A Combinatorial Polynomial Algorithm for Weighted Abstract Cut Packing

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Ga Tech, 22 March 2012

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- Begen and Queyranne (2011) use L-convexity to schedule stochastic appointments for, e.g., surgeries.

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   - Packing problems

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- This generic problem has many applications, e.g., flow is packing paths into arcs, connectivity is packing trees into edges, etc.

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\text{s.t.} \sum_{D \ni e} y_D \le u_e \quad \forall e \in E \qquad\qquad \text{s.t.} \sum_{e \in D} x_e \ge r_D \quad \forall D \in \mathcal{D}
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Big Question: When do these LPs have guaranteed integer optimal solutions?

An example packing LP

- Consider:

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$$
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1 & & 1 & & & 1 & & & \\
1 & 1 & & & 1 & & & & \\
 & 1 & 1 & & 1 & 1 & & & \\
 & 1 & & 1 & & & & 1 & \\
 & & 1 & 1 & 1 & 1 & & 1 & \\
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- Does this LP have an integer optimal solution?
- What if we change the RHS u? The objective r?

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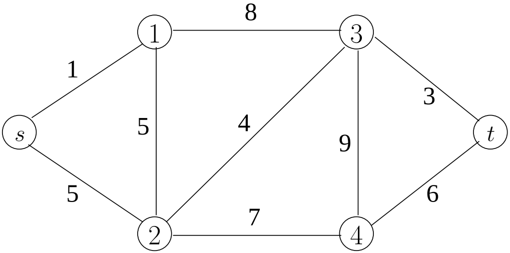
- How do I know that the first two objectives are "good" for all RHS?

How the example was constructed

- Consider the following graph:



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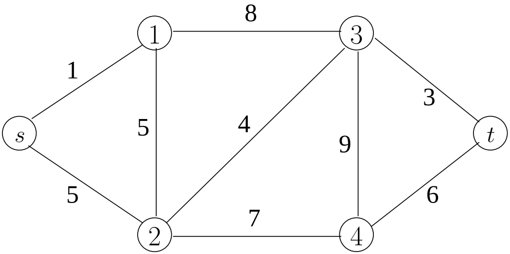

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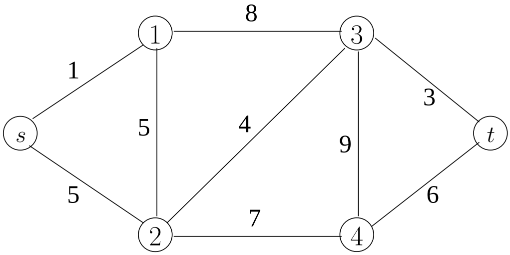

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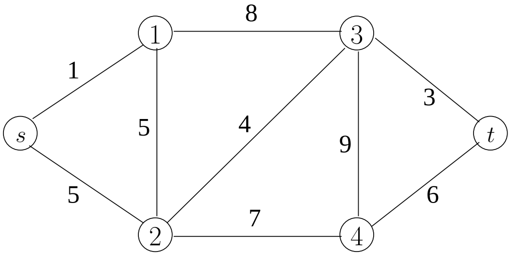

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- Why does this lead to integer optimal LP solutions?

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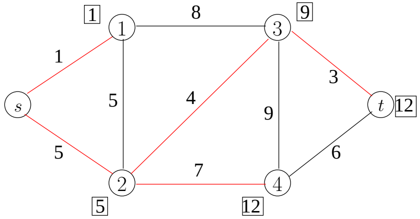
- Therefore the primal LP is just Shortest Path.
- And in fact Dijkstra's Algorithm gives an integer optimal solution to this form of Shortest Path.

Going back to the dual packing LP

- Here is the Dijkstra solution with its shortest path tree:



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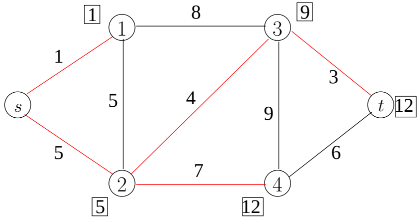
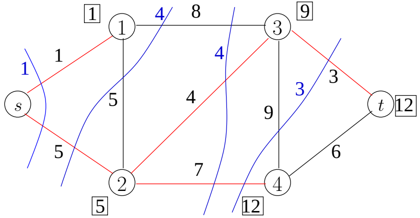


- Recall that we can greedily construct a tight cut packing that proves that this shortest path tree is optimal:



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- Hoffman did it . . .

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  - each $D \in \mathcal{L}$ has a per unit reward r_D (the *weight* of D)
- r satisfies a kind of supermodularity:

$$r_{D_i \wedge D_j} + r_{D_i \vee D_j} \geq r_{D_i} + r_{D_j}.$$

Understanding the lattice axioms

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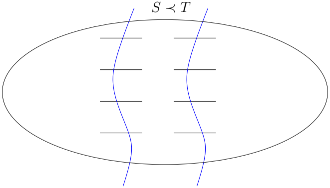

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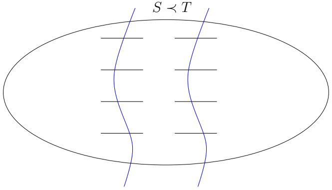

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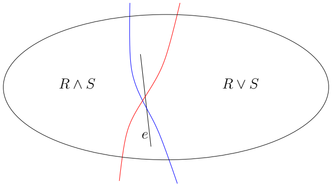

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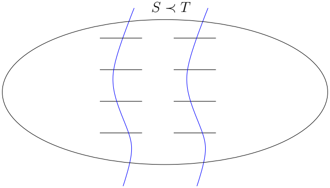

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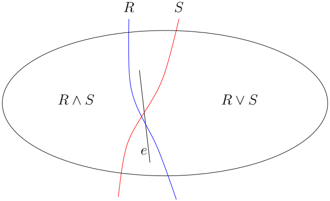
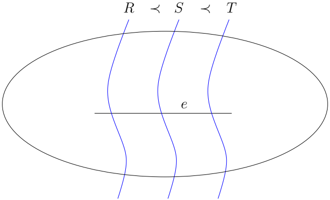


Ordinary cuts are consecutive ($e \in R \cap T \implies e \in S$):



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If $\mathcal{L}$ is just s–t cuts in a max flow network, and $r \equiv 1$, then this is just the usual blocking dual formulation of Dijkstra shortest path.

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Theorem (Hoffman & Schwartz '76)

When r and u are integral, (P) and (D) have integral optimal solutions.

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- Etc, etc ...

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Theorem (Hoffman '78)

If $\mathcal{L}$ is a submodular clutter, then the blocker of $\mathcal{L}$ is an abstract path system.

Outline

1. Combinatorial Optimization
   - Packing problems
2. Hoffman's Models
   - Lattice Polyhedra
   - Blocking
3. Algorithms
   - Primal-Dual Algorithm
   - P-D for WACP
4. Conclusion
   - Open questions

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  - There was a previous algorithm for the case where r is monotone (i.e., $D \preceq Q \implies r_D \leq r_Q$) by Frank, but this does not cover important applications such as polymatroid intersection.

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- Each iteration maintains that x and π are optimal for current flow value, so when x becomes a max flow, it is optimal.

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- But otherwise, the advantage of P-D is that it replaces the complicated objective $r^T y$ with a simple objective $\mathbb{1}^T y$.
- Due to R, the solution to the restricted dual could have -1 values in it, so the dual update need not be monotone.

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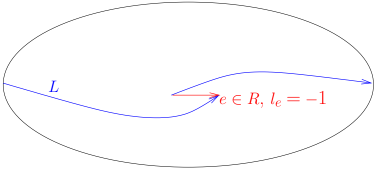

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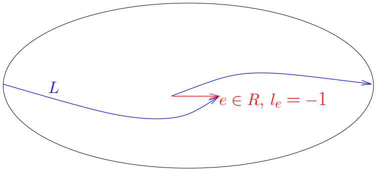

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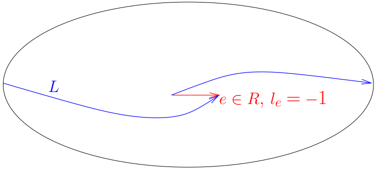

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4. If y and l are not complementary slack (i.e., if l is not tight on y's chain), update y along path l and return to Step 1.

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- This process is monotone, and so terminates in strongly polynomial time with CS solutions.

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- Corollary: new x and y are optimal for the new λ.

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- Theorem: This algorithm solves Weighted Abstract Cut Packing in $O((m \log C + m^2 \log r_{\max}(m + \mathrm{CO}))(m \cdot \mathrm{CO} + h(m + h)))$ (weakly polynomial) time ("CO" is # oracle calls; h is height of lattice, C is max u_e).

Outline

1. Combinatorial Optimization
   - Packing problems
2. Hoffman's Models
   - Lattice Polyhedra
   - Blocking
3. Algorithms
   - Primal-Dual Algorithm
   - P-D for WACP
4. Conclusion
   - Open questions

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Comments?