THE FORM OF A SOLUTION TO THE

INHOMOGENEOUS HEAT EQUATION

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CHAPTER I

INTRODUCTION

1.1 The Problem and Its Solution

Let W and g be functions of the cartesian coordinates $x_1, \ldots,$ x_n and the time variable y and let Δ denote the Laplacian operator with respect to the n cartesian coordinates. The partial differential equations

$$\text{(a)} \quad \Delta W - \frac{\partial W}{\partial y} = 0$$

and

$$\text{(b)} \quad \Delta W - \frac{\partial W}{\partial y} = g$$

arise in the investigation of heat flow in a isotropic solid body and the investigation of diffusion in a motionless medium [1]. The inhomogeneous equation (b) applies when there are sources in the medium.

In the discussion below we consider the special case of (b) which corresponds to an idealized physical situation of a one dimensional body. That is, we investigate the equation

$$\text{(c)} \quad \partial W = g(x, y)$$

where

$$\partial W = \frac{\partial^2 W}{\partial x^2}(x, y) - \frac{\partial W}{\partial y}(x, y)$$

for any real x, and y in the open interval (0, T). This specialization is no essential restriction on the generality of the results. The arguments that we give below carry over directly for the equation (b).

Herr in his paper [2] has shown that under certain growth and regularity conditions (namely, a Hölder condition with exponent between zero and one) on the function g, a solution to equation (c) exists. Specifically, it was shown that the function

$$\text{(d)}\quad u(x,y) = -\int_0^T \int_{-\infty}^{\infty} g(\xi,\gamma)\, e(x-\xi,\, y-\gamma)\, d\xi d\gamma \qquad \text{(Def.)}$$

satisfies the equation (c) where

$$\text{(e)}\quad e(s,t) = \begin{cases} \dfrac{1}{2\sqrt{\pi t}} \exp\left(-\dfrac{s^2}{4t}\right), & t > 0 \\ 0 & , t \leq 0. \end{cases} \qquad \text{(Definition)}$$

Herr has deduced with these conditions on the function g, that a general solution of equation (c) is given by the sum of the particular function u and a solution to the homogeneous equation. The problem of finding solutions to the homogeneous equation is treated in the literature (see e.g. [3]).

Hadamard has established that if the regularity condition on g is relaxed to requiring only continuity, it is not in general true that particular function u satisfies the inhomogeneous equation (c) (see [4]). A solution need not even exist.

We show below that if the function u defined by equation (d) is not a particular solution to the inhomogeneous equation (c) with the function g continuous and bounded, then there is no solution at all.

Adapting a procedure given by Hartman and Wintner in their paper [5], we prove that if there is a solution to equation (c), it is the sum of the particular function u and a solution to the homogeneous heat equation.

1.2 Method of Attack

We consider the equation

$$\text{(a)}\quad \lim_{h\to 0^+} \partial_h W(x,y) = g(x,y)$$

where

$$\text{(b)}\quad \partial_h W(x,y) = \frac{W(x+h,y)+W(x-h,y)+W(x,y-h^2)-3W(x,y)}{h^2} \qquad \text{(Def.)}$$

for $h>0$. The inhomogeneous equations (1.1.c) and (a) above are related. In fact, with certain regularity conditions on W

$$\lim_{h\to 0^+} \partial_h W(x,y) = \partial W(x,y).$$

We prove that the particular function u defined by equations (1.1.e) satisfies equation (a) above if the function g is continuous and bounded. If a solution to the inhomogeneous equation (1.1.c) exists, then it too satisfies equation (a). Thus the difference between this solution and the particular function satisfies the homogeneous version of (a). We show that if a function has certain regularity properties and satisfies the homogeneous version of (a), then it also is a solution to the equation

$$\partial W = 0.$$

This result implies that our particular function has the necessary derivatives, so it is a solution to our inhomogeneous equation

$$\bar{\partial} W = g.$$

The question of existence of solutions to the inhomogeneous partial differential equation (1.1.c) can then be reduced to an investigation of the properties of the particular function u (see [6]).

1.3 Conjecture

There are strong similarties between the properties of solutions to $\Delta W=0$ and $\bar{\partial} W=0$. The preliminaries in treating the corresponding inhomogeneous problems (compare [7] with the details below) are likewise similar. It seems likely that the procedure can be adapted to prove the corresponding theorems about Poisson's equation,

$$\Delta W \,(x_1, \ldots, x_n) = g\,(x_1, \ldots, x_n).$$

The author of [7] uses a method which he conjectures will not work for $n>3$ in Poisson's equation. If the procedure given below could be adapted, it would readily generalize to higher dimensions.

CHAPTER II

THE HEAT EQUATION

2.1 We begin our investigation with the following:

For some fixed $T>0$, let R be the strip

$$\{(x,y) \mid 0 \leq y \leq T\}.$$

Definition: The function w is said to be a solution to the equation

$$\partial W(x,y) = \frac{\partial^2 W}{\partial x^2}(x,y) - \frac{\partial W}{\partial y}(x,y) = g(x,y)$$

or the equation

$$\partial W(x,y) = 0$$

if w is continous on R, $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial w}{\partial y}$ exist in the interior of R, and if w satisfies the appropriate equation in the interior of R.

2.2 Now we state the most interesting result of this essay:

Theorem: If g is continuous and bounded on R, then the existence of a solution to the equation

$$\partial W(x,y) = g(x,y)$$

implies that the function

$$u(x,y) = -\iint_R e(x-\xi,y-\gamma)\, g(\xi,\gamma)\, d\xi d\gamma \qquad \text{(Def.)}$$

is a solution. Futhermore, any solution is the sum of this particular solution u and a solution to the homogeneous equation

$$\partial V(x,y) = 0.$$

Note that conversely if the function u defined above does not yield a solution to the inhomogeneous heat equation under the given conditions, then no such solution can exist.

With this theorem and note it becomes possible to investigate the existence of solutions to the inhomogeneous heat equation by examining the functions u and g (see [8]).

We undertake to prove this theorem with the following sequence of lemmas.

2.3 Lemma. If the function w is defined in some neighborhood of the point (x_o, y_o), if $\frac{\partial w}{\partial x}(x, y_o)$ exists for (x, y_o) in this neighborhood, and if $\frac{\partial^2 w}{\partial x^2}(x_o, y_o)$ and $\frac{\partial w}{\partial y}(x_o, y_o)$ exist, then

$$\lim_{h\to 0+} \partial_h W(x_o,y_o) = \lim_{h\to 0+} \frac{w(x_o+h,y_o)+w(x_o-h,y_o)+w(x_o,y_o-h^2)-3w(x_o,y_o)}{h^2}$$

$$= \partial w(x_o,y_o).$$

Proof. From the definition of partial differentiation it follows that

$$\text{(a)} \quad \lim_{h\to o+} \frac{w(x_o,y_o-h^2) - w(x_o,y_o)}{h^2} = -\frac{\partial w}{\partial y}(x_o,y_o).$$

By the Generalized Mean Value Theorem there is a number

θ, $0 < \theta < 1$, such that

$$\text{(b)}\quad \frac{w(x_o+h,y_o)+w(x_o-h,y_o)-2w(x_o,y_o)}{h^2}$$

$$= \frac{1}{2\theta h}\left[\frac{\partial w}{\partial x}\,(x_o+\theta h,y_o)-\frac{\partial w}{\partial x}\,(x_o-\theta h,y_o)\right]$$

Since the existence of $\frac{\partial w}{\partial x}\,(x,y_o)$ in a neighborhood of x_o guarantees the continuity of

$$\text{(c)}\quad \Delta_h w(x_o,y_o) = w(x_o+h,y_o)+w(x_o-h,y_o)-2w(x_o,y_o) \qquad \text{(Def.)}$$

as a function of h for sufficiently small h.

If we add and substract $\frac{\partial w}{\partial x}\,(x_o,y_o)$ to the numerator of the right hand side of equation (b) and allow h to approach zero through positive values, we see that the right hand side of (b) approaches

$$\frac{\partial^2 w}{\partial x^2}\,(x_o,y_o).$$

This result combined with equation (a) and the definition of $\partial_h W(x_o,y_o)$ and $\partial w(x_o,y_o)$ yield the proof of the lemma.

2.4 In order to facilitate the statement and proof of the next two lemmas we adopt the following convention:

Let y_o be a number strictly between zero and T and suppose the number δ is selected so that

$$0 < \delta < y_o \text{ and } y_o + \delta < T.$$

Let $\qquad$ (a) $\quad N = \left\{(x,y) \;\middle|\; |x-x_o| < \delta \text{ and } |y-y_o| < \delta\right\}$

and let $\overline{N}$ denote the usual closure of N.

$$\text{(b)}\quad \text{Let } A = \left\{(x,y) \;\middle|\; y-y_0 = -\delta \text{ and } |x-x_0| \le \delta \quad \text{or} \quad |x-x_0| = \delta \text{ and } |y-y_0| \le \delta\right\}.$$

Note that N is a square neighborhood and that A is the bottom and two sides of this neighborhood.

Let $N^* = \overline{N} - A$ (that is, N^* is the open neighborhood N with the "open" top adjoined).

Definition. With N as defined in (a) we say that w is a solution on N^* to the equation

$$\lim_{h\to 0^+} \partial_h W(x,y) = g(x,y)$$

or the equation

$$\lim_{h\to 0^+} \partial_h W(x,y) = 0$$

if w is defined and continuous on $\overline{N}$ and if w satisfies the appropriate equation on N^*.

2.5 We adapt a proof of the well known maximal principle for solutions to the homogeneous heat equation to establish the following maximal principle (See [9]).

Lemma. Suppose that w is a solution on N^* of the equation.

$$\text{(a)}\quad \lim_{h\to 0^+} \partial_h W(x,y) = 0.$$

Then w has its maximum on the set A.

Corollary. If w and v are two solutions on N^* to the equation (a) which agree on the set A, then they are identical on $\overline{N}$.

Note. This corollary states that solutions on N^* to the equation (a) are uniquely determined (if they exist) by their values on the bottom and two sides of $\overline{N}$ (that is, initial and boundary values).

Proof of Lemma. The set $\overline{N}$ is compact and w is continuous so the function assumes its maximum, say m, at some point $(\overline{x},\overline{y})$ in $\overline{N}$. Suppose that $(\overline{x},\overline{y})$ lies in N^* (i.e. $\overline{N}$ without the bottom and two sides). The set A (i.e. the bottom and two sides) is compact so w assumes its maximum, say M, on A. We have that $m \geq M$. Suppose that $m > M$. We show that this last supposition leads to a contradiction. This contradiction will establish the lemma.

Let $v(x,y) = w(x,y) + \frac{m - M}{6\delta^2}(x - \overline{x})^2$. For $(x,y) \epsilon A$ we have $v(x,y) < M + \frac{4}{6}(m - M) = \frac{1}{3}M + \frac{2}{3}m < m$. But $v(\overline{x},\overline{y}) = m$ so the maximum of v on $\overline{N}$ does not occur on A. Let (x^*,y^*) be the point in N^* at which v has its maximum. For $h > 0$ and sufficiently small

$$\frac{v(x^*+ h,y^*) + v(x^*- h,y^*) - 2v(x^*,y^*)}{h^2} \leq 0 \text{ and } \frac{v(x^*,y^*- h^2) - v(x^*, y^*)}{h^2} \leq 0$$

Thus, $\partial_h v(x^*,y^*) < 0$ for sufficiently small h, $h > 0$. Since $\lim_{h\to 0^+} \partial_h w(x^*,y^*)$ exists and since

$$\lim_{h\to 0^+} \partial_h\left\{\frac{m-M}{6\delta^2}(x - \overline{x})^2\right\}$$

exists by the lemma of Section 2.3, $\lim_{h\to 0^+} \partial_h v(x^*,y^*)$ exists and is no larger than zero. But by the lemma of Section 2.3 and our assumption that $\lim_{h\to 0^+} \partial_h w(x^*,y^*) = 0$, we have the relation

$$\lim_{h\to 0^+} \partial_h v(x^*,y^*) = \frac{m-M}{3\delta^2} > 0.$$

This last inequality contradicts the previous statement. Hence, our supposition that $(\overline{x},\overline{y})$ lies in N^* is false. The maximum of w occurs, therefore, on A and the lemma is proved.

<u>Proof of Corollary</u>. Let $K(x,y) = w(x,y)-v(x,y)$. Then K and $(-K)$ satisfy $\lim_{h\to 0^+} \partial_h W(x,y) = 0$. By the preceding lemma $K \leq 0$ and $(-K) \leq 0$ on N (recall that K vanishes on A). Hence, $K \equiv 0$ on N and the corollary is proved.

2.6 We state without proof the following well known result.

<u>Lemma</u>. Let f and h be defined and continuous on $[0,T]$. Let g be defined and continuous on $[0,1]$. Suppose that

$$f(0) = g(0) \text{ and } g(1) = h(0)$$

Then, there is an unique solution $v(x,y)$ to the equation $\partial w(x,y) = 0$ on $[0,1] \times [0,T]$ such that the following equations are satisfied:

$$\text{(a) } v(0,y) = f(y),$$

$$\text{(b) } v(x,0) = g(x),$$

$$\text{(c) } v(1,y) = h(y).$$

Remark. This theorem asserts the existence of a solution to the homogeneous heat equation on the rectangle [0,1] x [0,T] which has prescribed values on the bottom and two sides. Clearly, the result follows for any rectangle in the plane with sides parallel to the coordinate axes.

For a proof of this lemma the reader is referred to the literature (see e.g. [10] for an outline of the proof and [11] for some details similar to those needed to fill in this outline).

2.7 We use this result to show the connection between the solutions of our inhomogeneous limit difference equation and the inhomogeneous heat equation. This connection is stated in the following:

Lemma. If w is a solution on N^* to the equation $\lim_{h\to 0^+} \partial_h W(x,y) = 0$, then $\frac{\partial w}{\partial x}$, $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial w}{\partial y}$ exist and $\partial w(x,y) = 0$ on N.

Proof. Let $N(\theta) = \left\{ (x,y) \;\middle|\; |x - x_0| \leq \delta \text{ and for } \theta > 1,\ \delta < (y-y_0) < \theta\delta \right\}$. Clearly, $N(\theta) \supseteq N$. Let

$$w^*(x,y) = \begin{cases} w(x,y) & \text{for } (x,y) \in \overline{N} \\ w(x,y_0+\delta) & \text{for } (x,y) \in (N(\theta) - \overline{N}). \end{cases}$$

The function w^* is continuous on $N(\theta)$ (see the definition of Section 2.4). By the previous lemma there is a function w defined and continuous on N, which agrees with w^* on the bottom and two sides of the rectangle $N(\theta)$, such the $\partial W(x,y) = 0$ at each interior point of $N(\theta)$.

From the definition of w^*, W agrees with w on the bottom and two sides of $\overline{N}$. Since $\partial W = 0$ in the interior of $N(\theta)$, the lemma of Section 2.3 guarantees that $\lim_{h\to 0^+} \partial_h W(x,y) = 0$ on N^*. From Section 2.5

we know that W is uniquely determined on $\overline{N}$ by its values on A so $W \equiv w$. Hence, $\partial w = 0$ as claimed.

2.8 The previous lemmas have all dealt with generalities. Now we direct our efforts toward specific results involving the particular function u defined in Section 2.2. We will show (using arguments due to Hartman and Wintner, see [12]) that u satisfies the equation

$$\lim_{h \to 0^+} \partial_h W(x,y) = g(x,y)$$

in the interior of R subject to certain conditions on g.

In this direction we establish the following:

Lemma. Let
$$L(h) = \iint_R |\partial_h S(x_o,y_o)| d\xi d\gamma \qquad \text{(Def.)}$$

for $h > 0$, any real x_o and $0 < y_o < T$, where $S(x,y) = e(x-\xi,y-\gamma)$. (Def.) Then $L(h) = O(1)$ as $h \to 0^+$.

Proof. If $2h^2 < y_o$, we may write $L(h) = I_1 + I_2$

where
$$I_1 = \int_{y_o-2h^2}^{y_o} \int_{-\infty}^{\infty} |\partial_h S(x_o,y_o)| d\xi \, d\gamma$$

and
$$I_2 = \int_0^{y_o-2h^2} \int_{-\infty}^{\infty} |\partial_h S(x_o,y_o)| d\xi \, d\gamma.$$

Introducing the notation

$$\Delta^2 S(x_o,y_o) = S(x_o+h,y_o)+S(x_o-h,y_o)-2S(x_o,y_o)$$

and noting that $\partial S(x,y) = 0$ for $(y-\gamma) > 0$ (e(s,t) satisfies the heat equation for $t > 0$), we see that

$$(a)\quad |\partial_h S(x_o,y_o)| = |\partial_h S(x_o,y_o) - \partial S(x_o,y_o)|$$

$$\leq \left| \frac{\Delta^2 S(x_o,y_o)}{h^2} - \frac{\partial^2 S}{\partial x^2}(x_o,y_o) \right|$$

$$+ \left| \frac{\partial S}{\partial y}(x_o,y_o) - \frac{S(x_o,y_o) - S(x_o,y_o - h^2)}{h^2} \right|.$$

But

$$(b)\quad \frac{\Delta^2 S(x_o,y_o)}{h^2} - \frac{\partial^2 S}{\partial x^2}(x_o,y_o) = \frac{1}{h^2} \int_0^h \int_{-\theta}^{\theta} \int_{\theta}^{\rho} \frac{\partial^3 S}{\partial x^3}(x_o+\eta, y_o)\, d\eta\, d\rho\, d\theta$$

and

$$(c)\quad \frac{\partial S}{\partial y}(x_o,y_o) - \frac{S(x_o,y_o) - S(x_o,y_o - h^2)}{h^2} = \frac{1}{h^2} \int_0^h \int_0^{\rho} \frac{\partial^2 S}{\partial y^2}(x_o, y_o - \eta)\, d\eta\, d\rho.$$

Letting

$$I_{21} = \int_0^{y_o - 2h^2} \int_{-\infty}^{\infty} \frac{1}{h^2} \left| \int_0^h \int_{-\theta}^{\theta} \int_0^{\rho} \frac{\partial^3 e}{\partial x^3}(x_o + \eta - \xi, y_o - \gamma)\, d\eta\, d\rho\, d\theta \right| d\xi\, d\gamma,$$

and

$$I_{22} = \int_0^{y_o - 2h^2} \int_{-\infty}^{\infty} \frac{1}{h^2} \left| \int_0^{h^2} \int_0^{\rho} \frac{\partial^2 e}{\partial y^2}(x_o, y_o - \eta)\, d\eta\, d\rho \right| d\xi\, d\gamma,$$

we see that

$$L(h) \leq I_1 + I_{21} + I_{22}$$

follows from equations (a), (b), and (c) above.

We give the remainder of the proof in three parts. Parts one, two, and three will consist of showing that the integrals I_1, I_{21}, and

I_{22} respectively are bounded as $h \to 0^+$.

Part 1. We adopt the convention that if the upper or lower limit on an integral is missing it is understood to be ∞ or $-\infty$ respectively. Thus, we write

$$\text{(d)} \quad |I_1| \le \frac{1}{h^2} \int_{y_o-2h^2}^{y_o} \int S(x_o+h, y_o)\, d\xi\, d\gamma +$$

$$\frac{1}{h^2} \int_{y_o-2h^2}^{y_o} \int S(x_o-h, y_o) d\xi\, d\gamma + \frac{3}{h^2} \int_{y_o-2h^2}^{y_o} \int S(x_o, y_o) d\xi\, d\gamma$$

$$+ \frac{1}{h^2} \int_{y_o-2h^2}^{y_o} \int S(x_o, y_o-h^2)\, d\xi\, d\gamma.$$

Since $S(x_o, y_o-h^2) = 0$ for $y_o - h^2 - \gamma \le 0$ (recall the definition of S),

$$\frac{1}{h^2} \int_{y_o-2h^2}^{y_o} \int S\,(x_o, y_o-h^2)\, d\xi\, d\gamma = \frac{1}{h^2} \int_{y_o-2h^2}^{y_o-h^2} \int S(x_o, y_o - h^2)\, d\xi\, d\gamma$$

$$= \frac{1}{h^2} \int_{y_o-2h^2}^{y_o-h^2} \int e(x_o-\xi, y_o-h^2-\gamma)\, d\xi\, d\gamma$$

$$= \frac{1}{h^2} \int_{y_o-2h^2}^{y_o-h^2} \int \frac{\exp\,(-s^2)}{\sqrt{\pi}}\, ds\, dt = 1$$

where the change of variables $s = \dfrac{x_o - \xi}{2\sqrt{y_o-h^2-\gamma}}$ was made.

The other three integrals are similarly seen to be bounded as $h \to 0^+$.

Hence, $I_1 = O(1)$ as $h \to 0^+$ and part one is finished.

Part 2. Clearly from the definition of the integral I_{21}, we have

$$(e)\ I_{21} = \int_0^{y_0-2h^2} \int \frac{1}{h^2} \left| \int_0^h \left\{ \int_0^{\theta} - \int_0^{-\theta} \right\} \int_0^{\rho} \frac{\partial^3 e}{\partial x^3} (x_0 \pm \eta - \xi, y_0 - \gamma) d\eta d\rho d\theta \right| d\xi d\gamma.$$

With the change of variables $s = x_0 \pm \eta - \xi$ and $t = y_0 - \gamma$ we have

$$(f)\ \int_0^{y_0-2h^2} \int \left| \frac{\partial^3 e}{\partial x^3} (x_0 \pm \eta - \xi,\ y_0 - \gamma) \right| d\xi d\gamma = \int_{2h^2}^{y_0} \int \left| \frac{\partial^3 e}{\partial s^3} (s,t) \right| ds dt.$$

A simple calculation yields

$$\frac{\partial^3 e}{\partial s^3} (s,t) = \frac{1}{2\sqrt{\pi}} \left[\frac{3s}{4\sqrt{t^5}} - \frac{s^3}{8\sqrt{t^7}} \right] \exp\left(-\frac{s^2}{4t}\right)$$

which with the triangle inequality implies that the right hand side of (f) is dominated by

$$\frac{1}{\sqrt{\pi}} \int_{2h^2}^{y_0} \int_0 \frac{3s}{4\sqrt{t^5}} \exp\left(-\frac{s^2}{4t}\right) ds\ dt\ +$$

$$\frac{1}{\sqrt{\pi}} \int_{2h^2}^{y_0} \int \frac{s^3}{8\sqrt{t^7}} \exp\left(-\frac{s^2}{4t}\right) ds\ dt.$$

But if $r = \frac{s}{2\sqrt{t}}$,

$$(g)\ \int_{2h^2}^{y_0} \int_0 \frac{3s}{4\sqrt{t^5}} \exp\left(-\frac{s^2}{4t}\right) ds\ dt = \int_{2h^2}^{y_0} \frac{1}{\sqrt{t^3}} \int_0 3r \exp(-r^2)\ drdt$$

$$= \frac{3}{2} \int_{2h^2}^{y_o} \frac{1}{\sqrt{t^3}} \, dt = 3 \left[\frac{1}{h\sqrt{2}} - \frac{1}{\sqrt{y_o}} \right].$$

Similary there is a constant K such that

$$\int_{2h^2}^{y_o} \int_0 \frac{s^3}{8\sqrt{t^7}} \exp\left(- \frac{s^2}{4t}\right) ds \, dt = K \left[\frac{1}{h\sqrt{2}} - \frac{1}{\sqrt{y_o}} \right].$$

Consequently, the left hand side of (f) satisfies the relation

$$\int_0^{y_o - 2h^2} \int \left| \frac{\partial^3 e}{\partial x^3} (x_o \pm \eta - \xi, y_o - \gamma) \right| d\xi d\gamma \leq \frac{3+K}{\sqrt{\pi}} \left[\frac{1}{h\sqrt{2}} - \frac{1}{\sqrt{y_o}} \right].$$

Using the triangle inequality on (e), we have the relation

$$I_{21} \leq \int_0^{y_o - 2h^2} \int \frac{1}{h^2} \left| \int_0^h \int_0^\theta \int_0^\rho \frac{\partial^3 e}{\partial x^3} (x_o + \eta - \xi, y_o - \gamma) d\eta d\rho d\theta \right| d\xi d\gamma$$

$$+ \int_0^{y_o - 2h^2} \int \frac{1}{h^2} \left| \int_0^h \int_0^\theta \int_0^\rho \frac{\partial^3 e}{\partial x^3} (x_o - \eta - \xi, y_o - \gamma) \, d\eta d\rho d\theta \right| d\xi d\gamma$$

where in the last integral we have replaced η by $-\eta$ and ρ by $-\rho$. Taking the absolute values inside the integrals with respect to η, ρ and θ and then integrating first with respect to ξ and γ, we have from relation (i) that

$$I_{21} \leq \frac{1}{h^2} \int_0^h \int_0^\theta \int_0^\rho \frac{2(3+K)}{\sqrt{\pi}} \left[\frac{1}{h\sqrt{2}} - \frac{1}{\sqrt{y_o}} \right] d\eta d\rho d\theta.$$

Thus

$$I_{21} \leq \frac{1}{h^2} \frac{2(3+K)}{6\sqrt{\pi}} \, h^3 \left[\frac{1}{h\sqrt{2}} - \frac{1}{\sqrt{y_o}} \right] \qquad \text{or}$$

$$I_{21} = O(1) \text{ as } h \to 0^+.$$

Part 3. Taking the absolute values inside of the integrals on I_{22} and integrating first with respect to ξ and γ, we have that

$$I_{22} \leq \frac{1}{h^2} \int_0^{h^2} \int_0^{\rho} \int_0^{y_0 - 2h^2} \int \left| \frac{\partial^2 e}{\partial y^2} (x_0 - \xi, y_0 - \eta - \gamma) \right| d\xi \, d\gamma \, d\eta \, d\rho.$$

Making the change of variables $s = x_0 - \xi$, $t = y_0 - \eta - \gamma$,

$$\text{(j)} \quad \int_0^{y_0 - 2h^2} \int \left| \frac{\partial^2 e}{\partial y^2} (x_0 - \xi, y_0 - \eta - \gamma) \right| d\xi d\gamma = \int_{2h^2 - \eta}^{y_0 - \eta} \int \left| \frac{\partial^2 e}{\partial t^2} (s,t) \right| ds \, dt.$$

Now

$$\frac{\partial^2 e}{\partial t^2} (s,t) = \frac{1}{2\sqrt{\pi}} \left[\frac{3}{4\sqrt{t^5}} - \frac{3s^2}{4\sqrt{t^7}} + \frac{s^4}{16\sqrt{t^9}} \right] \exp\left(- \frac{s^2}{4t} \right),$$

so the left side of equation (j) is dominated by

$$\text{(k)} \quad \frac{1}{2\sqrt{\pi}} \int_{2h^2 - \eta}^{y_0 - \eta} \frac{1}{t^2} \int \left(\frac{3}{2} + 6 r^2 + 2r^2 \right) e^{-r^2} \, dr \, dt$$

where the transformation $r = \frac{s}{2\sqrt{t}}$ was made.

Letting C'' denote the inner integral of the right side of equation (k), we see that the left side of equation (j) is dominated by

$$\frac{C''}{2\sqrt{\pi}} \left[\frac{1}{2h^2 - \eta} - \frac{1}{y_0 - \eta} \right].$$

Thus

$$I_{22} \le \frac{1}{h^2} \int_0^{h^2} \int_0^{\rho} \frac{C''}{2\sqrt{\pi}} \left[\frac{1}{2h^2-\eta} - \frac{1}{y_0-\eta}\right] d\eta\, d\rho$$

$$= \frac{C''}{2h^2\sqrt{\pi}} \int_0^{h^2} \left[\ln\left(\frac{y_0-\rho}{2h^2-\rho}\right) - \ln\left(\frac{y_0}{2h^2}\right)\right] d\rho$$

$$\le \frac{C''}{2h^2\sqrt{\pi}} \int_0^{h^2} \left[\ln\left(\frac{y_0-h^2}{h^2}\right) - \ln\left(\frac{y_0}{2h^2}\right)\right] d\rho$$

$$= \frac{C''}{2\sqrt{\pi}} \ln\left(\frac{2y_0-2h^2}{y_0}\right) = O\ (1) \text{ as } h\to 0^+,$$

where we have used the fact that $\ln\left(\frac{y_0-\rho}{2h^2-\rho}\right)$ is an increasing function of ρ for ρ in $[0,h^2]$. That is,

$$\ln\left(\frac{y_0-\rho}{2h^2-\rho}\right) \le \ln\left(\frac{y_0-h^2}{h^2}\right) \text{ for } \rho \text{ in } [0,h^2].$$

From parts one, two, and three we have that $L(h) = O\ (1)$ as $h\to 0^+$ as claimed.

2.9 The previous lemma will now be applied to show that the particular function u satisfies our limit difference equation. If g has the constant value g_0 on R then

$$u(x,y) = -\iint_R g(\xi,\gamma)\ e(x-\xi,y-\gamma)\ d\xi d\gamma$$

$$= -g_0 \int_0^y \int e^{-r^2}\ drdt = -g_0 y$$

where we have made the change of variables $r = x-\xi, t = y-\gamma$. Thus for this function g, it is easy to see that

$$\lim_{h\to 0^+} \partial_h u(x,y) = g_o = g(x,y).$$

In a small neighborhood of the point (x_o, y_o), the continuous function g is essentially constant. If $g_o = g(x_o, y_o)$, then $g(x,y)$ is approximately equal to g_o in this neighborhood. It turns out that the values of g outside of this neighborhood appear in the expression for $\partial_h u(x_o, y_o)$ as quantities of small order, so again we will have

$$\lim_{h\to 0^+} \partial_h u(x_o, y_o) = g_o = g(x_o, y_o).$$

Using the special case of the constant function, we prove the following:

<u>Lemma</u>. Let g be continuous and bounded on R. Let u be defined as in Section 2.2. Then

$$\lim_{h\to 0^+} \partial_h u(x,y) = g(x,y)$$

for (x,y) in the interior of R.

<u>Proof</u>. Let (x_o, y_o) be any interior point of R and let $g_o = g(x_o, y_o)$. We show that

$$\lim_{h\to 0^+} \partial_h u(x_o, y_o) = g_o.$$

By the remarks preceding the statment of this lemma, it is sufficient to show that

$$\lim_{h\to 0^+} \partial_h v(x_o,y_o) = 0$$

where

$$\text{(a)} \quad v(x,y) = -\iint_R [g(\xi,\gamma)-g_o]e(x-\xi,y-\gamma)\, d\xi d\gamma.$$

Let $\epsilon > 0$, then there is a $\delta > 0$ so that $|g(x,y)-g_o| < \epsilon$ when $|x-x_o| < \delta$, $|y-y_o| < \delta$ and $(x,y) \epsilon R$. Since (x_o,y_o) is in the interior of R, we may suppose that δ is chosen so small that this last requirement is redundant. Let N denote this square neighborhood of the point (x_o,y_o).

Define the function g_1 so that it vanishes on N and is equal to $g(x,y) - g_o$ for every point (x,y) outside of N. Define g_2 so that it agrees with $g(x,y) - g_o$ on N and vanishes outside of N.

Note that $|g_2(x,y)| < \epsilon$ on R and that $g_1(x,y)+g_2(x,y) = g(x,y)-g_o$. Let

$$\text{(b)} \quad v_i(x,y) = -\iint_R g_i(\xi,\gamma)\, e\,(x-\xi,y-\gamma)\, d\xi d\gamma, i = 1,2,$$

then $v_1(x,y) + v_2(x,y) = v(x,y)$. Clearly

$$|\partial_h v_2(x_o,y_o)| \leq \epsilon \iint_R |\, \partial_h S(x_o,y_o)|\, d\xi d\gamma,$$

where $S(x,y) = e(x-\xi,y-\gamma)$. From the previous lemma we see that

$$\lim_{h\to 0^+} \sup \left| \partial_h v_2(x_o,y_o) \right| \le \epsilon\, C$$

for some constant $C > 0$.

If we could show that

$$\lim_{h\to 0^+} \sup \left| \partial_h v_1(x_o,y_o) \right| \le K\, \epsilon$$

for some fixed K and a certain choice of g_1, we could conclude that

$$\lim_{h\to 0^+} \sup \left| \partial_h v(x_o,y_o) \right| \le (K+C)\, \epsilon .$$

But this upper limit is independent of ϵ, hence,

$$\lim_{h\to 0^+} \partial_h v(x_o,y_o)$$

would exist and be zero. We now prove the first statement of this paragraph. The argument is presented in five parts.

<u>Part 1</u>. Using the change of variables $s = x_o - \xi, t = y_o - \gamma$ and the fact that $e(s,t) = 0$ for $t \le 0$, we have

$$h^2 \partial_h v_1(x_o,y_o) = -\int_{h^2}^{y_o}\!\!\int g_1\,(x_o - s, y_o - t)\,\{e(s+h,t) +$$

$$e(s-h,t) + e(s,t-h^2) - 3e(s,t)\}\; ds\, dt\; -$$

$$\int_0^{h^2}\!\!\int g_1(x_o - s, y_o - t)\,\{e(s+h,t) + e(s-h,t) - 3e(s,t)\}\; ds\, dt.$$

Let

$$I_1 = -\int_0^{h^2}\int g_1(x_o-s,y_o-t)\,\frac{\Delta_h e(s,t)}{h^2}\,dsdt,$$

$$I_2 = -\int_{h^2}^{y_o}\int g_1(x_o-s,y_o-t)\,\partial_h e(s,t)\,dsdt,$$

and

$$I_3 = \frac{1}{h^2}\int_0^{h^2}\int g_1(x_o-s,y_o-t)\,e(s,t)\,dsdt$$

where

$$\Delta_h\, e(s,t) = e(s+h,t) + e(s-h,t) - 2e(s,t).$$

From the equation for $h^2\partial_h\, v_1(x_o,y_o)$, we have $\partial_h v_1(x_o,y_o) = I_1 + I_2 + I_3$. We now deal with I_1.

Part 2. Choose h so that $0<2h<\delta$ and $2h^2<\delta$. Since g is bounded on R, g_1 is also bounded on R. Let M' be a bound on $|g_1(x,y)|$. Using the change of variables $r = s \pm h$, we find that

$$I_1 = -\frac{1}{h^2}\int_0^{h^2}\int e(r,t)\,\Delta_h\, g_1(x_o-r,y_o-t)\,drdt.$$

Let

$$\overline{I}_1 = \int e(r,t)\,\Delta_h\, g_1(x_o-r,y_o-t)\,dr, \text{ then}$$

$$|\overline{I}_1| \leq 4\,M'\int e(r,t)\,dr \leq 4\,M'.$$

Hence, $\bar{I}_1$ converges (uniformly in t for $0 \le t \le h^2 < \delta/2$). Therefore, we may write

$$\bar{I}_1 = \int_{-S}^{S} \Delta_h[g_1(x_o-r,y_o-t)]\ e(r,t)\,dr + H(S,t)$$

where $H(S,t) \to 0$ as $S \to \infty$ uniformly in t. Note that

$$\Delta_h\ g_1(x_o-r,y_o-t) = \Delta_h\ \{g_1(x_o-r,y_o-t)-[g(x_o-r,y_o-t)-g_o]\}$$

$$+ \Delta_h\ [g(x_o-r,y_o-t)-g_o].$$

Take S so large that $|H(s,t)| < \epsilon/2$. For this fixed S, $g(x_o-r,y_o-t)$ is uniformly continuous for r in $[-S,S]$. Consequently, for h sufficiently small $|\Delta_h[g(x_o-r,y_o-t)-g_o]| < \epsilon$.

But

$$|\ \Delta_h\ g_1(x_o-r,y_o-t)\ -[g(x_o-s,y_o-t)-g_o]\ | < 4\ \epsilon$$

from the definition of the approximating function g_1.

Hence

$$|\bar{I}_1| \le \int_{-S}^{S} 5\epsilon e(r,t)\ dr + \epsilon/2$$

$$\le 5\epsilon \int e(r,t)\ dr + \epsilon/2 \le \frac{11}{2}\ \epsilon.$$

Therefore

$$|I_1| \le \frac{1}{h^2} \int_0^{h^2} |\bar{I}_1|\ dt \le \frac{11}{2}\ \epsilon < 6\ \epsilon$$

(for h sufficiently small).

Part 3. Let

$$I_{21} = \int_{h^2}^{\delta}\int g_1(x_o-s,y_o-t)\ \partial_h\ e(x,t)\ dsdt$$

and let

$$I_{22} = \int_{\delta}^{y_o}\int g_1(x_o-s,y_o-t)\ \partial_h e(x,t)\ dsdt,$$

then $I_2 = -I_{21} - I_{22}$. In this part we consider only I_{21}. Since $g_1(x,y) = 0$ for $|x-x_o| < \delta$ and $|y-y_o| < \delta$ and since g_1 is bounded by M', we see that

$$|I_{21}| \le M' \int_{h^2}^{\delta}\int_{|s|\ge\delta} |\ \partial_h\ e(s,t)|dsdt.$$

From the proof of the lemma of Section 2.8 (see parts three and four) we find that

$$(c)\quad |I_{21}| \le \frac{M'}{h^2} \int_{h^2}^{\delta}\int_{|s|\ge\delta}\int_0^h\int_0^{\theta}\int_0^{\rho} \left|\ \frac{\partial^3 e}{\partial s^3}(s+\eta,t)\right| +$$

$$\left|\frac{\partial^3 e}{\partial s^3}(s-\eta,t)\ \right|\ d\eta d\rho d\theta dsdt + \frac{M'}{h^2}\int_{h^2}^{\delta}\int_{|s|\ge\delta}\int_0^{h^2}\int_0^{\rho}\left|\ \frac{\partial^2 e}{\partial t^2}(s,t-\eta)\right|d\eta d\rho dsdt.$$

But

$$\int_{|s|\ge\delta}\left|\frac{\partial^3 e}{\partial s^3}(s\pm\eta,t)\ \right|ds \le \frac{1}{2\sqrt{\pi}}\int_{|s|\ge\delta}\left\{\frac{3|s\pm\eta|}{4\sqrt{t^5}}\right. +$$

$$\frac{|s\pm\eta|^3}{8\sqrt{t^7}}\Big\} \exp\left[-\frac{(s\pm\eta)^2}{4t}\right] ds$$

$$\leq \frac{1}{2\sqrt{\pi}}\left[\int_{\delta} - \int^{-\delta}\right]\left\{\frac{3(s\pm\eta)}{4\sqrt{t^5}} + \frac{(s\pm\eta)^3}{8\sqrt{t^7}}\right\} \exp\left[-\frac{(s\pm\eta)}{4t}\right] ds$$

$$\leq \frac{1}{2t^2\sqrt{\pi}}\left[\int^{\frac{\delta\pm\eta}{2\sqrt{t}}} - \int_{\frac{-\delta\pm\eta}{2\sqrt{t}}}\right](3r + r^3)\exp(-r^2)\, dr$$

where we have substituted r for $\frac{s\pm\eta}{2\sqrt{t}}$.

Replacing r by $-r$ in the last integral and evaluating we obtain

$$\int_{|s|\geq\delta}\left|\frac{\partial^3 e}{\partial s^3}(s\pm\eta,t)\right| ds \leq \frac{1}{4\sqrt{\pi t^3}}\exp\left[-\frac{(\delta\pm\eta)^2}{4t}\right]$$

$$\left[2\frac{(\delta\pm\eta)^2}{4t} + 5\right] + \exp\left[-\frac{(\delta\mp\eta)^2}{4t}\right]\left[2\frac{(\delta\mp\eta)^2}{4t} + 5\right]$$

Since $0 \leq \eta \leq h < \delta/2$ (see the interval of integration for η), $\delta \pm \eta > 0$. Therefore

$$\lim_{t\to 0^+}\int_{|s|>\delta}\left|\frac{\partial^3 e}{\partial s^3}(s\pm\eta,t)\right| ds = 0$$

uniformly in η (δ fixed). Hence, there is a number $N > 0$ so that

$\int_{|s|>\delta}\left|\frac{\partial^3 e}{\partial s^3}(s\pm\eta,t)\right| ds \leq N$, for all η and t where $h^2 \leq t \leq \delta$ and

$$0 < \eta < h < \delta/2.$$

Performing the integration with respect to s we have

$$\text{(d)} \quad \frac{M'}{h^2}\int_{h^2}^{\delta}\int_{|s|\geq\delta}\int_0^h\int_0^\theta\int_0^\rho \left| \frac{\partial^3 e}{\partial s^3}(s-\eta,t)\right| d\eta\, d\rho\, d\theta\, ds\, dt$$

$$\leq \frac{M'}{h^2}\int_{h^2}^{\delta}\int_0^h\int_0^\theta\int_0^\rho 2N\, d\eta\, d\rho\, d\theta\, dt$$

$$= \frac{M'N}{3h^2}\; h^3(\delta-h^2) \to 0 \text{ as } h \to 0^+.$$

We show that the second part of the right side of (c) tends to zero as $h\to 0^+$. Again we will be repeating work done in the proof of the lemma of Section 2.9. We will need the following easily shown result:

For each nonnegative integer n, x^n erfc (x) is bounded on $[0,\infty]$ where erfc $(x) = \frac{2}{\sqrt{\pi}}\int_x e^{-r^2}\,dr$.

For $t > \eta$

$$\text{(e)} \quad \int_{|s|\geq\delta}\left| \frac{\partial^2 e}{\partial t^2}(s,t-\eta)\right| ds \leq \frac{1}{\sqrt{\pi}}\frac{1}{(t-\eta)^2}\int_{\frac{\delta}{2\sqrt{t-\eta}}}\left(\frac{3}{2}+6r^2+2r^4\right)e^{-r^2}\,dr$$

where we have substituted r for $\frac{s}{2\sqrt{t-\eta}}$. For $t \leq \eta$ and $s \neq 0$ the function vanishes identically and so does

$$\frac{\partial^2 e}{\partial t^2}.$$

In either case integrating the right side of equation (e)

$$\int_{|s|\geq\delta} \left|\frac{\partial^2 e}{\partial t^2}(s,t-\eta)\right| ds \leq \frac{1}{(t-\eta)^2}\Big[3\operatorname{erfc}\left(\frac{\delta}{2\sqrt{t-\eta}}\right) +$$

$$\frac{9}{2\sqrt{\pi}}\ \frac{\delta}{2\sqrt{t-\eta}}\ \exp\left(-\frac{\delta^2}{4(t-\eta)}\right) + \frac{\delta^3}{8\sqrt{\pi}(t-\eta)^3}\ \exp\left(-\frac{\delta^2}{4(t-\eta)}\right)\Big].$$

Thus

$$\int_{|s|\geq\delta} \left| \frac{\partial^2 e}{\partial t^2}(s,t-\eta) \right| ds$$

is bounded for η and t such that $0 \leq \eta \leq h$ and $h^2 \leq t \leq \delta$. Let P be a bound. Interchanging the order of integration we obtain

$$\frac{M'}{h^2}\int_{h^2}^{\delta}\int_0^{\rho} \left| \frac{\partial^2 e}{\partial t^2}(s,t-\eta) \right| d\eta d\rho ds dt \leq$$

$$\frac{M'}{h^2}\int_{h^2}^{\delta}\int_0^{h^2}\int_0^{\rho} P\ d\eta d\rho dt = \frac{M'P}{2h^2}\ h^4\ (\delta-h^2) \to 0$$

as $h \to 0^+$. Combining this last result with relations (c) and (d) we have that $I_{21} \to 0$ as $h \to 0^+$.

Part 4. We consider in this part the integral I_{22}. By a simple change of variables

$$I_{22} = \int_{\delta}^{y_o}\int_0 [g_1(x_o-s,y_o-t) + g_1(x_o+s,y_o-t)]\ \partial_h e(x,t)\ dsdt.$$

Since $t \geq \delta > 0$ and 0 and $0 < h < \delta/2$, Taylor's Theorem implies that there is a number α_+ in $(0,1)$ such that

$$e(s+h,t) = e(s,t) + \frac{\partial e}{\partial s}(s,t)\,h + \frac{1}{2}\frac{\partial^2 e}{\partial s^2}(s+\alpha_+ h,t)h^2.$$

Similarly, there are numbers α_- in $(0,1)$ and θ in $(0,1)$ such that

$$e(s-h,t) = e(s,t) - \frac{\partial e}{\partial s}(s,t)\,h + \frac{1}{2}\frac{\partial^2 e}{\partial s^2}(s-\alpha_- h,t)h^2$$

and

$$e(s,t-h^2) = e(s,t) - \frac{\partial e}{\partial t}(s,t-\theta h^2)\,h^2.$$

Therefore

$$\partial_h e(s,t) = \frac{1}{2}\left[\frac{\partial^2 e}{\partial t^2}(s+\alpha_+ h,t) + \frac{\partial^2 e}{\partial s^2}(s-\alpha_- h,t)\right] - \frac{\partial e}{\partial t}(s,t-\theta h^2).$$

Now $\delta \le t \le y_0$ and $\frac{\partial^2 e}{\partial s^2}(s,t)$ is a decreasing function of s for s sufficently large. In fact we find by elementary means that for $s > \sqrt{6y_0}$ the function $\frac{\partial^2 e}{\partial s^2}(s,t)$ is decreasing in s for all t in $[\delta,y_0]$. For s in $[0,\sqrt{6y_0}]$ and t in $[\delta,y_0]$, $\frac{\partial^2 e}{\partial s^2}(s,t)$ is continuous and hence bounded. Thus, $\frac{\partial^2 e}{\partial s^2} = (s+\alpha_+ h,t)$ is dominated by an integrable function; in fact one dominant function is given by the least upper bound of $\frac{\partial^2 e}{\partial s^2}(s,t)$ on $[0,\sqrt{6y_0}]x[\delta,y_0]$ and $\frac{\partial^2 e}{\partial s^2}$ otherwise. Similarly, $\frac{\partial^2 e}{\partial s^2}(s-\alpha_- h,t)$ is dominated by an integrable function.

Clearly

$$\left|\frac{\partial e}{\partial t}(s,t-\theta h^2)\right| \le \frac{1}{\sqrt{\pi}}\left[\frac{s^2}{8\sqrt{(t-\delta^2/4)^5}} + \frac{1}{4\sqrt{(t-\delta^2/4)^3}}\right]\exp\left(-\frac{s^2}{4y_0}\right)$$

for $\delta \le t \le y_o$ and $0 < \theta < 1$ (recall $h^2 < \delta/2$). From this inequality we see that $\frac{\partial e}{\partial t}(s,t-\theta h^2)$ is dominated by an integrable function. Hence $\partial_h e(s,t)$ is dominated by an integrable function for s in $[0,\infty]$ and t in $[\delta,y_o]$. Since

$$\lim_{h\to 0^+} \partial_h e(s,t) = 0 \text{ for } t \ge \delta,\ I_{22} \to 0 \text{ as } h\to 0^+ \text{ by the}$$

Dominated Convergence Theorem.

Part 5. We show now that $I_3 \to 0$ as $h\to 0^+$. Recall that

$$g_1(x_o-s,y_o-t) = 0 \text{ for } |s| < \delta \text{ and } |t| < \delta.$$

Since $h^2 < \delta/2$,

$$I_3 = \frac{1}{h^2}\int_0^{h^2}\int_{|s|\ge\delta} g_1(x_o-s,y_o-t)\ e(s,t)\ dsdt$$

so

$$|I_3| \le \frac{M'}{h^2}\int_{|s|\ge\delta} e(s,t)\ dsdt = \frac{2M'}{\sqrt{\pi}h^2}\int_0^{h^2}\int_{\frac{\delta}{2\sqrt{t}}} \exp(-r^2)\ drdt$$

$$= \frac{M'}{h^2}\int_0^{h^2} \operatorname{erfc}\left(\frac{\delta}{2\sqrt{t}}\right) dt$$

where we have made the change of variable $r = \frac{s}{2\sqrt{t}}$. But $\operatorname{erfc}\left(\frac{\delta}{2\sqrt{t}}\right)$ is an increasing function of t. Hence

$$|I_3| \le \frac{M'}{h^2}\operatorname{erfc}\left(\frac{\delta}{2h}\right)\int_0^{h^2} dt = M' \operatorname{erfc}\left(\frac{\delta}{2h}\right) \to 0 \text{ as } h\to 0^+.$$

Combining parts one through five we have

$$\limsup_{h \to 0^+} \left| \partial_h v_1(x_o,y_o) \right| \le 6 \epsilon$$

so by the argument preceding Part 1

$$\lim_{h \to 0^+} \partial_h v(x_o,y_o) = 0$$

as was to be shown. The lemma is therefore proved.

2.13 We are now in a position to prove the main theorem in this essay, namely, the theorem stated in Section 2.2.

Proof of Theorem. Let $y_o > 0$ and $y_o < T$. Let N, N^*, and $\overline{N}$ be defined as before (δ is assumed so small that $\overline{N}$ is a subset of the interior of R). The function $w-u$ is continuous on $\overline{N}$. By the lemma just proved and the lemma of Section 2.3

$$\partial_h \left[w(x,y) - u(x,y) \right] = 0$$

on N^*. By the lemma of Section 2.7

$$\partial \left[w(x,y) - u(x,y) \right] = 0.$$

Since w is a solution to $\partial W(x,y) = g(x,y)$ it follows that u has the necessary derivatives and is also a solution.

BIBLIOGRAPHY

1. Koshlyakov, Smirnov, and Gliner, Differential Equations of Mathematical Physics, Interscience Publishers, 1964, pp. 471-475.

2. Herr, G.H., The Inhomogeneous Heat Equation, Masters Thesis, Georgia Institute of Technology, 1960.

3. Koshlyakov, Smirnov, and Gliner, pp. 479-484.

4. Hadamard, J., "Équations du Type Parabolique Dépourvues de Solutions," Journal of Rational Mechanics and Analysis, 3, pp. 3-12.

5. Hartman and Wintner, "On the Inverse of the Parabolic Differential Operator $\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}$," American Journal of Mathematics, 75, pp. 598-610.

6. Hartman and Wintner, p. 602.

7. Cook, F. L., A Condition Concerning the Existence of Continuous Solutions to Poisson's Equation, Masters Thesis, Georgia Institute of Technology, 1963.

8. Hartman and Wintner, pp. 602-603.

9. Koshlyakov, Smirnov, and Gliner, p. 477.

10. Epstein, B., Partial Differential Equations, McGraw-Hill Book Company, 1962, pp. 226-231.

11. Churchill, R. V., Operational Mathematics, McGraw-Hill Book Company, 1958, pp. 199-206.

12. Hartman and Wintner, pp. 606-607.