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PDPI- HALE J K MATH (404)894-8769

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PROJECT ADMINISTRATIVE DATA

OCA contact: Melanie K. Couvillon (404) 894-4820

Sponsor technical contact:
DEBORAH LOCKHART
DIVISION OF MATHEMATICS
NATIONAL SCIENCE FOUNDATION
4201 WILSON BLVD
ARLINGTON VA
22230
Phone: 703-306-1882
Fax:
Email:

Sponsor issuing office:
MARY L. INABINET
DIVISION OF GRANTS AND AGREEMENTS
NATIONAL SCIENCE FOUNDATION
4201 WILSON BLVD. ROOM P60
ARLINGTON VA
22230
Phone: 703/306-1213
Fax:
Email:

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TOPICS IN DYNAMICAL SYSTEMS

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Shui-Nee Chow and Jack K. Hale, PI's

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Georgia Tech
Atlanta, GA 30332-0190

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Dr. Shui-Nee Chow, Co-Principal Investigator on this project wrote a paper with Hugo Leiva, a graduate student partially supported on this NSF Grant. Attached, please find a copy of these papers. They have been submitted to two journals for consideration of publication.

Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces

by S-N. Chow and H. Leiva

CDSNS93-156

Existence and Roughness of the Exponential Dichotomy for Skew-Product Semiflow in Banach Spaces *

Shui-Nee Chow †and Hugo Leiva ‡

Abstract

In this paper we introduce a concept of exponential dichotomy for skew-product semiflow in infinite dimensional Banach spaces which is an extension of the classic concept for evolution operators. This concept is used to study the roughness property of the skew-product semiflow. Also, we introduce the concept of discrete skew-product and give a necessary and sufficient condition for this discrete skew-product to have a Discrete Dichotomy. After that, we give necessary and sufficient conditions for the existence of exponential dichotomy for skew-product semiflow. Moreover we prove that the exponential dichotomy for skew-product semiflow is not destroyed by small perturbation. Finally, we apply these results to parabolic partial differential equations and functional differential equations.

Key words. skew-product semiflow, exponential dichotomy, discrete skew-product, discrete dichotomy, roughness.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

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†CDSNS Georgia Tech, Atlanta, Georgia 30332
‡CDSNS Georgia Tech and ULA-Venezuela: e-mail leiva@math.gatech.edu

Contents

1 Introduction

The concept of exponential dichotomy of linear differential equations was introduced by Perron in 1930 [24], which is concerned with the problem of conditional stability of a system

$$\dot{x} = A(t)x \tag{1.1}$$

and the conection with the existence of bounded solutions of the equation

$$\dot{x} = A(t)x + f(x,t) \tag{1.2}$$

where the state space is a Banch space X and $t \longrightarrow A(t) : I\!R \longrightarrow L(X)$ is bounded, continuous in the strong operator topology. For related work, see Massera and Shaffer [21], Hale [8], Levinson [15], Coppel [6], Sacker and Sell [28], [29], [30] and Palmer [22].

One of the important problems of exponential dichotomies of the equation (1.1) is its roughness. That is, they are not destroyed by small pertubations of the bounded operator $A(t)$. This was first proved by Massera and Shaffer [21] under the assuption that the original operator $A(t)$ is a bounded matrix; for the case that $A(t)$ is not a matrix the results still true if $A(t) \in L(X)$ where X is and infinite dimensional Banach space and can be found in Daleckii and Krein [7]. Also, Palmer [22] proved the following Lemma:

Lemma 1.1 *Let $A(t)$ and $B(t)$ be $n \times n$ matrix functions, bounded and continuous on $[t_0, \infty)$. Suppose the system (1.1) has an exponential dichotomy on $[t_0, \infty)$ with projection matrix function $P(t)$ and $B(t) \to 0$ as $t \to \infty$. Then the perturbed system*

$$\dot{x} = [A(t) + B(t)]x \tag{1.3}$$

also has an exponential dichotomy on $[t_0, \infty)$ and if $Q(t)$ is a corresponding projection matrix function

$$\|Q(t) - P(t)\| \to 0 \quad as \quad t \to \infty \tag{1.4}$$

The need for a new approach arose from the fact that for the linear differential equation (1.1) with unbounded operator $A(t)$, the solutions generally speaking, either can not be extended in the direction of the negative times, or can be extended, but not uniquely. For example, for parabolic partial differential equations many authors have studied these problems, (see for example, Henry [11], Kolesov [12] and Xiao-Biao Lin [17]). In the case of Henry, he studied the existence and roughness of the exponential dichotomy of the following linear differential equation

$$\dot{x} = (A_0 + A(t))x \tag{1.5}$$

where $t \to A(t) : I\!R \to L(X)$ is bounded, continuous in the strong operator topology and A_0 is the infinitesimal generator of an analitic semigroup of bounded linear operator in the Banach space X. Henry's results has been generalized by A. Carvalho [2], to the case when A_0 is the infinitesimal generator of a C_0 semigroup instead of an analitic semigroup. Both of them used the relation between discrete dichotomies and exponential dichotomies.

For the case of functional differential equations we can see the work done by Hale-X.B.Lin [9], H.Rodrigues [27] and M.Lizana [18].

In the case of M.Lizana, he proved that, if the linear nonautonomous functional differential equation

$$\dot{x}(t) = L(t)x_t, \qquad t \geq 0 \tag{1.6}$$

admits and exponential dichotomy on $I\!R_+$ where $L(t) : C = C[[-r, 0], I\!R^n] \to I\!R^n$ is a linear operator which is uniformly continuous and bounded with respect t on $I\!R_+$ in the operator norm of the space $L(C, I\!R^n)$, then it is not destroyed by small perturbation. Basically he proved the following:

Lemma 1.2 *If (1.6) has an exponential dichotomy on $I\!R_+$ and $B(t)$ is a linear bounded operator for $t \geq 0$ and continuous with respect to t in the operator norm of $L(C, I\!R^n)$, $\|B(t)\| \leq \epsilon$ for all $t \geq 0$, then*

$$\dot{x}(t) = (L(t) + B(t))x_t \tag{1.7}$$

has an exponential dichotomy on $I\!R_+$ if ϵ is sufficiently small.

The prove of the above Lemma is similar to that for ordinary differential equations (see [8], Lemma 5.2, p.p 125-127).

Also, Lizana proved the following Theorem

Theorem 1.1 *Suppose the above mentioned hypothesis on L are satisfied. If $L(t)$ is globally Lipschitz in t with a constant $\rho > 0$ sufficiently small and all the roots of the characteristic equation $det\Delta(t, \lambda) = 0$, where*

$$\Delta(t, \lambda) = \lambda I - L(t)(e^{\lambda \cdot} I) \tag{1.8}$$

verify the condition

$$|\mathbf{Re}\lambda(t)| \geq \alpha > 0, \qquad \forall t \geq 0 \tag{1.9}$$

where α is independent of t, then $\dot{x}(t) = L(t)x_t, \quad t \geq 0$ has an exponential dichotomy on $\mathbb{R}^+$, with some projection operators $P(\sigma)$ and $Q(\sigma)$, the subspace $Q(\sigma)X$ is finite dimensional and the dimension is independent of σ on $\mathbb{R}^+$.

All the problems above can be treated in the unifield setting of a linear skew-product semiflow. In [31] Sacker-Sell use a concept of exponential dichotomy for skew-product semiflow with the restriction that the unstable manifold has finite dimension and they give a sufficient condition for the existence of exponential dichotomies for skew-product semiflow, which is given by the following Theorem.

Theorem 1.2 *Let $\pi = (\Phi, \sigma)$ be a weakly hyperbolic skew-product semiflow on $\mathcal{E} = X \times \Theta$. If $codimS(\theta) = k$, $\theta \in \Theta$ ($S(\theta)$ -stable manifold), then π has an exponential dichotomy over Θ.*

This concept is also used by Magalhães in [19] and [20]. It is not hard to prove that the concept of exponential dichotomy used by Sacker-Sell and Magalhães is stronger than the copcept we use here.

A different characterization of the exponential dichotomy for skew-product flow in infinite Banach spaces appears in Rau [25] and [26]. He associates a strongly continuous group to the skew-product flow $\pi = (\Phi, \sigma)$ in the following way:

Given a skew-product flow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ we can associate a family $\{T(t)\}_{t \in \mathbb{R}}$ of linear operators on the Banach space $C(\Theta, X)$ defined by

$$T(t)f(\theta) = \Phi(\theta \cdot (-t), t) f(\theta \cdot (-t)), \ \ \forall t \in \mathbb{R} \tag{1.10}$$

for all $\theta \in \Theta$ and $f \in C(\Theta, X)$.

Proposition 1.1 *The operator family $\{T(t)\}_{t \in \mathbb{R}}$ given by (1.10) is a strongly continuous group on $C(\Theta, X)$.*

Since $\pi = (\Phi, \sigma)$ is a flow (two side flow) the definition of exponential dichotomy is the same as in the finite dimensional case; this allows Rau in [25] to give the following sufficient and necessary condition for the existence of exponential dichotomy for skew-product flow $\pi = (\Phi, \sigma)$.

Theorem 1.3 *(Theorema 12 in [25]) Let $\{T(t)\}_{t \in \mathbb{R}}$ be the strongly continuous group given by (1.10). Then the following statements are equivalent:*

(A) $\pi = (\Phi, \sigma)$ has an exponential dichotomy over Θ.

(B) $\sigma(T(t)) \cap \Gamma = \emptyset, \ \ \forall t \neq 0$
where Γ denotes the unit circle in C.

A similar result as Rau's can be found in Latushkin-Stepin [13], [14] and Antonevich [1].

In this paper, first of all, we introduce a concept of exponential dichotomy for skew-product semiflow weaker than the concepts used by Sacker-Sell and Magalhães. Here we allow the unstable manifold to have infinite dimension. On the other hand, ours concept is an extension of the classic concept of exponential dichotomy for evolution operators used by Henry in [11].

Second, we introduce the concept of Skew-Product Sequence and give a necessary and sufficient condition for this sequence to have discrete dichotomy.

Third, we apply the above results to prove roughness. We also give a necessary and sufficient condition for a linear skew-product semiflow to has an exponential dichotomy.

Finally, we consider some examples as applications of these results.

2 Preliminaries

In this section we shall present some definitions, notations and results about Skew Product Semiflow in infinite Banach spaces, as well as the definition of exponential dichotomy for the skew-product semiflow which is one of the most important concept of this paper.

2.1 Linear Skew–Product Semiflow

We begin with the notion of skew-product semiflow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$ where X is a fixed a Banach space (the state space) and Θ is a compact Hausdorff space.

Definition 2.1 Suppose that $\sigma(\theta, t) = \theta \cdot t$ is a flow on Θ, i.e., the mapping $(\theta, t) \to \theta \cdot t$ is continuous, $\theta \cdot 0 = \theta$ and $\theta \cdot (s+t) = (\theta \cdot s) \cdot t$, for all $s, t \in I\!R$.

A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is a mapping $\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta.t)$ for $t \geq 0$, with the following properties:

(1) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$

(2) $\lim_{t \to 0^+} \Phi(\theta, t)x = x$, uniformly in θ. This means that for every $x \in X$ and every $\epsilon > 0$ there is a $\delta = \delta(x, \epsilon) > 0$ such that $\|\Phi(\theta, t)x - x\| \leq \epsilon$, for all $\theta \in \Theta$ and $0 \leq t \leq \delta$.

(3) $\Phi(\theta, t)$ is a bounded linear operator from X into X that satisfies the cocycle identity:

$$\Phi(\theta, t+s) = \Phi(\theta.t, s)\Phi(\theta, t) \quad \theta \in \Theta, \quad 0 \leq s, t. \tag{2.1}$$

(4) for all $t \geq 0$ the mapping from $\mathcal{E}$ into X given by

$$(x, \theta) \to \Phi(\theta, t)x$$

is continuous.

The properties (2) and (3) imply that for each $(x, \theta) \in \mathcal{E}$ the solution operator $t \to \Phi(\theta, t)x$ is right continuous for $t \geq 0$. In fact :

$$\|\Phi(\theta, t+h)x - \Phi(\theta, t)x\| = \|[\Phi(\theta \cdot t, h) - I]\Phi(\theta, t)x\|$$

which goes to 0 as h goes to 0^+.

For any subset $\mathcal{F} \subset \mathcal{E}$ we define the fiber

$$\mathcal{F}(\theta) := \{x \in X : (x, \theta) \in \mathcal{F}\}, \quad \theta \in \Theta. \tag{2.2}$$

So $\mathcal{E}(\theta) = X \times \{\theta\}, \quad \theta \in \Theta$. If $U \subset \Theta$, then we define

$$\mathcal{F}(U) := \bigcup_{\theta \in U} \mathcal{F}(\theta).$$

Also we define

$$\mathcal{E}_0 = \{(x, \theta) \in \mathcal{E} : x = 0\}$$

the zero fiber.

Proposition 2.1 *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E}$. Then there exists constants $M \geq 1, \quad a > 0$ such that*

$$\|\Phi(\theta, t)\| \leq Me^{at} \quad \theta \in \Theta, \quad t \in \mathbb{R}_+.$$

Proof First we claim that there is $\delta > 0$ such that

$$M = \sup\{\|\Phi(\theta, t)\| : \quad \theta \in \Theta, \quad 0 \leq t \leq \delta\} < \infty$$

For the purpose of contradiction. Let us assume that there are sequences $\theta_n \in \Theta, \quad t_n \in \mathbb{R}_+$ such that $t_n \to 0^+$ and $\|\Phi(\theta_n, t_n)\| > n$. The Banach-Steinhaus Theorem (Uniform Boundedness Principle) implies that there is $x \in X$ such that

$$\{\|\Phi(\theta_n, t_n)x\| : n \in \mathbb{N}\}$$

is unbounded . This contradicts the fact that

$$\lim_{t \to 0^+} \Phi(\theta, t)x = x$$

uniformly in $\theta \in \Theta$. Therefore $M < \infty$.
On the other hand we have that $\Phi(\theta, 0) = I$, so $M \geq 1$.
Now fix $t \in \mathbb{R}_+$. Let m be an integer satisfying $m \leq t/\delta \leq m+1$, i.e., $\delta m \leq t \leq \delta m + \delta$. Then for any $\theta \in \Theta$ we have

$$\begin{aligned} \|\Phi(\theta, t)\| &= \|\Phi(\theta, t - \delta m + \delta m)\| \\ &= \|\Phi(\theta \cdot \delta, (t - \delta m) + \delta(m-1))\Phi(\theta, \delta)\| \end{aligned}$$

Now putting $\theta_0 = \theta, \quad \theta_1 = \theta_0 \cdot \delta, \quad \theta_2 = \theta_1 \cdot \delta, \cdots \theta_m = \theta_{m-1} \cdot \delta$; we get the following:

$$\begin{aligned} \|\Phi(\theta, t)\| &= \|\Phi(\theta_m, t - \theta m)\Phi(\theta_{m-1}, \theta) \cdots \Phi(\theta_1, \delta)\Phi(\theta_0, \delta)\| \\ &\leq M^{m+1} = M.M^m \leq M.M^{t/\delta}. \end{aligned}$$

If we put $a = \frac{1}{\delta} ln(M)$, then

$$\|\Phi(\theta, t)\| \leq M e^{at}.$$

□

Remark 2.1 The theory we present here extends easily to Banach bundles which are locally product spaces. A **Banach bundle** $\mathcal{E}$ with fiber X over a base space Θ with projection $\mathcal{P}$ is denoted by $(\mathcal{E}, X, \Theta, \mathcal{P})$, or $\mathcal{E}$ for short, and is defined as follows:

(1) X is a fixed Banach space and Θ is a compact Hausdorff space.

(2) The mapping $\mathcal{P} : \mathcal{E} \to \Theta$ is continuous.

(3) For each $\theta \in \Theta$, $\mathcal{P}^{-1}(\theta) = \mathcal{E}(\theta)$ is a Banach space, which is referred to as the fiber over θ.

(4) For each $\theta \in \Theta$, there is an open neighborhood U of θ in Θ and a homeomorphism $\tau : \mathcal{P}^{-1}(U) \to X \times U$ such that, for each $\eta \in U$, $\mathcal{P}^{-1}$ is mapped onto $X \times \{\eta\}$ and $\tau : \mathcal{P}^{-1}(\eta) \to X \times \{\eta\}$ is a linear isomorphism.

(5) The norms $\|\cdot\| = \|\cdot\|_\theta$ on the fiber $p^{-1}(\theta)$ vary continuously in θ. One can use the local coordinate notation (x, θ) to denote a typical point in a Banach bundle $\mathcal{E}$.

2.2 Projectors and Subbundles

A mapping $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is said to be a projector if $\mathbf{P}$ is continuous and has the form $\mathbf{P}(x, \theta) = (P(\theta)x, \theta)$, where $P(\theta)$ is a bounded linear projection on the fiber $\mathcal{E}(\theta)$.

For any projector $\mathbf{P}$ we define the range and null space by

$$\mathcal{R} = \mathcal{R}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = x\}$$

$$\mathcal{N} = \mathcal{N}(\mathbf{P}) = \{(x, \theta) \in E : P(\theta)x = 0\}$$

The continuity of $\mathbf{P}$ implies that the fibers $\mathcal{R}(\theta)$ and $\mathcal{N}(\theta)$ vary continuously in θ. This also means that $P(\theta)$ is strongly continuous in θ. The following result can be found in Sacker-Sell [31].

Lemma 2.1 *Let* $\mathbf{P}$ *be a projector on* $\mathcal{E}$, *then* $\mathcal{R}$ *and* $\mathcal{N}$ *are closed subsets in* $\mathcal{E}$ *and we have*

$$\mathcal{R}(\theta) \cap \mathcal{N} = \{0\}, \quad \mathcal{R}(\theta) + \mathcal{N}(\theta) = \mathcal{E}(\theta) \quad \theta \in \Theta.$$

Definition 2.2 A subset $\mathcal{V}$ is said to be a subbundle of $\mathcal{E}$, if there is a projector $\mathbf{P}$ on $\mathcal{E}$ with the property that $\mathcal{R}(\mathbf{P}) = \mathcal{V}$; in this case $\mathcal{W} = \mathcal{N}(\mathbf{P})$ is a complementary subbundle. i.e., $\mathcal{E} = \mathcal{V} + \mathcal{W}$ as a Whitney sum of subbundles.

For the proof of the following lemma see [31].

Lemma 2.2 *Let $\mathcal{V} \subset \mathcal{E}$ with the properties:*

(1) $\mathcal{V}$ is closed.

(2) $\mathcal{V}(\theta)$ is a linear subspace of $\mathcal{E}(\theta)$ for all $\theta \in \Theta$.

(3) $codim\mathcal{V}(\theta)$ is finite for all $\theta \in \Theta$.

(4) $codim\mathcal{V}(\theta)$ is locally constant on Θ.

Then $\mathcal{V}$ is a subbundle of $\mathcal{E}$.

2.3 The Stable, the Unstable and Initial Bounded Sets.

Definition 2.3 A point $(x,\theta) \in \mathcal{E}$ is said to have a negative continuation with respect to π if there exists a continuous functions $\phi : (-\infty, 0] \to \mathcal{E}$ satisfying the following properties:

(1) $\phi(t) = (\phi^x(t), \theta \cdot t)$ where $\phi^x : (-\infty, 0] \to X$

(2) $\phi(0) = (x,\theta)$

(3) $\phi(s) \in \mathcal{E}(\theta \cdot s)$ for each $s \leq 0$

(4) $\pi(\phi(s), t) = \phi(s+t)$ for each $s \leq 0$, and $0 \leq t \leq -s$

In this case the function ϕ is said to be a **negative continuation of the point** (x,θ). For any negative continuation ϕ and any $\tau \leq 0$, we define $\phi_\tau(t) = \phi(\tau + t)$ for $-\infty < t < -\tau$.

Now we shall define the following sets:

$\mathcal{M} := \{(x,\theta) \in \mathcal{E} : (x,\theta)$ has a negative continuation $\phi\}$

$\mathcal{U} := \{(x,\theta) \in \mathcal{M}$: there is a negative continuation ϕ of (x,θ) satisfying $\|\phi^x(t)\| \to 0$ as $t \to -\infty\}$

$\mathcal{B}^+ := \{(x,\theta) \in \mathcal{E} : \sup_{t\geq 0} \|\Phi(\theta,t)x\| < \infty\}$

$\mathcal{B}_u^- := \{(x,\theta) \in \mathcal{M} : (x,\theta)$ has a unique bounded negative continuation $\phi\}$

$\mathcal{B}^- := \{(x,\theta) \in \mathcal{M}$: there is a negative continuation ϕ of (x,θ) satisfying $\sup_{t\leq 0} \|\phi^x(t)\| < \infty\}$

$\mathcal{S} := \{(x,\theta) \in \mathcal{E} : \|\Phi(\theta,t)x\| \to 0$ as $t \to \infty\}$

$\mathcal{B} := \mathcal{B}^+ \cap \mathcal{B}^-$

The set $\mathcal{U}$ is called **unstable set**, $\mathcal{S}$ is the **stable set** and $\mathcal{B}$ is the **initial bounded set.**

Remark 2.2 The theory described here does allow for the possibility that the linear operator $\Phi(\theta, t)$ need not be one-to-one for some $t \leq 0$, i.e., $\Phi(\theta, t)$ may has a nontrivial null space. Because of this, it maybe possible for a point $(x, \theta) \in \mathcal{E}$ to have more than one negative continuation. It is easy to see that if $\Phi(\theta, t)$ is one-to-one for all $t > 0$,then every negative continuation is unique. Uniqueness of negative continuations is a common feature in the study of partial differential equations, see for example, Temam [32] and Hale [10].

For all $(x, \theta) \in \mathcal{B}_u^-$ we shall denote the unique bounded negative continuation, by $\Phi(\theta, t)x$, $t \leq 0$; this defines an extension of the mapping Φ. It is clear that for each $\theta \in \Theta$ the fiber $\mathcal{B}_u^-(\theta)$ is a linear subspaces of $\mathcal{E}(\theta)$ and $\Phi(\theta, t)x$ is linear in x for each $t \geq 0$, i.e., $\Phi(\theta, t)$ is a linear mapping from $\mathcal{B}_u^-(\theta)$ to $\mathcal{B}_u^-(\theta \cdot t)$ for $t \leq 0$.
Moreover, the cocycle identity

$$\Phi(\theta, t+s)x = \Phi(\theta \cdot t, s)\Phi(\theta, t)x \qquad s, t \in I\!R$$

is valid for all $(x, \theta) \in \mathcal{B}_u^-$.

2.4 Exponential Dichotomy for Linear Skew-Product Semiflow

Now we shall introduce a new concept of exponential dichotomy for skew-product semiflow in infinite dimensional Banach spaces which is an extension of the concept given by Henry in [11].

Definition 2.4 A projector **P** on $\mathcal{E}$ is said to be **invariant** if it satisfies the following property

$$P(\theta \cdot t)\Phi(\theta, t) = \Phi(\theta, t)P(\theta) \quad t \geq 0, \quad \theta \in \Theta \tag{2.3}$$

Definition 2.5 We shall said that a linear skew-product semiflow π on $\mathcal{E}$ has an **exponential dichotomy over an invariant set** $\hat{\Theta}$, where $\hat{\Theta} \subset \Theta$, if there are constants $k \geq 1$, $\beta > 0$ and invariant projector **P** such that for all $\theta \in \hat{\Theta}$ we have the following:

(1) $\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot t))$, $t \geq 0$ is an isomorphism with inverse:

$$\Phi(\theta \cdot t, -t) : \mathcal{N}(P(\theta \cdot t)) \to \mathcal{N}(P(\theta)), \quad t \geq 0$$

(2) $\|\Phi(\theta, t)P(\theta)\| \leq ke^{-\beta t}, \quad t \geq 0$

(3) $\|\Phi(\theta, t)(I - P(\theta)\| \leq ke^{\beta t}, \quad t \leq 0$

where $\Phi(\theta, t)(I - P(\theta))$ is well defined for $t \leq 0$ since $\mathcal{N}(P(\theta)) = \mathcal{R}(I - P(\theta))$.

Proposition 2.2 *If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ which admits an exponential dichotomy over Θ, then one has that $\mathcal{B} = \mathcal{E}_0$ and the corresponding projector* **P** *is such that*

$$\mathcal{R} = \mathcal{S}(\Theta), \qquad \mathcal{N} = \mathcal{U}(\Theta)$$

and

$$\mathcal{E} = \mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{P}) = \mathcal{S}(\Theta) + \mathcal{U}(\Theta)$$

(the Whitney sum of two subbundles)

Proof Let $\mathbf{P}$ be the corresponding projector and consider $(x, \theta) \in \mathcal{B}$. Set $y = P(\theta)x$ and $z = (I - P(\theta))x$. Then each one of the trajectories $\Phi(\theta, t)x$, $\Phi(\theta, t)y$ and $\Phi(\theta, t)z$ has a negative contiuation ϕ^x, ϕ^y and ϕ^z respectively. In fact, if ϕ^x is the negative continuation of $\Phi(\theta, t)$, then:

(1) $\phi^x(0) = x$

(2) $\phi^x(t+s) = \Phi(\theta \cdot s, t)\phi^x(s), \quad 0 \leq t \leq -s$

(3) $\Phi(\theta \cdot s, t)\phi^x(s) = \Phi(\theta, t+s)x, \quad 0 \leq -s \leq t.$

From here it is easy to prove that

$$\phi^y(s) = P(\theta \cdot s)\phi^x(s) \quad \text{and} \quad \phi^z(s) = (I - P(\theta \cdot s))\phi^x(s)$$

for all $s \leq 0$. Therefore

$$\phi^x(s) = \phi^y(s) + \phi^z(s), \qquad (-\infty, 0].$$

So

$$\|y\| \leq ke^{\beta t}\|\phi^x\|, \quad \forall t \leq 0.$$

Since $\phi^x(t)$ is bounded and $\beta > 0$, then $y = 0$.
From the definition of exponential dichotomy we have that

$$\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot t)), \quad t \geq 0$$

is an isomorphism with inverse:

$$\Phi(\theta \cdot t, -t) : \mathcal{N}(P(\theta \cdot t)) \to \mathcal{N}(P(\theta)), \quad t \geq 0.$$

Since $\mathcal{R}(I - P(\theta)) = \mathcal{N}(P(\theta))$, then we get the following: $z = (I - P(\theta))x \in \mathcal{N}(P(\theta))$ and

$$\begin{aligned} z &= \Phi(\theta \cdot t, -t)\Phi(\theta, t)z \\ &= \Phi(\theta \cdot t, -t)\Phi(\theta, t)(I - P(\theta))x \\ &= \Phi(\theta \cdot t, -t)(I - P(\theta \cdot t))\Phi(\theta, t)x, \quad t \geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \|z\| &\leq \|\Phi(\theta \cdot t, -t)(I - P(\theta \cdot t))\|\|\Phi(\theta, t)x\| \\ &\leq ke^{-\beta t}\|\Phi(\theta, t)x\|, \quad t \geq 0. \end{aligned}$$

Since $\Phi(\theta, t)x$ is bounded for $t \geq 0$ and $\beta > 0$, then $z = 0$. Therefore $x = y + z$. which means that $\mathcal{B} = \mathcal{E}_0$. This implies that $\mathcal{S} \cap \mathcal{U} = \mathcal{E}_0$.

Clearly we have that

$$\mathcal{R}(\mathbf{P}) \subset \mathcal{S}, \quad \mathcal{N}(\mathbf{P}) \subset \mathcal{U}$$

and

$$\mathcal{E} = X \times \Theta = \mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{P}).$$

Then

$$\mathcal{R}(\mathbf{P}) = \mathcal{S} \text{ and } \mathcal{N}(\mathbf{P}) = \mathcal{U}.$$

□

Corolary 2.1 *The projector* $\mathbf{P}$ *associated with the exponential dichotomy of* $\pi = (\Phi, \sigma)$ *is uniquely determined.*

Proof Assume that we have another projector $\mathbf{Q}$ associated with the exponential dichotomy of $\pi = (\Phi, \sigma)$. Then $\mathbf{Q}(x, \theta) = (Q(\theta)x, \theta)$ where $Q(\theta)$ is a linear bounded Projection on X.
From the previous lemma we get that

$$\mathcal{R}(P(\theta)) = \mathcal{R}(Q(\theta)), \quad \mathcal{N}(P(\theta)) = \mathcal{N}(Q(\theta)), \quad \theta \in \Theta.$$

This implies that $P(\theta) = Q(\theta), \quad \theta \in \Theta$. Therefore $\mathbf{P}(x, \theta) = \mathbf{Q}(x, \theta)$, $(x, \theta) \in \mathcal{E}$.

□

Lemma 2.3 *Assume* $\pi = (\Phi, \sigma)$ *has an exponential dichotomy over* Θ *with constants* β, k *and corresponding projector* $\mathbf{P}$. *Then the following holds:*

(1) $\|P(\theta))\| \le k, \quad \theta \in \Theta$

(2) If we define

$$G(\theta, t, s) = \begin{cases} \Phi(\theta \cdot s, t - s)P(\theta \cdot s), & \text{if } t \ge s \\ -\ \Phi(\theta \cdot s, t - s)(I - P(\theta \cdot s)), & \text{if } t \le s \end{cases} \tag{2.4}$$

then

$$\|G(\theta, t, s)\| \le k e^{-\beta|t-s|}, \quad t, s \in I\!R. \tag{2.5}$$

Proof Follows from the definition of exponential dichotomy. □

3 Discrete Skew-Product

Here we shall introduce the concept of discrete skew-product and present some results relative to discrete dichotomy for such skew-product, so as existence and preserving of the discrete dichotomy under small perturbation.

3.1 Discrete Dichotomy

We begin this subsection with the concept of discrete skew-product which appears in natural way when we discretize the skew-product semiflow on time.

Definition 3.1 Let $\sigma(\theta,t) = \theta \cdot t$, $t \in \mathbb{R}$ be a flow on Θ. A mapping $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ is called **discrete skew-product** if it can be written as follows:

$$\hat{\pi}(x,\theta,n) = (\Phi_n(\theta)x, \theta \cdot n), \qquad (x,\theta) \in \mathcal{E}, \quad n \in Z.$$

where $\Phi_n(\theta) \in L(X)$ and has the following properties

(1) there exists $\rho > 0$ such that

$$\|\Phi_n(\theta)\| < \rho, \quad \text{for all } n \in Z,\, \theta \in \Theta.$$

(2) for each $n \in Z$ the mapping from $\mathcal{E}$ to X given by $(x,\theta) \to \Phi_n(\theta)x$ is continuous.

Remark 3.1 In principle a discrete skew-product $\hat{\pi}$ is not a semiflow since there is not relation between $\Phi_n(\theta)$'s. However, in section 4 we shall define a discrete skew-product for which we do have relation between the $\Phi_n(\theta)$'s. Also, we do not need to assume that $\Phi_n(\theta)$ is invertible in whole the space X.

Now we shall introduce two concepts of discrete dichotomy over Θ.

Definition 3.2 (pointwise discrete dichotomy) We shall say that a discrete skew-product $\hat{\pi}$ has a pointwise discrete dichotomy over Θ if for each $\theta \in \Theta$, there exist M_θ, $\alpha_\theta < 1$ and a family of projections $P_n(\theta)$ on X such that

(1) $\Phi_n(\theta)P_n(\theta) = P_{n+1}(\theta)\Phi_n(\theta), \quad n \in Z.$

(2) $\Phi_n(\theta) : \mathcal{N}(P_n(\theta)) \to \mathcal{N}(P_{n+1}(\theta)), \quad n \in Z$
is an isomorphism with inverse:

$$\Phi_n(\theta)^{-1} : \mathcal{N}(P_{n+1}(\theta)) \to \mathcal{N}(P_n(\theta)).$$

(3) Define $\Phi_{n,m}(\theta) := \Phi_{n-1}(\theta)\Phi_{n-2}(\theta)\ldots\Phi_m(\theta)$, and $\Phi_{m,m}(\theta) := I$, $n > m$. Then

$$\|\Phi_{n,m}(\theta)P_m(\theta)x\| \leq M_\theta \alpha_\theta^{n-m}\|x\|, \quad n \leq m \in Z.$$

(4) $\|\Phi_{n,m}(\theta)(I - P_m(\theta))x\| \leq M_\theta \alpha_\theta^{m-n}\|x\|, \quad n < m; \quad n,m \in Z$
where $\Phi_{n,m}(\theta)(I - P_m(\theta))$ is well defined by (2) since $\mathcal{R}(I - P_m(\theta)) = \mathcal{N}(P_m(\theta))$.
In particular, if $m = 0$ we get

$$\|\Phi_{n,0}(\theta)P_0(\theta)x\| \leq M_\theta \alpha_\theta^{n}\|x\|, \quad n \geq 0,$$

and

$$\|\Phi_{n,0}(\theta)(I - P_0(\theta))x\| \leq M_\theta \alpha_\theta^{-n}\|x\|, \quad n \leq 0.$$

Definition 3.3 (uniform discrete dichotomy) We shall say that a discrete skew-product $\hat{\pi}$ has a uniform discrete dichotomy over Θ, if in definition 3.2 M_θ and α_θ are independent of $\theta \in \Theta$ ($M_\theta = M, \quad \alpha_\theta = \alpha$).

Remark 3.2 It is easy to check the following properties:

$$\Phi_{n,m}(\theta)\Phi_{m,k}(\theta) = \Phi_{n,k}(\theta); \quad \Phi_n(\theta)\Phi_{n,m}(\theta) = \Phi_{n+1,m}(\theta) \tag{3.1}$$

for $n > m \geq k$.

$$\Phi_{n,m}(\theta)P_m(\theta) = P_n(\theta)\Phi_{n,m}(\theta) \tag{3.2}$$

$$\Phi_{n,n-1}(\theta) = \Phi_{n-1}(\theta); \quad \Phi_{n,r}(\theta)\Phi_{r-1}(\theta) = \Phi_{n,r-1}(\theta). \tag{3.3}$$

for all $r \leq n, m, k \in Z$ and $\theta \in \Theta$.

Lemma 3.1 *Let $\hat{\pi} : \mathcal{E} \times Z \rightarrow \mathcal{E}$ be a discrete skew-product . If for a sequence $y = \{y_n\}$ in X and $\theta \in \Theta$ we have*

$$x_{n+1} = \Phi_n(\theta)x_n + y_n, \quad n \in Z. \tag{3.4}$$

Then

$$x_n = \Phi_{n,m}(\theta)x_m + \sum_{k=m}^{n-1} \Phi_{n,k+1}(\theta)y_k, \quad n > m \in Z \tag{3.5}$$

Proof

The case $m = n - 1$ is easy. In fact, from part (3) of definiton 3.2 we have that

$$x_n = \Phi_{n-1}(\theta)x_{n-1} + y_{n-1} = \Phi_{n-1}(\theta)x_{n-1} + \sum_{k=n-1}^{n-1} \Phi_{n,k+1}(\theta)y_k.$$

Assume that (3.5) is true for $m = r < n$. Then we shall prove it is true also for $m = r - 1$. In fact,

$$x_n = \Phi_{n,r}(\theta)x_r + \sum_{k=r}^{n-1} \Phi_{n,k+1}(\theta)y_k$$

and $x_r = \Phi_{r-1}x_{r-1} + y_{r-1}$. Therefore, using Remark 3.1 we get the following

$$\begin{aligned}
x_n &= \Phi_{n,r}(\theta)[\Phi_{r-1}(\theta)x_{r-1} + y_{r-1}] + \sum_{k=r}^{n-1} \Phi_{n,k+1}(\theta)y_k \\
&= \Phi_{n,r}(\theta)\Phi_{r-1}(\theta)x_{r-1} + \Phi_{n,r}(\theta)y_{r-1} + \sum_{k=r}^{n-1} \Phi_{n,k+1}(\theta)y_k \\
&= \Phi_{n,r-1}(\theta)x_{r-1} + \sum_{k=r-1}^{n-1} \Phi_{n,k+1}(\theta)y_k
\end{aligned}$$

□

Lemma 3.2 *Assume the discrete skew-product $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ has a pointwise discrete dichotomy over Θ with the corresponding M_θ, α_θ and projections $P_n(\theta)$. Then the following holds:*

(a) $\|P_n(\theta)\| \leq M_\theta, \quad n \in Z, \theta \in \Theta$

(b) We define the discrete Green's function as follows

$$G_{n,m}(\theta) = \begin{cases} \Phi_{n,m}(\theta) P_m(\theta), & n \geq m \\ -\ \Phi_{n,m}(\theta)(I - P_m(\theta)), & n < m. \end{cases} \tag{3.6}$$

Then

$$\|G_{n,m}(\theta)\| \leq M_\theta \alpha_\theta^{|n-m|}, \qquad \theta \in \Theta, n, m \in Z \tag{3.7}$$

Proof It follows from the definition 3.2 of pointwise discrete dichotomy.

□

Corolary 3.1 *Assume the discrete skew-product $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ has a uniform discrete dichotomy over Θ with constants M and $\alpha < 1$. Let $\{P_n(\theta)\}$, $n \in Z$, $\theta \in \Theta$ be the corresponding family of projections. Then we have:*

(a) $\|P_n(\theta)\| \leq M, \quad n \in Z, \quad \theta \in \Theta$

(b) $\|G_{n,m}(\theta)\| \leq M\alpha^{|n-m|}, \quad n \in Z, \quad \theta \in \Theta.$

Lemma 3.3 *Let $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ be a discrete skew-product which has a pointwise discrete dichotomy over Θ. Consider $\theta \in \Theta$ and $y = \{y_n\}$ in $l_\infty(Z, X)$. Then $\{x_n\} \in l_\infty(Z, X)$ is solution of*

$$x_{n+1} = \Phi_n(\theta) x_n + y_n, \qquad n \in Z \tag{3.8}$$

if and only if

$$x_n = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta) y_k. \tag{3.9}$$

Proof We have the following

$$x_{r+1} = \Phi_r(\theta) x_r + y_r$$

and using (3.5) we get

$$x_r = \Phi_{n,r}(\theta) x_n + \sum_{k=n}^{r-1} \Phi_{r,k+1}(\theta) y_k, \quad r > n.$$

Then

$$\begin{aligned} \Phi_{n,r}(\theta)[I - P_r(\theta)] x_r &= \Phi_{n,r}(\theta) x_r - \Phi_{n,r}(\theta) P_r(\theta) x_r \\ &= \Phi_{n,r}(\theta)[\Phi_{r,n}(\theta) x_n + \sum_{k=n}^{r-1} \Phi_{r,k+1}(\theta) y_k] - \\ &\quad \Phi_{n,r}(\theta) P_r(\theta)[\Phi_{r,n}(\theta) x_n + \sum_{k=n}^{r-1} \Phi_{r,k+1}(\theta) y_k] \end{aligned}$$

From Remark 3.1 we get

$$\begin{aligned}\Phi_{n,r}(\theta)[I-P_r(\theta)]x_r &= x_n+\sum_{k=n}^{r-1}\Phi_{n,k+1}(\theta)y_k-P_n(\theta)x_n\\ &- \sum_{k=n}^{r-1}\Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k.\end{aligned}$$

Hence

$$\Phi_{n,r}(\theta)[I-P_r(\theta)]x_r=[I-P_n(\theta)]x_n+\sum_{k=n}^{r-1}\Phi_{n,k+1}(\theta)[I-P_{k+1}(\theta)]y_k.$$

Since

$$\|\Phi_{n,r}(\theta)[I-P_r(\theta)]x_r\|\le M_\theta\alpha_\theta^{r-n}\|x_r\|$$

and $\|x_r\|\le C$ for all $r>n$, then

$$\|\Phi_{n,r}(\theta)[I-P_r(\theta)]x_r\|\to 0,\quad \text{as } r\to+\infty$$

and the series

$$\sum_{k=n}^{\infty}\Phi_{n,k+1}(\theta)[I-P_{k+1}(\theta)]y_k \quad \text{converges.}$$

Therefore

$$[I-P_n(\theta)]x_n=-\sum_{k=n}^{\infty}\Phi_{n,k+1}(\theta)[I-P_{k+1}(\theta)]y_k. \tag{3.10}$$

For $n>m$ we have the following:

$$x_n=\Phi_{n,m}(\theta)x_m+\sum_{k=m}^{n-1}\Phi_{n,k+1}(\theta)y_k.$$

Then

$$\begin{aligned}P_n(\theta)x_n &= P_n(\theta)\Phi_{n,m}(\theta)x_m+\sum_{k=m}^{n-1}P_n(\theta)\Phi_{n,k+1}(\theta)y_k\\ &= \Phi_{n,m}(\theta)P_m(\theta)x_m+\sum_{k=m}^{n-1}\Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k.\end{aligned}$$

On the other hand, from definition 3.2 we have

$$\|\Phi_{n,m}(\theta)P_m(\theta)x_m\|\le M_\theta\alpha_\theta^{n-m}\|x_m\|,\qquad n\ge m.$$

Since $\{x_n\}$ is bounded and $\alpha_\theta<1$, then

$$\|\Phi_{n,m}(\theta)P_m(\theta)x_m\|\to 0,\quad \text{as } m\to-\infty.$$

Therefore the series

$$\sum_{-\infty}^{n-1} \Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k \qquad \text{converges.}$$

Hence

$$P_n(\theta)x_n = \sum_{-\infty}^{n-1} \Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k \tag{3.11}$$

From (3.10) and (3.11) we get:

$$x_n = P_n(\theta)x_n + (I - P_n(\theta))x_n = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta)y_k.$$

Now we shall prove that the sequence $\{x_n\}$ given by (3.9) belongs to $l_\infty(Z, X)$ and is a solution of (3.8). In fact,

$$\begin{aligned} x_{n+1} &= \sum_{-\infty}^{\infty} G_{n+1,k+1}(\theta)y_k \\ &= \sum_{-\infty}^{n} \Phi_{n+1,k+1}(\theta)P_{k+1}(\theta)y_k - \sum_{n+1}^{\infty} \Phi_{n+1,k+1}(\theta)[I - P_{k+1}(\theta)]y_k. \end{aligned}$$

Then

$$\begin{aligned} x_{n+1} &= \sum_{-\infty}^{n-1} \Phi_{n+1,k+1}(\theta)P_{k+1}(\theta)y_k - \sum_{n}^{\infty} \Phi_{n+1,k+1}(\theta)[I - P_{k+1}(\theta)]y_k + y_n \\ &= \Phi_n(\theta)\left[\sum_{-\infty}^{n-1} \Phi_{n,k+1}(\theta)P_{k+1}(\theta)y_k - \sum_{n}^{\infty} \Phi_{n,k+1}(\theta)[I - P_{k+1}(\theta)]y_k\right] + y_n \\ &= \Phi_n(\theta)\sum_{-\infty}^{\infty} G_{n,k+1}(\theta)y_k + y_n. \end{aligned}$$

Hence

$$x_{n+1} = \Phi_n(\theta)x_n + y_n.$$

Now we see that $\{x_n\}$ is bounded respect $\theta \in \Theta$. In fact,

$$\|x_n\| \leq M_\theta \sum_{-\infty}^{\infty} \alpha_\theta^{|n-(k+1)|}\|y_k\| \leq M_\theta\|y\|\frac{1+\alpha_\theta}{1-\alpha_\theta}.$$

□

3.2 Necessary and Sufficient Conditions for Discrete Dichotomy

We begin this subsection with a theorem which is an extension of Theorem 7.6.5 [11], to the case of discrete skew-product.

Theorem 3.1 *Let $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ be a discrete skew-product in $\mathcal{E} = X \times \Theta$. Then the following statements are equivalent:*

(A) $\hat{\pi}$ has a pointwise discrete dichotomy over Θ.

(B) for $f = \{f_n\}$ in $l_\infty(Z, X)$ and $\theta \in \Theta$, there exists a unique bounded solution $\hat{x} = \hat{x}(\theta) = \{x_n\}$ of the equation

$$x_{n+1} = \Phi_n(\theta)x_n + f_n \tag{3.12}$$

Proof If (A) holds, then by using Lemma 3.3 we get that (B) holds with

$$x_n = x_n(\theta) = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta)y_k$$

and

$$\|x_n\| \leq M_\theta \frac{1+\alpha_\theta}{1-\alpha_\theta}\|f\|_\infty, \quad n \in Z.$$

Suppose (B) holds and put $I\!B = l_\infty(Z, X)$. Consider the linear operator $S_\theta : I\!B \to I\!B$ given by

$$\hat{x} = \{x_n\} \in I\!B \to \{x_{n+1} - \Phi_n(\theta)x_n\}.$$

Since $\|\Phi_n(\theta)\| < \rho$ for all $\theta \in \Theta$ and $n \in Z$, then $S_\theta \hat{x} \in I\!B$ for all $\hat{x} \in I\!B$. We shall prove the Theorem in several claims.

Claim 1. S_θ is a bounded linear isomorphism and

$$\|S_\theta\| \leq 1 + \rho, \quad \forall \theta \in \Theta.$$

It follows from (B) and Part (2) of the definition 3.1.

Claim 2. S_θ^{-1} is a bounded linear operator with

$$\|S_\theta^{-1}\| \geq \frac{1}{1+\rho}.$$

In fact. From the open mapping Theorem we get that S_θ^{-1} is a bounded linear operator. On the other hand, we have the following:

$$\|S_\theta^{-1}\|\|S_\theta\| \geq \|S_\theta^{-1}S_\theta\| = \|I\| = 1 \Rightarrow (1+\rho)\|S_\theta^{-}1\| \geq 1.$$

Claim 3. Define $G_\theta = S_\theta^{-1}$. Then G_θ can be written as follows

$$(G_\theta f)_n = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta)f_k, \quad n \in Z$$

at least for sequence $\{f_k\}$ with $f_k = 0$ for all large $|k|$. Where $G_{n,m}(\theta) \in L(X)$ with

$$\|G_{n,m}(\theta)\|_{L(X)} \leq \|G_\theta\|_{L(I\!B)}$$

and

(a) $G_{n+1,m}(\theta) - \Phi_n(\theta)G_{n,m}(\theta) = 0, \quad \text{if } n \neq m-1$

(b) $G_{m,m}(\theta) - \Phi_{m-1}(\theta)G_{m-1,m}(\theta) = I$

Proof of the claim 3. Consider the sequence $f = \{\cdots 0, x, 0, \cdots\}$ with x the $(m-1)th$ component of f. Define the following operator

$$G_{n,m}(\theta)x = (G_\theta f)_n, \quad \text{where} \quad G_\theta f = \{(G_\theta f)_n : n \in Z\}$$

Now consider $f = \{\cdots 0, x_1, x_2, 0, \cdots\}$ with x_1 the $(m-1)th$ component of f. Then

$$(G_\theta f)_n = (G_\theta f_1)_n + (G_\theta f_2)_n = G_{n,m}(\theta)x_1 + G_{n,m+1}(\theta)x_2$$

where $f_1 = \{\cdots 0, x_1, 0 \cdots\}$ and $f_2 = f - f_1$.
In this way for sequence $f \in I\!B$ such that $f_k = 0$ for all large $|k|$ we get that

$$(G_\theta f)_n = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta)f_k, \quad n \in Z.$$

It is easy to see that:

$$G_{n,m}(\theta) \in L(X) \quad \text{and} \quad \|G_{n,m}(\theta)\|_{L(X)} \leq \|G_\theta\|_{L(I\!B)}$$

Let $\hat{x} = \{x_n\}$ and $f = \{f_n\}$ in $I\!B$ such that $G_\theta f = \hat{x}$.
Then we have

$$x_{n+1} = \Phi_n(\theta)x_n + f_n \iff (G_\theta f)_{n+1} = \Phi_n(\theta)(G_\theta f)_n + f_n.$$

In particular if $f = \{\cdots 0, x, 0, \cdots\}$ with x the $(m-1)th$ component of f, then

$$G_{n+1,m}(\theta)x = \Phi_n(\theta)G_{n,m}(\theta)x + f_n$$

Therefore

$$\begin{aligned} G_{n+1,m}(\theta) - \Phi_n(\theta)G_{n,m}(\theta) &= 0 \ \text{ if } \ n \neq m-1 \\ G_{m,m}(\theta) - \Phi_{m-1}(\theta)G_{m-1,m}(\theta) &= I. \end{aligned}$$

Now consider

$$\begin{aligned} P_m(\theta) = G_{m,m}(\theta) = \Phi_{m-1}G_{m-1,m}(\theta) + I., i.e., \\ I - P_m(\theta) = -\Phi_{m-1}G_{m-1,m}(\theta). \end{aligned} \tag{3.13}$$

Claim 4.

(c) $G_{n,m}(\theta) = \Phi_{n,m}(\theta)P_m(\theta), \quad \text{if } \ n \geq m$

(d) $\Phi_{m,n}(\theta)G_{n,m}(\theta) = -(I - P_m(\theta)), \quad \text{if } \ n < m$

Proof of the claim 4. By induction and using parts (a) and (b) of claim 3. In fact. For $n = m$ (c) is true. Suppose that (c) is true for $k > m$, i.e., $G_{k,m}(\theta) = \Phi_{k,m}(\theta)P_m(\theta)$.

Since $k > m$, then $k \neq m-1$. Therefore

$$G_{k+1,m}(\theta) = \Phi_k(\theta)G_{k,m}(\theta) = \Phi_k(\theta)\Phi_{k,m}(\theta)P_m(\theta)$$

So

$$G_{k+1,m}(\theta) = \Phi_{k+1,m}(\theta)P_m(\theta).$$

(d) if $n = m-1$, then

$$G_{m,m}(\theta) = \Phi_{m-1}(\theta)G_{m-1,m}(\theta) + I$$

so

$$\Phi_{m-1}(\theta)G_{m-1,m}(\theta) = -(I - P_m(\theta)).$$

Since $\Phi_{m-1}(\theta) = \Phi_{m,m-1}(\theta)$, then we get

$$\Phi_{m,m-1}(\theta)G_{m-1,m}(\theta) = -(I - P_m(\theta)).$$

Suppose the relation (d) is true for $n = k < m$: $\Phi_{m,k}(\theta)G_{k,m}(\theta) = -(I - P_m(\theta))$.
Since $k-1 \neq m-1$, from Claim 3 we get

$$\begin{aligned} \Phi_{m,k-1}(\theta)G_{k-1,m}(\theta) &= \Phi_{m,k}(\theta)\Phi_{k-1}(\theta)G_{k-1,m}(\theta) \\ = \Phi_{m,k}(\theta)G_{k,m}(\theta) &= -(I - P_m(\theta)). \end{aligned}$$

Claim 5. If $x_{n+1} = \Phi_n(\theta)x_n, \ \ n \geq m$ defines a bounded sequence, then $(I - P_m(\theta))x_m = 0$.
Putting $x_n = 0$ for $n < m$ we get the following

$$x_{n+1} = \Phi_n(\theta)x_n, \quad \text{if} \quad n \neq m-1 \quad \text{and} \quad x_m - \Phi_{m-1}(\theta)x_{m-1} = x_m.$$

Therefore $\hat{x} = \{x_n\}$ is a bounded solution of the equation $x_{n+1} = \Phi_n(\theta)x_n + f_n$ where $f = \{\cdots, 0, x_m, 0, \cdots\}$ with x_m the $(m-1)th$ componet of f.
On the other hand we know that

$$y_n = (G_\theta f)_n = G_{n,m}(\theta)x_m$$

is also solution of $x_{n+1} = \Phi_n(\theta)x_n + f_n$. Hence , $x_n = G_{n,m}(\theta)x_m$.
In particular:

$$x_m = G_{m,m}(\theta)x_m = P_m(\theta)x_m \Rightarrow (I - P_m(\theta))x_m = 0,$$

and the claim 5 is proved.
Claim 6. $P_m(\theta)$ is a projection.
In fact. Consider $x \in X$ and $x_n = G_{n,m}(\theta)x$. Then $x_{n+1} = \Phi_n(\theta)x_n$ if $n \geq m$ and $\|x_n\| \leq \|G_\theta\|\|x\|$. It follows from the foregoing Claim that:

$$(I - P_m(\theta))G_{m,m}(\theta)x = (I - P_m(\theta))P_m(\theta)x = 0. \ \text{So} \ P_m(\theta)x = P_m^2(\theta)x, \ \ \forall x \in X$$

Claim 7. If $(I - P_m(\theta))x = 0$, then $\Phi_m(\theta)P_m(\theta)x = P_{m+1}(\theta)\Phi_m(\theta)x$.

If $(I - P_m(\theta))x = 0$, then

$$(I - G_{m,m}(\theta))x = 0 \iff x = G_{m,m}(\theta)x = x_m$$

Consider $x_n = G_{n,m}(\theta)x$. Then

$$x_{n+1} = \Phi(\theta)x_n \text{ if } n \geq m$$

and $\{x_n\}$ is bounded. Therefore

$$\begin{aligned} 0 = (I - P_{m+1}(\theta))x_{m+1} &= (I - P_{m+1}(\theta))\Phi_m(\theta)x_m \\ = (I - P_{m+1}(\theta))\Phi_m(\theta)x &= \Phi_m(\theta)(I - P_m(\theta))x_m \\ = \Phi_m(\theta)(I - P_m(\theta))x & \end{aligned}$$

So

$$P_{m+1}(\theta)\Phi_m(\theta)x = \Phi_m(\theta)P_m(\theta)x,$$

and the claim 7 is established.
Claim 8. $\Phi_m(\theta) : \mathcal{N}(P_m(\theta)) \to \mathcal{N}(P_{m+1}(\theta))$ is an isomorphism for all $\theta \in \Theta$.
In fact. Assume $\{x_n : n \leq m\}$ is bounded and $x_{n+1} = \Phi_n(\theta)x_n, \quad n < m$.
Then by putting $x_n = 0$ for $n > m$ we get that $\{x_n\}$ is solution of

$$x_{n+1} = \Phi_n(\theta)x_n + f_n \text{ with } f = \{\cdots 0, 0, -\Phi_m(\theta)x_m, 0, \cdots\}$$

where $-\Phi_m(\theta)x_m$ is the $m\,th$ component of f. Therefore

$$x_n = G_{n,m+1}(-\Phi_m(\theta)x_m) = -G_{n,m+1}(\theta)\Phi_m(\theta)x_m.$$

In particular

$$0 = x_{m+1} = -G_{m+1,m+1}(\theta)\Phi_m(\theta)x_m = -P_{m+1}(\theta)\Phi_m(\theta)x_m$$

so $\Phi_m(\theta)x_m \in \mathcal{N}(P_{m+1}(\theta))$.
Now suppose that $x \in \mathcal{N}(P_m(\theta))$ and let $y_n = G_{n,m}(\theta)x$. Then $\{y_n\}$ is bounded and

$$\begin{cases} y_{n+1} &= \Phi_n(\theta)y_n \text{ if } n \neq m-1 \\ y_{m-1} &= \Phi_{m-2}(\theta)y_{m-2} \\ y_m = &= \Phi_{m-1}(\theta)y_{m-1} + x. \end{cases}$$

Then the set $\{y_n : n \leq m-2\}$ is bounded and $y_{n+1} = \Phi_n(\theta)x_n, \quad n < m-2$.
Therefore, if we put $y_n = 0$ for $n > m-2$ we get that $\{y_n\}$ is solution of

$$y_{n+1} = \Phi_n(\theta)y_n + f_n \text{ with } f = \{\cdots 0, 0, -\Phi_{m-2}(\theta)y_{m-2}, 0, 0, \cdots\}$$

where $-\Phi_{m-2}(\theta)y_{m-2}$ is the $(m-2)th$ component of f.
So in the same way as before we get that $0 = y_{m-1} = -G_{m-1,m-1}(\theta)\Phi_{m-2}(\theta)y_{m-2}$, which implies

$$P_{m-1}(\theta)\Phi_{m-2}(\theta)y_{m-2} = 0 \iff y_{m-1} = \Phi_{m-2}(\theta)y_{m-2} \in \mathcal{N}(P_{m-1}(\theta))$$

Since $x \in \mathcal{N}(P_m(\theta))$, then we get: $y_m = G_{m,m}(\theta)x = P_m(\theta)x = 0$ and

$$\begin{aligned} -\quad x &= -(I - P_m(\theta))x = y_m - x = (G_{m,m}(\theta) - I)x \\ &= \Phi_{m-1}(\theta)G_{m-1,m}(\theta)x = \Phi_{m-1}y_{m-1} \in \Phi_{m-1}(\theta)\mathcal{N}(P_{m-1}(\theta). \end{aligned}$$

Then

$$x \in \Phi_{m-1}(\theta)\mathcal{N}(P_{m-1}(\theta), \quad \text{so} \quad \mathcal{N}(P_m(\theta) \subseteq \Phi_{m-1}(\theta)\mathcal{N}(P_{m-1}(\theta). \tag{3.14}$$

Now consider $\bar{y}_n = y_n = G_{n,m}(\theta)x$, for $n < m-1$ and

$$\bar{y}_{n+1} = y_n = \Phi_n(\theta)\bar{y}_n, \quad n+1 \geq m-1 \iff n \geq m-2.$$

Then

$$\bar{y}_{m-1} = \Phi_{m-2}(\theta)\bar{y}_{m-2} = \Phi_{m-2}(\theta)y_{m-2} = y_{m-1}$$

and

$$\bar{y}_m = \Phi_{m-1}(\theta)\bar{y}_{m-1} = \Phi_{m-1}(\theta)y_{m-1} = -x.$$

So we have that $\{\bar{y}_n : n \leq m\}$ is bounded and

$$\bar{y}_{n+1} = \Phi_n(\theta)\bar{y}_n, \quad n < m.$$

In the same way as before we get the following:

$$0 = -P_{m+1}(\theta)\Phi_m(\theta)\bar{y}_m = -P_{m+1}(\theta)\Phi_m(\theta)(-x)$$

then

$$\Phi_m(\theta)x \in \mathcal{N}(P_{m+1}(\theta)). \tag{3.15}$$

If $\Phi_m(\theta)x = 0$, then $\bar{y}_{m+1} = \Phi_m(\theta)\bar{y}_m = -\Phi_m(\theta)x = 0$. Therefore $\bar{y}_n = 0$ for all $n \leq m$.
Thus $\{\bar{y}_n\}$ is bounded for $n \geq m$ and $\bar{y}_{n+1} = \Phi_n(\theta)\bar{y}_n, \quad n \leq m$.
So from Claim 5 we get the following

$$(I - P_m(\theta))\bar{y}_m = (I - P_m(\theta))(-x) = 0 \Rightarrow x - P_m(\theta)x = x = 0. \tag{3.16}$$

In conclusion we have proved the following:
From (3.14) $\mathcal{N}(P_{m+1}(\theta)) \subseteq \Phi_m(\theta)(\mathcal{N}(P_m(\theta))$
From (3.15) $\Phi_m(\theta)(\mathcal{N}(P_m(\theta)) \subseteq \mathcal{N}(P_{m+1}(\theta))$
From (3.16) $\Phi_m(\theta)$ is one to one. Therefore

$$\Phi_m(\theta) : \mathcal{N}(P_m(\theta)) \to \mathcal{N}(P_{m+1}(\theta))$$

is an isomorphism.

Now we can write the Claim 4 in the following way

$$\begin{cases} (c) \ G_{n,m}(\theta) &= \quad \Phi_{n,m}(\theta)P_m(\theta), & \text{if } n \geq m \\ (d) \ G_{n,m}(\theta) &= - \quad \Phi_{n,m}(\theta)(I - P_m(\theta)), & \text{if } n < m. \end{cases}$$

Claim 9. If $P_m(\theta)x = 0$, then

$$\Phi_m(\theta)P_m(\theta)x = P_{m+1}(\theta)\Phi_m(\theta)x.$$

If $x \in \mathcal{N}(P_m(\theta)) \Rightarrow \Phi_m(\theta)P_m(\theta)x = 0$ and $\Phi_m(\theta)x \in \mathcal{N}(P_{m+1}(\theta))$, so $P_{m+1}(\theta)\Phi_m(\theta)x = 0$. Therefore

$$\Phi_m(\theta)P_m(\theta)x = P_{m+1}(\theta)\Phi_m(\theta)x.$$

Claim 10.

$$\Phi_m(\theta)P_m(\theta)x = P_{m+1}(\theta)\Phi_m(\theta)x, \quad \theta \in \Theta, \quad m \in Z.$$

It follows from Claims 7,9 and $I = P_m(\theta) + (I - P_m(\theta))$.
Now we are ready to prove part (3) and (4) of Definition 3.2 of pointwise discrete dichotomy.
Claim 11. For all $\theta \in \Theta$ we have:

$$\|\Phi_{n,m}(\theta)P_m(\theta)x\| \leq \|G_\theta\|^2(1 - \|G_\theta\|_{-1})^{n-m}, \quad n \geq m. \tag{3.17}$$

If $\Phi_{n,m}(\theta)P_m(\theta)x = 0$ for some $n \geq m$, then $\Phi_{p,m}(\theta)P_m(\theta)x = 0$, for $p \geq n$; it follows from: $\Phi_{n+1,m}(\theta)x = \Phi_n(\theta)\Phi_{n,m}(\theta)x$.
Assume that $\Phi_{n,m}(\theta)P_m(\theta)x \neq 0$. Then

$$\Phi_k^{-1}(\theta) := \|\Phi_{k,m}(\theta)P_m(\theta)x\| > 0, \qquad m \leq k \leq n.$$

From Claim 4 we have:

$$G_{n,k}(\theta) = \Phi_{n,k}(\theta)P_k(\theta) \text{ and } V_k(\theta) := \Phi_{k,m}(\theta)P_m(\theta)x.\phi_k(\theta), \quad m \leq k \leq n.$$

Then

$$\begin{aligned}
\sum_{k=m}^{n} G_{n,k}(\theta)V_k(\theta) &= \sum_{k=m}^{n} \Phi_{n,k}(\theta)P_k(\theta)\Phi_{k,m}(\theta)P_m(\theta)x.\phi_k(\theta) \\
&= \sum_{k=m}^{n} \Phi_{n,k}(\theta)\Phi_{k,m}(\theta)P_m(\theta)P_m(\theta)x.\phi_k(\theta) \\
&= \sum_{k=m}^{n} \Phi_{n,m}(\theta)P_m(\theta)x.\phi_k(\theta) \\
&= \Phi_{n,m}(\theta)P_m(\theta)x.\sum_{k=m}^{n} \phi_k(\theta).
\end{aligned}$$

Let $f = \{\cdots 0, V_m(\theta), \cdots, V_n(\theta), 0, \cdots\}$ with V_m the $(m-1)th$ component of f. Then

$$\|f\| = \sup_{k \in Z} \|f_k\| = \sup_{m \leq k \leq n} \|V_k(\theta)\| = 1$$

and

$$\begin{aligned}
\|\sum_{k=m}^{n} G_{n,k}(\theta)V_k(\theta)\| &= \|\sum_{k=m}^{n} G_{n,k}(\theta)f_k\| \\
&= \|(G_\theta f)_n\| \leq \|G_\theta\|\|f\| = \|G_\theta\|.
\end{aligned}$$

Therefore

$$\Phi_n^{-1}(\theta). \sum_{k=m}^{n} \Phi_k(\theta) = \|\Phi_{n,m}(\theta)P_m(\theta)x. \sum_{k=m}^{n} \Phi_k(\theta)\| \leq \|G_\theta\|.$$

Then we have:

$$\frac{-1}{\Phi_n^{-1}(\theta). \sum_{k=m}^{n} \Phi_k(\theta)} \leq -\|G_\theta\|^{-1}, \quad \text{and} \quad \|G_\theta\| > 1.$$

Define $\Psi_n(\theta) := \sum_{k=m}^{n} \Phi_k(\theta)$. Then we get the following

$$\Psi_{n-1}(\theta) = \sum_{k=m}^{n-1} \Phi_k(\theta) = \sum_{k=m}^{n} \Phi_k(\theta) - \phi_n(\theta)$$

$$= \Psi_n(\theta)(1 - \frac{1}{\phi_n^{-1}(\theta).\Psi_n(\theta)}) \leq \Psi_n(\theta)(1 - \|G_\theta\|^{-1}).$$

Then $\Psi_n(\theta) - \Psi_{n-1}(\theta) \geq \Psi_n(\theta)\|G_\theta\|^{-1}$. So we get

$$\phi_n(\theta) \geq \|G_\theta\|^{-1}(1 - \|G_\theta\|^{-1})^{-1}\Psi_{n-1}(\theta).$$

Thus

$$\phi_n(\theta) \geq \|G_\theta\|^{-1}(1 - \|G_\theta\|^{-1})^{-1})^{-(n-m)}\Psi_m(\theta)$$

i.e.,

$$\phi_n(\theta) \geq \|G_\theta\|^{-1}(1 - \|G_\theta\|^{-1})^{-1})^{-(n-m)}\phi_m(\theta)$$

i.e.,

$$\phi_n^{-1}(\theta) \leq \|G_\theta\|^{-1}(1 - \|G_\theta\|^{-1})^{-1})^{n-m}\phi_m^{-1}(\theta)$$

Hence

$$\begin{aligned} \|\Phi_{n,m}(\theta)P_m(\theta)x\| &\leq \|G_\theta\|(1 - \|G_\theta\|^{-1})^{n-m}\|P_m(\theta)x\| \\ &= \|G_\theta\|(1 - \|G_\theta\|^{-1})^{n-m}\|G_{n,m}(\theta)x\| \\ &\leq \|G_\theta\|^2(1 - \|G_\theta\|^{-1})^{n-m}\|x\|, \quad n \geq m. \end{aligned}$$

We have proved this by assuming that the left hand of (3.17) was not trivial, in other case it is evidently true. This completes the proof of Claim 11.

Claim 12. For all $\theta \in \Theta$ we have:

$$\|\Phi_{n,m}(\theta)(I - P_m(\theta))x\| \leq (1 + \|G_\theta\|)^2(\frac{\|G_\theta\|}{1 + \|G_\theta\|})^{m-n}\|x\|, \quad \text{if } n < m. \tag{3.18}$$

In fact, it can be proved in the same way as (3.17).

Finally if we define for all $\theta \in \Theta$ the following numbers:

$$M_\theta := (1 + \|G_\theta\|)^2 > \|G_\theta\|^2 \tag{3.19}$$

$$\alpha_\theta := \frac{\|G_\theta\|}{1+\|G_\theta\|} \geq 1 - \|G_\theta\|^{-1}. \tag{3.20}$$

Then (3.17) and (3.18) imply that $\hat{\pi} : \mathcal{E} \times Z \longrightarrow \mathcal{E}$ has pointwise discrete dichotomy over Θ with constants M_θ and α_θ given by (3.19) and (3.20).

□

3.3 Equivalence Between Pointwise and Uniform Discrete Dichotomy.

Now we want to know when pointwise discrete dichotomy and uniform discrete dichotomy are equivalent. The answer fo this question is given by the following lemmas.

Lemma 3.4 *Let $\hat{\pi} : \mathcal{E} \times Z \longrightarrow \mathcal{E}$ be a discrete skew-product and suppose there exists $1 < \delta < 2$ such that $\|\Phi_n(\theta)x\| \geq \delta\|x\|$ for all $n \in Z$, $\theta \in \Theta$, $x \in X$. Then pointwise discrete dichotomy and uniform discrete dichotomy of $\hat{\pi}$ over Θ are equivalent.*

Proof Clearly uniform discrete dichotomy implies pointwise discrete dichotomy .
Assume that $\hat{\pi}$ has pointwise discrete dichotomy with constants M_θ, $\alpha_\theta < 1$ for all $\theta \in \Theta$. Then from Theorem 3.1 we have that M_θ, α_θ are given by the formulas (3.19) and (3.20) respectively. Our goal is to find constants M, $\alpha < 1$ indenpendent of $\theta \in \Theta$.
Let $x \in I\!B = l_\infty(Z, X)$ and consider

$$\begin{aligned}
\|S_\theta\hat{x}\|_\infty &= \sup_{n\in Z}\{\|x_{n+1} - \Phi_n(\theta)x_n\|\} \\
&\geq \sup_{n\in Z}\{\|\Phi_n(\theta)x_n\| - \|x_{n+1}\|\} \\
&\geq \sup_{n\in Z}\{\delta\|x_n\| - \|x_{n+1}\|\} \\
&\geq (\delta - 1)\|\hat{x}\|_\infty = \gamma\|x\|_\infty, \quad \gamma = \delta - 1 < 1.
\end{aligned}$$

Therefore, $\|S_\theta\hat{x}\|_\infty \geq \gamma\|\hat{x}\|$, $\hat{x} \in I\!B$. Then

$$\|\hat{x}\| = \|S_\theta S_\theta^{-1}\hat{x}\| \geq \gamma\|S_\theta^{-1}\hat{x}\|, \quad \hat{x} \in I\!B.$$

So $\|S_\theta^{-1}\hat{x}\| \leq \frac{1}{\gamma}\|\hat{x}\| \Rightarrow \|S_\theta^{-1}\| \leq \frac{1}{\gamma}$, $\theta \in \Theta$.
Hence the operator $G_\theta = S_\theta^{-1}$ given by Theorem 3.1 has the following property

$$1 < \|G_\theta\| \leq \frac{1}{\gamma}.$$

Then, from (3.19) and (3.20) it is enough to take M and α as follows:

$$\begin{aligned}
M &:= (1 + \frac{1}{\gamma})^2 \geq (1 + \|G_\theta\|)^2 \geq \|G_\theta\|^2 \\
1 > \alpha &:= \frac{\frac{1}{\gamma}}{1 + \frac{1}{\gamma}} \geq \frac{\|G_\theta\|}{1 + \|G_\theta\|} \geq 1 - \|G_\theta\|^{-1}
\end{aligned}$$

or

$$M = \sup_{\theta\in\Theta}(1+\|G_\theta\|)^2 < \infty$$
$$\alpha = \sup_{\theta\in\Theta}\frac{\|G_\theta\|}{1+\|G_\theta\|} < 1.$$

□

Lemma 3.5 *Let $\hat{\pi} : \mathcal{E}\times Z \to \mathcal{E}$ be a discrete skew-product and suppose there exist $0<\delta_1<\delta_2<1$ such that*

$$\delta_1\|x\| \le \|\Phi_n(\theta)x\| \le \delta_2\|x\|, \quad n\in Z, \quad \theta\in\Theta, \quad x\in X. \tag{3.21}$$

Then pointwise discrete dichotomy and uniform discrete dichotomy of $\hat{\pi}$ over Θ are equivalent.

Proof Clearly uniform discrete dichotomy implies pointwise discrete dichotomy . Assume that $\hat{\pi}$ has pointwise discrete dichotomy with constants M_θ, $\alpha_\theta<1$ for all $\theta\in\Theta$. Then from Theorem 3.1 we have that M_θ, α_θ are given by the formulas (3.19) and (3.20) respectively. Again, our goal is to find constants M, $\alpha<1$ indenpendent of $\theta\in\Theta$. In order to do that, we shall consider the following linear bounded operators:

$$T(\theta): \mathbb{B}\to\mathbb{B}, \quad (T(\theta)\hat{x})_n = \Phi_n(\theta)x_n, \quad \theta\in\Theta. \tag{3.22}$$
$$L:\mathbb{B}\to\mathbb{B}, \quad (L\hat{x})_n = x_{n+1}, \quad \text{- the shift operator.}$$

Then the operator $S_\theta:\mathbb{B}\to\mathbb{B}$ given by Theorem 3.1 can be written as follows

$$S_\theta = L-\Phi(\theta) \tag{3.23}$$

It is easy to see that L is an isomorphism with $\|L\hat{x}\| = \|\hat{x}\|$, $\hat{x}\in X$. So $\|L\| = \|L^{-1}\| = 1$. On the other hand, we have that: $\|T(\theta)\hat{x}\|_\infty = \sup_{n\in Z}\|\Phi_n(\theta)x_n\|$. Hence, from (3.21) we get

$$\delta_1\le\|T(\theta)\|\le\delta_2<1, \quad \theta\in\Theta.$$

Therefore $\|L^{-1}T(\theta)\|<1$. It is well known that $(I-L^{-1}T(\theta))$ is invertible and

$$(I-L^{-1}T(\theta))^{-1} = \sum_{k=0}^{\infty}(L^{-1}T(\theta))^k.$$

From (3.23) we have the following

$$S_\theta^{-1} = \left[L(I-L^{-1}T(\theta))\right]^{-1} = (I-L^{-1}T(\theta))^{-1}L^{-1}.$$

Hence

$$\|S_\theta^{-1}\| \le \|(I-L^{-1}T(\theta))^{-1}\| \le \sum_{k=0}^{\infty}\|T(\theta)\|^k$$
$$= \frac{1}{1-\|T(\theta)\|} \le \frac{1}{1-\delta_1} = \frac{1}{\gamma}, \quad \gamma = 1-\delta_1, \quad \theta\in\Theta.$$

From here the proof follows as in Lemma 3.4. □

Corolary 3.2 *If $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ has pointwise discrete dichotomy over Θ, then the family of projections $\{P_n(\theta) : \theta \in \Theta,\ n \in Z\}$ associated with the pointwise discrete dichotomy of $\hat{\pi}$ is unique.*

Proof Assume that $\{Q_n(\theta)\}$ is another family of projection associated with the pointwise discrete dichotomy of $\hat{\pi}$. Let $x \in X$ and consider the sequence $f = \{f_n\}$ such that $f_{m-1} = x$, and $f_n = 0$ for $n \neq m-1$. For $\theta \in \Theta$ consider $\{x_n\}$ the unique bounded solution of

$$x_{n+1} = \Phi_n(\theta)x_n + f_n.$$

Then

$$x_n = \sum_{-\infty}^{\infty} G_{n,k+1}(\theta)f_k = \sum_{-\infty}^{\infty} \bar{G}_{n,k+1}(\theta)f_K$$

then $G_{n,m}(\theta)x = \bar{G}_{n,m}(\theta)x$. Where

$$G_{n,m}(\theta) = \begin{cases} \ \ \Phi_{n,m}(\theta)P_m(\theta), & \text{if } n \geq m \\ -\ \Phi_{n,m}(\theta)[I - P_m(\theta)], & \text{if } n < m \end{cases}$$

$$\bar{G}_{n,m}(\theta) = \begin{cases} \ \ \Phi_{n,m}(\theta)Q_m(\theta), & \text{if } n \geq m \\ -\ \Phi_{n,m}(\theta)[I - Q_m(\theta)], & \text{if } n < m \end{cases}$$

So

$$G_{m,m}(\theta)x = \bar{G}_{m,m}(\theta)x \iff P_m(\theta)x = Q_{m,m}(\theta)x$$

Therefore $P_m(\theta) = Q_m(\theta),\quad \theta \in \Theta,\quad m \in Z.$

□

Corolary 3.3 *For $\gamma > 0$ consider*

$$S_\gamma = \{\hat{x} = \{x_n\} \in \mathbb{B} : \sup_{n \in Z} \gamma^n \|x_n\| < \infty\}$$

where $\mathbb{B} = l_\infty(Z, X)$. Suppose that for each $\theta \in \Theta$ we have $0 < \alpha_\theta < 1$, $\alpha_\theta \leq \gamma \leq \frac{1}{\alpha_\theta}$ and for each $f \in S_\gamma$ there exists a unique solution of

$$x_{n+1} = \Phi_n(\theta)x_n + f_n \quad in \quad S_\gamma.$$

Then the skew-product sequence $\hat{\pi}$ has pointwise discrete dichotomy over Θ.
Moreover, if $\{G_{n,m}(\theta)\}$ is the corresponding Green's family, then $\|G_{n,m}(\theta)\| \leq M\alpha_\theta^{|n-m|}$.

3.4 Roughness for Discrete Dichotomy.

In this section we shall prove that discrete dichotomy for discrete skew-product s is not destroyed by small perturbations. For the proof of the following Theorem we shall use a Lemma from Henry [11], which says

Lemma 3.6 *If* $a \geq 0,\ \ b \geq 0,\ \ 0 < r < r_1,\ \ r_2 < 1,\ \ b < \frac{r_j - r}{1 + r r_j}$ *for* $j = 1, 2$ *and* $\{g_n\}$ *is a nonnegative sequence in* $I\!R$ *with* $g_n = O(r_2^{-|n|})$ *as* $|n| \to \infty$, *and*

$$g_n \leq a r_1^{|n|} + b \sum_{-\infty}^{\infty} r^{|n-k-1|} g_k, \quad n \in Z$$

then

$$g_n \leq \frac{a r_1^{|n|}}{\dfrac{1 - b(1 + r r_1)}{r_1 - r}}, \quad n \in Z.$$

Theorem 3.2 *Assume the discrete skew-product* $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ *has a pointwise discrete dichotomy over* Θ *with constants* M_θ, $\alpha_\theta < 1$. *If* $\tilde{M}_\theta > M_\theta$ *and* $\alpha_\theta < \tilde{\alpha}_\theta < 1$, *then there exists* $\epsilon_\theta > 0$, *such that for any discrete skew-product* $\hat{\pi}_1 : \mathcal{E} \times Z \to \mathcal{E},\ \ \hat{\pi}_1 = (\Psi_n(\cdot), \sigma)$ *with*

$$\sup_{n \in Z} \|\Phi_n(\theta) - \Psi_n(\theta)\| < \epsilon_\theta, \quad \theta \in \Theta$$

has a pointwise discrete dichotomy over $\theta \in \Theta$.

Proof From Lemma 3.6 the requirement on ϵ_θ is

$$\epsilon_\theta M_\theta < \frac{\tilde{\alpha}_\theta - \alpha_\theta}{1 + \alpha_\theta \tilde{\alpha}_\theta}$$

and

$$\tilde{M}_\theta \left(1 - \frac{\epsilon_\theta M_\theta (1 + \alpha_\theta \tilde{\alpha}_\theta)}{\tilde{\alpha}_\theta \alpha_\theta}\right) \geq M_\theta, \quad \theta \in \Theta.$$

For each $\theta \in \Theta$ and $f \in I\!B$, $x_{n+1} = \Psi_n(\theta) x_n + f_n$ has a unique solution $\hat{x} \in I\!B$ if and only if

$$x_n = \sum_{\infty}^{\infty} G_{n,k+1}(\theta)[(\Psi_k(\theta) - \Phi_n(\theta)) x_k + f_k]$$

is also solvable for each $f \in I\!B$; and it is true provided that

$$\sup_n \sum_{\infty}^{\infty} \|G_{n,K+1}(\theta)(\Psi_k(\theta) - \Phi_n(\theta))\| \leq \epsilon_\theta M_\theta \frac{1 + \alpha_\theta}{1 - \alpha_\theta} < 1.$$

In this case Theorem 3.1 shows that $\hat{\pi}_1$ has pointwise discrete dichotomy over Θ.

Let $\{\hat{G}_{n,m}(\theta)\}$ be the corresponding discrete Green'sfunction. Then for all n, m we have:

$$\hat{G}_{n,m}(\theta) = G_{n,m}(\theta) + \sum_{\infty}^{\infty} G_{n,K+1}(\theta)(\Psi_k(\theta) - \Phi_n(\theta))\hat{G}_{n,m}(\theta)$$

So

$$\|\hat{G}_{n,m}(\theta)\| \leq M_\theta \alpha_\theta^{|n-m|} + \epsilon_\theta M_\theta \sum_{-\infty}^{\infty} \alpha_\theta^{|n-k-1|} \|\hat{G}_{n,m}(\theta)\|$$

and $\|\hat{G}_{n,m}(\theta)\|$ is bounded.
From Lemma 3.6 above we get the following: If $\alpha_\theta < \tilde{\alpha}_\theta < 1$ and

$$\epsilon_\theta M_\theta < \frac{\tilde{\alpha}_\theta - \alpha_\theta}{1 + \alpha_\theta \tilde{\alpha}_\theta} < \frac{1 - \alpha_\theta}{1 + \alpha_\theta} < 1$$

then

$$\|\hat{G}_{n,m}(\theta)\| \leq \frac{M_\theta \tilde{\alpha}_\theta^{|n-m|}}{\dfrac{1 - \epsilon_\theta M_\theta (1 + \alpha_\theta \tilde{\alpha}_\theta)}{\tilde{\alpha}_\theta - \alpha_\theta}} \leq \tilde{M}_\theta \tilde{\alpha}_\theta^{|n-m|}, \quad \text{for small } \epsilon_\theta.$$

Corolary 3.4 *Assume the discrete skew-product $\tilde{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ has uniform discrete dichotomy over Θ with constants M, $\alpha < 1$, then there exist $\epsilon > 0$, such that any discrete skew-product $\tilde{\pi}_1 : \mathcal{E} \times Z \to \mathcal{E}$ with*

$$\sup_{n \in Z} \|\Phi_n(\theta) - \Psi_n(\theta)\| < \epsilon, \quad \theta \in \Theta$$

has uniform discrete dichotomy over Θ.

Theorem 3.3 *Assume that the discrete skew-product s $\tilde{\pi}_1$, $\tilde{\pi}_2$ have pointwise discrete dichotomy with discrete Green'sfunction satisfying*

$$\|G^i_{n,m}(\theta)\| \leq M_\theta \alpha_\theta^{|n-m|}, \quad i = 1, 2. \quad \alpha_\theta < 1.$$

Then

(a) if $\|\Phi^1_n(\theta) - \Phi^2_n(\theta)\| \leq \epsilon$ for $|n - m| \leq N$ and $\|\Phi^1_n(\theta) - \Phi^2_n(\theta)\| \leq B$ $n \in Z$; then

$$\|P^1_m(\theta) - P^2_m(\theta)\| \leq \frac{2M_\theta^2}{1 - \alpha_\theta^2}\left(\epsilon + B\alpha_\theta^{2N+1}\right)$$

(b) if $b < \alpha_\theta^{-2}$ and $\|\Phi^1_n(\theta) - \Phi^2_n(\theta)\| \leq \epsilon b^{|n-m|}$, $n \in Z$; then

$$\|P^1_m(\theta) - P^2_m(\theta)\| \leq \alpha_\theta(1 + b)\frac{M_\theta^2 \epsilon}{1 - b\alpha_\theta^2}.$$

Lemma 3.7 *Assume that $\hat{\pi} : \mathcal{E} \times Z_+ \to \mathcal{E}$ has a pointwise discrete dichotomy over Θ with constants M_θ, $\alpha_\theta < 1$ and $G_{n,m}(\theta)$, $n, m \geq 0$ the corresponding discrete Green'sfunction. Define the extension of $\hat{\pi}$ to $\mathcal{E} \times Z$ as follows: For all $\theta \in \Theta$ and $n < 0$*

$$P_n(\theta) = P_0(\theta) \quad and \quad \Phi_n(\theta) := \alpha_\theta(I - P_0(\theta)) + \alpha_\theta^{-1} P_0(\theta).$$

Then $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ has pointwise discrete dichotomy over Θ with constants M_θ, $\alpha_\theta < 1$.

Proof We shall check the conditions of Definition 3.2. In fact,

(a) $\Phi_n(\theta)P_n(\theta) = P_{n+1}(\theta)\Phi_n(\theta) \quad n \geq 0, \quad \theta \in \Theta.$
which is true by hypothesis. On the other hand for $n < 0$ we have:

$$\begin{gathered}\Phi_n(\theta)P_n(\theta) = \left(\alpha_\theta(I - P_0(\theta)) + \alpha_\theta^{-1} P_0(\theta)\right) P_0(\theta) = \\ P_0(\theta) \left(\alpha_\theta(I - P_0(\theta)) + \alpha_\theta^{-1} P_0(\theta)\right) = P_{n+1}(\theta)\Phi_n(\theta).\end{gathered}$$

(b) $\Phi_n(\theta) : \mathcal{N}(P_n(\theta)) \to \mathcal{N}(P_{n+1}(\theta))$ is an isomorphism for $n \geq 0$.
If $n < 0$, then $\Phi_n(\theta) : \mathcal{N}(P_n(\theta)) \to \mathcal{N}(P_{n+1}(\theta))$ is also an isomorphism. In fact, if $x \in \mathcal{N}(P_n(\theta)) = \mathcal{N}(P_0(\theta))$, then

$$\Phi_n(\theta)x = \alpha_\theta(I - P_0(\theta))x = \alpha_\theta x$$

which is an isomorphism.

(c) $\|\Phi_{n,m}(\theta)P_m(\theta)\| \leq M\alpha_\theta^{n-m}\|x\|, \quad n \geq m \geq 0.$

If $n \geq 0 > m$, then

$$\Phi_{n,m}(\theta) = \Phi_{n,0}(\theta)\Phi_{0,m}(\theta) \text{ and } P_m(\theta) = P_0(\theta)$$

$$\begin{aligned}\Phi_{0,m}(\theta)P_m(\theta) &= \Phi_{-1}(\theta)\Phi_{-2}(\theta)\cdots\Phi_m(\theta)P_0(\theta) \\ &= \Phi_{-1}(\theta)\Phi_{-2}(\theta)\cdots\Phi_{m+1}(\theta)\alpha_\theta^{-1}P_0(\theta) \\ &= \alpha_\theta^{-m}P_0(\theta)\end{aligned}$$

Then

$$\begin{aligned}\|\Phi_{n,m}(\theta)P_m(\theta)\| &= \|\Phi_{n,0}(\theta)\alpha_\theta^{-m}P_0(\theta)x\| \\ &\leq M_\theta\alpha_\theta^{n-m}\|x\|.\end{aligned}$$

If $0 > n \geq m$, then

$$\Phi_{n,m}(\theta)P_m(\theta)x = \alpha_\theta^{-m}P_0(\theta)x.$$

So

$$\|\Phi_{n,m}(\theta)P_m(\theta)x\| \leq \alpha_\theta^{|n-m|}\|x\|.$$

(d) If $m > n \geq 0$, then

$$\|\Phi_{n,m}(\theta)P_m(\theta)x\| \leq M_\theta \alpha_\theta^{m-n}\|x\|.$$

If $m \geq 0 > n$, then we have the following

$$\begin{aligned}\Phi_{n,m}(\theta)(I - P_m(\theta))x &= \Phi_{n,0}(\theta)\Phi_{0,m}(\theta)(I - P_m(\theta))x \\ &= \Phi_{0,n}^{-1}(\theta)\Phi_{0,m}(\theta)(I - P_m(\theta))x \\ &= \alpha_\theta^{-n}\Phi_{0,m}(\theta)(I - P_m(\theta))x.\end{aligned}$$

Therefore

$$\begin{aligned}\|\Phi_{n,m}(\theta)(I - P_m(\theta))x\| &\leq \alpha_\theta^{-1}\|\Phi_{0,m}(\theta)(I - P_m(\theta))x\| \\ &\leq M_\theta \alpha_\theta^{m-n}\|x\|.\end{aligned}$$

If $0 > m > n$, then

$$\Phi_{n,m}(\theta)(I - P_m(\theta))x = \alpha_\theta^{m-n}(I - P_0(\theta))x$$

So

$$\|\Phi_{n,m}(\theta)(I - P_m(\theta))x\| \leq M_\theta \alpha_\theta^{m-n}\|x\|.$$

□

Theorem 3.4 *Assume $\hat{\pi} : \mathcal{E} \times Z_+ \to \mathcal{E}$ has pointwise discrete dichotomy with constants M_θ, $\alpha_\theta < 1$. If $\tilde{M}_\theta > M_\theta$ and $\alpha_\theta < \tilde{\alpha}_\theta < 1$, then there exists $\mathcal{E}_\theta > 0$ such that for any discrete skew-product $\hat{\pi} : \mathcal{E} \times Z_+ \to \mathcal{E}$ with*

$$\sup_{n \geq 0} \|\Phi_n(\theta) - \Phi_n^1(\theta)\| < \mathcal{E}_\theta, \quad \textit{for } \theta \in \Theta$$

has pointwise discrete dichotomy over Θ, here $\hat{\pi} = (\Phi_n(\cdot), \sigma)$, $\hat{\pi}_1 = (\Phi_n^1(\cdot), \sigma)$.

4 Roughness

In this section we shall give a necessary and sufficient condition for skew-product semiflow to have an exponential dichotomy over Θ. Also, we will prove that the exponential dichotomy is not destroyed by small perturbation. In order to do so we will use the concept of discrete dichotomy introduced in the foregoing section.

We begin this section with a proposition on the relation between skew-product semiflow and skew-product sequence.

Proposition 4.1 *If $\pi = (\Phi, \sigma)$ is a skew-product semi-flow on $\mathcal{E}$, then the mapping $\hat{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ given by*

$$\hat{\pi}(x, \theta, n) := (\Phi_n(\theta), \theta \cdot n) \tag{4.1}$$

where

$$\Phi_n(\theta) = \Phi(\theta \cdot n, 1)$$

is a skew-product sequence.

Proof Follows directly from Definition 3.1 □

Remark 4.1 Even though $\Phi(\theta, t)$ is defined in principle only for $t \geq 0$, $\Phi_n(\theta) = \Phi(\theta \cdot n, 1)$ is well defined for all $n \in Z$ since $\sigma(\theta, t) = \theta \cdot t$ is defined for all $t \in I\!R$. This could be one of the advantage of using discrete skew-product.

Lemma 4.1 *Let $\pi = (\Phi, \sigma)$ be a skew-product semi-flow on $\mathcal{E}$. Suppose that π has an exponential dichotomy over Θ (with exponent β and k). Then the skew-product sequence $\hat{\pi}$ given by (4.1) has uniformdiscrete dichotomy over Θ, with constants $M = K$ and $\alpha = e^{-\beta}$.*

Proof Let **P** be the projector associated to the exponential dichotomy of π over Θ. Define the family of projections $\{P_n(\theta)\}$ as follows:

$$P_n(\theta) := P(\theta \cdot n), \quad \text{for all } \theta \in \Theta \text{ and } n \in Z$$

we shall prove the condition (a) - (d) of Definitions 3.2 and 3.3.
In order to see (a),consider

$$\Phi_n(\theta)P_n(\theta) = \Phi(\theta \cdot n, 1)P(\theta \cdot n) =$$

$$P(\theta \cdot n.1)\Phi(\theta \cdot n, 1) = P(\theta.(n+1))\Phi_n(\theta)$$

$$= P_{n+1}(\theta)P_n(\theta), \quad n \in Z, \quad \theta \in \Theta$$

(b) $\Phi_n(\theta) : \mathcal{N}(P_n(\theta)) \to \mathcal{N}(P_{n+1}(\theta))$ is an isomorphism.
In fact, from the Definition 2.5 we know that:

$$\Phi(\theta \cdot n, 1) : \mathcal{N}(P(\theta \cdot n)) \to \mathcal{N}(P(\theta.(n+1))$$

is an isomorphism.
Therefore

$$\Phi_n(\theta) : \mathcal{N}(P_n(\theta)) \to \mathcal{N}(P_{n+1}(\theta))$$

is an isomorphism.
(c) If we put $\Phi_{n,m}(\theta) := \Phi_{n-1}(\theta) \cdots \Phi_m(\theta)$, $n > m$ and $\Phi_{m,m}(\theta) := I$, $\theta \in \Theta$
Then

$$\|\Phi_{n,m}(\theta)P_m(\theta)\| = \|\Phi(\theta.(n-1), 1) \cdots \Phi(\theta.m, 1).P(\theta.m)\|$$
$$= \|\Phi(\theta.m, n-m)P(\theta.m)\| \leq Me^{\beta(n-m)} \text{ for all } \theta \in \Theta \text{ and } n \geq m.$$

Hence

$$\|\Phi_{n,m}(\theta)P_m(\theta)\| \leq M\alpha^{n-m}, \quad \text{for all } \theta \in \Theta \text{ and } n \geq m.$$

In the same way we can prove (d), this means

$$\|\Phi_{n,m}(\theta)(I - P_m(\theta))\| \leq M\alpha^{m-n}, \quad m > n \text{ and } \theta \in \Theta$$

This completes the proof of the Theorem. □

The following theorem give us a sufficient condition for the skew-product semiflow on $\mathcal{E}$ to have an exponential dichotomy over Θ.

Theorem 4.1 *Assume that $\pi = (\Phi, \sigma)$ is a skew-product semiflow on $\mathcal{E}$ and $M; \alpha$ are positive constant such that $\alpha = exp(-\beta) < 1$. Consider*

$$L := \sup\{\|\Phi(\theta, t)\| : 0 \leq t \leq 1, \quad \theta \in \Theta\} < \infty \tag{4.2}$$

If the skew-product sequence $\hat{\pi}$ given by (4.1) has uniform discrete dichotomy over Θ with constant M and α, then $\pi = (\Phi, \sigma)$ has exponential dichotomy with exponent β and constantKM where

$$K = \sup\{\|\Phi(\theta, t)\|e^{\beta t} : \quad 0 \leq t \leq 1, \quad \theta \in \Theta\} \tag{4.3}$$

Proof Suppose that $\{P_n(\theta)\}$ is the family of projections associated with the uniform discrete dichotomy of $\hat{\pi} = (\Phi_n(\cdot), \sigma)$. Define $P : \mathcal{E} \to \mathcal{E}$ as follows:

$$P(x, \theta) := (P(\theta)x, \theta); \quad P(\theta) := P_0(\theta).$$

Claim 1. $P_k(\theta) = P(\theta \cdot k) = P_0(\theta \cdot k)$ for all $\theta \in Z$ and $k \in Z$
In fact. Consider the skew-product sequence $\hat{\pi}_k : \mathcal{E} \times Z \to \mathcal{E}$ given by

$$\hat{\pi}_k(x, \theta, n) := (\Phi_n(\theta \cdot k), \theta \cdot n) \tag{4.4}$$

The $\hat{\pi}$ has uniform discrete dichotomy with family of projections $\{P_n(\theta \cdot k)\}$ and $\Phi_n(\theta \cdot k) = \Phi_{n+k}(\theta)$. Therefore $\hat{\pi}_k$ has uniform discrete dichotomy with family of projections $\{P_{n+k}(\theta)\}$. Then from the uniqueness we get that $P_{n+k}(\theta) = P_n(\theta \cdot k)$. In particular $P_k(\theta) = P_n(\theta.K) = P(\theta \cdot k)$

Claim 2. For all $n \geq K$ and $\theta \in \Theta$ we have the following:

$$P(\theta \cdot n)\Phi(\theta \cdot k, n - k) = \Phi(\theta \cdot k, n - k)P(\theta \cdot k)$$

In fact. It follows from the relation.

$$P_n(\theta)\Phi_{n,k}(\theta) = \Phi_k(\theta)P_k(\theta).$$

Claim 3. For any $t \geq 0$ and $\theta \in \Theta$ we have

$$\|\Phi(\theta, t)P(\theta)\| \leq KMe^{-\beta t}$$

In fact. Let $n \in Z_+$ be such that $n \leq t \leq n + 1$. Then

$$\begin{aligned}
&\|\Phi(\theta, t)P(\theta)\| = \|\Phi(\theta, t + n - n)P(\theta)\| \\
&= \|\Phi(\theta \cdot n, t - n)\Phi(\theta, n)P(\theta)\| \leq \|\Phi(\theta \cdot n, t - n)\|M\alpha^n \\
&= \|\Phi(\theta \cdot n, t - n)\|Me^{-\beta n} \\
&\leq \|\Phi(\theta \cdot n, t - n)\|e^{\beta(t-h)}Me^{-\beta t} \leq KMe^{-\beta t}.
\end{aligned}$$

Here we have used the fact that:

$$\Phi(\theta, n) = \Phi_{n,0}(\theta) = \Phi_{n-1}(\theta) \cdots \Phi_0(\theta)$$

For $x \in \mathcal{N}(P(\theta))$, $t \leq 0$ and $n \in Z$ with $n \leq t < n+1$. We shall define:

$$\Phi(\theta, t)x := \Phi(\theta \cdot n, t-n)\Phi(\theta, n)x \tag{4.5}$$

where

$$\Phi(\theta, n) = \Phi_{n,0}(\theta) = [\Phi_{0,n}(\theta)]^{-1}$$

and

$$\Phi_{0,n}(\theta) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta \cdot n))$$

is an isomorphism.

Claim 4. For $t \leq 0$, $\Phi(\theta, t) : \mathcal{N}(P(\theta)) \to X$ defined by (4.5) satisfies the following:

$$\|\Phi(\theta, t)(I - P(\theta)x\| \leq KMe^{\beta t}\|x\|.$$

In fact.

$$\begin{aligned}
\|\Phi(\theta,t)[I-P(\theta)]x\| &\leq \|\Phi(\theta\cdot n, t-n)\|\|\Phi(\theta,n)[I-P(\theta)]x\| \\
&\leq \|\Phi(\theta\cdot n, t-n)\|M\alpha^{-n}\|x\| = \|\Phi(\theta\cdot n, t-n)\|Me^{\beta n}\|x\| \\
&= \|\Phi(\theta\cdot n, t-n)\|e^{\beta n-\beta t}Me^{\beta t}\|x\| \\
&\leq \|\Phi(\theta\cdot n, t-n)\|e^{\beta(t-n)}Me^{\beta t}\|x\|KMe^{\beta t}
\end{aligned}$$

Claim 5. For all $\theta \in \Theta$ $\mathcal{B}^+(\theta) = \mathcal{R}(P(\theta))$.
In fact. From 2.3 we have that:

$$\mathcal{B}^+(\theta) = \{x \in X : \sup_{t \geq 0} \|\Phi(\theta, t)x\| < \infty\}$$

If $x \in \mathcal{R}(P(\theta))$, then $P(\theta)x = x$. So from Claim 3 we get

$$\|\Phi(\theta, t)x\| = \|\Phi(\theta, t)P(\theta)x\| \leq KMe^{-\beta t}$$

Hence $\Phi(\theta, \cdot)x$ is bounded for $t \geq 0$. This means that $\mathcal{R}(P(\theta)) \subset \mathcal{B}^+(\theta)$, $\forall \theta \in \Theta$.
Now we shall prove the following implication:

$$\text{If } x \in \mathcal{B}^+(\theta), \text{ then } x \in \mathcal{R}(P(\theta)).$$

In fact. If $x \notin \mathcal{R}(P(\theta))$, then

$$\Phi(\theta, t)x = \Phi(\theta, t)P(\theta)x + \Phi(\theta, t)(I - P(\theta))x$$

We already know that $\Phi(\theta, \cdot)P(\theta)x$ is bounded for $t \geq 0$. Hence, we only need to prove that $\Phi(\theta, t)(I - P(\theta))x$ is unbounded for $t \geq 0$. In fact. We have the following:

$$\|\Phi(\theta \cdot n, -n)(I - P(\theta \cdot n))v\| \leq Me^{-\beta n}\|v\|, \quad n \geq 0$$

where:

$$\Phi(\theta,n)=\Phi_{n,0}(\theta):\mathcal{N}(P(\theta))\to\mathcal{N}(P(\theta\cdot n))$$

is an isomorphism with inverse:

$$\Phi(\theta\cdot n,-n):=\Phi_{0,n}(\theta):\mathcal{N}(P(\theta\cdot n)\to\mathcal{N}(P(\theta)) \qquad (4.6)$$

Putting $v=\Phi(\theta,n)(I-P(\theta)x$ we get:

$$\begin{aligned}\|(I-P(\theta))x\| &= \|\Phi(\theta\cdot n,-n)(I-P(\theta\cdot n))\Phi(\theta,n)(I-P(\theta)x\| \\ &\leq Me^{-\beta n}\|\Phi(\theta,n)(I-P(\theta))x\|\end{aligned}$$

then

$$\|\Phi(\theta,n)(I-P(\theta))x\|\geq M^{-1}e^{\beta n}\|(I-P(\theta))x\|$$

Since $(I-P(\theta))x\neq 0$, then $\|\Phi(\theta,n)(I-P(\theta))x\|\to\infty$ as $n\to\infty$. So $x\notin\mathcal{B}^+(\theta)$.

Claim 6.

$$\Phi(\theta,t)\mathcal{R}(P(\theta))\subseteq\mathcal{R}(P(\theta\cdot t)),\quad t\geq 0$$

i.e.,

$$\Phi(\theta,t)\mathcal{R}(I-P(\theta))\subseteq\mathcal{R}(I-P(\theta\cdot t))$$

i.e.,

$$\Phi(\theta,t)\mathcal{N}(P(\theta))\subseteq\mathcal{N}(P(\theta\cdot t).$$

In fact. $x\in\mathcal{R}(P(\theta))\iff P(\theta)x=x$. Then $\Phi(\theta,t)x$ is bounded for $t\geq 0$ and

$$\Phi(\theta,s)x=\Phi(\theta\cdot t,s-t)\Phi(\theta,t)x,\quad s\geq t\geq 0$$

Putting $r=s-t\geq$ and $z=\Phi(\theta,t)x$. We get that $\Phi(\theta\cdot t,r)z$ is bounded for $r\geq 0$. Then from Claim 5 we get that

$$\Phi(\theta,t)x=z\in\mathcal{R}(P(\theta\cdot t)).$$

Claim 7.

$$\Phi(\theta,t):\mathcal{N}(P(\theta))\to\mathcal{N}(P)(\theta\cdot t)),\quad t\geq 0 \text{ is one to one}$$

In fact. For the purpose of contradiction let us suppose that there is $x\neq 0$ such that

$$x\in\mathcal{N}(P(\theta))\ \text{ and }\ \Phi(\theta,t)x=0$$

Consider $n\geq t$ and $\Phi(\theta,n)x=\Phi(\theta\cdot t,n-t)\Phi(\theta,t)x=0$. Hence

$$\Phi(\theta,n)=\Phi_{n,0}(\theta):\mathcal{N}(P(\theta))\to\mathcal{N}(P(\theta\cdot n))$$

is not one to one; which is a contradiction with the uniform discrete dichotomy of $\hat{\pi}$.

Claim 8.

$$\Phi(\theta\cdot t, s-t) : \Phi(\theta,t)\mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta\cdot s)) \quad s \geq t \geq 0 \text{ is one to one}$$

In fact. Suppose thre is $y \in \Phi(\theta,t)\mathcal{N}(P(\theta))$ such that $y \neq 0$ and $\Phi(\theta\cdot t, s-t)y = 0$. Let $n \geq s$ and $x \in \mathcal{N}(p(\theta))$ such that $\Phi(\theta,t)x = y$ Then

$$\begin{aligned}
\Phi(\theta,n)x &= \Phi(\theta\cdot s, n-s)\Phi(\theta,s)x \\
&= \Phi(\theta\cdot s, n-s)\Phi(\theta\cdot t, s-t)\Phi(\theta,t)x \\
&= \Phi(\theta\cdot s, n-s)\Phi(\theta\cdot t, s-t)y = 0
\end{aligned}$$

Hence, $\Phi(\theta,n)x = 0$ and $x \in \mathcal{N}(P(\theta))$ with $x \neq 0$. Which is a contradiction with the fact that $\Phi(\theta,n) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta\cdot n))$ is an isomorphism.

Let $x \in X$ and $t_1 < 0$. If there exists a unique $y \in X$ such that $\Phi(\theta\cdot t_1, -t_1)y = x$ we shall define: $\Phi(\theta,t_1)x := y$. In particular, if $x \in \mathcal{N}(P(\theta))$ and $t \leq 0$ we have the following

$$\Phi(\theta,t)x := \Phi(\theta\cdot n, t-n)\Phi(\theta,n)x = y$$

where $0 \leq t-n < 1$ and $\Phi(\theta,n) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta\cdot n))$ is an isomorphims. Indeed

$$\begin{aligned}
\Phi(\theta\cdot t, -t)y &= \Phi(\theta\cdot t, -t)\Phi(\theta\cdot n, t-n)\Phi(\theta,n)x \\
&= \Phi(\theta\cdot n\cdot(t-n), -t)\Phi(\theta\cdot n, t-n)\Phi(\theta,n)x \\
&= \Phi(\theta\cdot n, t-n-t)\Phi(\theta,n)x \\
&= \Phi(\theta\cdot n, -n)\Phi(\theta,n)x = x.
\end{aligned}$$

Claim 9. $\mathcal{N}(P(\theta)) = \mathcal{B}_u^-(\theta)$.
In fact. From 2.3 we have that

$$\mathcal{B}_u^-(\theta) = \{x \in X : \Phi(\theta,t)x \text{ is well defined and bounded for } t \leq 0\}$$

If $x \in \mathcal{N}(P(\theta))$, then

$$\Phi(\theta,t)x := \Phi(\theta\cdot n, t-n)\Phi(\theta,n)[I-P(\theta)]x$$

and from Claim 4 we get that

$$\|\Phi(\theta,t)x\| \leq KMe^{\beta t}\|x\|, \quad t \leq 0$$

Which implies that $\Phi(\theta,t)x$ is bounded for $t \leq 0$. Suppose that $x \notin \mathcal{N}(P(\theta))$, then putting

$$y = P(\theta)x \text{ and } z = (I-P(\theta))x \neq 0$$

we have two cases:
a) If $\Phi(\theta,t)x$ is not well defined, then $x \notin \mathcal{B}_u^-(\theta)$.

b) If $\Phi(\theta,t)x$ is well defined for $t \leq 0$, then $\Phi(\theta,t)z$ and $\Phi(\theta,t)x$ are well defined by using (4.5). Hence $\Phi(\theta,t)y$ is also well defined for $t \leq 0$. Therefore:

$$\Phi(\theta,t)x = \Phi(\theta,t)z + \Phi(\theta,t)y$$

We shall prove that $\Phi(\theta,t)y = \Phi(\theta,t)P(\theta)x$ is unbounded for $t \leq 0$. In fact

$$\begin{aligned} P(\theta)x &= \Phi(\theta\cdot n,-n)\Phi(\theta,n)P(\theta)x \\ &= \Phi(\theta\cdot n,-n)P(\theta\cdot n)\Phi(\theta,n)P(\theta)x, \quad n \leq 0 \end{aligned}$$

From Claim 3 we get that:

$$\|P(\theta)x\| \leq KMe^{\beta n}\|\Phi(\theta,n)P(\theta)x\|.$$

So

$$\|\Phi(\theta,n)P(\theta)x\| \geq (KM)^{-1}\|P(\theta)x\|e^{-\beta n} \quad \to \infty \text{ as } n \to -\infty$$

Since $\Phi(\theta,t)(I-P(\theta))x$ is bounded, we get that $\Phi(\theta,t)x$ is unbounded.
Thus

$$\mathcal{N}(P(\theta)) = \mathcal{B}_u^-(\theta), \quad \text{for all } \theta \in \Theta$$

Claim 10.

$$\Phi(\theta,t)\mathcal{N}(P(\theta)) \subset \mathcal{N}(P(\theta\cdot t)), \quad \text{for } t \leq 0$$

In fact. $x \in \mathcal{N}(P(\theta)) \iff x \in \mathcal{B}_u^-(\theta)$. Therefore, for all $x \in \mathcal{N}(P(\theta))$ and $s \leq t \leq 0$ we have

$$\Phi(\theta,s)x = \Phi(\theta\cdot t, s-t)\Phi(\theta,t)x.$$

So $\Phi(\theta\cdot t,s-t)\Phi(\theta,t)x$ is bounded for $s \leq t$. Then

$$\Phi(\theta,t)x \in \mathcal{B}_u^-(\theta\cdot t) = \mathcal{N}(P(\theta\cdot t)).$$

Claim 11.

$$\Phi(\theta,t) : \mathcal{N}(P(\theta)) \to \mathcal{N}(P(\theta\cdot t)), \quad t \geq 0$$

is an isomorphism. In fact, for all $x \in \mathcal{N}(P(\theta))$ we have

$$\Phi(\theta,t)x = \Phi(\theta\cdot n, t-n)\Phi(\theta,n)x$$

Where $0 \leq n \leq t < n+1$. Since $\Phi(\theta,n)\mathcal{N}(P(\theta)) = \mathcal{N}(P(\theta\cdot n))$ and using Claim 6 we get

$$\Phi(\theta,t)\mathcal{N}(P(\theta) = \Phi(\theta\cdot n,t-n)\mathcal{N}(P(\theta\cdot n)) \subseteq \mathcal{N}(P(\theta\cdot n\cdot(t-n))) = \mathcal{N}(P(\theta\cdot t)).$$

Now, if $x \in \mathcal{N}(P(\theta\cdot t)) = \mathcal{B}_u^-(\theta\cdot t)$, then

$$\Phi(\theta\cdot t, r+s)x = \Phi(\theta\cdot t\cdot r, s)\Phi(\theta\cdot t, r)x \text{ for all } r,s \in I\!\!R.$$

So

$$\Phi(\theta,t)\Phi(\theta\cdot t,-t)x=x.$$

If $y=\Phi(\theta\cdot t,-t)x \Rightarrow y\in\mathcal{N}(P(\theta))$ and $\Phi(\theta,t)y=x$.
Hence, from Claim 7 we get that

$$\Phi(\theta,t):\mathcal{N}(P(\theta))\to\mathcal{N}(P(\theta\cdot t)) \text{ is an isomorphism.}$$

Claim 12.

$$\Phi(\theta,t)P(\theta)=P(\theta\cdot t)\Phi(\theta,t), \quad \theta\in\Theta, \quad t\geq 0.$$

In fact. Clearly that for $t\geq 0$ we get

$$\Phi(\theta,t)P(\theta)x=P(\theta\cdot t)\Phi(\theta,t)x, \quad x\in\mathcal{N}(P(\theta)$$

$$\Phi(\theta,t)(I-P(\theta))x=(I-P(\theta\cdot t))\Phi(\theta,t)x, \quad x\in\mathcal{R}(P(\theta))$$

i.e.,

$$\Phi(\theta,t)P(\theta)x=P(\theta\cdot t)\Phi(\theta,t)x, \quad x\in\mathcal{R}(P(\theta)).$$

It follows that

$$\Phi(\theta,t)P(\theta)=P(\theta\cdot t)\Phi(\theta,t), \text{ for all } \theta\in\Theta \text{ and} t\geq 0$$

This completes the Proof of the Theorem

□

The following Theorem also gives a sufficient condition for the existence of exponential dichotomy for skew-product semiflow.

Theorem 4.2 *Let $\pi=(\Phi,\sigma)$ be a skew-product semiflow on $\mathcal{E}$. Assume there is an invariant projector* P *and for each $\theta\in\Theta$ there are constants $k_\theta\geq 1$, $\beta_\theta>0$ such that*
(1) $\Phi(\theta,t):\mathcal{N}(P(\theta))\to\mathcal{N}(P(\theta\cdot t))$, $t\geq 0$ is an isomorphism with inverse:

$$\Phi(\theta\cdot t,-t):\mathcal{N}(P(\theta\cdot t))\to\mathcal{N}(P(\theta)), \quad t\geq 0$$

(2) $\|\Phi(\theta,t)P(\theta)\|\leq k_\theta e^{-\beta_\theta t}$, $t\geq 0$
(3) $\|\Phi(\theta,t)(I-P(\theta)\|\leq k_\theta e^{\beta_\theta t}$, $t\leq 0$.
If one of the following conditions holds:
(4) there exists $1<\delta<2$ such that

$$\|\Phi(\theta,1)x\|\geq\delta\|x\|, \quad \theta\in\Theta, x\in X.$$

(5) there exist $0<\delta_1<\delta_2<1$ such that

$$\delta_1\|x\|\leq\|\Phi(\theta,1)x\|\leq\delta_2\|x\|, \quad \theta\in\Theta, \quad x\in X,$$

then π has an exponential dichotomy over Θ.

Proof It follows from Lemmas 3.4 and 3.5. □

The following Theorem says that the exponential dichotomy of the skew-product semiflow is not destroyed by small perturbation.

Theorem 4.3 *Suppose $\pi = (\Phi, \sigma)$ is a skew-product semiflow on $\mathcal{E}$ which has a exponential dichotomy (with exponent β and constant M). If*

$$L = \sup\{\|\Phi(\theta, t)\| : \quad 0 \le t \le 1, \quad \theta \in \Theta\}$$

and $Me^{-\beta} < e^{-\beta_1}$, $M_1 > M$, then there exists $\epsilon = \epsilon(\beta, \beta_1, M, M_1, L) > 0$ such that any skew-product semiflow $\tilde{\pi} = (\Psi, \sigma)$ on $\mathcal{E}$ satisfying

$$\sup\{\|\Phi(\theta, t) - \Psi(\theta, t)\| : \quad 0 \le t \le 1, \quad \theta \in \Theta\} \le \epsilon$$

has exponential dichotomy with exponent β_1 and constant M_1.

Proof Let $\hat{\pi}, \check{\pi} : \mathcal{E} \times Z \to \mathcal{E}$ be the skew-product sequences given by:

$$\hat{\pi}(x, \theta, n) := (\Phi_n(\theta)x, \theta \cdot n); \quad \check{\pi}(x, \theta, n) := (\Psi_n(\theta)x, \theta \cdot n)$$

where

$$\Phi_n(\theta) := (\Phi(\theta \cdot n, 1); \quad \Psi_n(\theta) := \Psi(\theta \cdot n, 1)$$

Clearly $\hat{\pi}$ has uniform discrete dichotomy with constants M and $\alpha = e^{-\beta}$. Consider for $t \ge 0$ the following:

$$\begin{aligned} \|\Phi(\theta, t) - \Psi(\theta, t)\| &= \|[\Phi(\theta \cdot k, t-k) - \Psi(\theta \cdot k, t-k)]\Psi(\theta, k) \\ &- \Phi(\theta \cdot k, t-k)[\Psi(\theta, k) - \Psi(\theta, k)]\| \end{aligned}$$

for $k \le t \le k+1$, $k \ge 0$. Since $0 \le t-k \le 1$, we get:

$$\|\Phi(\theta \cdot k, t-k) - \Psi(\theta \cdot k, t-k)\| \le \epsilon,$$

and

$$\begin{aligned} \|\Psi(\theta, k)\| &= \|\Psi(\theta \cdot (k-1), 1)\Psi(\theta \cdot (k-2), 1) \cdots \Psi(\theta, 1)\| \\ &= \|\Psi_{k-1}(\theta)\Psi_{k-2}(\theta) \cdots \Psi_0(\theta)\| \le (L+\epsilon)^k. \end{aligned}$$

Hence

$$\|\Phi(\theta, t) - \Psi(\theta, t)\| \le (L+\epsilon)^k . \epsilon + L\|\Psi(\theta, k) - \Phi(\theta, k)\|$$

In the same way we get that
$\|\Phi(\theta, k) - \Psi(\theta, k)\| =$

$$\|[\Phi(\theta.(k-1), 1) - \Psi(\theta.(k-1), 1)]\Psi(\theta, k-1) - \Phi(\theta.(k-1), 1)[\Psi(\theta, k-1) - \Phi(\theta, k-1]\|$$

$$\leq (L+\epsilon)^{k-1}\epsilon + L\|\Psi(\theta,k-1) - \Phi(\theta,k-1)\|$$

Therefore

$$\|\Phi(\theta,t) - \Psi(\theta,t)\| \leq [(L+\epsilon)^K + L(L+\epsilon)^{K-1} + \cdots + L^K]\epsilon$$

Then

$$\|\Psi(\theta,t) - \Psi(\theta,t)\| \leq C\epsilon, \quad 0 \leq t \leq 1, \quad \theta \in \Theta$$

Then

$$\|\Phi(\theta \cdot n, 1) - \Psi(\theta \cdot n, 1)\| < C\epsilon, \quad \theta \in \Theta$$

i.e.,

$$\|\Phi_n(\theta) - \Psi_n(\theta)\| < C\epsilon, \quad \theta \in \Theta$$

Since $M_1 > M$ and $1 > \alpha_1 = e^{-\beta_1} > e^{\beta} = \alpha$, then for ϵ small enough Corolary 3.6 implies that $\tilde{\pi}$ has uniform discrete dichotomy over Θ. Hence, it follows from Theorem 4.1 that $\tilde{\pi} = (\Psi, \sigma)$ has exponential dichotomy with constant KM_1 and exponent β_1.

□

5 Applications

In this section we shall consider a linear time dependent differential equation which generates a linear skew-product semiflow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$, where X is a Banach space and Θ is a compact topological Hausdorff space.

Consider the following linear time dependent system

$$\dot{x}(t) = A(\theta \cdot t)x(t), \quad t > 0 \tag{5.1}$$

where $A(\theta \cdot t) = A + B(\theta \cdot t)$, A is the infinitesimal generator of the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$; $\sigma(\theta,t) = \theta \cdot t$ is a flow on Θ and $B(\theta) \in L(X)$, $t \in I\!R$.

Lemma 5.1 *If $B(\cdot) : \Theta \to L(X)$ is strongly continuous, then the set $\{\|B(\theta)\| : \theta \in \Theta\}$ is bounded.*

Proof Consider the following sets

$$H = \{\|B(\theta)\| : \theta \in \Theta\}, \qquad H(x) = \{\|B(\theta)x\| : \theta \in \Theta\}$$

Since $\theta \to B(\theta)x$ is continuous and Θ is compact, then for each $x \in X$ we get that $H(x)$ is bounded. Hence, by the Uniform Boundedness Theorem we obtain that H is bounded.

□

Lemma 5.2 *If $B(\cdot) : \Theta \to L(X)$ is strongly continuous and $x(\cdot) : I\!R \to X$ is a continuous function, then for each $\theta \in \Theta$ the mapping $t \to B(\theta \cdot t)x(t)$ is continuous.*

Proof Fix $t \in I\!R$. Then

$$\begin{aligned} &\|B(\theta(t+h))x(t+h) - B(\theta \cdot t)x(t)\| \\ &= \|B(\theta(t+h))[x(t+h) - x(t)] - [B(\theta \cdot (t+h)) - B(\theta \cdot t)]x(t)\| \\ &\le L\|x(t+h) - x(t)\| + \|[B(\theta \cdot (t+h)) - B(\theta \cdot t)]x(t)\| \end{aligned}$$

where $L = \sup\{\|B(\theta)\| : \theta \in \Theta\}$ and $\theta \cdot t \in \Theta$ for $t \in I\!R$.
Therefore

$$\|B(\theta(t+h))x(t+h) - B(\theta \cdot t)x(t)\| \to 0, \quad \text{as} \quad h \to 0$$

□

To be precise in which sense the equation (5.1) generates a linear skew-product semiflow, we shall consider the following family of integral differential equations:

$$x(t) = T(t)x_0 + \int_0^t T(t-s)B(\theta \cdot s)x(s)\,ds. \quad t \ge 0 \quad \theta \in \Theta. \tag{5.2}$$

Definition 5.1 (Mild Solution). A solution $x(t) = x(t,\theta)$ of the equation (5.2) is called Mild Solution of (5.1).

The proof of the following theorem can be found in Chow and Leiva [3].

Theorem 5.1 *Let A be the infinitesimal generator of an strongly continuous semigroup $\{T(t)\}_{t\ge0}$ on X and $B(\cdot) : \Theta \to L(X)$ is also strongly continuous. Then for each $\theta \in \Theta$ and $x_0 \in X$ the problem*

$$\dot{x}(t) = A(\theta \cdot t)x = (A + B(\theta \cdot t))x(t); \quad x(0) = x_0 \tag{5.3}$$

has a unique mild solution $\Phi(\theta,t)x_0$ given by

$$\Phi(\theta,t)x_0 = T(t)x_0 + \int_0^t T(t-s)B(\theta \cdot s)\Phi(\theta,s)x_0 ds. \tag{5.4}$$

If

$$\|T(t)\| \le Me^{Wt}, \quad t \ge 0,$$

then

$$\|\Phi(\theta,t)\| \le Me^{(W+LM)t}, \quad t \ge 0 \tag{5.5}$$

where

$$L = \sup\{\|B(\theta)\| : \theta \in \Theta\}$$

Moreover, the mapping $\pi : \mathcal{E} \times I\!R_+ \to \mathcal{E}$ given by

$$\pi(x,\theta,t) = (\Phi(\theta,t)x, \theta \cdot t) \tag{5.6}$$

is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$.

Theorem 5.2 *Assume A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_0(t)\}_{t\geq 0}$, the mapping $\theta \to A(\theta) - A_0 : \Theta \to L(X)$ is strongly continuous and the equation $\dot{x}(t) = A(\theta \cdot t)x(t)$ has exponential dichotomy over Θ with exponent $\beta > 0$ and constant M. If $0 < \beta_1 < \beta$ and $M_1 > M$, then there exist $\epsilon > 0$ such that for any mapping $\theta \to B(\theta) : \Theta \to L(X)$ strongly continuous and $\|B(\theta)\| < \epsilon$, $\theta \in \Theta$ the equation*

$$\dot{x}(t) = (A(\theta \cdot t) + B(\theta \cdot t))\, x(t)$$

has exponential dichotomy over Θ with exponent β_1 and constant M_1.

Proof From Theorem 5.1 we get that the equations

$$\dot{x}(t) = A(\theta \cdot t)x(t) = (A_0 + (A(\theta \cdot t) - A_0))\, x(t)$$

and

$$\dot{x}(t) = (A(\theta \cdot t) + B(\theta \cdot t))\, x(t) = (A_0 + (A(\theta \cdot t) - A_0 + B(\theta \cdot t)))\, x(t)$$

generate respectively the skew-product semiflows:

$$\pi = (\Phi, \sigma), \tilde{\pi} = (\Psi, \sigma) : \mathcal{E} \times I\!R_+ \to \mathcal{E}, \quad \mathcal{E} = X \times \Theta$$

given by:

$$\Phi(\theta, t)x = T_0(t)x + \int_0^t T_0(t-s)(A(\theta \cdot s) - A_0)\Phi(\theta, s)x ds$$

$$\Psi(\theta, t)x = T_0(t)x + \int_0^t T_0(t-s)(A(\theta \cdot s)B(\theta \cdot s) - A_0)\Psi(\theta, s)x ds$$

Then

$$\|\Phi(\theta, t)x\| \leq M_0\|x\| + \int_0^t M_2\|\Phi(\theta, s)x\| ds, \quad 0 \leq t \leq 1$$

where

$$M_0 = \sup\{\|T_0(t)\| : \quad 0 \leq s \leq 1\}, \quad M_2 = M_0 \sup\{\|A(\theta) - A_0\| : \quad \theta \in \Theta\}.$$

From Gronwall's Lemma we get $\|\Phi(\theta, t)\| \leq M_0 e^{M_2} = L_1, \quad 0 \leq t \leq 1, \quad \theta \in \Theta$.
In the same way we get that $\|\Psi(\theta, t)\| \leq L_2, \quad 0 \leq t \leq 1, \quad \theta \in \Theta$. On the other hand, we have the followig

$$\begin{aligned}
\|\Phi(\theta, t)x - \Psi(\theta, t)x\| &\leq \int_0^t M_2\|\Phi(\theta, s)x - \Psi(\theta, s)x\| ds + \int_0^t M_0 L_2\|B(\theta \cdot s)x\| ds \\
&\leq M_0 L_2 \epsilon\|x\| + M_2 \int_0^t \|\Phi(\theta, s)x - \Psi(\theta, s)x\| ds, \quad 0 \leq t \leq 1.
\end{aligned}$$

Hence, from Gronwall's Lemma we get

$$\|\Phi(\theta, t) - \Psi(\theta, t)\| \leq M_0 L_2 \epsilon e^{M_2 t} \leq M_0 L_2 e^{M_2}\epsilon, \quad 0 \leq t \leq 1.$$

From Theorem 4.1 we get that the equation $\dot{x}(t) = (A(\theta \cdot t) + B(\theta \cdot t))\, x(t)$ has exponential dichotomy over Θ with exponent β_1 and constant M_1. □

Theorem 5.3 *Let $-A$ be a sectorial operator with a sector*

$$S_{-a,\phi} = \{\lambda \in C : \phi \leq |arg(\lambda + a)| \leq \pi, \quad \lambda \neq -a\} \subset \rho(-A)$$

where $\rho(-A)$ is the resolvent set of $-A$ and $a \in I\!R$, $0 < \phi < \pi/2$. Suppose that the spectrum $\sigma(A)$ does not intersect the strip region $\{\lambda \in C : \beta \leq Re\lambda \leq \alpha\}$, where $\beta < \lambda < a$. If for all $x \in X$ the mapping $\theta \to B(\theta)x : \Theta \to L(X)$ is continuous and $\sup\{\|B(\theta)\| : \theta \in \Theta\}$ *is small enough, then the equation*

$$\dot{x}(t) = (A + B(\theta \cdot t))x(t) \tag{5.7}$$

has an exponential dichotomy over $\theta \in \Theta$.

Proof It follows from Theorem 2.3 in [33] and Theorem 1.5.2 in [11] that the equation $\dot{x} = Ax$ has an exponential dichotomy over Θ. Finally, we can apply Theorem 5.2 to the perturbed equation (5.7). □

Now, consider the following family of functional differential equations which is more general than equation (1.6) studied in [18].

$$\dot{x}(t) = L(\theta \cdot t)x_t, \qquad \theta \in \Theta \tag{5.8}$$

where $\theta \cdot t$ is a flow on the compact Hausdorff set Θ which depend continuously on θ uniformly on compact interval of the time t; x_t denotes the function $s \to x(t+s)$, $-r \leq s \leq 0$. We assume that $L(\theta) : C = C\,[[-r,0], I\!R^n]] \to I\!R^n$ is linear and bounded operator, and for all $\phi \in C$ the mapping $\theta \to L(\theta)\phi$ is continuous.

Lemma 5.3 *If the equation (5.8) has an exponential dichotomy on Θ and $\theta \to M(\theta) \in L(C, I\!R^n)$ is strongly continuous. If $\|M(\theta)\|$ is small enough, then the equation*

$$\dot{x}(t) = (L(\theta \cdot t) + M(\theta \cdot t))\, x_t, \quad \theta \in \Theta, \tag{5.9}$$

has an exponential dichotomy over Θ.

Proof Using the same idea of [18] we can write the equation (5.9) in the following abstract way

$$\dot{z}(t) = (A(\theta \cdot t) + B(\theta \cdot t))\, z(t), \quad \theta \in \Theta, \quad z \in Z, \tag{5.10}$$

where $Z = I\!R^n \times L_2^n[-r,0]$ is a Hilbert space with the inner product

$$< \cdot, \cdot >_Z = < \cdot, \cdot >_{I\!R^n} + < \cdot, \cdot >_{L_2}.$$

The operators $A(\theta)$ are defined on a commun domain D given by

$$D = Dom\,(A(\theta)) = \{(v, \phi) \in Z : \phi \in W^{1,2}[-r,0], \;\; \phi(0) = v\}, \quad \theta \in \Theta,$$

by

$$A(\theta)(v, \phi) := \left(L(\theta)\phi, \dot{\phi}\right).$$

and $B(\theta)$ is a linear bounded operator from Z to Z given by

$$B(\theta)z = (M(\theta)\phi, 0)\,, \quad z = (v, \phi) \in Z.$$

So, for all $z \in Z$ mapping $\theta \longrightarrow B(\theta)z : \Theta \longrightarrow L(Z)$ is continuous and $\|B(\theta)\|$ is small enough. On the other hand, if we consider the operator

$$A_0(v, \phi) = (0, \dot{\phi}), \quad (v, \phi) \in D.$$

The operator A_0 is the infinitesimal generator of a strongly continuous semigroup $\{T_0(t)\}_{t\geq 0}$ and the mapping $\theta \longrightarrow A(\theta) - A_0 : \Theta \longrightarrow L(Z)$ is strongly continuous. Now, we finish the proof by applying the Theorem 5.2 to the equation (5.10). $\square$

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DYNAMICAL SPECTRUM FOR SKEW PRODUCT FLOW IN BANACH SPACES

S.N. CHOW AND H.LEIVA
School of Mathematics-CDSNS, Georgia Tech
Atlanta, Ga 30332

Abstract. In an earlier paper [1] we characterized the dynamical spectrum for Linear Skew-Product Semiflow in infinite dimensional Banach spaces. It was proved the spectrum is always a closed set, but it could be empty. Also we investigate the relation between the dynamical spectrum and the Lyapunov exponents. In this paper we shall characterize the dynamical spectrum for Linear Skew-Product Flow $\pi = (\Phi, \sigma)$ in a Banach space X. The fact that π is a flow allows us to prove that the spectrum is a nonempty compact set and get more information about it, also we can tell more about the Lyapunov exponents. Finally ours results can be applied to hyperbolic partial differential equations and neutral functional differential equations.

[1]

[1] *Key words.* time dependent linear differential equations, skew-product flow, dynamical spectrum, Lyapunov exponents.
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1 Introduction

In an earlier paper [1] we began an investigation of the dynamical spectrum for time dependent systems in infinite dimensional Banach spaces, using the concept of skew-product semiflow. That is the case for parabolic partial differential equations and functional differential equations; for that reason in [1] we use the concept of negative continuation and exponential dichotomy used by Sacker-Sell in [18]. In this definition of exponential dichotomy we assume that the unstable mainfold has finite dimension and is contained in the set of points that have a unique negative continuation. We characterized the dynamical spectrum and proved that the spectrum is always a closed set, but it could be empty; also, we investigate the relation between the dynamical spectrum and the Lyapunow exponents.

This paper is concerned with the dynamical spectrum for time dependent linear systems whose solutions are globally defined in $I\!R$. This is the case for hyperbolic partial differential equations, neutral functional differential equations and abstract ordinary differential equation $\dot{x} = A(t)x$ with bounded operator $A(t)$.

To study this problems we shall use the unified setting of a linear skew-product flow $\Phi(\theta,t) : \mathcal{E} \times I\!R \to \mathcal{E}$ (see Definition 2.1), where $\mathcal{E} = X \times \Theta$, X is a Banach space, Θ is a compact topological Hausdorff space and π is given by

$$\pi(x,\theta,t) = (\Phi(\theta,t)x, \theta.t), \quad t \in I\!R, \quad x \in X, \quad \theta \in \Theta$$

Many people have worked with skew-product flow, and semiflows in infinite dimensional Banach spaces. For example Sacker-Sell in [18] studied the existence of exponential dichotomy for the skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$.

Also, Magalhães in [11] give some characterization of the dynamical spectrum. A different characterization of the dynamical spectrum for skew-product flow in infinite Banach spaces appears in R. T. Rau [13]. He associates a strongly continuous group to the skew-product flow $\pi = (\Phi, \sigma)$ in the following way: Given a skew-product flow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ we can associate a family $\{T(t)\}_{t\in I\!R}$ of linear operators on the Banach space $C(\Theta, X)$ defined by

$$T(t)f(\theta) = \Phi(\theta \cdot (-t), t)f(\theta \cdot (-t)), \quad \forall t \in I\!R \tag{1.1}$$

for all $\theta \in \Theta$ and $f \in C(\Theta, X)$.

It is shown that the operator family $\{T(t)\}_{t\in I\!R}$ given by (1.1) is a strongly continuous group on $C(\Theta, X)$.

Since $\pi = (\Phi, \sigma)$ is a flow (two side flow) the definition of exponential dichotomy is the same as in the finite dimensional case; this allows Rau in [13] to give the following characterization of the dynamical spectrum

Theorem 1.1 *(Rau [13]) Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and denote G the infinitesimal generator of the evolution group $\{T(t)\}_{t\in I\!R}$ given by (1.1). Then*

$$\Sigma(\Theta) = ln|\sigma(T(1)) \setminus \{0\}| \supseteq G$$

A similar result as Rau's can be found in Latushki-Stepin [9]

In this paper we shall give a different characterization of the dynamical spectrum. Our characterization is an extension of the Sacker and Sell Theorem given in [18]. Here we proved that the spectrum can be written as a countable union of nonempty close disjoints intervals. we show the relation between the spectrum and the spectral subbundles associated with the corresponding spectral interval.

Also, this spectral decomposition can be used to study invariant manifold around an invariant set.

In this paper we follow closely the work done by Sacker and Sell in [18] for the finite dimensional case and the notation used for them in [19].

Since π is a flow (two side flow) the definitions of exponential dichotomy is very simple and we don't have to worry about negative continuation, like in [19], [1] and [11]. This allows us to prove that the spectrum is a nonempty compact set and we can give a simple characterization of the Lyapunov exponents in terms of the dynamical spectrum.

Finally we present some examples of Skew-Product Flow arising from hyperbolic partial differential equations and neutral functional differential equations.

2 Preliminaries

2.1 Linear Skew-Product Flow

In this section we shall present some definitions, notations, and results about Skew Product flows on Banach Bundles that we will use in the next sections.

Definition 2.1 Let $\mathcal{E} = X \times \Theta$ be given where X is a fixed Banach space (state space) and Θ is a compact Hausdorff Space. Assume that $\sigma(\theta, t) = \theta.t$ is a flow on Θ, i.e., the mapping $(\theta, t) \to \theta \cdot t$ is continuous, $\theta.0 = \theta$, and we have $\theta.(s+t) = (\theta \cdot s) \cdot t$ for all $s, t \in I\!R$. Then we shall call a Linear Skew-Product Flow $\pi = (\Phi, \sigma)$ *on* $\mathcal{E}$ as a mapping

$$\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t), \quad \forall t \in I\!R.$$

with the following properties:

(1) $\Phi(\theta, 0) = I$, the identity operator, for all $\theta \in \theta$.

(2) $\lim_{t \to 0} \Phi(\theta, t)x = x$, and this limit is uniform in θ. This mean that for every $x \in X$ and every $\epsilon > 0$ there is a $\delta = \delta(x, \epsilon) > 0$ such that $\|\Phi(\theta, t)x - x\| \leq \epsilon$, for all $\theta \in \Theta$ whenever $0 \leq t \leq \delta$.

(3) $\Phi(\theta, t)$ is a bounded linear mapping from X into X that satisfies the cocycle identity:

$$\Phi(\theta, s+t) = \Phi(\theta \cdot t, s)\Phi(\theta, t) \quad \theta \in \Theta; \quad s, t \in I\!R \tag{2.1}$$

(4) For each $t \in I\!R$ the mapping of $\mathcal{E}$ into X given by $(x, \theta) \to \Phi(\theta, t)x$ is continuous.

Properties (2) and (3) imply that for each $(x, \theta) \in \mathcal{E}$ the solution operator $t \longrightarrow \Phi(\theta, t)x$ is continuous for $t \in I\!R$. Indeed one has

$$\|\Phi(\theta, t+h)x - \Phi(\theta, t)x\| = \|[\Phi(\theta \cdot t, h) - I]\Phi(\theta, t)x\|$$

which goes to zero as $h \to 0$. The cocycle identity (2.1) implies that $\Phi(\theta, t)$ is an isomorphism with inverse

$$\Phi^{-1}(\theta, t) = \Phi(\theta \cdot t, -t) \quad \forall t \in I\!R$$

Proposition 2.1 *Let $\pi = (\Phi, \sigma)$ be a linear skew-product flow on $\mathcal{E}$. Then there exist constants $M \geq 1$, $a > 0$ such that*

$$\|\Phi(\theta, t)\| \leq Me^{a|t|}, \quad \theta \in \Theta, \quad t \in I\!R. \tag{2.2}$$

Proof First we claim that there is a $\delta > 0$ such that

$$M = \sup\{\|\Phi(\theta, t)\| : \theta \in \Theta, \quad -\delta \leq t \leq \delta\} < \infty$$

For the purpose of contradiction. Let us assume that there are sequences $\theta_n \in \Theta, t_n \in I\!R$ such that $t_n \to 0$ and $\|\Phi(\theta_n, t_n)\| > n$.

The Banach-Steinhaus Theorem (Uniform Boundedness Principle) implies that there is $x \in X$ such that

$$\{\|\Phi(\theta_n, t_n)x\| : n \in I\!N\}$$

is unbounded. This contradicts the fact that

$$\lim_{t \to 0} \Phi(\theta, t)x = x$$

unifomly for $\theta \in \Theta$. Therefore $M < \infty$. On the other hand we have that $\Phi(\theta, 0) = I$, so $M \geq 1$.

Now fix $t \in I\!R$. Assume that $t \geq 0$ (a similar argument will take care of the case $t \leq 0$). Let m be an integer satisfying $m \leq t/\delta \leq m+1 \Longleftrightarrow \delta_m \leq t \leq m\delta + \delta$. Then for any $\theta \in \Theta$ one has

$$\|\Phi(\theta,t)\| = \|\Phi(\theta, t-\delta m+\delta m)\| = \|\Phi(\theta.\delta,(t-\delta m)+\delta(m-1))\Phi(\theta,\delta)\|$$

Now putting:

$$\theta_0 = \theta, \quad \theta_1 = \theta_0 \cdot \delta, \quad \theta_2 = \theta_1 \cdot \delta, \cdots, \theta_m = \theta_{m-1} \cdot \delta$$

we get the following:

$$\|\Phi(\theta,t)\| = \|\Phi(\theta_m, t-\delta m)\Phi(\theta_{m-1},\delta)\ldots\Phi(\theta_0,\delta)\|$$

$$\leq M^{m+1} = M.M^m \leq M.M^{t/\delta}.$$

If we put

$$a = 1/\delta \ln(M)$$

then

$$\|\Phi(\theta,t)\| \leq Me^{at}$$

□

2.2 Projectors and Subbundles

A Banach bundle $\mathcal{E}$ with fiber X over a base space Θ with projection $\mathcal{P}$ is denoted by $(\mathcal{E}, X, \Theta, \mathcal{P})$, or $\mathcal{E}$ for short, and is defined as follows:

(1) X is a fixed Banach space and θ is a compact Hausdorff space.

(2) The mapping $\mathcal{P} : \mathcal{E} \to \Theta$ is a continuous mapping.

(3) For each $\theta \in \Theta, \mathcal{P}^{-1}(\theta) = \mathcal{E}(\theta)$ is a Banach space, which is referred to as the fiber over θ.

(4) For each $\theta \in \Theta$, there is an open neighborhood U of θ in Θ and a homeomorphism

$\tau : \mathcal{P}^{-1}(U) \to X \times U$ such that for each $\mathcal{N} \in U, \mathcal{P}^{-1}(\mathcal{N})$ is a mapped onto $X \times \{\mathcal{N}\}$ and $\tau : \mathcal{P}^{-1}(\mathcal{N}) \to X \times \{\mathcal{N}\}$ is a linear isomorphism.

(5) The norms $\|\cdot\| = \|\cdot\|_\theta$ on the fiber $\mathcal{P}^{-1}(\theta)$ vary continuously in θ.

One can use the local coordinate notation (x, θ) to denote a typical point in a Banach bundle $\mathcal{E}$. By this we mean that $(x, \theta) \in \mathcal{E}$. This is a shortened way to refer to property (4) above.

For any subset $\mathcal{F} \subset \mathcal{E}$ we define the fiber:

$$\mathcal{F}(\theta) := \{x \in X : (x, \theta) \in \mathcal{F}\}$$

$$\mathcal{E}(\theta) := X \times \{\theta\}$$

If $U \subset \Theta$, then we shall define the following set

$$\mathcal{F}(U) := \cup_{\theta \in U} \mathcal{F}(\theta)$$

$$\mathcal{E}_0 = \{(x, \theta) \in \mathcal{E} : x = 0\}$$

$\mathcal{E}_0$ is called the zero fiber.

A mapping $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ is said to be a projection if $\mathbf{P}$ is continuous and has the form $\mathbf{P}(x, \theta) = (P(\theta), \theta)$ where $P(\theta)$ is a bounded linear projection on the fiber $\mathcal{E}(\theta)$. For any projector $\mathbf{P}$ we define the range and the **null space** by

$$\mathcal{R} = \mathcal{R}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = x\}$$

and

$$\mathcal{N} = \mathcal{N}(\mathbf{P}) = \{(x, \theta) \in \mathcal{E} : P(\theta)x = 0\}.$$

Since **P** is continuous, this means that the fibers $\mathcal{R}(\theta)$ and $\mathcal{N}(\theta)$ vary continuously in θ. This also means that $P(\theta)$ varies continuous in the operator norm. The following result can be found in Sacker- Sell [18].

Lemma 2.1 *Let* $\mathbf{P}$ *be a projector on* $\mathcal{E}$*. Then* $\mathcal{R}$ *and* $\mathcal{N}$ *are closed in* $\mathcal{E}$*, and one has*

$$\mathcal{R}(\theta) \cap \mathcal{N}(\theta) = \{0\}, \quad \mathcal{R}(\theta) + \mathcal{N}(\theta) = \mathcal{E}$$

for all $\theta \in \Theta$.

Definition 2.2 A subset ν is said to be a **subbundle** of $\mathcal{E}$ if there is a projector P on $\mathcal{E}$ with the property that $\mathcal{R}(\mathbf{P}) = \mathcal{V}$

In this case $\mathcal{W} = \mathcal{N}(\mathbf{P})$ is a **Complementary** subbundle, *i.e.*, $\mathcal{E} = \mathcal{V} + \mathcal{W}$ as a Whitney Sum .

Lemma 2.2 *Let* $\mathcal{V} \subset \mathcal{E}$ *with the properties:*

(A) $\mathcal{V}$ *is closed.*

(B) $\mathcal{V}(\theta)$ *is a linear subspace of* $\mathcal{E}(\theta)$ *for all* $\theta \in \Theta$.

(C) codim $\mathcal{V}(\theta)$ *is finite for all* $\theta \in \Theta$.

(D) codim $\mathcal{V}(\theta)$ *is locally constamt on* Θ.

Then $\mathcal{V}$ *is a subbundle of* $\mathcal{E}$.

Proof See [18]. □

2.3 Stable, Unstable and the Initial Bounded Sets

Let $\pi = (\Phi, \sigma)$ be a given linear skew-product flow defined on $\mathcal{E} = X \times \Theta$. For $\lambda \in \mathbb{R}$ we define the shifted flow as follows:

$$\pi_\lambda = (\Phi_\lambda, \sigma), \quad \Phi_\lambda(\theta, t) = e^{\lambda t}\Phi(\theta, t) \text{ for } t \in \mathbb{R}, \quad \theta \in \Theta.$$

$$\mathcal{B}_\lambda = \{(x, \theta) \in \mathcal{E} : sup_{t \in \mathbb{R}} \|e^{-\lambda t}\Phi(\theta, t)x\| < +\infty\}$$

$$\mathcal{S}_\lambda = \{(x, \theta) \in \mathcal{E} : \|e^{-\lambda t}\Phi(\theta, t)x\| \to 0, \quad t \to +\infty\}$$

$$\mathcal{U}_\lambda = \{(x, \theta) \in \mathcal{E} : \|e^{-\lambda t}\Phi(\theta, t)x\| \to 0, \quad t \to -\infty\}$$

The set $\mathcal{U}_\lambda$ is the **unstable set**, $\mathcal{S}_\lambda$ is the **stable set**, and $\mathcal{B}_\lambda$ is the **initial bounded set** corresponding to π_λ. If $\lambda = 0$ we shall denote $\mathcal{B} = \mathcal{B}_0$, $\mathcal{U} = \mathcal{U}_0$ and $\mathcal{S} = \mathcal{S}_0$.

We are interested in knowing when $\mathcal{S}_\lambda$ and $\mathcal{U}_\lambda$ are complementary invariant subbundles of $\mathcal{E}$. The answer of this quation can be formulated in terms of dichotomies.

Definition 2.3 A project **P** on $\mathcal{E}$ is said to be invariant if we have

$$P(\theta.t)\Phi(\theta, t) = \Phi(\theta, t)P(\theta), \quad t \in \mathbb{R}, \quad \theta \in \Theta \tag{2.3}$$

which is equivalent to:

$$P(\theta.t) = \Phi(\theta, t)P(\theta)\Phi^{-1}(\theta, t), \quad t \in \mathbb{R}, \quad \theta \in \Theta \tag{2.4}$$

Definition 2.4 We shall say that a linear skew-product flow $\pi = (\Phi, \sigma)$ on $\mathcal{E}$ has an exponential dichotomy (ED) over an invariant set $\hat{\Theta}$, were $\hat{\Theta} \subset \Theta$, if there is an invariant Projector $\mathbf{P}$ on $\mathcal{E}$ and constants $k \geq 1$, $\beta > 0$ such that

$$\|\Phi(\theta, t)P(\theta)\Phi^{-1}(\theta, s)\| \leq ke^{-\beta(t-s)}, \quad s \leq t \tag{2.5}$$

$$\|\Phi(\theta, t)[I - P(\theta)]\Phi^{-1}(\theta, s)\| \leq ke^{\beta(t-s)}, \quad s \geq t \tag{2.6}$$

for all $\theta \in \hat{\Theta}$

Remark 2.1

(1) If $\hat{\Theta} = \{\theta\}$, then E.D corresponds to the usual concept of dichotomy

(2) If $\hat{\Theta} = \Theta$, then E.D over $\hat{\Theta}$ is equivalent to the splitting of $\mathcal{E}$.

(3) $P(\theta)$ varies continuously over $\hat{\Theta}$.

(4) k, β depend of $\hat{\Theta}$.

Proposition 2.2 *If π is a linear skew-product flow on $\mathcal{E} = X \times \Theta$ admits an exponential dichotony over Θ, then one has that the initial bounded set $\mathcal{B} = \mathcal{E}_0$ and the correspondent Projector* $\mathbf{P}$ *is such that:*

$$\mathcal{R}(\mathbf{P}) = \mathcal{S}(\hat{\Theta}), \quad \mathcal{N}(\mathbf{P}) = \mathcal{U}(\hat{\Theta})$$

$$\mathcal{E} = \mathcal{R}(\mathbf{P}) \oplus \mathcal{N}(\mathbf{P}) = \mathcal{S}(\hat{\Theta}) \oplus \mathcal{U}(\hat{\Theta})$$

(The whitney sum of two bundles)

Proof Let $\mathbf{P}$ be the correspondent Projector. Consider $(x, \theta) \in \mathcal{B}$ and set

$$y = P(\theta)x, \quad z = (I - P(\theta))x$$

Then we get the following:

$$\|\Phi(\theta,t)\| < N, \quad \forall t \in I\!R,$$

and

$$y = P(\theta)\Phi^{-1}(\theta,t)\Phi(\theta,t)x = \Phi(\theta,0)P(\theta)\Phi^{-1}(\theta,t)x$$

Therefore, from (2.5) we get:

$$\|y\| \leq ke^{\beta t}N, \qquad t < 0 \Rightarrow \|y\| = 0.$$

$$z = \Phi(\theta,0)[I - P(\theta)]\Phi^{-1}(\theta,t)\Phi(\theta,t)x$$

so

$$\|z\| \leq ke^{-\beta t}N, \quad t \in I\!R \Rightarrow \|z\| = 0$$

Hence

$$x = P(\theta)x + [I - P(\theta)]x = y + z = 0$$

It is easy to prove that :

$$\mathcal{R}(\mathbf{P}) \subset \mathcal{S} \quad \mathcal{N}(\mathbf{P}) \subset \mathcal{U}$$

Now, since $\mathcal{E} = \mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{P})$, then $\mathcal{S} = \mathcal{R}(\mathbf{P})$, $\mathcal{U} = \mathcal{N}(\mathbf{P})$.

□

3 The Dynamical Spectrum

Let $\hat{\Theta}$ be an invariant subset of Θ under the flow σ. Then the **resolvent** $\rho(\hat{\Theta})$ of $\hat{\Theta}$ under π is defined as follows:

$$\rho(\hat{\Theta}) := \{\lambda \in I\!R : \pi_\lambda \text{ admits and exponential dichotomy over } \hat{\Theta}\}.$$

The spectrum $\Sigma(\hat{\Theta})$ of $\hat{\Theta}$ under π is defined as follows

$$\Sigma(\hat{\Theta}) = I\!\!R \setminus \rho(\hat{\Theta})$$

Our main results are the following Theorems:

Theorem 3.1 *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and $\hat{\Theta}$ a compact connected invariant subset of Θ. Then the following statements are valid:*

(A) There is $a > 0$ such that

$$\|\Phi(\theta, t)\| \leq Me^{a|t|}, \quad \forall \theta \in \Theta \quad and \quad t \in I\!\!R,$$

and $\Sigma(\hat{\Theta}) \neq \emptyset, \quad \Sigma(\hat{\Theta}) \subset [-a, a]$

(B) For each set $\{\lambda_0 < \lambda_1 < \cdots < \lambda_m\} \subset \rho(\hat{\Theta})$ with $\lambda_0 < -a$ and *$a < \lambda_m$ such that $\Sigma(\hat{\Theta}) \cap (\lambda_{i_1}, \lambda_i) \neq \phi$ we get that*

$$\mathcal{V}_i := \mathcal{V}_i(\hat{\Theta}) = \mathcal{S}_{\lambda_i}(\hat{\Theta}), \quad i = 1, 2, \cdots, m$$

are invariant subbundles of $\mathcal{E}(\hat{\Theta})$.

(C) Let π^i be the restriction of π to $\mathcal{V}_i$ and $\Sigma_i(\hat{\Theta})$ the spectrum of $(\mathcal{V}_i, \pi^i)$ over $\hat{\Theta}$. Then one has that

$$\Sigma_i(\hat{\Theta}) = \Sigma(\hat{\Theta}) \cap (\lambda_{i-1}, \lambda_i), \quad i = 1, 2, \cdots, m.$$

(D) $\Sigma_i(\hat{\Theta}) = \Sigma(\hat{\Theta}) \cap (\lambda_{i-1}, \lambda_i), \quad i = 1, 2, \cdots, m$

(E) $\Sigma(\hat{\Theta}) = \bigcup_{i=1}^{m} \Sigma_i(\hat{\Theta})$

(F) $\mathcal{V}_i(\hat{\Theta}) \cap \mathcal{V}_j(\hat{\Theta}) = \mathcal{E}_0(\hat{\Theta}), \quad i \neq j$

(G) $\mathcal{E}(\hat{\Theta}) = \mathcal{V}_1(\hat{\Theta}) + \mathcal{V}_2(\hat{\Theta}) + \cdots + \mathcal{V}_m(\hat{\Theta})$ *(Whitney sum)*

In order to get more information about the spectrum we shall put some restriction on the unstable manifold $\mathcal{U}_\lambda$ for some $\lambda \in \rho(\hat{\Theta})$. Also we will need the following notation:

For $\lambda \in \rho(\hat{\Theta})$ we shall define:

$$\Sigma_\lambda(\hat{\Theta}) := \Sigma(\hat{\Theta}) \cap (-\infty, \lambda) \tag{3.1}$$

Theorem 3.2 *Assume that* $\dim \mathcal{E}(\hat{\Theta}) = \infty$, $\lambda \in \rho(\hat{\Theta})$, $a > 0$ *is as in part (A) of Theorem 3.1. Then the following statements are valid :*

(A) If $\dim \mathcal{U}_\lambda = n(\lambda) < \infty$, then $\lambda \geq -a$

(B) If $\dim \mathcal{S}_\lambda = m(\lambda) < \infty$, then $\lambda \leq a$

(C) If $\lambda \geq -a$ and $\mathcal{U}_\lambda \neq \mathcal{E}_0(\hat{\Theta})$, then $\lambda \leq a$

(D) If $\lambda \leq a$ and $\dim \mathcal{S}_\lambda \neq \mathcal{E}_0(\hat{\Theta})$, then $\lambda \geq -a$

(E) If $\dim \mathcal{U}_\lambda \to n(-a) < \infty$, as $\lambda \to -a^+$

then

$$-a \in \Sigma(\hat{\Theta}) \ \textit{and} \ \dim \mathcal{U}_\lambda = n(\lambda) < \infty \ \ \forall \lambda \in [-a + \infty) \cap \rho(\hat{\Theta})$$

(F) If $\dim \mathcal{S}_\lambda \to m(a) < \infty$, as $\lambda \to a^+$

then

$$a \in \Sigma(\hat{\Theta}) \ \textit{and} \ \dim \mathcal{S}_\lambda = m(\lambda) \ \ \forall \lambda \in (-\infty, a] \cap \rho(\hat{\Theta})$$

(G) If $1 \leq \dim \mathcal{U}_{\lambda_0} = n(\lambda_0) < \infty$, *then* $\lambda_0 \in [-a, a]$ *and*

$$\Sigma(\hat{\Theta}) = \Sigma_{\lambda_0}(\hat{\Theta}) \cup (\bigcap_{i=1}^{m} [a_i, b_i]) \tag{3.2}$$

$$m \leq \dim \mathcal{U}_{\lambda_0} = n(\lambda_0)$$

Moreover:

(H) $\Sigma_{\lambda_0}(\hat{\Theta}) = [a, \lambda_0) \cap \Sigma(\hat{\Theta})$

(I) If $\Sigma(\hat{\Theta}) \subset [\lambda_0, a] \Rightarrow \Sigma(\hat{\Theta}) = \bigcup_{i=1}^{m} [a_i, b_i]$.

(J) $\mathcal{E}(\hat{\Theta}) = \mathcal{S}_{\lambda_0}(\hat{\Theta}) + \mathcal{V}_1(\hat{\Theta}) + \ldots + \mathcal{V}_m(\hat{\Theta})$,

where

$$\mathcal{V}_i(\hat{\Theta}) = \mathcal{U}_{\lambda_i - 1} \cap \mathcal{S}_{\lambda_i}, \quad i = 1, 2, \ldots, m;$$

and

$$\{\lambda_0 < \lambda_1 < \ldots < \lambda_m\} \subset \rho(\hat{\Theta})$$

with

$$a < \lambda_m, \quad (\lambda_{i-1}, \lambda_i) \cap \Sigma(\hat{\Theta}) \neq \phi.$$

3.1 Lemmas

Here we shall derive a number of properties of the spectrum and the resolvent set which will be used in the proof of the main theorems.

Lemma 3.1 *Let $\hat{\Theta}$ be a compact invariant set in Θ and $\lambda \in \mathbb{R}$. Then the following statements are valid:*

(A) If $\|\Phi_\lambda(\theta, t)\| \to 0$ as $t \to +\infty$ for each $\theta \in \hat{\Theta}$, then $\lambda \in \rho(\hat{\Theta})$, $\Sigma(\hat{\Theta}) \subseteq (-\infty, \lambda)$, and $\mathcal{S}_\mu(\hat{\Theta}) = \mathcal{E}(\hat{\Theta})$ for all $\mu \geq \lambda$.

(B) If $\|\Phi_\lambda(\theta, t)\| \to 0$ as $t \to -\infty$ for each $\theta \in \hat{\Theta}$, then $\lambda \in \rho(\hat{\Theta})$, $\Sigma(\hat{\Theta}) \subseteq (\lambda, +\infty)$,

and $\mathcal{U}_\mu(\hat{\Theta}) = \mathcal{E}(\hat{\Theta})$ for all $\mu \leq \lambda$.

Proof We shall prove (A). The proof of (B) is similar. For each $\theta \in \hat{\Theta}$, the is $T(\theta) > 0$ such that $\|\Phi_\lambda(\theta, t)\| < \frac{1}{2}, \quad t \geq T(\theta)$.

Consider $x \in X$ fixed with $\|x\| = 1$. By the continuity of $\Phi_\lambda(\theta, t)x$ with respect to θ there exist a neighborhood $N_x(\theta)$ of θ such that $\|\Phi_\lambda(\tilde{\theta}, T(\theta))x\| < \frac{1}{2}$, for all $\tilde{\theta} \in N_x(\theta)$. Then by the compactness of $\tilde{\theta}$ we have the following:

$$\hat{\Theta} \subset \cup_{i=1}^{m} N_x(\theta_i), \quad T(\theta_1) \leq T(\theta_2) \leq \cdots \leq T(\theta_m)$$

We shall put $T_j = T(\theta_j), \quad J = 1, 2, \cdots m$.

We claim the following:

$$k = \sup\{\|\Phi_\lambda(\theta, t)\| : \theta \in \hat{\Theta}, \quad 0 \leq t \leq T_m\} < \infty.$$

In fact. Assume that $K = \infty$. Then there are sequences $\{\theta_n\} \subset \hat{\Theta}, \quad \{t_n\} \subset [0, T_m]$ such that

$$\|\Phi_\lambda(\theta_n), t_n\| > n, \quad n = 1, 2, 3 \ldots.$$

Since $\hat{\Theta}$ and $[0, T_m]$ are compact sets we can assume that $\{\theta_n\}$ converges to $\theta_0 \in \hat{\Theta}$ and $\{t_n\}$ converge to $t^* \in [0, T_m]$. Then by Banach-Steinhaus Theorem there must be an element $x_0 \in X$ so that the set:

$$\{\|\Phi_\lambda(\theta_n, t_n)x_0)\| : n = 1, 2, 3, \ldots\}$$

is unbounded. On the other hand the definition of skew-product flow implies that

$$\lim_{n \to \infty} \|\Phi_\lambda(\theta_n, t_n)x_0\| = \|\Phi(\theta_0, t^*)x_0\|$$

Which is a contradiction.

Now fix $t \geq 0$ and let $\theta \in \hat{\Theta}$. Then $\theta \in N_x(\theta_{j_1})$ for some J_1 and $\|\Phi_\lambda(\theta, T_{j_1})x\| < (\frac{1}{2})^2$. In the same way $\theta \cdot t_{j_1} \in N_x(\theta_{j_2})$ for some J_2 and

$$\|\Phi_\lambda(\theta, T_{j_1}+T_{j_2})x\| = \|\Phi_\lambda(\theta\cdot t_{j_1}, T_{j_2})\Phi(\theta, T_{j_2})x\| < (\frac{1}{2})^2.$$

Now continue this process until one has

$$\tau = T_{j_1}+\cdots+T_{j_l} \leq t < \tau + T_{j_{(l+1)}} \quad \text{and} \quad \|\Phi_\lambda(\theta, \mathrm{t})\mathrm{x}\| \leq (\frac{1}{2})^{l}.$$

Since $lT_1 \leq \tau \leq t$ and $0 \leq \mathrm{t}-\tau \leq \mathrm{T_m}$, we get the following

$$\|\Phi_\lambda(\theta,t)x\| = \|\Phi_\lambda(\theta.\tau, t-\tau)\Phi_\lambda(\theta,\tau)x\| \leq k(\frac{1}{2})^l \leq k(\frac{1}{2})^{t/T_1} = ke^{-\alpha t}$$

where $\alpha = -\frac{1}{T_1}\ln(\frac{1}{2}) > 0$.

Therefore we have gotten the following

$$\|\Phi_\lambda(\theta,t)x\| \leq ke^{-\alpha t}, \quad \theta \in \hat{\Theta}, \quad t \geq 0.$$

Since k and α do not depends of x we get that

$$\|\Phi_\lambda(\theta,t)\| \leq ke^{-\alpha t}, \quad \theta \in \hat{\Theta}, \quad t \geq 0.$$

From here we get that the skew-product flow $\pi_\lambda = (\Phi_\lambda, \sigma)$ has ED over $\hat{\Theta}$, with projections $P(\theta) = I$, e.i., $\lambda \in \rho(\hat{\Theta})$. On the other hand, if $\mu \geq \lambda$, then

$$\|\Phi_\mu(\theta,t)\| = e^{(\lambda-\mu)t}\|\Phi_\lambda(\theta,t)\| \leq ke^{-\alpha t}, \quad \theta \in \hat{\Theta}, \quad t \geq 0.$$

Therefore,

$$\mu \in \rho(\hat{\Theta}) \quad \text{and} \quad \mathcal{S}_\lambda(\hat{\Theta}) = \mathcal{S}_\mu(\hat{\Theta}) = \mathcal{E}(\hat{\Theta}).$$

□

Lemma 3.2 *Let $\hat{\Theta}$ be a compact invariant set in Θ. Then the resolvent $\rho(\hat{\Theta})$ is open. Moreover*

$$if \quad \lambda \in \rho(\hat{\Theta}), \quad then \quad \mathcal{S}_\lambda = \mathcal{S}_\mu \quad and \quad \mathcal{U}_\lambda = \mathcal{U}_\mu$$

for all μ in a neighborhood of λ.

Proof Fix $\lambda \in \rho(\hat{\Theta})$. Then by definition π_α admits an exponential dichotomy over $\hat{\Theta}$.

Hence there exists an invariant Projector $\mathbf{P} : \mathcal{E} \to \mathcal{E}$ and positive constants k and β such that

$$\|\Phi_\lambda(\theta,t)P(\theta)\Phi_\lambda^{-1}(\theta,s)\| \leq ke^{-\beta(t-s)}, \quad s \leq t,$$

$$\|\Phi_\lambda(\theta,t)[I - P(\theta)]\Phi_\lambda^{-1}(\theta,s)\| \leq ke^{\beta(t-s)}, \quad s \geq t.$$

Claim. If $\|\lambda - \mu\| < \alpha$ where $\alpha = \beta/2$, then

(a) $\|\Phi_\mu(\theta,t)P(\theta)\Phi^{-1}(\theta,s)\| \leq ke^{-\alpha(t-s)}, \quad t \geq s$

(b) $\|\Phi_\mu(\theta,t)[I - P(\theta)]\Phi_\mu^{-1}(\theta,s) \leq ke^{\alpha(t-s)}, \quad s \geq t.$

In fact:

$$\Phi_\mu(\theta,t)P(\theta)\Phi_\mu^{-1}(\theta,s) = e^{(\lambda-\mu)(t-s)}\Phi_\lambda(\theta,t)P(\theta)\Phi_\lambda^{-1}(\theta,s)$$

Therefore

$$\|\Phi_\mu(\theta,t)P(\theta)\Phi_\mu^{-1}(\theta,s)\| \leq ke^{(\lambda-\mu)(t-s)}.e^{\beta(t-s)}, \quad s \leq t$$

From the fact that $-\beta/2 < \lambda - \mu < \beta/2$ we get

$$\|\Phi_\mu(\theta,t)P(\theta)\Phi^{-1}(\theta,s)\| \leq ke^{-\alpha(t-s)}, \quad s \geq t.$$

In the same way, we get that

$$\|\Phi_\mu(\theta,t)[I - P(\theta)]\Phi^{-1}(\theta,s)\| \leq ke^{\alpha(t-s)}, \quad s \geq t.$$

Since the exponential dichotomies of $\pi_\lambda, \quad \pi_\mu$ involve the same projector $\mathbf{P}$ we have that

$$\mathcal{S}_\lambda(\hat{\Theta}) \text{ and } \mathcal{U}_\lambda(\hat{\Theta}) = \mathcal{U}_\mu(\hat{\Theta}) \text{ for } \|\lambda - \mu\| < \alpha.$$

□

Lemma 3.3 *Let $\hat{\Theta}$ be a compact invariant set in Θ. Then the spectrum $\Sigma(\hat{\Theta})$ is compact. More specifically, there exists an $a > 0$ such that, if $\lambda > a$, then $\lambda \in \rho(\hat{\Theta})$, and $\mathcal{S}_\lambda = \mathcal{E}$ and if $\lambda < -a$ then $\lambda \in \rho(\hat{\Theta})$ and $\mathcal{U}_\lambda = \mathcal{E}$.*

Proof Because of Lemma 3.2 we need only to prove that $\Sigma(\hat{\Theta})$ is bounded. Thanks to the Proposition 2.1 we get $k \geq 1$ and $a > 0$ such that

$$\|\Phi(\theta, t)\| \leq e^{a|t|}, \quad \theta \in \hat{\Theta}, \quad t \in \mathbb{R}.$$

Then if $\lambda > a$ we get

$$\|\Phi_\lambda(\theta, t)\| \leq ke^{(a-\lambda)t} \to 0, \quad as \quad t \to +\infty$$

For all $\theta \in \hat{\Theta}$. Therefore, by Lemma 3.1 one has that

$$(a, \infty) \subset \rho(\hat{\Theta}) \Leftrightarrow \Sigma(\hat{\Theta}) \subset (-\infty, a]$$

Similarly, if $\lambda < -a$ one has $\|\Phi_\lambda(\theta, t)\| \to 0$ as $t \to -\infty$ for all $\theta \in \hat{\Theta}$. Consequently

$$(-\infty, -a) \subset \rho(\hat{\Theta}) \Leftrightarrow \Sigma(\hat{\Theta}) \subset [-a, \infty)$$

Hence $\quad \Sigma(\hat{\Theta}) \subset [-a, a]$.

□

Lemma 3.4 *Let $\hat{\Theta}$ be a nonempty set in Θ and assume that* $\dim \mathcal{E} \geq 1$. *Then the spectrum $\Sigma(\hat{\Theta})$ is nonempty.*

Proof Pick $\theta_0 \in \hat{\Theta}$ and set $M_0 = H(\theta_0)$ where

$$H(\theta_0) = cl\{\theta_0.t : t \in \mathbb{R}\}$$

Then M_0 is a compact invariant set and clearly $\Sigma(M_0) \subseteq \Sigma(\hat{\Theta})$. It will be sufficient to show that $\Sigma(M_0)$ is noempty. From Proposition 2.1, we have $k \geq 1$ and $a > 0$ such that

$$||\Phi(\theta, t)|| \leq ke^{a|t|}, \quad \theta \in M_0, \quad t \in I\!R.$$

By Lemma 3.1 we get the following :

(a) If $\lambda > a$, then $\lambda \in \rho(M_0)$, $\mathcal{S}_\lambda(M_0) = \mathcal{E}(M_0)$ and $\mathcal{U}_\lambda(M_0) = \mathcal{E}_0(M_0)$.

(b) If $\lambda < -a$, then $\lambda \in \rho(M_0)$, $\mathcal{U}_\lambda(M_0) = \mathcal{E}(M_0)$ and $\mathcal{S}_\lambda(M_0) = \mathcal{E}_0(M_0)$.

Therefore $\Sigma(M_0) \subset [-a, a]$.

Next define

$$\lambda_0 = inf\{\lambda \in \rho(M_0) : \mathcal{S}_\lambda(M_0) = \mathcal{E}(M_0)\}$$

Then $-a \leq \lambda \leq a$.

For the purpose of contradiction, let us assume that $\lambda_0 \in \rho(M_0)$. Then there are two cases to consider:

(i) $\mathcal{S}_{\lambda_0}(M_0) = \mathcal{E}(M_0)$

(ii) $\mathcal{S}_{\lambda_0}(M_0) \neq \mathcal{E}(M_0)$.

For the case (i) the Lemma 3.2 implies that $\mathcal{S}_\lambda(M_0) = \mathcal{E}(M_0)$ for λ in a neighborhood of λ_0, which contradicts the definition of λ_0.

For the case (ii) one must have $\mathcal{U}_{\lambda_0}(M_0) \neq \mathcal{E}_0(M_0)$. From Proposition 2.2 we get that

$$\mathcal{E}(\Theta) = \mathcal{U}_{\lambda_0}(\theta) + \mathcal{S}_{\lambda_0}(\theta), \quad \forall\theta \in M_0.$$

Then by using Lemma 3.2 one again, we have $\mathcal{U}_\lambda(M_0) \neq \mathcal{E}_0(M_0)$ in a neighborhood of λ_0. This contradicts the fact that $\mathcal{U}_\lambda(M_0) = \mathcal{E}_0(M_0)$ for $\lambda > \lambda_0$ close enough to λ_0. Therefore $\lambda_0 \in \Sigma(M_0)$.

$\square$

Lemma 3.5 *Let $\hat{\Theta}$ be a compact invariant set in Θ. Consider $\lambda_1, \lambda_2 \in \rho(\hat{\Theta})$ with $\lambda_1 < \lambda_2$.*

$$If \quad \mathcal{S}_{\lambda_1}(\hat{\Theta}) = \mathcal{S}_{\lambda_2}(\hat{\Theta}) \ \ and \ \ \mathcal{U}_{\lambda_1}(\hat{\Theta}) = \mathcal{U}_{\lambda_2}(\hat{\Theta})$$

then

$$[\lambda_1, \lambda_2] \subset \rho(\hat{\Theta}) \ \ and \ \ \mathcal{S}_{\lambda}(\hat{\Theta}) = \mathcal{S}_{\lambda_1}(\hat{\Theta}), \ \ \mathcal{U}_{\lambda}(\hat{\Theta}) = \mathcal{U}_{\lambda_1}(\hat{\Theta})$$

for all $\lambda \in [\lambda_1, \lambda_2]$.

Proof In the same way as the proof of Lemma 8 in [17].

$\square$

The following Propositions are easy to prove.

Proposition 3.1 *Let A, B* and *C be subspaces of X. If $C \subseteq A$ then*

$$A \cap (B + C) = (A \cap B) + (A \cap C)$$

Proposition 3.2 *If $\lambda_1, \lambda_2 \in \rho(\hat{\Theta})$* and *$\lambda_1 < \lambda_2$, then*

$$\mathcal{E}(\hat{\Theta}) = \mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_2}(\hat{\Theta}) + \mathcal{U}_{\lambda_2}(\hat{\Theta}) + \mathcal{S}_{\lambda_1}(\hat{\Theta}) \tag{3.3}$$

Proof By simplicity we shall use the following notation: $\mathcal{S}_{\lambda} := \mathcal{S}_{\lambda}(\hat{\Theta})$ and $\mathcal{U} := \mathcal{U}_{\lambda}(\hat{\Theta})$.

It is easy to prove the following property:

$$\text{If } \ \lambda_1 \leq \lambda_2 \Rightarrow \mathcal{S}_{\lambda_1} \subseteq \mathcal{S}_{\lambda_2} \ \text{ and } \ \mathcal{U}_{\lambda_2} \subseteq \mathcal{U}_{\lambda_1} \tag{3.4}$$

From Proposition 2.2 we already know that

$$\mathcal{E}(\hat{\Theta}) = \mathcal{U}_{\lambda_i} + \mathcal{S}_{\lambda_i}, \ \ i = 1, 2.$$

Therefore, from Proposition 3.1 we get

$$\mathcal{U}_{\lambda_1} = \mathcal{U}_{\lambda_1} \cap (\mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_2}) = \mathcal{U}_{\lambda_2} + \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2}$$

and

$$\mathcal{S}_{\lambda_2} = \mathcal{S}_{\lambda_2} \cap (\mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1}) = \mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2}.$$

Then by using (3.2) we get that

$$\mathcal{E}(\hat{\Theta}) = \mathcal{U}_{\lambda_1} + \mathcal{S}_{\lambda_1} = \mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2}.$$

□

Lemma 3.6 *Let $\hat{\Theta}$ be a compact invariant set in Θ and $\lambda_1, \lambda_2 \in \rho(\hat{\Theta})$ with $\lambda_1 < \lambda_2$.*

Then the following statements are equivalent:

(A) There is a $\mu \in (\lambda_1, \lambda_2) \cap \Sigma(\hat{\Theta})$

(B) $\mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_2}(\hat{\Theta}) \neq \mathcal{E}_0(\hat{\Theta})$.

Moreover, $\mathcal{F} = \mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{U}_{\lambda_2}(\hat{\Theta})$ is an invariant subbundle of $\mathcal{E}$

Proof $(A) \Rightarrow (B)$

From Proposition 3.2 we have the following

$$\mathcal{E}(\hat{\Theta}) = \mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_2}(\hat{\Theta}) + \mathcal{U}_{\lambda_2}(\hat{\Theta}) + \mathcal{S}_{\lambda_1}(\hat{\Theta}).$$

If $\mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} = \mathcal{E}_0$, then we get

$$\mathcal{E} = \mathcal{U}_{\lambda_2}(\hat{\Theta}) + \mathcal{S}_{\lambda_1}(\hat{\Theta}) = \mathcal{U}_{\lambda_1}(\hat{\Theta}) + \mathcal{S}_{\lambda_1}(\hat{\Theta}) = \mathcal{U}_{\lambda_2}(\hat{\Theta}) + \mathcal{S}_{\lambda_2}\hat{\Theta})$$

Since $\mathcal{U}_{\lambda_2}(\hat{\Theta}) \subseteq \mathcal{U}_{\lambda_1}(\hat{\Theta})$ and $\mathcal{S}_{\lambda_1}(\hat{\Theta}) \subseteq \mathcal{S}_{\lambda_2}(\hat{\Theta})$

then

$$\mathcal{S}_{\lambda_1}(\hat{\Theta}) = \mathcal{S}_{\lambda_2}(\hat{\Theta}) \text{ and } \mathcal{U}_{\lambda_1}(\hat{\Theta}) = \mathcal{U}_{\lambda_2}(\hat{\Theta})$$

Now we can apply Lemma 3.5, it means that

$$\mathcal{U}_{\lambda_1}(\hat{\Theta}) = \mathcal{U}_{\lambda}(\hat{\Theta}) \text{ and } \mathcal{S}_{\lambda_2}(\hat{\Theta}) = \mathcal{S}_{\lambda}(\hat{\Theta})$$

for $\lambda \in [\lambda_1, \lambda_2] \subset \rho(\hat{\Theta})$. Therefore $(\lambda_1, \lambda_2) \cap \Sigma(\hat{\Theta}) = \emptyset$ which contradicts (A)

$(B) \Rightarrow (A)$. Define

$$\mu := \inf\{\lambda \in \rho(\hat{\Theta}) : calS_{\lambda}(\hat{\Theta}) = \mathcal{S}_{\lambda_2}(\hat{\Theta})\}.$$

Then $\lambda_1 < \mu < \lambda_2$ and $\mu \in \Sigma(\hat{\Theta})$.

In fact. Lemma 3.2 implies that $\mu < \lambda_2$.

For the purpose of contradiction, let us assume that $\mu \in \rho(\hat{\Theta})$. Then there exists a neighborhood of μ such that for all λ in that neighborhood $\mathcal{S}_{\lambda}(\hat{\Theta}) = \mathcal{S}_{\mu}(\hat{\Theta})$. Hence, $\mu \in \Sigma(\hat{\Theta})$.

Assume that $\mu < \lambda_1$. Then we get

$$\mathcal{U}_{\mu}(\hat{\Theta}) \subseteq \mathcal{S}_{\lambda_1}(\hat{\Theta}) \subseteq \mathcal{S}_{\lambda_2}(\hat{\Theta}).$$

Then applying Lemma 3.2 and the definition of μ we get that $\mathcal{S}_{\lambda_1}(\hat{\Theta}) = \mathcal{S}_{\lambda_2}(\hat{\Theta})$. From (B) we have that $\mathcal{S}_{\lambda_2}(\hat{\Theta}) \cap \mathcal{U}_{\lambda_1}(\hat{\Theta}) \neq \mathcal{E}_0(\hat{\Theta})$. So $\mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_1}(\hat{\Theta}) \neq \mathcal{E}_0(\hat{\Theta})$. Which is a contradiction with the fact that $\lambda_1 \in \rho(\hat{\Theta})$. Thus $\mu \in (\lambda_1, \lambda_2)$

Finally, since both $\mathcal{U}_{\lambda_1}(\hat{\Theta})$ and $\mathcal{S}_{\lambda_2}(\hat{\Theta})$ are invariant subbundles of $\mathcal{E}(\hat{\Theta})$, it follows that

$$\mathcal{F} = \mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_2}(\hat{\Theta})$$

is also invariant subbundle.

□

Lemma 3.7 *Let $\hat{\Theta}$ be a compact invariant set in Θ and let λ_1, λ_2 be chosen so $(\lambda_1, \lambda_2) \cup \Sigma(\hat{\Theta}) \neq \phi$. Let*

$$\mathcal{F} := \mathcal{U}_{\lambda_1}(\hat{\Theta}) \cap \mathcal{S}_{\lambda_2}(\hat{\Theta})$$

and $\hat{\pi}$ the restriction of π to $\mathcal{F}$. Let $\hat{\Sigma}(\hat{\Theta}) := \Sigma(\mathcal{F})$ denote the spectrum of $(\mathcal{F}, \hat{\pi})$ over $\hat{\Theta}$. Then

$$\hat{\Sigma}(\hat{\Theta}) = \Sigma(\hat{\Theta}) \cap (\lambda_1, \lambda_2).$$

Proof We shall give the proof in three steps.

Step 1. Consider $\lambda \in \rho(\hat{\Theta})$ and define

$$\mathcal{F}_s(\lambda) := \mathcal{F} \cap \mathcal{S}_\lambda(\hat{\Theta}) \quad \text{and} \quad \mathcal{F}_u(\lambda) := \mathcal{F} \cap \mathcal{U}_\lambda(\hat{\Theta}).$$

Then $\mathcal{F}_s(\lambda)$ and $\mathcal{F}_u(\lambda)$ are invariant subbundles and

$$\mathcal{F} = \mathcal{F}_s(\lambda) + \mathcal{F}_u(\lambda)$$

In fact, suppose that $\lambda < \lambda_1$. Then

$$\mathcal{S}_\lambda(\hat{\Theta}) \subset \mathcal{S}_{\lambda_1}(\hat{\Theta}) \quad \text{and} \quad \mathcal{U}_{\lambda_1}(\hat{\Theta}) \subset \mathcal{U}_\lambda(\hat{\Theta})$$

From now on we shall omit the argument $\hat{\Theta}$ if it is neccesary, in order to simplify the computation. So

$$\mathcal{F}_s(\lambda) = \mathcal{F} \cap \mathcal{S}_\lambda = \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} \cap \mathcal{S}_\lambda = \mathcal{E}_0 \cap \mathcal{S}_{\lambda_2} = \mathcal{F}_0$$

and

$$\mathcal{F}_u(\lambda) = \mathcal{F} \cap \mathcal{U}_\lambda = \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} \cap \mathcal{U}_\lambda = \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} = \mathcal{F}$$

Hence $\mathcal{F} = \mathcal{F}_s(\lambda) + \mathcal{F}_u(\lambda)$.

Similarly, if $\lambda > \lambda_2$, then

$$\mathcal{F}_s(\lambda) = \mathcal{F} \quad \text{and} \quad \mathcal{F}_u(\lambda) = \mathcal{F}_0$$

For all $\lambda \in [\lambda_1, \lambda_2]$ we shall use Proposition 3.2 and the fact that $\mathcal{E} = \mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_2} = \mathcal{U}_{\lambda_1} + \mathcal{S}_{\lambda_1}$. So we have:

$$\begin{array}{rcccl} \mathcal{U}_{\lambda_1} & = & \mathcal{U}_{\lambda_1} \cap (\mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_2}) & = & \mathcal{U}_{\lambda_2} + \mathcal{F} \\ \mathcal{U}_{\lambda_1} & = & \mathcal{U}_{\lambda_1} \cap (\mathcal{U}_{\lambda} + \mathcal{S}_{\lambda}) & = & \mathcal{U}_{\lambda} + \mathcal{F}_s(\lambda) \\ \mathcal{U}_{\lambda} & = & \mathcal{U}_{\lambda} \cap (\mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_2}) & = & \mathcal{U}_{\lambda_2} + \mathcal{F}_u(\lambda) \end{array}$$

Therefore

$$\mathcal{U}_{\lambda_1} = \mathcal{U}_{\lambda_2} + \mathcal{F} = \mathcal{U}_{\lambda_2} + \mathcal{F}_u(\lambda) + \mathcal{F}_s(\lambda)$$

Since $\mathcal{F}_u(\lambda) + \mathcal{F}_s(\lambda) \subseteq \mathcal{F}$, then $\mathcal{F} = \mathcal{F}_u(\lambda) + \mathcal{F}_s(\lambda)$

Step 2. $\rho(\hat{\Theta}) \subseteq \hat{\rho}(\hat{\Theta}) = I\!R \setminus \hat{\Sigma}(\hat{\Theta})$. In fact, if $\lambda \in \rho(\hat{\Theta})$, then there is a projector $\mathbf{P} : \mathcal{E}(\hat{\Theta}) \to \mathcal{E}(\hat{\Theta})$ and positive constants k and β such that

$$\mathcal{R}(\mathbf{P}) = \mathcal{S}_\lambda(\hat{\Theta}), \quad \mathcal{N}(\mathbf{P}) = \mathcal{U}_\lambda(\hat{\Theta})$$

and

$$\|\Phi_\lambda(\theta, t)P(\theta)\Phi_\lambda^{-1}(\theta, s)x\| \le k\|x\|e^{-\beta(t-s)}, \quad t \ge s, \quad \theta \in \hat{\Theta}$$

$$\|\Phi_\lambda(\theta, t)(I - P(\theta))\Phi_\lambda^{-1}(\theta, s)x\| \le k\|x\|e^{\beta(t-s)}, \quad t \le s, \quad \theta \in \hat{\Theta}$$

If $\hat{\mathbf{P}}$ is the restiction of $\mathbf{P}$ to $\mathcal{F}$, then

$$\mathcal{R}(\hat{\mathbf{P}}) = \mathcal{F}_s(\lambda), \quad \mathcal{N}(\hat{\mathbf{P}}) = \mathcal{F}_u(\lambda), \quad \mathcal{F} = \mathcal{R}(\hat{\mathbf{P}}) + \mathcal{N}(\hat{\mathbf{P}}).$$

Therefore, by restricting the above inequalities to all $(x, \theta) \in \mathcal{F}$ we obtain

$$\|\hat{\Phi}_\lambda(\theta, t)\hat{P}(\theta)\hat{\Phi}_\lambda^{-1}(\theta, s)x\| \le k\|x\|e^{-\beta(t-s)}, \quad t \ge s$$

$$\|\hat{\Phi}_\lambda(\theta, t)(I - \hat{P}(\theta))\hat{\Phi}_\lambda^{-1}(\theta, s)x\| \le k\|x\|e^{\beta(t-s)}, \quad t \le s$$

Thus $\hat{\pi}_\lambda$ admits an exponential dichotomy over $\hat{\Theta}$, i.e., $\lambda \in \hat{\rho}(\hat{\Theta})$. So $\rho(\hat{\Theta}) \subset \hat{\rho}(\hat{\Theta})$ and therefore $\hat{\Sigma}(\hat{\Theta}) \subset \Sigma(\hat{\Theta})$. This completes the step 1.

Since $\mathcal{F} \subseteq \mathcal{S}_{\lambda_2}(\hat{\Theta})$, then

$$\|\hat{\Phi}_{\lambda_2}(\theta, t)x\| \to 0 \quad \text{as} \quad t \to \infty$$

for all $(x, \theta) \in \mathcal{F}$ uniformly in x. Hence, applying Lemma 3.1 we obtain that $\hat{\Sigma}(\hat{\Theta}) \subseteq (-\infty, \lambda_2)$ and $\mathcal{F}_s(\lambda) = \mathcal{F}, \quad \lambda \geq \lambda_2$. In the same way we obtain $\mathcal{F} \subseteq \mathcal{U}_{\lambda_1}(\hat{\Theta})$. This implies that

$$\|\hat{\Phi}_{\lambda_1}(\theta, t)\| \to 0, \quad \text{as} \ t \to -\infty$$

Using Lemma 3.1 again, we get that

$\hat{\Sigma}(\hat{\Theta}) \subseteq (\lambda_1, \infty)$ and $\mathcal{F}_u = \mathcal{F}$. Then

$$\hat{\Sigma}(\hat{\Theta}) \subseteq (\lambda_1, \lambda_2) \cap \Sigma(\hat{\Theta})$$

In order to prove the opposite inclusion, it is sufficient to show that: if $\lambda \in \hat{\rho}(\hat{\Theta}) \cap (\lambda_1, \lambda_2)$, then $\lambda \in \rho(\hat{\Theta})$. In fact, suppose that

$$\hat{\rho}(\hat{\Theta}) \cap (\lambda_1, \lambda_2) \subset \rho(\hat{\Theta})$$

which is equivalent to :

$$\rho^c(\hat{\Theta}) \subset (\hat{\rho}(\hat{\Theta}))^c \cup (-\infty, \lambda_1] \cup [\lambda_2, \infty) \Leftrightarrow \Sigma(\hat{\Theta}) \subset \hat{\Sigma}(\hat{\Theta}) \cup (-\infty, \lambda_1] \cup [\lambda_2, \infty).$$

On the other hand, we already know that $\hat{\Sigma}(\hat{\Theta}) \subset (\lambda_1, \lambda_2)$. Therefore,

$$\Sigma(\hat{\Theta}) \cap (\lambda_1, \lambda_2) \subset \hat{\Sigma}(\hat{\Theta}).$$

Step 3. $\hat{\rho}(\hat{\Theta}) \cap (\lambda_1, \lambda_2) \subset \rho(\hat{\Theta})$. In fact, if $\lambda \in \hat{\rho}(\hat{\Theta}) \cap (\lambda_1, \lambda_2)$, then there is a projector $\mathbf{Q} : \mathcal{F} \to \mathcal{F}$ and positive contants k and β such that

$$\|\hat{\Phi}_\lambda(\theta,t)Q(\theta)\hat{\Phi}_\lambda^{-1}(\theta,s)x\| \le k\|x\|e^{-\beta(t-s)}, \quad t \ge s \tag{3.5}$$

$$\|\hat{\Phi}_\lambda(\theta,t)(I-Q(\theta))\hat{\Phi}_\lambda^{-1}(\theta,s)x\| \le k\|x\|e^{\beta(t-s)}, \quad t \le s \tag{3.6}$$

and for all $\theta \in \hat{\Theta}$

Consider the projectors :

$$\mathbf{P}_1, \mathbf{P} : \mathcal{E}(\hat{\Theta}) \to \mathcal{E}(\hat{\Theta}) \quad \text{such that} \quad \mathcal{R}(\mathbf{P}_1) = \mathcal{S}_{\lambda_1}, \;\; \mathcal{N}(\mathbf{P}_1) = \mathcal{U}_{\lambda_1}$$

and

$$\mathcal{R}(\mathbf{P}) = \mathcal{F} = \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2}, \quad \mathcal{N}(\mathbf{P}) = \mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_1}.$$

From the Proposition 3.2 we have

$$\mathcal{E} = \mathcal{U}_{\lambda_2} + \mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} = \mathcal{R}(\mathbf{P}) + \mathcal{N}(\mathbf{P})$$

$\mathbf{P}_\lambda = \mathbf{P}_1 + \mathbf{QP}$ is a projector on $\mathcal{E}$ such that

$$\mathcal{R}(\mathbf{P}_\lambda) = \mathcal{S}_{\lambda_1} + \hat{\mathcal{S}}_\lambda, \quad \mathcal{N}(\mathbf{P}_\lambda) = \hat{\mathcal{U}}_\lambda + \mathcal{U}_{\lambda_2}$$

where $\hat{\mathcal{S}}_\lambda = \mathcal{R}(\mathbf{Q})$ and $\hat{\mathcal{U}}_\lambda = \mathcal{N}(\mathbf{Q})$.

Using (3.5) and (3.6) and the fact that $\lambda \in (\lambda_1, \lambda_2)$, λ_1, $\lambda_2 \in \rho(\hat{\Theta})$ one can show that there are positive constants l and α such that

$$\|\Phi_\lambda(\theta,t)P_\lambda(\theta)\Phi_\lambda^{-1}(\theta,s)\| \le le^{-\alpha(t-s)}, \quad t \ge s, \;\; \theta \in \hat{\Theta}$$

$$\|\Phi_\lambda(\theta,t)(I-P_\lambda(\theta))\Phi_\lambda^{-1}(\theta,s)\| \le le^{\alpha(t-s)}, \quad t \le s, \;\; \theta \in \hat{\Theta}$$

noindent which implies that $\lambda \in \rho(\hat{\Theta})$.

□

3.2 Proof of Main Theorems

Proof of Theorem 3.1

The statement (A), (B) and (C) follow from Lemma 3.3, 3.4 and 3.6 respectively.

The statement (D) follows from Lemma 3.7.

(E) From Lemma 3.3 we have that

$$\Sigma(\hat{\Theta}) \subset [-a, a]$$

Therefore

$$\Sigma(\hat{\Theta}) = \Sigma(\hat{\Theta}) \cap (\bigcup_{i=1}^{m}(\lambda_{i-1}, \lambda_i)) = \bigcup_{i=1}^{m} \Sigma(\hat{\Theta}) \cap (\lambda_{i-1}, \lambda_i) = \bigcup_{i=1} \Sigma_i(\hat{\Theta}).$$

(F) Consider $i + 1 \leq j$

$$\mathcal{V}_i = \mathcal{U}_{\lambda_i-1} \cap \mathcal{S}_{\lambda_i} \Rightarrow \mathcal{V}_i \subseteq \mathcal{S}_{\lambda_i},$$

by the monotocity of $\mathcal{U}_\lambda$ we get that $\mathcal{V}_j \subset \mathcal{U}_{\lambda_i}$. On the other hand we know that $\mathcal{U}_{\lambda_i} \cap \mathcal{S}_{\lambda_i} = \mathcal{E}_0$.

(G) From Lemma 3.3 we have that:

if $\lambda > a \Rightarrow \lambda \in \rho(\hat{\Theta})$, $\mathcal{S}_\lambda(\hat{\Theta}) = \mathcal{E}(\hat{\Theta})$ and $\mathcal{U}_\lambda(\hat{\Theta}) = \mathcal{E}_0(\hat{\Theta})$.

if $\lambda < -a \Rightarrow \lambda \in \rho(\hat{\Theta})$, $\mathcal{U}_\lambda(\hat{\Theta}) = \mathcal{E}(\hat{\Theta})$ and $\mathcal{S}_\lambda(\hat{\Theta}) = \mathcal{E}_0(\hat{\Theta})$.

Also we know that

$$\mathcal{E}(\hat{\Theta}) = \mathcal{S}_{\lambda_i}(\hat{\Theta}) + \mathcal{U}_{\lambda_i}(\hat{\Theta})$$

Therefore

$$\mathcal{E}(\hat{\Theta}) \quad = \quad \mathcal{U}_{\lambda_0} = \mathcal{U}_{\lambda_0} \cap (\mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1})$$

$$
\begin{aligned}
&= \mathcal{U}_{\lambda_0} \cap \mathcal{S}_{\lambda_1} + \mathcal{U}_{\lambda_1} \\
&= \mathcal{V}_1 + \mathcal{U}_{\lambda_1} = \mathcal{V}_1 + \mathcal{U}_{\lambda_1} \cap (\mathcal{S}_{\lambda_2} + \mathcal{U}_{\lambda_2}) \\
&= \mathcal{V}_1 + \mathcal{U}_{\lambda_1} \cap \mathcal{S}_{\lambda_2} + \mathcal{U}_{\lambda_2} \\
&= \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{U}_{\lambda_2} \cap (\mathcal{S}_{\lambda_3} + \mathcal{U}_{\lambda_3}) \\
&= \cdots\cdots\cdots\cdots\cdots \\
&= \cdots\cdots\cdots\cdots\cdots \\
&= \cdots\cdots\cdots\cdots\cdots \\
&= \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 + \ldots + \mathcal{V}_m + \mathcal{U}_{\lambda_m} \\
&= \mathcal{V}_1 + \mathcal{V}_2 + \ldots + \mathcal{V}_m. \quad (\mathcal{U}_{\lambda_m} = \mathcal{E}_0(\hat{\Theta}))
\end{aligned}
$$

□

Proof of Theorem 3.2

(A) - (F) follow easily from Lemmas 3.2 and 3.3. Proof of (G), from Lemma 3.3 we have the following:

If $\lambda_0 < -a$, then $\mathcal{U}_{\lambda_0} = \mathcal{E}$ and $\dim \mathcal{U}_{\lambda_0} = \infty$

If $\lambda_0 > a$, then $\mathcal{U}_{\lambda_0} = \mathcal{E}_0$ and $\dim \mathcal{U}_{\lambda_0} = 0$

Therefore $\lambda_0 \in [-a, a]$.

Now consider the set

$$\{\lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_m\} \subset \rho(\hat{\Theta})$$

such that

$$a < \lambda_m, \quad (\lambda_{i-1}, \lambda_i) \cap \Sigma(\hat{\Theta}) \neq \phi$$

From Proposition 2.2. we get that: $\mathcal{E}(\hat{\Theta}) = \mathcal{S}_{\lambda_0}(\hat{\Theta}) + \mathcal{U}_{\lambda_0}(\hat{\Theta})$. Then let us denote by π^{λ_0} the restriction of Π to $\mathcal{U}_{\lambda_0}$ and $\Sigma^{\lambda_0}(\hat{\Theta})$ the spectrum of $(\mathcal{U}_{\lambda_0}, \pi^{\lambda_0})$ over $\hat{\Theta}$.

So by using Lemma 3.7 we obtain that

$$\Sigma^{\lambda_0}(\hat{\Theta}) = (\lambda_0, +\infty) \cap \Sigma(\hat{\Theta}) = (\lambda_0, a] \cap \Sigma(\hat{\Theta}).$$

Therefore $\dim \mathcal{V}_i \geq 1$, which implies that $m \leq \dim \mathcal{U}_{\lambda_0}$. Now we shall show that the resolvent $\rho(\hat{\Theta})$ of $(\mathcal{U}_{\lambda_0}, \pi^{\lambda_0})$ consist of $(k+1)$ intervals where $k \leq n(\lambda_0)$. If $\rho_{\lambda_0}(\hat{\Theta})$ consists of $(n(\lambda_0)+2)$ intervals :

$$\rho_{\lambda_0}(\hat{\Theta}) = \bigcup_{i=1}^{n(\lambda_0)+2} (c_i, d_i)$$

So we get that

$$\Sigma_{\lambda_0}(\hat{\Theta}) \cap (\lambda_{i-1}, \lambda_i) \neq \phi$$

Then

$$\mathcal{V}_i = \mathcal{U}_{\lambda_{i-1}} \cap \mathcal{S}_{\lambda_i} \neq \mathcal{E}_0(\hat{\Theta}), \quad \text{in} \quad (\mathcal{U}_{\lambda_0}, \pi^{\lambda_0})$$

Therefore

$$\dim \mathcal{U}_{\lambda_0} \geq n_1 + n_2 + \ldots + N_{n+1} \geq n+1$$

Which is a contradiction. Hence $\rho_{\lambda_0}(\hat{\Theta})$ consists of $k+1$ intervals whith $k \leq n$. So $\Sigma_{\lambda_0}(\hat{\Theta})$ is the union of k compact intervals. The remainder of the proof is easy . $\square$

4 Lyapunov Exponents

In this section we shall investigate the relation between the Dynamic Spectrum and the Lyapunov characteristic exponents.

For this purpose we shall assume that there exists $\lambda_0 \in \rho(\hat{\Theta})$ such that $1 \leq \dim \mathcal{U}_{\lambda_0} < \infty$.

Consider $\{\lambda_0 < \ldots < \lambda_m\} \subset \rho(\hat{\Theta})$, such that $0 < \lambda_m$ and $\Sigma(\hat{\Theta}) \cap (\lambda_{i-1}, \lambda_i) \neq \phi$. Then from Theorem 3.2 we get:

$$\Sigma_{\lambda_0}(\hat{\Theta}) = (\lambda_0, a] \cap \Sigma(\hat{\Theta}) \tag{4.1}$$

$$\Sigma(\hat{\Theta}) = \Sigma_{\lambda_0}(\hat{\Theta}) \cup (U_{i=1}^{m}[a_i, b_i]) \tag{4.2}$$

$$m \leq \dim \mathcal{U}_{\lambda_0}, \quad \lambda_0 \in [-a, a] \tag{4.3}$$

$$\mathcal{E}(\hat{\Theta}) = \mathcal{S}_{\lambda_0}(\hat{\Theta}) + \mathcal{V}_1(\hat{\Theta}) + \ldots + \mathcal{V}_m(\hat{\Theta}) \tag{4.4}$$

$$\mathcal{V}_i = \mathcal{S}_{\lambda_i} \cap \mathcal{U}_{\lambda_{i-1}}, \quad i = 1, 2, \ldots, m \tag{4.5}$$

$$\text{If } \ \Sigma(\hat{\Theta}) \subset [\lambda_0, a] \Rightarrow \Sigma(\hat{\Theta}) = U_{i=1}^{m}[a_i, b_i] \tag{4.6}$$

The spectral intervals $[a_i, b_i]$ have been ordered so that $b_i \leq a_{i+1}, \quad i = 1, 2, \ldots, m-1$.

Let $\mathbf{P}_i : \mathcal{E}(\hat{\Theta}) \rightarrow \mathcal{E}(\hat{\Theta})$ denote a projector corresponding with the descomposition (4.4) such that Range $(\mathbf{P}_i) = \mathcal{R}(\mathbf{P}_i) = \mathcal{V}_i$ and the null space being the sum of the remain $\mathcal{V}_j$ and $\mathcal{S}_{\lambda_0}$ for $j \neq i$

Then if $\mathbf{P}_{\lambda_0} : \mathcal{E}(\hat{\Theta}) \to \mathcal{E}(\hat{\Theta})$ is the projector on $\mathcal{E}$ such that

$$\mathcal{R}(\mathbf{P}_{\lambda_0}) = \mathcal{S}_{\lambda_0} \text{ and } \mathcal{N}(\mathbf{P}_{\lambda_0}) = \mathcal{V}_1 + \mathcal{V}_2 + \ldots + \mathcal{V}_m$$

Hence

$$I = P_{\lambda_0}(\theta) + P_1(\theta) + \ldots + P_m(\theta), \quad \forall \theta \in \hat{\Theta}.$$

Given a point $(x, \theta) \in \mathcal{E}$, $x \neq 0$, we shall define the four **Lyapunov exponents** of (x, θ) as follows:

$$\lambda_s^+(x, \theta) = \overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| \tag{4.7}$$

$$\lambda_i^+(x, \theta) = \underline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| \tag{4.8}$$

$$\lambda_s^-(x, \theta) = \overline{\lim_{t \to -\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| \tag{4.9}$$

$$\lambda_i^-(x, \theta) = \underline{\lim_{t \to -\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| \tag{4.10}$$

Theorem 4.1 *(A) If $(x, \theta) \in \mathcal{V}_i$ where $\mathcal{V}_i$, is the spectral subbundle associated with λ_0 and $[a_i, b_i]$, and $x \neq 0$, then the four Lyapunov exponents agree and the limits*

$$\lim_{t \to +\infty} \frac{1}{t} \ln \|\Phi(\theta, t)x\| = \lim_{t \to -\infty} \frac{1}{t} \ln \|\Phi(\theta, t)x\|$$

exist and are equal to a_i.

(B) If $(x, \theta) \in \mathcal{S}_{\lambda_0}$, $x \neq 0$ then the two Lyapunov exponents (4.7) - (4.8) agree and the limits

$$\overline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| = \underline{\lim_{t \to +\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| = \lambda_0.$$

Proof The proof of (A) is similar to the prove of Theorem 3 in [ ]. In order to prove (B) let us consider $(x, \theta) \in S_{\lambda_0}$ with $x \neq 0$. Then

$$\lim_{t \to \infty} \|\Phi_{\lambda_0}(\theta, t)x\| = \lim_{t \to +\infty} \|e^{-\lambda_0 t}\Phi(\theta, t)x\| = 0$$

Therefore, there is a constant $M > 0$ such that

$$\ln \|e^{-\lambda_0 t}\Phi(\theta, t)x\| \leq M, \quad t \geq 0$$

Then

$$0 = \lim \frac{1}{t} \ln \|e^{-\lambda_0 t}\Phi(\theta, t)x\| \;\; = \lim_{t \to \infty} [-\lambda_0 + \frac{1}{t} \ln \|\Phi(\theta, t)x\|]$$

So

$$\lim_{t \to \infty} \frac{1}{t} \ln \|\Phi(\theta, t)x\| = \lambda_0$$

□

Definition 4.1 For all $\theta \in \Theta$ we define the upper Lyapunov exponent $\lambda_s^+(\theta)$ and the lower Lyapunov exponent $\lambda_i^+(\theta)$ as follows:

$$\lambda_s^+(\theta) := sup\{\lambda_s^+(x, \theta) : \quad x \in X, \quad x \neq 0\}$$

$$\lambda_s^+(\theta) := inf\{\lambda_i^+(x, \theta) : \quad x \in X, \quad x \neq 0\}$$

Theorem 4.2 *The upper Lyapunov exponent $\lambda_s^+(\theta)$ and the lower Lyapunov exponent $\lambda_i^+(\theta)$ associated to $\theta \in \Theta$ are given by :*

$$\lambda_s^+(\theta) = \overline{\lim_{t \to \infty}} \frac{1}{t} \ln \|\Phi(\theta, t)\| \tag{4.11}$$

$$\lambda_i^+(\theta) = -\overline{\lim_{t \to \infty}} \frac{1}{t} \ln \|\Phi(\theta.(-t), t)\| \tag{4.12}$$

Proof We shall denote by $\gamma(\theta)$ the right side of (4.11). We have the following

$$\frac{1}{t}\ln\|\Phi(\theta,t)x\| \le \frac{1}{t}\ln\|\Phi(\theta,t)\| + \frac{1}{t}\ln\|x\|$$

which implies that

$$\overline{\lim_{t\to\infty}}\frac{1}{t}\ln\|\Phi(\theta,t)x\| \le \overline{\lim_{t\to\infty}}\frac{1}{t}\ln\|\Phi(\theta,t)\|$$

Hence $\lambda_s^+(x,\theta) \le \gamma(\theta)$, for all $x \in X$, $x \neq 0$; therefore $\lambda_s^+(\theta) \le \gamma(\theta)$.

In order to prove the opposite inequality we shall use the following fact: For each $\epsilon > 0$ and $x \in X$ there is $N_{\epsilon,x} > 0$ such that

$$\|\Phi(\theta,t)x\| \le N_{\epsilon,x}e^{(\lambda_s^+(\theta)+\epsilon)t}, \qquad t \ge 0$$

The above inequality can be writing as follows

$$\|\Phi(\theta,t)xe^{-(\lambda_s^+(\theta)+\epsilon)t}\| \le N_{\epsilon,x}$$

Then the operators family

$$\left\{\ \Phi(\theta,t)e^{-(\lambda_s^+(\theta)+\epsilon)t}: \quad t \ge 0\ \right\}$$

is bounded for each $x \in X$. It follows from Uniform Boundness Theorem that there exists $N_\epsilon > 0$ such that

$$\|\Phi(\theta,t)e^{-(\lambda_s^+(\theta)+\epsilon)t}\| \le N_\epsilon, \qquad t \ge 0$$

Hence

$$\|\Phi(\theta,t)\| \le N_\epsilon e^{(\lambda_s^+(\theta)+\epsilon)t}, \qquad t \ge 0$$

Therefore

$$\frac{1}{t}\ln\|\Phi(\theta,t)x\| \le \frac{1}{t}\ln(N_\epsilon) + (\lambda_s^+(\theta) + \epsilon)$$

So

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|\Phi(\theta, t)x\| \leq \lambda_s^+(\theta) + \epsilon$$

Hence

$$\gamma(\theta) \leq \lambda_s^+(\theta).$$

The proof of (4.12) is not hard to do.

□

5 Examples

In this section we shall present some examples in which we are able to locate the Dynamical Spectrum.

The following example is sufficiently general to be considered here.

Example 5.1 In this example we shall consider a linear time dependent differential equation that generates a linear skew-product flow on the trivial Banach bundle $\mathcal{E} = X \times \Theta$ where X is a Banach space and Θ is a compact topological Hausdorff space.

More precisely, we shall study the following linear time dependent system

$$\dot{x}(t) = A(\theta \cdot t)x(t), \quad t \in I\!R \tag{5.1}$$

where $A(\theta \cdot t) = A + B(\theta \cdot t)$, A is the infinitesimal generator of the strongly continuous group $\{T(t)\}_{t \in I\!R}$; $\sigma(\theta, t) = \theta \cdot t$ is a flow on Θ and $B(\theta) \in L(X)$, $t \in I\!R$.

Lemma 5.1 *If $B(\cdot) : \Theta \to L(X)$ is strongly continuous, then the set $\{\|B(\theta)\| : \theta \in \Theta\}$ is bounded.*

Proof Consider the following sets

$$H = \{\|B(\theta)\| : \theta \in \Theta\}, \qquad H(x) = \{\|B(\theta)x\| : \theta \in \Theta\}$$

Since $\theta \to B(\theta)x$ is continuous and Θ is compact, then for each $x \in X$ we get that $H(x)$ is bounded. Hence, by the Uniform Boundedness Theorem we obtain that H is bounded. $\square$

Lemma 5.2 *If $B(\cdot) : \Theta \to L(X)$ is strongly continuous and $x(\cdot) : I\!R \to X$ is a continuous function, then for each $\theta \in \Theta$ the mapping $t \to B(\theta \cdot t)x(t)$ is continuous.*

Proof Fix $t \in I\!R$. Then

$$\|B(\theta(t+h))x(t+h) - B(\theta \cdot t)x(t)\|$$

$$= \|B(\theta(t+h))[x(t+h) - x(t)] - [B(\theta.(t+h)) - B(\theta \cdot t)]x(t)\|$$

$$\leq L\|x(t+h) - x(t)\| + \|[B(\theta.(t+h)) - B(\theta \cdot t)]x(t)\|$$

where $L = \sup\{\|B(\theta)\| : \theta \in \Theta\}$ and $\theta.t \in \Theta$ for $t \in I\!R$.
Therefore

$$\|B(\theta(t+h))x(t+h) - B(\theta \cdot t)x(t)\| \to 0, \quad \text{as} \quad h \to 0$$

□

To be precise in which sense the equation (5.1) generates a linear skew-product flow, we shall consider the following family of integral differential equations:

$$x(t) = T(t)x_0 + \int_0^t T(t-s)B(\theta \cdot s)x(s)\,ds. \;\; t \in I\!R \;\; \theta \in \Theta. \tag{5.2}$$

Definition 5.1 A solution $x(t) = x(t, \theta)$ of the equation (5.2) is called a Mild Solution of (5.1).

Theorem 5.1 *Let A be the infinitesimal generator of an strongly continuous group $\{T(t)\}_{t \in I\!R}$ on X and $B(\cdot) : \Theta \to L(X)$ is also strongly continuous. Then for each $\theta \in \Theta$ and $x_0 \in X$ the problem*

$$\dot{x}(t) = A(\theta \cdot t)x = (A + B(\theta \cdot t))x(t); \quad x(0) = x_0 \tag{5.3}$$

has a unique mild solution $\Phi(\theta, t)x_0$ *given by*

$$\Phi(\theta, t)x_0 = T(t)x_0 + \int_0^t T(t-s)B(\theta \cdot s)\Phi(\theta, s)x_0 ds. \tag{5.4}$$

If

$$\|T(t)\| \leq Me^{W|t|}, \quad t \in \mathbb{R},$$

then

$$\|\Phi(\theta, t)\| \leq Me^{(W+LM)|t|}, \quad t \in \mathbb{R} \tag{5.5}$$

where

$$L = \sup\{\|B(\theta)\| : \theta \in \Theta\}$$

Moreover, the mapping $\pi : \mathcal{E} \times \mathbb{R} \to \mathcal{E}$ *given by*

$$\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta \cdot t) \tag{5.6}$$

is a linear skew-product flow on $\mathcal{E} = X \times \Theta$ *, maybe without the condition (4) of Definition 2.1.*

The following Proposition corresponds to the part (4) of Definition 2.1

Proposition 5.1 *If the flow* $\sigma(\theta, t) = \theta \cdot t$ *depends continuously on* θ *in compacts intervals, then for all fixed* $t \in \mathbb{R}$*, the mapping from* $\mathcal{E}$ *to* X *given by* $(x, \theta) \to \Phi(\theta, t)x$ *is continuous.*

Proposition 5.2 *Let $\pi = (\Phi, \sigma)$ be the linear skew-product flow defined by (5.6). Then*

$$\Sigma(\hat{\Theta}) \subset [-(W + LM), W + LM]$$

where

$$L = \sup\{\|B(\theta)\| : \theta \in \Theta\}, \qquad \|T(t)\| \leq Me^{W|t|}, \quad t \in \mathbb{R}.$$

Example 5.2 (The perturbed wave equation). The simplest one dimensional hyperbolic equation with a time dependent perturbation of the form

$$\begin{aligned} u_{tt} &= u_{xx} + b(t)u_x, \qquad t \in \mathbb{R}, \quad -\infty < x < \infty \qquad (5.7) \\ u(0,x) &= u_1(x) \\ u_t(0,x) &= u_2(x) \end{aligned}$$

where $b(\cdot) : \mathbb{R} \to \mathbb{R}$ is uniformly continuous and bounded function.

Let $u_1, u_2 \in L_2(\mathbb{R})$. Define:

$$z_1(t,x) = u_t(t,x) \text{ and } z_2(t,x) = u_x(t,x).$$

Then the equation can be rewritten as follow

$$\frac{\partial z_1(t,x)}{\partial t} = \frac{\partial z_2(t,x)}{\partial x} + b(t)z_2(t,x) \qquad (5.8)$$

$$\frac{\partial z_2(t,x)}{\partial t} = \frac{\partial z_1(t,x)}{\partial x} \qquad (5.9)$$

Hence, letting $z(t,x)$ denote the column vector with components $z_1(t,x), z_2(t,x)$, we can write the equations (5.8) and (5.9) as follow

$$\frac{\partial z(t,x)}{\partial t} = Az(t,x) + B(t)z(t,x) \qquad (5.10)$$

where

$$A = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \qquad B(t) = \begin{bmatrix} 0 & b(t) \end{bmatrix}$$

Therefore the operator A is now defined on the class of functions $\mathrm{col}[z_1, z_2]$ in the product Hilbert spaces $Z = L_2(\mathbb{R}) \times L_2(\mathbb{R})$ such that their derivatives are also in Z and $t \to B(t) \in L(Z)$ is a uniformly continuous and bounded function.

It is well known that A generates a strongly continuous contraction semigroup $\{T(t)\}_{t\geq} \subset L(Z)$. Actually for $z \in D(A)$

$$\frac{d}{dt}[T(t)z, T(t)z] = 0.$$

Since the domain of A is dense this means that $\|T(t)z\| = \|z\|$ for every $z \in Z$ or $T(t)$ is an "isometry". We have actually a group; $T(t)$ has bounded inverse for every t. Either by working with Fourier transforms (or formally taking e^{At}) it can be verified that

$$T(t)z(x) = \begin{bmatrix} \frac{z_1(x+t)+z_1(x-t)}{2} + \frac{z_2(x+t)-z_2(x-t)}{2} \\ \frac{z_2(x+t)+z_2(x-t)}{2} + \frac{z_1(x+t)-z_1(x-t)}{2} \end{bmatrix}$$

The function $B(\cdot)$ belong to the space W of the function $C : \mathbb{R} \to \mathbb{R}^2$ which are continuous and bounded, endow with the topology of the uniform convergence on compact sets of $\mathbb{R}$. For each $C \in W$ and $\tau \in \mathbb{R}$ we define the $\tau-$ translation of C as the function $C_\tau \in W$ given by $C_\tau(t) = C(\tau + t)$. Then we consider the Hull of $B(\cdot)$ as the following set

$$\Theta = \mathrm{Hull}(B) = cl\{B_\tau : \tau \in \mathbb{R}\}$$

where cl denotes the closure in the topology of W. The set Θ is compact in W due to Proposition 1.1 in [1].

We define on Θ the flow $\sigma(C,t) = C_t = \theta \cdot t, \quad \theta = C$. Then Θ is invariant under σ.

Instead of concentrating on the single aquation (5.10) we consider the family of equations

$$\frac{d}{dt}z(t) = Az(t) + B(\theta \cdot t)z(t), \quad \theta \in \Theta \tag{5.11}$$

Here we have abused notation and written $B(\theta \cdot t) := \sigma(\theta \cdot t)$.

Then using Theorema 5.1 we get that the mapping $\pi : Z \times \Theta \times I\!R \to \Theta \times Z$ given by

$$\pi(x,\theta,t) = (\Phi(\theta,t)x, \theta.t)$$

where $\Phi(\theta,t)$ is given by

$$\Phi(\theta,t)z_0 = T(t)z_0 + \int_0^t T(t-s)B(\theta \cdot s)\Phi(\theta,s)z_0 ds$$

is an skew-product flow on $Z \times \Theta$.

Moreover, if $L = \sup\{\|B(t)\| : t \in I\!R\}$, then we get that

$$\Sigma(\hat{\Theta}) \subset [-L, L]$$

since $\|T(t)\| = 1$.

Example 5.3 Using the same idea as in [2] we can consider the equation (5.9) with more general perturbation function $b(t,x)$ which is only bounded and measurable depending on t, x in. In fact. Consider the equation

$$u_{tt} = u_{xx} + b(t,x)u_x, \quad t \in I\!R, \quad -\infty < x < \infty \tag{5.12}$$

$$
\begin{aligned}
u(0,x) &= u_1(x) \\
u_t(0,x) &= u_2(x)
\end{aligned}
$$

In the same way as equation (5.9) we can write equation (5.12) as follow

$$
\frac{\partial z_1(t,x)}{\partial t} = \frac{\partial z_2(t,x)}{\partial x} + b(t,x)z_2(t,x) \tag{5.13}
$$

$$
\frac{\partial z_2(t,x)}{\partial t} = \frac{\partial z_1(t,x)}{\partial x} \tag{5.14}
$$

Hence, letting $z(t,x)$ denote the column vector with components $z_1(t,x), z_2(t,x)$, we can write the equations (5.13) and (5.14) as follow

$$
\frac{\partial z(t,x)}{\partial t} = Az(t,x) + B(t)z(t,x) \tag{5.15}
$$

where

$$
A = \begin{bmatrix} 0 & \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & 0 \end{bmatrix} \qquad B(t)z(x) = \begin{bmatrix} 0 & b(t,x)z_2(x) \end{bmatrix}
$$

Therefore the operator A is now defined on the class of functions col$[z_1, z_2]$ in the product Hilbert spaces $Z = L_2(\mathbb{R}) \times L_2(\mathbb{R})$ such that their derivatives are also in Z and the function $B(t,x) = \begin{bmatrix} 0 & b(t,x) \end{bmatrix} = b(t,x)$ belong to the space

$$
L^\infty = l^\infty(\mathbb{R}^2) = \{b : \mathbb{R}^2 \to \mathbb{R}^2 \ : \ b(\cdot,\cdot) \text{ is measurable and bounded almost everywhere}\}
$$

with the essential supremum norm, i.e.,

$$
\|b\|_\infty = \text{ess.sup}\{\|b(t,x)\| : \ t,\, x \in \mathbb{R}\}.
$$

For $R > 0$ we consider

$$
B_R = \{b(\cdot,\cdot) \in L^\infty : \ \|b\|_\infty \le R\}
$$

So, B_R denote the close ball in L^∞ of radiu R. It is basic fact that the set B_R endowed with the weak* topology of L^∞ is a compact metrizable topological space (Interesting, the whole space L^∞ with the weak* topology is not metrizable).
Since $L_1^* = L_\infty$, then a sequence $\{f_n\}$ converges to $f \in L_\infty$ if and only if, for every $g \in L_1$ one has:

$$< f_n, f > \to < f, g > \iff \int f_n . g d\mu \to \int f . g d\mu, \quad \text{as} \quad n \to \infty$$

where $< \cdot, \cdot >$ is the duality between L_∞ and L_1.
Moreover, the translation mapping $\sigma(\cdot,\cdot) : R_R \times I\!R \to B_R$ given by

$$\sigma(b, t) = b_t = b(t + \cdot, \cdot),$$

is continuous in this topoly and the mapping σ is a flow on B_R. Then we shall consider the Hull of b as the following set:

$$\Theta = \text{Hull}(b) = cl\{b_t : \quad t \in I\!R\} \subseteq B_R$$

where cl denotes the closure in the topology of B_R. Clearly the set Θ is compact in B_R.
We define on Θ the flow $\sigma(b, t) = b_t = \theta \cdot t, \quad \theta = b$. Then Θ is invariant under σ.
Instead of concentrating on the single aquation (5.15) we consider the family of equations

$$\frac{d}{dt} z(t) = Az(t) + B(\theta \cdot t)z(t), \quad \theta \in \Theta \tag{5.16}$$

Here we have abused notation and written $B(\theta \cdot t) := \sigma(\theta \cdot t)$.

Then using Theorema 5.1 we get that the mapping $\pi : Z \times \Theta \times I\!R \to \Theta \times Z$ given by

$$\pi(x, \theta, t) = (\Phi(\theta, t)x, \theta.t)$$

where $\Phi(\theta,t)$ is given by

$$\Phi(\theta,t)z_0 = T(t)z_0 + \int_0^t T(t-s)B(\theta\cdot s)\Phi(\theta,s)z_0 ds$$

is an skew-product flow on $Z\times\Theta$.

Moreover, if $L = \|b(t,x)\|_\infty$, then we get that

$$\Sigma(\hat{\Theta}) \subset [-L, L]$$

since $\|T(t)\| = 1$.

Example 5.4 Linear skew-product flow defined by a Linear Neutral Functional Differential Equation.

For a fixed $r > 0$, the space of continuous functions $g : [-r,0] \to I\!R$ with the usual uniform norm is denoted by $C = C([-r,0], I\!R)$.

Then we shall consider the folling Linear Neutral Functional differential Equation (LNFDE)

$$\frac{d}{dt}[x(t) - cx(t-r)] = bx(t-r) + a(t)x(t) \tag{5.17}$$

where $c, b \in I\!R$ and $a : I\!R \to I\!R$ is a uniformly continuous and bounded function.

The equation (5.17) can be written as follow

$$\frac{d}{dt}[Dx(t)] = Lx_t + f(t,x_t) \tag{5.18}$$

where $x_t(\theta) = x(t+\theta)$, for $-r \le \theta \ge 0$, $D\phi = \phi(0) - c\phi(-r)$, $L\phi = b\phi(-r)$ and $f(t,x_t) = a(t)x(t)$.

From Theorem 7.2 in [3] we have that for each $\phi \in C$ there is a unique solution $x(t,\phi,a)$ of (5.18) defined on $I\!R$ which satisfies the initial condition $x_0 = \phi$.

Moreover, if we consider the equation

$$\frac{d}{dt}[Dx(t)] = Lx_t \ \ (a = 0) \tag{5.19}$$

and the corresponding solution map

$$T(t) : C \to C, \ \ T(t)\phi = x_t(\cdot, \phi), \ \ t \in I\!R,$$

then from Theorem 7.4 of [3] we get that the solutions of (5.18) are given by the following variational constant formula

$$x(t, \phi, a) = T(t)\phi(0) + \int_0^t X(t-s)a(s)x(s, \phi, a)ds, \tag{5.20}$$

where $X(t)$ is the solution of the equation

$$\dot{X}(t) - c\dot{X}(t-r) = bX(t-r)$$

with the initial condition $X(t) = 0, \ \ t < 0, \ \ X(0) = 1$. From Theroem 7.6 and corollary 7.2 in [3] we get the following: If $\alpha_0 = \sup\{\mathrm{Re}\lambda : \lambda(1 - ce^{-\lambda r}) = be^{-\lambda r}\}$, then for any $\alpha > \alpha_0$ there is $k = k(\alpha)$ such that

$$\|T(t)\| \leq ke^{\alpha|t|}\|\phi\| \ \text{ and } \ |X(t)| \leq ke^{\alpha|t|}, \ \ \forall t \in I\!R \tag{5.21}$$

where $\|\phi\| = sup_{-r\leq s\leq 0}|\phi(s)|$.

The function $a(\cdot)$ belong to the space W of the functions $g : I\!R \to I\!R$ which are continuous and bounded, endow with the topology of the uniform convergence on compact sets of $I\!R$. For each $g \in W$ and $\tau \in I\!R$ we define the $\tau-$ translation of g as the function $g_\tau \in W$ given by $g_\tau(t) = g(\tau + t)$. Then we consider the Hull of $a(\cdot)$ as the following set

$$\Theta = \text{Hull}(a) = cl\{a_\tau : \tau \in I\!R\}$$

where cl denotes the closure in the topology of W. The set Θ is compact in W and invariant under the flow $\sigma(g,t) = g_t = \theta \cdot t, \quad \theta = g$, due to Proposition 1.1 in [1].

Instead of concentrating on the single aquation (5.18) we consider the family of equations

$$\frac{d}{dt}[Dx(t)] = Lx_t + a(\theta \cdot t)x(t), \quad \theta \in \Theta; \tag{5.22}$$

here we have abused notation and written $a(\theta \cdot t) := \sigma(\theta \cdot t)$.

Then we get that the mapping $\pi : C \times \Theta \times I\!R \to C \times \Theta$ given by

$$\pi(\phi, \theta, t) = (\Phi(\theta, t)\phi, \theta \cdot t)$$

where $\Phi(\theta, t)\phi$ is given by

$$\Phi(\theta, t)\phi = T(t)\phi + \int_0^t X_{t-s}a(s)\Phi(\theta, s)\phi ds$$

is an skew-product flow on $C \times \Theta$.

Moreover, if $L = \sup\{\|a(t)\| : t \in I\!R\}$, then from (5.21) we get that

$$\|\Phi(\theta, t)\| \leq ke^{(\alpha + kL)|t|}, \quad \forall t \in I\!R \quad \text{and} \quad \Sigma(\hat{\Theta}) \subset [-(\alpha + kL), \alpha + kL].$$

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TOPICS IN DYNAMICAL SYSTEMS

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Shui-Nee Chow and Jack K. Hale, PI's

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Georgia Tech
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NSF Annual Progress Report

TOPICS IN DYNAMICAL SYSTEMS

Summary of Progress:

Two papers were written that received full or partial support from this project. Please find attached copies of these papers and written below are titles and abstracts. Also, J. Pinto finished his Ph.D. thesis on slow motion manifolds. A manuscript is in preparation.

V.S. Afraimovich and S-N. Chow, *Synchronizations in lattices of nonlinear Duffing's oscillators*

Abstract: In this work we prove the possibility of stochastic synchronization in two-dimensional lattice of coupled Duffing's oscillators with external periodic forces. The synchronization occurs provided coupling is dissipative and the coefficients of coupling is greater than some critical values. These values depend on parameters of individual subsystems and on the size of the lattice.

J.K. Hale and W. Huang, *Periodic solutions of singularly perturbed delay equations*

Abstract: We consider the singularly perturbed delay differential system

$$\begin{aligned} \epsilon\dot{x}(t) &= f_\lambda(x(t), y(t)) \\ y(t) &= g_\lambda(x(t), y(t), x(t-1), y(t-1)), \end{aligned} \tag{1.1}$$

where $\epsilon > 0$, λ are small real parameters, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors and the functions $f_\lambda(x, y) = f(x, y, \lambda)$ and $g_\lambda(x, y, z, w) = g(x, y, z, w, \lambda)$ are smooth vector valued functions which vanish for all variables equal to zero.

In many situations, for fixed $\lambda > 0$, there is an $\epsilon(\lambda) > 0$ such that (1.1) undergoes a generic Hopf bifurcation at $(\lambda, \epsilon, x, y) = (\lambda, \epsilon(\lambda), 0, 0)$ to a periodic solution $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$. Let us assume that there is a sectorial region S in a neighborhood U of $(\lambda, \epsilon) = (0, 0)$ such that $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ exists for all $(\lambda, \epsilon > 0)$ in S. Our objective is to understand the behavior of the profile of $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ as $\epsilon \to 0$. We are not able to do this in the general context described, but we can say something if we impose more conditions on the functions f_λ, g_λ.

It is to be expected that the limiting profile is in some way related to the equation obtained by putting $\epsilon = 0$ in (1.1). For $\epsilon = 0$, we suppose that the resulting equation (1.1) defines a map on $\mathbb{R}^m \times \mathbb{R}^n$:

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto T_\lambda(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \tag{1.2}$$

for which the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. Let us also suppose that $T_\lambda(x, y)$ undergoes a generic period doubling bifurcation at $(x, y, \lambda) = (0, 0, 0)$ with the period two points being $d_{j\lambda} \in \mathbb{R}^m \times \mathbb{R}^n$, $j = 1, 2$. If the bifurcation is supercritical, the period two orbit is stable and, if the bifurcation is subcritical, the period two orbit is unstable. In the supercritical case, there is a natural stable periodic function of period two given by the square wave $(x^s_\lambda(t), y^s_\lambda(t))$ which alternately takes the values $d_{1\lambda}$ and $d_{2\lambda}$ on intervals of length one. In the subcritical case, the period two points are unstable and the natural periodic function of period two is a pulse wave $(x^p_\lambda(t), y^p_\lambda(t))$ which is zero except on the integers and it alternately takes the values $d_{1\lambda}$ and $d_{2\lambda}$ on the integers.

Under some conditions on the functions f_λ, g_λ, the generic period doubling bifurcation of $T_\lambda(x, y)$ leads to a generic Hopf bifurcation in (1.1) which is supercritical (resp. subcritical) if the period doubling bifurcation is supercritical (resp. subcritical). A natural question is the following: Is it possible that the limiting profile of the periodic solution $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ obtained through the Hopf bifurcation is either the square wave or pulse wave? We present situations in which this is true.

Periodic solutions of singularly perturbed delay equations

by

J.K. Hale and W. Huang

CDSNS94-201

Periodic solutions of singularly perturbed delay equations

by

Jack K. Hale * and Wenzhang Huang

1. Introduction. In this paper, we consider the singularly perturbed delay differential system

$$\begin{aligned} \epsilon\dot{x}(t) &= f_\lambda(x(t), y(t)) \\ y(t) &= g_\lambda(x(t), y(t), x(t-1), y(t-1)), \end{aligned} \tag{1.1}$$

where $\epsilon > 0$, λ are small real parameters, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors and the functions $f_\lambda(x, y) = f(x, y, \lambda)$ and $g_\lambda(x, y, z, w) = g(x, y, z, w, \lambda)$ are smooth vector valued functions which vanish for all variables equal to zero.

In many situations, for fixed $\lambda > 0$, there is an $\epsilon(\lambda) > 0$ such that (1.1) undergoes a generic Hopf bifurcation at $(\lambda, \epsilon, x, y) = (\lambda, \epsilon(\lambda), 0, 0)$ to a periodic solution $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$. Let us assume that there is a sectorial region S in a neighborhood U of $(\lambda, \epsilon) = (0, 0)$ such that $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ exists for all $(\lambda, \epsilon > 0)$ in S. Our objective is to understand the behavior of the profile of $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ as $\epsilon \to 0$. We are not able to do this in the general context described, but we can say something if we impose more conditions on the functions f_λ, g_λ.

It is to be expected that the limiting profile is in some way related to the equation obtained by putting $\epsilon = 0$ in (1.1). For $\epsilon = 0$, we suppose that the resulting equation (1.1) defines a map on $\mathbb{R}^m \times \mathbb{R}^n$:

$$(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto T_\lambda(x, y) \in \mathbb{R}^m \times \mathbb{R}^n \tag{1.2}$$

for which the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. Let us also suppose that $T_\lambda(x, y)$ undergoes a generic period doubling bifurcation at $(x, y, \lambda) = (0, 0, 0)$ with the period two points being $d_{j\lambda} \in \mathbb{R}^m \times \mathbb{R}^n$, $j = 1, 2$. If the bifurcation is supercritical, the period two orbit is stable and, if the bifurcation is subcritical, the period two orbit is unstable. In the supercritical case, there is a natural stable periodic function of period two given by the square wave $(x^s_\lambda(t), y^s_\lambda(t))$ which alternately takes the values $d_{1\lambda}$ and $d_{2\lambda}$ on intervals of length one. In the subcritical case, the period two points are unstable and the natural periodic function of period two is a pulse wave $(x^p_\lambda(t), y^p_\lambda(t))$ which is zero except on the integers and it alternately takes the values $d_{1\lambda}$ and $d_{2\lambda}$ on the integers.

Under some conditions on the functions f_λ, g_λ, the generic period doubling bifurcation of $T_\lambda(x, y)$ leads to a generic Hopf bifurcation in (1.1) which is supercritical (resp. subcritical) if the period doubling bifurcation is supercritical (resp. subcritical). A natural question is the following: Is it possible that the limiting profile of the periodic solution $(\tilde{x}^*_{\epsilon\lambda}, \tilde{y}^*_{\epsilon\lambda})$ obtained through the Hopf bifurcation is either the square wave or pulse wave? We will give some situations in which this is true.

Let us mention that equations of the form (1.1) occur frequently in the applications. The delay differential equation

$$\epsilon\dot{x}(t) = -x(t) + f_\lambda(x(t-1)). \tag{1.3}$$

* Partially supported by NSF and DARPA

has often served as a model for physiological control systems [6], [14], [20] and for the transmission of light through a ring cavity [1], [5], [12], [13]. This can be considered as a special case of (1.1) by letting $y(t) = x(t-1)$. Under the conditions that the map $x \mapsto f_\lambda(x)$ undergoes a generic supercritical periodic doubling bifurcation at $(x, \lambda) = (0,0)$, it was shown in [2] that there is a periodic solution of (1.3) which approaches the square wave mentioned above. If the bifurcation is subcritical, it was shown in [7] that there is a periodic solution of (1.3) which approaches a pulse wave but the pulses are larger that the values $d_{1\lambda}, d_{2\lambda}$ mentioned above. The explanation for this will be given later. The limiting behavior of large amplitude periodic solutions of (1.3) also has been investigated in [15], [16] when the limiting profile is a square wave.

The more general equation

$$(\epsilon_m \frac{d}{dt} + 1) \cdots (\epsilon_1 \frac{d}{dt} + 1) y(t) = h_\lambda(y(t-1)), \tag{1.4}$$

where each $\epsilon_j > 0$ is a small parameter, also has been proposed as a model for transmission of light through a ring cavity (see [18], [19]). Equation (1.4) is equivalent to the system

$$\begin{aligned} \epsilon_1 \dot{x}_1(t) + x_1(t) &= x_2(t) \\ &\cdots\cdots \\ \epsilon_{m-1} \dot{x}_{m-1}(t) + x_{m-1}(t) &= x_m(t) \\ \epsilon_m \dot{x}_m(t) + x_m(t) &= h_\lambda(x_1(t-1)). \end{aligned} \tag{1.5}$$

If we scale the ϵ_j as $\epsilon_j = \epsilon \alpha_j^{-1}$, $j = 1, 2, \ldots, m$, then we obtain an equivalent matrix equation

$$\epsilon \dot{x}(t) + Ax(t) = Af_\lambda(x(t-1)), \tag{1.6}$$

where A and f are given by

$$A = \begin{bmatrix} \alpha_1 & -\alpha_1 & 0 & \ldots & 0 & 0 \\ 0 & \alpha_2 & -\alpha_2 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & \alpha_{m-1} & -\alpha_{m-1} \\ 0 & 0 & 0 & \ldots & 0 & \alpha_m \end{bmatrix}, \qquad f_\lambda(x) = \begin{bmatrix} h_\lambda(x_1) \\ h_\lambda(x_1) \\ \cdot \\ \cdot \\ \cdot \\ h_\lambda(x_1) \\ h_\lambda(x_1) \end{bmatrix}.$$

This also is a special case of (1.1) with $y(t) = x_1(t-1)$.

Under the assumption that the map $x \mapsto h_\lambda(x)$ undergoes a generic period doubling, it was shown in [8] that there are periodic solutions of (1.5) resembling either a square wave or a pulse wave depending upon whether the bifurcation was supercritical or subcritical. For the general matrix equation (1.6) without assuming that it arises from (1.5) and under generic conditions on the function f_λ in (1.6), similar results were given in [8].

The proofs of the above results exploited the properties of flows on center manifolds for some scaled equations which will be mentioned below.

The equation

$$(1.7) \qquad \begin{aligned} \epsilon\dot{x}(t) + Ax(t) &= Af_\lambda(y(t)) \\ y(t) &= g_\lambda(x(t-1),\, y(t-1))\,, \end{aligned}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors, the $m \times m$ matrix A has an inverse and the functions $f_\lambda(y)$ and $g_\lambda(x,\, y)$ are smooth vector valued functions, has been used in [12], [13] as a model of a ring cavity containing a nonlinear dielectric medium for which part of the transmitted light is fed back into the medium. For $\epsilon = 0$, the map $T_\lambda(x,y) = (f_\lambda(y), g_\lambda(f_\lambda(y), y)$. Under the assumption that there is a supercritical period doubling bifurcation of the map $y \to g_\lambda(f_\lambda(y), y)$ at $(y, \lambda) = (0,0)$ and some other generic conditions, it was shown in [3] that there is a periodic solution of (1.7) which is similar to the square wave for ϵ small. The proof of this result used functional analytic methods based on exponential dichotomies.

Our objective in this paper is to show that center manifold techniques can be applied to discuss (1.7) when the period doubling bifurcation is either supercritical or subcritical. The limiting profile of the corresponding periodic solutions is either a square wave or a pulse wave. The precise statements of the results on convergence to square waves or pulse waves for the case (1.7) are given in Section 2. Sections 3 and 4 are devoted to the proof. The hypotheses imposed imply the existence of periodic solutions which are related to the period two points of the map. However, that analysis does not address the important question of the relataionship between the existence of generic period doubling of the map and the existence and stability of a generic Hopf bifurcation. These problems are addressed in Sections 5 and 6.

We remark that the equation (1.1) arises also in the theory of transmission lines. If the lines are lossless and described by the telegraph equations with the boundary conditions reflecting Kirchoff's laws, it has been known for a long time that the flow can be described by an equivalent set of neutral delay differential equations (see, for example, [11] for a discussion and references). Many of these same problems also can be written in the form (1.1). For example, in [17], the equations for a transmission line with a tunnel diode and a lumped parallel capacitance can be written as

$$(1.8) \qquad \begin{aligned} \epsilon\dot{x}(t) &= y(t) - g(x(t)) \\ y(t) &= \alpha + Ky(t-1) - x(t) - Lx(t-1)\,, \end{aligned}$$

where $(x(t), y(t))$ represent the voltage and current at one end of the line, all constants are positive and represent physical parameters. Under reasonable assumptions in the model, the parameter ϵ can be considered to be very small. In [17], several wave forms were observed numerically which compared reasonably well with experimental results. Some of these wave forms were very similar to square waves.

It is natural to enquire if it is theoretically possible to prove that, for some values of the parameters, there are periodic solutions of (1.8) which are similar to a square wave for $\epsilon > 0$ small. If we try to follow the same procedure as before, we first investigate the possibility of a supercritical period doubling bifurcation of some map. For $\epsilon = 0$, we must have $y(t) = g(x(t))$ and thus

$$(1.9) \qquad x(t) + g(x(t)) = \alpha + Kg(x(t-1)) - Lx(t-1).$$

For a given function g similar to the cubic polynomial, it is possible to show that there are values of the parameters in the equation and a constant x_0 such that (1.9) defines a map which undergoes a supercritical period doubling bifurcation near x_0. With the methods that we develop below, it should be possible to prove that there is a solution of (1.8) which approaches as $\epsilon \to 0$ either a square wave or a pulse wave. However, we do not discuss this problem here.

2. A ring cavity model. In this section, we consider the ring cavity model (1.7). We will impose some generic conditions on the functions f, g which will ensure that the associated map for $\epsilon = 0$ undergoes a generic period doubling at $(y, \lambda) = (0, 0)$. We search for periodic solutions of (1.7) of period approximately 2 and impose additional generic conditions on f, g which will permit the reduction of the problem to a discussion of certain periodic solutions of a two-dimensional vector field on a center manifold of specific scaled equations. We discuss also how the conditions on f, g are related to the generic Hopf bifurcation of periodic solutions of (1.7) for fixed $\epsilon > 0$ and λ being the bifurcation parameter. Since this section is devoted entirely to (1.7), we repeat some of the previous formulas and hypotheses since some are stated in other terms.

We consider the equation

$$\begin{aligned} \epsilon \dot{x}(t) + Ax(t) &= Af_\lambda(y(t)) \\ y(t) &= g_\lambda(x(t-1),\, y(t-1))\,, \end{aligned} \tag{2.1}$$

where $\epsilon > 0$, λ are small parameters, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$ are vectors, A is an $m \times m$ constant matrix and the functions $f_\lambda(y)$ and $g_\lambda(x,\, y)$ are smooth vector valued functions. We suppose that

(H1) $$A^{-1} \text{ exists} \quad f_\lambda(0) = 0, \quad g_\lambda(0,0) = 0.$$

For $\epsilon = 0$, there is the map

$$y \in \mathbb{R}^n \mapsto \mathcal{F}_\lambda(y) \equiv g_\lambda(f_\lambda(y), y) \in \mathbb{R}^n. \tag{2.2}$$

We first want to impose some conditions on the functions f_λ, g_λ to ensure that $\mathcal{F}_\lambda$ undergoes a generic period doubling at $(y, \lambda) = (0, 0)$ and that the period two point is either stable or of index 1; that is, has an unstable manifold of dimension 1. For notation, we let

$$\begin{aligned} A_2(\lambda) = D_y f_\lambda(0), \quad B_1(\lambda) &= D_x g_\lambda(0,0), \quad B_2(\lambda) = D_y g_\lambda(0,0) \\ C(\lambda) &\equiv B_1(\lambda)A_2(\lambda) + B_2(\lambda) = D_y \mathcal{F}_\lambda(0)\,. \end{aligned} \tag{2.3}$$

We also let $\sigma(C)$ denote the spectrum of a square matrix C and let $B_\rho = \{z \in \mathbb{C} \colon |z| \leq \rho\}$. We assume that

(H2) $$\begin{aligned} &-(1+\lambda) \in \sigma(C(\lambda)) \text{ is a simple eigenvalue,} \\ &\sigma(C(\lambda)) \setminus \{-(1+\lambda)\} \subset B_\rho \text{ with } \rho < 1. \end{aligned}$$

We remark that the theory below could be developed for the case in which $\sigma(C(\lambda)\setminus\{-(1+\lambda)\}$ has no eigenvalues of modulus 1, but in this case, if there is an eigenvalue with modulus > 1, then the period two bifurcation always will be unstable, a situation that is not the desirable one from the point of view of applications.

If **(H2)** is satisfied, then we can introduce a change of variables in y to obtain

$$C(\lambda) = \begin{bmatrix} -(1+\lambda) & 0 \\ 0 & H_0(\lambda) \end{bmatrix}, \quad \sigma(H_0(\lambda) \subset B_\rho, \ \rho < 1. \tag{2.4}$$

If we let $y = \text{col}\,(y_1, y_2) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$, $\mathcal{F}_\lambda = \text{col}\,(\mathcal{F}_{1\lambda}, \mathcal{F}_{2\lambda}) \in \mathbb{R}^1 \times \mathbb{R}^{n-1}$, then we can use (2.4) to write

$$\begin{aligned} \mathcal{F}_{1\lambda}(y) &= -(1+\lambda)y_1 + k_1(\lambda)y_1^2 + y_1 k_2(\lambda) y_2 + k_3(\lambda) y_1^3 + O(\|y_2\|^2 + y_1^2\|y_2\| + y_1^4) \\ \mathcal{F}_{2\lambda}(y) &= H_0(\lambda)y_2 + y_1^2 H_1(\lambda) + y_1 H_2(\lambda) + y_2 O(\|y_2\|^2 + y_1^2\|y_2\| + \|y\|^3) \end{aligned} \tag{2.5}$$

as $(y,\lambda) \to (0,0)$.

We assume that

(H3) $$R_1 \equiv k_2(0)[I_{n-1} - H_0(0)]^{-1} H_1(0) + k_1^2(0) + k_3(0) \neq 0,$$

where, for any integer k, I_k denotes the identity matrix in $\mathbb{R}^k$. In the following, we often omit this index if there is no reason for confusion.

Lemma 2.1. *If* (**H1**)-(**H3**) *are satisfied, then there is a generic period two doubling bifurcation of $\mathcal{F}_\lambda$ at $(y,\lambda) = (0,0)$. More precisely, if $R_1\lambda > 0$, then there are period two points $d_{1\lambda}, d_{2\lambda}$ of $\mathcal{F}_\lambda$ such that $\mathcal{F}_\lambda(d_{1\lambda}) = d_{2\lambda}, \mathcal{F}_\lambda(d_{2\lambda}) = d_{1\lambda}$, which is stable for $R_1 > 0$ (supercritical bifurcation) and unstable for $R_1 < 0$ (subcritical bifurcation).*

The proof is a standard application of the method of Lyapunov-Schmidt and is omitted.

To relate the period doubling of the map $\mathcal{F}$ to (2.1), we seek periodic solutons of (2.1) with a period $2 + 2(r_0 + h)\epsilon$, where r_0 is a fixed parameter (to be determined later) which depends only upon the matrices $A, A_2(0), B_1(0), B_2(0)$ and h will be determined as a function of ϵ. If $(x(t), y(t))$ is such a solution of (2.1), we introduce the transformation (originally used in [4] for a scalar equation)

$$\begin{aligned} u_1(t) &= x(-\epsilon(r_0+h)t), \quad u_2(t) = x(-\epsilon(r_0+h)t + 1 + \epsilon(r_0+h)) \\ v_1(t) &= y(-\epsilon(r_0+h)t), \quad v_2(t) = y(-\epsilon(r_0+h)t + 1 + \epsilon(r_0+h)) \end{aligned}. \tag{2.6}$$

Since $x(t)$ and $y(t)$ have period $2 + 2(r_0 + h)\epsilon$, we see that

$$\begin{aligned} u_2(t-1) &= x(-\epsilon(r_0+h)t - 1) \\ v_2(t-1) &= y(-\epsilon(r_0+h)t - 1). \end{aligned} \tag{2.7}$$

If we use (2.6) and (2.7) in (2.1), we deduce that

$$
\begin{aligned}
\dot{u}_1(t) &= (r_0+h)Au_1(t) - (r_0+h)Af_\lambda(v_1(t)) \\
\dot{u}_2(t) &= (r_0+h)Au_2(t) - (r_0+h)Af_\lambda(v_2(t)) \\
v_1(t) &= g_\lambda(u_2(t-1), v_2(t-1)) \\
v_2(t) &= g_\lambda(u_1(t-1), v_1(t-1))\,.
\end{aligned} \tag{2.8}
$$

This equation is now independent of ϵ and we can consider h, λ as the bifurcation parameters, assuming, of course, that we know r_0.

It is convenient to write these equations in a more compact form by letting $u = \text{col}\,(u_1, u_2)$, $v = \text{col}\,(v_1, v_2)$,

$$
\begin{aligned}
\hat{A} &= \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} A_2(0) & 0 \\ 0 & A_2(0) \end{bmatrix}, \\
\hat{B}_j &= \begin{bmatrix} 0 & B_j(0) \\ B_j(0) & 0 \end{bmatrix}, j = 1, 2.
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
F_{h\lambda}(u,v) &= -\begin{bmatrix} (r_0+h)f_\lambda(v_1) - r_0 D_y f_\lambda(0)v_1 - hAu_1 \\ (r_0+h)f_\lambda(v_2) - r_0 D_y f_\lambda(0)v_2 - hAu_2 \end{bmatrix} \\
G_{h\lambda}(u,v) &= \begin{bmatrix} g_\lambda(u_2,v_2) - D_x g_\lambda(0,0)u_2 - D_y g_\lambda(0,0)v_2 \\ g_\lambda(u_1,v_1) - D_x g_\lambda(0,0)u_1 - D_y g_\lambda(0,0)v_1 \end{bmatrix}.
\end{aligned} \tag{2.10}
$$

If we use (2.9) and (2.10), then (2.8) can be written as

$$
\begin{aligned}
\dot{u}(t) &= r_0\hat{A}u(t) - r_0\hat{A}\hat{A}_2 v(t) + F_{h\lambda}(u(t), v(t)) \\
v(t) &= \hat{B}_1 u(t-1) + \hat{B}_2 v(t-1) + G_{h\lambda}(u(t-1), v(t-1)).
\end{aligned} \tag{2.11}
$$

The linear variational equation of (2.11) for $(h, \lambda) = (0, 0)$ is given by

$$
\begin{aligned}
\dot{u}(t) &= r_0\hat{A}u(t) - r_0\hat{A}\hat{A}_2 v(t) \\
v(t) &= \hat{B}_1 u(t-1) + \hat{B}_2 v(t-1).
\end{aligned} \tag{2.12}
$$

The eigenvalues of (2.12) are the solutions of the characteristic equation

$$
\det \Delta(\mu, r_0) = 0, \quad \Delta(\mu, r_0) = \begin{bmatrix} \mu - r_0\hat{A} & r_0\hat{A}\hat{A}_2 \\ -\hat{B}_1 e^{-\mu} & I - \hat{B}_2 e^{-\mu} \end{bmatrix}. \tag{2.13}
$$

Because of (2.3), (2.4) and the symmetry in (2.12), zero always is an eigenvalue. We impose conditions on the coefficients to ensure that, for a suitable choice of r_0, 0 is an eigenvalue of multiplicity two and there are no other eigenvalues on the imaginary axis. To do this, we need some additional notation. Let

$$
-B_1(0)A^{-1}A_2(0) \equiv S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad -B_1(0)A^{-2}A_2(0) \equiv W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \tag{2.14}
$$

where S_{ij}, W_{ij} are matrices with S_{11} and W_{11} scalars and the other matrices of obvious dimension. We suppose that

$$\textbf{(H4)} \qquad R_0 \equiv -S_{11}^2 + 2[W_{11} + S_{12}(I + H_0(0))^{-1}S_{21}] \neq 0, \quad S_{11} \neq 0,$$

and that

$$\textbf{(H5)} \qquad \det \begin{bmatrix} i\omega I_{2m} - S_{11}\hat{A} & S_{11}\hat{A}\hat{A}_2 \\ -\hat{B}_1 e^{-i\omega} & I_{2n} - \hat{B}_2 e^{-i\omega} \end{bmatrix} \neq 0 \quad \text{for } \omega \in \mathbb{R} \setminus \{0\}.$$

We make the final hypothesis that

$$\textbf{(H6)} \qquad \sigma(B_2(0)) \subset B_\rho, \quad \rho < 1.$$

Lemma 2.2. *If* (**H4**)*-*(**H6**) *are satisfied and* $r_0 = S_{11}$*, then* $\mu = 0$ *is an eigenvalue of (2.12) of multiplicity two and there is a* $\delta > 0$ *such that the remaining eigenvalues satisfy* $|\mathrm{Re}\mu| \geq \delta > 0$ *and there are only a finite number of eigenvalues with positive real parts.*

Theorem 2.3. *Suppose that* $\mathcal{F}_\lambda(x)$ *satisfies (2.5) and* (**H1**)*-*(**H6**)*. Then there are a neighborhood* V *of zero in* $\mathbb{R}^m \times \mathbb{R}^n$ *and a neighborhood* U *of* $(0,\, 0)$ *in the* $(\lambda,\, \epsilon)$ *plane and a sectorial region* S *in* U *such that, if* $(\lambda,\, \epsilon) \in U$*, then there is a periodic solution of (2.1) in* V *given by a periodic function* $(\tilde{x}_{\lambda,\,\epsilon}, \tilde{y}_{\lambda,\,\epsilon})$ *with period* $2\tau(\lambda,\, \epsilon) = 2 + 2S_{11}\epsilon + O(|\epsilon|(|\lambda| + |\epsilon|))$ *as* $(\lambda,\, \epsilon) \to (0,\, 0)$ *if and only if* $(\lambda,\, \epsilon) \in S$*. Furthermore, this solution is unique.*

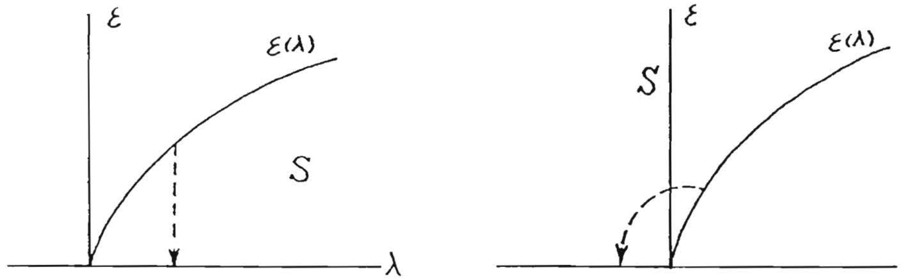
Of course, the sector S must belong to the set $\epsilon > 0$ in the $(\lambda,\, \epsilon)$ plane. We actually will show that, if $R_1 > 0$ (the supercritical case of period doubling of the map) and $R_0 > 0$, then the sector $S \subset \{(\lambda,\, \epsilon) : \epsilon > 0,\, \lambda > 0\}$ and, for $\lambda = \lambda_0 > 0$, fixed, the set $\{\epsilon : (\epsilon,\, \lambda_0) \in S\}$ is an interval $\lambda_0 \times (0,\, \epsilon_0(\lambda_0))$, where

$$\epsilon_0(\lambda_0) = \frac{1}{\pi}\left(\frac{2\pi\lambda_0}{R_0}\right)^{\frac{1}{2}} + O(\lambda_0)$$

as $\lambda_0 \to 0$. For any $\epsilon \in (0, \epsilon_0(\lambda_0))$, the periodic solution $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon})$ approaches a square wave as $\epsilon \to 0$; that is, the periodic solution $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon})$ has the property that $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon}) \to d_{1\lambda}$ (respectively, $d_{2\lambda}$) as $\epsilon \to 0$ uniformly on compact sets of $(0,\, 1)$ (respectively, $(1,\, 2)$).

If $R_1 < 0$ (the subcritical case of period doubling of the map) and $R_0 > 0$, the sector S contains points (ϵ, λ) with λ both negative and positive and the periodic orbits have a different structure as $\epsilon \to 0$. More precisely, for $\lambda = \lambda_0 > 0$, fixed, the set $\{\epsilon : (\epsilon,\, \lambda_0) \in S\}$ is an interval $\lambda_0 \times (\epsilon_0(\lambda_0),\, \beta_0(\lambda_0))$. For $\lambda = \lambda_0 < 0$, fixed, the set $\{\epsilon : (\epsilon,\, \lambda_0) \in S\}$ is an interval $\lambda_0 \times (0,\, \alpha_0(\lambda_0))$. For any $\epsilon \in (0, \alpha_0(\lambda_0))$, the unique periodic solution $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon})$ becomes pulse like as $\epsilon \to 0$ in the following sense: the periodic solution $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon})$ has the property that $(\tilde{x}_{\lambda_0,\epsilon}(t), \tilde{y}_{\lambda_0,\,\epsilon}) \to 0$ as $\epsilon \to 0$ uniformly on compact sets of $(0,\, 1) \cup (1,\, 2)$. In the pulse like solution, the pulses in the solution occur near the integers and are opposite in direction. However, the magnitude of the pulse near the integers exceeds the magnitude of the corresponding period two point of the map.

The part of the boundary of the sector S described by the curve $\Gamma = \{(\lambda, \epsilon(\lambda)\}$ corresponds to a Hopf bifurcation curve. In the supercritical case, the Hopf bifurcation at $(\lambda_0, \epsilon(\lambda_0))$ is in the direction of $\epsilon < \epsilon(\lambda_0)$ and the periodic orbit has a unique continuation to the interval $\{(\lambda_0, \epsilon) : 0 < \epsilon < \epsilon(\lambda_0)$. In the subcritical case, the Hopf bifurcation is in the direction of $\epsilon > \epsilon(\lambda_0)$ and the periodic orbit has a unique continuation in all of S (see Figure 2.1).



$R_0 > 0, \quad R_1 > 0$ $\qquad$ $R_0 > 0, \quad R_1 < 0$

Figure 2.1. Bifurcation Diagram in the (λ, ϵ)-plane

Lemma 2.2 will be proved in Section 3 and Theorem 2.3 in Section 4. As remarked earlier, Sections 5 and 6 will be devoted to relating the hypotheses in Theorem 2.3 to those that are required to obtain a generic Hopf bifurcation on the curve Γ.

3. Proof of Lemma 2.2. For any integer N, Let $C_N = C([-1,0];\mathbb{R}^N)$. Let $\hat{\Omega}_0 = \{(\varphi_1, \varphi_2) \in C_{2m} \times C_{2n} : \varphi_2(0) = \hat{B}_1\varphi_1(-1) + \hat{B}_2\varphi_2(-1)\}$. Equation (2.12) generates a C^0-semigroup $\hat{S}(t)$ on $\hat{\Omega}_0$. If $\hat{A}$ is the infinitesimal generator of $\hat{S}(t)$, then

$$\begin{aligned} \mathcal{D}(\hat{A}) &= \{(\varphi_1, \varphi_2) \in \hat{\Omega}_0 : (\dot{\varphi}_1, \dot{\varphi}_2) \in \hat{\Omega}_0, \dot{\varphi}_1(0) = S_{11}\hat{A}\varphi_1(0) - S_{11}\hat{A}\hat{A}_2\varphi_2(0)\} \\ \hat{A}(\varphi_1, \varphi_2) &= (\dot{\varphi}_1, \dot{\varphi}_2). \end{aligned}$$

In this section, we are going to prove a stronger statement than Lemma 2.2.

Lemma 3.1. *If $r_e(\sigma(\hat{S}(t)))$ is the radius of the essential spectrum of $\hat{S}(t)$, then there is an $\alpha > 0$ such that $r_e(\sigma(\hat{S}(t))) < e^{-\alpha t}$ for all $t \geq 0$. Also, if $e^{\mu t}$ is an eigenvalue of $\hat{S}(t)$, then μ is an eigenvalue of $\hat{A}$. Furthermore, there is a $\delta > 0$ such that 0 is an eigenvalue of $\hat{A}$ of multiplicity two and no elements of $\sigma(\hat{A}) \setminus \{0\}$ belongs to the set $|\mathrm{Re}\,\mu| \geq \delta$.*

Proof. It is known (see [11]) that $r_e(\sigma(\hat{S}(t))) \leq e^{-\alpha t}$, where α is such that the eigenvalues of $B_2(0)$ have modulus $\leq e^{-\alpha}$. From **(H6)**, we obtain the first statement of the theorem. It also is known that the eigenvalues of $\hat{S}(t)$ are given by $e^{\mu t}$, where μ is an eigenvalue of $\hat{A}$, which coincide with the solutions of (2.13). The multiplicity of an eigenvalue of $\hat{A}$ is the same as the multiplicity of the eigenvalues of (2.13) (see [11]). Thus, there is a $\delta > 0$ such that, if μ is an eigenvalue of $\hat{A}$ with $\mathrm{Re}\mu \in (-\delta, \delta)$, then μ is purely imaginary and there are only a finite number of eigenvalues with $\mathrm{Re}\mu \geq \delta$. From **(H5)**, the only possible purely imaginary eigenvalue is $\mu = 0$. Therefore, we need only show that $\mu = 0$ is an eigenvalue of multiplicity two.

Using (2.3), (**H2**), and (2.4), it is easy to verify that a basis for $\mathcal{N}(\hat{\mathcal{A}})$ is given by

$$\varphi^1 = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}, \quad d_2 = \begin{bmatrix} e \\ -e \end{bmatrix}, \quad d_1 = \hat{A}_2 d_2 = \begin{bmatrix} A_2 e \\ -A_2 e \end{bmatrix}, \quad e = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n. \tag{3.1}$$

Also, a basis of $\mathcal{N}(\hat{\mathcal{A}}^2)$ is given by φ^1, φ^2, where φ^2 is defined by the following relations:

$$\varphi^2(\theta) = \begin{bmatrix} \theta d_1 + d_1^* \\ \theta d_2 + d_2^* \end{bmatrix}, \quad d_2^* = \frac{1}{S_{11}} \begin{bmatrix} 0 \\ (I + H_0(0))^{-1} S_{21} \\ 0 \\ -(I + H_0(0))^{-1} S_{21} \end{bmatrix}, \quad d_1^* = \hat{A}_2 d_2^* + \frac{1}{S_{11}} d_1. \tag{3.2}$$

We now prove that $\mathcal{N}(\hat{\mathcal{A}}^3) \subset \mathcal{N}(\hat{\mathcal{A}}^2)$. If $(\varphi_1, \varphi_2) \in \mathcal{N}(\hat{\mathcal{A}}^3)$, then $\hat{\mathcal{A}}(\varphi_1, \varphi_2) \in \mathcal{N}(\hat{\mathcal{A}}^2)$ and there exist constants α, β such that $\hat{\mathcal{A}}(\varphi_1, \varphi_2) = (\dot{\varphi}_1, \dot{\varphi}_2) = \alpha\varphi^1 + \beta\varphi^2$. Using (3.1), (3.2) and integrating, we deduce that there are vectors $h_1 \in \mathbb{R}^{2m}$, $h_2 \in \mathbb{R}^{2n}$ such that

$$\varphi_i(\theta) = h_i + \theta(\alpha d_i + \beta d_i^*) + \frac{1}{2}\beta\theta^2 d_i, \quad i = 1, 2.$$

Since $(\varphi_1, \varphi_2) \in \mathcal{D}(\hat{\mathcal{A}})$, we must have

$$\alpha d_1 + \beta d_1^* = \dot{\varphi}_1(0) = S_{11} \hat{A}[\varphi_1(0) - \hat{A}_2 \varphi_2(0)] = S_{11} \hat{A}[h_1 - \hat{A}_2 h_2], \tag{3.3}$$

$$h_2 = \varphi_2(0) = \hat{B}_1 \varphi_1(-1) + \hat{B}_2 \varphi_2(-1) = \Sigma_{i=1}^2 \hat{B}_i [h_i - \beta d_i^* + (\frac{1}{2}\beta - \alpha) d_i]. \tag{3.4}$$

From (3.3), we have

$$h_1 = \hat{A}_2 h_2 + \frac{1}{S_{11}} \hat{A}^{-1} (\alpha d_1 + \beta d_1^*). \tag{3.5}$$

If we observe that $d_2 = \Sigma_{i=1}^2 \hat{B}_i d_i$, $d_2^* = \Sigma_{i=1}^2 \hat{B}_i (d_i^* - d_i)$, and substitute (3.5) into (3.4), we have

$$h_2 = (\hat{B}_1 \hat{A}_2 + \hat{B}_2) h_2 + \frac{1}{S_{11}} \hat{B}_1 \hat{A}^{-1} (\alpha d_1 + \beta d_1^*) - \beta d_2^* - (\frac{1}{2}\beta + \alpha) d_2. \tag{3.6}$$

If we let $h_2 = \mathrm{col}\,(h_{21}, h_{22}) \in \mathbb{R}^n \times \mathbb{R}^n$, use (3.6) and the definitions of $d_i, d_i^*, \hat{A}_i, \hat{B}_i, i = 1, 2$, we deduce that $h_{21} = -h_{22}$ and

$$\begin{aligned} h_{21} &= \begin{bmatrix} 1 & 0 \\ 0 & -H_0(0) \end{bmatrix} h_{21} + \frac{\alpha}{S_{11}} S e \\ &+ \frac{\beta}{S_{11}^2} S \begin{bmatrix} 0 \\ (I + H_0(0))^{-1} S_{21} \end{bmatrix} + \frac{\beta}{S_{11}^2} W e - \begin{bmatrix} \frac{1}{2}\beta + \alpha \\ \frac{\beta}{S_{11}} (I + H_0(0))^{-1} S_{21} \end{bmatrix}, \end{aligned}$$

or

$$\begin{bmatrix} 0 \\ I+H_0(0) \end{bmatrix} h_{21} = \begin{bmatrix} \left(\frac{1}{S_{11}^2}[S_{12}(I+H_0(0))^{-1}S_{21}+W_{11}]-\frac{1}{2}\right)\beta \\ \frac{\alpha S_{21}}{S_{11}} + \frac{\beta}{S_{11}^2}S_{22}(I+H_0(0))^{-1}S_{21} + \frac{\beta}{S_{11}^2}W_{21} - \frac{\beta}{S_{11}}(I+H_0(0))^{-1}S_{21} \end{bmatrix}.$$

From (**H4**), this implies that $\beta = 0$ and, thus, $\hat{\mathcal{A}}(\varphi^1, \varphi^2) = \alpha\varphi^1$. Since φ^1 is a constant function, this implies that $\hat{\mathcal{A}}^2(\varphi^1, \varphi^2) = 0$ and $(\varphi^1, \varphi^2) \in \mathcal{N}(\hat{\mathcal{A}}^2)$. This completes the proof of Lemma 3.1.

4. Proof of Theorem 2.3. Since 0 is an eigenvalue of (2.12) of multiplicity two and the spectrum of the corresponding semigroup $\hat{S}(t)$ for $t > 0$ intersects the unit circle only at the point 1, there should be a center manifold of dimension two of (2.11) which contains all of the periodic orbits. We show that such a manifold exists and we compute explicitly the vector field on this manifold up through terms of order 3. The method of computation is through the variation of constants formula which was developed in [9]. We first outline the ideas behind the variation of constants formula and then give the results of the computations without supplying the details which are rather long.

The idea is simple and consists in embedding the equation (2.8) into a neutral functional differential equation for which a variation of constants formula is well known. From the corresponding linear equation, we can determine a special decomposition of the space $C_{2m} \times C_{2n}$ which yields the variation of constants for the flow defined by (2.8). We consider the linear equation

$$\frac{d}{dt}D(u_t, v_t) = L(u_t, v_t), \tag{4.1}$$

where

$$D(\varphi_1, \varphi_2) = \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) - \Sigma_{i=1}^2 \hat{B}_i\varphi_i(-1) \end{bmatrix}, \quad L(\varphi_1, \varphi_2) = \begin{bmatrix} S_{11}(\hat{A}\varphi_1(0) - \hat{A}\hat{A}_2\varphi_2(0)) \\ 0 \end{bmatrix}.$$

Equation (4.1) defines a C^0-semigroup $S(t)$ on $C_{2m} \times C_{2n}$, whose infinitesimal generator $\mathcal{A}_S$ is given by

$$\begin{aligned} &\mathcal{D}(\mathcal{A}_S) = \{(\varphi_1, \varphi_2) \in C_{2m} \times C_{2n} : (\dot{\varphi}_1, \dot{\varphi}_2) \in C_{2m} \times C_{2n}, D(\dot{\varphi}_1, \dot{\varphi}_2) = L(\varphi_1, \varphi_2)\} \\ &\mathcal{A}_S(\varphi_1, \varphi_2) = (\dot{\varphi}_1, \dot{\varphi}_2). \end{aligned}$$

We know that $\mu = 0$ is an eigenvalue of $\mathcal{A}_S$ of multiplicity $2 + 2n$. If φ^1, φ^2 are defined in (3.1), (3.2), then we can obtain a basis of $\mathcal{N}(\mathcal{A}_S^{2+2n})$ given by

$$\tilde{\Phi} = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} \end{bmatrix},$$

where, for $\theta \in [-1, 0]$,

$$\begin{aligned} &\tilde{\Phi}_{11}(\theta) = [d_1 \quad \theta d_1 + d_1^*], \quad \tilde{\Phi}_{12}(\theta) = [\tfrac{1}{S_{11}}\hat{A}^{-1}d_1^* + \theta d_1^* + \tfrac{\theta^2}{2}d_1 \quad \hat{A}_2 Q] \\ &\tilde{\Phi}_{21}(\theta) = [d_2 \quad \theta d_2 + d_2^*], \quad \tilde{\Phi}_{22}(\theta) = [\theta d_2^* + \tfrac{\theta^2}{2}d_2 \quad Q], \end{aligned} \tag{4.2}$$

and Q is the $2n \times (2n-1)$ matrix given by

(4.3)
$$Q = \begin{bmatrix} 0 & 0 & 0 \\ (I+H_0(0))^{-1}(I-H_0(0))^{-1} & 0 & (I+H_0(0))^{-1}H_0(0)(I-H_0(0))^{-1} \\ 0 & 1 & 0 \\ (I+H_0(0))^{-1}H_0(0)(I-H_0(0))^{-1} & 0 & (I+H_0(0))^{-1}(I-H_0(0))^{-1} \end{bmatrix}.$$

Furthermore, a basis for $\mathcal{N}([\mathcal{A}_S^*]^{2+2n})$ is given by

$$\tilde{\Psi} = \begin{bmatrix} \tilde{\Psi}_{11} & \tilde{\Psi}_{12} \\ 0 & I_{2n} \end{bmatrix},$$

where, for $s \in [0,1]$,

(4.4)
$$\tilde{\Psi}_{11}(s) = \begin{bmatrix} c_1 \\ c_1^* + sc_1 \end{bmatrix}, \quad \tilde{\Psi}_{12}(s) = \begin{bmatrix} sc_2 \\ -sc_2^* + \frac{s^2}{2}c_2 \end{bmatrix},$$

and

(4.5)
$$\begin{aligned} c_2 &= [\, e^T \quad -e^T \,], \quad c_1 = \frac{1}{S_{11}} c_2 \hat{B}_1 \hat{A}^{-1} \\ c_2^* &= \frac{1}{S_{11}} [0 \quad S_{12}(I+H_0(0))^{-1} \quad 0 \quad -S_{12}(I+H_0(0))^{-1}\,], \\ c_1^* &= \frac{1}{S_{11}} (c_2 - c_2^*)\hat{B}_1\hat{A}^{-1} - \frac{c_2}{S_{11}^2}\hat{B}_1\hat{A}^{-2}. \end{aligned}$$

If we define the bilinear form $(\cdot,\cdot)$ for the neutral equation (4.1) 12 as

$$\begin{aligned} (\psi,\varphi) =& \psi_1(0)\varphi_1(0) + \psi_2(0)[\varphi_2(0) - \Sigma_{i=1}^2 \hat{B}_i\varphi_i(-1)] \\ & - \int_{-1}^{0} \dot{\psi}_2(\xi+1)\Sigma_{i=1}^2 \hat{B}_i\varphi_i(\xi)d\xi, \end{aligned}$$

where $\psi = (\psi_1,\psi_2) \in C_{2m}^* \times C_{2n}^*$, $\varphi = (\varphi_1,\varphi_2) \in C_{2m} \times C_{2n}$, $C_k^* = C([0,1];\mathbb{R}^{k*})$, then, with a few computations, we deduce that

(4.6)
$$(\tilde{\Psi},\tilde{\Phi}) = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix},$$

where

(4.7)
$$P_{11} = \begin{bmatrix} 0 & \frac{R_0}{S_{11}^2} \\ -\frac{R_0}{S_{11}^2} & c_1^*d_1^* + c_2^*d_2^* - \frac{2}{3} \end{bmatrix},$$

$$P_{12} = \begin{bmatrix} p_{11} & \frac{1}{S_{11}} S_{12}(I + H(0))^{-1} & 0 & -\frac{1}{S_{11}} S_{12}(I + H(0))^{-1} \\ p_{21} & \bar{R}_0 & \frac{R_0}{2S_{11}^2} & -\bar{R}_0 \end{bmatrix}, \tag{4.8}$$

$$P_{22} = \begin{bmatrix} -\frac{R_0}{2S_{11}^2} & 0 & 1 & 0 \\ R_1^* & I_{n-1} & 0 & 0 \\ \frac{R_0}{2S_{11}^2} & 0 & 1 & 0 \\ -R_1^* & 0 & 0 & I_{n-1} \end{bmatrix}, \tag{4.9}$$

and

$$\begin{aligned} p_{11} &= \frac{1}{S_{11}} c_1 \hat{A}^{-1} \hat{A}_2 d_2^* + \frac{1}{S_{11}^2} c_1 \hat{A}^{-2} \hat{A}_2 d_2 - \frac{1}{S_{11}} c_2 \hat{B}_1 \hat{A}^{-1} d_1^* + \frac{2}{3}, \\ p_{21} &= \frac{1}{S_{11}} [c_1^* \hat{A}^{-1} d_1^* + c_2^* \hat{B}_1 \hat{A}^{-1} d_1^* + c_2^* d_2^* + \frac{1}{4}], \\ \bar{R}_0 &= \frac{1}{S_{11}^2} [S_{11} S_{12} - (S_{12}(I + H(0))^{-1} S_{22} + W_{21})](I + H(0))^{-1}, \\ R_1^* &= \frac{1}{S_{11}^2} [(I + H(0))^{-1} S_{21} - W_{21} - S_{22}(I + H(0))^{-1} S_{21}]. \end{aligned} \tag{4.10}$$

Since $R_0 \neq 0$, the matrices P_{11}, P_{22} are nonsingular. As a consequence, $(\tilde{\Psi}, \tilde{\Phi})$ is nonsingular. If we change the bases by the transformation

$$\Psi = \begin{bmatrix} P_{11}^{-1} & -P_{11}^{-1} P_{12} P_{22}^{-1} \\ 0 & I_{2n} \end{bmatrix} \tilde{\Psi} = \begin{bmatrix} P_{11}^{-1} \tilde{\Psi}_{11} & P_{11}^{-1} \tilde{\Psi}_{12} - P_{11}^{-1} P_{12} P_{22}^{-1} \\ 0 & I_{2n} \end{bmatrix}, \tag{4.11}$$

$$\Phi = \tilde{\Phi} \begin{bmatrix} I_2 & 0 \\ 0 & P_{22}^{-1} \end{bmatrix} = \begin{bmatrix} \tilde{\Phi}_{11} & \tilde{\Phi}_{12} P_{22}^{-1} \\ \tilde{\Phi}_{21} & \tilde{\Phi}_{22} P_{22}^{-1} \end{bmatrix}, \tag{4.12}$$

then

$$(\Psi, \Phi) = I_{2+2n}. \tag{4.13}$$

The explicit expressions for P_{11}^{-1} and P_{22}^{-1} are given by

$$P_{11}^{-1} = \begin{bmatrix} p^* & -\frac{S_{11}^2}{R_0} \\ \frac{S_{11}^2}{R_0} & 0 \end{bmatrix}, \quad p^* = \frac{S_{11}^4}{R_0^2} (c_1^* d_1^* + c_2^* d_2^* - \frac{2}{3}),$$

$$P_{22}^{-1} = \begin{bmatrix} -\frac{S_{11}^2}{R_0} & 0 & \frac{S_{11}^2}{R_0} & 0 \\ \frac{S_{11}^2 R_1^*}{R_0} & I_n & -\frac{S_{11}^2 R_1^*}{R_0} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ -\frac{S_{11}^2 R_1^*}{R_0} & 0 & \frac{S_{11}^2 R_1^*}{R_0} & I_n \end{bmatrix}.$$

A few elementary calculations also yield

$$\mathcal{A}_S\Phi = \Phi B, \tag{4.14}$$

where

$$B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix},$$
$$B_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -\frac{S_{11}^2}{R_0} & 0 & \frac{S_{11}^2}{R_0} & 0 \end{bmatrix}. \tag{4.15}$$

We now decompose the solution space $\hat{\Omega}_0$ of (2.12) using the above bases Ψ and Φ. Let Q^+ denote the union of the generalized eigenspaces of the eigenvalues of $\hat{\mathcal{A}}$ with real parts > 0, let Q^{+*} denote the union of the generalized eigenspaces of the eigenvalues of $\hat{\mathcal{A}}^*$ with real parts > 0 and define

$$Q^- = \{\varphi = (\varphi_1, \varphi_2) \in C_{2m} \times C_{2n} : (\Psi, \varphi) = 0, (\psi, \varphi) = 0 \text{ for all } \psi \in Q^{+*}\}.$$

It is shown in [9] that

$$\hat{\Omega}_0 = Q^- \oplus [\Phi_1] \oplus Q^+,$$

where $[\cdot]$ denotes span and

$$\Phi_1 = \begin{bmatrix} \tilde{\Phi}_{11} \\ \tilde{\Phi}_{21} \end{bmatrix} \tag{4.16}$$

and $\tilde{\Phi}_{11}, \tilde{\Phi}_{21}$ are defined in (4.2). Relations (4.14), (4.15) imply that

$$\tilde{\Phi}_1(\sigma) = \Phi_1(0)e^{B_{11}\sigma}, \quad \sigma \in \mathbb{R}. \tag{4.17}$$

We introduce the notation

$$\bar{F}_{h\lambda}(\varphi) = \begin{bmatrix} 0 \\ F_{h\lambda}(\varphi_1(0), \varphi_2(0)) \end{bmatrix}, \quad \bar{G}_{h\lambda}(\varphi) = \begin{bmatrix} 0 \\ G_{h\lambda}(\varphi_1(-1), \varphi_2(-1)) \end{bmatrix},$$

$X_0(\theta) = 0$ for $\theta \in [-1, 0)$, $X_0(0) = I_{2m+2n}$ and

$$\Theta(\varphi) = \varphi + X_0\bar{G}_{h\lambda}(\varphi), \tag{4.18}$$

where $\varphi = \operatorname{col}(\varphi_1, \varphi_2) \in C_{2m} \times C_{2n}$. Using the variation of constants formula in [9] and the standard methods in the theory of center manifolds, it is possible to show that a center manifold of (2.1) in a neighborhood of 0 and h, λ sufficiently small, has the form

$$\varphi = \Phi_1 U + W_{h\lambda}(U) + X_0\bar{G}_{h\lambda}(\Phi_1 U), \quad W_{h\lambda}(U) \in Q = Q^- \oplus Q^+, \ U \in \mathbb{R}^2.$$

Furthermore, if we make the transformation of variables

$$\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \Phi_1 U(t) + W_{h\lambda}(U(t)) + X_0 \bar{G}_{h\lambda}(\Phi_1 U(t))$$

in (2.11), then the flow on the center manifold is given by

$$\begin{aligned} \frac{dU(t)}{dt} =& B_{11}U(t))\Psi_{11}(0)\bar{F}_{h\lambda}(\Theta(\Phi_1 U(t) + W_{h\lambda}(U(t)))) \\ &+ [B_{11}\Psi_{12}(0) + B_{12}]\bar{G}_{h\lambda}(\Phi_1 U(t) + W_{h\lambda}(U(t)). \end{aligned} \tag{4.19}$$

Furthermore, the function $W_{h\lambda}(U)$ satisfies the integral equation

$$\begin{aligned} W_{h\lambda}(U(t)) =& -\int_{-\infty}^{t} d_\sigma \hat{S}(t-\sigma)X_0^{Q^-}\bar{G}_{h\lambda}(\Phi_1 U(\sigma) + W_{h\lambda}(U(\sigma))) \\ &+ \int_{-\infty}^{t} \hat{S}(t-\sigma)X_0^{Q^-}\bar{F}_{h\lambda}(\Theta(\Phi_1 U(\sigma) + W_{h\lambda}(U(\sigma))))d\sigma \\ &+ \int_{t}^{\infty} d_\sigma \hat{S}(t-\sigma)X_0^{Q^+}\bar{G}_{h\lambda}(\Phi_1 U(\sigma) + W_{h\lambda}(U(\sigma))) \\ &- \int_{t}^{\infty} \hat{S}(t-\sigma)X_0^{Q^+}\bar{F}_{h\lambda}(\Phi_1 U(\sigma) + W_{h\lambda}(U(\sigma))))d\sigma. \end{aligned} \tag{4.20}$$

We now want to obtain an approximation of the vector field on the center manifold. It turns out that the only terms in $W_{h\lambda}(U)$ that are important are the cubic terms in U of the Taylor series of the function $W_{00}(U)$. The necessary terms in h, λ are obtained directly from the vector field in (2.1).

It is clear that

$$W_{h\lambda}(U) = O((|h| + |\lambda|)|U| + |U|^2)$$

as $(|h|, |\lambda|, |U|) \to (0,0,0)$. By using the same argument as in [], we deduce that

$$\begin{aligned} W_{00}(U) =& -\int_{-\infty}^{0} d_\sigma \hat{S}(-\sigma)X_0^{Q^-}[\frac{1}{2}D_\varphi^2\bar{G}_{00}(0)(\Phi_1(\sigma)U)^2] \\ &+ \int_{-\infty}^{0} \hat{S}(-\sigma)X_0^{Q^-}[\frac{1}{2}D_\varphi^2\bar{F}_{00}(0)(\Phi_1(\sigma)U)^2]d\sigma \\ &+ \int_{0}^{\infty} d_\sigma \hat{S}(-\sigma)X_0^{Q^+}[\frac{1}{2}D_\varphi^2\bar{G}_{00}(0)(\Phi_1(\sigma)U)^2] \\ &- \int_{0}^{\infty} \hat{S}(-\sigma)X_0^{Q^+}[\frac{1}{2}D_\varphi^2\bar{F}_{00}(0)(\Phi_1(\sigma)U)^2]d\sigma \\ &+ O(|U|^3) \end{aligned} \tag{4.21}$$

as $|U| \to 0$.

As in [7], we can show that, for any $\zeta \in \mathbb{R}^{2n+2m}$ and any integer $k \geq 0$,

$$\int_0^\infty \hat{S}(\sigma)\sigma^k X_0^{Q^-}\zeta d\sigma - \int_{-\infty}^0 \hat{S}(\sigma)\sigma^k X_0^{Q^+}\zeta d\sigma \\ = k!(-1)^{k+1}(\mathcal{A}_{\hat{S}}|Q)^{-(k+1)}X_0^Q\zeta. \tag{4.22}$$

If d_1, d_2 are defined as in (3.2) and we let

$$\xi = \operatorname{col}(d_1, d_2), \quad U = \operatorname{col}(U_1, U_2),$$

use (4.16), apply (4.22) for $k = 0$, and perform several computations, we conclude that

$$W_{00}(U) = \operatorname{col}(W_1^0(U), W_2^0(U)) + O(|U_2||U| + |U|^3) \tag{4.23}$$

as $|U| \to 0$, where

$$\begin{aligned} W_1^0(U) &= \frac{1}{2}U_1^2 \begin{bmatrix} A_2\gamma + b \\ A_2\gamma + b \end{bmatrix}, \quad W_2^0(U) = \frac{1}{2}U_1^2 \begin{bmatrix} -X_0 a + \gamma \\ -X_0 a + \gamma \end{bmatrix}, \\ a &= D^2_{(x,y)}g_0(0)(A_2e, e), \quad b = D_y^2 f_0(0)(e,e), \\ \gamma &= M[a + B_1 b], \quad M = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & (I + H_0(0))^{-1} \end{bmatrix}. \end{aligned} \tag{4.24}$$

With the above estimate on $W_{00}(U)$, we can estimate the vector field in (4.19) for $(h, \lambda) = (0, 0)$. Using the definitions of Φ_1 in (4.16) and Θ in (4.18), we can verify that

$$\begin{aligned} &\Psi_{11}(0)\bar{F}_{00}(\Theta(\Phi_1 U + W_{00}(U))) \\ &= \begin{bmatrix} O(|U|^3) \\ \frac{2S_{11}^2}{R_0} e^T B_1[f^*(e, \frac{1}{2}\gamma) + \frac{1}{3!}D_y^3 f_0(0)(e)^3]U_1^3 \end{bmatrix}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &(B_{11}\Psi_{12}(0) + B_{12})\bar{G}_{00}(\Phi_{10}U + W_{00}(U)) \\ &= \begin{bmatrix} O(|U|^3) \\ \frac{2S_{11}^2}{R_0} e^T[g^*((A_2e, e), \frac{1}{2}(A_2\gamma + b, \gamma)) + \frac{1}{3!}D^3_{(x,y)}g_0(0)(A_2e, e)^3]U_1^3 \end{bmatrix}, \end{aligned} \tag{4.26}$$

where the second component in each of the formulas (4.25), (4.26) are $O(|U_2||U|^2 + |U|^4)$ as $|U| \to 0$, and where f^*, g^* are the quadratic forms defined by

$$\begin{aligned} f^*(y, \bar{y}) &= y^T D_y^2 f_0(0)\bar{y} \\ g^*((\bar{x}, \bar{y}), (\tilde{x}, \tilde{y})) &= [\,\bar{x}^T \quad \bar{y}^T\,]\, D^2_{(x,y)}g_0(0) \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}. \end{aligned} \tag{4.27}$$

For $(h, \lambda) = (0, 0)$, equations (4.25)-(4.27) give us the relevant terms in the Taylor series expansion of the vector field in (4.19). In the next result, we show that the coefficient of U_1^3 is related to the hypothesis **(H3)** imposed for the generic period doubling of the map associated with the system (2.1). In fact, except for a nonzero constant factor, the sum of the terms in U_1^3 in (4.25), (4.26) is equal to the constant R_1 in **(H3)**.

Lemma 4.1. *If R_1 is defined as in* (**H3**)*, then*

$$
\begin{aligned}
R_1 =& e^T B_1 [f^*(e, \frac{1}{2}\gamma) + \frac{1}{3!} D_y^3 f_0(0)(e)^3] \\
&+ e^T [g^*((A_2 e, e), \frac{1}{2}(A_2\gamma + b, \gamma)) + \frac{1}{3!} D^3_{(x,y)} g_0(0)(A_2 e, e)^3]
\end{aligned} \tag{4.28}
$$

Proof. From the definition of $\mathcal{F}_\lambda(y)$ in (2.2), we deduce that

(4.29)

$$
\begin{aligned}
\mathcal{F}_0(y) = (B_1 A_2 + B_2) y + \frac{1}{2} B_1 D_y^2 f_0(0)(y)^2 + \frac{1}{2} D^2_{(x,y)} g_0(0)(A_2 y, y)^2 \\
+ g^*((A_2 y, y), (\frac{1}{2} D_y^2 f_0(0)(y)^2, 0)) + \frac{1}{3!} B_1 D_y^3 f_0(0)(y)^3 + \frac{1}{3!} D^3_{(x,y)} g_0(0)(A_2 y, y)^3.
\end{aligned}
$$

From (4.29), we deduce that

$$
\begin{aligned}
k_3(0) =& e^T \{ g^*((A_2 e, e), (\frac{1}{2} D_y^2 f_0(0)(e)^2, 0)) \\
&+ \frac{1}{3!} B_1 D_y^3 f_0(0)(e)^3 + \frac{1}{3!} D^3_{(x,y)} g_0(0)(A_2 e, e)^3 \},
\end{aligned} \tag{4.30}
$$

$$
k_2(0) = e^T [B_1 f^*(e, I_*) + g^*((A_2 e, e), (A_2 I_*, I_*))], \tag{4.31}
$$

where I_* is the $m \times (m-1)$ matrix $\operatorname{col}(0, I_{m-1})$,

$$
k_1(0) = \frac{1}{2} e^T [B_1 f^*(e, e) + g^*((A_2 e, e), (A_2 e, e))], \tag{4.32}
$$

$$
H_1(0) = \frac{1}{2} I_*^T [B_1 f^*(e, e) + g^*((A_2 e, e), (A_2 e, e))]. \tag{4.33}
$$

From the definition of M and γ in (4.24), we see that

$$
\begin{aligned}
\gamma &= \frac{1}{2} e^T (a + B_1 b) e + I_* (I - H_0(0))^{-1} I_*^T (a + B_1 b) \\
&= k_1(0) e + 2 I_* (I - H_0(0))^{-1} H_1(0),
\end{aligned}
$$

which together with (4.32) allows us to conclude that

$$
\begin{aligned}
& e^T B_1 [f^*(e, \frac{1}{2}\gamma) + \frac{1}{3!} D_y^3 f_0(0)(e)^3] \\
=& \frac{1}{2} k_1(0) e^T B_1 f^*(e, e) + e^T B_1 f^*(e, I_*)(I - H(0))^{-1} H_1(0) \\
&+ \frac{1}{3!} e^T B_1 D_y^3 f_0(0)(e)^3.
\end{aligned} \tag{4.34}
$$

In a similar way, we verify that

$$
\begin{aligned}
&e^T[g^*((A_2e,e),\frac{1}{2}(A_2\gamma+b,\gamma)))+\frac{1}{3!}D^3_{(x,y)}g_0(0)(A_2e,e)^3]\\
&=\frac{1}{2}k_1(0)e^Tg^*((A_2e,e),(A_2e,e))\\
&+\frac{1}{2}e^Tg^*((A_2e,e),(A_2I_*,I_*))(I-H_0(0))^{-1}H_1(0)\\
&+e^Tg^*(((A_2e,e),(\frac{1}{2}D^2_yf_0(0)(e)^2,0))+\frac{1}{3!}D^3_{(x,y)}g_0(0)(A_2e,e)^3.
\end{aligned}
\tag{4.35}
$$

If we now use (4.30)-(4.35), we obtain the relation (4.28) and the lemma is proved.

As remarked above, the relevant terms in h,λ in the Taylor series of the vector field in (4.16) are easier to obtain since they will not depend upon any specific knowledge of the function $W_{h\lambda}(U)$. After several rather straightforward but lengthy computations, we deduce that

$$
\begin{aligned}
&\Psi_{11}(0)[\bar{F}_{h\lambda}(\Theta(\Phi_1U+W_{h\lambda}(U)))-\bar{F}_{00}(\Theta(\Phi_1U+W_{00}(U)))\\
&+[B_{11}\Psi_{12}(0)+B_{12}[\bar{G}_{h\lambda}(\Phi_1U+W_{h\lambda}(U))-\bar{G}_{00}(\Phi_1U+W_{00}(U))\\
&=A_{h\lambda}U+O((|\lambda|+|h|)|U|^2+(|\lambda|+|h|)^2|U|)
\end{aligned}
\tag{4.36}
$$

as $(h,\lambda,U)\to(0,0,0)$, where

$$
A_{h\lambda}=\begin{bmatrix}\alpha\lambda & \beta_1\lambda+\beta_2h\\ -\frac{2S_{11}^2}{R_0}\lambda & \frac{2S_{11}}{R_0}h+\ell^*\lambda\end{bmatrix}
\tag{4.37}
$$

and $\alpha,\beta_1,\beta_2,\ell^*$ are constants.

From Lemma 4.1, formulas (4.25)-(4.27) and (4.36), (4.37), we conclude that the vector field in (4.19) is given by

$$
\begin{aligned}
\dot{U}_1&=\alpha\lambda U_1+(1+\beta_1\lambda+\beta_2h)U_2+O((|\lambda|+|h|)|U|^2+(|\lambda|+|h|)^2|U|+|U|^3)\\
\dot{U}_2&=-\frac{2S_{11}^2}{R_0}\lambda U_1+(\frac{2S_{11}}{R_0}h+\ell^*\lambda)U_2+\frac{2S_{11}^2R_1}{R_0}U_1^3+\ell_1^*U_1^2U_2+\ell_2^*U_1U_2^2+\ell_3^*U_2^3\\
&+O((|\lambda|+|h|)|U|^2+(|\lambda|+|h|)^2|U|+|U|^4),
\end{aligned}
\tag{4.38}
$$

where $\ell^*,\ell_1^*,\ell_2^*,\ell_3^*$ are constants.

To analyze the periodic solutons of (4.38), it is convenient to write them in a different form. Since $R_0\neq 0$, $S_{11}\neq 0$, we can make a linear change of variables of the form $U\mapsto Z=(I+P(h,\lambda))U$ and change the time scale $t\mapsto(1+\delta(h,\lambda))t$, where $P(0)=0,\delta(0)=0$ such that the new equation for Z is given by

$$
\begin{aligned}
\dot{Z}_1&=Z_2+O((|\lambda|+|h|)|Z|^2+(|\lambda|+|h|)^2|Z|+|Z|^3)\\
\dot{Z}_2&=-\frac{2S_{11}^2}{R_0}\lambda Z_1+(\frac{2S_{11}}{R_0}h+\ell^*\lambda)Z_2+\frac{2S_{11}^2R_1}{R_0}Z_1^3+\ell_1^*Z_1^2Z_2+\ell_2^*Z_1Z_2^2+\ell_3^*Z_2^3\\
&+O((|\lambda|+|h|)|Z|^2+(|\lambda|+|h|)^2|Z|+|Z|^4).
\end{aligned}
\tag{4.39}
$$

This equation now has the same form as the one considered in [9]. The discussion there for the existence of periodic solutions corresponding to periodic solutons of (2.1) can be repeated verbatum to complete the proof of Theorem 2.3.

5. Hopf bifurcation curve. For the periodic orbit whose existence is given by Theorem 2.3, we want to impose additional conditions on the vector field in (2.1) which will ensure that this orbit occured through a Hopf bifurcation and that supercritical (subcritical) period doubling of the map corresponds to stability (instability) of the periodic orbit of (2.1). In this section, we specify conditions which will imply the existence of such a Hopf bifurcation curve and, in the next section, we consider the stability properties.

We consider the linear system

$$\begin{aligned} \epsilon\dot{x}(t) + Ax(t) &= AA_2(\lambda)y(t) \\ y(t) &= B_1(\lambda)x(t-1) + B_2(\lambda)y(t-1), \end{aligned} \tag{5.1}$$

for which the characteristic matrix is

$$\Delta(\lambda,\epsilon,\mu) = \begin{bmatrix} \epsilon\mu I_m + A & -AA_2(\lambda) \\ -B_1(\lambda)e^{-\mu} & I_n - B_2(\lambda)e^{-\mu} \end{bmatrix} \tag{5.2}$$

It is convenient to introduce the following definition. We say that a curve Γ in the (λ,ϵ)-plane, $\epsilon > 0$, is a *Hopf Bifurcation Curve* of (5.1) if there is an $\epsilon^* > 0$ and a continuous function $\lambda = \lambda(\epsilon), 0 < \epsilon \le \epsilon^*$, $\lambda(\epsilon) \to 0$ as $\epsilon \to 0$ such that, if $\Gamma_{\epsilon^*} = \{(\lambda,\epsilon) : \lambda = \lambda(\epsilon), \epsilon \in (0,\epsilon^*]\}$, then, for any $(\lambda_0,\epsilon_0) \in \Gamma_{\epsilon^*}$, there are two purely imaginary solutions $\pm i\beta_0$ of the characteristic equation $\det \Delta(\lambda_0,\epsilon_0,\mu) = 0$ and the remaining solutions μ satisfy $\mathrm{Re}\,\mu \neq 0$. We say that a Hopf Bifurcation Curve is a *Generic Hopf Bifurcation Curve with respect to* ϵ if, for fixed λ_0, the two eigenvalues $\mu(\lambda_0,\epsilon)$, $\bar{\mu}(\lambda_0,\epsilon)$, $\mu(\lambda_0,\epsilon_0) = i\beta_0$, satisfy $\mathrm{Re} d\mu(\lambda_0,\epsilon_0)/d\epsilon < 0$. This type of transversal crossing of the imaginary axis of the eigenvalue $\mu(\lambda_0,\epsilon)$ implies that there will be a Hopf bifurcation with respect to ϵ at ϵ_0. We say that a Hopf Bifurcation Curve is the *First Hopf Bifurcation Curve* if, for each fixed $\epsilon_0 \in (0,\epsilon^*]$, all eigenvalues μ of (5.1) with $\lambda = \lambda(\epsilon_0)$ and $\epsilon > \epsilon_0$ have $\mathrm{Re}\,\mu < 0$. The Generic First Hopf Bifurcation Curve is the most interesting because there is a transfer of stability of the origin at $\epsilon = \epsilon_0$; that is, the origin is stable for $\epsilon > \epsilon_0$ and unstable for $\epsilon < \epsilon_0$. From the physical origins of the problem, this is natural because we expect that the origin is stable for large ϵ (by a change of time scale, this is small delay) and eventually becomes unstable for small ϵ (large delay). The First Hopf Bifurcation Curve represents the first change in the stability properties of the origin.

We want to determine the First Hopf Bifurcation Curve for (5.1) which is generic with respect to ϵ.

We retain the condition (**H2**) which corresponds to $\lambda = 0$ being a period doubling for the map. To have all of the relevant hypotheses close at hand, we repeat this hypothesis.

(H2) $B_1(\lambda)A_2(\lambda) + B_2(\lambda) \equiv C(\lambda) = \begin{bmatrix} -(1+\lambda) & 0 \\ 0 & H_0(\lambda) \end{bmatrix}$, $\sigma(H_0(0) \subset \{z : |z| < 1\}$.

With the notation,

$$(5.3)\quad -B_1(0)A^{-1}A_2(0) \equiv S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}, \quad -B_1(0)A^{-2}A_2(0) \equiv W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix},$$

we also retain the hypothesis

$$\textbf{(H4)}\qquad R_0 \equiv -S_{11}^2 + 2[W_{11} + S_{12}(I + H_0(0))^{-1}S_{21}] \neq 0, \quad S_{11} \neq 0.$$

Of course, we must keep the hypothesis

$$\textbf{(H6)}\qquad \sigma(B_2(0)) \subset B_\rho, \quad \rho < 1.$$

The additional hypotheses that we need are

$$\textbf{(H7)}\qquad \min\{\operatorname{Re} z : z \in \sigma(A)\} > 0,$$

$$\textbf{(H8)}\quad \operatorname{Det}\begin{bmatrix} i\theta I_m + A & -AA_2(0) \\ -B_1(0)e^{-iv} & I_n - B_2(0)e^{-iv} \end{bmatrix} \neq 0, \text{ for } \theta > 0, 0 \leq v \leq 2\pi, (\theta, v) \neq (0, \pi),$$

$$\textbf{(H9)}\qquad \operatorname{Det}\begin{bmatrix} \mu I_m + A & -AA_2(0) \\ -B_1(0) & I_n - B_2(0) \end{bmatrix} \neq 0, \text{ for all } \mu \in \mathbb{C}, \operatorname{Re}\mu \geq 0.$$

The main result of this section is

Theorem 5.1. *Under the assumptions* (**H2**), (**H4**), (**H6**) *and* (**H7**), *the hypotheses* (**H8**) *and* (**H9**) *are necessary and sufficient for the existence of the First Hopf Bifurcation Curve which is generic with respect to* ϵ.

The proof will be given in terms of several lemmas which bring out the role of each hypothesis.

Lemma 5.2. *Under the hypotheses* (**H2**), (**H4**), (**H6**) *and* (**H7**), *there is an* $\epsilon^* > 0$ *and a* C^2*-curve* $\Gamma_{\epsilon^*} = \{((\lambda(\epsilon), \epsilon), \epsilon \in (0, \epsilon^*\}$ *such that (i)*$\lambda(\epsilon) \to 0$ *as* $\epsilon \to 0$, *(ii) for any* $(\lambda_0, \epsilon_0) \in \Gamma_{\epsilon^*}$, *there is a unique pair of purely imaginary eigenvalues* $\pm iv(\lambda_0, \epsilon_o)$ *with* $v \in (0, 2\pi)$. *Furthermore,* Γ_{ϵ^*} *is the unique curve satisfying (i) and (ii).*

Proof. If

$$G(\alpha, \lambda) = B_1(\lambda)(\alpha I_m + A)^{-1}A_2(\lambda),$$

then, under assumption (**H7**), we can verify that det $\Delta(\lambda, \epsilon, iv) = 0$ if and only if det $[e^{iv}I_n - C(\lambda) + i\epsilon vG(i\epsilon v, \lambda)] = 0$; that is, if and only if there is a nonzero vector $h \in \mathbb{C}^n$ such that

$$(5.4)\qquad [e^{iv}I_n - C(\lambda) + i\epsilon vG(i\epsilon v, \lambda)]h = 0.$$

If we set $\lambda = \epsilon = 0$, then (5.4) implies that $[e^{iv}I_n - C(0)]h = 0$, which, from (**H2**), implies that $v = \pi$ and $h = \gamma \,\mathrm{col}\,[1,0]$, where γ is a constant. For λ, ϵ small, it follows that the first component of h in (5.4) is not zero and we can take $h = \mathrm{col}\,[1,\xi]$. With this observation, we can determine the solutions of (5.4) near $(\lambda, \epsilon, h) = (0, 0, \mathrm{col}\,[1, 0])$ by determining the zeros of the function

$$\begin{aligned}&\mathcal{F} : \hat{I} \times I \times \mathbb{C}^{n-1} \to \mathbb{C}^n \\ &\mathcal{F}(v, \lambda, \epsilon, \xi) = [e^{iv}I_n - C(\lambda) + i\epsilon v G(i\epsilon v, \lambda)] \begin{bmatrix} 1 \\ \xi \end{bmatrix},\end{aligned} \tag{5.5}$$

where $\hat{I} \subset \mathbb{R}$ is a neighborhood of π and $I \subset \mathbb{R}$ is a neighborhood of 0.

We have $\mathcal{F}(\pi, 0, 0, 0) = 0$. If we define

$$T \equiv \frac{\partial \mathcal{F}(v, \lambda, \epsilon, \xi)}{\partial(v, \lambda, \xi)}|_{(\pi,0,0,0)} : \mathbb{R} \times \mathbb{R} \times \mathbb{C}^{n-1} \to \mathbb{C}^n l,$$

then

$$T(v, \lambda, \xi) = \begin{bmatrix} -iv + \lambda \\ -(I + H_0(0))\xi \end{bmatrix}.$$

Since $I + H_0(0)$ is invertible, the map T is invertible. Therefore, the Implicit Function Theorem implies that there are C^2-functions $v(\epsilon), \lambda(\epsilon), \xi(\epsilon)$ defined for $\epsilon \in (-\epsilon^*, \epsilon^*)$, $\epsilon^* > 0$, such that $v(0) = \pi$, $\lambda(0) = 0$ $\xi(0) = 0$, and

$$\mathcal{F}(v(\epsilon), \lambda(\epsilon), \epsilon, \xi(\epsilon)) = 0, \quad \epsilon \in (-\epsilon^*, \epsilon^*). \tag{5.6}$$

Differentiating (5.6) with respect to ϵ, we obtain

$$\begin{aligned}&\dot{v}[ie^{iv}I_n + i\epsilon G(i\epsilon v, \lambda) - \epsilon^2 v \frac{\partial}{\partial \alpha} G(i\epsilon v, \lambda)] \begin{bmatrix} 1 \\ \xi \end{bmatrix} + \dot{\lambda}[-\dot{C}(\lambda) + i\epsilon v \frac{\partial}{\partial \lambda} G(i\epsilon v, \lambda)] \begin{bmatrix} 1 \\ \xi \end{bmatrix} \\ &+ [e^{iv}I_n - C(\lambda) + i\epsilon v G(i\epsilon v, \lambda)] \begin{bmatrix} 0 \\ \dot{\xi} \end{bmatrix} + [ivG(i\epsilon v, \lambda) - \epsilon v^2 \frac{\partial}{\partial \alpha} G(i\epsilon v, \lambda)] \begin{bmatrix} 1 \\ \xi \end{bmatrix} = 0.\end{aligned} \tag{5.7}$$

If we set $\epsilon = 0$ and use the fact that $-G(0,0) = S$, we obtain

$$\begin{bmatrix} -i\dot{v}(0) + \dot{\lambda}(0) \\ -(I + H_0(0))\dot{\xi}(0) \end{bmatrix} = -iv(0)G(0,0) \begin{bmatrix} 1 \\ \xi(0) \end{bmatrix} = i\pi \begin{bmatrix} S_{11} \\ S_{21} \end{bmatrix}.$$

Hence,

$$\dot{v}(0) = -\pi S_{11}, \quad \dot{\lambda}(0) = 0, \quad \dot{\xi}(0) = -i\pi(I + H_0(0))^{-1} S_{21}. \tag{5.8}$$

Differentiating (5.7) with respect to ϵ, setting $\epsilon = 0$ and using (5.8), we deduce that

$$\begin{aligned}&[(\dot{v}^2(0) - i\ddot{v}(0))I_n + 2i\dot{v}(0)G(0,0) - \ddot{\lambda}(0)\dot{C}(0) - 2v^2(0)\frac{\partial}{\partial \alpha} G(0,0)] \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= 2[i\dot{v}(0)I_n - iv(0)G(0,0)] \begin{bmatrix} 0 \\ \dot{\xi} \end{bmatrix} + [I_n + C(0)] \begin{bmatrix} 0 \\ \ddot{\xi} \end{bmatrix}.\end{aligned}$$

From this relation and the fact that $\frac{\partial}{\partial\alpha}G(0,0) = W$, we have

$$\dot{v}^2(0) + \ddot{\lambda}(0) - 2v^2(0)W_{11} = 2i[\dot{v}(0) - v(0)S_{12}\dot{\xi}(0)].$$

From this relation and (5.8), we have

$$\ddot{\lambda}(0) = 2\pi^2 W_{11} - \pi^2 S_{11}^2 + 2\pi^2 S_{12}(I + H_0(0))^{-1} S_{21} = \pi^2 R_0 \neq 0.$$

As a consequence, for small ϵ, we have the Hopf bifurcation curve $\Gamma_{\epsilon^*} = \{(\lambda(\epsilon), \epsilon), \epsilon \in (0, \epsilon^*)]\}$, where

$$\lambda(\epsilon) = \frac{1}{2}\pi^2 R_0 \epsilon^2 + o(\epsilon^2) \tag{5.9}$$

as $\epsilon \to 0$. This completes the proof of the lemma.

Remark 5.1. The expression (5.9) shows that the graph of the curve Γ_{ϵ^*} is over the positive (resp. negative) λ-axis if $R_0 > 0$ (resp. $R_0 < 0$).

Lemma 5.3. *Suppose that the hypotheses* (**H2**)*,* (**H4**)*,* (**H6**)*,* (**H7**) *and* (**H8**) *are satisfied and let* $\Gamma_{\epsilon^*} = \{(\lambda(\epsilon), \epsilon), \epsilon \in (0, \epsilon^*]\}$ *be given by Lemma 5.2. Then there is an* $\epsilon_* > 0$ *such that* Γ_{ϵ_*} *is a Generic Hopf Bifurcation Curve.*

Proof. We first show that there is an $\epsilon_* > 0$ such that, for each $\epsilon_0 \in (0, \epsilon_*]$, we have

$$\det \Delta(\lambda(\epsilon_0), \epsilon, i\beta) \neq 0, \quad \text{for } \epsilon > \epsilon_0,\ \beta \geq 0. \tag{5.10}$$

If this is not the case, then we can find sequences $\{\epsilon_n\}$, $\{\hat{\epsilon}_n\}$, $\{\beta_n\}$ in $(0, \infty)$ such that $\epsilon_n < \hat{\epsilon}_n$, $\epsilon_n \to 0$ as $n \to \infty$, and

$$0 = \det(\lambda(\epsilon_n), \hat{\epsilon}_n, i\beta_n) \equiv \begin{bmatrix} i\hat{\epsilon}_n\beta_n I_m + A & -AA_2(\lambda(\epsilon_n)) \\ -B_1(\lambda(\epsilon_n))e^{-i\beta_n} & I_n - B_2(\lambda(\epsilon_n))e^{-i\beta_n} \end{bmatrix}. \tag{5.11}$$

From (5.11), it follows that $\hat{\epsilon}_n\beta_n$, $n \geq 1$, is bounded. Since $\lambda(\epsilon_n) \to 0$ as $n \to \infty$ and we are assuming (**H8**), we must have $\hat{\epsilon}_n\beta_n \to 0$, $e^{-i\beta_n} \to -1$ as $n \to \infty$. If we let $k_n \geq 0$ be an integer and $v_n \in (0, 2\pi]$ be such that $\beta_n = 2k_n\pi + v_n$, then we have $v_n \to \pi$ as $n \to \infty$. Also, if $\epsilon'_n = \hat{\epsilon}_n\beta_n/v_n$, then

$$\lim_{n\to\infty} \epsilon'_n = 0, \quad \epsilon'_n \geq \hat{\epsilon}_n > \epsilon_n, \tag{5.12}$$

and

$$\det \Delta(\lambda(\epsilon_n), \epsilon'_n, iv_n) = \det \Delta(\lambda(\epsilon_n), \hat{\epsilon}_n, i\beta_n) = 0. \tag{5.13}$$

From (5.13) and Lemma 5.2, it follows that $\lambda(\epsilon_n) = \lambda(\epsilon'_n)$ for sufficiently large n. From (5.9), the function $\lambda(\epsilon)$ is monotone for ϵ near 0 and, thus, we must have $\epsilon_n = \epsilon'_n$, which contradicts (5.12). This completes the proof of relation (5.10).

We next show that, if $\mu = iv(\epsilon_0)$ is a zero of $\Delta(\lambda(\epsilon_0), \epsilon_0, \mu)$, then it is a simple zero. For fixed $\epsilon_0 \in (0, \epsilon_*]$, let $\lambda_0 = \lambda(\epsilon_0), \mu_0 = iv(\epsilon_0)$ and $\xi_0 = \xi(\epsilon_0)$, where $\lambda(\epsilon), v(\epsilon), \xi(\epsilon)$ are defined in (5.6). We define the function

$$
\begin{aligned}
&\mathcal{K} : \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{R} \to \mathbb{C}^n \\
&\mathcal{K}(\mu, \eta, \epsilon) = [e^{\mu} I_n - C(\lambda_0) + \epsilon\mu G(\epsilon\mu, \lambda_0)] \begin{bmatrix} 1 \\ \eta \end{bmatrix}.
\end{aligned} \tag{5.14}
$$

We have $\mathcal{K}(\mu_0, \xi_0, \epsilon_0) = 0$. Using (5.8) and (5.9), we deduce that

$$
\frac{\partial \mathcal{K}(\mu_0, \xi_0, \epsilon_0)}{\partial(\mu, \eta)} = \begin{bmatrix} -1 & 0 \\ 0 & -(I + H_0(0)) \end{bmatrix} + O(\epsilon_0)
$$

as $\epsilon_0 \to 0$. Thus, if ϵ_* is sufficiently small, this matrix has nonzero determinant and the Implicit Function Theorem implies that there are C^2-functions $\mu(\epsilon), \eta(\epsilon)$ defined for ϵ in a neighborhood U of 0 such that $\mu(\epsilon_0) = iv(\epsilon_0)$, $\eta(\epsilon_0) = \xi_0$, and

$$
\mathcal{K}(\mu(\epsilon), \eta(\epsilon)) = 0, \quad \epsilon \in U. \tag{5.15}
$$

In particular, $\mu = iv_0$ is a simple zero of det $\Delta(\lambda_0, \epsilon_0, \mu)$.

We next determine how $\mu(\epsilon)$ depends upon ϵ. Differentiating (5.15) with respect to ϵ, we obtain

$$
\begin{aligned}
&\dot{\mu}(\epsilon_0)[e^{iv_0} I_n + \epsilon_0 G(i\epsilon_0 v_0, \lambda_0)] \begin{bmatrix} 1 \\ \xi_0 \end{bmatrix} + [e^{iv_0} I_n - C(\lambda_0) + i\epsilon_0 v_0 G(i\epsilon_0 v_0, \lambda_0)] \begin{bmatrix} 0 \\ \dot{\eta}(\epsilon_0) \end{bmatrix} \\
&\quad = -[iv_0 G(i\epsilon_0 v_0, \lambda_0) - \epsilon_0 v_0^2 \frac{\partial}{\partial \alpha} G(i\epsilon_0 v_0, \lambda_0)] \begin{bmatrix} 1 \\ \xi_0 \end{bmatrix} + O(\epsilon_0^2).
\end{aligned} \tag{5.16}
$$

If we use (5.8), the relations $G(0,0) = -S$, $\partial G(0,0)/\partial \alpha = W$, (5.16) and Talyor's theorem around ϵ_0, we conclude that

$$
\begin{aligned}
\dot{\mu}(\epsilon_0)[-(1 + \epsilon_0 S_{11}) + i\epsilon_0 \pi S_{11}] &= i\epsilon_0 \pi S_{12} \dot{\eta}(\epsilon_0) + i\pi S_{11} + i\pi S_{12} \xi_0 + 2\epsilon_0 \pi^2 W_{11} \\
&\quad + O(\epsilon_0^2) \\
-[I + H_0(0)] \dot{\eta}(\epsilon_0) &= i\pi S_{21} + O(\epsilon_0).
\end{aligned} \tag{5.17}
$$

The second relation of (5.17) implies that

$$
\dot{\eta}(\epsilon_0) = -i\pi [I + H_0(0)]^{-1} S_{21} + O(\epsilon_0).
$$

If we substitute this expression and $\xi_0 = -i\pi(1 + H_0(0))^{-1} S_{21} + O(\epsilon_0)$ into (5.17), we conclude that

$$
\begin{aligned}
&\dot{\mu}(\epsilon_0)[-(1 + \epsilon_0 S_{11}) + i\epsilon_0 \pi S_{11}] \\
&\qquad = i\pi S_{11} + 2\epsilon_0 \pi^2 [W_{11} + S_{12}(I + H_0(0))^{-1} S_{21}] + O(\epsilon_0^2).
\end{aligned}
$$

Using this relation and the definition of R_0 in (**H4**), we have

$$\mathrm{Re}\,\dot{\mu}(\epsilon_0) = -\epsilon_0 R_o \pi^2 + O(\epsilon_0^2) \tag{5.18}$$

as $\epsilon_0 \to 0$. For ϵ_* sufficiently small, this shows that the Hopf Bifurcation Curve is Generic and the lemma is proved.

We now complete the proof of sufficiency in Theorem 5.1 using these two lemmas and hypothesis (**H9**). We only need to prove that, for each $\epsilon_0 \in (0, \epsilon_*]$, and all $\epsilon > \epsilon_0$, there is no zero $\mu(\epsilon)$ of the characteristic matrix in (5.2) with $\mathrm{Re}\,\mu(\epsilon) > 0$. If $\det \Delta(\lambda(\epsilon_0), \hat{\epsilon}, \mu) = 0$ has a solution $\mu(\hat{\epsilon})$ with $\mathrm{Re}\ \mu(\hat{\epsilon}) > 0$ for some $\hat{\epsilon} > \epsilon_0$, then we can extend this function to a continuous function $\mu(\epsilon)$ for $\epsilon \in (\epsilon_0, \infty)$ and have $\det \Delta(\lambda(\epsilon_0), \epsilon, \mu(\epsilon)) = 0$ for $\epsilon \in (\epsilon_0, \infty)$. From Lemma 5.3 and, in particular, the implication (5.10), we must have $\mathrm{Re}\ \mu(\epsilon) > 0$ for all $\epsilon \in (\epsilon_0, \infty)$. If we let $z(\epsilon) = \epsilon\mu(\epsilon)$, then

$$\det \begin{bmatrix} z(\epsilon)I_m + A & -AA_2(\lambda(\epsilon_0)) \\ -B_1(\lambda(\epsilon_0))e^{-\frac{z(\epsilon)}{\epsilon}} & I_n - B_2(\lambda(\epsilon_0))e^{-\frac{z(\epsilon)}{\epsilon}} \end{bmatrix} = \det \Delta(\lambda(\epsilon_0), \epsilon, \mu(\epsilon)) = 0. \tag{5.19}$$

Since $\mathrm{Re}\ \frac{z(\epsilon)}{\epsilon} > 0$, relation (5.19) implies that $z(\epsilon)$ is bounded in (ϵ_0, ∞). Hence there is a subsequence $\epsilon_j \to \infty$ as $j \to \infty$ such that $z(\epsilon_j) \to z^*$. It is clear that $\mathrm{Re}\, z^* \geq 0$. By letting $j \to \infty$ in (5.19), we deduce that

$$\det \begin{bmatrix} z^* I_m + A & -AA_2(\lambda(\epsilon_0)) \\ -B_1(\lambda(\epsilon_0)) & I_n - B_2(\lambda(\epsilon_0)) \end{bmatrix} = 0.$$

This contradicts (**H9**) if ϵ_* is small enough. This completes the proof of the existence of the First Hopf Bifurcation Curve under the hypotheses of the theorem.

The proof of the necessity can be supplied by carefully considering the arguments that were used in the proof of sufficiency. We do not give the details.

Let us now consider some special cases. The equation (1.3) can be written as a system

$$\begin{aligned} \epsilon\dot{x}(t) + x(t) &= f_\lambda(y(t)) \\ y(t) &= x(t-1). \end{aligned} \tag{5.20}$$

The hypotheses (**H1**)-(**H3**) for generic period doubling of the map $x \mapsto f_\lambda(x)$ are equivalent to saying that the function $f_\lambda(x)$ can be written as

$$f_\lambda(x) = -(1+\lambda)x + \alpha x^2 + \beta x^3, \tag{5.21}$$

where $\alpha^2 + \beta \neq 0$. The linear variational equation about $x = 0$ for (5.21) is a special case of (5.1) with $A = 1$, $A_2(\lambda) = -(1+\lambda)$, $B_1(\lambda) = 1$, $B_2(\lambda) = 0$. It is now obvious that the hypotheses (**H2**), (**H4**), (**H6**), (**H7**) are satisfied. A simple computation shows that hypotheses (**H8**) is equivalent to $i(\theta - \sin v) + 1 + \cos v \neq 0$ for $\theta > 0$, $0 \leq v \leq 2\pi$, $v \neq \pi$. This is clearly satisfied. Also, (**H9**) is equivalent to $\mu + 2 \neq 0$ for $\mathrm{Re}\ \mu \geq 0$, which is true. Therefore, there is a Generic First Bifurcation Curve with respect to ϵ.

Let us next discuss equation (1.4) with $\epsilon_j = \epsilon\alpha_j, j = 1, 2, \ldots, m$, and the function $f_\lambda(x)$ satisfying (5.21). It is possible to write this equation in a matrix form (5.1) and verify directly the hypotheses for the existence of a Generic Hopf Bifurcation Curve are satisfied. Rather than do this, we will show this curve exists by analyzing the characteristic equation. The proof is not difficult and also is a simple illustration of the ideas used in the proof of Theorem 2.3. The characteristic equation for the linearization of (1.4) about $x = 0$ is given by

$$E(\epsilon, \lambda, \mu) \equiv (\epsilon\alpha_m\mu + 1)\cdots(\epsilon\alpha_1\mu + 1) + (1 + \lambda)e^{-\mu} = 0. \tag{5.22}$$

In this particular situation, the determinant in (**H8**) is given by

$$E_1(\theta, v) \equiv E(\frac{\theta}{v}, 0, iv) = \prod_{j=1}^{m}(i\theta\alpha_j + 1) + e^{-iv}.$$

If $\theta \geq 0, 0 \leq v < 2\pi$, are such that $E_1(\theta, v) = 0$, then we have $|\prod_{j=1}^{m}(i\theta\alpha_j + 1)| = 1$; that is, $\prod_{j=1}^{m}(\theta^2\alpha_j^2 + 1) = 1$. Since $\alpha_j > 0$, we must have $\theta = 0$ and hence $1 + e^{-iv} = 0$; that is, $v = \pi$. Therefore, the condition (**H8**) is satisfied.

Next, the function in (**H9**) has the form

$$C_2(\mu) \equiv \prod_{j=1}^{m}(\mu\alpha_j + 1) + 1.$$

If there is a $\mu = u + iv, u \geq 0$, such that $C_2(u + iv) = 0$, then it follows that $1 = \prod_{j=1}^{m}((u\alpha_j + 1)^2 + v^2)$. Therefore, we must have $u = v = 0$, which leads to the assertion that $0 = C_2(0) = 1 + 1$, which is a contradiction. As a consequence, $C_2(\mu) \neq 0$ for all $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu \geq 0$; that is, (**H9**) is satisfied.

We can now show that, for fixed $\epsilon_0 > 0$, there is a unique $\lambda^*(\epsilon_0) > 0$ such that (5.22) has exactly two purely imaginary roots and the remaining ones have negative real parts for $(\lambda, \epsilon) = (\lambda(\epsilon_0), \epsilon_0)$. For $\epsilon > \epsilon_0$, the origin is asymptotically stable and, for $0 < \epsilon < \epsilon_0$, the origin is unstable. In this way, we obtain the existence of a Generic First Hopf Bifurcation Curve.

In the general matrix case, the problem becomes more complicated. More specifically, we can have the map associated with the differential equation undergo generic period doubling and have the linearization about the origin in the differential equation not possess a Generic First Hopf Bifurcation Curve. For example, let us consider the system

$$\begin{aligned} \epsilon\dot{x}_1(t) + ax_1(t) &= af_\lambda(x_1(t-1)) \\ \epsilon\dot{x}_2(t) + Bx_2(t) &= B\nu x_2(t-1)), \end{aligned} \tag{5.23}$$

where f_λ is the same function as before, $x_2 \in \mathbb{R}$, B, ν are constants, $B < 0$, $|\nu| < 1$. The map obtained for $\epsilon = 0$ is given by $(x_1, x_2) \mapsto (f_\lambda(x_1), \nu x_2)$ and it undergoes a generic period doubling at $(x_1, x_2, \lambda) = (0, 0, 0)$. The hypotheses (**H2**), (**H4**) and (**H6**) are satisfied, but we have not satisfied hypothesis (**H7**).

If we want to relate the period doubling bifurcation to a Generic First Hopf Bifurcation Curve, then we must have the solutions of the linear variational equation for $\lambda = 0$ approach zero as $t \to \infty$; that is, the eigenvalues must have negative real parts. This is equivalent to having the solutions of the equation

$$\Delta(\mu) \equiv \mu + B(1 - \nu e^{-\mu}) = 0 \tag{5.24}$$

having negative real parts. Since $\Delta(0) = B(1 - \mu) < 0$ and $\Delta(\mu) \to \infty$ as $\mu \to \infty$, there is a positive zero for $\Delta(\mu)$.

6. The Hopf bifurcation. Let us suppose that the hypotheses of Theorem 2.3 and Theorem 5.1 are satisfied. Then, as shown in Section 5, there is a first Hopf Bifurcation Curve $\Gamma_{\epsilon^*} = \{((\lambda(\epsilon), \epsilon), \epsilon \in (0, \epsilon^*\}$. Furthermore, if $R_0 > 0$, then, for each fixed λ, the zero solution is stable if $\epsilon < \epsilon^*$ and unstable if $\epsilon > \epsilon^*$. At $\epsilon = \epsilon^*$, the linearization of (2.4) at the origin has a pair of purely imaginary simple eigenvalues. As a consequence, we expect that there is a generic Hopf bifurcation arising at ϵ. This is true and, in fact, we have

Theorem 6.1. *Suppose that the hypotheses of Theorem 2.3 and Theorem 5.1 are satisfied. If, in addition, $R_0 > 0$, then there exists an $\epsilon^* > 0$ such that, for each fixed $(\lambda_0, \epsilon_0) \in \Gamma_{\epsilon^*}$, $\lambda_0 = \lambda(\epsilon_0)$, where Γ_{ϵ^*} is the First Hopf Bifurcation Curve, system (2.1) has a generic Hopf bifurcation at $(\lambda, \epsilon, x, y) = (\lambda_0, \epsilon_0, 0, 0)$ which is supercritical (resp. subcritical) if $R_1 > 0$ (resp. $R_1 < 0$).*

We remark that this result implies that the generic period doubling point $\lambda = 0$ for the mapping $\mathcal{F}_\lambda$ is supercritical (resp. subcritical) if and only if the generic Hopf bifurcation at (λ_0, ϵ_0) is supercritical (resp. subcritical).

To prove Theorem 6.1, we are going to use some results from [10] for more general hybrid systems in which they gave a specific formula for a constant α^* depending upon the second and third derivatives of the nonlinearities in (2.1) evaluated at zero which determines the direction of the Hopf bifurcation. More specifically, the bifurcation is supercritical (resp. subcritical) if $\alpha^* > 0$ (resp. $\alpha^* < 0$). We now describe this constant, attempting to motivate each step and refer the reader to [10] for details.

Let

$$\tilde{f}(y, \lambda) = f(y, \lambda) - A_2(\lambda)y,$$

$$\tilde{g}(x, y, \lambda) = g(x, y, \lambda) - B_1(\lambda)x - B_2(\lambda)y.$$

The linearized system of (2.1) is

$$\begin{aligned} \epsilon\dot{x}(t) &= -Ax(t) + AA_2(\lambda)y(t) \\ y(t) &= B_1(\lambda)x(t-1) + B_2(\lambda)y(t-1) \end{aligned} \tag{6.1}$$

From our hypotheses on f and g, there is an $\epsilon^* > 0$ such that, for each $0 < \epsilon_0 < \epsilon^*$, there is a unique $\lambda_{\epsilon_0} > 0$ such that the linear system (6.1) for $\epsilon = \epsilon_0$, $\lambda = \lambda_{\epsilon_0}$ has an imaginary eigenvalue $\mu_0 = i\nu_0 = i\pi + O(\epsilon_0)$ and it is simple. For fixed λ_{ϵ_0}, the eigenvalue $\mu_0(\epsilon)$ of (6.1) with $\mu_0(\epsilon_0) = \mu_0$ satisfies the inequality $\dot{\mu}_0(\epsilon_0) < 0$.

Along with (6.1), we consider the linear neutral equation

$$\begin{aligned}&\epsilon\dot{x}(t) = -Ax(t) + AA_2(\lambda)y(t)\\ &\frac{d}{dt}[y(t) - B_1(\lambda)x(t-1) - B_2(\lambda)y(t-1)] = 0.\end{aligned} \tag{6.2}$$

This equation plays an important role in the variation of constants formula for a perturbation of (6.1) and, thus, an important role in the explicit computation of approximation of periodic orbits. The nonzero eigenvalues of (6.1) and (6.2) as well as their multiplicies coincide.

The characteristic matrix for (6.2) is

$$\Delta(\epsilon_0,\mu) = \begin{bmatrix} \mu I + \frac{1}{\epsilon_0}A & -\frac{1}{\epsilon_0}AA_2(\lambda_{\epsilon_0}) \\ -\mu B_1(\lambda_{\epsilon_0})e^{-\mu} & \mu(I - B_2(\lambda_{\epsilon_0})e^{-\mu}) \end{bmatrix}.$$

It is not difficult to show that there is a vector η_{ϵ_0} which satisfies $\Delta(\epsilon_0,\mu_0)\eta_{\epsilon_0} = 0$ and has the form

$$\eta_{\epsilon_0} = \mathrm{col}[\eta^1_{\epsilon_0},\eta^2_{\epsilon_0}], \quad \eta^1_{\epsilon_0} = A_2(\lambda_{\epsilon_0})e_1 + O(\epsilon_o), \quad \eta^2_{\epsilon_0} = e_1 + O(\epsilon_0), \quad e_1 = \mathrm{col}(1,0) \in \mathbb{R}^n.$$

As a consequence,

$$\Phi^{\epsilon_0}(\theta) = \left[\eta_{\epsilon_0}e^{\mu_0\theta}, \bar{\eta}_{\epsilon_0}e^{-\mu_0\theta}\right]E$$

is a basis for the eigenfunction space for μ_0 and $\bar{\mu}_0$ considered as an eigenvalue of either (6.1) or (6.2). If Φ^{1,ϵ_0} is the first column of Φ^{ϵ_0}, we have

$$\lim_{\epsilon_0\to 0}\Phi^{1,\epsilon_0}(\theta) = \lim_{\epsilon_0\to 0}\begin{bmatrix}\Phi^{11,\epsilon_0}(\theta)\\ \Phi^{21,\epsilon_0}(\theta)\end{bmatrix} = \frac{1}{2}(e^{i\pi\theta} + e^{-i\pi\theta})\begin{bmatrix}A_2(0)e_1\\ e_1\end{bmatrix}. \tag{6.3}$$

Now, if $[b^1_0, b^2_0] \in \mathbb{C}^{m+n^*}$ is such that $[b^1_0,b^2_0]\Delta(\epsilon_0,\mu_0) = 0$, then we have

$$\begin{aligned}b^1_0 &= \epsilon_0\mu_0 b^2_0 B_1(\lambda_{\epsilon_0})e^{-\mu_0}(\mu_0\epsilon_0 + A)^{-1},\\ b^2_0[B^1_0 e^{-\mu_0}&(\mu_0\epsilon_0 + A)^{-1}AA_2(\lambda_{\epsilon_0}) - (I - B_2(\lambda_{\epsilon_0})e^{-\mu_0})] = 0.\end{aligned}$$

So we can choose b^2_0 such that

$$b^2_0 = e^*_1 + O(\epsilon_0), \quad e^*_0 = [1,0] \in \mathbb{R}^{n*}.$$

If we let

$$\gamma_{\epsilon_0} = [\gamma^1_{\epsilon_0},\gamma^2_{\epsilon_0}], \quad \gamma^i_{\epsilon_0} = \frac{b^i_0}{b^1_0\eta^1_{\epsilon_0} + \mu_0 b^2_0\eta^2_{\epsilon_0}}, \quad i = 1,2,$$

then one is able to verify that

$$\Psi^{\epsilon_0}(s) = E^{-1}\begin{bmatrix}\gamma_{\epsilon_0}e^{-\mu_0 s}\\ \bar{\gamma}_{\epsilon_0}e^{\mu_0 s}\end{bmatrix}$$

is a basis for the eigenfunction space corresponding to μ_0 and $\bar{\mu}_0$ for the formal adjoint operator of (6.2) (not (6.1)). Moreover, we have

$$
(6.4)\qquad
\begin{aligned}
&\lim_{\epsilon_0 \to 0} \Psi^{11,\epsilon_0}(s)\frac{A}{\epsilon_0} = -(e^{i\pi s} + e^{-i\pi s})e_1^* B_1(0) \\
&\lim_{\epsilon_0 \to 0} \delta\Psi^{12,\epsilon_0}(s) = -(e^{i\pi s} + e^{-i\pi s})e_1^*,
\end{aligned}
$$

where $\Psi^{1,\epsilon_0} = [\Psi^{11,\epsilon_0}, \Psi^{12,\epsilon_0}]$ and Ψ^{1,ϵ_0} is the first row lof Ψ^{ϵ_0}.

To obtain the direction of bifurcation, it is first necessary to determine the influence of the second order terms in (2.1) on the periodic orbit. These second order terms give rise to corrections in the solution of the form $\xi(\epsilon_0)e^{2i\nu_0 t} + \bar{\xi}(\epsilon_0)e^{-2i\nu_0 t} + \xi_0(\epsilon_0)$. It is shown in [10] that the quantity ξ is given by

$$
\xi(\epsilon_0) = \frac{1}{8}\begin{bmatrix} 2\mu_0 + \frac{A}{\epsilon_0} & -\frac{AA_2(\lambda(\epsilon_0))}{\epsilon_0} \\ -B_1(\lambda(\epsilon_0))e^{-2\mu_0} & I - B_2(\lambda(\epsilon_0))e^{-2\mu_0} \end{bmatrix}^{-1} \begin{bmatrix} \frac{A}{\epsilon_0} D_y^2 f(0, \lambda(\epsilon_0))\langle \eta_{\epsilon_0}^2 \rangle^2 \\ D_z^2 g(0, \lambda(\epsilon_0))\langle \eta_{\epsilon_0} \rangle^2 \end{bmatrix}
$$

where $D_z = D_{(x,y)}$. By using (6.4) and letting $\xi(\epsilon_0) = \operatorname{col}[\xi^1(\epsilon_0), \xi^2(\epsilon_0)]$, we have

$$
(6.6)\qquad
\begin{aligned}
&\lim_{\epsilon_0 \to 0} \xi^2(\epsilon_0) = \xi_0^2, \\
&\lim_{\epsilon_0 \to 0} \xi^1(\epsilon_0) = \xi_0^1,
\end{aligned}
$$

where

$$
\begin{aligned}
\xi_0^1 &\equiv A_2(0)\xi_0^2, \\
\xi_0^2 &\equiv \frac{1}{8}\begin{bmatrix} \frac{1}{2} e_1^* [B_1(0) D_y^2 f(0,0)\langle e_1 \rangle^2 + D_z^2 g(0,0)\langle \begin{bmatrix} A_2(0)e_1 \\ e_1 \end{bmatrix} \rangle^2] \\ (I - H_0(0))^{-1} E_2 [B_1(0) D_y^2 f(0,0)\langle e_1 \rangle^2 + D_z^2 g(0,0)\langle \begin{bmatrix} A_2(0)e_1 \\ e_1 \end{bmatrix} \rangle^2] \end{bmatrix}
\end{aligned}
$$

and E_2 is the $(n-1) \times n$ matrix:

$$
E_2 = \begin{bmatrix}
0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
0 & 0 & 0 & \cdot & \cdot & \cdot & 1
\end{bmatrix}.
$$

Similary, the constant $\xi_0(\epsilon_0)$ satisfies

$$
(6.7)\qquad \lim_{\epsilon_0 \to 0} \xi_0(\epsilon_0) = \lim_{\epsilon_0 \to 0} \begin{bmatrix} \xi_0^1(\epsilon_0) \\ \xi_0^2(\epsilon_0) \end{bmatrix} = 2\begin{bmatrix} \xi_0^1 \\ \xi_0^2 \end{bmatrix}.
$$

From [10], the number $\alpha^*(\epsilon_0)$ for system (2.1) at $(\lambda(\epsilon_0), \epsilon_0)$ is given by

$$\alpha^*(\epsilon_0) = -\int_0^{\frac{2\pi}{\mu_0}} \Psi^{11,\epsilon_0}(s)\frac{A}{\epsilon_0}D_y^2 f(0,\lambda(\epsilon_0))\langle\Phi^{21,\epsilon_0}(s), \xi(\epsilon_0)e^{2\mu_0 s} + \bar{\xi}(\epsilon_0)e^{-2\mu_0 s} + \xi_0(\epsilon_0)\rangle ds$$
$$-\frac{1}{3!}\int_0^{\frac{2\pi}{\mu_0}} \Psi^{11,\epsilon_0}(s)\frac{A}{\epsilon_0}D_y^3 f(0,\lambda(\epsilon_0))\langle\Phi^{21,\epsilon_0}(s)\rangle^3 ds$$
$$+\int_0^{\frac{2\pi}{\mu_0}} \dot{\Psi}^{12,\epsilon_0}(s)D_z^2 g(0,\lambda(\epsilon_0))\times$$
$$\langle\Phi^{1,\epsilon_0}(s-1), \xi(\epsilon_0)e^{2\mu_0(s-1)} + \bar{\xi}(\epsilon_0)e^{-2\mu_0(s-1)} + \xi_0(\epsilon_0)\rangle ds$$
$$+\frac{1}{3!}\int_0^{\frac{2\pi}{\mu_0}} \dot{\Psi}^{12,\epsilon_0}(s)D_z^3 g(0,\lambda(\epsilon_0))\langle\Phi^{1,\epsilon_0}(s-1)\rangle^3 ds.$$

By applying (4.4)-(4.7), we obtain

$$\begin{aligned}
&\lim_{\epsilon_0\to 0}\alpha^*(\epsilon_0)\\
=&e_1^*\int_0^2 (e^{i\pi s}+e^{-i\pi s})^2(2+e^{2i\pi s}+e^{-2i\pi s})D_y^2 f(0,0)\langle\frac{1}{2}e_1, \xi_0^2\rangle ds\\
&+\frac{e_1^*}{3!}\int_0^2 (e^{i\pi s}+e^{-i\pi s})^4 D_y^3 f(0,0)\langle\frac{1}{2}e_1\rangle^3 ds\\
&+e_1^*\int_0^2 (e^{i\pi s}+e^{-i\pi s})^2(2+e^{2i\pi s}+e^{-2i\pi s})D_z^2 g(0,0)\langle\frac{1}{2}\begin{bmatrix} A_2(0)e_1 \\ e_1\end{bmatrix}, \begin{bmatrix}\xi_0^1\\ \xi_0^2\end{bmatrix}\rangle ds\\
&+\frac{e_1^*}{3!}\int_0^2 (e^{i\pi s}+e^{-i\pi s})^4 D_z^3 f(0,0)\langle\frac{1}{2}\begin{bmatrix} A_2(0)e_1 \\ e_1\end{bmatrix}\rangle^3 ds\\
=&12e_1^*[D_y^2 f(0,0)\langle\frac{1}{2}e_1, \xi_0^2\rangle + D_z^2 g(0,0)\langle\frac{1}{2}\begin{bmatrix} A_2(0)e_1 \\ e_1\end{bmatrix}, \begin{bmatrix} A_2(0)\xi_0^2 \\ \xi_0^2\end{bmatrix}\rangle\\
&+\frac{1}{3!}D_y^3 f(0,0)\langle\frac{1}{2}e_1\rangle^3 + \frac{1}{3!}D_z^3 g(0,0)\langle\frac{1}{2}\begin{bmatrix} A_2(0)e_1 \\ e_1\end{bmatrix}\rangle^3]\\
=&6R_1,
\end{aligned}$$

where the last equality follows from relation (2.5) and (**H3**).

The last equality shows that, for ϵ_0 small, the quantity $\alpha^*(\epsilon_0)$ has the same sign as R_1. Therefore, from Lemma 2.1 and the above mentioned result from [10], we see that the Hopf bifurcation is supercritical if $R_1 > 0$ and subcritical if $R_1 < 0$. This completes the proof of Theorem 6.1.

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Synchronizations in lattices of nonlinear

Duffing's oscillators

by

V.S. Afraimovich and S.N. Chow

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Synchronizations in Lattices of Nonlinear Duffing's Oscillators

V. S. Afraimovich[1] and S.-N. Chow[2]

Abstract

In this work we prove the possibility of stochastic synchronization in two dimensional lattice of coupled Duffing's oscillators with external periodic forces. The synchronization occurs provided coupling is dissipative and the coefficients of coupling is greater than some critical values. These values depend on parameters of individual subsystems and on the size of lattice.

1. Introduction

The phenomenon of stochastic synchronization was first observed in [1] for identical coupled subsystems and in [2] for different systems. In [2] a rigorous definition of synchronization was introduced and a theory of synchronization of parametrically excited, diffusively coupled nonlinear oscillators was suggested. Later, this phenomenon was observed in different fields – electrical engineering, biology, lasers systems, etc. (see, for example, [3–11]). Roughly speaking, synchronization occurs when two or more different systems exhibit similar behavior provided some dissipative coupling exists between them and a coefficient of this coupling is greater than some critical value.

A mathematical foundation of synchronization of two slightly different subsystems was first introduced in [9]. It seems that there are no rigorous mathematical results related to this except for [1, 2, 9, 10], even though this phenomenon is important in applications and a mathematically interesting topic.

Our work deals with synchronization in large lattices. For this case, some features of continuous nonequilibrium media become evident: for example, boundary conditions play an essential role. Of course, it is not surprising that a large lattice of coupled oscillators is a model of nonequilibrium media and must reflect its features. But mathematical difficulties increase in comparison to two coupled subsystems. Therefore, we restrict ourself in this

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paper to the case of specific subsystems – Duffing's oscillator with periodic forces. It is well-known that each of them may manifest chaotic behavior for a wide range of parameters.

2. Formulation of the Problem

Let us consider the following system

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -k_i y_i + \alpha_i x_i - \beta_i x_i^3 + a_i + b_i \cos \omega t + c_1 \Delta y_i + c_2 \Delta x_i \end{cases} \tag{1}$$

where i is a p-dimensional integer vector, i.e., $i \in \mathbb{Z}^p$, $k_i > 0$, $\alpha_i > 0$, $\beta_i > 0$, a_i, b_i, $c_1 > 0$ and $c_2 > 0$ are constants. In this paper, for the sake of definiteness and in order to avoid cumbersome calculations and formulas, we consider only the case $p = 2$. But our technique works and all results hold for an arbitrary value of $p \geq 2$. The symbol Δ denotes a discrete version of the Laplace operator:

$$\Delta \xi_i = \xi_{i_1+1,i_2} + \xi_{i_1,i_2+1} + \xi_{i_1-1,i_2} + \xi_{i_1,i_2-1} - 4\xi_{i_1,i_2}, \qquad i = (i_1, i_2).$$

Our assumptions are:

(i) Let $N > 0$ be a fixed and large integer. We consider an $N \times N$-lattice, $1 \leq i_1 \leq N$, $1 \leq i_2 \leq N$, with the *Dirichlet boundary conditions:* $x_i \equiv 0$, $y_i \equiv 0$ if $i = (i_1, i_2)$ satisfies at least one of the following inequalities:

$$i_1 \leq 0, \quad i_2 \leq 0, \quad i_1 \geq N+1, \quad i_2 \geq N+1.$$

(ii) The subsystems are *almost identical*, i.e.,

$$|k_i - k_j| \leq \epsilon \quad |\alpha_i - \alpha_j| \leq \epsilon, \quad |\beta_i - \beta_j| \leq \epsilon$$

$$|a_i - a_j| \leq \epsilon, \quad |b_i - b_j| \leq \epsilon$$

where $1 \leq i_1 \leq N$, $1 \leq i_2 \leq N$; the parameter ϵ characterizes scattering of the oscillators and is small; the parameters c_1 and c_2 are responsible for the magnitude of the coupling and play crucial roles in our study.

It is possible to show (see Section 5) that system (1) is dissipative in the following sense. Let $D > 0$ be sufficiently large and $0 < \omega < k_i$. Let

$$U(x,y) = \sum_i \left(\frac{y_i^2}{2} - \frac{\alpha_i x_i^2}{2} + \frac{\beta_i x_i^4}{y} + \omega x_i y_i \right).$$

Define

$$S = \{(x,y) \,|\, U(x,y) = D\}.$$

Then

$$\left.\frac{dU}{dt}\right|_S = (\mathrm{grad}\, U, Z)\Big|_S < 0$$

where Z is the vector field defined by the right hand side of (1). Thus, there exists a global attractor of (1) which belongs to a bounded region independent of c_1 and c_2. We note that there are many results related to existence of global attractors for infinite or finite dimensional dissipative systems (see, for example, [20], [21]). In our case, the proof of dissipativeness is nontrivial.

For any solution $x_i = x_i(t)$, $y_i = y_i(t)$ belonging to the global attractor the following inequalities hold

$$|x_i(t)| \leq M, \qquad |y_i(t)| \leq M \tag{2}$$

where M is a constant independent of c_1, c_2. Fix an arbitrary solution $\{x = x_i(t), y = y_i(t)\}$ belonging to the global attractor. Set

$$u_{ij}(t) = x_i(t) - x_j(t), \qquad v_{ij}(t) = y_i(t) - y_j(t).$$

The main result of our paper is the following.

Theorem. *Let $\underline{k} = \min_i k_i$. There exist constants $c_1^* > 0$, $c_2^* > 0$ and $0 < q < 1$ such that if*

$$c_2 > c_2^*, \qquad c_1 > c_1^* \qquad c_1 > \frac{c_2}{q\underline{k}}$$

then

$$\overline{\lim_{t\to\infty}} |u_{ij}(t)| \leq K\epsilon, \qquad \overline{\lim_{t\to\infty}} |v_{ij}(t)| \leq K\epsilon$$

where K is some constant independent of ϵ, c_1 and c_2.

It follows from the Theorem that the projections of solutions on the coordinate subspaces are close to each other for an infinite interval of time. That is, the synchronization occurs.

3. Proof of Theorem

Introduce the mean values of the parameters:

$$k = \frac{1}{N^2}\sum_i k_i, \quad \alpha = \frac{1}{N_2}\sum_i \alpha_i, \quad \beta = \frac{1}{N^2}\sum_i \beta_i, \quad a = \frac{1}{N^2}\sum_i a_i, \quad b = \frac{1}{N^2}\sum_i b_i$$

and rewrite the system (1) in the following form

$$\begin{cases} \dot{x}_i = y_i \\ \dot{y}_i = -ky_i + \alpha x_i - \beta x_i^3 + a + b\cos\omega t + c_1\Delta y_i + c_2\Delta x_i + \delta_i, \end{cases} \tag{3}$$

where

$$\delta_i = -(k_i - k)y_i(t) + (\alpha_i - \alpha)x_i(t) - (\beta_i - \beta)x_i^3(t) + (a_i - a) + (b_i - b)\cos\omega t\,.$$

Taking (2) into account, we have $|\delta_i| \le K_1 \cdot \epsilon$, for some constant $K_1 > 0$.

The system for $u_{ij}(t), v_{ij}(t)$ takes the following form:

$$\begin{cases} \dot{u}_{ij} = v_{ij} \\ \dot{v}_{ij} = -kv_{ij} + \alpha u_{ij} - \beta\gamma_{ij}(t)u_{ij} + c_1\mathcal{L}v_{ij} + c_2\mathcal{L}u_{ij} + \epsilon_{ij}(t) \end{cases} \tag{4}$$

where

$$\gamma_{ij}(t) = x_i^2(t) - x_i x_j + x_j^2(t), \qquad \epsilon_{ij}(t) = \delta_i(t) - \delta_j(t),$$

$$\mathcal{L}v_{ij} = \Delta y_i - \Delta y_j, \qquad \mathcal{L}u_{ij} = \Delta x_i - \Delta x_j\,.$$

Note that

$$\begin{aligned} \mathcal{L}u_{ij} &= \Delta x_i - \Delta x_j \\ &= (x_{i_1+1,i_2} - x_{j_1+1,j_2}) + (x_{i_1-1,i_2} - x_{j_1-1,j_2}) + (x_{i_1,i_2+1} - x_{j_1,j_2+1}) \\ &\qquad + (x_{i_1,i_2-1} - x_{j_1,j_2-1}) - 4(x_{i_1,i_2} - x_{j_1,j_2}) \\ &= u_{i_1+1,i_2,j_1+1,j_2} + u_{i_1-1,i_2,j_1-1,j_2} + u_{i_1,i_2+1,j_1,j_2+1} + u_{i_1,i_2-1,j_1,j_2-1} - 4u_{i_1,i_2,j_1,j_2}. \end{aligned}$$

Denoting $\{u_{ij}\}$ by u, $\{v_{ij}\}$ by v, $\{\gamma_{ij}\}$ by γ, $\{\epsilon_{ij}\}$ by $\underline{\epsilon}$, we can write

$$\begin{cases} \dot{u} = v \\ \dot{v} = -kv + \alpha u - \beta\gamma * u + c_1\mathcal{L}v + c_2\mathcal{L}u + \underline{\epsilon} \end{cases} \tag{5}$$

where $(\gamma * u)_{ij} = \gamma_{ij} \cdot u_{ij}$.

We consider the system (5) for the values of (i,j) belonging to the index set $J = \bigcup_{|k_1|\leq N, |k_2|\leq N} J_{k_1} \times J_{k_2}$ such that the set J_{k_α}, $\alpha = 1,2$ is a discrete interval. $J_{k_\alpha} = \{(i_\alpha, j_\alpha) \mid j_\alpha = i_\alpha + k_\alpha$ where $-k_\alpha \leq i_\alpha \leq N$ if $k_\alpha \geq 0$ and $0 \leq i_\alpha \leq N - k_\alpha$ if $k_\alpha \leq 0\}$. It is simple to see that

$$d^2 \stackrel{\text{def}}{=} \operatorname{card}\left(\bigcup_{|k_\alpha|\leq N} J_{k_\alpha}\right) = (N+1) + 2[(N+2) + (2N+1)] \cdot \frac{N}{2} = (N+1)(3N+1).$$

Therefore, $\{u_{ij}\}$ and $\{v_{ij}\}$ belong to $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$, and $u_{ij} \equiv 0$, $v_{ij} \equiv 0$ if $(i,j) \notin J$. Let us remark that $\epsilon_{ij} \equiv 0$ if $(i,j) \notin J$.

We prove below that u_{ij} is small for any $(i,j) \in J$. In particular, it will be so for i and j belonging to the original $N \times N$ lattice. We will essentially use below the following result.

Main Lemma. *(i) The operator $\mathcal{L} : \mathbb{R}^{d^2} \times \mathbb{R}^{d^2} \to \mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$ is self-adjoint. (ii) Let $\sigma(\mathcal{L})$ be the spectrum of $\mathcal{L}$ and*

$$\lambda_0 = -8 \sin^2 \frac{\pi}{2(2N+2)} .$$

If $\lambda_s \in \sigma(\mathcal{L})$, then $\lambda_s \leq \lambda_0$.

The main Lemma will be proved in Section 4. The main Theorem will be proved by using several changes of variables. The first change of variables is linear.

Proposition 1. *We assume that*

$$\alpha - 8c_2 \sin^2 \frac{\pi}{2(2N+2)} < 0 . \tag{6}$$

There exists a nonsingular self-adjoint linear map $A : \mathbb{R}^{d^2} \times \mathbb{R}^{d^2} \to \mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$ such that the following change of variables

$$v = w + Au \tag{7}$$

applied to the system (5) *gives the following*

$$\begin{cases} \dot{w} = -kw - Aw + \beta\gamma * u + c_1\mathcal{L}w + \underline{\epsilon} \\ \dot{u} = w + Au \end{cases} \qquad (8)$$

Furthermore, for any $\rho_s \in \sigma(A)$ *there exists* $\lambda_s \in \sigma(\mathcal{L})$ *such that*

$$\rho_s = \frac{1}{2}\left(-k + c_1\lambda_s + \sqrt{(-k + c_1\lambda_s)^2 + 4(\alpha + c_2\lambda_s)}\right) < 0\,. \qquad (9)$$

Proof. Differentiating (7), we obtain $\dot{v} = \dot{w} + A\dot{u}$, and it follows from (5) and (7) that

$$\dot{w} + Aw + A^2u = -kw - kAu + \alpha u - \beta\gamma * u + c_1\mathcal{L}w + c_1\mathcal{L}Au + c_2\mathcal{L}u + \underline{\epsilon}$$

Claim that

$$A^2u = -kAu + \alpha u + c_1\mathcal{L}Au + c_2\mathcal{L}u \qquad (10)$$

If (10) is satisfied, then (8) will also be satisfied. We will show that there is an operator A satisfying (10). Let h^s be an eigenvector corresponding to the eigenvalue λ_s of the operator $\mathcal{L}$. Hence, $\mathcal{L}h^s = \lambda_s h^s$, and the set $\{h^s\}$ forms a system of coordinates in $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$. Each element u can be represented as $u = \sum_s u^s h^s$. Define A as follows

$$Au = \sum_s u_s\rho_s h^s \qquad (11)$$

where ρ_s is as in (9). Then

$$\begin{aligned} A^2u &= \sum_s u_s\rho_s^2 h^s, \qquad -kAu = \sum_s -k\rho_s u_s h^s, \qquad \alpha u = \sum_s \alpha u_s h^s, \\ c_2\mathcal{L}u &= \sum_s c_2\lambda_s u_s h^s, \qquad c_2\mathcal{L}Au = \sum_s c_1\lambda_s\rho_s u_s h^s \end{aligned} \qquad (12)$$

Note that $\rho_s^2 = -k\rho_s + \alpha + c_2\lambda_s + c_1\lambda_s\rho_s$. Substituting (12) into (10) we obtain the desired result. It is simple to check that A is self-adjoint. □

Now, we work with the system (8).

The second change of variables. Note that $\|\underline{\epsilon}\| < K_2 \cdot \epsilon$ where $K_2 > 0$ is a constant. If we show that

$$\limsup_{t\to\infty} \|w(t)\| = O(\epsilon)\,,$$

then it will imply the desired result. We are looking for a change of variables in the form

$$w_{ij} = \eta_{ij} + g_{ij}(t) \cdot u_{ij} \tag{13}$$

where $\eta = \{\eta_{ij}\}$ are new variables and $g(t) = \{g_{ij}(t)\}$ is some function to be determined. Differentiating (13),

$$\dot{w}_{ij} = \dot{\eta}_{ij} + \dot{g}_{ij}u_{ij} + g_{ij} \cdot \dot{u}_{ij}.$$

Substituting (8) and (13) into the above expression, we obtain

$$\begin{aligned} &\dot{\eta}_{ij} + \dot{g}_{ij}u_{ij} + g_{ij}(\eta_{ij} + g_{ij}u_{ij}) + g_{ij}(Au)_{ij} \\ &\qquad = -k\eta_{ij} - kg_{ij}u_{ij} - \beta\gamma_{ij}u_{ij} + c_1\mathcal{L}\eta_{ij} + c_1\mathcal{L}g_{ij}u_{ij} + \epsilon_{ij} \end{aligned} \tag{14}$$

If $g_{ij}(t)$ satisfies the equation

$$\dot{g}_{ij}u_{ij} + g_{ij}^2u_{ij} = -kg_{ij}u_{ij} - \beta\gamma_{ij}u_{ij} - g_{ij}(Au)_{ij} + c_1\mathcal{L}g_{ij}u_{ij} \tag{15}$$

then the equation (14) becomes

$$\dot{\eta}_{ij} = -k\eta_{ij} - g_{ij}(t)\eta_{ij} + c_1\mathcal{L}\eta_{ij} + \epsilon_{ij} \tag{16}$$

and

$$\dot{u}_{ij} = \eta_{ij} + g_{ij}(t)u_{ij} + (Au)_{ij} \tag{17}$$

Solution of the equation (15). Rewrite the equation (15) in an operator form. For this, take into account that an arbitrary element $p = \{p_{ij}\}$ corresponds to the operator (diag p) which acts on the space $\{u\}$ as follows: $((\text{diag } p)(u))_{ij} = p_{ij}u_{ij}$. So, the equation (15) can be written in the form

$$\begin{aligned} &(\text{diag } \dot{g})(u) + (\text{diag } g^2)(u) \\ &\qquad = -k(\text{diag } g)(u) - \beta(\text{diag } \gamma)(u) - (\text{diag } g)(Au) + c_1\mathcal{L}(\text{diag } g)(u) \end{aligned} \tag{18}$$

For the sake of convenience set $\underline{\bullet} = (\text{diag } \bullet)$ ($\bullet$ will be g, γ, $\dot{g}$, or g^2), then (18) becomes

$$\underline{\dot{g}}u + \underline{g}^2u = -k\underline{g}u - \beta\underline{\gamma}u - \underline{g}Au + c_1\mathcal{L}\underline{g}u \tag{19}$$

The equation (19) will be satisfied if the operator Riccati equation

$$\underline{\dot{g}} + \underline{g}^2 = -k\underline{g} - \beta\underline{\gamma} - \underline{g}A + c_1\mathcal{L}\underline{g}$$

is satisfied. Rewrite it as follows

$$\dot{\underline{g}} = -\underline{g}^2 - \beta\underline{\gamma} + \mathcal{R}\underline{g} \tag{20}$$

where $\mathcal{R}$ is a linear operator:

$$\mathcal{R}\underline{g} = -k\underline{g} - \underline{g}A + c_1\mathcal{L}\underline{g} \tag{21}$$

Proposition 2. *We have that $\mathcal{R}$ is self-adjoint and the spectrum $\sigma(\mathcal{R}) = \{r_s\}$, where*

$$r_s = \frac{1}{2}\left(-k + c_1\lambda_s - \sqrt{(-k + c_1\lambda_s)^2 + 4(\alpha + c_2\lambda_s)}\right), \quad \lambda_s \in \sigma(\mathcal{L})$$

Proof. By definition, the operator $\mathcal{R}$ acts on the space $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$. We will show that the eigenvectors $\{h^s\}$ of $\mathcal{L}$ are also the eigenvectors of $\mathcal{R}$. By Proposition 1 and (21), we have

$$\begin{aligned} \mathcal{R}h^s &= -kh^s - h^s\rho_s + c_1\lambda_s h^s = (-k - \rho_s + c_1\lambda_s)h^s \\ &= (-k + c_1\lambda_s - \frac{1}{2}\left(-k + c_1\lambda_s + \sqrt{(-k + c_1\lambda_s)^2 + 4(\alpha + c_2\lambda_s)}\right) h^s = r_s h^s. \end{aligned}$$

Corollary. *(i)*

$$r = \max_s r_s \le -\frac{1}{2}\left(k + 8c_1 \sin^2 \frac{\pi}{2(2N+2)}\right) < 0$$

(ii) If $S(t)$ is the fundamental matrix of the system $\dot{\xi} = \mathcal{R}\xi$ then

$$\|S(t - t_0)\| \le e^{qr(t-t_0)} \tag{22}$$

where $0 < q < 1$ is some constant independent of c_1, c_2.

We will look for a solution of (20) which is a solution of the following integral equation

$$\underline{g}(t) = -\int_0^t S(t-\tau)(\underline{g}^2(\tau) + \beta\underline{\gamma}(\tau))d\tau \tag{23}$$

Evidently, any continuous solution of (23), $t \in \mathbb{R}^+$, satisfies the equation (20). We will solve the equation (23) using the contraction mapping principle. For any $T > 0$ and $m > 0$, consider a metric space $\mathcal{H}_{T,m}$ of continuous vector functions

$$\underline{h} : [0, T] \to \mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$$

satisfying

$$\mathcal{H}_{T,m} = \{\underline{h}(\cdot) \mid \underline{h}(0) = 0, \|\underline{h}(t)\| \le m, 0 \le t \le T\}$$

The space $\mathcal{H}_{T,m}$ is endowed with the C^0-metrics:

$$\mathrm{dist}(\underline{h}'(t), \underline{h}''(t)) = \sup_{0 \le t \le T} \|h'(t) - h''(t)\|.$$

Denote by Γ the maximal norm: $\Gamma = \sup_t \|\gamma(t)\|$. By the uniform boundedness of the global attractor, Γ is independent of c_1, c_2.

Proposition 3. *Suppose*

$$q^2 r^2 > 4\beta\Gamma \tag{24}$$

and

$$m = m_0 \equiv \frac{1}{2}\left(q|r| - \sqrt{q^2 r^2 - 4\beta\Gamma}\right). \tag{25}$$

Let Q be an operator defined by:

$$Q\underline{h}(t) = -\int_0^t S(t-\tau)(\underline{h}^2(\tau) + \beta\underline{\gamma}(\tau))d\tau. \tag{26}$$

Then the following statements hold

(i) $Q\mathcal{H}_{T,m_0} \subset \mathcal{H}_{T,m_0}$;

(ii) Q *is a contraction.*

Proof. (i) It follows from the definition (26) that

$$\|Q\underline{h}(t)\| \le \int_0^t e^{qr(t-\tau)} \cdot (\|\underline{h}^2(\tau)\| + \beta\Gamma)d\tau$$

By the definition of the norm $\|\cdot\|$, $\|\underline{h}^2(\tau)\| \le \|\underline{h}(\tau)\|^2$. Thus,

$$\begin{aligned}\|Q\underline{h}(t)\| &\le \int_0^t e^{qr(t-\tau)} \cdot (m_0^2 + \beta\Gamma)d\tau \\ &= \frac{1}{qr}(1 - e^{qrt}) \cdot (m_0^2 + \beta\Gamma) \\ &< \frac{m_0^2 + \beta\Gamma}{q|r|}\end{aligned}$$

If (24) and (25) are satisfied, then $\frac{m_0^2+\beta\Gamma}{q|r|} = m_0$ and $\|Q\underline{h}(t)\| \le m_0$ for any $t \in [0, T]$.

(ii) Let $\underline{h}_1(t), \underline{h}_2(t) \in \mathcal{H}_{T,m_0}$ be arbitrary. By using the scalar product in $\mathbb{R}^{d^2} \times \mathbb{R}^{d^2}$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|Q\underline{h}_1(t) - Qh_2(t)\| &\le \int_0^t e^{qr(t-\tau)} \cdot \|h_1^2(\tau) - h_2^2(\tau)\| d\tau \\ &\le \int_0^t e^{qr(t-\tau)} \cdot \|\underline{h}_1(\tau) - \underline{h}_2(\tau)\| \cdot 2m_0 d\tau \\ &\le \frac{1}{q|r|}(1 - e^{qrt}) \cdot 2m_0 \cdot \operatorname{dist}(\underline{h}_1(t), \underline{h}_2(t)) \\ &< \frac{2m_0}{q|r|} \cdot \operatorname{dist}(\underline{h}_1(t), \underline{h}_2(t)|.\end{aligned}$$

Since

$$\frac{2m_0}{q|r|} = p \equiv \frac{q|r| - \sqrt{q^2r^2 - 4\beta\Gamma}}{q|r|} < 1,$$

we have

$$\operatorname{dist}(Q\underline{h}_1(t), Q\underline{h}_2(t)) \le p \operatorname{dist}(\underline{h}_1(t), \underline{h}_2(t))$$

then statement (ii) is proved. □

The unique fixed point of Q, say $\underline{h}_T(t)$, is the desired solution. Evidently, if $Q\underline{h}_{T_1}(t) = \underline{h}_{T_1}(t)$, $Q\underline{h}_{T_2}(t) = \underline{h}_{T_2}(t)$ and $T_2 > T_1$, then $\underline{h}_{T_1}(t) \equiv \underline{h}_{T_2}(t)$ for all $0 \le t \le T_1$. Denote this fixed point by $\underline{g}(t) = \{g_{ij}(t)\}$. It satisfies the equation (15). Therefore the original system is represented in the form (16), (17). It follows from Proposition 3 that

$$\|\underline{g}(t)\| \le m_0 = \frac{2\beta\Gamma}{q|r| + \sqrt{q^2r^2 - 4\beta\Gamma}} \to 0 \quad \text{as } c_1 \to \infty.$$

Moreover, since $g_{ij}(t)$ satisfies the equation

$$g_{ij}(t) = -\int_0^t (S(t-\tau)(\underline{g}^2(\tau) + \beta\underline{\gamma}(\tau))_{ij} d\tau \tag{27}$$

and $\mathcal{R}$ is self-adjoint, we have that $0 \le -g_{ij}(t) \le m_0$.

Solution of equations (16), (17). Denote by $T_{16}(t_0, t)$ the fundamental matrix of the linear system

$$\dot{\eta}_{ij} = -k\eta_{ij} - g_{ij}(t)\eta_{ij} + c_1\mathcal{L}\eta_{ij}.$$

It follows from (27) then

$$\|T_{16}(t_0, t)\| \le a \cdot e^{(-k + m_0 - c_1 8 \sin^2 \frac{\pi}{2(2N+2)})(t - t_0)}$$

where a is some constant. Therefore, if

$$k + c_1 8 \sin^2 \frac{\pi}{2(2N+2)} > m_0 = \frac{2\beta\Gamma}{q|r| + \sqrt{q^2r^2 - 4\beta\Gamma}} \tag{28}$$

then the solution of (16) satisfies the inequality

$$\|\eta(t)\| \leq \|\eta(0)\| \cdot e^{(-k+m_0+c_1\lambda_0)t} + \frac{b}{|-k + c_1\lambda_0 + m_0|} \cdot \epsilon_1 \tag{29}$$

where $\epsilon_1 = \sup \|\underline{\epsilon}(t)\|$ and b is some constant.

Denote by $T_{17}(t_0, t)$ the fundamental matrix of the linear system

$$\dot{u}_{ij} = g_{ij}(t)u_{ij} + (Au)_{ij}.$$

By Proposition 1 and (27), we have

$$\|T_{17}(t, t_0)\| \leq a_1 e^{q\rho_0(t-t_0)} \tag{30}$$

where

$$0 > \rho_0 = \max_s \rho_s = \frac{1}{2}\left(-k + c_1\lambda_0 + \sqrt{(-k + c_1\lambda_0)^2 + 4(\alpha + c_2\lambda_0)}\right)$$

and a_1 is some constant. Therefore, we obtain from (17) and (29) the following estimate

$$\begin{aligned} \|u(t)\| \leq & e^{\rho_0 t}\|u(0)\| \\ & + \|\eta(0)\| \cdot a_1 |e^{(-k+m_0+c_1\lambda_0)t} - e^{q\rho_0 t}| \cdot \frac{1}{(-q\rho_0 + k - c\lambda_0 - m_0)} \\ & + \frac{a_1 b}{q|\rho_0||-k + c_1\lambda_0 + m_0|} \cdot \epsilon_1. \end{aligned} \tag{31}$$

Since $\epsilon_1 \leq \text{const} \cdot \epsilon$, (29) and (31) imply the validity of Theorem. The constant c_2^* can be found from (6):

$$c_2^* = \frac{\alpha}{8\sin^2 \frac{\pi}{2(2N+2)}} \tag{32}$$

and the constant c_1^* can be found from (24), (28):

$$c_1^* \leq \frac{2\sqrt{\beta\Gamma} - k}{8\sin^2 \frac{\pi}{2(2N+2)}} \tag{33}$$

4. Spectra of Some Operators and Proof of Main Lemma

Since the operator $\mathcal{L}$ is determined by the discrete Laplace operator Δ, we consider, first, the spectrum of Δ denoted by $\sigma(\Delta)$.

(i) For the case $p = 1$, under Dirichlet boundary conditions the operator Δ can be represented by the following $M \times M$ matrix

$$\begin{pmatrix} -2 & 1 & 0 & \cdots & \\ 1 & -2 & 1 & 0 & \\ \cdots & \cdots & \cdots & & \\ 0 & & 1 & -2 & 1 \\ 0 & & 0 & 1 & -2 \end{pmatrix}$$

where M is the number of points in the one-dimensional lattice. It is well-known (see, for example, [19]) that

$$\sigma(\Delta) = \left\{ \lambda_s = -2 - 2\cos\frac{\pi s}{M+1} \mid s = 1, \ldots, M \right\}.$$

Thus,

$$\max_s \lambda_s = -4\sin^2 \frac{\pi}{2(M+1)}. \tag{34}$$

Furthermore, this operator is self-adjoint and its eigenvectors are mutually orthogonal in the corresponding Euclidean space.

(ii) For the case $p = 2$, we will find the eigenvectors and eigenvalues of Δ by using separation of variables. This method was used in [12] for the case of periodic boundary conditions.

Let M_1, $M_2 > 1$ be fixed integers. We consider Δ acting on the $M_1 \times M_2$ lattice. Let

$$\Delta y_{i_1, i_2} = \lambda y_{i_1 i_2}, \qquad \text{where} \quad 1 \le i_1 \le M_1, \; 1 \le i_2 \le M_2 .$$

Setting $y_{i_1 i_2} = Y_{i_1} \cdot Y_{i_2}$, we have

$$Y_{i_2} \Delta_1 Y_{i_1} + Y_{i_1} \Delta_1 Y_2 = \lambda Y_{i_1} Y_{i_2} ,$$

where Δ_1 is the one-dimensional Laplace operator. If $\Delta_1 Y_{i_1} = \mu_1 Y_{i_1}$ and $\Delta_1 Y_{i_2} = \mu_2 Y_{i_2}$, then $\lambda = \mu_s + \mu_t$, where

$$\mu_1 = -2 - 2\cos\frac{\pi s}{M_1 + 1}, \qquad \mu_2 = -2 - 2\cos\frac{\pi t}{M_2 + 1}. \tag{35}$$

Let

$$\lambda_{s,t} = -4 - 2\left(\cos\frac{\pi s}{M_1+1} + \cos\frac{\pi t}{M_2+1}\right), \qquad s = 1,\dots,M_1, \quad t = 1,\dots,M_2\,.$$

It is clear that Δ is self-adjoint for $d = 2$, as well. If follows from (35) that

$$\max_{s,t} \lambda_{s,t} \le -8\sin^2\frac{\pi}{2(M+1)} \tag{36}$$

where $M = \max\{M_1, M_2\}$. Let $\mu_s Y^{(s)} = \Delta_1 Y^{(s)}$, $s = 1,\dots,M_1$, $\mu_t Y^{(t)} = \Delta_1 Y^{(t)}$, $t = 1,\dots,M_2$, and the eigenvectors $\{Y^{(s)}\}$, $(\{Y^{(t)}\})$ form an orthonormal basis in $\mathbb{R}^{M_1}$ $(\mathbb{R}^{M_2})$. Let $Z^{s,t}_{i_1 i_2} = Y^s_{i_1}\cdot Y^t_{i_2}$, $Z^{s,t} = \{Y^{s,t}_{i_1 i_2}\}$, $i_1 = 1,\dots,M_1$, $i_2 = 1,\dots,M_2$. We proved that $\Delta_2 Z^{s,t} = \lambda_{s,t} Z^{s,t}$. It remains to show that $M_1\cdot M_2$ vectors $Z^{s,t}$ form a basis in $\mathbb{R}^{M_1}\times\mathbb{R}^{M_2}$. It is simple to check that $Z^{s,t}\ne 0$. Let us check that they are linearly independent. It will be so if they are orthogonal. Show it. Fix two pairs (s,t) and (s_0,t_0). Then

$$\sum_{i_1=1}^{M_1}\sum_{i_2=1}^{M_2} Z^{s,t}_{i_1 i_2}\cdot Z^{s_0,t_0}_{i_1 i_2} = \sum_{i_1=1}^{M_1} Y^s_{i_1}\cdot Y^{s_0}_{i_1}\sum_{i_2=1}^{M_2} Y^t_{i_2}\cdot Y^{t_0}_{i_2} = 1$$

if $s = s_0$, $t = t_0$ and 0 otherwise. Thus, $Z^{s,t}$ form an orthonormal basis in the Euclidean space $\mathbb{R}^{M_1}\times\mathbb{R}^{M_2}$.

(iii) Let us now look at the operator $\mathcal{L}$:

$$\mathcal{L}v_{i_1 i_2 j_1 j_2} = v_{i_1-1 i_2 j_1-1 j_2} + v_{i_1+1 i_2 j_1+1, j_2} + v_{i_1 i_2-1 j_1 j_2-1} + v_{i_1 i_2+1 j_1 j_2+1} - 4v_{i_1 i_2 j_1 j_2}, \qquad i_k = 1,\dots,N, \quad k = 1,2,3,4,$$

with the boundary conditions described above: $v_{ij}\equiv 0$ if $(i,j)\notin J$. The simple observation shows that for $(i_1,j_1)\in J_{k_1}$, $(i_2,j_2)\in J_{k_2}$, $-N\le k_{1,2}\le N$, if we introduce the notations $w_{i_1 i_2 k_1 k_2} = v_{i_1 i_2 j_1 j_2}$ then

$$\mathcal{L}v_{i_1 i_2 j_2 j_2} = \Delta_2 w_{i_1 i_2 k_1 k_2} = w_{i_1-1 i_2 k_1 k_2} + w_{i_1+1 i_2 k_1 k_2} + w_{i_1 i_2+1 k_1 k_2} + w_{i_1 i_2-1 k_1 k_2} - 4w_{i_1 i_2 k_1 k_2}$$

where

(i) $0\le i_1\le N-k_1$ if $-N\le k_1\le 0$,

(ii) $-k_1\le i_1\le N$ if $0\le k_1\le N$,

(iii) $0\le i_2\le N-k_2$ if $-N\le k_2\le 0$,

(iv) $-k_2 \le i_2 \le N$ if $0 \le k_2 \le N$.

Therefore, the eigenvalues of $\mathcal{L}$ are:

$$-4-2\left(\cos\frac{s\pi}{N-k_1+2}+\cos\frac{t\pi}{N-k_2+2}\right),\quad s=1,\dots,N-k_1+1,$$
$$t=1,\dots,N-k_2+1,\ \text{in the case (i), (iii);}$$
$$-4-2\left(\cos\frac{s\pi}{N-k_1+2}+\cos\frac{t\pi}{N+k_2+2}\right),\quad s=1,\dots,N-k_1+1,$$
$$t=1,\dots,N+k_2+1,\ \text{in the case (i), (iv);}$$
$$-4-2\left(\cos\frac{s\pi}{N+k_1+2}+\cos\frac{t\pi}{N-k_2+2}\right),\quad s=1,\dots,N+k_1+1,$$
$$t=1,\dots,N-k_2+1,\ \text{in the case (ii), (iii);}$$
$$-4-2\left(\cos\frac{s\pi}{N+k_1+2}+\cos\frac{t\pi}{N+k_2+2}\right),\quad s=1,\dots,N+k_1+1,$$
$$t=1,\dots,N+k_2+1,\ \text{in the case (ii), (iv).}$$

In the "worse" case ($k_{1,2}=\pm N$)

$$\max\{\lambda \mid \lambda\in\sigma(\mathcal{L})\} = -4\left(1+\cos\frac{(1+N)\pi}{2N+2}\right) = -8\sin^2\frac{\pi}{2(2N+2)} \tag{37}$$

The support of the eigenvector, for example, corresponding to the eigenvalue

$$-4-2\left(\cos\frac{s\pi}{N-k_1+2}+\cos\frac{t\pi}{N-k_2+2}\right)$$

is a piece of the two dimensional discrete plane:

$$\{(i_1,i_2,j_1,j_2) \mid j_1-i_1=k_1, j_2-i_2=k_2\}$$

in the four-dimensional discrete set J. On this piece the operator $\mathcal{L}$ coincides with the Laplacian Δ with the Dirichlet boundary conditions. Thus, all eigenvectors with the support on this piece are mutually orthogonal. Two-dimensional planes corresponding to different pairs (k_1,k_2) do not intersect. So, the eigenvectors corresponding to different pairs (k_1,k_2) are mutually orthogonal. Therefore, all eigenvectors are mutually orthogonal and form a basis in $\mathbb{R}^{d^2}\times\mathbb{R}^{d^2}$. It follows from this that $\mathcal{L}$ is self-adjoint. (Though, it is possible to prove directly from the definition of $\mathcal{L}$.) This proves the Main Lemma. We denoted the spectrum $\sigma(\mathcal{L})$ by $\{\lambda_s\}$ and the corresponding eigenvectors by $\{h^s\}$.

5. Dissipativeness of the System

Denote by $\{\xi^s\}$ the eigenvectors of the discrete Laplace operator Δ for $d = 2$, and by $\{\lambda_s\}$ the corresponding eigenvalues, see (35). It was shown in Section 4 that $\xi^{s_1} \perp \xi^{s_2}$. We assume that $\{\xi^s\}$ are chosen so that they form an orthonormal basis in $\mathbb{R}^{N^2} \times \mathbb{R}^{N^2}$: $\sum_i \xi_i^{s_1} \xi_i^{s_2} = \delta_{s_1 s_2}$. Let $x = \{x_i\}$, $y = \{y_i\}$ be elements of $\mathbb{R}^{N^2} \times \mathbb{R}^{N^2}$ and $\|x\|$ be the Euclidean norm, and the scalar product $(x, y) = \sum_i x_i y_i$. Consider the following representation of x, y:

$$x_i = \sum_s X_s \xi_i^s, \qquad y_i = \sum_s Y_s \xi_i^s.$$

Then, the following formulas hold:

$$\sum_s X_s^2 = \|x\|^2, \qquad \sum_i x_i y_i = \sum_s X_s Y_s, \qquad \sum_s Y_s^2 = \|y\|^2 \tag{38}$$

$$\sum_i y_i \Delta y_i = \sum_s \lambda_s Y_s^2, \qquad \sum_i x_i \Delta x_i = \sum_s \lambda_s X_s^2 \tag{39}$$

$$\sum_i x_i \Delta y_i = \sum_s \lambda_s X_s Y_s, \qquad \sum_i y_i \Delta x_i = \sum_s \lambda_s X_s Y_s \tag{40}$$

Moreover, we have

$$\sum_i x_i y_i = \sum_i \left(\sum_{s_1} X_{s_1} \xi_i^{s_1} \right) \left(\sum_{s_2} Y_{s_2} \xi_i^{s_2} \right) = \sum_{s_1, s_2} \sum_i X_{s_1} Y_{s_2} \xi_i^{s_1} \xi_i^{s_2}$$
$$= \sum_{s_1, s_2} X_{s_1} Y_{s_2} \delta_{s_1 s_2} = \sum_s X_s Y_s.$$

$$\sum_i y_i \Delta y_i = \sum_i \left(\sum_{s_1} Y_{s_1} \xi_i^{s_1} \right) \sum_{s_2} Y_{s_2} \lambda_{s_2} \xi_i^{s_2} = \sum_{s_1, s_2} \lambda_{s_2} \sum_i Y_{s_1} \xi_i^{s_1} Y_{s_2} \xi_i^{s_2}$$
$$= \sum_{s_1, s_2} \lambda_{s_2} Y_{s_1} Y_{s_2} \delta_{s_1 s_2} = \sum_s \lambda_s Y_s^2.$$

Assume that the conditions of the Theorem are satisfied, i.e.,

$$c_1 > \frac{c_2}{q\underline{k}}.$$

Thus,

$$c_2 - \omega c_1 = 0 \tag{41}$$

for some positive ω such that

$$0 < \omega < q\underline{k} < k_i. \tag{42}$$

Consider now the function

$$U(x,y) = \sum_i \frac{y_i^2}{2} - \frac{\alpha_i x_i^2}{2} + \frac{\beta_i x_i^4}{4} + \omega x_i y_i\,.$$

We will show that $\frac{dU}{dt} < 0$ on the surface $U(x,y) = D$ if D is large enough. A direct calculation shows that

$$\begin{aligned}\frac{dU}{dt} = &\sum_i (\omega - k_i)y_i^2 + \omega\alpha_i x_i^2 - \omega\beta_i x_i^4 - \omega k_i x_i y_i + (y_i + \omega x_i)(a_i + b_i \cos\omega t) \\ &+ (y_i + \omega x_i)(c_1 \Delta y_i + c_2 \Delta x_i)\end{aligned} \tag{43}$$

It follows from (39) and (40) that

$$\sum_i (y_i + \omega x_i)(c_1 \Delta y_i + c_2 \Delta x_i) = \sum_s \lambda_s (c_1 Y_s^2 + \omega c_2 X_s^2 + (c_2 + c_1\omega) X_s Y_s) \tag{44}$$

By (41), the right hand part of (44) can be written as

$$\sum_s \lambda_s c_1 \left(Y_s + \frac{c_2 + c_1\omega}{2c_1} X_s \right)^2 = \sum_s \lambda_s c_1 (Y_s + \omega X_s)^2 \le 0$$

Therefore, it follows from (43) that

$$\begin{aligned}\frac{dU}{dt} &\le \sum_i (\omega - k_i)y_i^2 + \omega\alpha_i x_i^2 - \omega\beta_i x_i^4 - \omega k_i x_i y_i + (y_i + \omega x_i)(a_i + b_i \cos\omega t) \\ &\le -\sum_i (k_i - \omega)y_i^2 + \omega\beta_i x_i^4 - \omega\alpha_i x_i^2 + \omega k_i x_i y_i - (|a_i| + |b_i|)(|y_i| + \omega|x_i|) \\ &= V(x,y), \qquad \text{say.}\end{aligned}$$

We need to show that $V(x,y) < 0$ provided $U(x,y) = D$, $D \gg 1$. Set $\mu = D^{-1/4}$, $\eta_i = y_i D^{-1/2}$, $\xi_i = x_i D^{-1/4}$. Then the equality $U(x,y) = D$ can be rewritten as

$$\sum_i \frac{1}{2}\eta_i^2 + \frac{1}{4}\beta_i \xi_i^4 - \frac{1}{2}\mu^2 \alpha_i \xi_i^2 + \mu\omega\xi_i\eta_i = 1 \tag{45}$$

Proposition 4. *There exist $\mu_0 > 0$ and $C > 0$ such that for any $0 < \mu \le \mu_0$ and any solution (ξ,η) of (45), we have*

$$\|\xi\| \le C, \qquad \|\eta\| \le C\,.$$

Proof. Suppose not. Then there exist sequences $\{\mu_n\}$, $\{\xi^{(n)}\}$, $\{\eta^{(n)}\}$ such that $\mu_n \to 0$ and $\|\xi^{(n)}\| + \|\eta^{(n)}\| \to \infty$ as $n \to \infty$ and $(\xi^{(n)}, \eta^{(n)})$ is a solution of (45) for $\mu = \mu_n$.

One of the following holds:

(i) $\|\eta^{(n)}\| \leq \text{constant} \stackrel{\text{def}}{=} C$ for all n;

(ii) $\|\eta^{(n)}\| \to \infty$ as $n \to \infty$.

Suppose (i) is true. Then, $\|\xi^{(n)}\| \to \infty$ as $n \to \infty$. Let $p(n) = \max_i |\xi_i^{(n)}|$, $\underline{\beta} = \min_i \beta_i$, $\overline{\alpha} = \max_i \alpha_i$. Thus, $p(n) \to \infty$ as $n \to \infty$. Represent (45) as

$$\sum_i \frac{1}{4}\beta_i \xi_i^{(n)4} + \sum_i \frac{1}{2}\eta_i^{(n)2} = 1 + \sum_i \frac{1}{2}\mu_n^2 \alpha_i \xi_i^{(n)2} + \sum_i \mu_n \omega \xi_i^{(n)} \eta_i^{(n)} \tag{46}$$

It follows from (46) that

$$\frac{1}{4}\underline{\beta} p^4(n) + \frac{1}{2}\|\eta^{(n)}\|^2 < 1 + \frac{1}{2}\mu_n^2 \overline{\alpha} \|\xi^{(n)}\|^2 + \mu_n \omega \|\xi^{(n)}\| \cdot \|\eta^{(n)}\| \tag{47}$$

Dividing (47) by $p^4(n)$ and taking into account the inequality $\|\xi^{(n)}\| \leq N^2 p(n)$, we obtain

$$\frac{1}{4}\underline{\beta} + \frac{1}{2p^4(n)}\|\eta^{(n)}\|^2 < \frac{1}{p^4(n)} + \frac{N^4}{2p^2(n)}\mu_n^2 \overline{\alpha} + \frac{\mu_n N^2}{p^3(n)} \cdot C \tag{48}$$

The inequality (48) for sufficiently large n contradicts (i).

Consider now the case (ii). Dividing (46) by $\|\eta^{(n)}\|^2$ and using notations above, we have

$$\frac{1}{2} + \frac{1}{4}\underline{\beta}\frac{p^4(n)}{\|\eta^{(n)}\|^2} < \frac{1}{\|\eta^{(n)}\|^2} + \frac{1}{2}\mu_n^2 \overline{\alpha} \cdot \frac{N^4 p^2(n)}{\|\eta^{(n)}\|^2} + \mu_n \omega \frac{N^2 p(n)}{\|\eta^{(n)}\|} \tag{49}$$

If $p(n)$ is a bounded sequence, then (49) cannot be satisfied for n large enough, and we have a contradiction. Assume now that $p(n) \to \infty$ as $n \to \infty$. Then denoting by ζ_n the ratio $\frac{p(n)}{\|\eta^{(n)}\|}$, we obtain from (49)

$$\frac{1}{2} - \frac{1}{\|\eta^{(n)}\|^2} - \mu_n \omega N^2 \zeta_n + \left(\frac{1}{4}\underline{\beta} p^2(n) - \frac{1}{2}\mu_n^2 \overline{\alpha} N^4\right)\zeta_n^2 < 0 \tag{50}$$

If n is large, then the discriminant of the quadratic polynomial (with respect to ζ_n) on the left hand side of (50) is negative and the leading coefficients are positive. Thus, (50) cannot be satisfied. □

Corollary. *There exists* μ_1, $0 < \mu_1 \leq \mu_0$ *and a constant* $c_0 > 0$ *such that for any* $0 < \mu \leq \mu_1$ *and any solution* (ξ, η) *of the equation* (45) *the following inequality holds*

$$\sum_{i=1}^{\infty} \eta_i^2 + \xi_i^4 > c_0 \tag{51}$$

Proof. Let $\widetilde{\beta} = \min\left\{\frac{1}{2}, \min_i \frac{1}{4}\beta_i\right\}$. Then it follows from (45) and Proposition 4 that for $\mu < \mu_0$

$$\sum_i \eta_i^2 + \xi_i^4 \geq \frac{1}{\widetilde{\beta}}(1 - \mu\omega C^2).$$

Setting $\mu_1 \leq 1/2\omega C^2$ we obtain the desired result with $c_0 \geq 1/2\widetilde{\beta}$. □

It remains to show that $V(x, y) < 0$. We have,

$$\begin{aligned} V(x,y) &\leq -\left[(\underline{k} - \omega)\sum_i y_i^2 + \omega\underline{\beta}\sum_i x_i^4 - \omega\overline{\alpha}\|x\|^2 \right. \\ &\quad \left. + \omega\sum_i k_i x_i y_i - |a_i| - |b_i|(|y_i| + \omega|x_i|)\right] \\ &= -D\left[(\underline{k} - \omega)\|\eta\|^2 + \omega\underline{\beta}\sum_i \xi_i^4 + \mu F(\xi, \eta, \mu)\right] \end{aligned}$$

where F is a bounded function (as $\mu \searrow 0$). Therefore

$$V(x,y) \leq -D\left[\min\{(\underline{k} - \omega), \omega\underline{\beta}\}c_0 + \mu F(\xi, \eta, \mu)\right] < 0.$$

So, we prove the dissipativeness of the system and the inequality (2).

6. Concluding Remarks

1. Our method works for any nonlinearity $f(x)$ satisfying the following condition

$$f(x_1) - f(x_2) = (x_1 - x_2)g(x_1, x_2)$$

   where $g(x_1, x_2) > 0$ for $x_1 \neq x_2$.

2. In the work of T. L. Carrol and L. M. Pecora (see [3, 4, 11] and bibliography in [11]), one approach for stochastic synchronization was suggested. It is based on the

following two points: (i) in the simplest case, all subsystems are identical and coupling admits the existence of spatially homogeneous solutions; (ii) stability of solutions is based on the "transversal" Lyapunov exponent technique (see also [16, 17]). We note that for (i), the synchronization phenomenon becomes the stability property of spatially-homogeneous solutions. For non-identical subsystems, synchronization means stability of some invariant submanifolds. In case (ii), one can give only "local" synchronization. In general, synchronization means that we have the realization of a synchronous regime for a wide region of initial conditions. It seems that the approach of these authors is probably only an interstitial step in the study of synchronization phenomena.

3. As a criterion of synchronization, it is suggested in [11] to find negative transversal Lyapunov exponents. But it is known from the theory of stability in ordinary differential equations (see, for example, [22]) that an invariant set may be destroyed under small perturbations if negative Lyapunov exponents in transversal directions are larger than negative Lyapunov exponents in the tangent directions of the invariant set. This indicates that "structurally stable" synchronization comes when the transversal Lyapunov exponents turn out to be smaller than the tangent ones.
4. It is shown in [13–18] and other works that chains (or lattices) of coupled oscillators may exhibit a complex spatio-temporal behavior. In particular, they may have attractors of different dimensions. It seems to us, the dimension of attractors depends directly on the number of excited modes. We hope to study this elsewhere.

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PART IV -- FINAL PROJECT REPORT -- SUMMARY DATA ON PROJECT PERSONNEL

(To be submitted to cognizant Program Officer upon completion of project)

The data requested below are important for the development of a statistical profile on the personnel supported by Federal grants. The information on this part is solicited in response to Public Law 99-383 and 42 USC 1885C. All information provided will be treated as confidential and will be safeguarded in accordance with the provisions of the Privacy Act of 1974. You should submit a single copy of this part with each final project report. However, submission of the requested information is not mandatory and is not a precondition of future award(s). Check the "Decline to Provide Information" box below if you do not wish to provide the information.

Please enter the numbers of individuals supported under this grant.
Do not enter information for individuals working less than 40 hours in any calendar year.

	Senior Staff		Post-Doctorals		Graduate Students		Under-Graduates		Other Participants[1]	
	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.	Male	Fem.
A. Total, U.S. Citizens	2									
B. Total, Permanent Residents										
U.S. Citizens or Permanent Residents[2]: American Indian or Alaskan Native										
Asian..........................										
Black, Not of Hispanic Origin.........										
Hispanic........................										
Pacific Islander..................										
White, Not of Hispanic Origin........										
C. Total, Other Non-U.S. Citizens					3					
Specify Country 1.										
2.										
3.										
D. Total, All participants (A + B + C)	2				3					
Disabled[3]										

☐ Decline to Provide Information: Check box if you do not wish to provide this information (you are still required to return this page along with Parts I-III).

[1] Category includes, for example, college and precollege teachers, conference and workshop participants.

[2] Use the category that best describes the ethnic/racial status for all U.S. Citizens and Non-citizens with Permanent Residency. *(If more than one category applies, use the one category that most closely reflects the person's recognition in the community.)*

[3] A person having a physical or mental impairment that substantially limits one or more major life activities; who has a record of such impairment; or who is regarded as having such impairment. *(Disabled individuals also should be counted under the appropriate ethnic/racial group unless they are classified as "Other Non-U.S. Citizens.")*

AMERICAN INDIAN OR ALASKAN NATIVE: A person having origins in any of the original peoples of North America and who maintains cultural identification through tribal affiliation or community recognition.

ASIAN: A person having origins in any of the original peoples of East Asia, Southeast Asia or the Indian subcontinent. This area includes, for example, China, India, Indonesia, Japan, Korea and Vietnam.

BLACK, NOT OF HISPANIC ORIGIN: A person having origins in any of the black racial groups of Africa.

HISPANIC: A person of Mexican, Puerto Rican, Cuban, Central or South American or other Spanish culture or origin, regardless of race.

PACIFIC ISLANDER: A person having origins in any of the orignal peoples of Hawaii; the U.S. Pacific territories of Guam, American Samoa, and the Northern Marinas; the U.S. Trust Territory of Palau; the islands of Micronesia and Melanesia; or the Philippines.

WHITE, NOT OF HISPANIC ORIGIN: A person having origins in any of the original peoples of Europe, North Africa, or the Middle East.

PART III - TECHNICAL INFORMATION

1. Attractors in inhomogeneous conservation laws and parabolic regularizations. Trans. Am. Math. Soc 347(1995), 1239-1254.
2. Limits of semigroups depending on a parameter. Resenhas 1(1993), 1-45.
3. Attractors and dynamics in partial differential equations. Nato-MSI Lectures, Cambridge, 1995.
4. (with Fusco and Xun) Traveling waves as limits of solutions of bounded domains. SIAM J. Math. Anal. 27(1996), 1554-1558.
5. (with Huang) Hopf bifurcation analysis for hybrid systems. In Differential Equations and Applications to Biology and Industry (Eds. Martell et al) World Scientific (1996), 157-172.
6. Coupled oscillators on a circle. Resenhas 1(1994), 441-457.
7. (with Huang) Periodic solutions of singularly perturbed delay equations. ZAMP 47(1996), 57-88.
8. Effects of delays on dynamics. In Topological Methods in Differential Equations and Inclusions (EDS. Granas and Frigon), Kluwer(1995), 191-238.
9. (with Afraimovich and Chow) Synchronization in lattices of coupled oscillatiors. Physcia D 103(1997), 442-451.
10. Diffusive coupline, dissipation and synchronization. J. Dyn. Diff. Eqns. 9(1997),1-52.
11. Dynamics of a scalar parabolic equation. Canadian Appl. Math. Quart. 5(3)(1997), 209-305.
12. 'Long-Time Behavior as A-0 of Parabolic Equations depending on At', by Anatoli Babin and Shui-Nee Chow.
13. 'Two definitions of Exponential Dichotomy for Skew-Product Semiflow in Banach Spaces' by Shui-Nee Chow and Hugo Leiva.
14. 'Floquet Bundles for Scalar Parabolic Equations' by Shui-Nee Chow, Kening Lu, and John Mallet-Paret.