stuart has moved you back to the Main Room.
To:
Main Room
People (1)
Mayer__Gregory_S
Center for the Enhancement of Teaching and Learning

If you have questions during recitation

Voice

- use microphone any time
- to use mic, pres TALK, or press and hold
  - Windows: control
  - Mac: command

Text

- use chat window to send message
- can send a message to me, another student, or to “main room”

Purpose of Recitations

Currently, the purpose of our recitations:

help students understand course material so that they can complete assignments and prepare for quizzes and exams.

QH6 in a Nutshell

- students in Math 1502 are divided into 6 sections
- ours is the only section that
  - doesn’t have campus students
  - uses Wimba for recitations
- Why Wimba?
  - you can interact with students at other schools

Wimba Technical Problems?

You can:

- reload your browser
- log in/out of Wimba
- use a different web browser (don’t use Chrome)
- get help from another student and/or your TA
- contact Wimba tech support
  http://support.blackboardcollaborate.com

Tablets and Mics

Students in QH6 can borrow tablets and mics

If you already have a tablet and/or mic you want to use, that's ok too

Equipment need to be returned

If you don’t have a tablet and/or mic and want to borrow one, email me

Tablets come with CD, use it to configure tablet settings

Grading Weights

	QH6 (%)	All other sections (%)
Homework	10	10
Final	25	25
Quizzes	60	65
Recitations	5	0
Total	100	100

Grades will be made available through T- Square

Grading

Activities not be graded for first few recitations.

Each recitation activity is worth 5% / N, where N = number of graded activities in the semester.

Related Websites

- Recordings of recitations and lectures: tegrity.gatech.edu
- Discussion forum: piazza.com
- Lectures: gtcourses.gatech.edu
- Homework: www.mymathlab.com

First homework due August 22, 11:59 PM

gtcourses.gatech.edu/201308/fall_2013/math_1502/math_1502-20130819-074610/main.htm?layout=default&type=ms&bandwidth=high&audioonly=no&trac

MATH 1502 - Calculus II 08/19/2013 07:46 AM

Thomas Morley

Georgia Tech Professional Education

$$3+\frac{3}{4}+\frac{3}{4^2}+\cdots$$

$$-\frac{3}{1-1/4}$$

$$3+\frac{3}{4}+\frac{3}{4^2}+\cdots$$

YOUR QUESTION

SUBMIT

Data posted successfully

00:32:43 of Live

Your TA: Greg

- Canadian, eh
- mon français est tres mauvais
- moved to the US ~1 year ago
- post-doctoral fellow
- PhD in applied math (image processing), MSc in Electrical Engineering

- email or call me with any questions you have
- greg.mayer@ceismc.gatech.edu
- 404-894-8599

Questions?

Any questions before we discuss

- Geometric Series,
- Alternating Series,
- Taylor Series, and
- Taylor Polynomials?

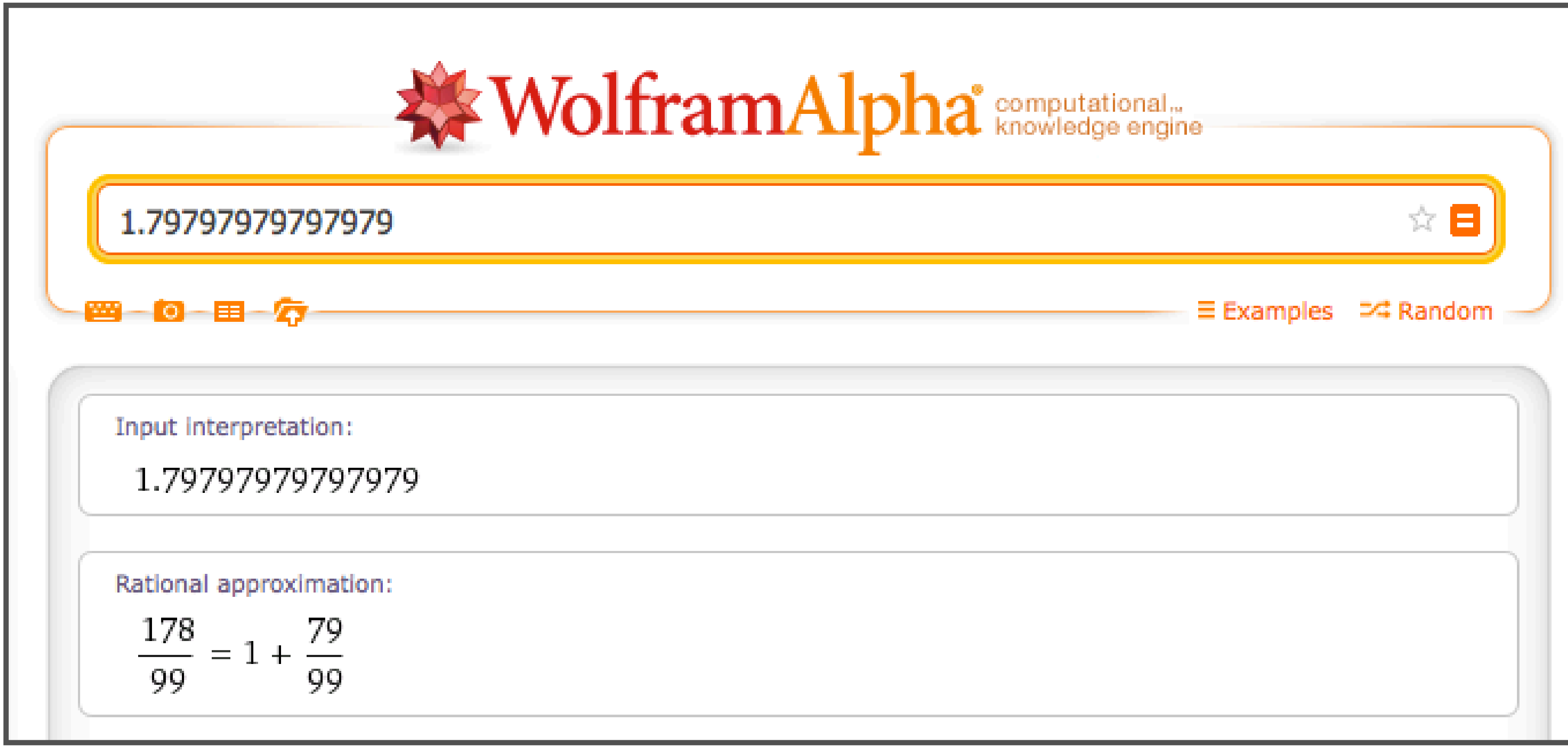
Geometric Series

The sum of the geometric series is equal to

$$\sum_{k=1}^{\infty} ar^{k-1} =$$

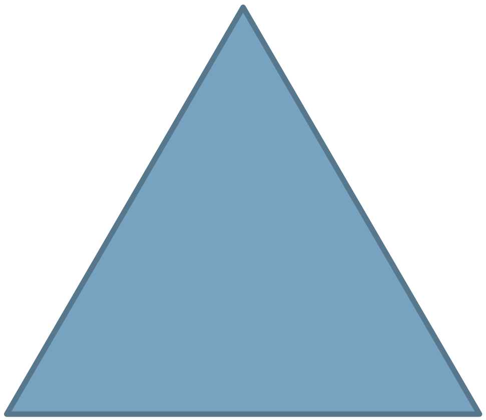
Application

Express 1.79797979… as a rational number.



What other methods can you use to check your answer?

Koch Snowflake



Let’s create a Koch Snowflake and find its area.
Start with an equilateral triangle with unit area.

Taylor Polynomials and Series

The formula for the ________________ is

$$P_N = \sum_{k=0}^{N} \frac{f^{(k)}(\quad)}{k!} x^k = f(\quad) + \frac{f'(\quad)}{1!} x + \frac{f''(\quad)}{2!} x^2 + \ldots + \frac{f^{(N)}(\quad)}{N!} x^N$$

Formula for the ________________ is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\quad)}{k!} x^k = f(\quad) + \frac{f'(\quad)}{1!} x + \frac{f''(\quad)}{2!} x^2 + \ldots$$

How are these formulas different?

Formula for exp(x)

In class you saw (or will see that)

$$\exp(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Find the Taylor expansions of:

$$e^{-x} =$$

$$e^{-x^2} =$$

First Question on Quiz 1 Last Year

Use series to find the limit as x → 0 of $$\frac{e^{2x^2}-1-x^2}{x^4}$$

RECITATION 01

WRITE

$$1.\overline{79} = 1 + \frac{79}{\underset{(100)}{?}} + \frac{79}{\underset{(10000)}{?}} + \frac{79}{\underset{(1000000)}{?}} + \ldots$$

? = get students to tell you, or screw it up & ask them to fix

$$= 1 + \frac{79}{100}\left[1 + \frac{1}{100} + \frac{1}{10000} + \ldots\right]$$

$$= 1 + \frac{79}{100}\left[\frac{1}{(100)^0} + \frac{1}{100^1} + \frac{1}{100^2} + \ldots\right]$$

$$= 1 + \frac{79}{100}\sum_{k=?}^{\infty}\left(\frac{1}{100}\right)^{?}$$

where should sum start?
1, then exp is k-1
0, then exp is k

$$= 1 + \frac{79}{100}\left(\frac{1}{1-?}\right)$$

1/100, why?

ASK WHAT DOES THIS WORK OUT TO?

$$= 1 + \frac{79}{99}$$

$$= \frac{178}{99}$$

WE HAVE: EXPRESSED A REPEATING DECIMAL AS A RAT. NUMBER.

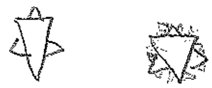
~~1.79~~ ($1.\overline{abc}$ is similar

RECITATION 02

2

SAY

1, divide each side into 3 equal segments
2, make equ. Δ's on middle 3^{RD}.
REPEAT



dd-dd-dd

$\Delta \; \Delta \; \Delta$

each iteration has different color.

area of (blue) $\Delta = 1$

area of (red) $\Delta = 1/9$

(green) $\Delta = \frac{1}{9^2}$

recall: area of Δ is $\frac{\sqrt{3}}{4} s^2$
(s = length one side)

area of snowflake

(trust me)

$$= 1 + 3\left(\frac{1}{9}\right) + 12\left(\frac{1}{9^2}\right) + 48\left(\frac{1}{9^3}\right) + \dots$$

How many red Δs?
How many green Δs?

$$= 1 + 3\left(\frac{1}{9} + 4^1 \frac{1}{9^2} + 4^2 \frac{1}{9^3} + \dots\right)$$

$$= 1 + 3 \sum_{k=1}^{\infty} \frac{4^{k-1}}{9^k}$$

$$= 1 + \frac{3}{9}\left(\sum_{k=1}^{\infty} \frac{4^{k-1}}{9^{k-1}}\right)$$

$$= 1 + \frac{1}{3}\left(\frac{1}{1 - 4/9}\right)$$

what is this?

$$= 8/5$$

FINITE AREA! WHOA.

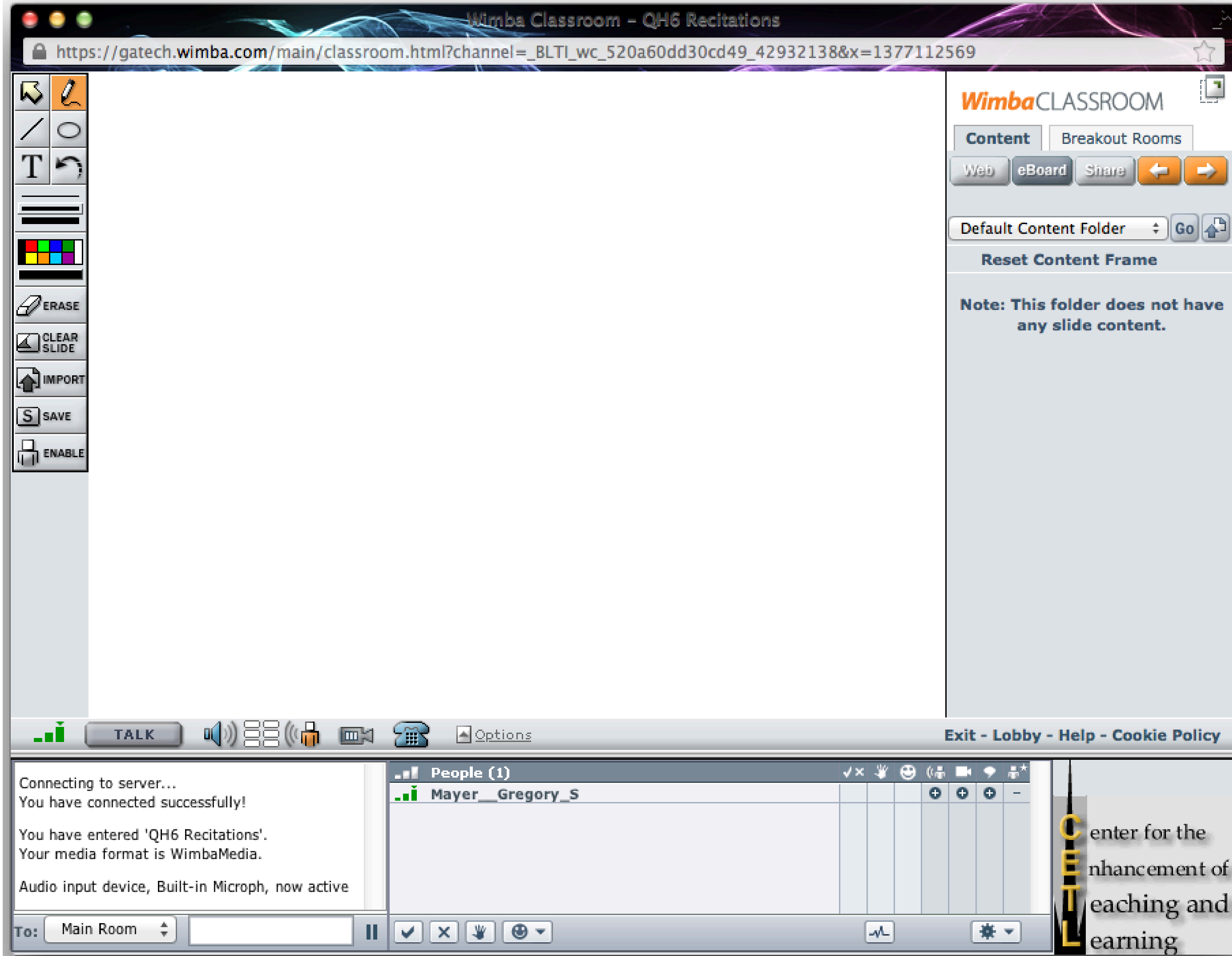
QH6 Recitation 02

Today:

1. Geometric Series (10.2)
2. Wimba
3. Taylor Series, Taylor Polynomials (10.8)

While we're waiting to start:

$$\sum_{k=0}^{\infty} \frac{10}{3^k} =$$


Wimba Classroom – QH6 Recitations
https://gatech.wimba.com/main/classroom.html?channel=_BLTI_wc_520a60dd30cd49_42932138&x=1377112569
ERASE
CLEAR SLIDE
IMPORT
SAVE
ENABLE
WimbaCLASSROOM
Content
Breakout Rooms
Web
eBoard
Share
Default Content Folder
Go
Reset Content Frame
Note: This folder does not have any slide content.
TALK
Options
Exit - Lobby - Help - Cookie Policy
Connecting to server...
You have connected successfully!
You have entered 'QH6 Recitations'.
Your media format is WimbaMedia.
Audio input device, Built-in Microph, now active
To:
Main Room
People (1)
Mayer__Gregory_S
Center for the Enhancement of Teaching and Learning

Wimba Status

clear away approve disapprove surprise
confusion clap laughter go faster go slower

Taylor Polynomials and Series

Definition of the ________________________________ is

$$P_N = \sum_{k=0}^{N} \frac{f^{(k)}(\quad)}{k!} x^k = f(\quad) + \frac{f'(\quad)}{1!} x + \frac{f''(\quad)}{2!} x^2 + \ldots + \frac{f^{(N)}(\quad)}{N!} x^N$$

Definition of the ___________________________________ is

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\quad)}{k!} x^k = f(\quad) + \frac{f'(\quad)}{1!} x + \frac{f''(\quad)}{2!} x^2 + \ldots$$

How are these formulas different? ______________________

Taylor Expansion of e^x

The Taylor expansion of e^x, about x = 0, is:

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(k)}(0)}{k!} x^k =$$

Find the Taylor expansion of e^{-x} about x = 0.

Taylor Polynomials $\exp(-x^2)$, about $x = 0$

Checking our result

WolframAlpha computational knowledge engine

taylor expansion of e^(-x^2)

Examples Random

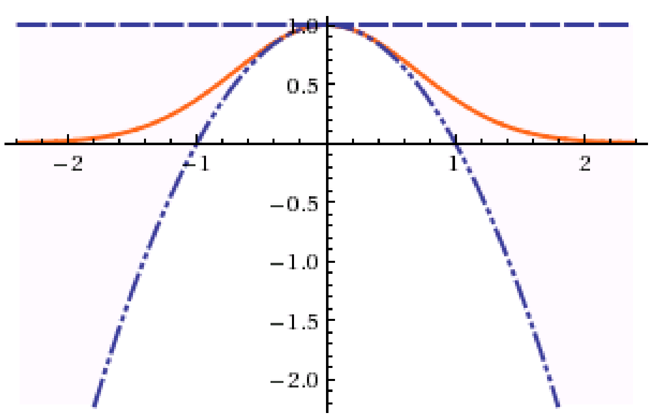
Input interpretation:

series	e^{-x^2}

Series expansion at x=0: More terms

$$1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + O(x^9)$$

Approximations about x=0 up to order 2: More terms



(order n approximation shown with n dots)

Enable interactivity

First HW

- When is it due?
- If you get stuck, send me an email.
- We can meet on Wimba. Send me an email to set up a time.

Taylor Polynomials of exp(x), about x = 3

Checking our result

Taylor expansion of e^(x) at 3

Examples Random

Input interpretation:

series	e^x	point	$x = 3$

Series expansion at x=3:

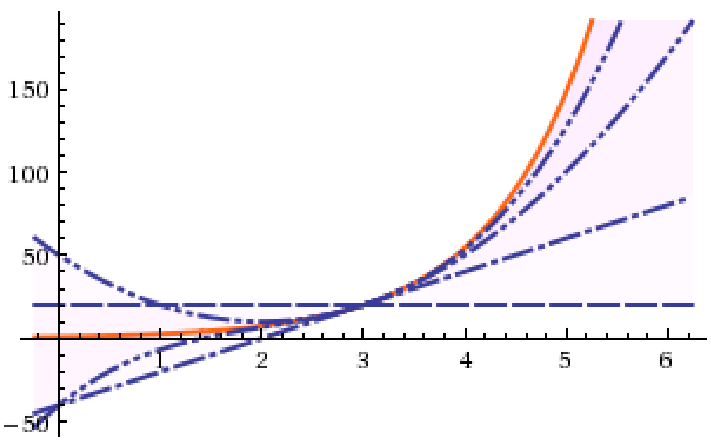
More terms

$$e^3 + e^3 (x-3) + \frac{1}{2} e^3 (x-3)^2 + \frac{1}{6} e^3 (x-3)^3 + \frac{1}{24} e^3 (x-3)^4 + \frac{1}{120} e^3 (x-3)^5 + O\left((x-3)^6\right)$$

(converges everywhere)

Approximations about x=3 up to order 3:

More terms



Remainder Theorems

Recall: the N[th] Taylor Polynomial

$$P_N = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k$$

Formula for the Taylor Series

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \ldots$$

$$= f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + R_3(x)$$

Two useful expressions for the remainder

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \qquad \left|R_n(x)\right| \le \left(\max_{t \in [0,x]} \left|f^{(n+1)}(t)\right|\right) \frac{|x|^{n+1}}{(n+1)!}$$

Example 1: Find the Lagrange form of the remainder, $R_n(x)$, if $n = 4$, and $f(x)=e^{2x}$.

R02

GEOMETRIC SERIES

$$\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$$

ASK

what does $|r|$ have to be?

$|r| < ?$

EXAMPLE

"a"

$$\sum_{k=0}^{\infty} \frac{10}{3^k} = 10 \sum_{k=0}^{\infty} \frac{1}{3^k}$$

LET $k = u-1$

THEN, when $k=0, u=1$

$$= 10\left(\sum_{u=1}^{\infty} \frac{1}{3^{u-1}}\right)$$

$$= 10\left(\frac{1}{1-\frac{1}{3}}\right)$$

$$= 10\left(\frac{3}{2}\right)$$

$$= 15$$

ASK STUDENTS:

DOES THIS CONVERGE?

WHY?

USE SAME TRICK ON FORMULA

$$\Rightarrow \sum_{k=1}^{\infty} ar^{k-1} = \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

N^{th} ORDER
TAYLOR POLY AT 0

$$P_N = \sum_{k=0}^{N} \frac{f^{(k)}(0)}{k!} x^k$$

TAYLOR EXPANSION AT $x=0$

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$$

e^x: $f^{(0)}(0) = ?$ 1

$f^{(1)}(0) = ?$ 1

$f^{(2)}(0) = ?$ 1

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

TAYLOR ABOUT $x=0$ OF e^{-x}

e^{-x} let $u=-x$,

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3}$$

e^{-x^2} let $u=-x^2$

$$e^u = \sum_{k=0}^{\infty} \frac{u^k}{k!} = \sum_{k=0}^{\infty} -$$

ASK

WHAT DOES "ABOUT $x=0$" MEAN?

TAYLOR POLY e^{-x^2}, about $x=0$

4

WRITE START WITH TAYLOR EXPANSIONS

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$$

$$P_0 = \sum_{k=0}^{0} \frac{(-x^2)^k}{k!} = ?$$

1

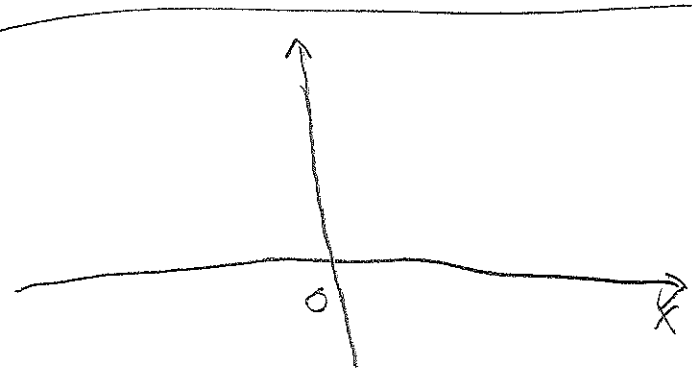
$$P_1 = \sum_{k=0}^{1} \frac{(-x^2)^k}{k!} = ?$$

$1 - x^2$

$1 - x^2 + \frac{x^2}{2} -$

$$P_2 = ?$$

lets try something different



CAN SOMEONE FROM ____ DRAW P_0

" " " " " P_1

(5)

e^x at $x=3$

TAYLOR EXPANSION OF $f(x)$ at $x=a$ is

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)(x-a)^k}{k!},$$

EXPANSION OF e^x at $x=3$ is

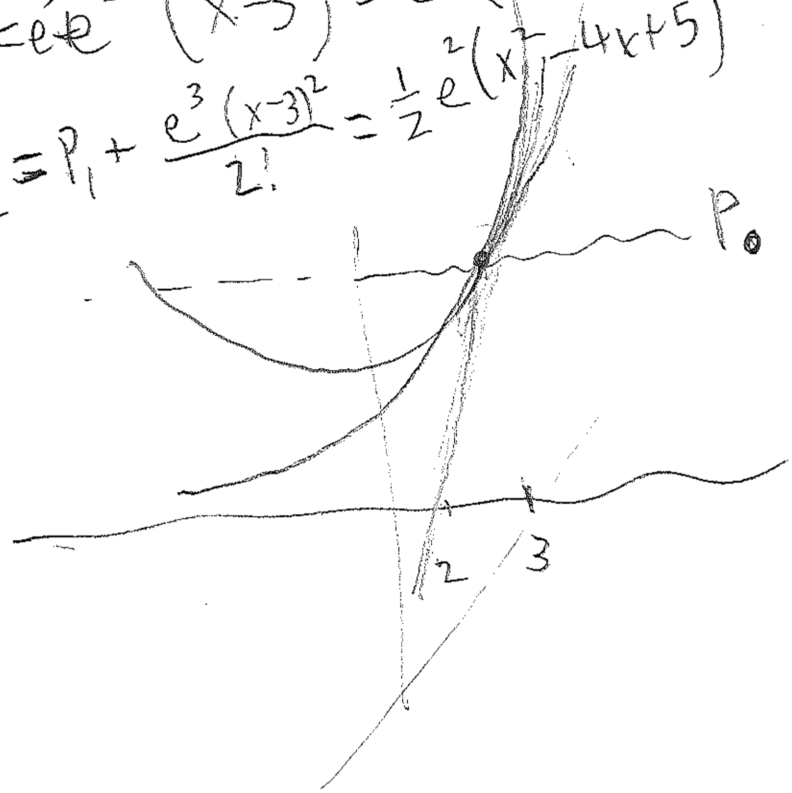
$$e^x=\sum_{k=0}^{\infty}\frac{e^3(x-3)^k}{k!}$$

e^3

$$P_0=e^3$$

$$P_1=e^3 e^3(x-3)=e^3(x-2)$$

$$P_2=P_1+\frac{e^3(x-3)^2}{2!}=\frac{1}{2}e^2(x^2-4x+5)$$



QH6 Recitation 03

1. $P_2(x)$ of $1/(1-x)^2$
2. Lagrange remainder (10.9)
3. HW questions?
4. Announcements
5. Group work

While we're waiting to start:

a) Plot a rough sketch of $f(x) = 1/(1 - x)^2$ for x between -1 and +1.
b) Use the Taylor expansion for $1/(1 – x)$ to find $P_2(x)$ of $f(x)$ about $x = 0$.

The Taylor expansion of $1 / (1 – x)$ about $x = 0$ is

$$\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \ldots$$

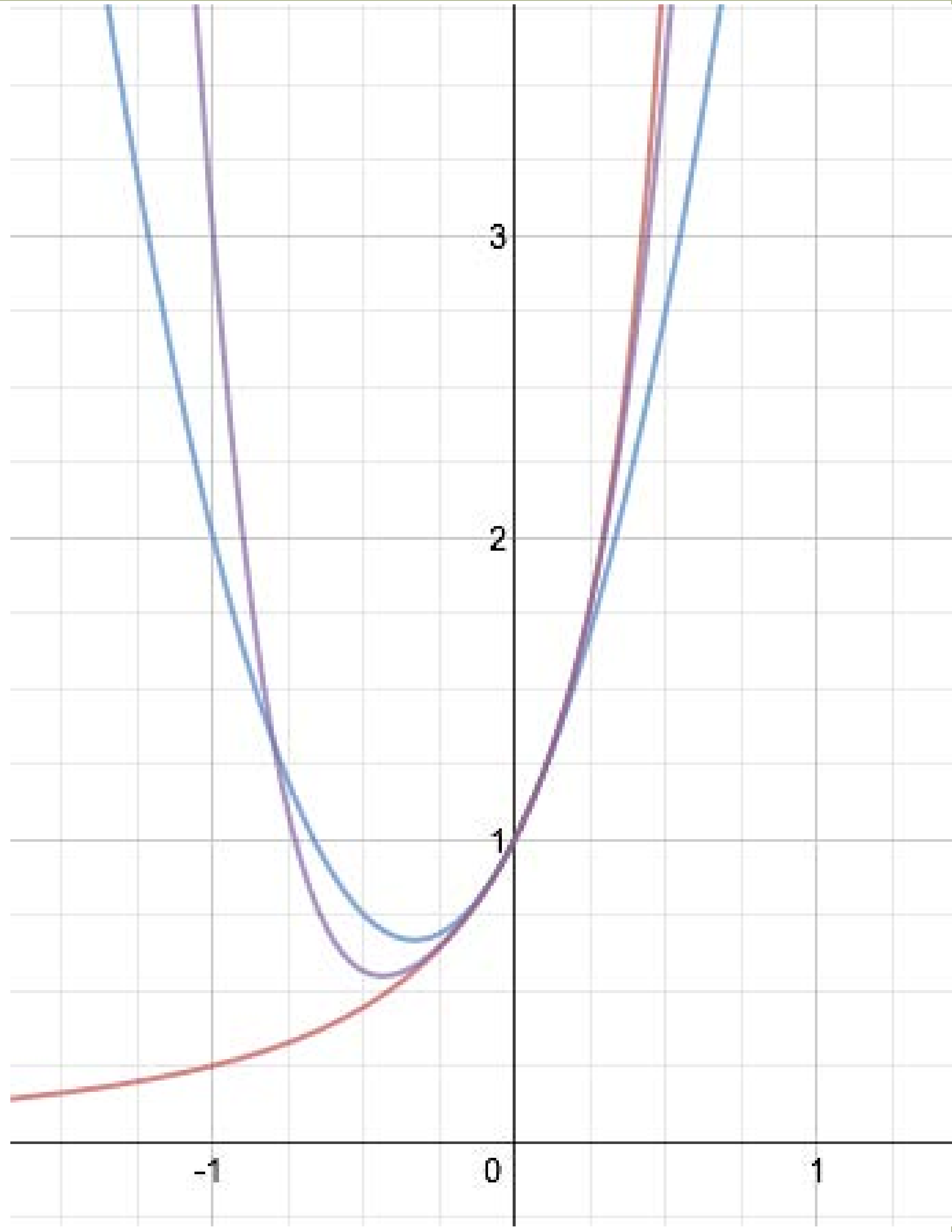
$P_2(x)$ of $1 / (1 - x)^2$

a) Plot a rough sketch of $f(x) = 1/(1 - x)^2$ for x between -1 and + 1.

b) Use the Taylor expansion for $1/(1 – x)$ to find $P_2(x)$ of $f(x)$ about $x = 0$.

The Taylor expansion of $1 / (1 – x)$ about $x = 0$ is $\dfrac{1}{(1-x)} = 1 + x + x^2 + x^3 + \ldots$

$P_2(x)$ of $1\ /\ (1 - x)^2$



Lagrange Remainder

Recall: the N^{th} order Taylor Polynomial about x = 0 is:

$$P_N(x) = \sum_{k=0} \frac{f^{(k)}(0)}{} x$$

The Taylor Series about x = 0 is:

$$f(x) = \sum_{k=0} \frac{f^{(k)}(0)}{} x =$$

Questions about HW1?

Announcements

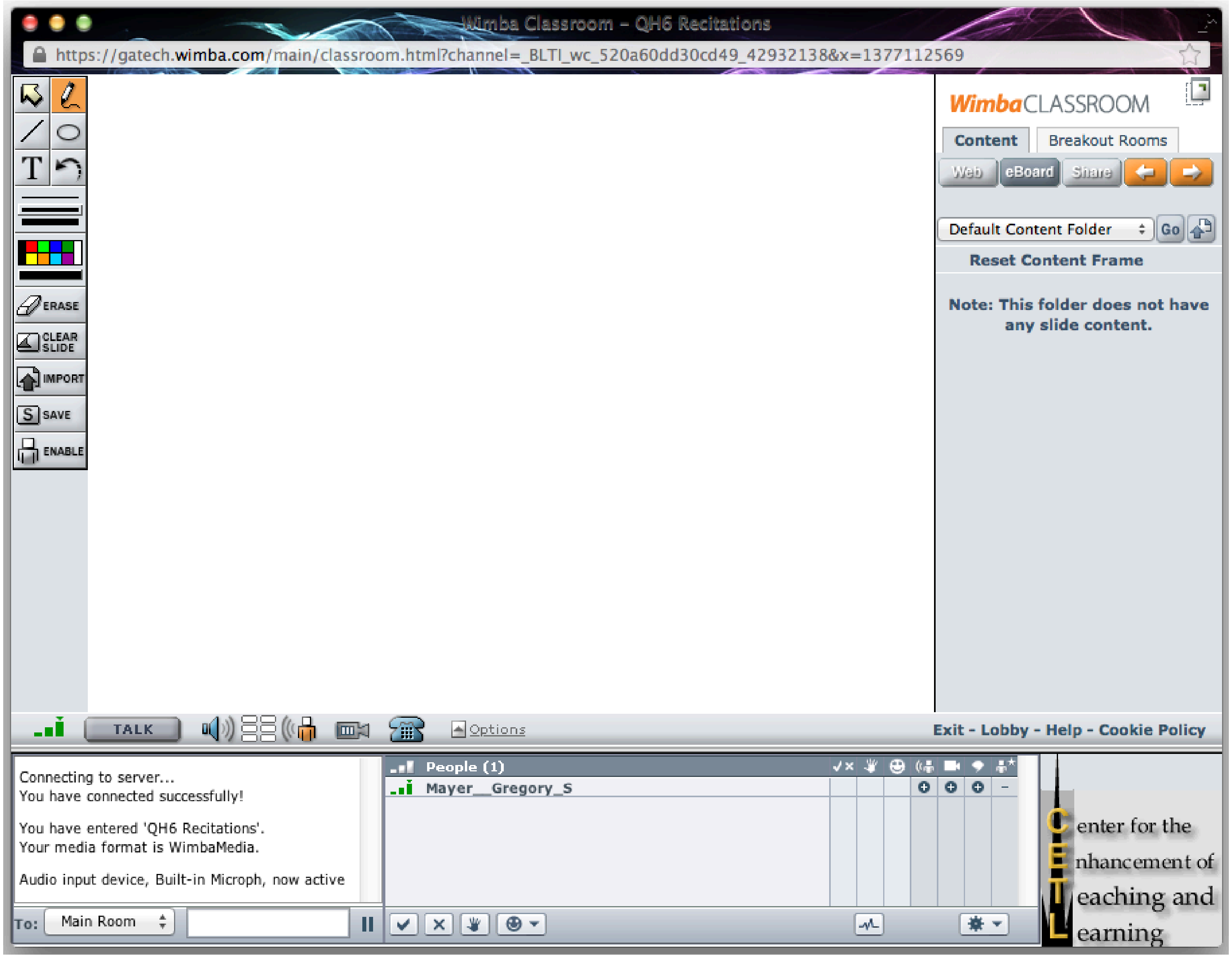
Online Survey

- I need your input on the recitations
- you'll receive an email invitation today for an online survey
- we will discuss survey results next week

Technical problems viewing lectures? Call Distance Education at ___________.

QH6 recitations are for QH6 students.

Let's Try Group Work



First Question on Quiz 1 Last Year

Use series to find the limit as x → 0 of $$\frac{e^{2x^2} - 1 - x^2}{x^4}$$

Example 3: Find the Taylor expansion for $f(x) = \dfrac{2x}{1-x^2}$

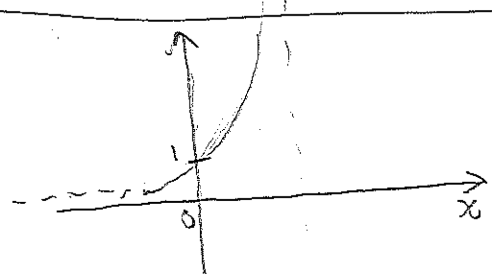
RO3

a) Rough sketch of $f=\frac{1}{(1-x)^2}$, $(-\infty, 1)$

b) P_2 of $f(x)$ using $\frac{1}{1-x}=1+x+x^2+\dots$

a)

x	f
-1	1/4
0	1
+1	$+\infty$

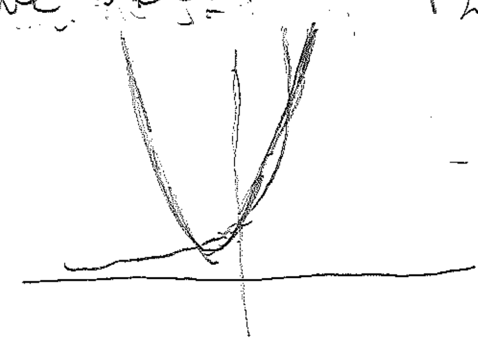


b) $f=\frac{d}{dx}\left(\frac{-1}{1-x}\right) = 1+2x+3x^2+\dots$

$$P_2(x)=1+2x+3x^2$$

what does x have to be for this to work?
$-1<x<+1$

SAY ~~If~~ we sketch P_2



SAY P_2 approximates $f(x)$: well when x small and poorly when x is large.

Q How can we improve the approximation?
- increase N
- keep interval of approx small

A

for small x P_4 better approx then P_∞

LAGRANGE (FROM 10.9)

T.P. $P_N = \sum_{k=0}^{N} \frac{f^{(n)}(0)}{k!} x^k$

T.S. $f(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots$

$= P_n(x) + R_N(x)$?

R_N = remainder $= f(x) - P_n(x)$

LAGRANGE FORM OF REMAINDER $= \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$, c some number $\in [0, x]$

PROOF IN CLASS?

SAY

WRITE "size" of remainder

$|P_n + R_N| = |R_N| = |f^{(n+1)}(c)| \frac{|x^{n+1}|}{|(n+1)!|}$, TOO FAR

$\leq \max_{c \in [0,x]} f^{(n+1)}(c) \frac{|x^{n+1}|}{|(n+1)!|}$

QH6 Recitation 04

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

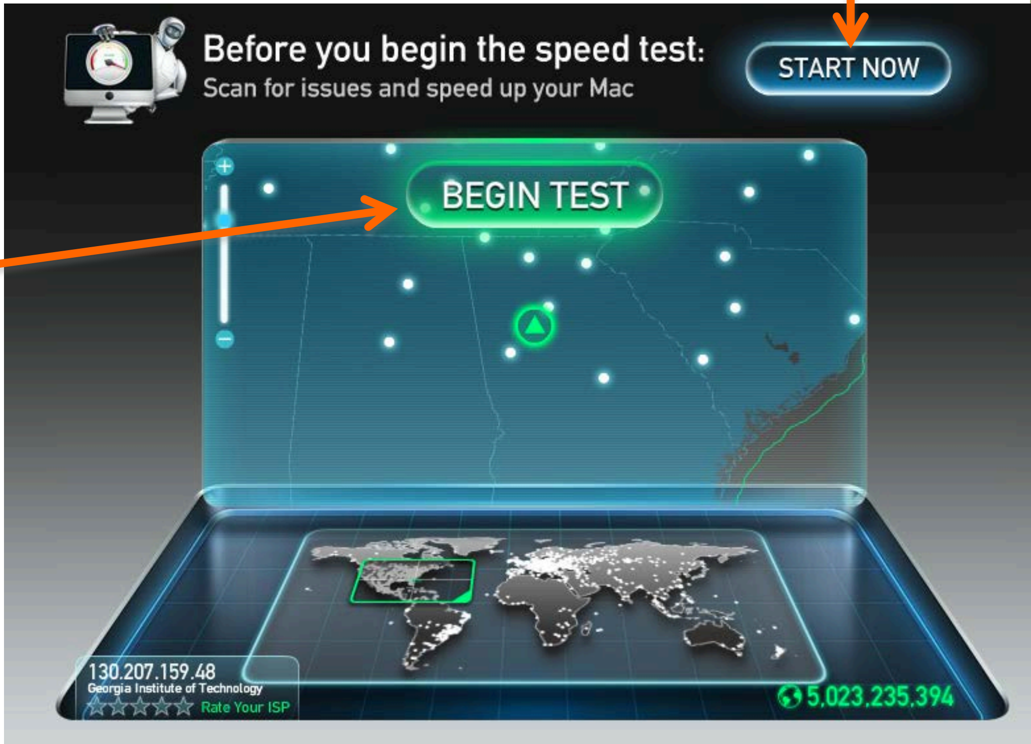
Today:

1. Group work
2. Remainder Theorems (10.9)

don't click the ad

While we're waiting to start:

- if possible, visit www.speedtest.net
- click BEGIN TEST
- write your download speed on this board, or type it in the chat window



Group Work

- For today, I’ll assign everyone randomly into breakout rooms
- You’ll have about 5 min
- A few suggestions:
  - discuss a solution strategy before solving
  - solve the question in 4 to 6 lines
- I will move between rooms and will move everyone back to main room

A Question from Quiz 1 Last Year

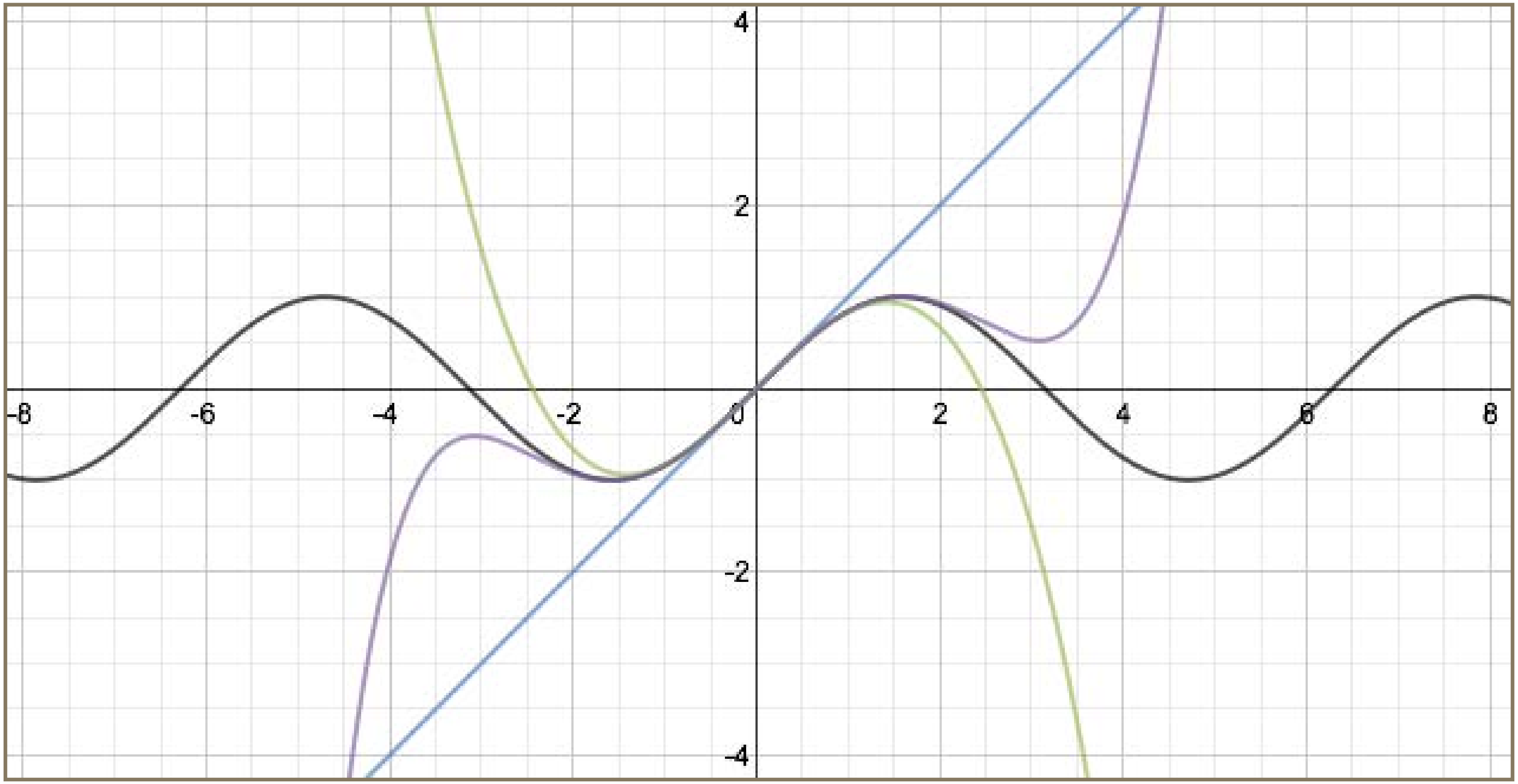
Use series to find the limit as $x \to 0$ of
Don't use l'Hospital's rule.

$$\frac{e^{2x^2} - 1 - x^2}{x^4}$$

If you can’t make it to a recitation or lecture ...

- A video archive of recitations and lectures in Math 1502 can be viewed at:

Taylor Polynomials of sin(x)



Another Question from Quiz 1 Last Year

Estimate $e^{3/2}$ to within 10^{-4}.

Estimate the integral to within 0.01 by using series

$$\int_0^1 x^4 e^{-x^2}\, dx$$

RS14

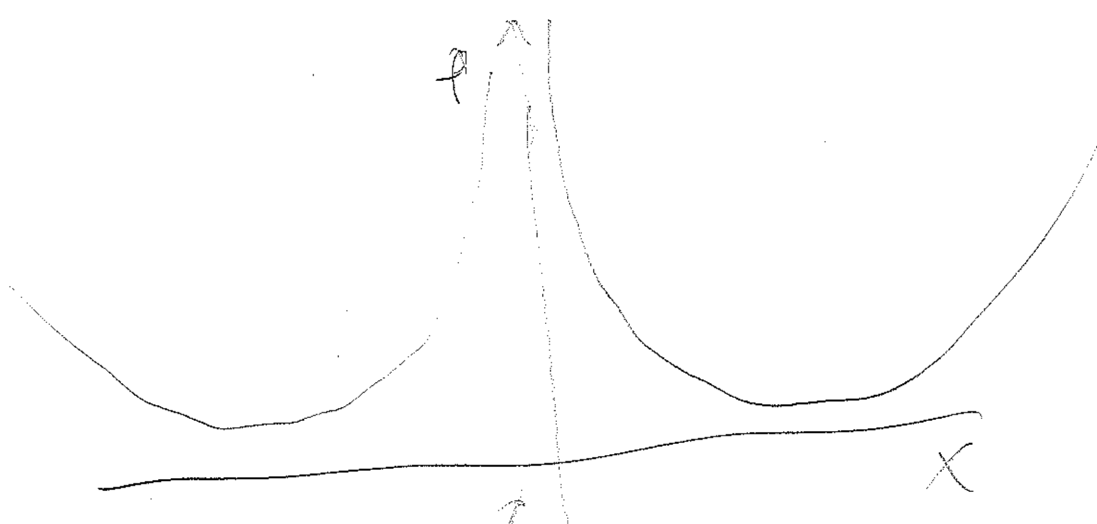
$$\frac{e^{2x^2}-1-x^2}{x^4}$$

$$e^{2x^2} = 1 + \frac{2x^2}{1!} + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \cdots$$

$$\frac{e^{2x^2}-1-x^2}{x^4} = \frac{x^2 + \frac{4x^4}{2} + \frac{8x^8}{3!} + \cdots}{x^4}$$

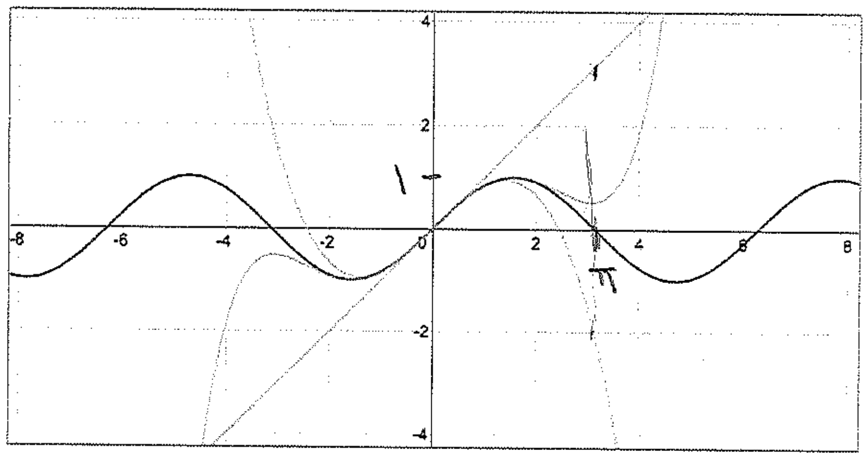
$$= x^{-2} + 2 + \frac{8}{3!}x^4 + \cdots$$

$$\lim_{x \to 0} = \infty$$



- how can you check answer?
- l'Hospital requires 4 derivatives, sort of

Taylor Polynomials of sin(x)



We have a graph of sin(x) and a few Taylor Polynomials.
What color is sin(x)?
What color is P1(x)? How do we know? Is P1 always linear?
What color is P3(x)? Why?
P5 is purple.
Use the graph to estimate R1(π), R3(π) and R5(π).
What could we do to get a better approximation of sin(x) at π?

Recall: $f(x) = P_N(x) + R_N(x)$

so: $R_N = f(x) - P_N(x)$.

using the graphs $|R_1(\pi)| \approx +3.14...$, $|R_3(\pi)| \approx 2$

$|R_5(\pi)| \approx -0.5$

Another Question from Quiz 1 Last Year

Estimate $e^{3/2}$ to within 10^{-4}. , $f = e^x$, $f^{(k)} = e^x \;\forall k$

$R_N(x) = f(x) - P_N(x)$

$= \frac{f^{(n)}(c)}{} \frac{x^{n+1}}{(n+1)!}$ (Lagrange), $c \in [0, x]$

$|R_N(x)| \leq \underbrace{\max f^{(n)}(c)}_{e^x} \frac{|x|^{n+1}}{(n+1)!}$,

but $x = 3/2$, so

$|R_n(\tfrac{3}{2})| = e^{3/2} \frac{|3/2|^{n+1}}{(n+1)!} \leq 10^{-4}$

n	$e^{3/2} \frac{(3/2)^{n+1}}{(n+1)!}$	
1	5	< 0.00001? NO!
2	2.5	
4	0.28	
8	? 0.0004	< 0.0001 NO!
9	0.00007	YES

$\Rightarrow \boxed{N = 9}$

ASK: what is the 1st deriv of e^x?

2nd

3rd ?

is e^x an increasing funct. of x?

QH6 Recitation 5

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Alternating Series, Partial Fractions, Integration of Series

Another Question from Quiz 1 Last Year

Estimate $e^{3/2}$ to within 10^{-4}.

$$|R_N(x)| = |f(x) - P_N(x)|$$

$$= \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x)^{n+1} \right|$$

(Lagrange remainder)

$c \in [0, x]$

$$\le \max f^{(n+1)} \frac{|x|^{n+1}}{(n+1)!}$$

Announcements

Online Survey

- I need your input on the recitations
- you’ll receive an email invitation today for an online survey
- we will discuss survey results

Group Work

- For today, I'll assign everyone randomly into breakout rooms
- You'll have about 10 min
- A few suggestions:
  - discuss a solution strategy before solving
  - solve the question in 4 to 6 lines
- I will move between rooms and will move everyone back to main room

Estimate the integral to within 0.01 by using series

$$\int_0^1 x^4 e^{-x^2}\, dx$$

Sum of a Series

Find the sum of $$\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$$

R05

ESTIMATE $e^{3/2}$ TO WITHIN 10^{-4}

$$|R_N(x)| = |f - P_N|$$

$$= \left| \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} \right|, \quad c \in [0, x]$$

$$\le \max_c f^{(n+1)}(c) \frac{|x|^{n+1}}{(n+1)!}$$

What do we use for $\max_c f^{(n+1)}(c)$?

5 works (because $e^{3/2} \approx 4.48$)

(choose a # $> e^{3/2}$)

n	$5 \cdot \frac{(3/2)^{n+1}}{(n+1)!}$
1	5.625
2	2.8125
⋮	⋮
8	0.00052969796 ....
9	0.00008 ✓ ($<10^{-4}$)

$$\Rightarrow e^{3/2} \approx P_9\left(\tfrac{3}{2}\right) = 1 + x + \frac{x^2}{2} + \dots + \frac{x^9}{9!} \Big|_{x=3/2}$$

$$= 4.48167\ldots$$

NASTY!

R05

$\int_0^1 x^4 e^{-x^2} dx$ within 0.01

$$I = \int_0^1 \left(x^4 \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} \right) dx$$

$$= \int_0^1 x^4 - x^6 + \frac{x^8}{2!} - \frac{x^{10}}{3!} + \dots dx$$

$$= \frac{1}{5} - \frac{1}{7} + \frac{1}{9 \cdot 2} - \frac{1}{11 \cdot 3 \cdot 2} + \frac{1}{288}$$

$$= \underset{a_1}{\frac{1}{5}} - \underset{a_2}{\frac{1}{7}} + \underset{a_3}{\frac{1}{18}} - \underset{a_4}{\frac{1}{66}} + \underset{a_5}{\frac{1}{288}} - \dots$$

why do we not want to use Lagrange?

By A.S.R.T.,

6.

$$|R_n| \leq |a_{n+1}|$$

$I_1 = \frac{1}{5}$, $|R_1| = \frac{1}{7} = 0.14 \dots > 0.01$

$I_2 = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$, $|R_2| = \frac{1}{18} = 0.06 > 0.01$

$\vdots$ $|R_3| = |\frac{1}{66}| = 0.0\overline{15} > 0.01$

$I_4 = \frac{1}{5} - \frac{1}{2} + \frac{1}{18} - \frac{1}{66} = 0.0975$

$|R_4| = |a_5| = \frac{1}{288} = 0.003472 < 0.01$

$\Rightarrow I = 0.0975 \pm 0.01$

~~R05, 06~~ (DIDN'T HAVE TIME TO FINISH IN R05) ~~3~~ 7

Find the sum of $\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$.

~~WHAT IS ON RHS?~~

$$\frac{1}{(3k-2)}\frac{1}{(3k+1)} = \frac{A}{3k-2} + \frac{B}{3k+1}$$

distinct linear factors, not repeated

what do we do now?

$$1 = A(3k+1) + B(3k-2)$$

describe using text chat the next step

VTCTDNS

$$1 = 3k(A+B) + (A-2B)$$

$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ A-2B=1 \end{array}\right\} \begin{array}{l} B=-1/3 \\ A=+1/3 \end{array}$$

$$S = \sum \frac{1/3}{3k-2} - \frac{1/3}{3k+1}$$

$$= \frac{1}{3}\left((1-\tfrac{1}{4}) + (\tfrac{1}{4}-\tfrac{1}{7}) + (\tfrac{1}{7}-\tfrac{1}{10}) + \dots\right)$$

$$= 1/3$$

AT ~~fill in the blanks when you have them~~ have them fill in blanks

because We found a ___ series.

– SEND LINK –

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Partial Fractions, Integration of Series, Convergence

Find the sum of: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$

Quiz

To prepare for quiz:
-review the **integral test** for convergence (Section 10.3)
-pay close attention to tomorrow’s lecture
-complete practice quiz
-email me with questions
Bring a calculator.
Questions during quiz:
•can call/text me on my cell,
•can call Dr. Morley (his number provided to your facilitator),
•if you can connect to Wimba, you can ask questions that way.

What are you putting on your formula sheet?

Integration of Series

Find a series representation of $\int_0^x \sin \frac{\pi t^2}{2} dx$

Find at least the first 3 non-zero terms.

Integral Test

Determine whether the series converges: $$\sum_{k=1}^{\infty} \frac{1}{k \ln(1+k)}$$

Bound on Finite Sum

Find an upper bound for $\displaystyle\sum_{k=1}^{N} \frac{1}{k^2}$

~~R05,~~ 06 (DIDN'T HAVE TIME TO FINISH IN R05) ~~3~~ 3 1.

Find the sum of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$.

~~WHAT IS ON RHS?~~

$$\frac{1}{(3k-2)}\frac{1}{(3k+1)} = \frac{A}{3k-2} + \frac{B}{3k+1}$$

distinct linear factors, not repeated

what do we do now?

$$1 = A(3k+1) + B(3k-2)$$

describe using text chat the next step

UTCTDNS

$$1 = 3k(A+B) + (A-2B)$$

$$\Rightarrow \left.\begin{array}{l} A+B=0 \\ A-2B=1 \end{array}\right\} \begin{array}{l} B=-1/3 \\ A=+1/3 \end{array}$$

$$S = \sum \frac{1/3}{3k-2} - \frac{1/3}{3k+1}$$

$$= \tfrac{1}{3}\left((1-\tfrac14) + (\tfrac14 - \tfrac17) + (\tfrac17 - \tfrac1{10}) + \dots\right)$$

$$= 1/3$$

because We found a ___ series.

CAT fill in the blanks when you have then

have then fill in blanks

— SEND LINK —

R06

2

Series for $\int_0^x \sin \frac{\pi t^2}{2}\, dx$
(1st 3 terms)

$$\int_0^x \left[\left(\frac{\pi t^2}{2}\right) - \left(\frac{\pi t^2}{2}\right)^3 / 3! + \frac{\left(\frac{\pi t^2}{2}\right)^5}{5!} - \dots \right] dt$$

get them to fill in

$$= \int_0^x \frac{\pi t^3}{2 \cdot 1!} - \frac{\pi^3 t^6}{2^3 \cdot 3!} + \frac{\pi^5 t^{10}}{2^5 \cdot 5!} - \dots dt$$

$$= \frac{\pi x^4}{4 \cdot 2 \cdot 1!} - \frac{\pi^3 x^7}{7 \cdot 2^3 \cdot 3!} + \frac{\pi^5 x^{10}}{11 \cdot 2^5 \cdot 5!} - \dots$$

$4+3k$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \pi^{2k+1} x^{4+3k}}{(2^{2k+1})(2k+1)!\,(4+3k)}$$

(not necessary)

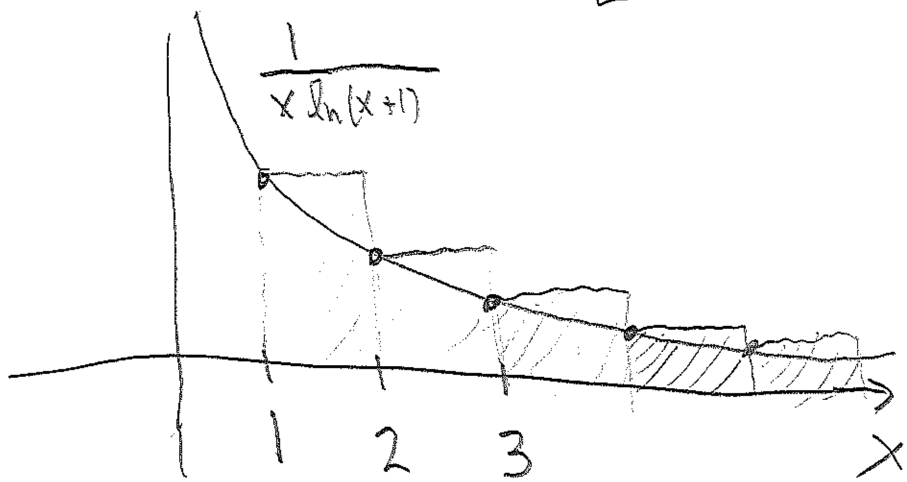
k	2k+1	4+3k
0	1	4
1	3	7
2	5	10
3	7	

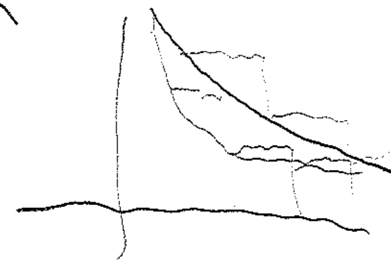
(if asked)

ROG

$$\sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)}$$

LEFT-SUM





$$\sum \frac{1}{k \ln(k+1)} > \int_1^{\infty} \frac{1}{x \ln(x+1)} dx$$

$$> \int_1^{\infty} \frac{1}{(x+1)\ln(x+1)} dx$$

$$= \ln\left(\ln(x+1)\right)\Big|_1^{\infty} \quad (= \infty - 1)$$

which DNE.

$$\Rightarrow \sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)} \text{ DNE}$$

QH6 Quiz 3

Good luck on Quiz 3!

If you have any questions, you can message me through wimba. I’ll be on wimba until 8:55.

QH6 Recitation 8

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Alternating Series Test, Quiz 1, Limits, Survey Results

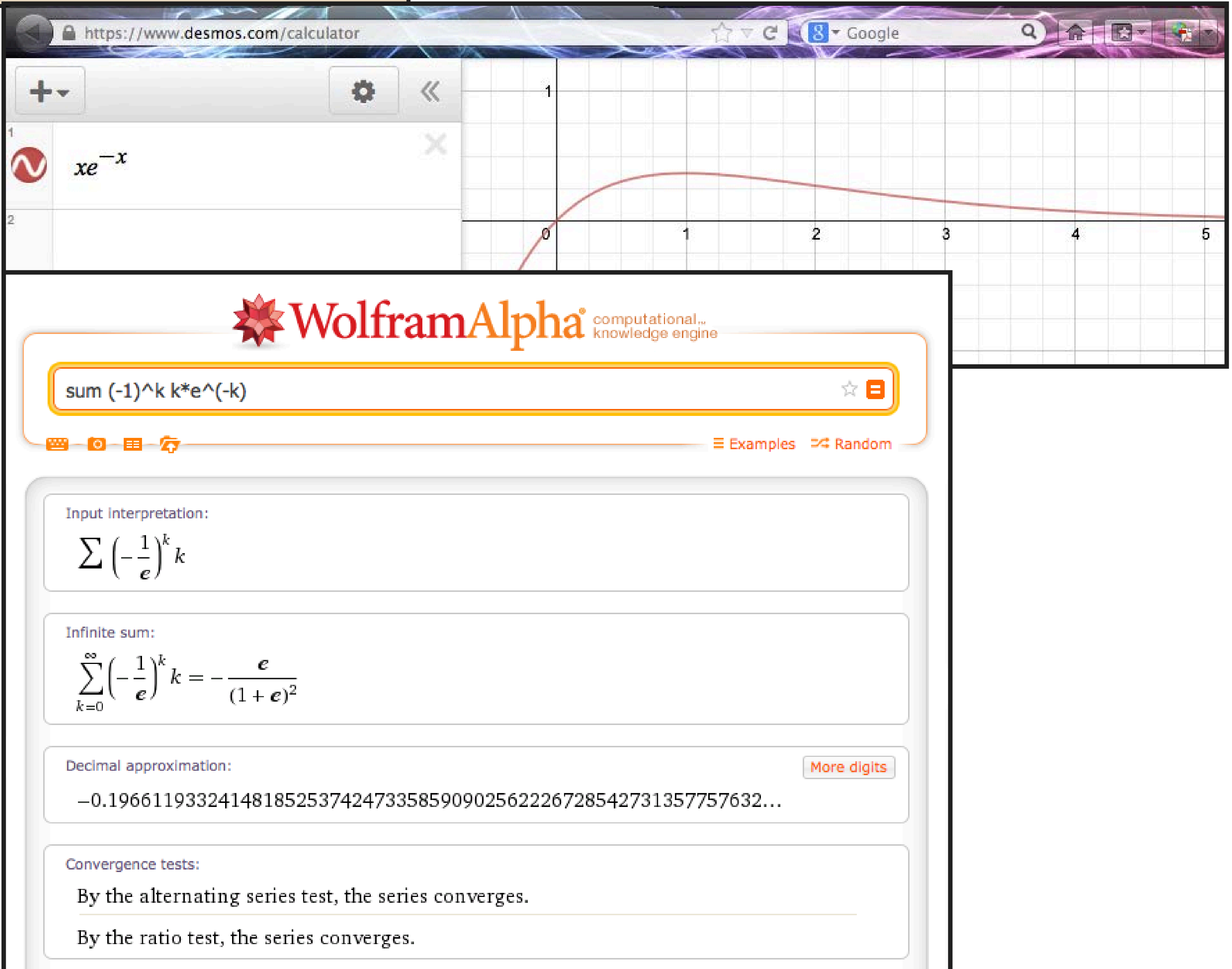
Suppose we have the series $\sum (-1)^k a_k$

What conditions do we need to apply the alternating series test?

Does the following series converge? Why/why not?

$$\sum (-1)^k k e^{-k}$$

Checking Our Work



How was Quiz 1?

What challenges, if any, did you encounter when writing Quiz 1?

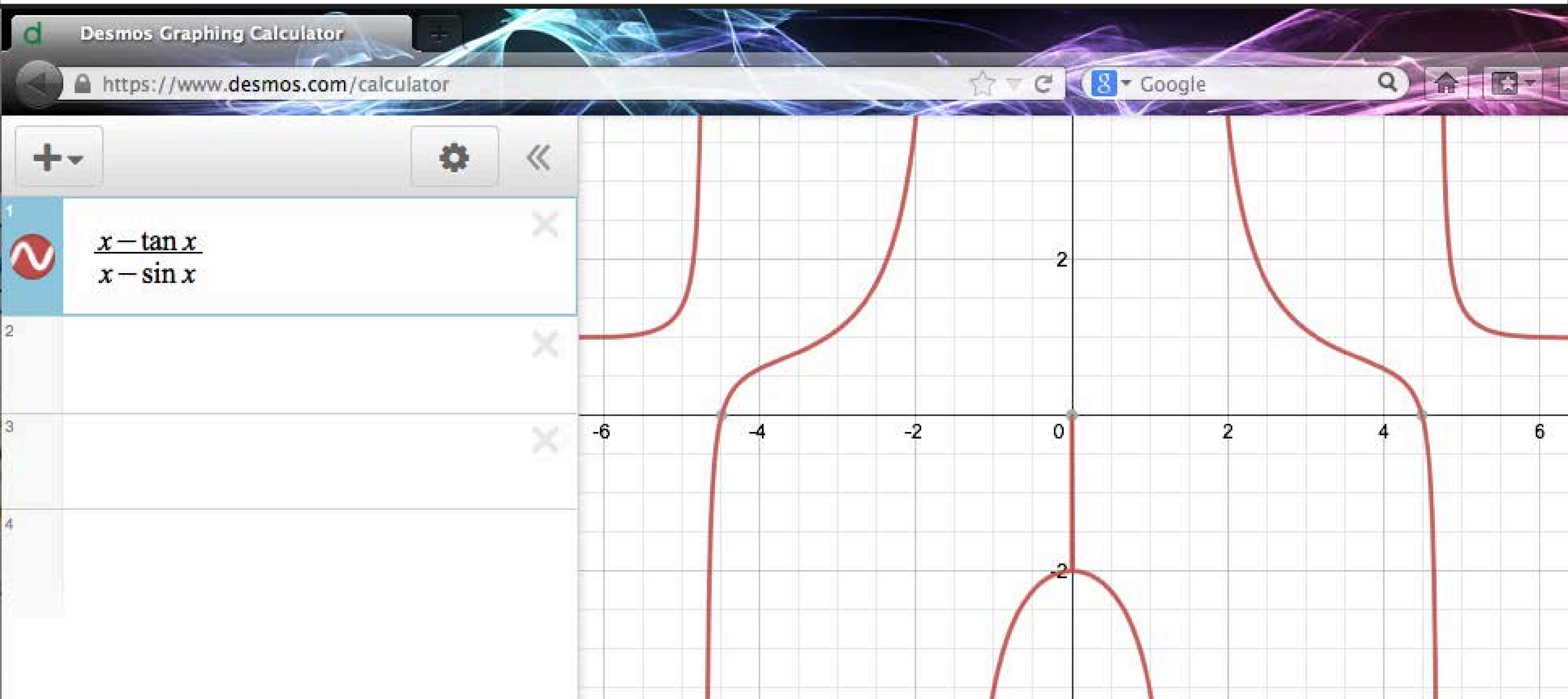
Using Series to Evaluate a Limit

The Taylor expansion of tan(x) is

$$\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$$

Evaluate $$\lim_{x \to 0} \frac{x - \tan x}{x - \sin x}$$

Checking Our Work



Using Series to Evaluate a Limit

Evaluate $$\lim_{x \to 0}\left(\frac{1}{\sin(x)} - \frac{1}{x}\right)$$

Using Series to Evaluate a Limit

Evaluate $$\lim_{x \to \frac{\pi}{2}} \frac{\cos x}{\sin(2x)}$$

R08

(1)

If $\sum (-1)^k a_k$, what conditions do we need to apply the ASCT?

1/ $a_k \geq 0 \quad \forall k$

2/ a_k decreasing

not $a_k \to 0$

Does $\sum (-1)^k k e^{-k}$ converge?

- $k e^{-k} \geq 0 \quad \forall k$
- is $k e^{-k}$ decreasing?

Ask: how do we know?

$$\frac{d}{dx} x e^{-x} = e^{-x} - x e^{-x}$$
$$= e^{-x}(1-x) < 0 \text{ for } x > 1$$

$\Rightarrow k e^{-k}$ is decreasing function of k

$\Rightarrow \sum (-1)^k k e^{-k}$ converges.

Sometimes: take derivative to verify ASCT can be applied

R08

$$\lim_{x\to 0} \frac{x-\tan x}{x-\sin x}$$

2

$$= \lim_{x\to 0} \frac{x-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}+\ldots\right)}{x-\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots\right)}$$

it won't matter what comes after 1st 3 terms, yay!

$$= \lim_{x\to 0} \frac{-\frac{x^3}{3}-\frac{2x^5}{15}+\ldots}{\frac{x^3}{6}-\frac{x^5}{5!}+\ldots}$$

(What trick can we use?
– poly. division, yes, but hard.
– multiply by $\frac{1/x^3}{1/x^3} = \frac{x^{-3}}{x^{-3}}$)

Gives us:

$$= \lim_{x\to 0} \frac{-\frac{1}{3}-\frac{2}{15}x^2+\ldots}{\frac{1}{6}-\frac{x^3}{5!}+\ldots}$$

$$= \frac{1/3}{1/6}$$

$$= 2$$

R08

③

$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$. THERE ARE AT LEAST TWO APPROACHES.

COMMON DENOMINATOR

$= \lim_{x \to 0} \left(\frac{x - \sin x}{x \sin x} \right)$

$= \lim_{x \to 0} \frac{-x^3/3! + x^5/5! - \ldots}{x^2 - x^4/3! + x^6/5!}$

$= 0/1+0 = 0$

(multiply by $\frac{x^{-2}}{x^{-2}}$)

EXPAND CSC X

$\csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \ldots$

$\Rightarrow \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right)$

$= \lim_{x \to 0} \left(\frac{x}{6} + \frac{7x^3}{360} + \ldots \right)$

$= 0$

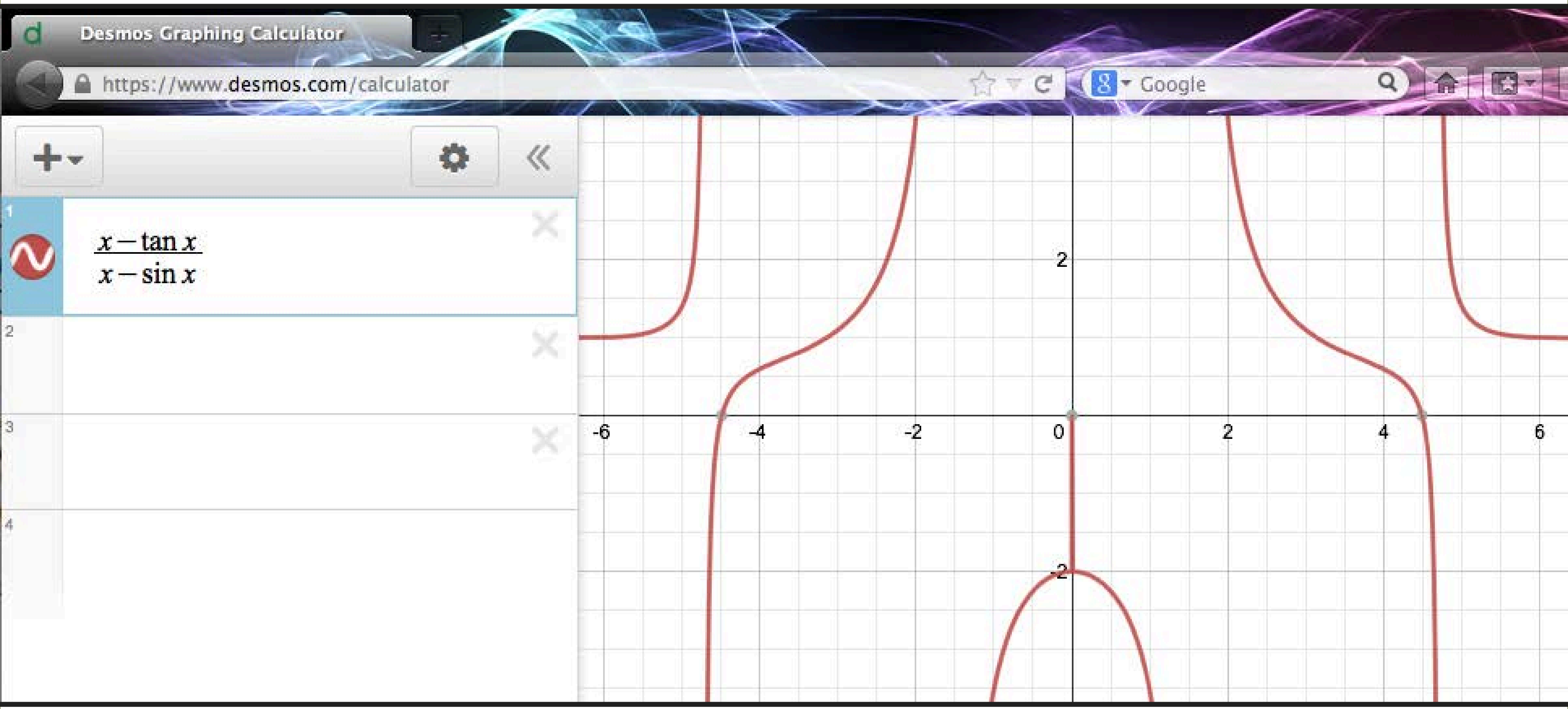
QH6 Recitation 9

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: l'Hopital's Rule, Quiz 2 Notes, Ice Breaker, Dot Products

Use l'Hopitals rule to evaluate: $\lim_{x \to 0} \frac{x - \tan x}{x - \sin x}$

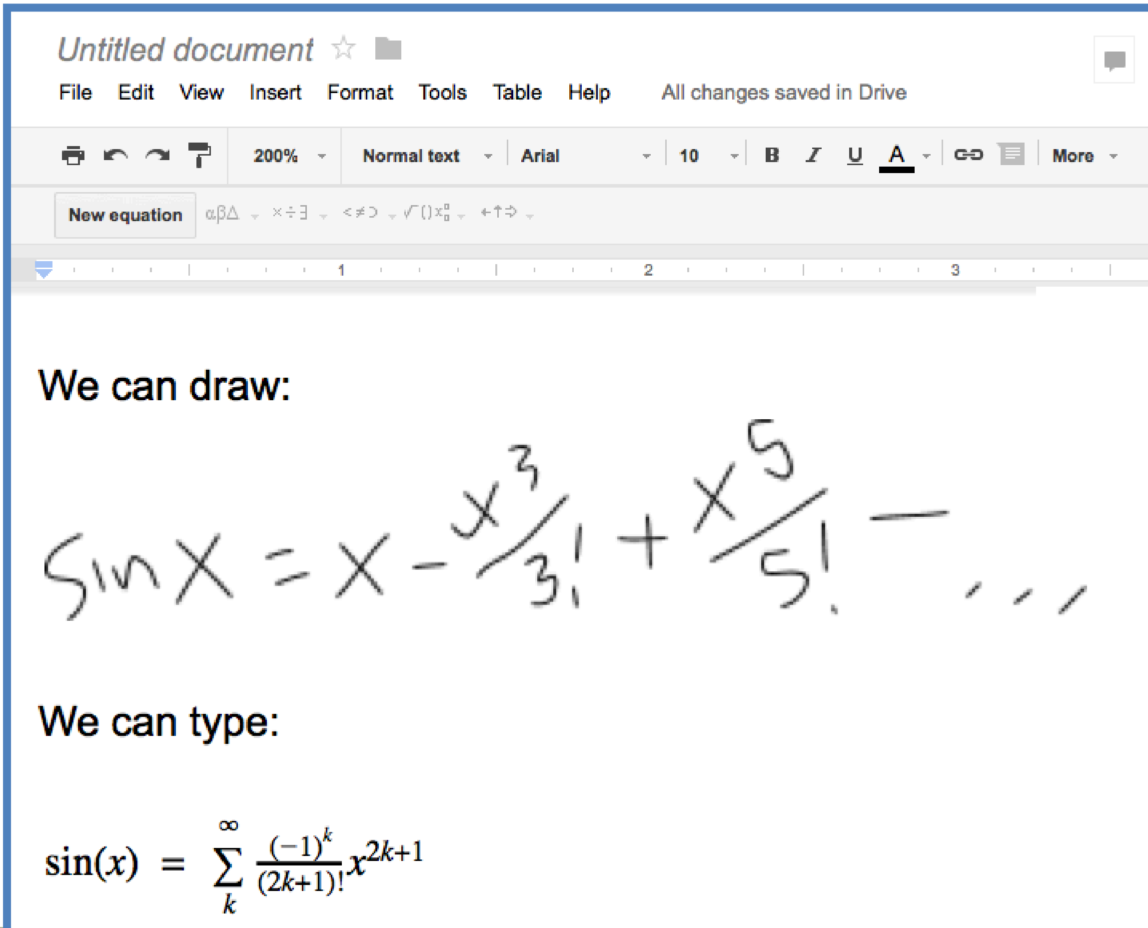
Checking Our Work



Quiz 2 Formula Page Collaboration

Let’s work together on creating formula pages for Quiz 2!

Participation is optional, not graded.



Google Docs

- can add drawings
- can see who wrote what
- has a chat feature
- can export documents to Word and other formats
- revision history
- no log-in required to view

Quiz 2 Formula Page Collaboration

Vote using text chat: which technology would you prefer?

1. Google Docs
2. Piazza

Rules:

a) errors in contributions can be corrected by anyone
b) don’t delete other students contributions
c)

(First?) Icebreaker

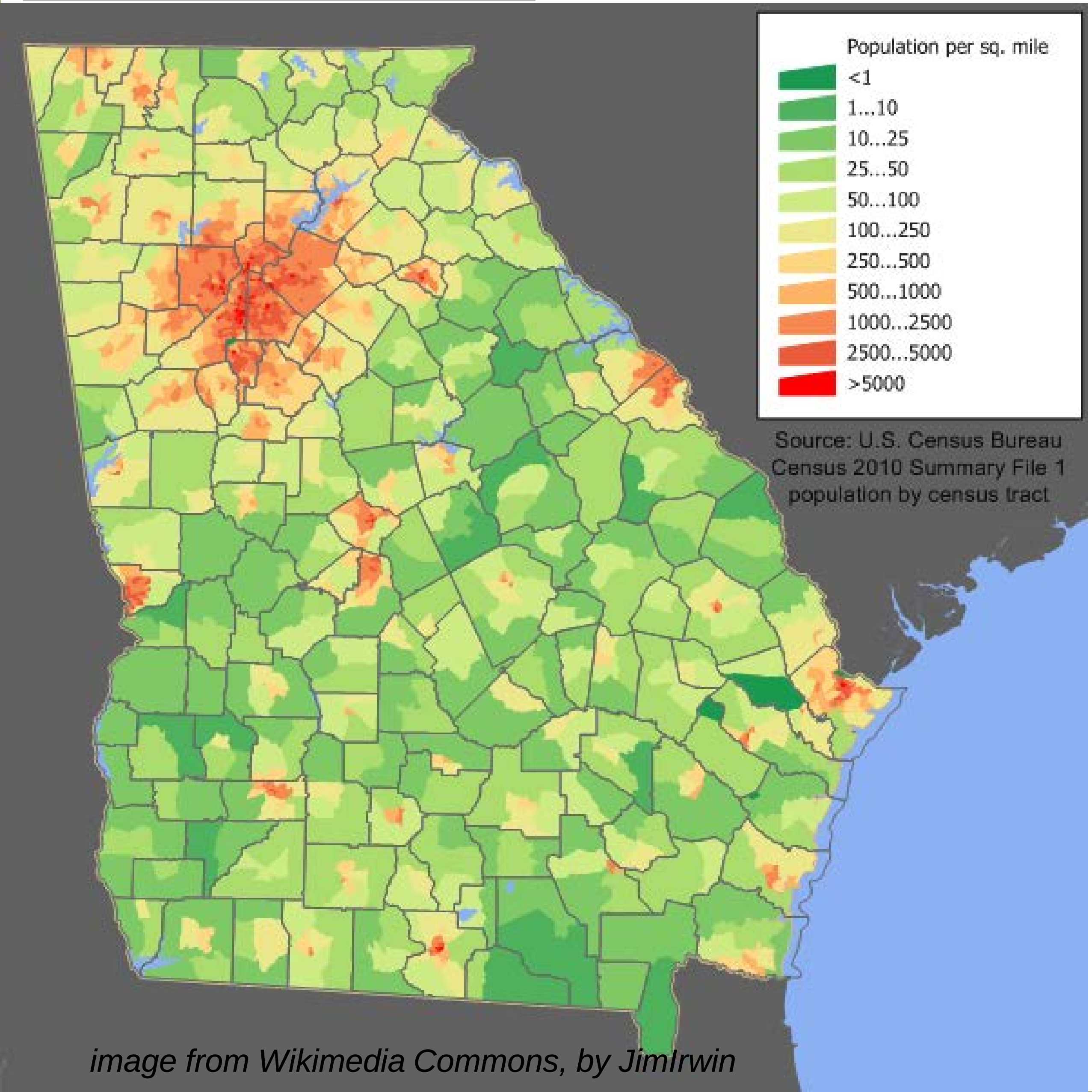


image from Wikimedia Commons, by JimIrwin

To use mic: press and hold the talk button.

Everyone:

- say your name,
- one thing about yourself,
- place a dot on the map that approximates your current location.

The Dot Product

If **u** = 5**i** + 3**j** + 6**k**, and **v** = 3**i** + 2**j** – 4**k**,

then **u • v** = ________+ ________ + ________ = _____ .

<u>Properties</u>

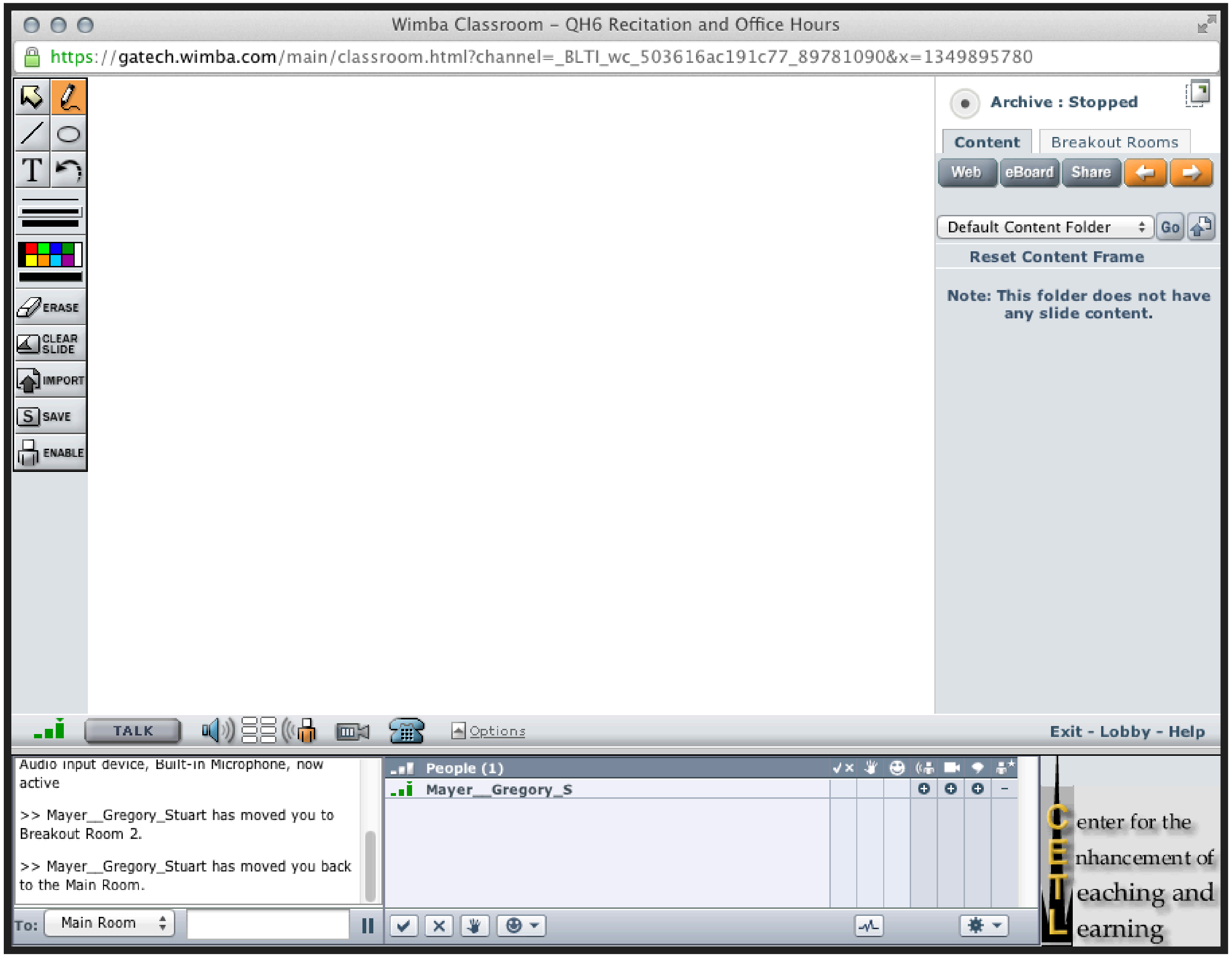
$$\mathbf{a}\cdot\mathbf{a}=\left\|\mathbf{a}\right\|^2$$, which is the _______________ of vector **a**.

$$\vec{a}\cdot\vec{b}=\vec{b}\cdot\underline{\qquad\qquad}$$

$$\vec{a}\cdot\vec{b}=\left\|\vec{a}\right\|\left\|\vec{b}\right\|\underline{\qquad\qquad\qquad}$$

Presenter Status

- allows you to move yourself and others in/out of breakout rooms
- allows you to give/remove others presenter status



Dot Products

a) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

b) Show that $4\left(\vec{a}\cdot\vec{b}\right)=\left\|\vec{a}+\vec{b}\right\|^2-\left\|\vec{a}-\vec{b}\right\|^2$

Projections

Given any two vectors **u** and **v**, find $\mathbf{u}_{||} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$.

QH6 Recitation 9

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: l'Hopital's Rule, Quiz 2 Notes, Ice Breaker, Dot Products

Use l'Hopitals rule to evaluate: $\lim_{x\to 0} \dfrac{x-\tan x}{x-\sin x}$

$$\overset{H}{=} \lim \frac{1-\sec^2 x}{1-\cos x}$$

$$\overset{H}{=} \lim \frac{-2\sec^2 x \tan x}{\sin x}$$

$$\overset{H}{=} \lim \frac{-4\sec^2 x \tan^2 x - 2\sec^4 x}{\cos x}$$

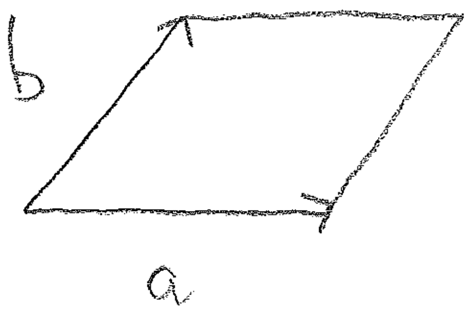
$$= \frac{0-2}{1} = -2$$

Dot Products

a) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

b) Show that $4\left(\vec{a}\cdot\vec{b}\right)=\left\|\vec{a}+\vec{b}\right\|^2-\left\|\vec{a}-\vec{b}\right\|^2$

a)



diagonals are $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$.

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b}$$

$$= \|\vec{a}\| - \|\vec{b}\|$$

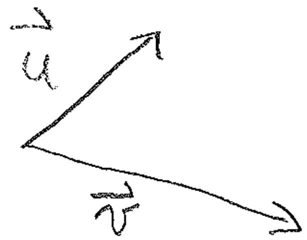
$$= 0$$

b) $\|\vec{a}+\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2 = (\vec{a}+\vec{b})^2 - (\vec{a}-\vec{b})^2 = \ldots$

Projections

We want an expression for $u_{\parallel} = \text{proj}_{\vec{v}}\,\vec{u}$.

Given any two vectors **u** and **v**, find $\mathbf{u}_{\parallel} = \text{proj}_{\mathbf{v}}\,\mathbf{u}$.



Define $\vec{u}_{\parallel}$ and $\vec{u}_{\perp}$ as:

Then $\vec{u}_{\parallel} + \vec{u}_{\perp} = \vec{u}$.

By construction, $\vec{u}_{\parallel} \cdot \vec{u}_{\perp} = 0$

To find expression,

try $\vec{u}_{\parallel} = (\vec{u} \cdot \vec{v}) \dfrac{\vec{v}}{\|\vec{v}\|}$

To see if this works, check to see if

$0 = \vec{u}_{\parallel} \cdot \vec{u}_{\perp} = \vec{u}_{\parallel} \cdot (\vec{u} - \vec{u}_{\parallel})$

It should be.

Today: Dot Products

A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

Dot Products

Show that $$4\left(\vec{a}\cdot\vec{b}\right)=\left\|\vec{a}+\vec{b}\right\|^2-\left\|\vec{a}-\vec{b}\right\|^2$$

Application of Angles

Suppose vectors **u** = **i** + x**j** + **k**, and **v** = 2**i** – **j** **+** y**k**. Find x and y so that **u** and **v** are perpendicular.

Dot Products

What can we conclude about vectors **a** and **b** if:

a) $\left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 = \left\|\vec{a}+\vec{b}\right\|^2$

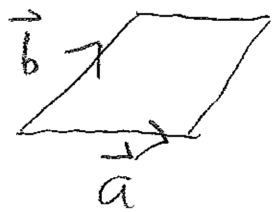
b) $\left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 = \left\|\vec{a}-\vec{b}\right\|^2$

QH6 Recitation 10

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Dot Products

A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.



$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{b} = \|\vec{a}\| - \|\vec{b}\| = 0$$

Ask students (why is $\|\vec{a}\| = \|\vec{b}\|$?)

Dot Products

Show that $$4(\vec{a}\cdot\vec{b}) = \|\vec{a}+\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2$$

$$\|\vec{a}+\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2 = (a+b)^2 + (a-b)^2$$

$$= (a^2 + 2a\cdot b + b^2) - (a^2 - 2a\cdot b + b^2)$$

$$= 4\vec{a}\cdot\vec{b}$$

$$= LHS$$

Application of Length and Angles

Suppose vectors **u = i** + x**j** + **k**, and **v** = 2**i** – **j** + y**k**. Find x and y so that **u** and **v** are perpendicular ~~and have the same length.~~

$$\vec{0} = \vec{u} \times \vec{v} = \begin{vmatrix} i & j & k \\ 1 & x & 1 \\ 2 & -1 & y \end{vmatrix} = \begin{bmatrix} xy+1 \\ 2-y \\ -1-2x \end{bmatrix}$$

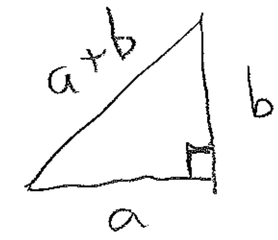
$$\Rightarrow \begin{array}{l} xy+1=0 \\ 2-y=0 \\ -1-2x=0 \end{array} \quad \Rightarrow \begin{array}{l} y=2 \\ x=-\tfrac{1}{2} \end{array}$$

$$\|\vec{u}\| = \sqrt{1^2+(-\tfrac{1}{2})^2+1^2} = \sqrt{5/4}$$

$$\|\vec{v}\| = \sqrt{2^2+(-1)^2+}$$

Dot Products

What can we conclude about vectors **a** and **b** if:



a) $\|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{a}+\vec{b}\|^2$

$\|\vec{a}+\vec{b}\|^2 = (a+b)\cdot(a+b) = a\cdot a + b\cdot b + 2a\cdot b = \|\vec{a}\|^2 + \|b\|^2 + 2a\cdot b$

$\Rightarrow a\cdot b = 0 \Rightarrow$ a and b are $\perp$.

b) $\|\vec{a}\|^2 + \|\vec{b}\|^2 = \|\vec{a}-\vec{b}\|^2$

same as a).

QH6 Recitation 11

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Projections, Cross Products, Planes

Given any two vectors $\mathbf{u}$ and $\mathbf{v}$, find $\mathbf{u}_{||} = \text{proj}_{\mathbf{v}}\, \mathbf{u}$.

Quiz 2, Midterm Grades

<u>Quiz 2</u>

Thursday October 10.

Questions?

<u>Midterm Grades</u>

- Submitted on Friday at noon
- Calculated as
    - 50% quiz 1
    - 50% homework
- Let me know if any grades in t-square are not correct
- Did you get Quiz 1 back? How did you get it back?

Cross Product Properties

If **a** = **i** + 6**k**, and **b** = 3**i** + 2**j**,

then **a × b** =

Properties

$\vec{a} \times \vec{b}$ is ______________ to vectors $\vec{a}$ and $\vec{b}$

$\left\|\vec{a} \times \vec{b}\right\| = \left\|\vec{a}\right\| \left\|\vec{b}\right\|$ __________

Properties

If $\mathbf{a} \times \mathbf{b} = 0$ and $\mathbf{a} \bullet \mathbf{b} = 0$, what can we conclude about vectors **a** and **b**?

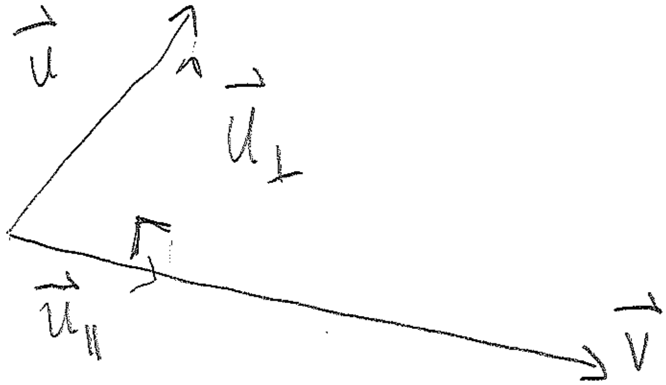
Planes

Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a unit normal vector to the plane that contains the three points.
b) Find an equation of the plane.

R11

1/



We need an expression for $\vec u_\parallel = \text{proj}_{\vec v}\vec u$.

($\vec u_\parallel + \vec u_\perp = \vec u$, or $\vec u_\perp = \vec u - \vec u_\parallel$

$\hat v = \vec v / \|\vec v\|$.) DON'T WRITE

Let's try: $\vec u_\parallel = (\vec u \cdot \hat v)\,\vec v$. $\hat v$

Then $0 = \vec u_\parallel \cdot \vec u_\perp$

$= (\vec u \cdot \vec v)\,\vec v \cdot \vec u_\perp$, by our def'n of $\vec u_\parallel$

$= (\vec u \cdot \vec v)\,\vec v \cdot (\vec u - \vec u_\parallel)$, because $\vec u_\perp = \vec u - \vec u_\parallel$

$= (\vec u \cdot \vec v)(\vec u \cdot \vec v) - (\vec u \cdot \vec v)(\vec v \cdot \vec u_\parallel)$, because dot prod. are distributive.

$= (\vec u \cdot \vec v)(\vec u \cdot \vec v) - (\vec u \cdot \vec v)(\vec u \cdot \vec v)\,\underbrace{\vec v \cdot \vec v}_{\text{need this to be } 1}$

How can we change our def'n of $\vec u_\parallel$ s.t. we do have zero?

$\Rightarrow \boxed{\vec u_\parallel = \text{proj}_{\vec v}\vec u = (\vec u \cdot \hat v)\,\hat v}$, because it works

R11 & R12

- If $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$, then what can we conclude about $\vec{a}$ and $\vec{b}$?

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ or at least one of $\vec{a}, \vec{b}$ zero

$\vec{a} \times \vec{b} = 0 \Rightarrow \vec{a} \parallel \vec{b}$ or one of $\vec{a}, \vec{b}$ zero

Q

$P(1,2,3)$, $Q(1,3,4)$, $R(2,2,2)$

Q/ How do we find two vectors in the plane?

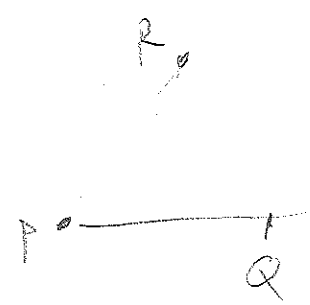
$$\overrightarrow{PQ} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \overrightarrow{PR} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Q/ How do we find a vector normal to the plane?

$$\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-1-0)\hat{i} - (0-1)\hat{j} + (0-1)\hat{k}$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

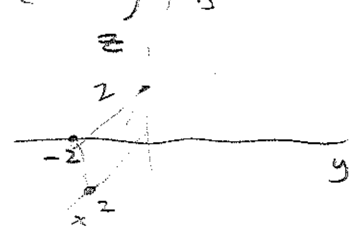


$$\|\vec{N}\| = \sqrt{3} \Rightarrow \hat{N} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

any point in plane $= (x,y,z)$, specific point in plane $= P(1,2,3)$, a vector in plane $= \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$

$$0 = \hat{N} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = \left(-(x-1) + (y-2) - (z-3)\right)\frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{x - y + z = 2}$$



- What did we learn?
- what concepts did we use?

QH6 Recitation 12

If you are not in QH6, please attend your recitation at gtcourses.gatech.edu. This recitation is for QH6 students.

Today: Cross Products, Planes, Lines

Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a unit normal vector to the plane that contains the three points.

b) Find an equation of the plane.

Dot and Cross Products

Which of the following make sense? Explain why/why not.

1) $\mathbf{a} \times (\mathbf{b} \bullet \mathbf{c})$

2) $\mathbf{a} \bullet (\mathbf{b} \bullet \mathbf{c})$

3) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

4) $\mathbf{a} \bullet (\mathbf{b} \times \mathbf{c})$

1) What do you get when you cross an elephant and a grape?

2) What do you get when you cross a mountain-climber with a mosquito?

Coplanar

Determine whether the vectors are co-planar:

$\mathbf{j} - \mathbf{k}$

$3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$

$3\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

Scalar Parametric Form

What do these equations represent?

$$x_1 = 1 + t \qquad x_2 = 1 - u$$

$$y_1 = -1 - t \qquad y_2 = 1 + 3u$$

$$z_1 = -4 + 2t \qquad z_2 = -2u$$

Planes

Find the equation for the line that is perpendicular to the yz-plane, and also passes through P(1,4,3).

R11 & R12

2/

- If $\vec{a} \times \vec{b} = \vec{0}$ and $\vec{a} \cdot \vec{b} = 0$, then what can we conclude about $\vec{a}$ and $\vec{b}$?

$\vec{a} \cdot \vec{b} = 0 \Rightarrow \vec{a} \perp \vec{b}$ or at least one of $\vec{a}, \vec{b}$ zero

$\vec{a} \times \vec{b} = 0 \Rightarrow \vec{a} \parallel \vec{b}$ or one of $\vec{a}, \vec{b}$ zero

Q

$P(1,2,3)$, $Q(1,3,4)$, $R(2,2,2)$

Q/ How do we find two vectors in the plane?

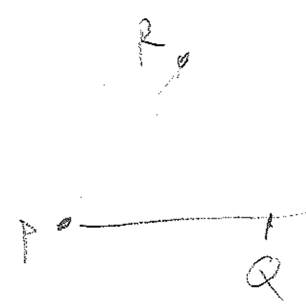
$$\vec{PQ} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{PR} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Q/ How do we find a vector normal to the plane?

$$\vec{N} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = (-1-0)\hat{i} - (0-1)\hat{j} + (0-1)\hat{k}$$

$$= -\hat{i} + \hat{j} - \hat{k}$$

$$= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

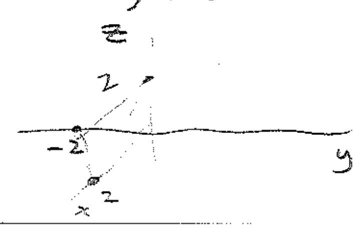


$$\|\vec{N}\| = \sqrt{3} \Rightarrow \hat{N} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

any point in plane $= (x,y,z)$, specific point in plane $= P(1,2,3)$, a vector in plane $= \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix}$

$$0 = \hat{N} \cdot \begin{bmatrix} x-1 \\ y-2 \\ z-3 \end{bmatrix} = \left(-(x-1) + (y-2) - (z-3)\right) \frac{1}{\sqrt{3}}$$

$$\Rightarrow \boxed{x - y + z = 2}$$



- What did we learn?
- what concepts did we use?

R12 13.6, #22

Q WHAT DOES CO-PLANAR MEAN?

$$\begin{bmatrix}0\\1\\-1\end{bmatrix}\times\begin{bmatrix}3\\-1\\2\end{bmatrix}\times\begin{bmatrix}3\\-2\\3\end{bmatrix}=\begin{vmatrix}i & j & k\\0 & 1 & -1\\3 & -1 & 2\end{vmatrix}\times\begin{bmatrix}3\\-2\\3\end{bmatrix}=\begin{vmatrix}i & j & k\\3 & -3 & -3\\3 & -2 & 3\end{vmatrix}$$ TOO HARD

* u, v, w are co-planar they are in the same plane.

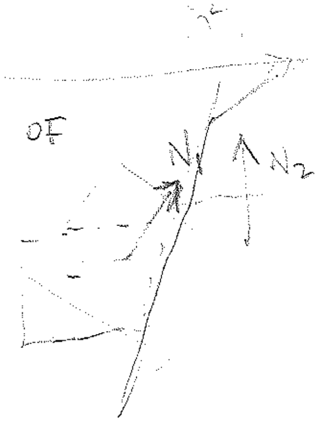
(1) if u, v, w are co-planar, then $(u\times v)\times w\neq 0$ X

(2) u, v, w " " $\Longleftrightarrow$ $(u\times v)\cdot w=0$

Q: why iff?

USE (2): $\begin{bmatrix}3\\-3\\-3\end{bmatrix}\cdot\begin{bmatrix}3\\-2\\3\end{bmatrix}=9+6-9\neq 0$

$\Rightarrow$ u, v, w are not co-planar.



EQU LINE FORMED FROM INTERSECTION OF

(1) $x+2y+3z=0$

(2) $-3x+4y+z=0$

(1) is $0=\begin{bmatrix}1\\2\\3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\underbrace{\begin{bmatrix}1\\2\\3\end{bmatrix}}_{N_1}\cdot\underbrace{\begin{bmatrix}x-0\\y-0\\z-0\end{bmatrix}}_{\text{vector in plane}}$

$0=\underbrace{\begin{bmatrix}-3\\4\\1\end{bmatrix}}_{N_2}\cdot\begin{bmatrix}x-0\\y-0\\z-0\end{bmatrix}$

$\vec{N}_1\times\vec{N}_2=\begin{vmatrix}1 & 2 & 3\\-3 & 4 & 1\end{vmatrix}=\begin{bmatrix}-10\\-10\\+10\end{bmatrix}$, $\Rightarrow$ l: $x_1=x_0+$
$y_1=y_0+$
$z_1=z_0+$

QH6 Recitation 13

Today: Cross Products, Planes, Lines

Do these lines intersect each other? Why?

$$x_1 = 1 + t \qquad x_2 = 1 - u$$

$$y_1 = -1 - t \qquad y_2 = 1 + 3u$$

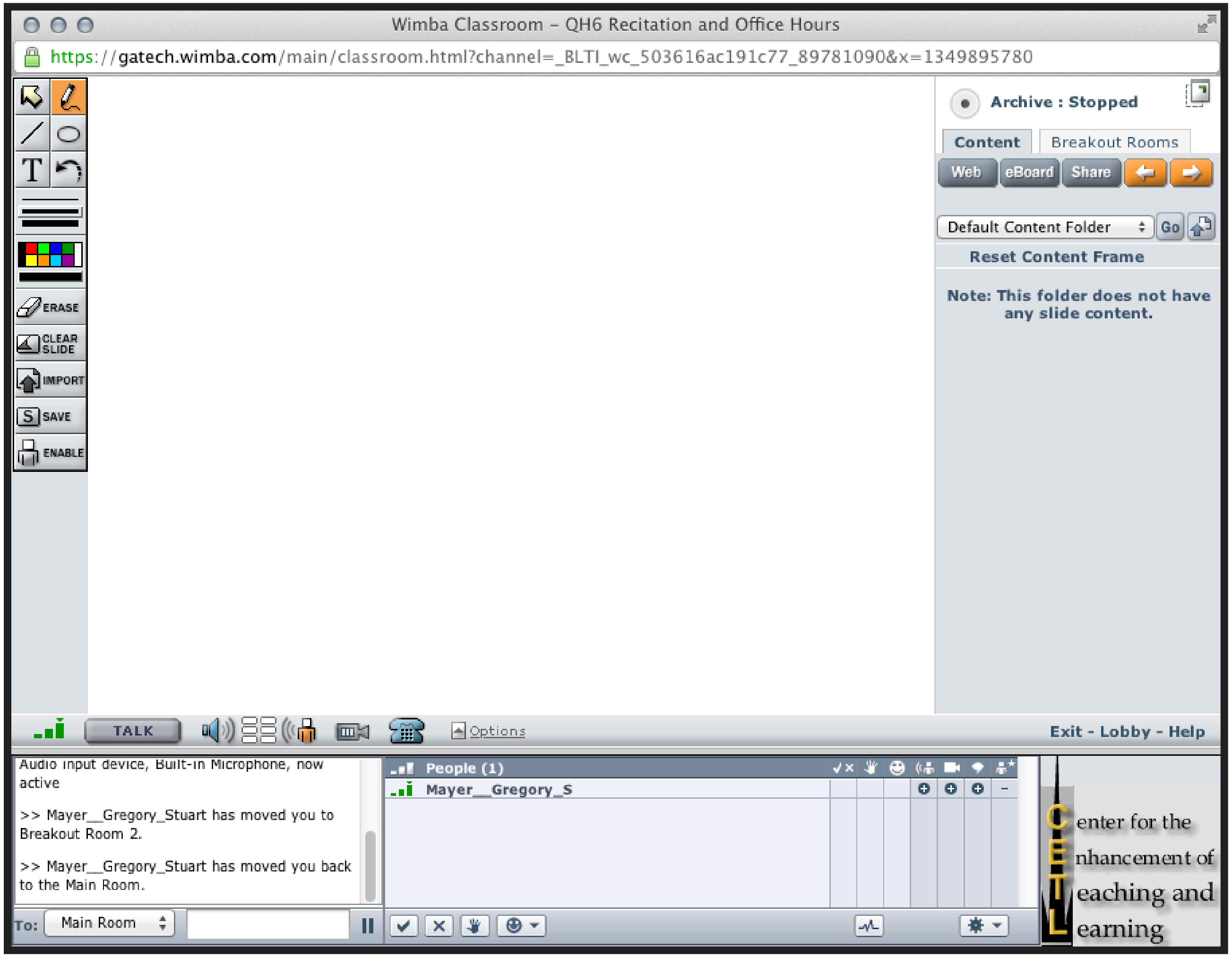
$$z_1 = -4 + 2t \qquad z_2 = -2u$$

Announcements

- Fall Recess: October 14, 15 – no lectures and recitations?
- Technology online survey: everyone please complete the survey!
- Any questions about HW7?

Presenter Status

- allows you to move yourself and others in/out of breakout rooms
- allows you to give/remove others presenter status



Q1, Last Year's Quiz 2

Find a parametrization of the line that is the intersection of the planes

P: $x - 2y + z = 3$

Q: $2x + y + z = 1$

R13

$$\left.\begin{aligned} x_1 &= 1+t \\ y_1 &= -1-t \\ z_1 &= -4+2t \end{aligned}\right\} l_1 \qquad \left.\begin{aligned} x_2 &= 1-u \\ y_2 &= 1+3u \\ z_2 &= -2u \end{aligned}\right\} l_2$$

why? If $\left.\begin{aligned} t=0, &\quad P(1,-1,-4) \\ t=1, &\quad Q(2,-2,-2) \end{aligned}\right\} \overrightarrow{PQ} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$\overrightarrow{PQ}$ is a vector $\parallel$ to l_1

$\left.\begin{aligned} u=0, &\quad R(1,1,0) \\ u=1, &\quad S(0,4,-2) \end{aligned}\right\} \overrightarrow{RS} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}$

~~if~~ Are l_1 & l_2 $\parallel$? No, $\overrightarrow{PQ} \times \overrightarrow{RS} = 0$

not scalar multiples

Do l_1 & l_2 intersect?

if they do, $\exists$ t,u that satisfy

$$\left\{\begin{aligned} 1+t &= 1-u \quad \Rightarrow \quad t=-u \\ -1-t &= 1+3u \quad \Rightarrow \quad -1+u = 1+3u \\ -4+2t &= -2u \qquad\qquad u=-1,\ t=+1 \end{aligned}\right.$$

$$\boxed{-6 = +2}$$

NO.

$\Rightarrow$ LINES ARE "SKEW"

R13&14

P $\quad x - 2y + z = 3$

Q $\quad 2x + y + z = 1$

2 EQUATIONS, 3 UNKNOWNS.
Let $y = t$, Then $x + 3t = -2 \Rightarrow x = -2 - 3t$
$y = t$
$z = 1 - t + 2 + 6t$
$= 3 + 5t$

P can be written in the form

$$\underbrace{(1,-2,1)}_{\text{a vector } \perp \text{ to plane P!}} \cdot \underbrace{(x - \ , y - \ , z - \ )}_{\text{vector in plane}} = 0$$

we need $-x_0 + 2y_0 - z_0 = +3$
choose $z_0 = 0$, $y_0 = 1 \Rightarrow x_0 = -1$

Likewise $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector $\perp$ Q.

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector $\parallel$ to line of intersection

$$= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

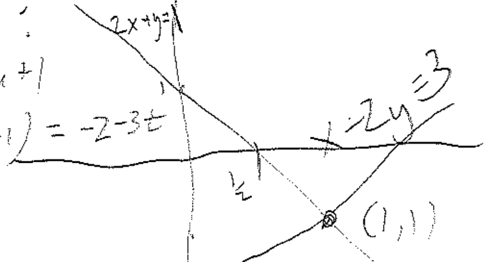
$$= \begin{bmatrix} -3 \\ +1 \\ 5 \end{bmatrix}$$

test $x = 1$: $y = -1$ ✓

ℓ is $\boxed{\begin{aligned} x &= 1 - 3t \\ y &= -1 + t \\ z &= 0 + 5t \end{aligned}}$

let $t+1 = u$, $t = u + 1$
$\Rightarrow x = 1 - 3(u+1) = -2 - 3t$
$z = 5(u+1)$

graph:



choose $z = 0$, then $\begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 & 3 \\ 0 & -3 & 0 & -5 \end{bmatrix}$

or plot

ALTERNATE SOLUTION

$x - 2y + z = 3$ ①

$2x + y + z = 1$ ②

Parametric form is

$x = x_0 + t\, d_x$

$y = y_0 + t\, d_y$

$z = z_0 + t\, d_z$

$\vec{d}$ = vector ∥ to line

② − ① is:

why did can we eliminate one variable!

$x + 3y = -2$

- One equ., two unknowns
- line in $\mathbb{R}^2$.
- let $y = t$, $x = -2 - 3t$

what does $y = t$ mean?

Then ① yields

$(-2-3t) - 2t + z = 3$

$z = 5 + 5t$

$$\Rightarrow \begin{aligned} x &= -2-3t \\ y &= t \\ z &= 5+5t \end{aligned}$$

QH6 Recitation 14

Today: Cross Products, Planes, Lines

Q1, Last Year's Quiz 2

(Both we denote you guys?)

Find a parametrization of the line that is the intersection of the planes

(P: x - 2y + z = 3) $\cdot -1$

Q: 2x + y + z = 1

$$-x+2y-z=-3$$
$$2x+y+z=1$$
$$x+3y=-2 \rightarrow x+3t=-2$$

$y=t$ $\quad$ $x=-3t-2$

$$z=-x+2y+3=-(-3t-2)+2t+3= \quad z=5t+5$$

For $y=t$

$$(x,y,z)=(-3t-2,\ t,\ 5t+5)$$

Announcements

- Quiz 2 Office Hours are:

Planes and Lines

Line L is determined by $P_1(4,-3,1)$ and $P_2(2,-2,3)$.
Plane Q is determined by $Q_1(2,0,-4)$, $Q_2(1,2,3)$, $Q_3(-1,2,1)$.
Do L and Q intersect? If so, where?

Planes and Lines

Find the equation for the line that is perpendicular to the yz-plane, and also passes through Q(1,1,7). Explain your process.

Q2, Last Year's Quiz 2 (basically)

For what values of b is **w** in the plane determined by vectors **u** and **v**?

$$\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} \qquad \vec{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}$$

R13&14

P $x - 2y + z = 3$

Q $2x + y + z = 1$

2 EQUATIONS, 3 UNKNOWNS.

Let $y = t$, Then $x + 3t = -2 \Rightarrow x = -2 - 3t$

$y = t$

$z = 1 - t + 2 + 6t$

$= 3 + 5t$

P can be written in the form

$\underbrace{(1,-2,1)}_{\text{a vector } \perp \text{ to plane P}} \cdot \underbrace{(x - \ \ , y - \ \ , z - \ \ )}_{\text{vector in plane}} = 0$

we need $-x_0 + 2y_0 - z_0 = +3$

choose $z_0 = 0$, $y_0 = 1 \Rightarrow x_0 = -1$

Likewise $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector $\perp$ Q.

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a vector $\parallel$ to line of intersection

$= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

$= \begin{bmatrix} -3 \\ +1 \\ 5 \end{bmatrix}$

test $x = 1$: $y = -1$ ✓

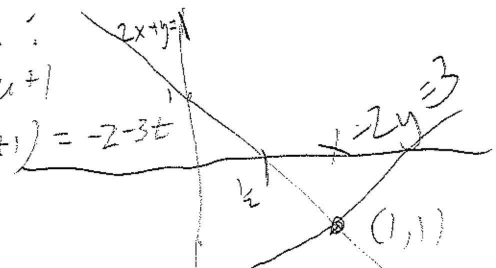
ℓ is $\boxed{\begin{aligned} x &= 1 - 3t \\ y &= -1 + t \\ z &= 0 + 5t \end{aligned}}$

let ~~$t-1$~~ $= u$, $t = u + 1$

$\Rightarrow x = 1 - 3(u+1) = -2 - 3t$

$z = 5(u+1)$

graph:



$\boxed{\text{choose } z = 0, \text{ then}}$ $\left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & 3 \\ 0 & -3 & 0 & -5 \end{array}\right]$

or plot

ALTERNATE SOLUTION

$$x-2y+z=3 \quad (1)$$
$$2x+y+z=1 \quad (2)$$

Parametric form is

$$x = x_0 + t\,d_x$$
$$y = y_0 + t\,d_y$$
$$z = z_0 + t\,d_z$$

$\vec{d}$ = vector ∥ to line

(2) − (1) is:

why did can we eliminate one variable!

$$x + 3y = -2$$

- one equ., two unknowns
- line in $\mathbb{R}^2$.
- let $y = t$, $x = -2-3t$

what does $y=t$ mean?

Then (1) yields

$$(-2-3t)-2t+z=3$$

$$z = 5+5t$$

$$\Rightarrow \quad \begin{aligned} x &= -2-3t \\ y &= t \\ z &= 5+5t \end{aligned}$$

R14

P2

d: $\begin{bmatrix} -2 \\ +1 \\ +2 \end{bmatrix}$

normal to plane, $\vec{N}$

ASK

- take some time to think about, then ~~give it a go~~. describe how you would start in text or misc.

what do we need to find to do this?

if $d \cdot \vec{N} = 0$, then $\boxed{d \parallel \text{plane}}$

$$\vec{N} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & +7 \\ -3 & +2 & +5 \end{vmatrix}$$

$$= \begin{bmatrix} -4 \\ -16 \\ +4 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

$$\vec{d} \cdot \vec{N} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

$$= -2 + 4 - 2 = 0$$

$\Rightarrow \vec{d} \parallel \vec{N}$. Is L in Q?

Q is $\vec{N} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$ is P_1 is in plane, then

$\vec{N} \cdot \begin{bmatrix} x - 4 \\ y + 3 \\ z - 1 \end{bmatrix} = 0$, which it isn't

If it does intersect:

$x(t) = 4 - 2t$

$y(t) = -3 + t$

$z(t) = 1 + 2t$

Find t s.t. equ. of plane satisfied.

Equ of plane:

$$0 = \vec{N} \cdot (\vec{x} - \vec{Q}_1)$$

$$0 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{pmatrix} x - 2 \\ y - 0 \\ z + 4 \end{pmatrix}$$

$$\boxed{6 = x + 4y - z}$$

subs: LHS $= 6$

RHS $= (4 - 2t) + 4(-3 + t) - (1 + 2t)$

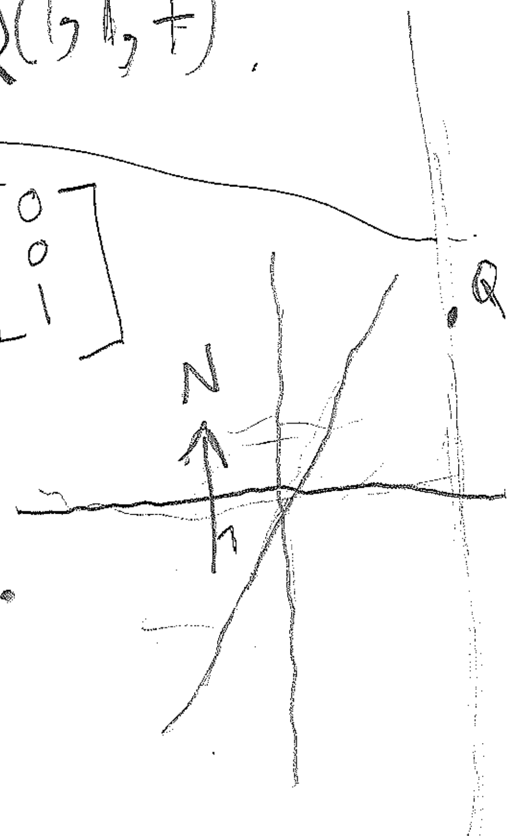
$= -9 - 2t - 2t + 4t$

$= -9$

EQU LINE $\perp$ TO XY-PLANE &

PASSES THROUGH $Q(1, 5, 7)$.

- NORMAL TO XY-PLANE: $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



- LINE, PARAMETRIC, IS

$$x = 1 + 0t$$

$$y = 5 + 0t$$

$$z = 7 + t$$

$$\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}$$

X If $\vec{w}$ is in the plane, then

If $\underline{\vec{N} \cdot \vec{w} = 0}$, then $\vec{w}$ is in the plane.

$$\vec{N} = \vec{u} \times \vec{v} = \begin{vmatrix} 4 & 2 & 3 \\ 1 & 1 & b \end{vmatrix} = \begin{bmatrix} 2b-3 \\ -4b+3 \\ 2 \end{bmatrix}$$

$$0 = \vec{N} \cdot \vec{w} = (2b-3)(-1) + (-4b+3)(1) + (2)(3)$$

$$0 = -2b + 3 + 3 - 4b + 6$$

$$6b = 12$$

$$b = 2$$

Today: Review for Quiz 2

For what values of z does the series converge?

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^z}$$

Announcements

- Quiz 2 Office Hours
    - I’ll be in Wimba
        - Tuesday: 7:30 pm to 9:00 pm
        - Wednesday: 7:30 pm to 9:00 pm
    - May need to get my attention with mic ☺
- Last year’s Quiz 2, Q2 and Q3, will be more helpful for your Quiz 3
- Google Doc notes
- Bring calculator
- Any questions about Quiz 2?

Planes and Lines

a) Find the angle between the planes:

$$x + y + z = 1$$

$$x - 2y + 3z = 1$$

b) Find the symmetric equation of the line between the planes.

Distance Formula

Vector **r** is || to line L. Find the distance between L and an arbitrary point P.

Lines

L is the line $\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{d}$.

a)Find the scalar t_0 so that $\mathbf{r}(t_0) \perp L$

b)Find the parameterization $\mathbf{R}(t) = \mathbf{R}_0 + t\mathbf{D}$ for L, where $\mathbf{R}_0 \perp L$, and $||\mathbf{D}|| = 1$.

$\boxed{R15}$

12.3, 37

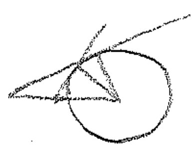
FOR WHAT VALUES OF z DOES THE SERIES CONVERGE?

$$\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^z}$$

$$\int_2^{\infty} \frac{1}{x(\ln x)^z}\,dx = \int_2^{\infty} \frac{1}{x u^z}\, x\,du$$

$u = \ln x$
$du = \frac{1}{x} dx$

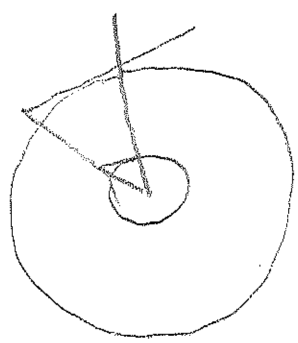
$$= \int_2^{\infty} u^{-z}\,du$$



$$= \frac{1}{1-z} u^{1-z} \Big|_2^{\infty}$$

$$\Rightarrow z > 1$$

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^?}$$

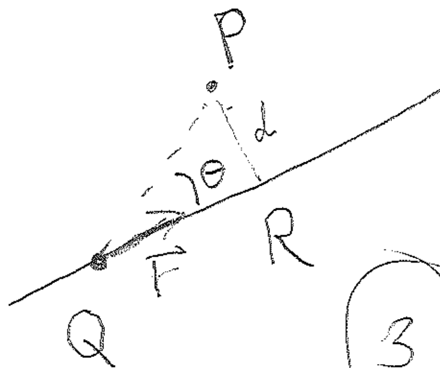


Why can we apply integral test?

$\vec{r}$ is $\parallel$ to line l.
Find the distance
between an arbitrary point P
and l.

1. DRAW DIAGRAM

2. INTRODUCE VARS



3. Given: $\vec{r}$, l, P
Need: d

$$\|PR\| = d = \|\overrightarrow{QP}\| \sin\theta$$

We don't have θ

but: $\|\vec{r} \times \overrightarrow{QP}\| = \|\vec{r}\|\|\overrightarrow{QP}\| \sin\theta$

$$= \|\vec{r}\| \, d$$

$$d = \frac{\|\vec{r} \times \overrightarrow{QP}\|}{\|\vec{r}\|}$$

$x+y+z=1$
$x-2y+3z=1$

a) $N_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}, \quad N_2 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ $\boxed{\text{why?}}$

$$\cos\theta = \frac{1-2+3}{\sqrt{3}\sqrt{1+4+9}} = \frac{2}{\sqrt{3}\sqrt{14}} \Rightarrow \theta \approx 72°$$

b) DID YOU SEE S.F.?

Symmetric form: $\dfrac{x-x_0}{r_1} = \dfrac{y-y_0}{r_2} = \dfrac{z-z_0}{r_3}$

$\vec{r}$ is $\parallel$ to line; (x_0, y_0, z_0) is any point.

$$\vec{r} = \vec{N}_1 \times \vec{N}_2 = \begin{bmatrix}5\\-2\\-3\end{bmatrix}$$

~~x~~ NEED POINT: choose $z=0$, then

$$\left.\begin{array}{l} x+y=1 \\ x-2y=1 \end{array}\right\} y=0,\ x=1$$

$$\Rightarrow \frac{x-1}{5} = \frac{y+0}{-2} = \frac{z-0}{-3}$$

$= \cancel{h+p - x^2p} \Leftarrow \cancel{\frac{3p}{oh-h}} = \cancel{\frac{1p}{ox-x}}$

OR: get $x(t) = x_0 + 5t$
$y(t) =$
$\}$ plug and go.

ℓ is the line $\vec{r}(t) = \vec{r}_0 + t\vec{d}$.

a) Find the scalar t_0 s.t. $\vec{r}(t_0) \perp L$

b)

a) $0 = \vec{d} \cdot \vec{r}(t_0) = 0$ why?
$= \vec{d} \cdot (\vec{r}_0 + t_0 \vec{d})$
$= \vec{d} \cdot \vec{d}\, t + \vec{d} \cdot \vec{r}_0$

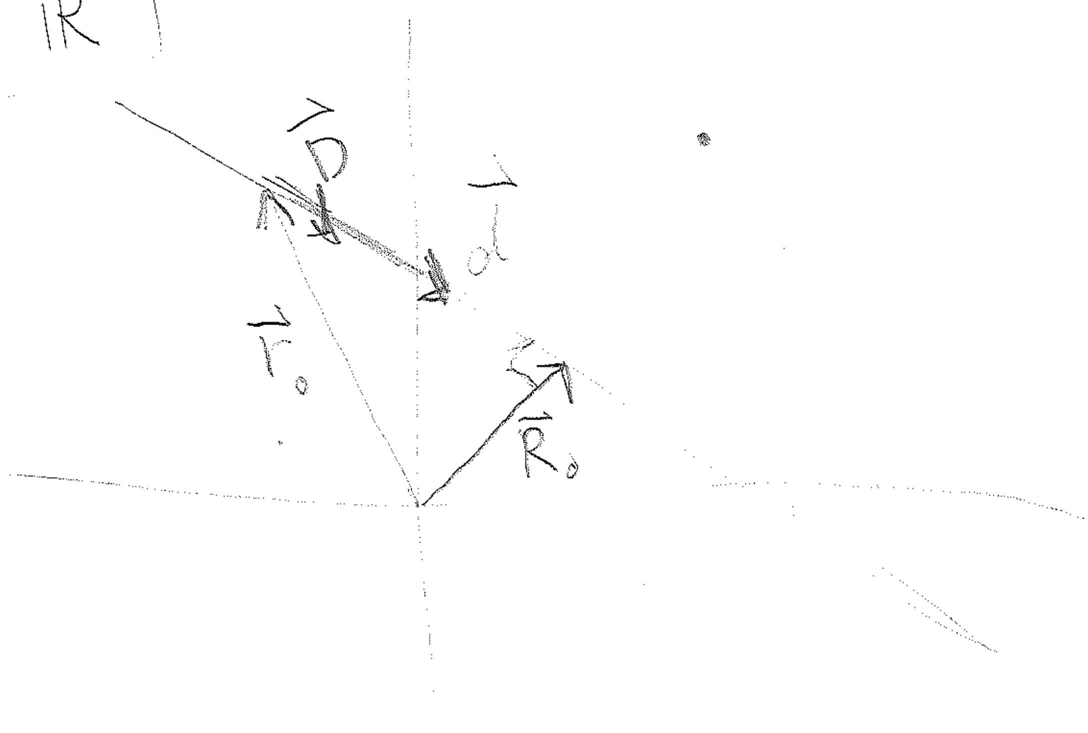
$$t_0 = \frac{\vec{d} \cdot \vec{r}_0}{\|\vec{d}\|^2}$$

b) NEED $\vec{R} = R_0 + t\vec{D}$
s.t. $\vec{R}_0 \perp \ell$, $\|D\| = 1$,

$\vec{R}_0 \perp \ell \Rightarrow \vec{R}_0 = \vec{r}(t_0)$

$\vec{D} \parallel \vec{d}$,

IN $\mathbb{R}^2$



QH6 Quiz 4

Good luck on Quiz 4!

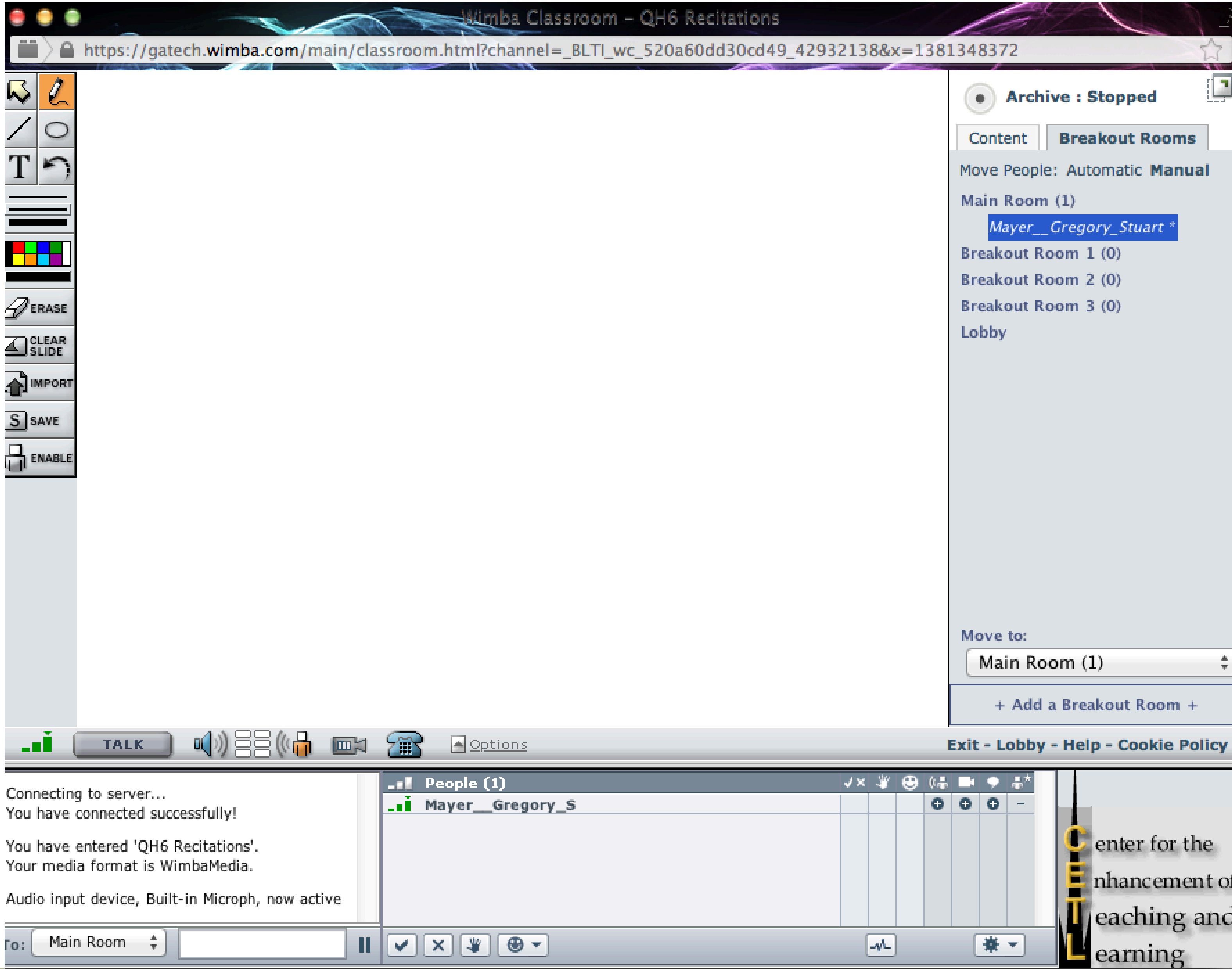
If you have any questions, you can message me through wimba. I'll be on wimba until 8:55.

QH6 Recitation 17

Today: Pop Quiz, Linear Systems

- Pop quiz grading
  - Correct 5 points
  - Name on page 3 points
  - Did not take: 0 points.
- Time: 10 minutes
- To submit your work, either
  a) work on whiteboard in breakout room, press the **SAVE** button
  b) work on paper,
     - give to facilitator,
     - leave 2 inch margin,
     - write your name and QH6 at the top

Moving to Breakout Room



Center for the Enhancement of Teaching and Learning

Pop Quiz

For what values of **a** does

$$\begin{aligned} 7x + 2y - 3z &= 25 \\ y + 3z &= 5 \\ 3y + \mathbf{a}z &= 3 \end{aligned}$$

Have a solution?

Linear Combinations

For what values of *h* is *b* a linear combination of vectors v_1, v_2?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$$

Announcements

Solutions to a System

Find *h* and *k* such that the system has a) no sol’n, b) a unique sol’n, and c) many solutions.

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

Two Fundamental Questions

If

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

then $\vec{y}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

If we are only given $\vec{y}, \vec{v}_1, \vec{v}_2, \vec{v}_3,$ then:

1. **can** we find the c's?
2. **how** do we find the c's?

Example 1

Determine whether *b* is a linear combination of the vectors formed from the columns of A.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

QH6 Recitation 17

Today: Pop Quiz, Linear Systems

- Pop quiz grading
  - Correct 5 points
  - Name on page 3 points
  - Did not take: 0 points.
- Time: 10 minutes
- To submit your work, either
  a) work on whiteboard in breakout room, press the **SAVE** button
  b) work on paper,
     - give to facilitator,
     - leave 2 inch margin,
     - write your name and QH6 at the top

Pop Quiz

For what values of **a** does

$$\begin{aligned} 7x + 2y - 3z &= 25 \\ y + 3z &= 5 \\ 3y + \mathbf{a}z &= 3 \end{aligned}$$

Have a solution?

$$\left[\begin{array}{ccc|c} 7 & 2 & -3 & 25 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & a & 3 \end{array}\right] \overset{R_3 - 3R_2}{\sim} \left[\begin{array}{cccc} 7 & 2 & -3 & 25 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 9-a & 12 \end{array}\right]$$

$\Rightarrow$ if $a = 9$, no solution, because R_3 is $0x + 0y + 0z = 12$

if $a \neq 9$, $z = \frac{12}{9-a}$, and the system has a solution.

Linear Combinations

For what values of h is $\vec{b}$ a linear combination of vectors v_1, v_2?

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}$$

$\vec{b}$ is a lin. combination of $\vec{v}_1, \vec{v}_2$ if $\exists$ c_1, c_2 s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{b}$$

$$\Rightarrow \begin{bmatrix} 1 & -5 \\ 3 & -8 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix} \Rightarrow \begin{aligned} c_1 - 5c_2 &= 3 \\ 3c_1 - 8c_2 &= -5 \\ -c_1 + 2c_2 &= h \end{aligned}$$

$7c_2 = -14$, $c_2 = -2$

$c_1 = -7$

$$\Rightarrow -(-7) + 2(-2) = h$$

$$\boxed{h = 3}$$

Solutions to a System

Find *h* and *k* such that the system has a) no sol'n, b) a unique sol'n, and c) many solutions.

$$x_1 + hx_2 = 2$$

$$4x_1 + 8x_2 = k$$

$$\left[\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right] \sim \begin{bmatrix} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{bmatrix} \Rightarrow x_2 = \frac{k-8}{8-4h}$$

if $k \neq 8$, $h = 2$, no sol'n.

if $k = 8$, $h = 2$, then $0x_1 + 0x_2 = 0$ has ∞ sol'ns

if $k =$ anything and $h \neq 2$, unique sol'n.

Two Fundamental Questions

If

$$\vec{y} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$$

then $\vec{y}$ is a linear combination of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

If we are only given $\vec{y}, \vec{v}_1, \vec{v}_2, \vec{v}_3$, then:

1. **can** we find the c's?
2. **how** do we find the c's?

QH6 Recitation 18

Today: Linear Dependence

Determine whether *b* is a linear combination of the vectors formed from the columns of A.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

The vector b is a linear combination of the columns of matrix A if:

Pop Quiz

For what values of **a** does

$$\begin{aligned} 7x + 2y - 3z &= 25 \\ y + 3z &= 5 \\ 3y + \mathbf{a}z &= 3 \end{aligned}$$

Have a solution?

Linear Dependence (1.7)

Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 ..., \vec{v}_N$ are linearly dependent (LD) if $\exists\ c_1, c_2, c_3, ..., c_N$ *not* all ____________ , such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ...c_N\vec{v}_N = \vec{0}$$

If the vectors are not LD, they are _____________________.

To determine whether a set of vectors are _____, we solve:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ... + c_N\vec{v}_N = \vec{0}$$

which has the same solution as the linear system whose augmented matrix is $\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & ... & \vec{v}_N & \vec{0} \end{bmatrix}$.

Announcements

Conceptual Question

Determine whether the following vectors are LI.

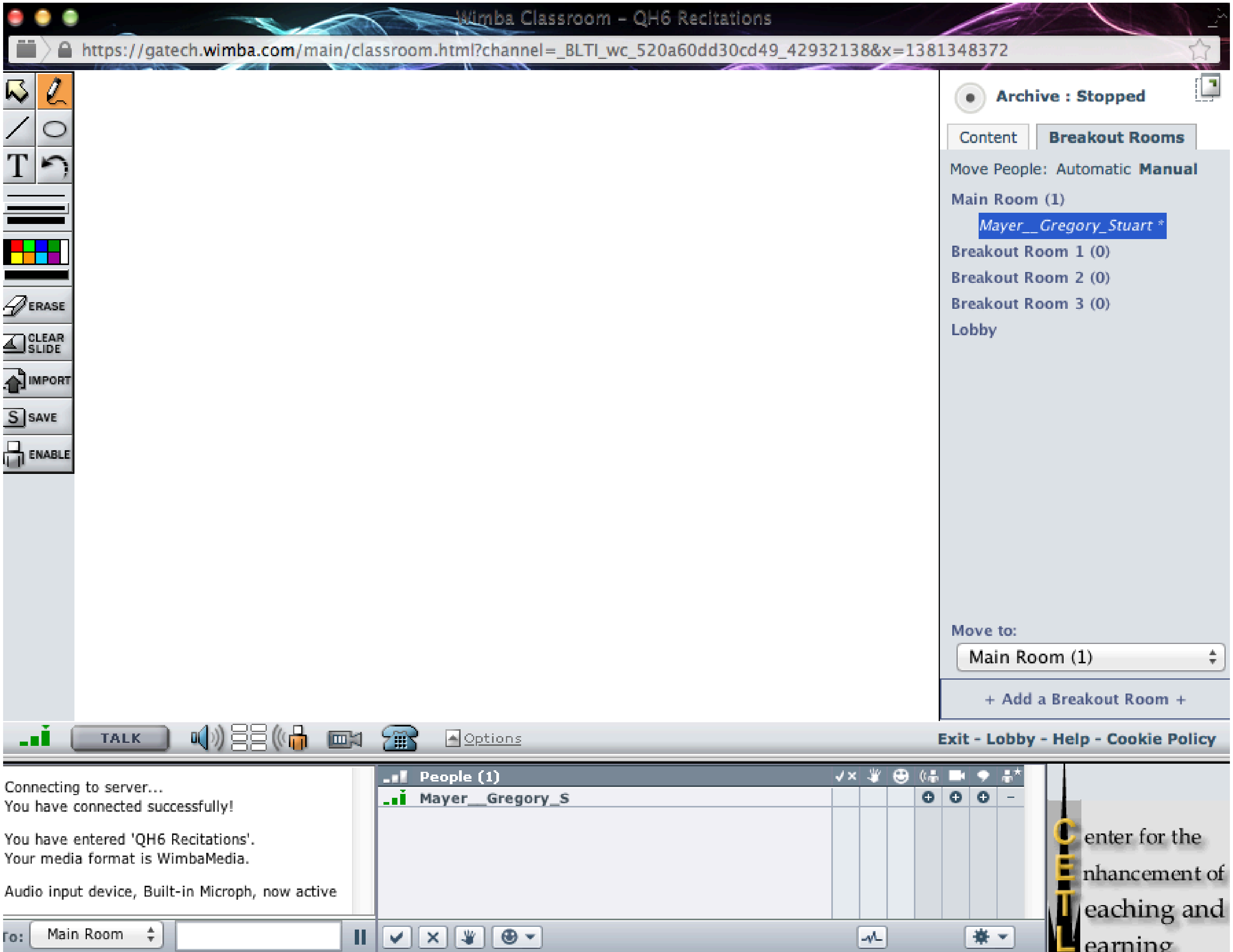
$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

Conceptual Question

Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$$

Moving to Breakout Room



Example 1 Find the values of h s.t. the following vectors are LD.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

R18 (THE NOTES FOR R17 WERE TOSSED)

DETERMINE IF $\vec{b}$ CAN BE EXPRESSED AS A L.C. OF COLS OF A.

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix} \quad \& \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

A $\vec{b}$ is a linear comb of cols of A if!

$\boxed{c_1, c_2, c_3 \text{ exist s.t.:}}$

$$c_1 \vec{a}_1 + c_2 \vec{a}_2 + c_3 \vec{a}_3 = \vec{b} \quad (1)$$

W We can write this as a matrix eqn:

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{b}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{b} \quad (2)$$

Q Thus: ~~this~~ $\vec{b}$ is a LC of cols of A if equation (2) has a solution.

i.e. c_1, c_2, c_3 exist

Q ~~Form aug mx~~ what is the question asking us to do?

A - find c_1, c_2, c_3

2

$$A = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$$

Form aug. matrix:

$$\begin{array}{c} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right] \end{array} \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \sim \left[\right.$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 1 & 4 & 3 \end{bmatrix} \begin{array}{l} \\ R_2 + 2R_1 \\ R_3/2 \end{array}$$

$$\sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - R_2 \end{array}$$

$\Rightarrow$ c_3 is <u>free</u>.

$c_2 = \underline{3 - 4c_3}$

$c_1 = \underline{2 - 5c_3}$

$\Rightarrow \exists\, c_1, c_2, c_3$ s.t. (1) is satisfied.

(non-unique)

R 18 ③

system:

$$7x + 2y - 3z = 25 \quad ①$$
$$y + 3z = 5 \quad ②$$
$$3y + az = 3 \quad ③$$

$3② - ③$: $(9-a)z = 12$

$$\Rightarrow \begin{cases} \text{if } a = 9, & 0 \cdot z = 12 \\ \text{if } a \neq 9, & z = \frac{12}{9-a} \end{cases}$$

$\Rightarrow$ if $a = 9$, system has no sol'n.

if $a \neq 9$, system has at least one sol'n.

Q why do we avoid ①?

Q what is the Q asking us to do?

Conceptual Question

Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

<u>NOT</u> LI because vectors are in $\mathbb{R}^2$ and there are 4 vectors.

Students should explain why the vectors are LD.

Conceptual Question

Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$$

One of the vectors is the zero vector,
so vectors can't be LI.

QH6 Recitation 19

Today: Linear Dependence, Transforms

Find the values of h s.t. the following vectors are LD.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

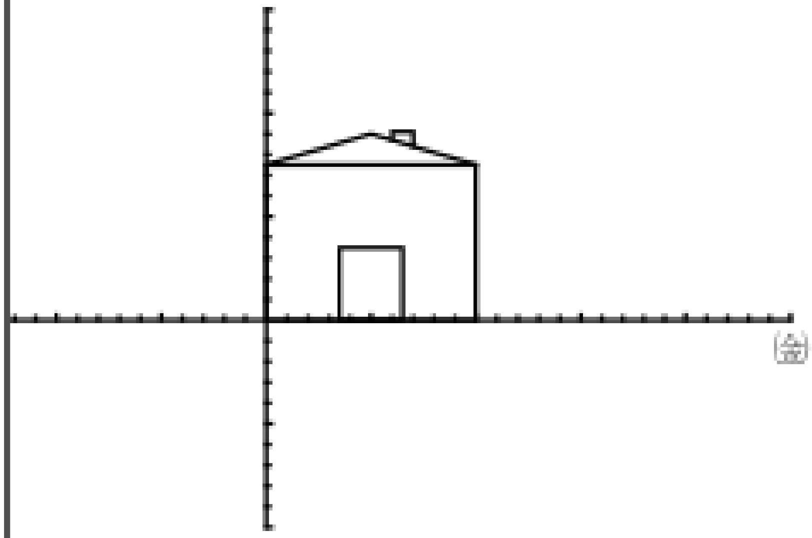
If ____________ , then the three vectors are LD.

Tuesday: Graded Activity

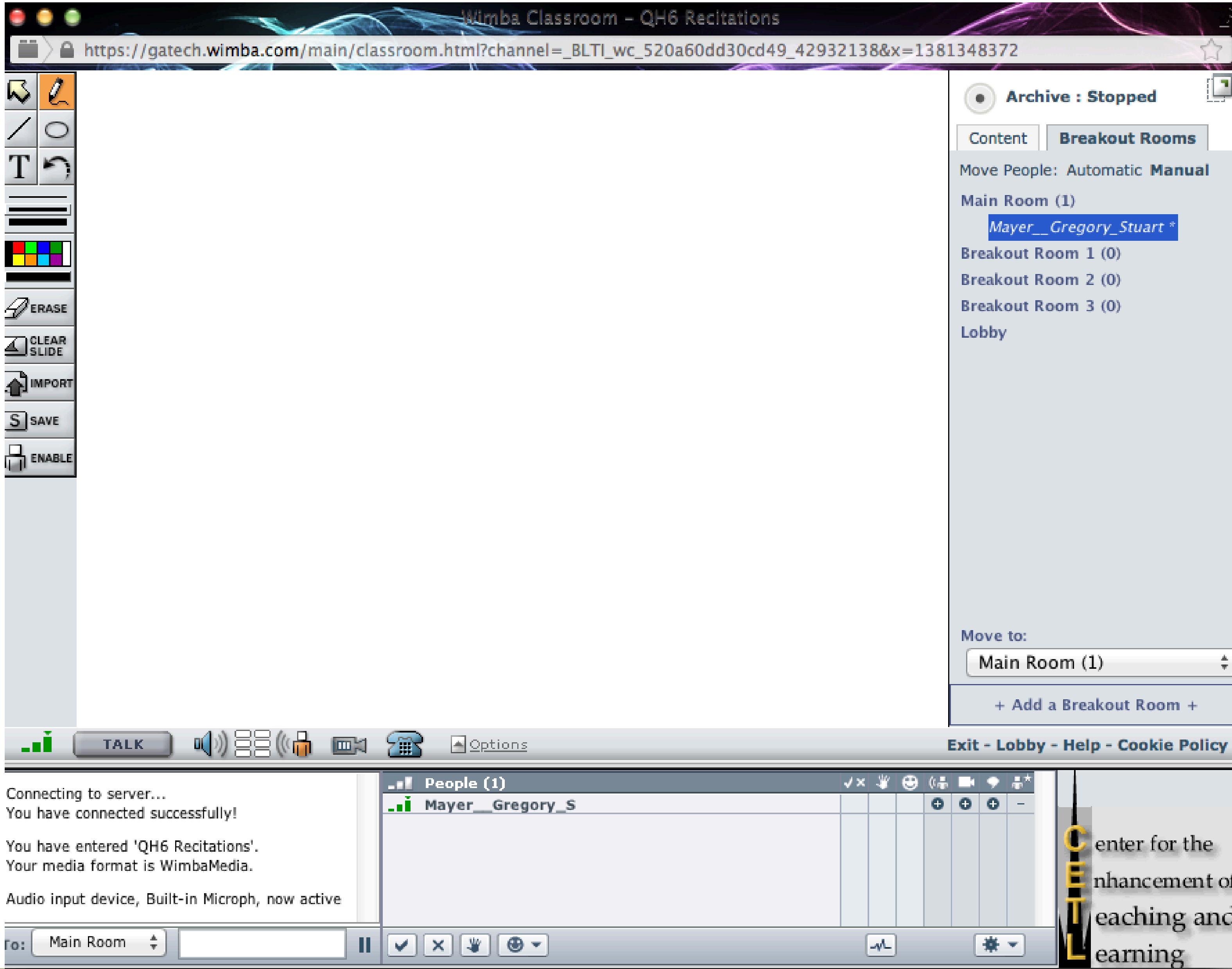
- You'll have about 10 minutes to solve question 3 from last year's quiz
- Can work by yourself or in a group of up to 3 students
- You'll need to submit your work through wimba in a breakout room
- Press the SAVE button to create screen shot of your work

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \text{Cos}\left[\frac{-\pi}{3}\right] & -\text{Sin}\left[\frac{-\pi}{3}\right] \\ \text{Sin}\left[\frac{-\pi}{3}\right] & \text{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC. Show the intermediate steps.



Moving to Breakout Room



Transforms

Let $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, y_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, y_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$, and T be a linear transformation that maps e_1 onto y_1, and e_2 onto y_2.

Find the image of $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ under T.

$$\begin{bmatrix}3\\-6\\1\end{bmatrix}, \begin{bmatrix}6\\4\\-3\end{bmatrix}, \begin{bmatrix}9\\h\\3\end{bmatrix}$$

Find h s.t. vectors are LD.

$$\begin{bmatrix}3 & 6 & 9 & 0\\-6 & 4 & h & 0\\+1 & -3 & 3 & 0\end{bmatrix} \underset{\text{swap } R1 \& R3}{\sim} \begin{bmatrix}1 & -3 & 3 & 0\\-6 & 4 & h & 0\\3 & 6 & 9 & 0\end{bmatrix}$$

$$\underset{R3/3}{\sim} \begin{bmatrix}1 & -3 & 3 & 0\\-6 & 4 & h & 0\\1 & 2 & 3 & 0\end{bmatrix}$$

$$\underset{\substack{R3-R_1\\R2+6R_1}}{\sim} \begin{bmatrix}1 & -3 & 3 & 0\\0 & -14 & h+18 & 0\\0 & 5 & 0 & 0\end{bmatrix}$$

(different if we use $R2+6R3$)

If $\_\_\_$ then the vectors are LD.

$\exists\, c_1, c_2, c_3$ not all zero, s.t.

$A\vec{x}=\vec{0}$, or $\begin{bmatrix}3 & 6 & 9\\-6 & 4 & h\\1 & -3 & 3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

or $\vec{0} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$

To determine whether ① is satisfied, form aug. matrix, and solve.

From R3: $x_2 = 0$

From R2: $(h+18)x_3 = 0$

Two CASES

$\Rightarrow$ 1, If $h \neq -18$, $x_3 = 0$. But $x_2 = 0$. So $x_1 = 0$.

So $x_1 = x_2 = x_3 = 0$. Thus, vectors are LI.

2, If $h = -18$, $0 \cdot x_3 = 0$. Thus x_3 is free!

$x_2 = 0$, $x_1 = -3x_3$, AND vectors are LD.

$\Rightarrow h = -18$ for vectors to be LD.

R19

EXAMPLE 3

If: $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{y}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{y}_2 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

Let T map $\vec{e}_1$ to $\vec{y}_1$, and $\vec{e}_2$ to $\vec{y}_2$.

Find image of $\begin{bmatrix} 5 \\ -3 \end{bmatrix}$ ~~and $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$~~ ← TOO REDUNDANT

Let $T(\vec{x}) = A\vec{x} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow \begin{cases} a_{11} = 2 \\ a_{21} = 5 \end{cases}$

Similarly, $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} a_{12} = -1 \\ a_{22} = 6 \end{cases}$

$\Rightarrow A \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 13 \\ 7 \end{bmatrix}$

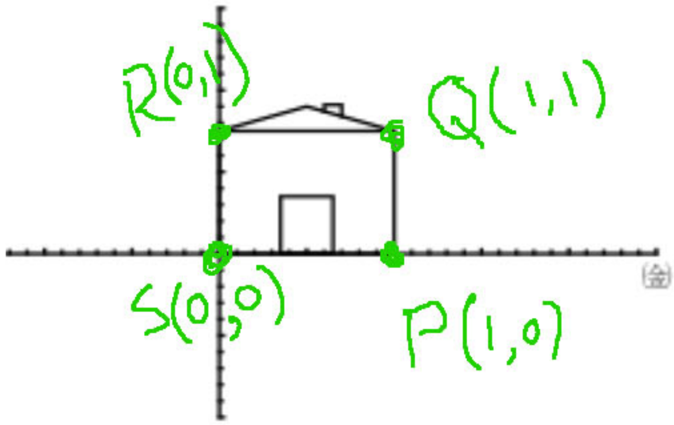
$\boxed{\#19, 1.8}$

2/2

- ask students how to start this one.
- "Now find elements of A using given info"

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \cos\left[\frac{-\pi}{3}\right] & -\sin\left[\frac{-\pi}{3}\right] \\ \sin\left[\frac{-\pi}{3}\right] & \cos\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC . Show the intermediate steps.



For convenience, let's label the 4 corners of the house as above.

The transformation matrix is the product of three matrices:

$$T = ABC$$

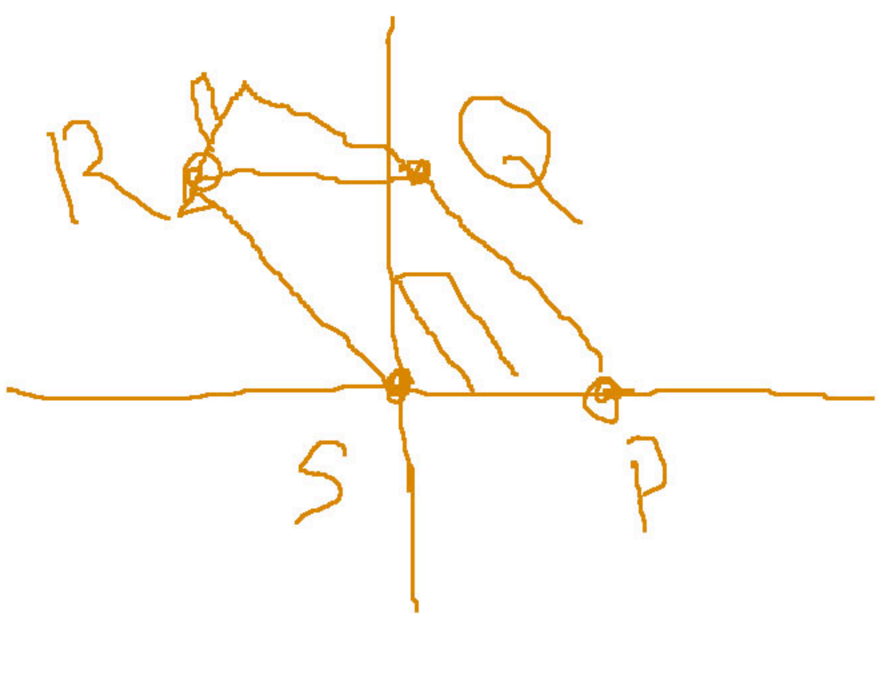
Matrix C gets applied 1st:

P: $C\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

Q: $C\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

R: $C\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

S: $C\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$



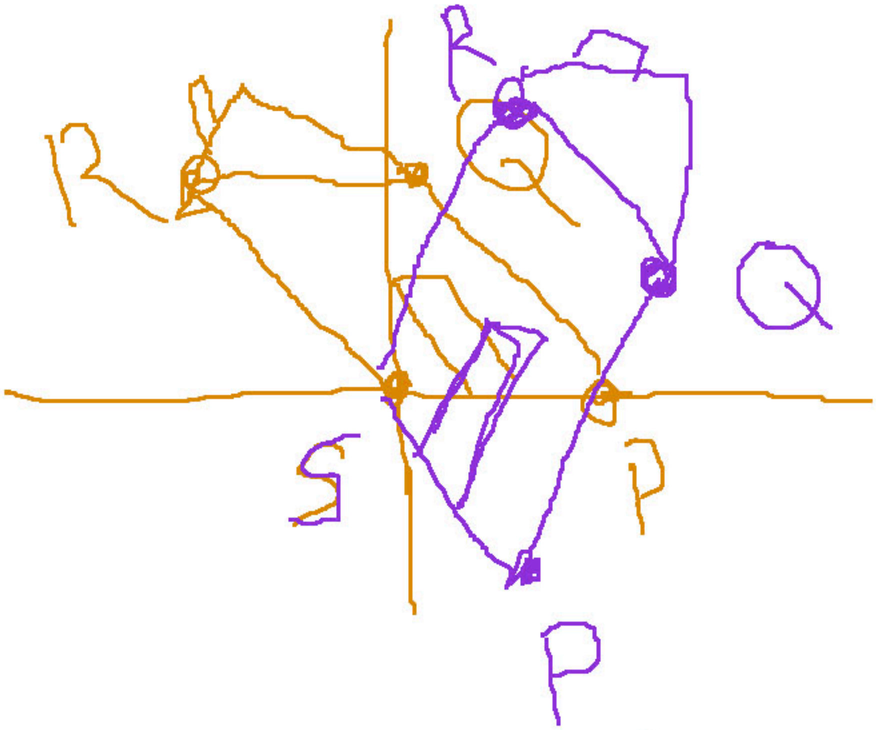
Now we apply matrix B.
By inspection, we know that B is a rotation matrix, and that it will rotate the house by $\pi/3$ rad clockwise.

$$P: BC\begin{pmatrix}1\\0\end{pmatrix} = B\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

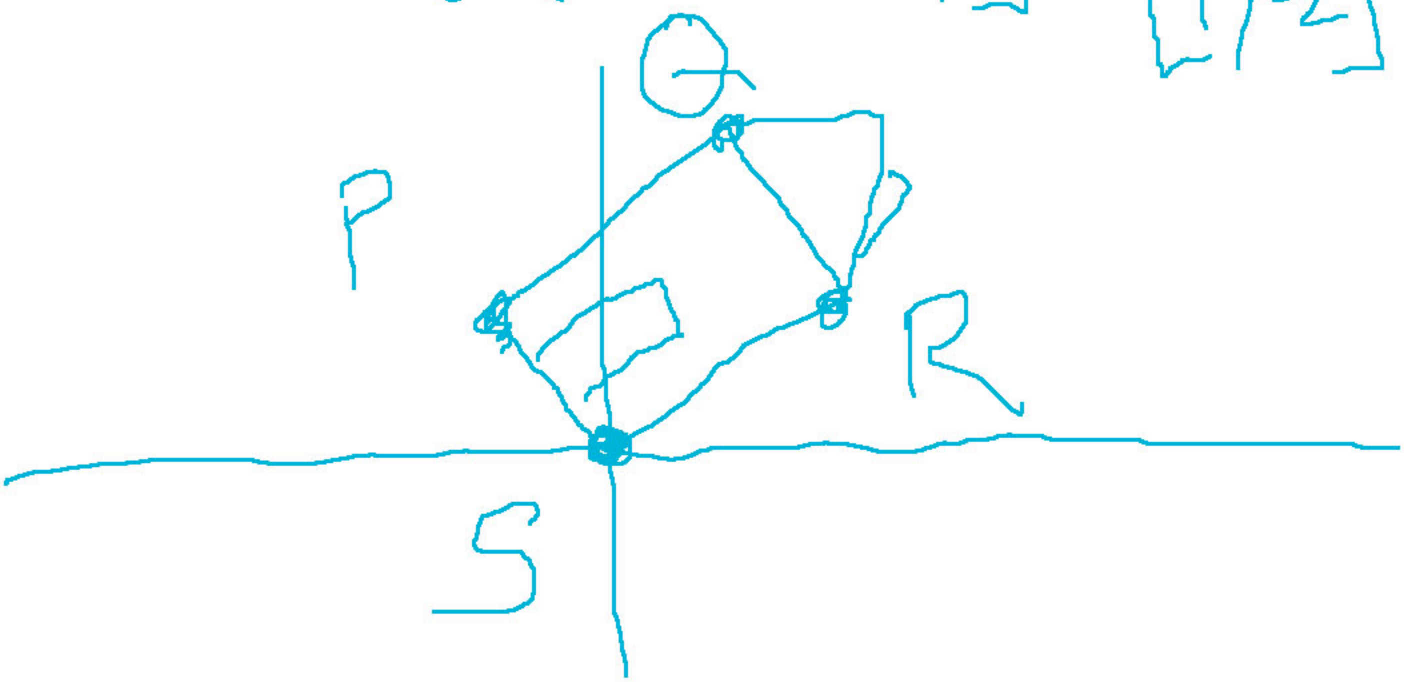
$$= \begin{pmatrix}\frac{1}{2}\\ -\sqrt{3}/2\end{pmatrix}$$

Calculations for other points similar.



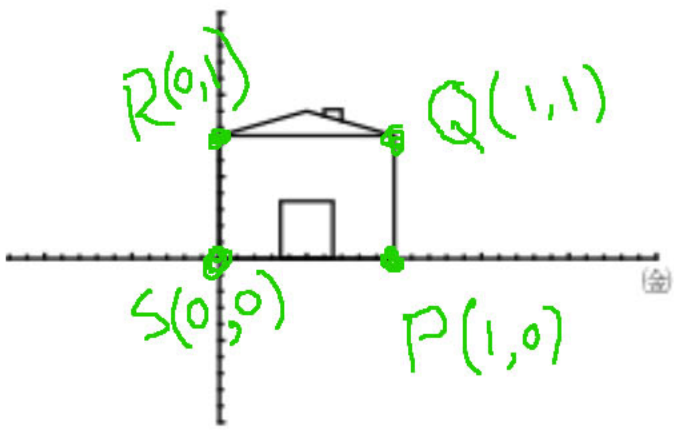
Matrix A reflects each point about the line $y=x$:

$$P: AB\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = A\begin{bmatrix}1/2\\-\sqrt{3}/2\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2\\1/2\end{bmatrix}$$



Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \text{Cos}\left[\frac{-\pi}{3}\right] & -\text{Sin}\left[\frac{-\pi}{3}\right] \\ \text{Sin}\left[\frac{-\pi}{3}\right] & \text{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC . Show the intermediate steps.



For convenience, let's label the 4 corners of the house as above.

The transformation matrix is the product of three matrices:

$$T = ABC$$

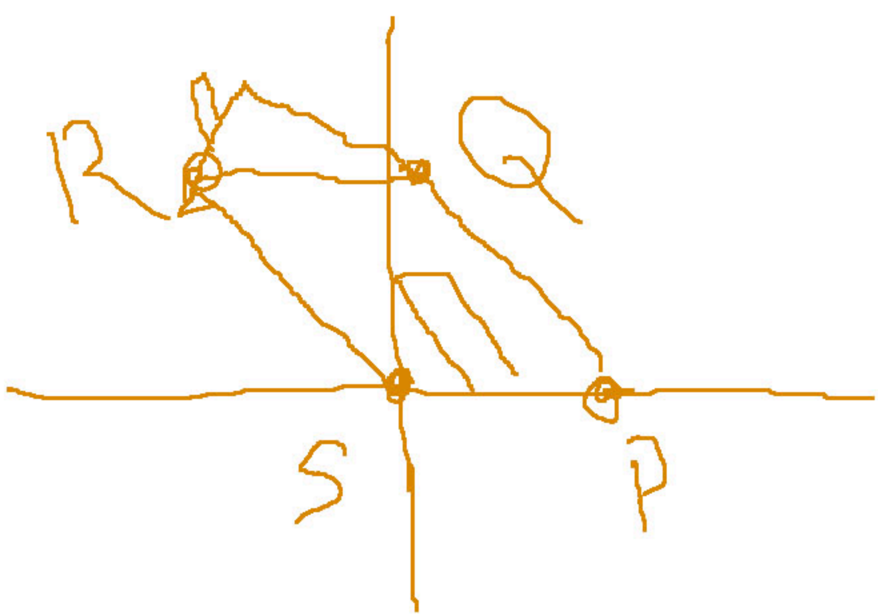
Matrix C gets applied 1st:

$P: C\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

$Q: C\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$

$R: C\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$S: C\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix}$



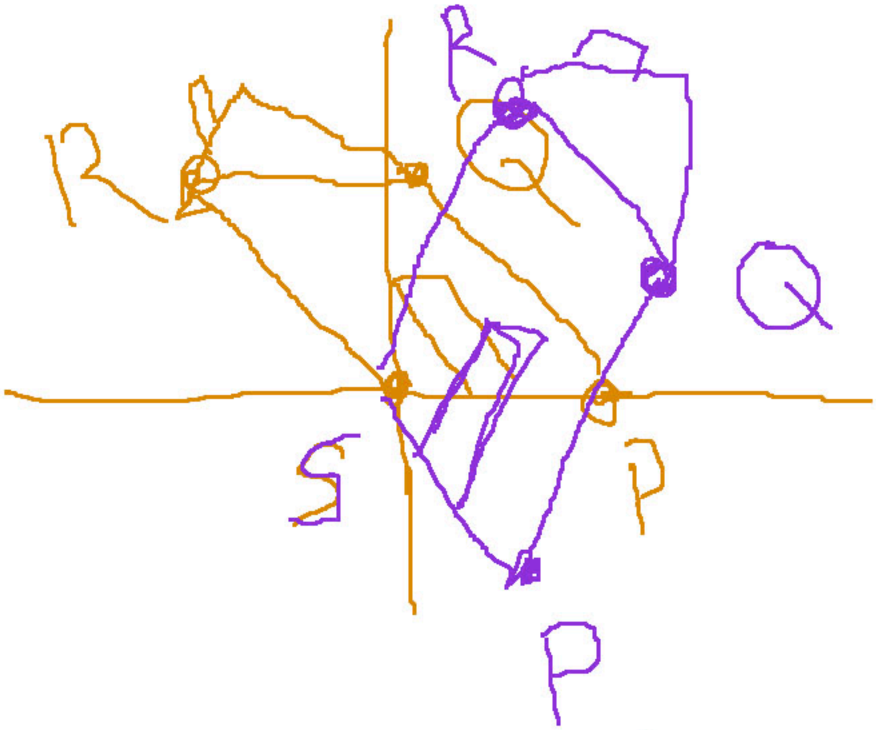
Now we apply matrix B. By inspection, we know that B is a rotation matrix, and that it will rotate the house by $\pi/3$ rad clockwise.

$$P: BC\begin{pmatrix}1\\0\end{pmatrix} = B\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

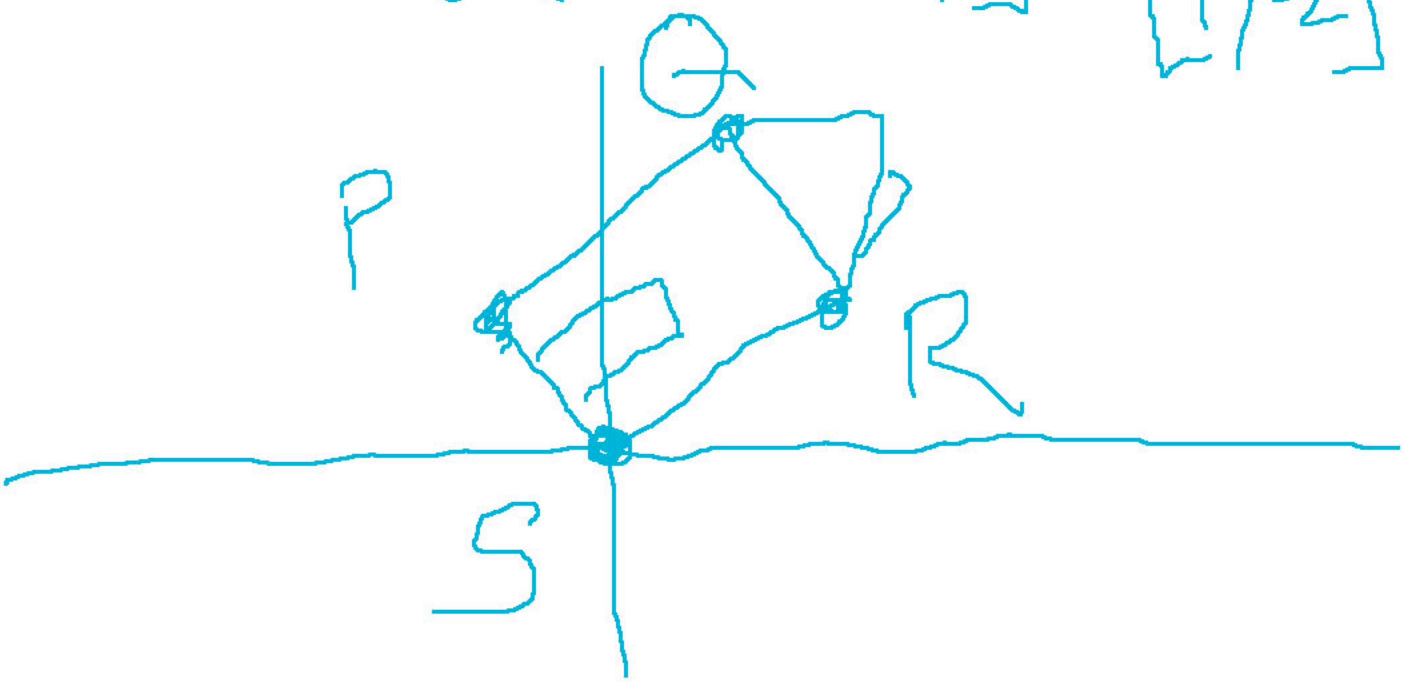
$$= \begin{pmatrix}\frac{1}{2}\\ -\sqrt{3}/2\end{pmatrix}$$

Calculations for other points similar.



Matrix A reflects each point about the line $y=x$:

$$P: AB\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = A\begin{bmatrix}1/2\\-\sqrt{3}/2\end{bmatrix} = \begin{bmatrix}-\sqrt{3}/2\\1/2\end{bmatrix}$$



QH6 Recitation 20

Today: Transforms (1.8), LU Decomposition (2.5)

Find an LU factorization of A, if possible.

$$A = \begin{bmatrix} 9 & 12 \\ 18 & 21 \end{bmatrix}$$

Announcements

- HW due tonight on inverses
- Quiz on Thursday
- Office hours tonight and tomorrow, 8:00 pm to 10:00 pm, on Wimba
- I'll email you last year's Quiz 3 today. Only questions 2 and 3 from it are relevant.

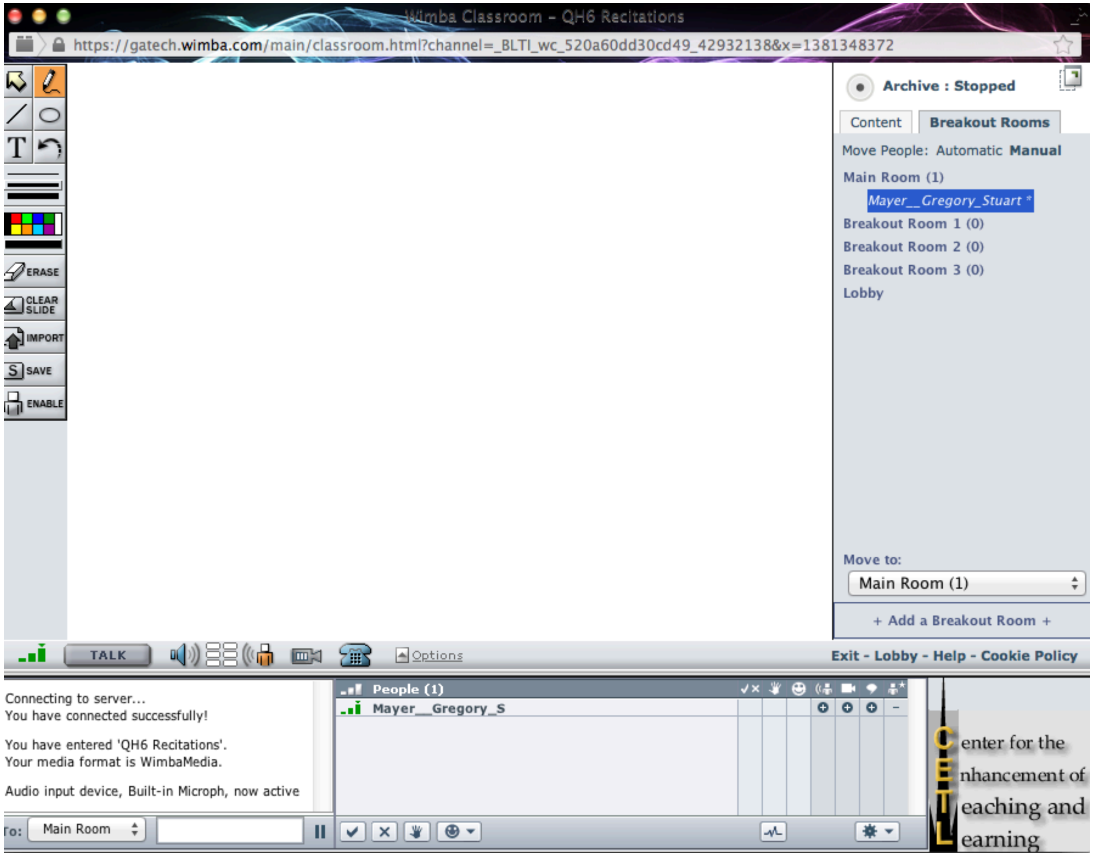
Graded Activity

- You'll have about 10 minutes to solve question 3 from last year's quiz
- Can work by yourself or in a group of up to 3 students
- You'll need to submit your work through wimba in a breakout room
- **Press the SAVE button to create screen shot of your work**

Moving in/out of Breakout Rooms

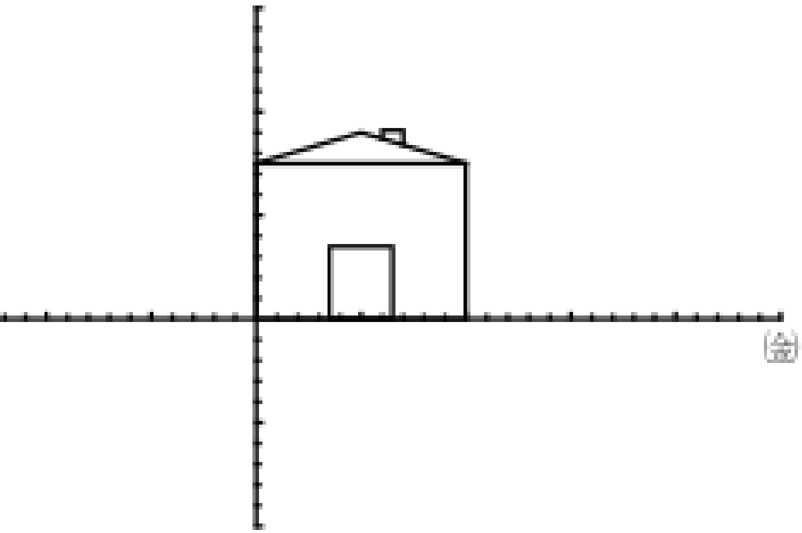
To Move Yourself Into a Breakout Room:

1. Select the **Breakout Rooms** tab
2. Select **Manual**
3. Select **your name**
4. Move to: select a room



Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \mathrm{Cos}\left[\frac{-\pi}{3}\right] & -\mathrm{Sin}\left[\frac{-\pi}{3}\right] \\ \mathrm{Sin}\left[\frac{-\pi}{3}\right] & \mathrm{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC . Show the intermediate steps.



LU Factorization (2.5)

Suppose we have matrix A and vector b, and we want to find x, where

$$Ax=b$$

If we can find matrices, L, U, such that

- L is lower triangular
- U is upper triangular
- where A=LU

then we can solve Ax=b by solving

$$Ly = b$$
$$Ux = y$$

LU Decomposition (From Homework)

Solve the equation $\mathbf{Ax} = \mathbf{b}$ by using the LU factorization given for A.

$$A = \begin{bmatrix} 3 & -6 & 3 \\ -6 & 10 & 0 \\ 6 & -10 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -6 & 3 \\ 0 & -2 & 6 \\ 0 & 0 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 9 \\ -8 \\ 8 \end{bmatrix}$$

Inverse Matrix

Find the inverse matrix of: $\begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$

Example 3 Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$$

QH6 Recitation 20

Today: Transforms (1.8), LU Decomposition (2.5)

Find an LU factorization of A, if possible.

$$A=\begin{bmatrix} 9 & 12 \\ 18 & 21 \end{bmatrix}$$

Since $A = LU$, we need L and U, where

$$U = \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix}$$

Use row operations to find U:

$$\begin{pmatrix} 9 & 12 \\ 18 & 21 \end{pmatrix} \overset{R_2-2R_1}{\sim} \begin{pmatrix} 9 & 12 \\ 0 & -3 \end{pmatrix} \Rightarrow \boxed{U = \begin{pmatrix} 9 & 12 \\ 0 & -3 \end{pmatrix}}$$

Since $LU = A$, $L_{2,1} \cdot 9 = 18 \Rightarrow L_{21} = 2$

$$\Rightarrow L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$, see text for method for finding L.

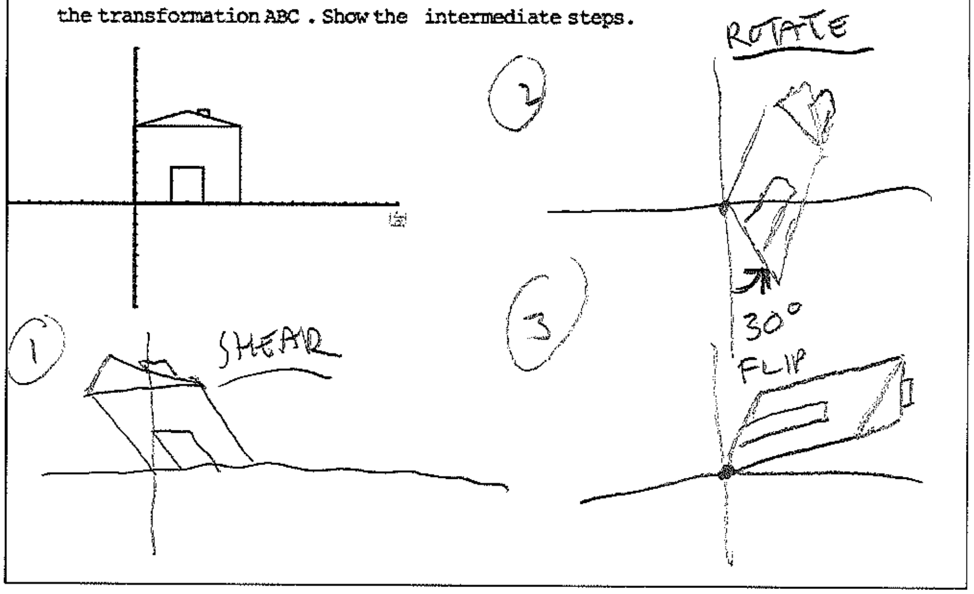
Ask: what if $A_{11} = 0$, could we find U?

Announcements

- HW due tonight on inverses
- Quiz on Thursday
- Office hours tonight and tomorrow, 8:00 pm to 10:00 pm, on Wimba
- I'll email you last year's Quiz 3 today. Only questions 2 and 3 from it are relevant.

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \mathrm{Cos}\left[\frac{-\pi}{3}\right] & -\mathrm{Sin}\left[\frac{-\pi}{3}\right] \\ \mathrm{Sin}\left[\frac{-\pi}{3}\right] & \mathrm{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC . Show the intermediate steps.



LU Factorization (2.5)

Suppose we have matrix A and vector b, and we want to find x, where

Ax=b

If we can find matrices, L, U, such that

- L is lower triangular
- U is upper triangular
- where A=LU

then we can solve Ax=b by solving

Ly = b
Ux = y

Example 3 Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$$

YES BECAUSE C_2 is arbitrary.

QH6 Recitation 21

Today: Subspaces of R^n (2.8)

Find i) a nonzero vector in Nul A, and ii) a vector in Col A.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \\ 3 & 3 & 4 \end{bmatrix}$$

Announcements

- Quiz 3 graded on Friday, you’ll get it back next week
- Graded Activity Thursday
  - You'll have about 10 minutes to solve question 1 from last year's quiz 3
  - group size: 1 to 3 students
  - submit your work through wimba, in a breakout room
  - **Press the SAVE button to submit your work**
  - **Write name on board**
  - Everyone in group use a different color

Definitions (From Section 2.8)

Let’s fill in the blanks:

Col A is the set of all linear combinations of the _________ of A.

Nul A is the set of all solutions to _____________ .

The ____________ columns of the matrix A form a basis for the column space of A.

Example 2

a) how many vectors are there in the set $\{v_1, v_2, v_3\}$?

b) how many vectors are there in Col A?

c) is p in Col A?

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}, \vec{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}, A = [\vec{v}_1, \vec{v}_2, \ \vec{v}_3]$$

Example 3 Determine whether the vectors form a basis in R^2

a) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

Example 6

Construct a 3x3 matrix A and a nonzero vector **b** s.t. **b** is not in Col A.

RECITATION #2~~1~~ (~~THURSDAY~~ – ~~TUES WAS A HOLIDAY~~)

#14 from 2.8

GRADED

$3 - \frac{10}{3} = -\frac{1}{3}$

① FIND NONZERO VECTOR IN Nul A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -5 & -1 & 0 \\ 2 & 7 & 11 \\ 3 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -5 \\ 0 & 9 & 15 \\ 0 & 3 & 5 \\ 0 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow x_3$ is free.

Let $x_3 = 1$. Then $\begin{bmatrix} +1/3 \\ -5/3 \\ 1 \end{bmatrix}$ is in Nul A.

$-3 + 5/3 = \frac{-9}{3} + \frac{5}{3} = -4/3$

- we chose $\vec{x}$ s.t. $A\vec{x} = \vec{0}$
- is $\vec{x}$ unique?

(Would any value of x_3 suffice? What about $x_3 = 0$?
$\Rightarrow$ No. We are asked for a nonzero vector)

② FIND A VECTOR IN THE COLUMN SPACE OF A

ANY column of A will do. Eg- $\begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \\ 3 \end{bmatrix}$

R21

$\boxed{\#7 \text{ from } 2.8}$

$$\vec{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 6 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -3 \\ 8 \\ -7 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} -4 \\ 6 \\ -7 \end{bmatrix}.$$

$$\vec{p} = \begin{bmatrix} 6 \\ -10 \\ 11 \end{bmatrix}, \quad A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$$

a) how many vectors are in $\{v_1, v_2, v_3\}$?

Can someone tell me "how many ...."?

3

b) how many vectors are in Col A? GOOD QUESTION TO DO → distractor is 3

- Col A is a set
- Can someone tell me "how many ...."

∞

(there is an ∞ number of linear combinations of the columns of A)

c) is $\vec{p}$ in Col A?

Form augmented matrix and see if $\exists\, \vec{x}$ s.t. $A\vec{x} = \vec{p}$.

$$\left[\begin{array}{ccc|c} 2 & -3 & -4 & 6 \\ -8 & 8 & 6 & -10 \\ 6 & -7 & -7 & 11 \end{array}\right] \sim \begin{bmatrix} 2 & -3 & -4 & 6 \\ 0 & -4 & -10 & 14 \\ 0 & 2 & 5 & -7 \end{bmatrix}$$

$$\sim \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ at least one free variable

$\Rightarrow$ system is consistent

$\Rightarrow \vec{p} \in \text{Col } A.$

Example 3 Determine whether the vectors form a basis in R^2

a) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

if ~~they~~ are LI, they form a basis.
if they are LD, they do not form a basis.
The vectors are LI, so they form a basis in $\mathbb{R}^2$.

b) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

These vectors are LD and so do not form a basis.

Example 6 Construct a 3x3 matrix A and a nonzero vector **b** s.t. **b** is not in Col A.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

there does not exist an $\vec{x}$ s.t.

$$A\vec{x} = \vec{b}$$

QH6 Quiz 4

Good luck on Quiz 4!

If you have any questions, you can message me through wimba. I'll be on wimba until 8:55.

QH6 Recitation 23

Today: Null Space Example, Pop Quiz, Graded Activity (2.8)

Find i) a basis for Col A, and ii) a basis for Nul A.

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pop Quiz Instructions

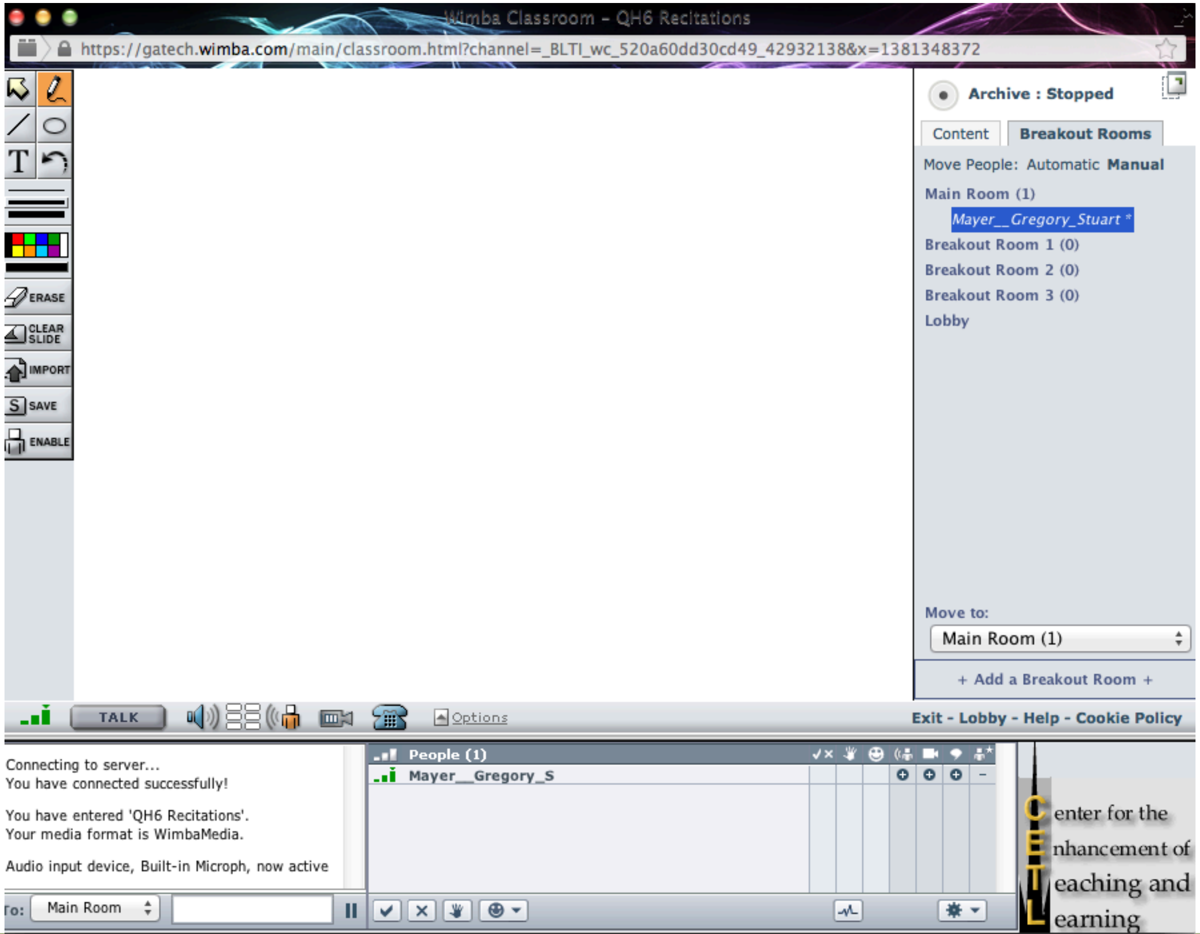
- Pop quiz grading
  - Correct 5 points
  - Name on page 3 points
  - Did not take: 0 points.
- Time: 10 minutes
- To submit your work, either
  a) work on whiteboard in breakout room:
    - write name on board
    - press the **SAVE** button

  b) work on paper
    - give to facilitator,
    - leave 2 inch margin,
    - write your name and QH6 at the top

Moving in/out of Breakout Rooms

To Move Yourself Into a Breakout Room:

1. Select the **Breakout Rooms** tab
2. Select **Manual**
3. Select **your name**
4. Move to: select a room



Pop Quiz

Find the coordinates of the vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ with respect to the basis

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Graded Activity

- Quiz 3 graded on Friday, you’ll get it back next week
- Graded Activity Thursday
  - You'll have about 10 minutes to solve question 1 from last year's quiz 3
  - group size: 1 to 3 students
  - for full marks: identify basis vectors, state nullity and rank
  - submit your work through wimba, in a breakout room
  - **Press the SAVE button to submit your work**
  - **Write name on board**
  - Everyone in group use a different color

Graded Activity

Find a basis for the null space of A, where $A = \begin{pmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{pmatrix}$

Find the rank of A and the nullity of A

QH6 Recitation 23

Today: Null Space Example, Pop Quiz, Graded Activity (2.8)

Find i) a basis for Col A, and ii) a basis for Nul A.

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

ii) x_3, x_5 are free,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_5 - x_3 + \frac{x_2}{3} \\ 2x_5 - 3x_3 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4/3 \\ 2 \\ 6 \\ 2 \\ 1 \end{bmatrix}$$

basis for Nul A

Pop Quiz

Find the coordinates of the vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ with respect to the basis

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\left.\begin{aligned} x - y &= 1 \\ 2x + y &= 4 \end{aligned}\right\} \quad 2 - 2\,1 \text{ is: } \quad 0x + 3y = 2 \Rightarrow y = 2/3$$

$$\Rightarrow x = 5/3$$

Coordinates are $\left(\frac{5}{3}, \frac{2}{3}\right)$

Graded Activity

Find a basis for the null space of A, where $A = \begin{pmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{pmatrix}$

Find the rank of A and the nullity of A

$$\sim \begin{pmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & 0 & 4 & 1 & 6 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Nullity is 3. To find a basis for nullspace,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{4}{3}x_3 - \frac{x_4}{3} - 2x_5 \\ -3x_3 - 2x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -4/3 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

QH6 Recitation 24

Today: Col A Example from Thursday, Eigenvalues (5.1, 5.2)

Construct a 3x3 matrix A and a nonzero vector **b** s.t. **b** is not in Col A.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

$$b: \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

How do we know if this is correct?

Announcements

Pop Quiz

Find the coordinates of the vector $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ with respect to the basis

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

5.1

Is λ = 2 an eigenvalue of matrix A? Why/why not?

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$$

5.2

Find a basis for the eigenspace of A, for the eigenvalue λ = -5.

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$$

5.2 Find the characteristic polynomial and e-values of:

a) $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

b) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

R24

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

b is $\notin$ Col A if $\nexists\ \vec{x}$ s.t. $A\vec{x} = \vec{b}$;

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 4 \end{array}\right] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

row 3 is $0 \cdot x_1 + 0x_2 + 0x_3 = +1$, not consistent.

$\Rightarrow$ $\vec{b}$ not in Col A.

POP

Q: $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is in basis $\{v_1, v_2\}$ if

$$\exists\ \vec{x} \text{ s.t. } x_1 [v_1] + x_2 v_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

SOLVE SYSTEM:

..... $x_1 = 5/3$

$x_2 = 2/3$

$\Rightarrow$ coords are $\left(5/3, 2/3\right)$

RECITATION 24: ~~TUES. NOV 6~~

#1, 5.1

Q Is $\lambda = 2$ an eval of $A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$? why/why not?

Q) If $\lambda = 2$ is an evect, then equation does λ have to satisfy?

A) If λ is an eval, $A\vec{v} = \lambda\vec{v}$, or $(A - \lambda I)\vec{v} = 0$

Q) what is $A - \lambda I$? (CALCULATE IT)

A) $(A - \lambda I)\vec{v} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

Q) are columns of $A - \lambda I$ L.I.?

A) no: $2\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\Rightarrow$ columns are L.D.

SAY) Let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} + v_2\begin{bmatrix} 2 \\ 6 \end{bmatrix} = 0$$

$\Rightarrow$ has a non-trivial sol'n

$\Rightarrow$ can find evects

$\Rightarrow$ $\lambda = 2$ is an eval

COMPREHENSION CHECK:

Q if col's of $A - \lambda I$ were LI, what would $\vec{v}$ have to be?

A, $\vec{0}$

4

GOOD GROUP WORK PROBLEM: (TRY TO DO THIS ON THURSDAY)

FIND A BASIS FOR THE EIGENSPACE OF

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}, \quad \text{FOR } \lambda = -5$$

3 #15, 5.1

Q) what are the corresponding e-vectors of $\lambda = -5$?

A) Sol'ns to:

$$(A - \lambda I)\vec{v} = \vec{0}$$

or:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}$$

or:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right) \Rightarrow \quad x_2, x_3 \text{ free}$$

$\Rightarrow$ let $x_2 = -1, x_3 = 0 \Rightarrow x_1 = +1, \quad \vec{V}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

$x_2 = 0, \ x_3 = -1, \Rightarrow x_1 = +1, \ \vec{V}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$\vec{V}_1, \vec{V}_2$ are LI e-vectors that form a basis for e-space of A for $\lambda = -5$.

Today: Diagonalization (5.3)

An matrix A is diagonalizable if it can be written in the form:

____

where

P is ___________________

D is ___________________

Suppose A is $N \times N$. To diagonalize A:

1. find all _________________ of A
2. find N _______ eigenvectors of A
3. construct ______ from vectors in step 2
4. construct ______ from values of step 1

5.3 Diagonalize the following matrices, if possible.

$$A=\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix};\ B=\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix};\ C=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \text{ where } \lambda\text{'s of } C \text{ are } 2,2,5$$

5.2

Find the characteristic polynomial and e-values of:

a) $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

b) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

5.1

Is $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $B = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$?

QH6 Recitation 25

Today: Diagonalization (5.3)

An matrix A is diagonalizable if it can be written in the form:

$$PDP^{-1}$$

where

P is a matrix of eigenvectors

D is a diagonal matrix of eigenvalues

Suppose A is N×N. To diagonalize A:

1. find all eigenvalues of A
2. find N LI eigenvectors of A
3. construct P from vectors in step 2
4. construct D from values of step 1
5. Find P^{-1}.

5.3 Diagonalize the following matrices, if possible.

$A = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix}$; $C = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ where λ's of C are 2,2,5

A) $0 = \det(A - \lambda I)$

$= \det\begin{pmatrix} 1-\lambda & 3 \\ 4 & 2-\lambda \end{pmatrix}$

$= \lambda^2 - 3\lambda - 10$

$\lambda = -2, 5$

$\Rightarrow D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}$ or $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$

if $\lambda_1 = -2$, $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

if $\lambda_2 = 5$, $V_2 = \begin{bmatrix} 3/4 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 3/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -9/7 & 3/7 \\ 4/7 & 4/7 \end{bmatrix}$

B) $\lambda_{1,2} = 3$

$v_1 = V_2 = \begin{bmatrix} c \\ 0 \end{bmatrix}$, c is arbitrary

$P = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, P^{-1} DNE.

c) e-vectors are $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & & \\ & 2 & \\ & & 5 \end{bmatrix}$

P^{-1} is too hard to calculate.

(ANOTHER NICE GROUP WORK PROBLEM)

#1, 5.2

FIND THE CHARACTERISTIC POLYNOMIAL AND THE ~~REAL~~ e-VALUES OF

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

Q) What equation do we need to solve?

A) $\det(A - \lambda I) = 0$

Q) what values of λ solve it?

A) $0 = \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 7^2$

$= 4 - 4\lambda + \lambda^2 - 49$

$= \lambda^2 - 4\lambda - 45$

$0 = (\lambda - 9)(\lambda + 5)$

$\Rightarrow \lambda = 9, -5$

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$. Find the e-values of A, by inspection. $\boxed{\#17,\ 5.1}$

~~Q) what equation must λ satisfy?~~

~~A) $A\vec{v} = \lambda\vec{v}$~~

Q) What equation <u>would</u> we solve?

A) $\det(A - \lambda I) = 0$

$\Rightarrow \lambda = +3, -2, 0$

they might forget this one

Q) how many e-values are there?

A) 3 (not necessarily unique)

Is $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an evect of $A = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$?

#4, 5.1

Q) What equation must $\vec{v}$ satisfy?

A) $A\vec{v} = \lambda\vec{v}$

Q) What is $A\vec{v}$?

A) $A\vec{v} = \begin{bmatrix} -3 \\ +3 \end{bmatrix}$

Q) Is $A\vec{v}$ a scalar multiple of $\vec{v}$?

A) of course

WRITE) $\exists$ a scalar, λ, s.t.

$A\vec{v} = \lambda\vec{v}$

$\Rightarrow \vec{v}$ is an evect of A

COMPREHENSION

1. Q) what is the eval corresponding to $\vec{v}$?

   A) $\lambda = 3$, of course

2. Q) is $\vec{v}$ the only evect of A?

   A) No. $2\vec{v}$ is also an evect.

QH6 Recitation 26

Today: Orthogonality, Quiz Review

Let $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Show that these are (pairwise) orthogonal. If

$$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

where the v_i's are as above and the a_i's are scalars, FIND a_2

Quiz 4

I'll have office hours tonight, tomorrow, 8 to 10 PM, on Wimbaaaa.

1. What topics does Quiz 4 cover?

2. What would you like to us to review now?

2012 Quiz 4, Question 2

6 is an eigenvalue of $\begin{pmatrix} 12 & 3 & -3 \\ 2 & 9 & 1 \\ -6 & 3 & 15 \end{pmatrix}$. Find a corresponding eigenvector.

6.4 Orthog Basis, QR Decomposition

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Are the columns of A LI?

Do the columns of A form a basis for R^4?

Are the columns of A mutually orthogonal?

6.4 Orthog Basis, QR Decomposition

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Find an orthogonal basis for the column space of A.

6.4 QR

A=QR. R is an upper triangular matrix. Find R.

$$A=\begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix}, Q=\begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

2012 Quiz 4, Question 1

Find a the eigenvalues and corresponding eigenvectors for the matrix $A = \begin{pmatrix} 5 & 2 \\ 4 & 7 \end{pmatrix}$. Use this to find a formula for A^k

(you may leave it as a product of three 2×2 matrices) in terms of these quantifies.

E-Values & E-Vectors True or false:

a) Eigenvalues must be nonzero scalars.

a) Eigenvectors must be nonzero vectors.

5.1

Is $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ an eigenvector of $B = \begin{bmatrix} 5 & 2 \\ 3 & 6 \end{bmatrix}$?

Answer this question without finding eigenvalues.

5.2

Find the characteristic polynomial and e-values of:

a) $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

b) $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

Basis Vectors

Determine whether the vectors form a basis in R^2. Explain your reasoning.

a) $\begin{bmatrix} 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -3 \end{bmatrix}$

b) $\begin{bmatrix} -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ -10 \end{bmatrix}$

R 26

1

$$V_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) PAIRWISE ORTH:

$V_1 V_2 = 0$

$V_1 V_3 = 0$

$V_2 V_3 = 0$

- if two vectors are not orthogonal, then what is their dot product?
- why is dot product of two orthog vectors zero?

---

b) $$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 V_1 + a_2 V_2 + a_3 V_3$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

hard!

easy: $$\begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = a_1 V_1 \cdot V_2 + a_2 V_2 \cdot V_2 + a_3 V_2 \cdot V_3$$

$$1 = 0 + a_2(1+4+1)$$

$$\Rightarrow a_2 = 1/6$$

R5

6 is an e-val of $\begin{pmatrix} 12 & -3 & -3 \\ 2 & 9 & 1 \\ -6 & 3 & 15 \end{pmatrix}$. Find evect.

2

Solve: $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 6 & 3 & -3 \\ 2 & 3 & 1 \\ -6 & 3 & 9 \end{pmatrix} \vec{v} = \vec{0}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 6 & 3 & -3 & 0 \\ 2 & 3 & 1 & 0 \\ -6 & 3 & 9 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 6 & 3 & -3 & 0 \\ 6 & 9 & 3 & 0 \\ 0 & 6 & +6 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 2 & +1 & -1 & 0 \\ 0 & 6 & +6 & 0 \\ 0 & 1 & +1 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & +1 & 0 \\ 0 & 1 & +1 & 0 \end{array}\right)$$

$$\Rightarrow \quad X_3 = 1, \quad X_2 = -1, \quad X_1 = +1 \quad \Rightarrow \quad v = \begin{bmatrix} 1 \\ -1 \\ +1 \end{bmatrix}$$

R 26
EXAMPLE 2: ASK IF STUDENTS WOULD LIKE TO DO IN GROUPS.

FIND AN ORTHOG. BASIS FOR COLSPACE OF A. #9, 6.4

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

WRITE

Let X_1, X_2, X_3 be columns of A. ALL PROCEDURAL

$$V_1 = X_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

$$V_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{-40}{20} \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$$

THM 11, p. 355

$$= \begin{bmatrix} 9 \\ 3 \\ -3 \\ 9 \end{bmatrix}$$

$$V_3 = X_3 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} V_1 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2$$

$$= \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2} V_1 - \left(\frac{-1}{2}\right) V_2$$

DO NOT DO, JUST WRITE.

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$\Rightarrow$ orthog basis is $\{V_1, V_2, V_3\}$

COMPREH. Q what would happen if dim Colspace $A = 2$?

A if (dim Colspace $A = 2$), #LI vectors in basis = 2, we would find only two nonzero vectors. ie: $V_3 = 0$ (or $V_2 = 0$)

EXAMPLE 3: QR FACTORIZATION

13, 6.4

- don't give algorithm, rather, explore the process.

Find R.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -5 \\ 1 & 5 \end{bmatrix}, \quad Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

Q What dimensions does R have?

A A is 4×2, Q is 4×2, so R is 2×2

Q R is [ ], and is upper-tri. What is bottom-left element?

A 0. We need to find three elements: $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

- we have a little puzzle here

METHOD 1

Q what is a?

A a: $\frac{5}{6}a - \frac{1}{6}\cdot 0 = 5 \Rightarrow a = 6$

Q What are b & c?

$$\left.\begin{aligned} \tfrac{5}{6}b - \tfrac{c}{6} &= 9 \\ \tfrac{1}{6}b + \tfrac{5}{6}c &= 7 \end{aligned}\right\} \Rightarrow \left(\begin{array}{cc|c} 5 & -1 & 54 \\ 1 & 5 & 42 \end{array}\right)$$

$$\ldots \sim \begin{pmatrix} 1 & 0 & 12 \\ 0 & 1 & 6 \end{pmatrix}$$

CHECK:

$$QR = \frac{1}{6}\begin{bmatrix} 5 & -1 \\ 1 & 5 \\ -3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 30 & 9 \\ 6 & 42 \\ -18 & \\ 6 & \end{bmatrix} \text{etc}$$

METHOD 2

Q are col's of A orthonormal?

A yes, by def'n

- students may not know this definition

Q $Q^TQ = ?$ (and have you seen transpose?)

A $Q^TQ = \begin{bmatrix} v_1\cdot v_1 & v_1\cdot v_2 \\ v_1\cdot v_2 & v_2\cdot v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

WRITE

$$A = QR$$
$$Q^TA = Q^TQR$$
$$R = Q^TA = \frac{1}{6}\begin{bmatrix} 5 & 1 & -3 & 1 \\ -1 & 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} \\ \\ \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

scale, if you like (not recommended)

- either approach works.
- method 2 requires less thinking!

QH6 Quiz 4

Good luck on Quiz 4!

If you have any questions, you can message me through wimba. I’ll be on wimba until 8:55.

QH6 Recitation 28

Today: Announcements, Least Squares

Dec 2: Quiz 4 graded, Least Squares HW due
Dec 6: last day of classes
Dec   : final exam
Jan 6: first day of Math 2401, please complete online survey

Exemptions from final exam? I have no idea. I hope so.

Motivation

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

1) Are the columns of A LI?

2) Do the columns of A form a basis for R^3?

3) Is b in Col A?

4) Is there a solution to Ax = b?

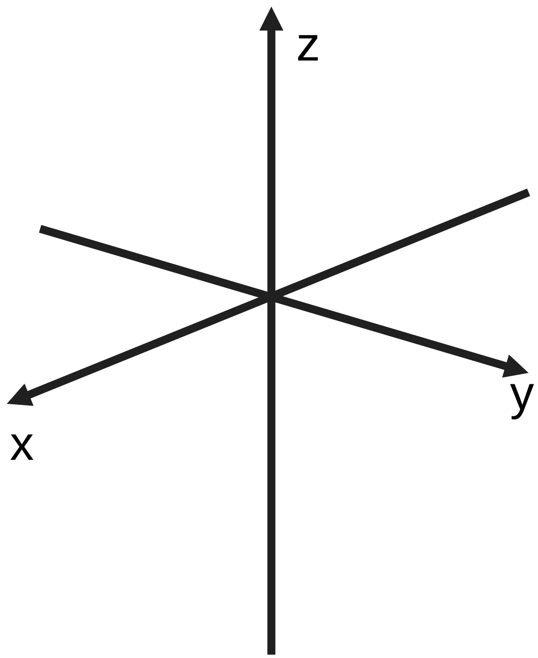
5) Therefore, we will:

Essential Idea

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

Our solution to this system:

find a vector x such that || b – Ax || ≤ || b – Ax ||



Solve

Consider the system Ax = b, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

A Special Case

Find a least squares sol’n to Ax=b, where:

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \; b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$$

Example True or false:

a) The LS problem is to find an x that makes Ax as close as possible to b.

a) A LS solution of Ax=b is a vector x such that:

Example Describe all LS solutions to

$$x+y=2$$
$$x+y=4$$

Example True or false:

a) If $\{v_1,v_2,v_3\}$ form an orthogonal basis, then $\{v_1,v_2,cv_3\}$ is another, different orthogonal basis

a) If $A=QR$, and Q has orthonormal columns, then $R=Q^TA$

a) If x is in subspace W, then $x-\text{proj}_W x \neq 0$

QH6 Recitation 28

Today: Announcements, Least Squares

Dec 2: Quiz 4 graded, Least Squares HW due
Dec 6: last day of classes
Dec   : final exam
Jan 6: first day of Math 2401, please complete online survey

Exemptions from final exam? I have no idea. I hope so.

Motivation

Consider the system $Ax = b$, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

1) Are the columns of A LI?

YES

2) Do the columns of A form a basis for R^3?

YES

3) Is b in Col A?

NO

4) Is there a solution to $Ax = b$?

NO

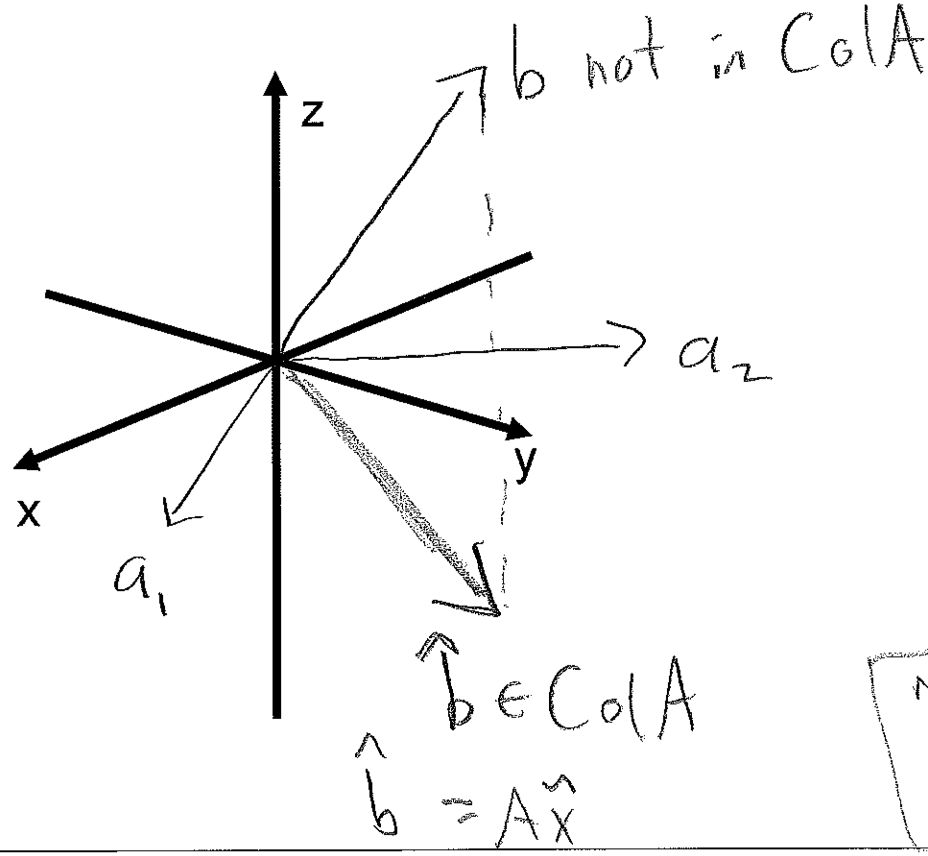
5) Therefore, we will: FIND THE CLOSEST SOLUTION TO $Ax=b$, $\hat{x}$.

Essential Idea

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

Our solution to this system:

find a vector $\hat{x}$ such that $\| b - A\hat{x} \| \le \| b - Ax \|$



- a_1, a_2 not necessarily $\perp$
- $(b-\hat{b}) \perp \hat{b}$, and $(b-\hat{b}) \perp$ colA

So: $A^T(b-\hat{b}) = A^T(b - A\hat{x})$

$\Rightarrow A^TA\hat{x} = A^Tb$

$$\boxed{\hat{x} = (A^TA)^{-1}A^Tb}$$

NOTE

$A^TA = I$ if A is orthonormal

Solve

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\hat{X} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84} \begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

RECITATION 28: LEAST-SQUARES

HW: 10, 12, 18 FROM 6.5 **2**

EXAMPLE: FIND A L.S. SOL'N OF $Ax=b$: $A=\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}$, $b=\begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$ $\boxed{9, 6.5}$

Q Yes or no: Are columns of A orthogonal?

A Yes.

SAY If they were not orthogonal, we would need to approach this problem differently.

Q What is the orthog proj of b onto column space of A?

A $\hat{b} = \frac{b\cdot a_1}{a_1\cdot a_1} a_1 + \frac{b\cdot a_2}{a_2\cdot a_2} a_2 = \dots = \frac{2}{7}a_1 + \frac{1}{7}a_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Q what is $\hat{x}$?

A $\hat{x} = \begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix}$, because $A\hat{x}=\hat{b}$

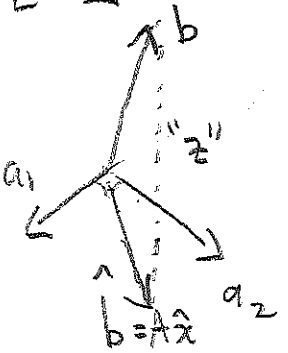
$\begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} 2/7 \\ 1/7 \end{bmatrix} = \hat{b}$

we solve $\begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}\hat{x} = \hat{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

TA DA!

WRITE

SCHEMATIC



$\hat{b} \in \text{span}\{a_1, a_2\}$

$\boxed{\hat{x} = \text{L.S. sol'n to } Ax=b}$

inconsistent system.

ie: $\boxed{A\hat{x} = \hat{b}}$

or: $\hat{x}$ is the vector that minimizes

$$\|b-\hat{b}\| = \|b - A\hat{x}\|$$

(why does it minimize? Hopefully covered in class!)

CONCEPTUAL QUESTIONS

Q The LS problem is to find an $\vec{x}$ that makes $A\vec{x}$ as close as possible to $\vec{b}$. 17a

A True. (This is the def'n of the LS. problem)

Q A L.S. sol'n of $A\vec{x}=\vec{b}$ is a vector $\hat{x}$ s.t. 17b

$$\|b-Ax\| \leq \|b-A\hat{x}\| \quad \forall x \text{ in } \mathbb{R}^n.$$

A F! inequality in wrong direction

Q The LS sol'n of $Ax=b$ is point in the col space of A closest to $\vec{b}$. 18b

A F. The LS sol'n is $\hat{x}$.
The closest point in the colspace of A is $\hat{b}$.

don't do

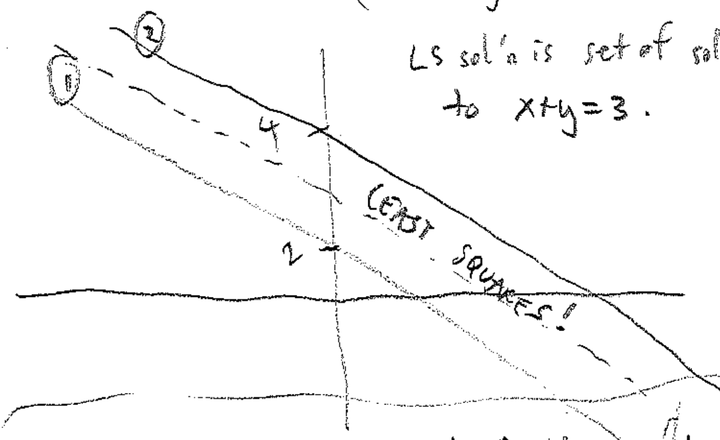
Q Describe all LS sol'ns to $x+y=2$ ①
$x+y=4$ ②

A NORMAL EQUATIONS: $A^T A\vec{x}=A^T\vec{b}$

A $A=\begin{pmatrix}1&1\\1&1\end{pmatrix} \Rightarrow A^TA=\begin{pmatrix}2&2\\2&2\end{pmatrix}$

and $A^Tb=\begin{pmatrix}6\\6\end{pmatrix}$

$2x+2y=6$
or $x+y=3$



LS sol'n is set of sol'ns to $x+y=3$.

LS solution, $\hat{x}$, is the vector $\hat{x}$ s.t. $\|b-A\hat{x}\| \leq \|b-A\vec{x}\|$

EXAMPLE 4: CONCEPTUAL QUESTIONS

17a, 6.4

Q TRUE/FALSE

IF $\{v_1, v_2, v_3\}$ form an orthog basis,

$\{v_1, v_2, cv_3\}$ is a <u>different</u> orthog basis

A F, A basis is a set of all possible linear combinations of vectors.

$\Rightarrow$ The two bases are the same.

---

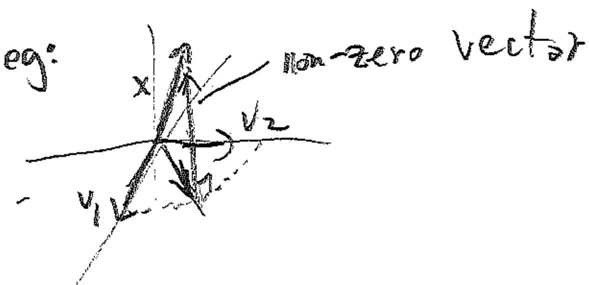
17 c)

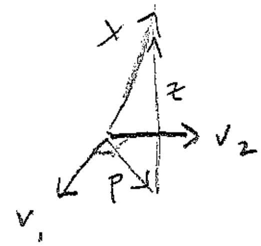
Q IF $A = QR$, Q has orthonormal columns, then $R = Q^T A$

A T. Because $Q^T Q = I$

Q IF x is not in subspace W, then $\vec{x} - \text{proj}_W x \neq 0$.

A T. eg:





But:
IF x is in W, then $\vec{x} - \text{proj}_W \vec{x} = 0$. ← get them to say this

QH6 Recitation 28

Today: Least Squares, Initial Value Problems

Describe all least squares solutions to

$$x + y = 2$$
$$x + y = 4$$

Dates

Dec 2: Quiz 4 graded, Least Squares HW due
Dec 6: last day of classes
Dec   : final exam
Jan 6: first day of Math 2401, please complete online survey

Can students be exempted from writing the final exam?

True or False

a) If $\{v_1,v_2,v_3\}$ form an orthogonal basis, then $\{v_1,v_2,cv_3\}$ is another, different orthogonal basis.

a) If $A=QR$, and Q has orthonormal columns, then $R=Q^TA$.

a) If x is in subspace W, then $x - \text{proj}_W x \neq 0$.

Initial Value Problems

$$x_1'(t) = ax_1 + bx_2$$

$$x_2'(t) = cx_1 + dx_2$$

a) We can write this system as:

b) If w is a solution to x'=Ax, then w'=

c) If b = c = 0, what is the solution to x' = Ax?

d) If A has e-value λ and e-vector v, then show that

Example 3 Let A be some 2 x 2 matrix.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = -3, \lambda_2 = -1, \vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

If x(t) is the position of a particle at time t, and x'(t)=Ax, find an expression for x(t).

Example 4 Solve the IVP x=Ax, where

$$A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}, \lambda_1 = -1, \lambda_2 = 3, \vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

CONCEPTUAL QUESTIONS

Q The LS problem is to find an $\vec{x}$ that makes $A\vec{x}$ as close as possible to $\vec{b}$. 17a

A True. (This is the def'n of the LS. problem)

Q A L.S. sol'n of $A\vec{x}=\vec{b}$ is a vector $\hat{x}$ s.t. 17b

$$\|b - Ax\| \leq \|b - A\hat{x}\| \quad \forall x \text{ in } \mathbb{R}^n.$$

A F! inequality in wrong direction

Q The LS sol'n of $Ax=b$ is point in the col space of A closest to $\vec{b}$. 18b

A F. The LS sol'n is $\hat{x}$.

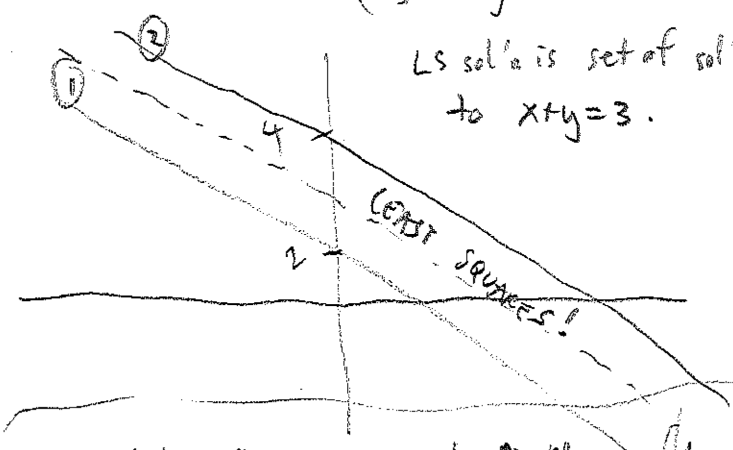
The closest point in the colspace of A is $\hat{b}$.

for 4 do

Q Describe all LS sol'ns to $\begin{matrix} x+y=2 & (1) \\ x+y=4 & (2) \end{matrix}$

A NORMAL EQUATIONS: $A^TA\vec{x}=A^T\vec{b}$

A $A=\begin{pmatrix}1&1\\1&1\end{pmatrix} \Rightarrow A^TA=\begin{pmatrix}2&2\\2&2\end{pmatrix}$ and $A^Tb=\begin{pmatrix}6\\6\end{pmatrix}$ $\Bigg\}$ $2x+2y=6$ or $x+y=3$



LS sol'n is set of sol'ns to $x+y=3$.

LS solution, $\hat{x}$, is the vector $\hat{x}$ s.t. $\|b - A\hat{x}\| \leq \|b - A\vec{x}\|$

EXAMPLE 4: CONCEPTUAL QUESTIONS

Q | TRUE/FALSE
IF $\{v_1, v_2, v_3\}$ form an orthog basis, $\{v_1, v_2, cv_3\}$ is a different orthog basis

17a, 6.4

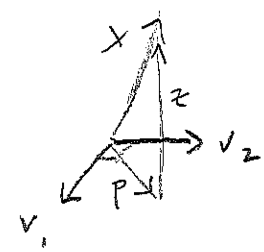
A | F, A basis is a set of all possible linear combinations of vectors.
⇒ The two bases are the same.

Q | IF $A = QR$, Q has orthonormal columns, then $R = Q^T A$

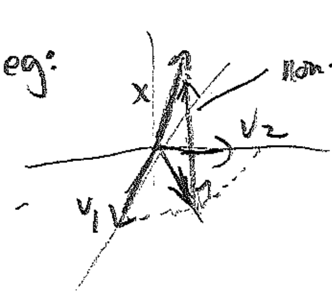
17c)

A | T. Because $Q^T Q = I$

Q | IF x is not in subspace W, then $\vec{x} - \text{proj}_W x \neq 0$.



A | T. eg:



But:
IF x is in W, then $\vec{x} - \text{proj}_W \vec{x} = 0.$ ← got then to say this

(FROM R25)

EXAMPLE 2 : TERMINOLOGY & IVP DEF'N

SUPPOSE $x_1'(t) = ax_1 + bx_2$
$x_2'(t) = cx_1 + dx_2$

(LET'S JUST MAKE SURE WE'RE ALL SPEAKING THE SAME LANGUAGE.)

Q Then $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} ? \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

A $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(SHOULDN'T SURPRISE ANYONE)

Let $\vec{x}' = A\vec{x}$

Q Let $\vec{w}$ be a "solution" to $\vec{x}' = A\vec{x}$. Then $\vec{w}' = ?$

A $\vec{w}' = A\vec{w}$.

(when we sub w into the DE, LHS = RHS)

Q IF $b = c = 0$, what is the sol'n?

A $x_1' = ax_1 \Rightarrow x_1 = e^{at}$
$x_2' = dx_2 \Rightarrow x_2 = e^{bt}$
$\Rightarrow$ sol'ns are (real) exponentials

Q If A has e-val λ and e-vect $\vec{v}$, show $\vec{x} = \vec{v}e^{\lambda t}$ is A SOLUTION.

A $\frac{d}{dt}\vec{x} = \frac{d}{dt}(\vec{v}e^{\lambda t}) = \lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t}$
$\Rightarrow \vec{x}' = A\vec{x}$

$\vec{v}$ is constant (because $x = ve^{\lambda t}$, $Ax = Ave^{\lambda t}$)

SAY GENERAL SOLUTION TO $\vec{x}' = A\vec{x}$ IS $\vec{x} = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$

EXAMPLE 3

#2, 5.7

LET $\vec{x}(t)$ BE THE POSITION OF A PARTICLE AT TIME t.

IF $\vec{x}' = A\vec{x}$, WHERE

A is 2x2

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_1 = -3$, $\lambda_2 = -1$

$\vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

FIND POSITION AT TIME t.

Q | What are two sol'ns? (to $\vec{x}' = A\vec{x}$)

A | $\vec{v}_1 e^{-3t}$, $\vec{v}_2 e^{-t}$

Q | YES OR NO: IS THE SUM OF SOL'NS ALSO A SOLUTION? $\vec{x}' = Ax$

A | YES. (DO YOU KNOW THIS, OR ARE YOU GUESSING?)

Q | THE LIN. COMB OF SOLUTIONS IS A SOLUTION, SO

$$c_1\vec{v}_1 e^{-3t} + c_2\vec{v}_2 e^{-t} = \vec{x}$$

IS GEN SOL'N.

HOW DO WE FIND c_1, c_2?

PROOF:
$x' = Ax$
$\lambda_1 c_1 v_1 e^{\lambda_1 t} + \lambda_2 c_2 v_2 e^{\lambda_2 t}$
- etc.
diff & matrix multip are linear.

A | USE $\vec{x}(0)$:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^0 + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^0$$

or $\left(\begin{array}{cc|c} -1 & -1 & 2 \\ 1 & -1 & 3 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \end{array}\right)$

$$\Rightarrow \vec{x} = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + \frac{5}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

Q | as $t \to 0$, what happens to $\vec{x}$?

A | $\vec{x} \Rightarrow \vec{0}$.

ASK STUDENTS IF THEY'D LIKE TO DO THIS IN GROUPS (Y/N)

SOLVE THE IVP $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -2 & -5 \\ 1 & 4 \end{bmatrix}$ #4,5,7

$\vec{x}_0 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

e-vals are $-1, +3$

Q What are (the) e-values?

A $0 = (-2-\lambda)(4-\lambda) + 5$

$= \lambda^2 - 2\lambda - 3$

$= (\lambda+1)(\lambda-3) \Rightarrow \lambda = -1, 3$

Q WHAT IS THE e-vect for $\lambda = -1$?

$\left(\begin{array}{cc|c} -1 & -5 & 0 \\ 1 & +5 & 0 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{v}_1 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$

Q WHAT IS THE evect for $\lambda = +3$?

$\left(\begin{array}{cc|c} -5 & -5 & 0 \\ 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \vec{v}_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$

Q What is general sol'n?

$$\vec{x} = c_1 \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$$

Q We still need c's, how do we get them?

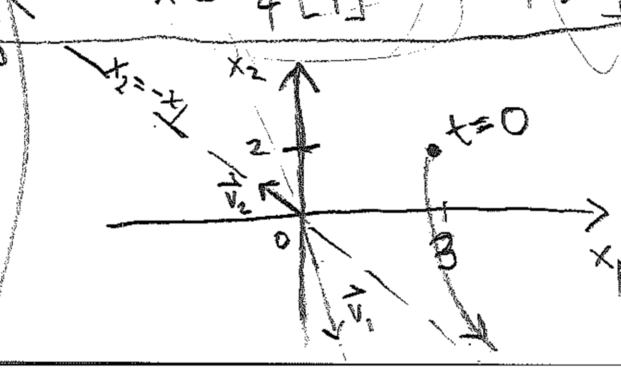
$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} -5 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\left(\begin{array}{cc|c} -5 & -1 & 3 \\ 1 & 1 & 2 \end{array}\right) \sim \left(\begin{array}{ccc} 1 & 0 & 13/4 \\ 0 & 1 & -5/4 \end{array}\right)$$

Q Sol'n is?

$$\vec{x} = \frac{13}{4} \begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t} + \frac{-5}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t}$$

COMPREHENSION



- SADDLE, ORIGIN
- TRAJECTORY IN SOLUTION SPACE
- IF t IS LARGE, $\vec{x} \approx \frac{5}{4} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} = \begin{bmatrix} 5/4\, e^{3t} \\ -5/4\, e^{3t} \end{bmatrix}$

QH6 Recitation 29

Today: Differential Equations

If You Are Writing the Final

- exam has two parts, each part is 50 minutes
- Work with your facilitator to schedule a time on either the 9th or 10th to write the final
- Your facilitator has complete instructions for writing final
- I will not be on wimba during the final
- Grady students write on campus on Dec 11
- Office hours: Sun 3:00 pm to 4:00 pm & 8:00 pm to 9:00 pm

Grades: after this recitation I will

1. enter your HW grades in t-square
2. apply recitation grades in t-square
3. send you an email to indicate if you are/aren’t exempt from final

Example

a) Solve the IVP: $ty' + 2y = 4t^2$, $y(0) = y_0$.
b) Plot the solution for various values of y_0.

Example

Solve the BVP:

$y'' + 4y' + 13y = 0$, $y(0) = 2$, $y(\pi/2) = 1$.

Example

12 is an eigenvalue of $\begin{pmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{pmatrix}$. Find as many linearly independent eigenvectors for this eigenvalue as possible.

Initial Value Problems

$$x_1'(t) = ax_1 + bx_2$$
$$x_2'(t) = cx_1 + dx_2$$

a) We can write this system as:

b) If w is a solution to x'=Ax, then w'=

c) If b = c = 0, what is the solution to x' = Ax?

d) If A has e-value λ and e-vector v, then show that

Example 3 Let A be some 2 x 2 matrix.

$$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda_1 = -3, \lambda_2 = -1, \vec{x}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

If x(t) is the position of a particle at time t, and x'(t)=Ax, find an expression for x(t).

QH6 Office Hours

Thank you for coming to office hours! If you have a question, you may need to get my attention by typing something in the chat window. Or by saying something with the mic.

Today’s office hours run from 8:00 PM to 10:00 PM.