Dynamics of Cyclic Feedback Systems

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Abstract

The dynamics of cyclic feedback systems are described. The emphasis is both in showing the diversity of possible dynamics in these sytems and in showing that there is a underlying dynamic structure possessed by all these systems. In particular, for the special class of monotone cyclic feedback systems, the dynamics is fairly simple; the recurrent sets can only consist of fixed points or periodic orbits and in many cases can be shown to be Morse-Smale. This is contrasted with the general cyclic feedback systems for which chaotic dynamics can occur.

The general properties which large subclasses of these systems have in common include periodic orbits and a semi–conjugacy onto a simple,

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but non-trivial, model dynamical system. To describe all systems simultaneously, a purely topological description of the invariant sets is introduced.

1 Introduction

Cyclic feedback systems ($\mathcal{CFS}$) are systems of ordinary differential equations of the form

$$\dot{x}_i = f_i(x_i, x_{i-1}) \quad i = 1, \dots, n \quad (x_0 = x_n) \tag{1}$$

where for all $\zeta \neq 0$

$$\delta_i f_i(0, \zeta)\zeta > 0 \tag{2}$$

for some $\delta_i = \pm 1$. To simplify the notation we shall sometimes write

$$\dot{x} = f(x).$$

The cyclicity of these systems is obvious from (1). The constraint (2) is called a feedback condition and, in particular, is a *positive feedback* if $\delta_i = 1$ and a *negative feedback* if $\delta_i = -1$. We make a simple observation that via a change of variables of the form $x_i \to \lambda_i x_i$ where $\lambda_i = \pm 1$ it is possible to assume *without loss of generality* that

$$\begin{aligned} \delta_1 &= \Delta = \pm 1 \\ \delta_1 &= 1, \quad i = 2, \dots, n \end{aligned}$$

where $\Delta = \delta_1 \delta_2 \dots \delta_n$ is expressed in the terms of the original feedback conditions.

This suggests, and indeed it is the case, that Δ is one of the defining characteristics of these systems. With this in mind we let $\mathcal{CFS}^+$ and $\mathcal{CFS}^-$ denote the class of $\mathcal{CFS}$ when $\Delta = +1$ or $\Delta = -1$, respectively. Another important quantity, at least as far as the global dynamics is concerned, is the parity of the dimension of the system n. We shall employ the notation $\mathcal{CFS}_{odd}$ and $\mathcal{CFS}_{even}$ to denote $\mathcal{CFS}$ with n odd or even.

Systems of this form appear in a variety of applications and, also, are of mathematical interest in their own right.

The idea of using a feedback in models of cell mechanisms goes back to the paper of Jacob and Monod [16]. They modeled genetic regulatory mechanism in bacteria using feedback systems. To mention other models, Morales and McKay [25] used cyclic feedback systems to model metabolic pathways in bacteria and Weiss and Kavanau [31] used the models from this class to describe the control mechanism of the growth of cells. For a more comprehensive list of models we refer the reader to the paper of Hastings et.al. [10].

We would like to remark that these models were built to explain how the cells are able to stabilize certain 0 in ever changing 0. The analogy with a control theory comes to mind 0. It is well known that a negative feedback has stabilizing properties and these were explored in the papers [16, 25, 31]. However, it is also well known that if one imposes too large a negative feedback, the system usually starts to 0.

In recent years there was an increased effort to understand more about 0 phenomena in general. We are keenly aware of the fact that no living organism is in the state of 0 and most processes around us are periodic (or even more complicated). In neural networks, if one considers a ring architecture of the neurons where the neurons are connected to each other in a cyclic fashion, then one naturally arrives to the description of the dynamics using $\mathcal{CFS}$. This architecture 0 attention in recent years for its ability to support stable oscillations, which may be viewed as stored spatio-temporal information. As an example, we mention the work of Atyia and Baldi [2], who used the models from the class of $\mathcal{CFS}$ to explore the so called "labeling hypotheses", which is related to the question how the brain processes the visual information.

From the mathematical point of view $\mathcal{CFS}$ are interesting, because in their study one comes across many important ideas which have been used in last decade in the dynamical systems.

There is a direct link to a scalar delay-differential equation. Let us consider the equation of the form

$$\dot{x}(t) = f(x(t), x(t-1)).$$

If we divide the interval $[-1, 0]$ into n equal subintervals and then use the linear approximation of the solution on each subinterval, we obtain a cyclic feedback system

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_n) \\ \dot{x}_i &= \frac{1}{n}(x_{i-1} - x_i). \end{aligned}$$

In the next section we define a discrete Liapunov function for $\mathcal{CFS}$. Such a function was used to study scalar parabolic equations [1, 3, 11, 20] and (not 0) scalar delay-differential equations with negative feedback [17].

There is a important subclass of $\mathcal{CFS}$, called *monotone cyclic feedback systems* ($\mathcal{MCFS}$), which are obtained by imposing the additional assumption

$$\delta_i \frac{\partial f_i(\eta,\zeta)}{\partial \zeta} > 0$$

for all $(\eta,\zeta) \in \mathbf{R}^2$. We want to remark that if $\Delta = 1$ in a $\mathcal{MCFS}$, then the flow, given by (1), generates a *monotone dynamical system.* These systems have been studied extensively over the last decade by many authors, for instance by H. Matano [19], M. Hirsch [12, 13, 14, 15], H. Smith [27] and others. An important property of these systems is that almost all trajectories converge to a fixed point (Hirsch [13, 14]), which was used in the applications to neural networks.

The intent of this article is to demonstrate both that $\mathcal{CFS}$ display a wide variety of dynamics and that these systems share important common dynamic properties. We begin by making several assumptions.

A1 $f_i \in C^1(\mathbf{R}^2, \mathbf{R}), \qquad i = 1, \ldots, n.$

A2

$$\delta_i \frac{\partial f_i(\eta,\zeta)}{\partial \zeta}|_{(0,0)} > 0 \quad i = 1, \ldots, n$$

A3 *The exists a global compact attractor $\mathcal{A}$. In other words, there exists a compact set $\mathcal{A}$ such that for every $R >> 1$, the omega limit set of the ball of radius R about the origin is $\mathcal{A}$, i.e.*

$$\omega(B_R(0)) = \mathcal{A}.$$

The hypotheses (**A1**) - (**A3**) shall be assumed throughout this paper.

With these assumptions we are ready to discuss the dynamics of $\mathcal{CFS}$. In the next section we shall define a discrete Liapunov function which is at the heart of most of the results described in this article. In Section 3, the concept of a Morse decomposition will be introduced to provide an abstract framework with which we will deal with the information provided by the Liapunov function. This is followed by Section 4 in which the Conley indices for the Morse sets are described. In Section 5 we discuss the dynamics of $\mathcal{MCFS}$. As will be made clear, the dynamics for these systems is surprisingly simple. This will be contrasted with the general $\mathcal{CFS}$, for which chaotic dynamics can occur. It is hoped that by the end of this section the reader is aware of the diversity of the dynamics possible in $\mathcal{CFS}$. The final two sections strive to demonstrate that there are common factors within this diversity. For example, in Section 6, reasonably general conditions which 0 the existence of periodic orbits will be given. Even more generally, it will be shown that some Morse sets must have the topology of at least a circle. Finally, in Section 7, the global dynamics will be discussed. In particular, it will be shown that the dynamic structure of $\mathcal{CFS}$ can be mapped onto the dynamic structures of a simple (but not trivial) model dynamical system.

2 The Liapunov Function

Observe that the subspaces

$$Y_i := \{x \in \mathbf{R} \mid x_i = 0\} \setminus \{0\} \subset \mathbf{R}^n$$

are sections to the flow generated by $\mathcal{CFS}$. In particular, on Y_i,

$$\dot{x}_i = f_i(x_i, x_{i-1}) = f_i(0, x_i).$$

Thus, by the feedback condition

$$\dot{x}_i \quad \begin{cases} > 0 & \text{if } \delta_i = 1,\ x_{i-1} > 0 \text{ or } \delta_i = -1,\ x_{i-1} < 0 \\ < 0 & \text{if } \delta_i = 1,\ x_{i-1} < 0 \text{ or } \delta_i = -1,\ x_{i-1} > 0. \end{cases} \tag{3}$$

The complement of these regions $\mathbf{R}^n \setminus \cup Y_i$ can be expressed as

$$\bigcup Q(\sigma_1, \ldots, \sigma_n) \bigcup \{0\}$$

where for $\sigma_i = \pm 1$,

$$Q(\sigma_1, \ldots, \sigma_n) := \{x \in \mathbf{R}^n \mid \sigma_i x_i > 0\}.$$

The Qs are of course open cones which correspond to "orthants" in $\mathbf{R}^n$. Observe that (3) indicates how orbits move from one Q to another. Now define $\mathcal{N} : \bigcup Q(\sigma_1, \ldots, \sigma_n) \to \mathbf{Z}$ by

$$\mathcal{N}(x) = cardinality\{i \mid \delta_i x_i x_{i-1} < 0\}. \tag{4}$$

Clearly, $\mathcal{N}$ is constant on each Q, and hence, is a continuous function. Finally, let

$$X_i = \{x \in R^n \mid x_i = 0,\ \delta_{i+1}\delta_i x_{i+1} x_{i-1} < 0\}.$$

One can check, by examining the flow on X_i, that it is possible to extend the domain of definition of $\mathcal{N}$ to

$$X := (\cup X_i)) \cup (\cup Q(\sigma_1, \dots, \sigma_n))$$

while preserving the continuity of $\mathcal{N}$ (see Figure 1). Furthermore, except for the origin, the subsets of $\mathbf{R}^n$ where $\mathcal{N}$ is not defined, i.e. $\mathbf{R}^n \setminus (X \cup \{0\})$ are precisely the boundaries between the regions on which $\mathcal{N}$ assumes different values. Thus, $\mathcal{N}$ is left undefined on the complement of X.

The following result justifies the name Liapunov functional for N.

Proposition 2.1 (Mallet-Paret and Smith [18]) *Let $x(t)$ be a nontrivial solution of (1). Then*

a. $x(t) \in X$ except at isolated values of t.

b. $\mathcal{N}(x(t))$ is locally constant for $x(t) \in X$.

c. if $x(t_0) \notin X$ then $\mathcal{N}(x(t_0^+)) < \mathcal{N}(x(t_0^-))$, where $t_0^+ > t_0$ and $t_0^- < t_0$.

d. if $x(t) \in X$ then $(x_i(t), x_{i-1}(t)) \neq (0,0)$ for $1 \leq i \leq n$.

Remark 2.2 An immediate implication of Proposition 2.1 is that any recurrent dynamics which occurs in a $\mathcal{CFS}$ must be contained in the open sets of X on which $\mathcal{N}$ is constant.

One final comment; observe that for those $x \in R^n$ with each $x_i \neq 0,\ 1 \leq i \leq n$

$$(-1)^{\mathcal{N}(x)} = sign \prod_{i=1}^{n} \delta_i x_i x_{i-1} = \prod_{i=1}^{n} \delta_i = \delta_1 = \Delta \tag{5}$$

so $\mathcal{N}$ takes only odd values if $\Delta = -1$ and only even values if $\Delta = 1$.

Figure 1: ($\Delta = -1$ and $n = 3$) $\mathcal{N} = 3$ on the two orthants $Q(1,-1,1)$ and $Q(-1,1,-1)$, and $\mathcal{N} = 1$ elsewhere. Observe that $\mathcal{N}$ is not defined only on $\partial(Q(1,-1,1) \cup Q(-1,1,-1))$. On these partial hyperplanes the vector field points from the open sets where $\mathcal{N} = 3$ to the open set where $\mathcal{N} = 1$. Finally, on the set where $\mathcal{N} = 1$ the vector field on the faces X_i indicate the possibility of trajectories passing through the orthant in the following order $Q(1,1,1) \to Q(-1,1,1) \to Q(-1,-1,1) \to Q(-1,-1,-1) \to Q(1,-1,-1) \to Q(1,1,-1) \to Q(1,1,1)$.

3 Morse Decompositions

Let $\varphi : \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ denote the flow generated by the $\mathcal{CFS}$ and let $\alpha(Y)$ and $\omega(Y)$ denote the alpha and omega limit sets of Y under φ. Recall (**A3**) that $\mathcal{A}$ denotes the global attractor for φ, and hence, is a compact invariant set.

Definition 3.1 A *Morse decomposition* of $\mathcal{A}$ is a finite collection of mutually disjoint compact invariant subsets of $\mathcal{A}$

$$\mathcal{M}(\mathcal{A}) := \{M(p) \mid p \in (\mathcal{P}, >)\}$$

indexed by a partially ordered set $\mathcal{P}$ such that if $x \in \mathcal{A} \setminus \cup_{p\in\mathcal{P}} M(p)$, then there exists $p > q$ such that $\alpha(x) \subset M(p)$ and $\omega(x) \subset M(q)$.

The individual invariant subsets $M(p)$ are called *Morse sets*, and the remaining portion, $\mathcal{A} \setminus \bigcup M(p)$, is referred to as the set of *connecting orbits.*

Remark 3.2 Observe that the existence of a partial ordering on the Morse decomposition implies that the recurrent dynamics of $\mathcal{A}$ must lie entirely in the Morse sets.

The similarities between Remarks 2.2 and 3.2 suggest that $\mathcal{N}$ can be used to define a Morse decomposition for $\mathcal{CFS}$. For example, if $\Delta = -1$ one could set

$$M(p) := \{x \in X \mid \mathcal{N}(\varphi(t,x)) = 2p+1 \ \forall t \in \mathbf{R}\}, \quad p = 0, 1, 2, \ldots$$

This almost works as a definition of a Morse decomposition. The problem is the origin $0 \in \mathbf{R}^n$. Observe that 0 is a fixed point for any $\mathcal{CFS}$ and $0 \in \mathcal{A}$. Therefore, 0 must lie in a Morse set. However, $0 \notin X$, and hence, cannot lie in any set of the form $M(p)$ as defined above.

Deciding how to include the origin into the Morse decomposition requires an understanding of the spectral properties of $Df(0)$. These spectral properties are at the heart of many of the results associated with this Liapunov function, and in fact for similar Liapunov functions. For a full account the reader is referred to [5, 7, 9, 17, 18]. For our purposes it is sufficient to state the following definitions and results.

Let J represent the number of the eigenvalues with positive real part of the matrix $Df(0)$.

Assume $0 \leq i < n$ and that $J > 0$.

$$\text{If } \Delta = -1 \text{ and } n \text{ is odd, then} \quad P = \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i & \text{if j=2i,2i+1.} \end{cases}$$

$$\text{If } \Delta = 1 \text{ and } n \text{ is odd, then} \quad P = \begin{cases} \frac{n+1}{2} & \text{if } J = n \\ i & \text{if j=2i-1,2i.} \end{cases}$$

$$\text{If } \Delta = -1 \text{ and } n \text{ is even, then} \quad P = \begin{cases} \frac{n}{2} & \text{if } J = n \\ i & \text{if j=2i,2i+1.} \end{cases}$$

$$\text{If } \Delta = 1 \text{ and } n \text{ is odd, then} \quad P = \begin{cases} \frac{n+2}{2} & \text{if } J = n \\ i & \text{if j=2i-1,2i.} \end{cases}$$

As a preliminary step in the construction of the Morse sets we make the following definitions.

$$\begin{aligned} \text{If } \Delta = -1 \text{ then } \tilde{M}(p) &= \{x(t) : \mathcal{N}(x(t)) = 2p+1 \text{ for all } t\}. \\ \text{If } \Delta = 1 \text{ then } \tilde{M}(p) &= \{x(t) : \mathcal{N}(x(t)) = 2p \text{ for all } t\}. \end{aligned}$$

Now, for $p = 0, \dots, P-1$ set

$$M(p) = \tilde{M}(p)$$

and

$$M(P) = \{0\} \cup \bigcup_{i \geq P} \tilde{M}(p).$$

Proposition 3.3 ([9, Proposition 3.4]) *The collection*

$$\mathcal{M}(\mathcal{A}) = \{M(p) \mid p = 0, \dots, P\}$$

is a Morse decomposition of the global attractor $\mathcal{A}$ with an admissible ordering $p > p-1$.

4 The Conley Indices

As the reader may have realized by now the existence of Morse decompositions is equivalent to the existence of a discrete Liapunov function. The primary reason for insisting on using the framework of Morse decompositions is that it permits us to use the algebraic machinery associated with the Conley index [4, 24, 26, 28].

Recall that the Conley index of an isolated invariant set S is the homotopy type of a pointed topological space, i.e.

$$h(S) \sim (A, a_0).$$

For our purposes it is more convenient to use the cohomological Conley index (which we shall refer to from now on as the Conley index)

$$CH^*(S) := H^*(A, a_0)$$

where H^* denotes Alexander–Spanier cohomology [21, 29].

Since each Morse set is an isolated invariant set, it has a Conley index. The following proposition indicates what these indices are.

Proposition 4.1 *The cohomological Conley indices of the Morse sets for the $\mathcal{CFS}$ are as follows.*

$$CH^k(M(P)) = \begin{cases} \mathbf{Z} & k = 2P \\ 0 & otherwise \end{cases}$$

If $J < n$ and $\Delta = 1$, then

$$\begin{aligned} CH^k(M(0)) &= \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & k = 0 \\ 0 & otherwise \end{cases} \\ CH^k(M(p)) &= \begin{cases} \mathbf{Z} & k = 2p, 2p+1 \\ 0 & otherwise \end{cases} \qquad p = 1, \ldots, P-1. \end{aligned}$$

If $J < n$ and $\Delta = -1$, then

$$CH^k(M(p)) = \begin{cases} \mathbf{Z} & k = 2p, 2p+1 \\ 0 & otherwise \end{cases} \qquad p = 0, \ldots, P-1.$$

If $J = n$, then the indices of the Morse sets $M(p)$, $p \neq P-1$ are as above. The remaining index is as follows.
If $\Delta = 1$ and n is even or if $\Delta = -1$ and n is odd, then

$$CH^k(M(P-1)) = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & k = n-1 \\ 0 & otherwise. \end{cases}$$

If $\Delta = 1$ and n is odd or if $\Delta = -1$ and n is even, then

$$CH^k(M(P-1)) = \begin{cases} \mathbf{Z} & k = n-2, n-1 \\ 0 & otherwise. \end{cases}$$

Though the details of the proof are rather complicated, the idea is fairly simple. Let

$$L(s) = \begin{pmatrix} s & 0 & 0 & \cdots & \pm 1 \\ 1 & s & 0 & \cdots & 0 \\ 0 & 1 & s & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & s \end{pmatrix}$$

and let

$$h(x) = \begin{pmatrix} x_1^3 \\ \vdots \\ x_n^3 \end{pmatrix}$$

Observe that

$$\dot{x} = L(s)x + h(x) \tag{6}$$

is a $\mathcal{CFS}$ where Δ is determined by the upper right hand entry of $L(s)$. In analogy with the definition of J as the number of eigenvalues with a positive real part of $Df(0)$, we denote by $J(s)$ the number of eigenvalues with a positive real part of $L(s)$. We defined P to be the number of Morse sets in a Morse 0 for the system $\dot{x} = f(x)$ as a function of J, n and Δ. Let us denote $P_f := P$. Let $P(s)$ be defined in the same way using $J(s)$, n and Δ for the system (6). Varying the parameter s allows one to vary $J(s)$ from 0 to n. Furthermore, the nonlinearity h guarantees the existence of a global attractor. For (6) it is reasonably straightforward to compute the index of the Morse sets as a function of s. Now recall that as long as the isolating neighborhoods are preserved, the Conley index remains unchanged. Therefore, the strategy is as follows. For the $\mathcal{CFS}$ of interest,

$$\dot{x} = f(x),$$

one determines Δ and chooses $L(s)$ accordingly. Next one computes P_f from the spectrum of $Df(0)$, and then chooses s such that $P(s) = P_f$. Finally, and this is the most technical part, one creates a homotopy from $L(s) + h(x)$ to $f(x)$ which does not essentially change the spectral properties of the linearized operator at the origin and which preserves the existence of a global attractor.

5 The Range of Dynamics

In this section we will discuss the range of dynamics which $\mathcal{CFS}$ can exhibit. To do this we begin by describing the results for $\mathcal{MCFS}$ and then contrasting these results with those of non–monotone systems.

Theorem 5.1 (Mallet-Paret and Smith, [18]) *Let us consider a $\mathcal{MCFS}$ in R^n. Consider any point x and its omega limit set $\omega(x)$. Then $\omega(x)$ is one of the following:*

i) a fixed point

ii) a limit cycle

iii) a set $H = E \cup C$ where E is set of equilibria and C is the set of connecting orbits between the equilibria in E.

The main tool used in the proof of this result is the Liapunov function $\mathcal{N}$ defined in Section 2. However, for $\mathcal{MCFS}$, the function $\mathcal{N}$ is non–increasing along the difference $y(t) := \tilde{x}(t) - x(t)$ of any two solutions $\tilde{x}(t)$ and $x(t)$ and along $y(t) := \dot{x}(t)$ for any solution $x(t)$. This can be used to examine the structure of trajectories in the neighborhood of a periodic orbit.

The discrete Liapunov function can be also used to show transversality of the intersection of the stable and unstable manifolds of the critical elements i.e. fixed points and periodic orbits. For the scalar parabolic equation this was realized by Henry [11] and Angenent [1] using a zero number as a Liapunov function. Fusco and Oliva [6] used the function $\mathcal{N}$ to show that for $\mathcal{MCFS}$ with $\Delta = 1$ stable and unstable manifolds of two critical elements intersect 1 provided at least one of them is a periodic orbit.

These results show that the dynamics of $\mathcal{MCFS}$ is 0 simple taking into account the fact that these are high dimensional dynamical systems. A natural question arises, whether the dynamics of a general $\mathcal{CFS}$ is also as simple.

The answer is negative.

Gedeon [8] has constructed a class of $\mathcal{CFS}$ which exhibits a chaotic 0. We proceed to describe these results.

Let us consider the following $\mathcal{MCFS}$ with negative feedback ($\Delta = -1$)

$$\begin{aligned} \dot{x}_1 &= -a_1 x_1 - b_1 f(x_3) \\ \dot{x}_2 &= -a_2 x_2 + b_2 x_1 \\ \dot{x}_3 &= -a_3 x_3 + b_3 x_2 \end{aligned} \tag{7}$$

where f is a monotone C^1 function satisfying the feedback condition

$$x f(x) > 0 \qquad \text{if } x \neq 0.$$

We also assume without loss of generality that $f'(0) = 1$. It can be shown, that if one fixes the 0 a_1, a_2, a_3, b_2, b_3 then, under some additional conditions, there is a value $b_1^* = b_1(a_1, a_2, a_3, b_2, b_3)$ at which the origin undergoes a subcritical Hopf bifurcation. Thus for the value of b_1 such that $b_1 - b_1^* < 0$,

$|b_1 - b_1^*| << 1$ the system (7) admits a hyperbolic periodic orbit γ. It also can be shown that, if $f \in C^3$, $f''(0) = 0$ and $f'''(0) > 0$, then γ has a one-dimensional unstable manifold which for us will mean that there is one Floquet multiplier with absolute value bigger than one.

We shall assume from now that the hyperbolic periodic orbit γ with one dimensional unstable manifold of the system (7) is given. Let $[\gamma(t)]_3$ denote the third coordinate of the point $\gamma(t)$.

Definition 5.2 Given a periodic orbit γ, let $M = (M_1, M_2, M_3) \in \gamma$ such that $M_3 = \max_{0 \le t \le \bar{T}} [\gamma(t)]_3$, where $\bar{T}$ is the minimal period of $\gamma(t)$, be a point with maximal value of the third coordinate on γ.

Let us consider the following class of nonlinearities (Figure 2)

$$\begin{array}{lcl} g(x) & = & f(x) \text{ if } x \in (-\infty, M_3 + \delta] \\ g'(x) & < & 0,\ 0 < g(x) \le L \text{ if } x \in (M_3 + \delta + \eta, \infty) \\ g(x) & & \text{has a unique maximum in } y \in (M_3 + \delta, M_3 + \delta + \eta] \\ & & \text{with } g(y) < f(y) \end{array} \tag{8}$$

Observe, that there are three constants δ, η, L in this definition. We will assume that $0 < L < f(M_3)$ so that the second line of the definition makes sense. We will take $\delta, \eta > 0$.

Note, that every function g from this class coincides with the function f in the range of γ and therefore γ is a periodic orbit of the system

$$\begin{array}{lcl} \dot{x}_1 & = & -a_1 x_1 - b_1 g(x_3) \\ \dot{x}_2 & = & -a_2 x_2 + b_2 x_1 \\ \dot{x}_3 & = & -a_3 x_3 + b_3 x_2. \end{array} \tag{9}$$

We denote by Σ the space of biinfinite sequences of 0s and 1s and by σ a shift map defined by

$$\sigma(\omega)_i = \omega_{i-1}$$

for $\omega \in \Sigma$ where ω_i is the i-th entry in ω.

Now we follow the following strategy to show the existence of complicated dynamics in $\mathcal{CFS}$. We show that there is a nonlinearity g of the type (8) such that the Poincarè map Π_g corresponding to the periodic orbit γ for the system (9) admits the intersection of the stable and unstable manifolds. This is done in the following theorem.

Figure 2: The graph of the function $g_{\delta,\eta}(x)$ has to lie in a L-shaped region, which is shown here for two different values of δ, η.

Theorem 5.3 (Gedeon [8]) *Assume that the system (7) admits a hyperbolic periodic orbit γ with one-dimensional unstable manifold. Fix L such that $0 < L < f(M_n)$ and choose a two dimensional family $\mathcal{G} : (\delta, \eta) \to g_{\delta,\eta}$ of functions of the form (8).*

Fix any one dimensional family $\mathcal{F} \subset \mathcal{G}$ parameterized by a continuous curve of the form $(\delta, \eta(\delta))$ in the neighborhood of $(0,0)$ such that

$$\eta(\delta) = O(\delta^q), \qquad q > \frac{3}{2} \qquad \text{as } \delta \to 0. \tag{10}$$

There exist $\bar{\delta} = \bar{\delta}(\mathcal{F}, \mathcal{G}) > 0$ and a hyperplane H with the following properties.

For any $\delta \leq \bar{\delta}$ the system (9) with $g_{\delta,\eta} \in \mathcal{F}$ admits H as a Poincarè section with a Poincarè map $\Pi_{g_{\delta,\eta}}$ and there exist an invariant set $S \subset H$, a continuous surjective map $\rho : S \to \Sigma$ and an integer d such that the following diagram commutes

$$\begin{tikzcd} \mathbf{S} \arrow[r, "\Pi^d_{g_{\delta,\eta}}"] \arrow[d, "\rho"'] & \mathbf{S} \arrow[d, "\rho"] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd}$$

i.e. $(S, \Pi^d_{g_{\delta,\eta}})$ is semi–conjugate to (Σ, σ).

Observe, that since the condition (10) is open, the set of (δ, η) for which the Theorem holds is open in $\mathbf{R}^2$. Note, that the result does assert the existence of a semi–conjugacy and not a conjugacy which means that the map ρ is not necessarily one-to-one.

Let us remark, that the system (9) with $g_{\delta,\eta}(x) \in \mathcal{F}$ and $\delta \leq \bar{\delta}$ is the simplest possible system which may have a chaotic behavior, since the phase space is 3-dimensional and there is only one nonlinear term on the right-hand side.

The result was obtained by altering a $\mathcal{MCFS}$ into a $\mathcal{CFS}$ by changing the function $f(x)$ into a function $g_{\delta,\eta}(x)$. However, the functions f and $g_{\delta,\eta}(x)$ are not close in any function space. A natural question is, whether we can achieve the same result by a small perturbation of the function f. The answer is positive.

Figure 3: Function $h(x)$.

Theorem 5.4 (Gedeon [8]) *Assume that the system (7) admits a hyperbolic periodic orbit* γ *with one-dimensional unstable manifold.*

For every ϵ *there is a function* $h \in C^1(\mathbf{R}, \mathbf{R})$ *with*

$$||f - h||_{C^0} < \epsilon$$

and a Poincarè section H *with the following properties.*

The system (7) with f *replaced by* h *admits* H *as a Poincarè section with a Poincarè map* π *and there exist an invariant set* $S \subset H$*, a continuous surjective map* $\rho : S \to \Sigma$ *and an integer* d *such that the following diagram commutes*

$$\begin{tikzcd} \mathbf{S} \arrow[r, "\pi^d"] \arrow[d, "\rho"'] & \mathbf{S} \arrow[d, "\rho"] \\ \Sigma \arrow[r, "\sigma"] & \Sigma \end{tikzcd}$$

i.e. (S, π^d) *is semi–conjugate to* (Σ, σ)*.*

This result is interesting because, as we saw in Theorem 5.1, for $\mathcal{MCFS}$ a Poincarè-Bendixon trichotomy holds and by a result of Tereščák [30] a C^1 perturbation of a $\mathcal{MCFS}$ will preserve the Poincaré-Bendixon properties of the flow.

In more limited setting the C^1-perturbation result is due to M. Hirsch. Observe that if n is odd and we change $t \to -t$ in the flow generated by (7) then all the feedbacks δ_i change the sign and so we get a $\mathcal{MCFS}$ with $\Delta = 1$. Such a flow defines a monotone dynamical system, as was mentioned in Introduction. For $n = 3$ such a system cannot exhibit chaotic dynamics; furthermore, this property is stable under C^1 perturbations of the flow (Hirsch [13],[14]). Again Theorem 5.4 provides a concrete example of the fact that this property is not stable under C^0 perturbation.

As we see the dynamics of a general $\mathcal{CFS}$ may be very complicated which should be contrasted with the simple dynamics of the subclass of $\mathcal{MCFS}$.

6 The Structure of Morse Sets

In the previous section it was remarked that the recurrent dynamics for $\mathcal{MCFS}$ can consist of at most fixed points, periodic orbits, or heteroclinic cycles. Of course, as we saw in section 5, for the general $\mathcal{CFS}$, the dynamics within the Morse sets can be much more complicated, however, it is reasonable to ask whether, even in these more general systems, one can be assured that the Morse sets contain fixed points or periodic orbits. Part of the purpose of this section is to demonstrate that the Conley index can be used to answer these questions. Thus, we begin with some abstract existence theorems.

Theorem 6.1 (McCord [22]) *Let S be an isolated invariant set and assume*

$$CH^k(S) \approx \begin{cases} \mathbf{Z} & \text{if } k = j \\ 0 & \text{otherwise.} \end{cases}$$

Then, S contains a fixed point.

The next theorem provides for the existence of periodic orbits. First, however, we need the following concept. Ξ is a *Poincaré section* for an isolating neighborhood if Ξ is a local section, i.e. $\varphi((-\epsilon,\epsilon),\Xi)$, an open subset of $\mathbf{R}^n$, is 0 to $\Xi\times(-\epsilon,\epsilon)$ for $\epsilon>0$, $\Xi\cap N$ is closed, and for every $x\in N$ there exists $t_x>0$ such that

$$\varphi(t_x,x)\in\Xi.$$

Theorem 6.2 (McCord, Mischaikow, and Mrozek [23]) *If S is an isolated invariant set with isolating neighborhood N such that N has a Poincaré section and*

$$CH^k(S)\approx\begin{cases}\mathbf{Z} & \textit{if } k=j,j+1\\ 0 & \textit{otherwise}\end{cases}$$

for some j, then S contains a periodic orbit.

Returning now to $\mathcal{CFS}$ we have the following theorem.

Theorem 6.3 *1. If $\Delta=1$, then $M(0)$ contains at least two fixed points.*

2. Let $J=n$. For all $\mathcal{CFS}^+_{even}$ and $\mathcal{CFS}^-_{odd}$, $M(P-1)$ contains at least two fixed points.

Proof. We give the proof of (1) and claim that the proof of (2) is similar. We assume without loss of generality that $\delta_i=1$ for all i. Then it is easy to check that

$$M(0)\subset Q(1,1,\ldots,1)\cup Q(-1,-1,\ldots,-1)=\mathcal{N}^{-1}(0).$$

From the details of the proof of Proposition 4.1 it is easy to check that

$$CH^k(\mathrm{Inv}Q(\pm1,\ldots,\pm1))\approx\begin{cases}\mathbf{Z} & \text{if } k=0\\ 0 & \text{otherwise.}\end{cases}$$

Now one applies Theorem 6.1. □

The next theorem is concerned with the existence of periodic orbits and as the reader may have guessed will be proven with the aid of Theorem 6.2. Again, Proposition 4.1 provides us with the appropriate indices, so all that remains is to find Poincaré sections for the isolating neighborhoods. However, for $\mathcal{CFS}$ the X_is are ideal candidates for Poincaré sections. Recall that on X_i, $\dot{x}_i\neq0$, and hence, compact subsets of X_i are local sections for the flow. The most general conditions under which X_i acts as a Poincaré section is not known, however, if it does then a periodic orbit $x(t)$ in $M(p)$ can be characterized as follows. For every $i = 1, \ldots, n$ there exist times t_i and t_i' such that $x_i(t_i) > 0$ and $x_i(t_i') < 0$. We shall refer to periodic orbits with this property as *large periodic orbits.* If X_i is not a Poincaré section for $M(p)$, then it appears possible that there exist periodic orbits which remain in an orthant $Q(\sigma_1, \ldots, \sigma_n)$.

Theorem 6.4 (Gedeon and Mischaikow [9]) *If $J < n$ then for $\mathcal{CFS}^+$ let $p = 1, \ldots, P-1$ and for $\mathcal{CFS}^-$ let $p = 0, \ldots, P-1$*

1. *If, for some $i = 1, \ldots, n$, some X_i is a Poincaré section of $M(p)$, then $M(p)$ contains a large periodic orbit.*
2. *If, in addition, one considers a $\mathcal{MCFS}$ and if $M(p)$ contains no fixed points, then the appropriate X_i acts as a Poincaré section, and hence, $M(p)$ contains a large periodic orbit.*

While completely general conditions on the existence of Poincaré sections are not known, the following theorem provides reasonable sufficient conditions.

Theorem 6.5 (Gedeon and Mischaikow [9]) *Consider a $\mathcal{CFS}$ of the following form*

$$\dot{x}_i = \alpha_i g_i(x_i) + \beta_i f_i(x_{i-1}), \quad i = 1, \ldots, n$$

where $\alpha_i, \beta_i \in \{\pm 1\}$ and we assume that for every i, $x_i g_i(x_i) > 0$ and $x_{i-1} f_{i-1}(x_{i-1}) > 0$. If

$$\prod_{i=1}^{n} \alpha_i \beta_i = (-1)^{n+1}$$

then for every i X_i is a Poincaré section.

While these theorems are rather general in nature, they fall short of the stated goal of this section which was to show that there are dynamic structures that are shared by the Morse sets for all $\mathcal{CFS}$. Obviously, if there are fixed points in the Morse sets then these theorems are not applicable. On the other hand as the following result shows, even when there are fixed points, and hence, when there need not be periodic orbits, there is a set which topologically is similar (though it may be more complicated) to the large periodic orbit.

Theorem 6.6 (Gedeon and Mischaikow [9]) *There exists an essential continuous surjective map*

$$\theta_p : M(p) \to S^1$$

where S^1 *is a 1 circle embedded in* $\mathcal{N}^{-1}(p) \subset \mathbf{R}^n$.

Observe that large periodic orbits in $M(p)$ can also be characterized as 1 circle embedded in $\mathcal{N}^{-1}(p) \subset \mathbf{R}^n$.

7 The Global Structure

Our goal in this final section is to describe dynamic structures which are common to all $\mathcal{CFS}$. The first point which needs to be addressed is what is meant by describe? From the topological point of view conjugacy provides the most complete description of the dynamics. Recall that a conjugacy between a flow $\varphi : \mathbf{R} \times Z \to Z$ and a flow $\psi : \mathbf{R} \times Y \to Y$ is given by a homeomorphism $h : Z \to Y$ and a time reparameterization $\tilde{\varphi}$ of φ such that

$$\begin{array}{ccc}
\mathbf{R} \times Z & \xrightarrow{id \times h} & \mathbf{R} \times Y \\
\tilde{\varphi} \downarrow & & \downarrow \psi \\
Z & \xrightarrow{h} & Y
\end{array}$$

commutes. Observe that if one does not allow for the time reparameterization of one of the flows, then two periodic orbits which differ only in their period can not be made conjugate. Clearly, if the flow ψ is completely understood, then all the topological properties of φ are also understood. In this case ψ can referred to as the model dynamics.

The results of section 5 should convince the reader that this notion of equivalence is far too strong for our purpose of trying to express a uniform structure for all the possible $\mathcal{CFS}$. A weak notion, and the first we shall employ, is that of a semi–conjugacy. In particular, φ is said to be *semi–conjugate* to ψ if the above diagram commutes where the 0 h is replaced by a continuous surjection $\rho : Z \to Y$.

The model dynamics ψ for the $\mathcal{CFS}$ which we shall use depends on P and is defined as follows. Let A be a square matrix of the form

$$A = \begin{bmatrix} A_0 & 0 & \dots & 0 \\ 0 & A_2 & & \\ & & \ddots & \\ 0 & & & A_{P-1} \end{bmatrix}.$$

The submatrices A_p, $p = 0, \dots, P-1$ have two forms:

$$A_p = \frac{1}{p+1} \qquad \text{(Type I)}$$

and

$$A_p = \begin{bmatrix} (p+1)^{-1} & 2\pi \\ -2\pi & (p+1)^{-1} \end{bmatrix}. \qquad \text{(Type II)}$$

Let $z = (z_0, \dots, z_{k-1}) \in \mathbf{R}^k$. Then in polar coordinates $z = r\zeta$ where $r \geq 0$ and $\zeta \in S^{k-1}$, the unit sphere in $\mathbf{R}^k$. Let $D^k = \{z = (z_0, \dots, z_{k-1}) \mid \sum_{p=0}^{k-1} z_p^2 \leq 1\}$ be the closed unit ball in $\mathbf{R}^k$. Consider the flow

$$\psi : \mathbf{R} \times D^k \to D^k \tag{11}$$

generated by the equations

$$\dot{\zeta} = A\zeta - \langle A\zeta, \zeta \rangle \zeta \tag{12}$$

$$\dot{r} = r(1-r). \tag{13}$$

The dynamics of ψ is most easily understood if one observes that (12) is obtained by projecting the linear system $\dot{z} = Az$ onto the unit sphere.

The choice of Type I or Type II matrices is determined by the $\mathcal{CFS}$. The specific choices for the A_p's as a function of the type of $\mathcal{CFS}$ are as follows:

$\mathcal{CFS}^-_{odd}$: A_p, $p = 0, \dots P-1$ are of Type II unless $n = 2P+1$ when A_p, $p = 0, \dots, P-2$ are of Type II and A_{P-1} is of Type I.

$\mathcal{CFS}^+_{odd}$: A_0 is of Type I and A_p, $p = 1, \dots P-1$ are of Type II.

$\mathcal{CFS}^-_{even}$: A_p, $p = 0, \dots P-1$ are of Type II.

$\mathcal{CFS}^+_{even}$: A_0 is of Type I and A_p, $p = 1, \ldots, P-1$ are of Type II unless $n = 2P$ when A_p, $p = 1, \ldots, P-2$ are of the Type II and A_P is of Type I.

When it is necessary to distinguish between the model flows we shall let $\psi^\pm_*$ denote the corresponding flow where $*$ denotes *even* or *odd*.

Let $\Pi(p)$, $p = 0, \ldots, P-1$ denote the invariant set of ψ in the invariant subspace corresponding to A_p and let $\Pi(P) := \mathbf{0}$, the origin. Observe that $\{\Pi(p) \mid p = 0, \ldots, P\}$ forms a Morse decomposition of ψ on D^k.

Theorem 7.1 *Consider $\mathcal{CFS}^\pm_*$. Assume that if A_p is of Type II, then $M(p)$ has a Poincaré section. Then, there exist a continuous surjective function*

$$\rho : \mathcal{A} \to D^K$$

for which $M_p = \rho^{-1}(\Pi(p))$ $(p = 0, \ldots, P)$ and a continuous flow $\tilde{\varphi} : \mathbf{R} \times \mathcal{A} \to \mathcal{A}$ obtained via an order preserving time reparameterization of φ such that the following diagram commutes

$$\begin{tikzcd} \mathbf{R} \times \mathcal{A} \arrow[r, "id \times \rho"] \arrow[d, "\tilde{\varphi}"'] & \mathbf{R} \times D^K \arrow[d, "\psi^\pm_*"] \\ \mathcal{A} \arrow[r, "\rho"] & D^K \end{tikzcd}$$

i.e. φ is semi–conjugate to $\psi^\pm_*$.

Immediate corollaries of Theorem 7.1 are as follows.

Corollary 7.2 *Consider $\mathcal{MCFS}^\pm_*$ and assume that if A_p is of Type II, then $M(p)$ has no fixed points. Then, there exists a semi–conjugacy from $\tilde{\varphi}$ to $\psi^\pm_*$.*

Corollary 7.3 *Consider a $\mathcal{CFS}$ of the following form*

$$\dot{x}_i = \alpha_i g_i(x_i) + \beta_i f_i(x_{i-1}), \quad i = 1, \ldots, n$$

where $\alpha_i, \beta_i \in \{\pm 1\}$ and we assume that for every i, $x_i g_i(x_i) > 0$ and $x_{i-1} f_{i-1}(x_{i-1}) > 0$. If

$$\prod_{i=1}^{n} \alpha_i \beta_i = (-1)^{n+1},$$

then there exists a semi–conjugacy from $\tilde{\varphi}$ to $\psi^\pm_*$.

These results all depend upon the appropriate Morse sets not containing fixed points. In the last section, we dealt with this problem, by showing that these Morse sets could always be mapped onto a circle in a non-trivial manner. We will employ the same idea here, but on the level of Morse decompositions.

Recall that given a Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ an *interval* $I \subset \mathcal{P}$ satisfies the property that if $p, q \in I$ and $p > r > q$, then $r \in I$. The importance of intervals is that given a Morse decomposition all coarser Morse decompositions involve isolated invariant set of the form

$$M(I) := \left(\bigcup_{p \in I} M(p)\right) \bigcup \left(\bigcup_{p,q \in I} C(p,q)\right)$$

where I is an interval and

$$C(p,q) := \{x \in \mathcal{A} \mid \omega(x) \subset M(q) \text{ and } \alpha(x) \subset M(p)\}$$

is the set of connecting orbits from $M(p)$ to $M(q)$.

Definition 7.4 A Morse decomposition $\mathcal{M}(\mathcal{A}) = \{M(p) \mid p \in (\mathcal{P}, >)\}$ is *topologically semi-equivalent* to $\mathcal{M}(\mathcal{B}) = \{M(q) \mid q \in (\mathcal{Q}, >)\}$ if there exists

1. an order preserving bijection $\bar{\rho} : \mathcal{P} \to \mathcal{Q}$, and
2. a continuous surjection $\rho : \mathcal{A} \to \mathcal{B}$ such that

$$M(I) = \rho^{-1}(M(\bar{\rho}(I)))$$

for every interval $I \subset \mathcal{P}$.

Theorem 7.5 *Given $\mathcal{CFS}_*^{\pm}$. The Morse decomposition $\mathcal{M}(\mathcal{A})$ is topologically semi-equivalent to $\mathcal{M}(D^K, \psi_*^{\pm})$.*

Observe that in this description the dynamics is almost completely ignored. In particular, we lose all information concerning individual orbits. On the other hand, what is preserved is the purely topological structure of the invariant sets. Observe, that unstable manifolds get mapped to unstable manifolds and the same for stable manifolds. As an example of the information this description provides, let us assume that $\Delta = -1$, $n \geq 4$ and $P \geq 2$. Then the set $M(0,1) := M(0) \cup M(1) \cup C(M(1), M(0))$ is a pre-image of an essential map onto $\Pi(0,1) := \Pi(0) \cup \Pi(1) \cup C(\Pi(1), \Pi(0))$. One can easily check that $\Pi(0,1)$ is homeomorphic to S^3 a 3-dimensional sphere.

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