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ALTERNATIVE PROCESS MODELS FOR THE

ECONOMIC DESIGN OF T^2 CONTROL CHARTS

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SUMMARY

The purpose of this thesis is to report the development of three cost models for the T^2 control chart for multiple quality characteristics. One of the models is for processes that possess the Markov property, i.e. the time in the in-control state follows the geometric distribution. The other two models are for processes that do not possess the Markov property and assume Poisson and logarithm series distribution for the time in the in-control state. A general method for choosing the sample size and critical region parameter for the T^2 chart is presented.

A grid search numerical procedure is developed to determine the optimal sample size, critical region parameter, and the average time cost. Plots are given of the cost functions for various values of the cost coefficients. Also 162 numerical examples are presented and analyzed. It is concluded that the Markov assumption is important in the design of a multivariate quality control model.

CHAPTER I

INTRODUCTION

This chapter gives an overview of quality control procedure for multiple quality characteristics, the purpose and scope of this research, and a survey of the pertinent literature. The development of the various cost models and the solution procedures employed will be discussed in later chapters.

Multiple Quality Characteristics

Statistical testing procedures are used to control the quality of products produced by many types of industrial processes. A product is considered to be of acceptable quality if some measurable characteristics of the product fall within prescribed limits. Otherwise, the product is considered to be unacceptable or defective. The function of quality control procedure is to determine if a process is in or out of control.

Many industrial processes are characterized by two or more quality characteristics, and the joint effect of these characteristics describes product quality. For example, in the production of steel bars, the strength and the diameter are both important quality characteristics. They are also jointly distributed random variables. The use of univariate methods in an attempt to control a process like the one mentioned above would result in substantial error from the independent testing and control of each variable. Jackson (11), Montgomery and

Klatt (19), and other authors have shown how the control procedure can be distorted if one uses a univariate quality control procedure on a process with more than one quality characteristic. Hence control procedures for these kind of processes must be based on multivariate statistical techniques.

Purpose and Scope of the Thesis

Considerable research in quality control methodology has been directed towards improving techniques involving only a single quality characteristic. This research often involves the development of a cost model relating the test parameter to some measure of effectiveness. In most of the univariate cost models developed, the usual test parameters explicated are the sample size, the time interval between samples and the statistics defining the critical region. There are two limitations in these models; first, the quality characteristics used in the control procedure must be univariate in nature, and second, the population variance is assumed known. A very basic assumption made in these models is the Markovian nature of the process shift from the in-control to the out-of-control state. That is, given that the system remains in control at a certain point in time, the probability of its deterioration by some future time is independent of the past history of the process. For discrete time, this implies that the duration in the in-control state follows the geometric distribution.

The objective of this research is to develop alternative quality control models of a process characterized by at least two quality characteristics. It is assumed that these quality characteristics are continuous random variables. Furthermore these models should not require prior knowledge of the population covariance structure. The time in the in-control state is assumed to follow the geometric distribution, Poisson distribution, and the logarithmic series or "logseries" distribution. These latter two distributions are considered since for some processes, the basic assumption of the Markovian nature of the process shift from the in-control to out-of-control state may not be valid. For example, in the rolling of steel bolts, a common process shift occurs as a result of heat due to friction in the continual production of the bolts. When a false alarm induces a search for an assignable cause, the machine has an opportunity to cool down. The occurrence of the true shift is thus postponed. As a consequence it might be more realistic to assume that the distribution of the duration in control pertains only to the interval since the previous search. The reason for choosing the Poisson and the logseries distribution is mainly for convenience. The primary multivariate quality control technique is assumed to be the Hotelling T^2 control chart, and the economic selection of the test parameter (sample size and critical region) will be considered. It is assumed that the optimal sample size and optimal critical region will yield the minimum total cost of operating a multivariate quality control system.

Survey of the Literature

The importance of multivariate statistical procedures was recognized by Hotelling (9) as early as 1931. Generalization of Student's t distribution provided Hotelling with a basis for development of a unique statistical procedure. This procedure could be utilized in situations involving more than one variable, and in addition, those instances in which the variance must be estimated from a sample. Hotelling showed that if N observations from a multivariate normal population with mean vector $\underline{u}$ and covariance matrix $\underline{V}$ have been recorded, and the covariance matrix is estimated from the sample by the Statistic $\underline{S}$, then the quadratic form

$$T^2 = N(\underline{\bar{x}}-\underline{u}_0)'\underline{S}^{-1}(\underline{\bar{x}}-\underline{u}_0)$$

has the probability density function

$$f(T^2) = \frac{2\Gamma(N/2)T^{p-1}}{\Gamma(p/2)\Gamma[(N-p)/2](N-1)[1+T^2/(N-1)]}$$

This density function is called the Hotelling T^2 distribution with p and N-1 degrees of freedom, and is usually denoted by $T^2_{p,N-1}$. Hotelling's 1931 offering has more recently been discussed by Hicks (8) and Jackson (11,12), and Kramer and Jensen (15-18). These authors have unanimously endorsed the importance of Hotelling's technique and have demonstrated the utility of these techniques in practical situations.

Several authors have developed cost models for univariate statistical control procedures. Cowden (4) developed a cost model to study the economic design of tests for the mean of a process. The form of his model is

$$C = C_0 + C_1 + C_2,$$

where C_0, C_1, and C_2 represent the operating cost associated with the test procedure, the engineering cost of investigating a process when there has been an apparent shift in the mean, and the cost incurred when the process is not in control and defective items are produced. The assumptions made in Cowden's model are:

1. The process is considered to be out-of-control at the beginning of each day.

2. Once the assignable cause of equality variation is detected, it is corrected quickly and no further trouble can occur that day.

3. The cost of looking for the assignable cause is proportional to the shift in the mean.

4. The probability of finding the assignable cause is a function of the cost of looking for trouble.

Because of the unrealistic nature of some of these assumptions, the value of the model is minimized.

Duncan (5) has developed a cost model of the univariate $\bar{X}$ chart that maximizes the long run average net income per unit of time for the process. His model is of the form

$$C = C_0 + C_1 + C_2,$$

where C_0 is the average cost per hour for operating the quality control procedure, C_1 is the average cost per hour of looking for the assignable cause and C_2 is the average cost per hour of producing defectives.

Duncan assumes that the process may shift from the in-control state to a single out-of-control state any time during the day. He further assumed that the time the process remains in the in-control state before going out of control is an exponential random variable with mean $1/\lambda$ hours.

Knappenberger and Grandage (14) developed a very comprehensive cost model for the univariate quality control. Their general cost model is of the form

$$E(C) = E(C_0) + E(C_1) + E(C_2)$$

where $E(C_0)$ is the expected cost per unit associated with sampling and testing procedures, $E(C_1)$ is the expected cost per unit associated with rejecting the null hypothesis of statistical control, and $E(C_2)$ is the expected cost per unit associated with the production of defective products.

In this model, it is assumed that the process may have more than two quality states, and the process parameter is a continuous random variable which can be satisfactorily approximated by a discrete random variable. The cost of investigating and correcting a process which is out of control is assumed to be a random variable. When the process goes out of control, the model assumes that it remains in the out-of-control state until detected. However, the process can shift farther out of control before being detected. The only limitation in

Knappenberger and Grandage's work is the complex method required for determining the various probabilities involved.

Baker (3) developed two cost models of an $\bar{X}$ chart assuming a discrete time process in which the output quality characteristic of interest is measurable on a continuous scale. His models are of the general form

$$ATC = a_1 + a_2 + a_3$$

where a_1 is the cost of taking an individual sample from the process, a_2 is the cost of shutting down the process and searching for an assignable cause, a_3 is the cost of operating out of control for one period, and ATC is the average time cost, i.e. the expected rate at which the three components, (a_1,a_2,a_3) of costs are incurred.

The assumption made in both of Baker's models is that the shift out of control occurs at the beginning of a period and the process remains out of control for at least one full period before the sampling procedure can indicate that corrective action should be taken. That is, conceptually, sampling is done at the end of each period. In one of Baker's models, he assumes that the duration of the process is not affected by the occurrence of false alarms. In the other model, Baker assumes that the time in control is dependent of the number of false alarms which occurred. Baker's results indicated that the inappropriate use of the geometric distribution may be significantly misleading in certain cost cases.

Montgomery and Klatt (19) have recently developed a cost model capable of dealing with multiple quality characteristics. Their model is a multivariate analog of several well-known models for the univariate $\bar{X}$ chart. The Hotelling T^2 control chart was used in their model and they further assume that only one assignable cause of variation exists, that monitoring will be carried out by taking successive samples of constant size at fixed sample intervals and that corrective action is taken when one sample produces a value of the test statistic which falls outside the control limits.

In the next chapter, the development of the various models used in this research will be presented. In Chapter III, the solution procedure will be discussed, the analysis of results will be presented in Chapter IV, followed by the conclusions and recommendations in Chapter V.

CHAPTER II

DEVELOPMENT OF THE MODELS

In this chapter, the development of the three cost models used in this research will be presented as well as a brief discussion of the Hotelling T^2 control chart.

The T^2 Control Chart

Suppose that the output of a process is described by p quality characteristics, and that $\underline{X}$ is a (p×1) random vector whose *jth* element is the *jth* quality characteristic (a continuous, measurable variable). We assume that $\underline{X}$ is distributed according to the p-variate normal, i.e.

$$f(\underline{x}) = 1/[(2\pi)^{p/2}|\underline{\Sigma}|^{\frac{1}{2}}] \exp\{-\tfrac{1}{2}(\underline{x}-\underline{\mu})'\underline{\Sigma}^{-1}(\underline{x}-\underline{\mu})\}, \tag{1}$$

where $E(X) = \underline{\mu}$ is the (p×1) mean vector of the quality characteristics, $Cov(\underline{X}) = \underline{\Sigma}$ is the (p×p) covariance matrix of $\underline{X}$, and the prime (') denotes the transpose operation. In general, $\underline{\mu}$ and $\underline{\Sigma}$ are unknown.

The control procedure for $\underline{X}$ is due to Hotelling (10) and Jackson (11,12). From a random sample of size n from $\underline{X}$, say $\underline{X}_1, \underline{X}_2, \ldots, \underline{X}_n$, we compute the sample mean vector and sample covariance matrix as

$$\bar{\underline{X}} = \frac{1}{n} \sum_{i=1}^{n} \underline{X}_i \tag{2}$$

$$\underline{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\underline{X}_i - \bar{\underline{X}})(\underline{X}_i - \bar{\underline{X}})' . \tag{3}$$

Let $\underline{\mu}_0$ denote the value of $\underline{\mu}$ corresponding to the in-control state. Thus if $\underline{\mu}_0 = \underline{\mu}$, the statistic

$$T^2 = n(\bar{\underline{X}} - \underline{\mu}_0)' \underline{S}^{-1} (\bar{\underline{X}} - \underline{\mu}_0) \tag{4}$$

is distributed as Hotelling's T^2 with p and n-p degrees of freedom, denoted by $T^2_{p,n-p}$. The statistic T^2 in quation (4) forms the boundary of an ellipsoid in the p-dimensional space of $\underline{\mu}_0$ with center at $\bar{\underline{X}}$ as shown in Figure 1 for the case p = 2.

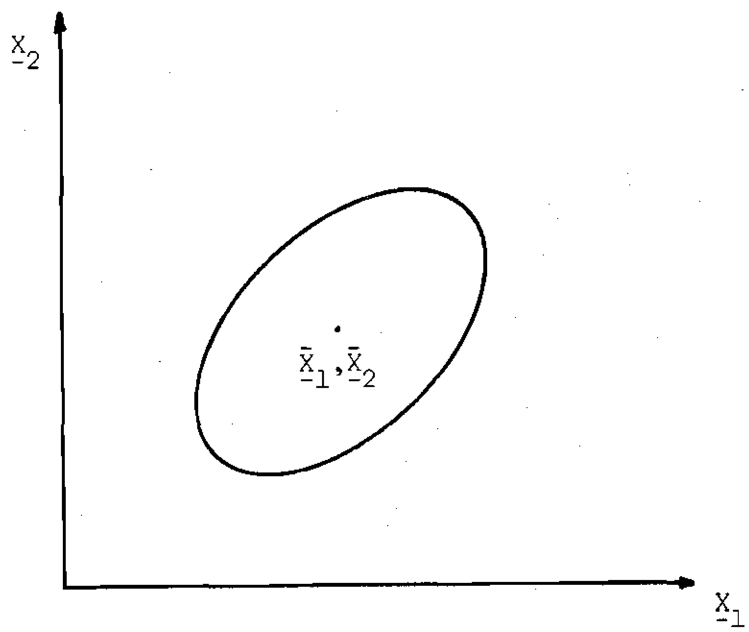

Figure 1. Two-Dimensional Ellipsoid Formed by the Statistic T^2

Define $T^2_{\alpha,p,n-p}$ as the upper α percentage point of Hotelling's T^2 distribution such that $\Pr\{T^2>T^2_{\alpha,p,n-p}\} = \alpha$. Then if $T^2 > T^2_{\alpha,p,n-p}$ we would conclude that $\underline{\mu} \neq \underline{\mu}_0$, i.e., the process is out of control. The distribution of the random variable $\{(n-p)/(n-1)p\}T^2$ is noncentral F with p and n-p degrees of freedom and noncentrality parameter τ, where

$$\tau = n(\underline{\mu}-\underline{\mu}_0)'\underline{\Sigma}^{-1}(\underline{\mu}-\underline{\mu}_0). \tag{5}$$

If $\underline{\mu} = \underline{\mu}_0$, then $\tau = 0$ and the F distribution is said to be central. Hence

$$T^2_{\alpha,p,n-p} = \frac{(n-1)p}{(n-p)} F_{\alpha,p,n-p} \tag{6}$$

Therefore, the percentage point of Hotelling's T^2 distribution can be found from tables of the cumulative F distribution (2).

The T^2 control chart has only an upper control limit of $T^2_{\alpha,p,n-p}$. Samples of size n are taken periodically, the quantity T^2 computed according to Equation (4), and T^2 plotted in a time-oriented sequence on the chart. It is only necessary to compute $\underline{S}^{-1}$ only (usually from preliminary data taken when the process is in control), as this control procedure is for means only. If $T^2 > T^2_{\alpha,p,n-p}$, we conclude that the process is out of control, and appropriate action is taken. Jackson (11) discusses how to determine which quality characteristics (component of $\underline{\mu}$) have assumed out-of-control values. Figure 2 shows an example of a T^2 control chart.

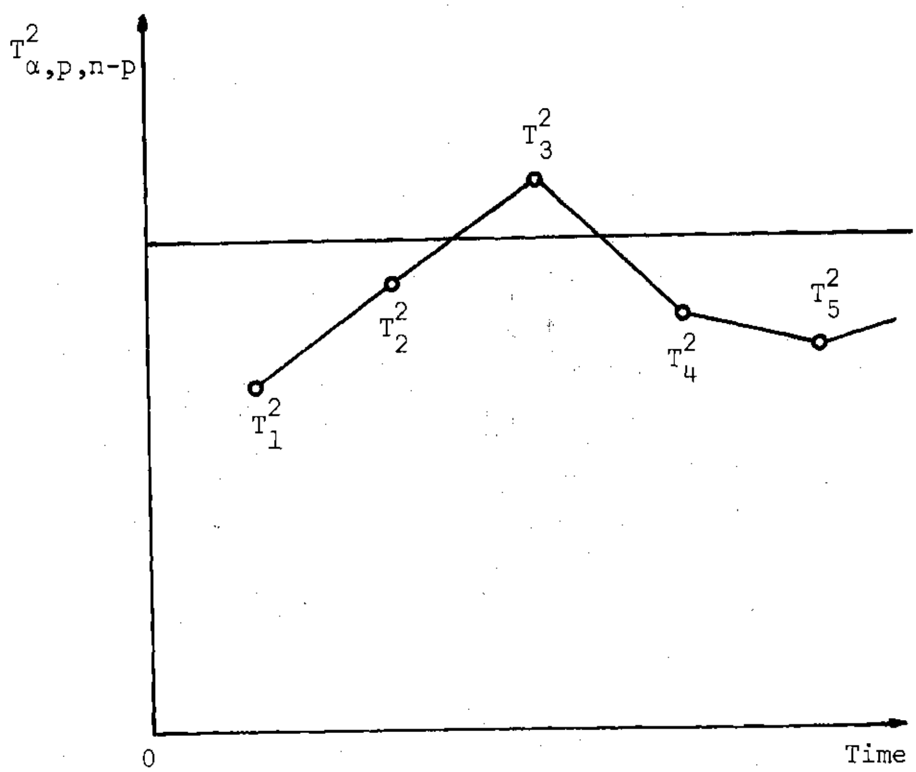

Figure 2. T^2 Control Chart

In this example, $T_3^2 > T^2_{\alpha,p,n-p}$, from which we conclude that $\underline{\mu} \neq \underline{\mu}_0$, i.e. the process mean vector is different from the desired standard.

If the population covariance matrix $\underline{\Sigma}$ were known with certainty, then $\underline{S}^{-1}$ in (4) can be replaced with $\underline{\Sigma}^{-1}$, and the statistic T^2 would be distributed as χ^2 with p degrees of freedom. Under these conditions, a control procedure could be based on the χ^2 distribution. Only rarely, however, could this be done, as knowledge of $\underline{\Sigma}$ would be unusual.

Economic Models of the T^2 Control Chart

Consider a discrete time process in which the output quality characteristics of interest are measurable on a continuous scale (e.g. length, weight, etc.). The system is assumed to remain in control for T periods, where T is a random variable with a discrete probability distribution, say

$$\Pr\{T=t\} = p(t) \qquad t=0,1,2,\ldots,$$

the process is assumed to shift out of control at the start of a period and sampling and plotting is done at the end of a period. That is, if a shift occurs, defective items will be produced for at least one period.

With the use of a T^2 control chart, the test statistic employed is given by Equation (4). For a single assignable cause model, $\underline{\mu}_0$ represents the in-control state and $\underline{\mu}_0 + \underline{\delta} = \underline{\mu}_1$ represents the out-of-control state, where $\underline{\delta}$ is the magnitude of the shift and is assumed known.

In any period, the probability α that a false alarm will occur, i.e. a type 1 error, is

$$\begin{aligned} \alpha &= \Pr\{\text{Type 1 error}\} \\ &= \Pr\{\text{False alarm}\} \\ &= \Pr\{T^2 > T^2_{\alpha,p,n-p} \mid \underline{\mu} = \underline{\mu}_0\} \\ &= \int_{T^2_{\alpha,p,n-p}}^{\infty} f(T^2)\,dT^2 \end{aligned} \tag{7}$$

and the probability $1-\beta$ of detecting an out-of-control state is

$$1 - \beta = \Pr\{\text{Detect out-of-control state}\}$$

$$= \int_{T^2_{\alpha,p,n-p}}^{\infty} f(T^2,\tau)dT^2 \tag{8}$$

where $\tau = n\pi$ with $\pi = \underline{\delta}'\underline{S}^{-1}\underline{\delta}$. Note that $1-\beta$ is just the power of the test.

The cost model used in this research is comprised of the following three components:

a_1 = Variable cost of sampling in the process. This is taken to be proportional to sample size.

a_2 = Cost of searching for assignable causes. This may be either real or false alarms, and reflects the shutdown and startup costs associated with the interruption, or the labor, overhead, and opportunity costs associated with testing and adjusting some facet of production.

a_3 = Cost of operating out of control for one period. This may reflect such factors as added corrections resulting from subsequent inspection, waste of input material, delay cost due to later disassemblies, customer dissatisfaction, etc.

The measure of effectiveness which we shall employ is the average time cost of the inspection and control procedure. This is the expected cost as which the three components of cost are incurred.

The Geometric (Markov) Model

In this model, an assumption made is that the duration of the process in control is not affected by the occurrence of false alarms. For discrete time process, this implies that the time in control follows the geometric distribution.

Let the random variable O be the run length out of control, and

T_i = number of periods the process is in control following the (i-1)*th* repair until the *ith* shift, and

O_i = number of periods process remains out of control from the *ith* shift until detected.

The system may then be characterized over a sequence of alternate intervals $\{T_1, O_1, T_2, O_2, ...\}$ and Figure 3 shows the behavior of the model.

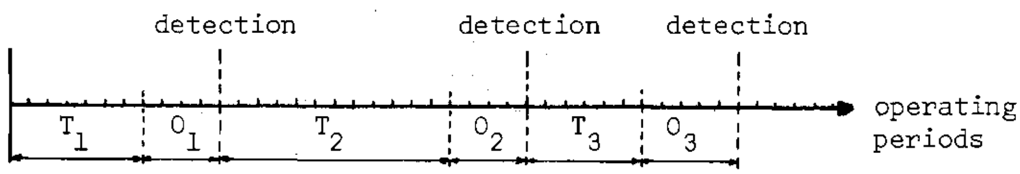

Figure 3. Behavior of the Geometric Model

The length of the *ith* in-control-shift-detection cycle is $T_i + O_i$, and the expected value of the *ith* cycle length is

$$E\{\text{Cycle length}\} = E\{T\} + E\{O\},$$

therefore the expected sampling cost per cycle is $a_1 n[E\{T\}+E\{O\}]$, the expected cost of investigating and correcting the process is $a_2[1+E\{X\}]$, where X is the number of false alarm per cycle, and the expected cost of producing defective is $a_3 E\{O\}$. Thus combining these elements, the long run average time cost per cycle is:

$$C = a_1 n + \frac{a_2[1+E\{X\}] + a_3 E\{O\}}{E\{T\} + E\{O\}} \tag{9}$$

But,

$$E\{O\} = \text{Number of Bernoulli trials with the first success}$$

$$= \frac{1}{1 - \beta} \tag{10}$$

where $1-\beta$ is the probability of success, and

$$E\{X\} = \sum_t E\{X/T=t\}Pr\{T=t\}$$

$$= \sum_t \alpha t p(t)$$

$$= \alpha \sum_t t p(t),$$

but

$$E\{T\} = \sum_t t p(t),$$

therefore,

$$E\{X\} = \alpha E\{T\}. \tag{11}$$

On substituting (10) and (11) into (9), we have

$$C = a_1 n + \frac{a_2[1+\alpha E\{T\}] + a_3/(1-\beta)}{E\{T\} + (1/1-\beta)} . \tag{12}$$

Under the Markov assumption made in this model, T follows the geometric distribution, hence

$$p(t) = \begin{cases} \theta(1-\theta)^t & t=0,1,2,\ldots \\ 0 & \text{otherwise} \end{cases}$$

and the expected value of T is

$$E\{T\} = \frac{1-\theta}{\theta} .$$

Thus

$$C = a_1 n + \frac{a_2(1-\beta)[\theta+\alpha(1-\theta)] + a_3\theta}{(1-\beta)(1-\theta) + \theta} . \qquad (13)$$

The Poisson Model

In the development of this model, we assume that the distribution of the duration in control depends on the time interval since the previous search. To assist in describing the system for this and the following (logseries) model, some additional notation is needed. Let

T' = Number of periods the process remains in control from the conclusion of the previous search (whether or not the search was induced by a false alarm).

O' = Number of periods the process is out of control prior to a particular signal for action (O' is analogous to O except that for a cycle in which the signal is a falm alarm, O' = 0), and

D = Number of periods following the conclusion of a search until the next signal for action.

In this and the following model, a cycle is defined as the interval between two signals, i.e. an interval T_i' in control followed by an interval O_i' out of control. Figure 4 shows the behavior of these two models.

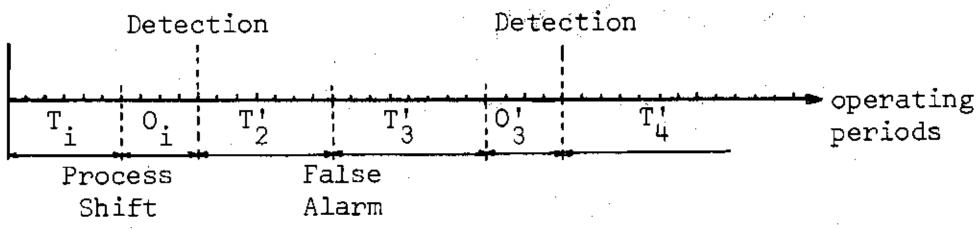

Figure 4. Behavior of the Poisson and the logseries model.

The cost incurred during the *ith* cycle is

$$a_1 n D_i + a_2 + a_3 O_i' . \tag{14}$$

Following the same procedure as in the geometric model, the average time cost is

$$C = a_1 n + \frac{a_2 + a_3 E\{O'\}}{E\{D\}} . \tag{15}$$

In this model, assume that the duration in control has a Poisson distribution with mean θ, that is

$$\Pr\{T=t\} = \begin{cases} \dfrac{\theta^t e^{-\theta}}{t!} & t=0,1,2,\ldots \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\alpha = \Pr\{D=t+1/D>t \text{ and } T'\geq t+1\} \tag{16}$$

$$1 - \beta = \Pr\{D=t+1/D>t \text{ and } T'\leq t\}. \tag{17}$$

If D > T', then the associated signal is valid, and the probability of such a cycle is

$$\Pr\{D>T'\} = \sum_{t=0}^{\infty} \Pr\{D>T'/t''=t\}\Pr\{T'=t\}$$

$$= \sum_{t=0}^{\infty} (1-\alpha)^t \, \theta^t e^{-\theta}/t! \tag{18}$$

or upon simplifying,

$$\Pr\{D>T'\} = e^{-\alpha\theta}. \tag{19}$$

The quantity E{O'} may be written as

$$E\{O'\} = E[O'/D\leq T']\Pr\{D\leq T'\} + E[O'/D>T']\Pr\{D>T'\} \tag{20}$$

where it can be observed that $E[O'/D\leq T'] = 0$ and

$$E[O'/D\leq T'] = 1/(1-\beta).$$

Hence

$$E\{O'\} = \left(1/(1-\beta)\right)e^{-\alpha\theta}. \tag{21}$$

The mean cycle length, conditioned on the length of time in control, can be written as

$$\begin{aligned} E[D/T'=t] &= \sum_{n=1}^{t} n\alpha(1-\alpha)^{n-1} + (1-\alpha)^t \sum_{n=t+1}^{\infty} n(1-\beta)\left(1-(1-\beta)\right)^{n-t-1} \\ &= \frac{1-(1-\alpha)^t - t\alpha(1-\alpha)^t}{\alpha} + (1-\alpha)^t\left(t + \frac{1}{1-\beta}\right) \\ &= \frac{1}{\alpha} - (1-\alpha)^t\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right) . \end{aligned} \tag{22}$$

The condition can then be removed as follows:

$$\begin{aligned} E\{D\} &= \sum_t E[D/T'=t]Pr\{T'=t\} \\ &= \sum_{t=0}^{\infty} \left[\frac{1}{\alpha} - (1-\alpha)^t \left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right)\right]\theta^t e^{-\theta}/t!, \end{aligned}$$

or

$$E\{D\} = \frac{1}{\alpha} - e^{-\alpha\theta}\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right). \tag{23}$$

On substituting into (15) we find

$$C = a_1 n + \frac{a_2\alpha(1-\beta) + a_3\alpha e^{-\alpha\theta}}{(1-\beta)(1-e^{-\alpha\theta}) + \alpha e^{-\alpha\theta}} \tag{24}$$

The Logseries Model

The last model that will be discussed is almost identical to the Poisson model except that the duration in control is assumed to have a logseries distribution, i.e.

$$P_r\{T=t\} = \begin{cases} P_0 & t=0 \\ \dfrac{\gamma\theta^t(1-P_0)}{t} & t=1,2,3,\ldots \end{cases} \tag{25}$$

where $0 < \theta < 1$ and $\gamma = -[\log(1-\theta)]^{-1}$.

Hence, using Equation (25), we have

$$\begin{aligned} Pr\{D>T'\} &= \sum_{t=0}^{\infty} Pr\{D>T'/T'=t\}Pr\{T'=t\} \\ &= 1(P_0) + \sum_{t=1}^{\infty}(1-\alpha)^t \frac{\gamma\theta^t}{t}(1-P_0) \\ &= P_0 + (1-P_0)\gamma \sum_{t=1}^{\infty} \frac{\gamma\theta^t}{t}(1-\alpha)^t \end{aligned}$$

but since

$$\sum_{t=1}^{\infty} \frac{\{\theta(1-\alpha)\}^t}{t} = -\ln\left(1-\theta(1-\alpha)\right),$$

therefore,

$$\Pr\{D>T'\} = P_0 + (1-P_0)\gamma[-\ell n\left(1-\theta(1-\alpha)\right)] \tag{26}$$

Similar to the Poisson model, the quantity $E\{O'\}$ may be written as

$$E\{O'\} = \frac{1}{1-\beta}\{P_0 + (1-P_0)\gamma \sum_{t=1}^{\infty} [\theta(1-\alpha)]^t/t\}$$

$$= \frac{1}{1-\beta}\{P_0 + (1-P_0)\gamma[-\ell n\left(1-\theta(1-\alpha)\right)]\}. \tag{27}$$

The mean cycle length is the same as given by Equation (23), and

$$E\{D\} = \sum_t E[D/T'=t]\Pr\{T'=t\}$$

$$= \sum_{t=0}^{\infty} \left[\frac{1}{\alpha} - (1-\alpha)^t\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right)\Pr\{T'=t\}\right]$$

$$= \frac{1}{1-\beta} P_0 + \sum_{t=1}^{\infty} \left[\frac{1}{\alpha} - (1-\alpha)^t\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right)\right] \frac{\gamma\theta^t}{t}(1-P_0)$$

$$= \frac{1}{1-\beta} P_0 + (1-P_0)\left[\frac{\gamma}{\alpha}\sum_{t=1}^{\infty}\frac{\theta^t}{t} - \gamma\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right)\sum_{t=1}^{\infty}\frac{\{(1-\alpha)\theta\}^t}{t}\right]$$

$$= \frac{1}{1-\beta} P_0 + (1-P_0)\left[\frac{\gamma}{\alpha}\left(-\ell n(1-\theta) - \gamma\left(\frac{1}{\alpha} - \frac{1}{1-\beta}\right)\left(-\ell n\ 1-\theta(1-\alpha)\right)\right]\right.$$

$$= \frac{1}{1-\beta} P_0 + (1-P_0)\left[\frac{\gamma}{\alpha}\left(\left(-\ell n(1-\theta) - \left(-\ell n\left(1-\theta(1-\alpha)\right)\right)\right) + \right.\right.$$

$$\left.\frac{\gamma}{1-\beta}\left(-\ell n\left(1-\theta(1-\alpha)\right)\right)\right]. \tag{28}$$

The average time cost is found by substituting Equations (27) and (28) into (15) as

$$C = a_1 n + \frac{a_2 + a_3\left[\frac{1}{1-\beta}\{P_0 + \gamma(1-P_0)\left(-\ell n\left(1-\theta(1-\alpha)\right)\right)\}\right]}{\frac{1}{1-\beta}P_0 + (1-P_0)\left[\frac{\gamma}{\alpha}\left(-\ell n(1-\theta) - \left(-\ell n\left(1-\theta(1-\alpha)\right)\right)\right) + \frac{\gamma}{\alpha}\left(-\ell n\left(1-\theta(1-\alpha)\right)\right)\right]} \quad (29)$$

Equations (13), (24), and (29) are the equations that will be used in this research. The next chapter will discuss the solution procedure used in this work.

CHAPTER III

SOLUTION PROCEDURE

The purpose of this chapter is to formulate a method for selecting the sample size (n) and the critical region parameter (T_α^2) which minimizes the cost functions given in Chapter II. The optimization technique employed is a direct search method and makes use of a digital computer program.

Optimization Method

The solution technique used in selecting the sample size and critical region parameter which minimizes the cost function is a simple grid search technique. The methods are for a discrete time process and the sample sizes are integer valued. Hence, for a particular set of cost parameter, and for a particular sample size, one can search through a range of probability for detecting the out-of-control state (α) to find the optimal average time cost and the critical region parameter associated with it for that particular sample size. The procedure consists of dividing the range of α into small intervals and calculating the average time cost corresponding to each interval for a particular sample size. In this fashion, the optimum average time cost for a range of sample sizes can be found. The optimum total cost corresponding to a set of cost parameters will be the lowest of the set of optimum average time costs.

The above procedure is used for all three models discussed in Chapter II. Computer programs coded in Fortran V for the UNIVAC 1108 were developed for all three models. These programs are essentially the same, differing only in the calculation of the average time cost which is done in a subroutine. A listing of the main program and the three subroutines can be found in the Appendix. A flow chart of one of the programs is shown in Figure 5.

The method described in this section are generally applicable to any number of quality characteristics. The computer programs, shown in the Appendix, are for two quality characteristics only. Minor modification in calculating T^2_α will be needed before the program can deal with more than two quality characteristics.

In order to calculate the probability 1-β as given by Equation (8) in Chapter II with a digital computer, numerical approximations are required. The following sections will be devoted to the discussion of this approximation.

Calculation of the Power of the Test (1-β)

As given by Equation (8), the power is

$$1 - \beta = \int_{T^2_{\alpha,p,n-p}}^{\infty} f(T^2,\tau)dT^2, \qquad (30)$$

and also from Equation (6) we have

$$T^2_{\alpha,p,n-p} = \frac{(n-1)p}{(n-p)} F_{\alpha,p,n-p}$$

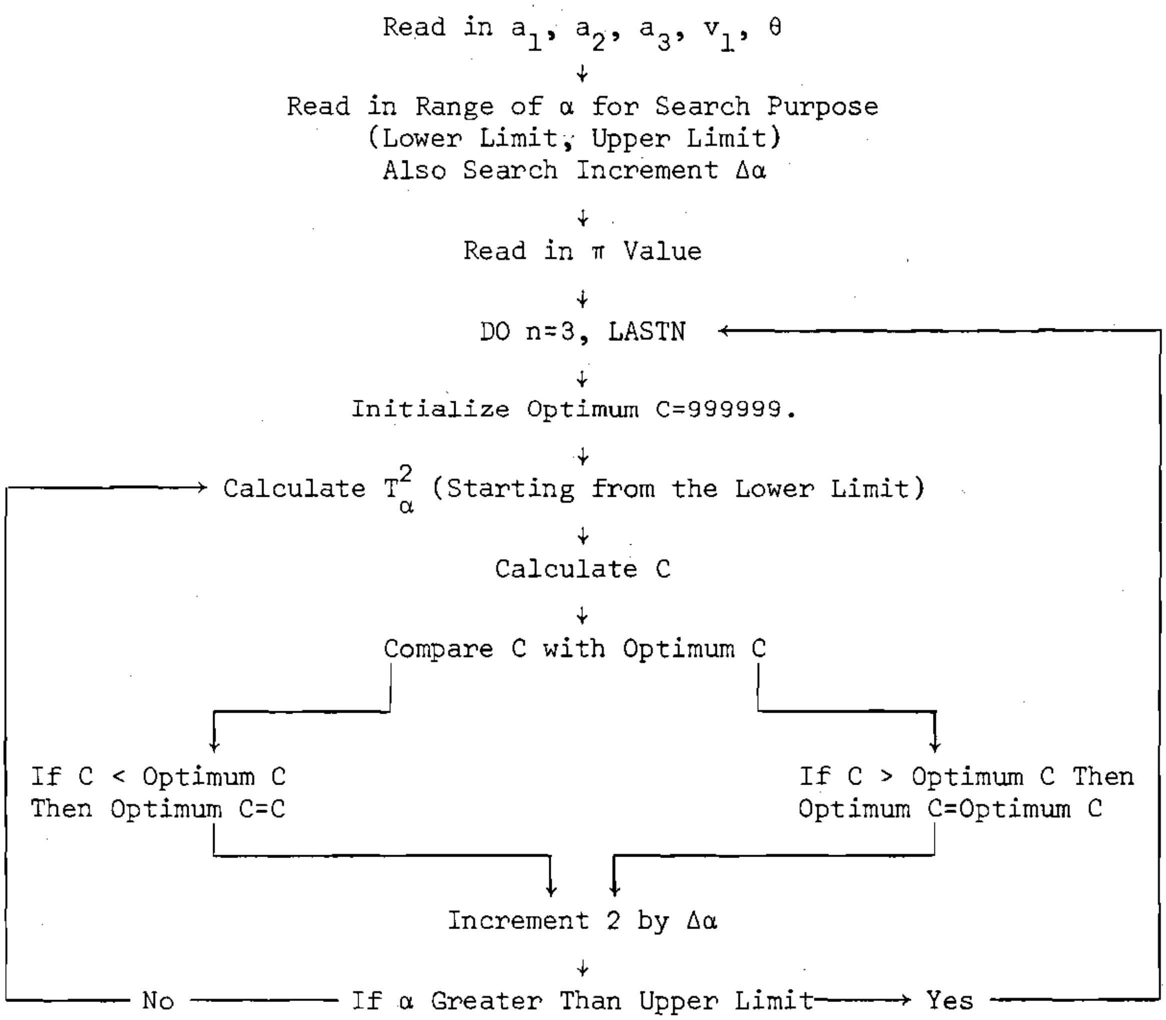

Figure 5. Flow Chart

or

$$F_{\alpha,p,n-p} = \frac{(n-p)}{(n-1)p} T^2_{\alpha,p,n-p}. \tag{31}$$

when τ, the noncentrality parameter, is not equal to zero, which is the case when $\mu \neq \mu_0$, Equation (31) becomes

$$F'_{\alpha,p,n-p} = \frac{(n-p)}{(n-1)p} T'^2_{\alpha,p,n-p}. \tag{32}$$

This quantity has the noncentral F distribution with degrees of freedom p and n-p and noncentrality parameter τ. Thus Equation (31) can be replaced with

$$1 - \beta = \int_{\frac{(n-p)}{(n-1)p} T'^2_{,p,n-p}}^{\infty} f(F',p,n-p,\tau)dF', \tag{33}$$

where $f(F',p,n-p,\tau)$ is the noncentral F distribution with p and n-p degrees of freedom and noncentrality parameter τ. The frequency function for the noncentral F distribution is given by

$$f(F',p,n-p,\tau) =$$

$$\sum_{i=0}^{\infty} \frac{\Gamma[(2i+n)/2][p/(n-p)]^{[(2i+p)/2]}\tau^i e^{-\tau} F^{[(2i+p-2)/2]}}{\Gamma[(n-p)/2]\Gamma[(2i+p)/2]i![1+pF/(n-p)]^{[(2i+n)/2]}}. \tag{34}$$

The approximation of 1-β is due to Klatt (13) and is an extension of

Paulson's (20) approximation. Paulson (20) developed an approximation for the probability integral of the central F distribution by defining a statistic x, a function of F, such that x is nearly normally distributed with zero mean and unit variance. That is

$$\int_{F_{\alpha,p,n-p}}^{\infty} f(F,p,n-p)dF \doteq \int_{x}^{\infty} f(z)dz, \tag{35}$$

and if we substitute v_1 and v_2 for p and n-p, respectively, Equation (35) becomes

$$\int_{F_{\alpha,v_1,v_2}}^{\infty} f(F,v_1,v_2)dF \doteq \int_{x}^{\infty} f(z)dz, \tag{36}$$

where $f(F,v_1,v_2)$ is the frequency function of the central F distribution and f(z) is the frequency function of the normal distribution. In (36) x is given by

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)F^{1/3} - \left(1 - \frac{2}{9v_1}\right)}{\left(\frac{2}{9v_2}\,F^{2/3} + \frac{2}{9v_1}\right)^{1/2}} \tag{37}$$

where $1 - \frac{2}{9v_2}$ and $\frac{2}{9v_2}$ is the mean and variance of anormally and independently distributed variate w and $1 - \frac{2}{9v_1}$ and $\frac{2}{9v_1}$ is the mean and variance of another normally and independently distributed variate y. Paulson (20) has found that x as given by Equation (37) is nearly normally distributed with zero mean and unit variance. In developing this approximation, Paulson has regarded $F^{1/3}$ as the ratio of two normally distributed variates.

Paulson's work was extended by Klatt (13) to obtain an approximation for the probability integral of the noncentral F distribution. The noncentral F is given by

$$F' = \frac{\chi_1^{2'}/v_1}{\chi_2^2/v_2} \tag{38}$$

which is the ratio of a noncentral Chi-square distribution ($\chi_1^{2'}$) to a central Chi-square distribution (χ_2^2) with v_1 and v_2 degrees of freedom, respectively. Wilson and Hilferty (21) have shown that $(\chi_2^2/v_2)^{1/3}$ is nearly normally distributed with mean $1 - \frac{2}{9v_2}$ and variance $\frac{2}{9v_2}$, also, Adbel-Aty (1) have shown that $[\chi_1^{2'}/(v_1+\tau)]^{1/3}$ is nearly normally distributed with mean $1 - \frac{2(v_1+2\tau)}{9(v_1+\tau)}$ and variance $\frac{2(v_1+2\tau)}{9(v_1+\tau)^2}$. If we consider the ratio

$$\left[\frac{\chi_1^{2'}/v_1}{\chi_2^2/v_2}\right]^{1/3} = F'^{1/3} = \left[\frac{v_1F'}{v_1}\right]^{1/3}, \tag{39}$$

then the ratio

$$\left[\frac{\chi_1^{2'}/(v_1+\tau)}{\chi_2^2/v_2}\right]^{1/3} = \left[\frac{v_1F}{v_1+\tau}\right]^{1/3} \tag{40}$$

Using this relationship and the mean and variance of $\chi_1^{2'}/(v_1+\tau)$ and

χ_2^2/v_2 as given previously, Equation (37) becomes

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)\left(\frac{v_1 F}{v_1+\tau}\right)^{1/3} - \left(1 - \frac{2(v_1+2\tau)}{9(v_1+\tau)}\right)}{\left\{\frac{2}{9v_2}\left[\frac{v_1 F}{v_1+\tau}\right]^{2/3} + \frac{2(v_1+2\tau)}{9(v_1+\tau)}\right\}^{1/2}}, \tag{41}$$

and since

$$F = \frac{n - p}{(n-1)p} T^2 = \frac{v_2}{(n-1)v_1} T^2,$$

Equation (41) becomes

$$x = \frac{\left(1 - \frac{2}{9v_2}\right)\left(\frac{v_1 T^2}{(v_1+\tau)(n-1)}\right)^{1/3} - \left(1 - \frac{2(v_1+2\tau)}{9(v_1+\tau)}\right)}{\left\{\frac{2}{9v_2}\left[\frac{v_1 T^2}{(v_1+\tau)(n-1)}\right]^{2/3} + \frac{2(v_1+2\tau)}{9(v_1+\tau)}\right\}^{1/2}}. \tag{42}$$

The quantity x as given by Equation (41) or (42) is nearly normally distributed with zero mean and unit variance.

The approximation for $1-\beta$ is

$$1 - \beta = \int_x^\infty \frac{e^{-\frac{z^2}{2}}}{(2\pi)^{1/2}}\, dZ, \tag{43}$$

where x is as given by Equation (42).

To evaluate the normal integral as given by Equation (43), Hasting's (7) approximation is used in conjunction with the above approximations. Hasting found that

$$\int_0^x \frac{2}{(\pi)^{1/2}} e^{-\frac{t^2}{2}} dK$$

can be approximated by

$$1 - \frac{1}{(1+b_1x+b_2x^2+b_3x^3+b_4x^4+b_5x^5+b_6x^6)^{16}}, \tag{44}$$

where $b_1 = .0705230784$

$b_2 = .0422820123$

$b_3 = .0092705272$

$b_4 = .001520143$

$b_5 = .002765672$

$b_6 = .000430638.$

Based on Equation (44), it can be shown that Equation (43) can be written as

$$1 - \beta = \frac{0.5}{(1+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6)^{16}}, \tag{45}$$

where $c_1 = b_i/2^{i/2}$, and the value of χ is as given by Equation (42). The values of c_i are:

$c_1 = .0498673470$

$c_2 = .0211410062$

$c_3 = .0032776263$

$c_4 = .0000380036$

$c_5 = .0000488900$

$c_6 = .0000053800.$

This approximation of the normal integral has an accuracy of 0.000003. The approximation of x is accurate to within 0.005 of the true value.

CHAPTER IV

ANALYSIS OF RESULTS

This chapter will present the results obtained by making use of the theory developed in Chapter II and the solution procedure suggested in Chapter III. A total of 54 different problems were analyzed for each of the three models. Six different sets of cost parameters (a_1, a_2, a_3), the noncentrality parameters, and three different probability distribution parameters were used to develop the 54 problems.

The cost parameters used in this study are:

$$a_1 = 0.4,\ 1.0$$

$$a_2 = 50,\ 100,\ \text{and}$$

$$a_3 = 100,\ 1000$$

The noncentrality parameters are based on values of π of 5, 10, and 25. The probability distribution parameters used are shown in Table 1.

Figures 6 to 14 illustrate the probability distributions for all these cases. In selecting these parameters, care has been exercised to make sure that the expected values are the same for all three distributions for each case. For example, the expected value of the geometric random variable is

$$E[x] = \frac{1-\theta}{\theta}, \tag{46}$$

the expected value of the Poisson random variable is

$$E[x] = \theta, \tag{47}$$

and the expected value of the logseries random is

$$E[x] = [\gamma\theta(1-\theta)^{-1}][1-P_0] \tag{48}$$

Thus, when θ of the geometric distribution is 0.025 as in case 1, the expected value E[x] is (1-0.025)/0.025=39, and since E[x] for the Poisson random variable is θ, therefore θ = 39 for case 1, and similarly for the logseries distribution where P_0 = 0.025 and θ = 0.98689. Note that P_0 for the logseries distribution is assumed to be equal to the geometric probability that x = 0.

Table 1. Probability Distribution Parameters Used in the Example Problems

	CASES		
Distribution	1	2	3
Geometric (θ)	0.025	0.05	0.15
Poisson (θ)	39	19	5.6667
Logseries (P_0,θ)	0.025, 0.98689	0.05, 0.9675	0.15, 0.8426

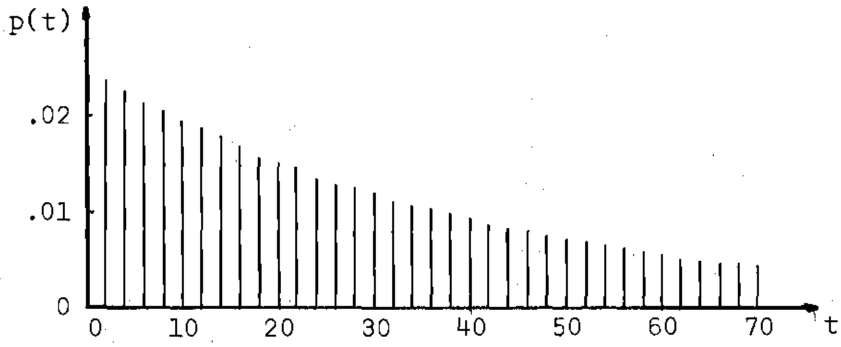

Figure 6. Geometric Distribution (θ=0.025)

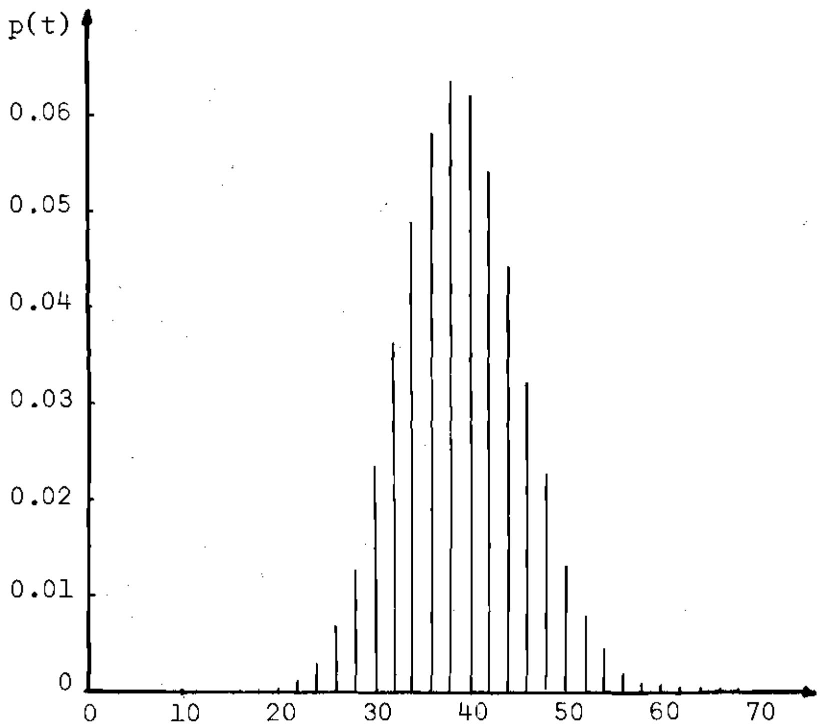

Figure 7. Poisson Distribution (θ=39)

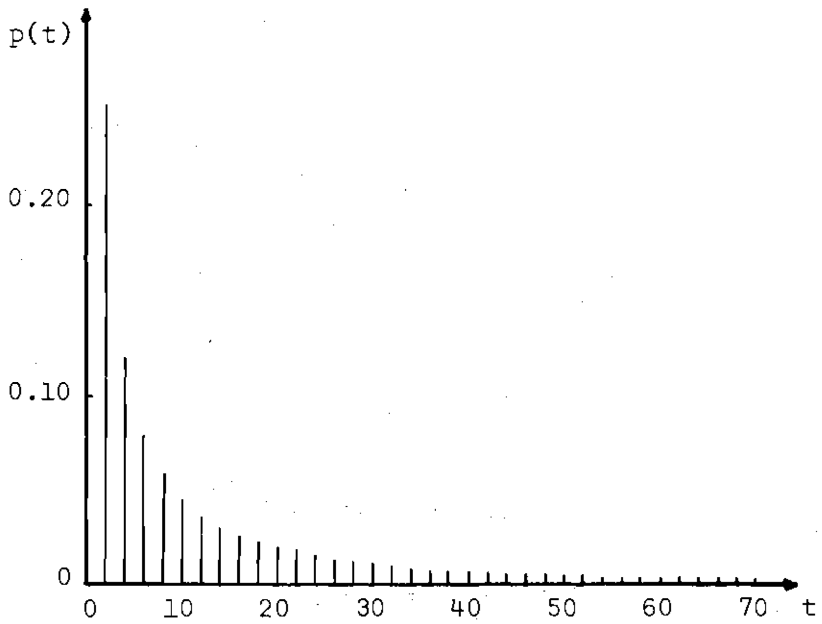

Figure 8. Logseries Distribution (P_0=0.025, θ=0.98689)

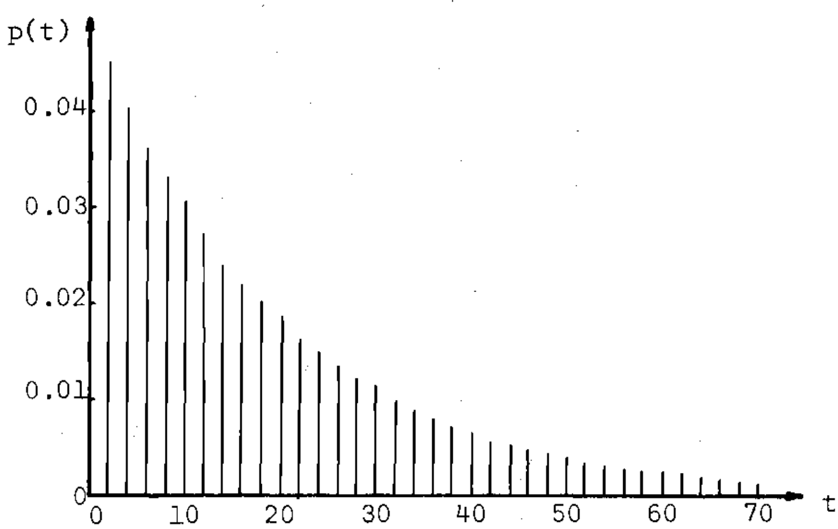

Figure 9. Geometric Distribution (θ=0.05)

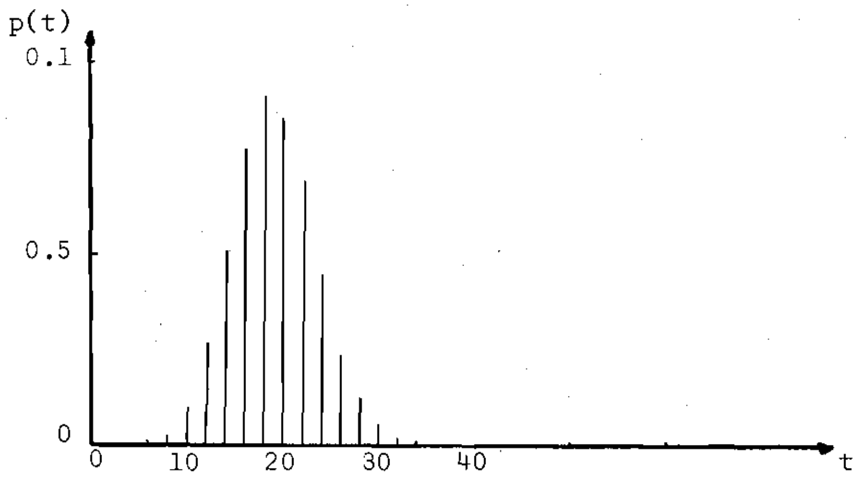

Figure 10. Poisson Distribution (θ=19)

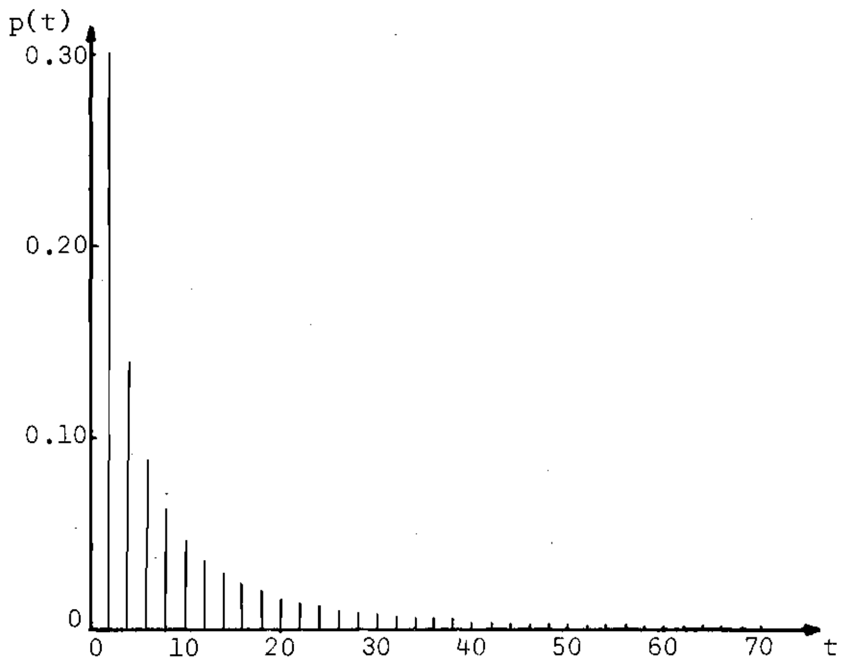

Figure 11. Logseries Distribution (P_0=0.05, θ=0.9675)

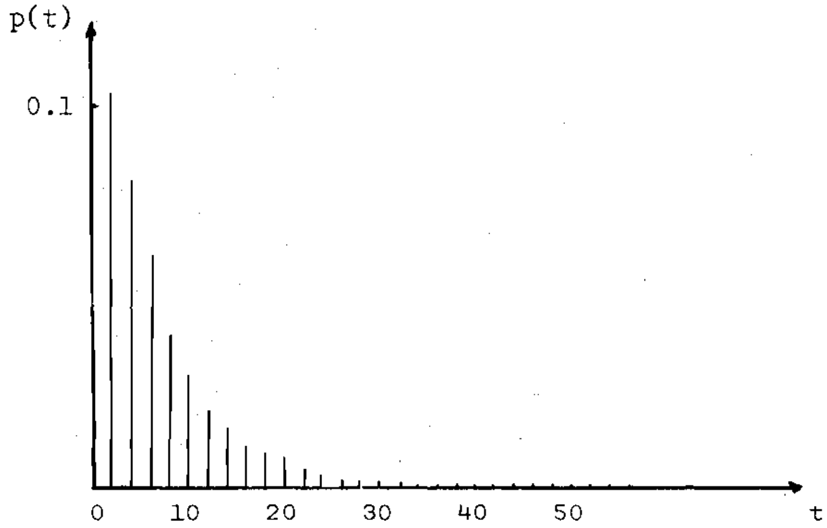

Figure 12. Geometric Distribution (θ=0.15)

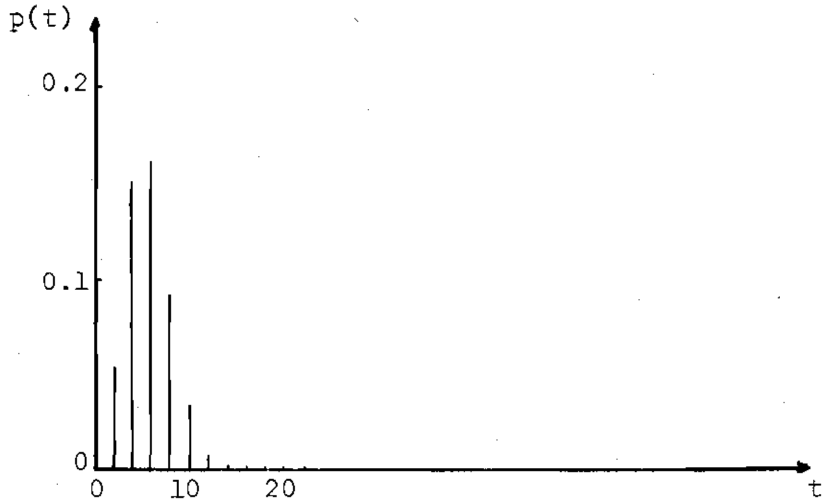

Figure 13. Poisson Distribution (θ=5.6667)

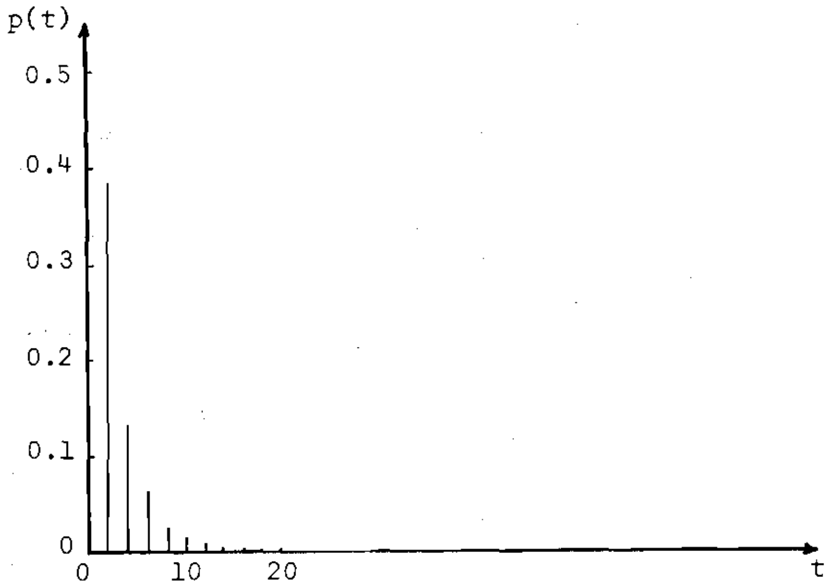

Figure 14. Logseries Distribution (P_0=0.15, θ=0.8426)

The next few sections will discuss the effect of each cost coefficient, the effect of the noncentrality parameter, the effect of the probability distribution parameter, and the distributions themselves on the sample size (n), the critical region parameter (T^2_α) and the average time cost (C).

Effect of a_1 on C, n, and T^2_α

Tables 2, 3 and 4 show a sample list of the results obtained. (The remaining results can be found in Tables 5 to 12.)

As a_1 increases, the optimal values of n decrease in all three models. In some cases, there is a significant decrease in sample size. The effect of a_1 on the optimal average time cost is very apparent. As would be expected, an increase in the variable cost of sampling causes an increase in the optimal average time cost.

Table 2. Effect of a_1 on the Optimal Value of C, n, and T^2_α (Case 1 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T^2_α	C	n_1	T^2_α	C	n	T^2_α	C
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	5	9	29.463	9.385	17	13.020	10.689	6	26.311	8.304
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 100$	5	7	31.857	13.839	8	28.000	18.596	5	33.384	11.543
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 1000$	10	14	18.438	8.818	14	17.092	8.735	13	15.981	8.398
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 100$	10	12	15.021	16.339	12	13.302	16.259	10	12.037	15.212
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	25	9	32.252	6.659	9	28.352	6.585	8	27.300	6.387
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 100$	25	5	96.000	11.231	8	24.475	11.590	4	104.143	9.960

Table 3. Effect of a_1 on the Optimal Value of C, n, and T_α^2 (Case 2 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n_1	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	5	17	13.984	13.527	18	12.707	13.071	16	12.331	12.911
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 100$	5	7	26.629	19.406	14	9.184	22.411	5	24.172	16.992
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	10	14	17.497	11.624	14	16.069	11.008	13	15.026	11.193
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 1000$	10	12	13.997	19.340	11	10.868	18.471	11	11.963	18.231
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	25	9	29.463	9.446	9	26.507	8.871	8	24.475	9.196
$a_1 = 1.0$, $a_2 = 100$, $a_3 = 100$	25	8	26.051	14.509	8	22.398	13.851	7	18.577	14.300

Table 4. Effect of a_1 on the Optimal Value of C, n, and T_α^2 (Case 3 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
a_1 = 0.4, a_2 = 100, a_3 = 100	5	19	13.766	26.389	17	9.963	21.021	17	11.237	25.764
a_1 = 1.0, a_2 = 100, a_3 = 100	5	15	10.619	36.525	13	6.536	29.880	14	8.963	34.884
a_1 = 0.4, a_2 = 100, a_3 = 1000	10	14	16.069	24.203	13	13.040	18.943	13	13.751	23.745
a_1 = 0.4, a_2 = 100, a_3 = 1000	10	13	14.430	32.235	11	9.775	25.862	11	10.563	31.058
a_1 = 0.4, a_2 = 100, a_3 = 100	25	9	26.507	21.973	8	19.710	16.966	9	25.020	21.659
a_1 = 1.0, a_2 = 100, a_3 = 100	25	8	23.162	27.220	7	16.078	21.556	8	21.706	26.545

Finally, we observe that the effect of a_1 on the optimal critical region parameter is dependent upon a_3. No generalization will be made at this time.

Effect of a_2 on C, n, and T^2_α

Tables 5, 6 and 7 show another set of numerical results. It can be seen that as the cost of searching for assignable causes (a_2) increases, the optimal sample size increases. This increase, in general, is not very significant. This should be the case since an increase in a_2 without adjusting the control limit would reduce the type one and type two error.

An increase in a_2 also causes an increase in the optimal critical region parameter. This should be expected, for the same reason as in the case of the sample size. That is, since false alarms are more costly, we wish to make the test procedure less sensitive.

Taking the total effect of increasing a_2 on the optimal n and T^2_α into account, we find no unexpected results. That is, the effect of increasing a_2 is to increase n and T^2_α, and thus reduce both error probabilities. This forces the test parameter to generate fewer false out-of-control signals, and to be less sensitive so far as detecting a true out-of-control state is concerned.

We observe that an increase in a_2 causes an increase in the optimal average time cost.

Table 5. Effect of a_2 on the Optimal Value of C, n, and T_α^2 (Case 1 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	5	18	11.396	9.605	18	10.027	9.586	15	9.061	8.898
$a_1 = 0.4$ $a_2 = 100$ $a_3 = 1000$	5	19	13.131	11.515	20	12.694	11.516	18	11.737	10.873
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	10	11	16.721	6.588	11	13.853	6.655	5	31.389	5.666
$a_1 = 0.4$ $a_2 = 100$ $a_3 = 100$	10	12	20.140	8.263	13	18.905	8.317	6	34.528	7.756
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	25	9	24.375	5.333	9	22.732	5.263	8	20.224	5.049
$a_1 = 0.4$ $a_2 = 100$ $a_3 = 1000$	25	10	29.456	6.926	10	28.002	6.813	9	24.375	6.622

Table 6. Effect of a_2 on the Optimal Value of C, n, and T_α^2 (Case 2 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 1000$	5	18	10.636	11.167	17	8.535	10.688	16	8.715	10.505
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	5	20	13.067	14.438	19	11.071	13.790	18	10.900	13.800
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 100$	10	12	16.631	8.135	11	12.699	7.791	10	12.904	7.706
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 100$	10	13	19.445	11.170	13	17.210	10.627	12	16.916	10.759
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 1000$	25	9	23.238	6.733	9	21.408	6.377	8	18.788	6.450
$a_1 = 0.4$, $a_2 = 100$, $a_3 = 1000$	25	10	28.002	9.670	9	22.261	9.088	9	23.238	9.370

Table 7. Effect of a_2 on the Optimal Value of C, n, and T^2_α (Case 3 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T^2_α	C	n	T^2_α	C	n	T^2_α	C
a_1 = 0.4, a_2 = 50, a_3 = 1000	5	19	10.520	17.670	16	6.822	14.320	17	8.340	16.987
a_1 = 0.4, a_2 = 100, a_3 = 1000	5	21	13.036	27.155	18	8.991	21.612	19	10.605	26.495
a_1 = 0.4, a_2 = 50, a_3 = 100	10	12	14.652	14.557	11	11.064	11.613	11	12.224	14.107
a_1 = 0.4, a_2 = 100, a_3 = 100	10	13	17.579	23.791	12	13.566	18.620	12	14.833	23.260
a_1 = 0.4, a_2 = 50, a_3 = 1000	25	9	25.020	12.841	8	18.788	10.120	8	20.224	12.536
a_1 = 0.4, a_2 = 100, a_3 = 1000	25	10	26.787	22.133	9	20.653	17.092	9	21.821	21.839

Effect of a_3 on C, n, and T_α^2

Tables 8, 9 and 10 provide some additional numerical results. These results are arranged to show the effect of a change in a_3 on the optimal value of C, n, and T_α^2.

By carefully studying these tables, it can be seen that an increase in the cost of operating out of control for one period (a_3) will increase the optimal sample size (n), but the optimal critical region parameter (T_α^2) will be decreased. Intuitively, this is what should be expected, because if it costs more to operate out of control for one period, the test procedure should become more sensitive. This is precisely what is accomplished by increasing n and decreasing T_α^2 simultaneously.

As before, the effect of an increase in a_3 is to cause an increase in the optimal average time cost.

Effect of the Noncentrality Parameter on C, n, and T_α^2

Recall that in Chapter II in the discussion of the power of the test, the noncentrality parameter was defined as $\tau = n\underline{\delta}'\underline{S}^{-1}\underline{\delta} = n\pi$. In this work, three different values of π were used; namely 5, 10 and 25. Varying the values of π is equivalent to varying the difference between the in-control and out-of-control states. Tables 11 and 12 are sample results which show the effect of variation in π on the optimal values of C, n, and T_α^2.

Based on the results shown on Tables 2 to 12, it can be seen that as π ranges from 5 to 25, or when there is a bigger shift in the process, the optimal value of n and C decreases for all three models, whereas the

Table 8. Effect of a_3 on the Optimal Value of C, n, and T_α^2 (Case 1 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
a_1 = 0.4, a_2 = 50, a_3 = 100	5	8	26.051	7.769	16	10.792	8.847	5	25.872	6.020
a_1 = 0.4, a_2 = 50, a_3 = 1000	5	18	11.396	9.605	18	10.027	9.586	15	9.061	8.898
a_1 = 1.0, a_2 = 100, a_3 = 100	10	6	49.233	12.430	10	13.634	15.134	5	42.784	10.813
a_1 = 1.0, a_2 = 100, a_3 = 1000	10	12	15.021	16.339	12	13.302	16.259	10	12.037	15.212
a_1 = 0.4, a_2 = 50, a_3 = 100	25	8	26.051	5.108	8	22.398	5.078	7	22.691	4.849
a_1 = 0.4, a_2 = 50, a_3 = 1000	25	9	24.375	5.333	9	22.732	5.263	8	20.224	5.049

Table 9. Effect of a_3 on the Optimal Value of C, n, and T_α^2 (Case 2 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	5	16	11.961	10.323	16	9.874	10.024	5	17.367	9.560
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	5	18	10.636	11.167	17	8.535	10.688	16	8.715	10.505
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 100$	10	6	40.644	17.814	10	12.197	17.475	5	32.024	15.974
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	10	12	13.997	19.340	11	10.868	18.471	11	11.963	18.231
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	25	8	24.012	6.566	8	21.075	6.194	8	22.770	6.257
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	25	9	23.238	6.733	9	21.408	6.377	8	18.788	6.450

Table 10. Effect of a_3 on the Optimal Value of C, n, and T_α^2 (Case 3 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	5	17	11.047	16.983	15	7.555	13.781	15	8.718	16.339
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	5	19	10.520	17.670	16	6.822	14.320	17	8.340	16.987
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 100$	10	11	14.418	31.124	10	10.228	25.036	10	11.883	30.012
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	10	13	14.430	32.235	11	9.775	25.862	11	10.563	31.058
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	25	9	25.020	12.841	8	18.788	10.120	8	20.224	12.536
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	25	9	21.408	13.013	8	16.191	10.239	8	17.101	12.726

Table 11. Effect of π on the Optimal Value of C, n, and T_α^2 (Case 1 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_α^2	C	n	T_α^2	C	n	T_α^2	C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	5	18	11.396	9.605	18	10.027	9.586	15	9.061	8.898
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	10	13	15.721	7.111	12	12.494	7.042	11	12.407	7.111
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$	25	9	24.375	5.333	9	22.732	5.263	8	20.224	5.049
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	5	16	10.481	22.050	16	8.674	22.104	4	25.986	17.228
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	10	12	15.021	16.339	12	13.302	16.259	10	12.037	15.212
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	25	8	21.706	12.316	8	19.467	12.125	8	21.075	11.576

Table 12. Effect of π on the Optimal Value of C, n, and T_{α}^{2}
(Case 3 of Probability Distribution Parameter)

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3	π	n	T_{α}^{2}	C	n	T_{α}^{2}	C	n	T_{α}^{2}	C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	5	17	11.047	16.983	15	7.555	13.781	15	8.718	16.339
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	10	12	14.652	14.557	11	11.064	11.633	11	12.224	14.107
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 100$	25	9	25.020	12.841	8	18.788	10.120	8	20.224	12.536
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	5	18	10.168	38.632	15	8.402	31.397	16	8.012	36.913
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	10	13	14.430	32.235	11	9.775	25.862	11	10.563	31.058
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$	25	9	22.261	27.723	8	16.940	22.007	8	17.982	26.941

optimum value for T_α^2 increases. That is, a less costly and less selective procedure is required to detect out-of-control states that differ considerably from the in-control state.

Effect of Probability Distribution Parameter on C, n, and T_α^2

By varying the parameter of the probability distributions, the expected number of periods that the process remains in control is varied. Three different cases of probability distribution parameters were used in this work. Most of the numerical results are shown in Table 13.

The results obtained show that as the expected number of periods the process remains in control decreases, the optimal sample size increases slightly, the optimal T_α^2 decreases and the optimal average time cost increases. Intuitively, this is what should happen since decreasing the expected number of periods the process remains in control would require a more sensitive control procedure.

Effect of the Markov Assumption

From the results obtained, it can be seen that in general, the geometric (Markov) model differs somewhat from the Poisson and the log-series models, and requires a larger sample size, larger critical region parameter, and results in a higher average time cost than the Poisson model. If we consider all the problems with case 3 of probability distribution parameter, notice in these problems the results of the optimum average time cost between the logseries model and the geometric model differed by as much as 20 per cent in some cases.

Table 13. Effect of Probability Distribution Parameter on the Optimal Value of C, n, and T^2_α

Cost Parameter a_1, a_2, a_3, π	Case	Geometric n	Geometric T^2_α	Geometric C	Logseries n	Logseries T^2_α	Logseries C	Poisson n	Poisson T^2_α	Poisson C
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 1000$, $\pi = 10$	1	11	16.721	6.588	11	13.853	6.655	5	31.389	5.666
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 100$, $\pi = 10$	2	12	16.631	8.135	11	12.699	7.791	10	12.904	7.706
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 100$, $\pi = 10$	3	12	14.642	14.557	11	11.064	11.633	11	12.224	14.107
$a_1 = 0.4$, $a_2 = 50$, $a_3 = 1000$, $\pi = 10$	1	13	15.721	7.111	12	12.494	7.042	11	12.407	6.661

Table 13. Continued

Cost Parameter a_1, a_2, a_3, π	Case	Geometric n	Geometric T^2	Geometric C	Logseries n	Logseries T^2	Logseries C	Poisson n	Poisson T^2	Poisson C
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$ $\pi = 10$	2	13	14.818	8.557	12	11.913	8.135	12	12.820	8.118
$a_1 = 0.4$ $a_2 = 50$ $a_3 = 1000$ $\pi = 10$	3	13	13.751	14.918	11	9.290	11.916	12	11.570	14.449
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 100$ $\pi = 10$	1	6	49.233	12.430	10	13.634	15.134	5	42.784	10.813
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 100$ $\pi = 10$	2	6	40.644	17.814	10	12.197	17.475	5	32.024	15.974

Table 13. Continued

Cost Parameter		Geometric			Logseries			Poisson		
a_1, a_2, a_3, π	Case	n	T^2	C	n	T^2	C	n	T^2	C
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 100$ $\pi = 10$	3	11	14.418	31.124	10	10.228	25.036	10	11.883	30.012
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$ $\pi = 25$	1	8	21.706	12.316	8	19.467	12.125	8	21.075	11.576
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$ $\pi = 25$	2	9	24.375	15.168	8	18.577	14.300	8	19.710	14.382
$a_1 = 1.0$ $a_2 = 100$ $a_3 = 1000$ $\pi = 25$	3	9	22.261	27.723	8	16.940	22.007	8	17.982	26.941

Therefore care must be exercised in making the assumption that a process possesses the Markov property.

Baker (3) in this work demonstrated that the results of the geometric model and the Poisson model differed by as much as 55 per cent when the $\bar{X}$ control chart is used. The results of this work show that the T^2 control chart is not as sensitive to the Markov assumption as the $\bar{X}$ control chart.

General Behavior of the Cost Functions

In order to study the general behavior of the cost functions, the response surface of three sample problems of each model were plotted. This is done using the CALCOMP GENERAL PURPOSE CONTOUR PLOTTING PACKAGE, and the plots are shown in Figures 15 to 23.

From these contours, it can be seen that in general these cost functions are well-behaved. For the geometric model, as n increases from 3 and α increases from 0.001, the surface slopes steeply towards the optimum. Similar behavior is exhibited by the Poisson model. It is also observed that the cost functions for the logseries model are not as steep in this vicinity as those for the geometric and Poisson models. This leads us to conclude that if we must estimate the optimum parameters rather than determine them analytically, it would be best to overestimate n and α for all three models. The effect of poorly estimated test parameters is much more serious in the geometric model than in the Poisson model, and much less serious in the logseries model.

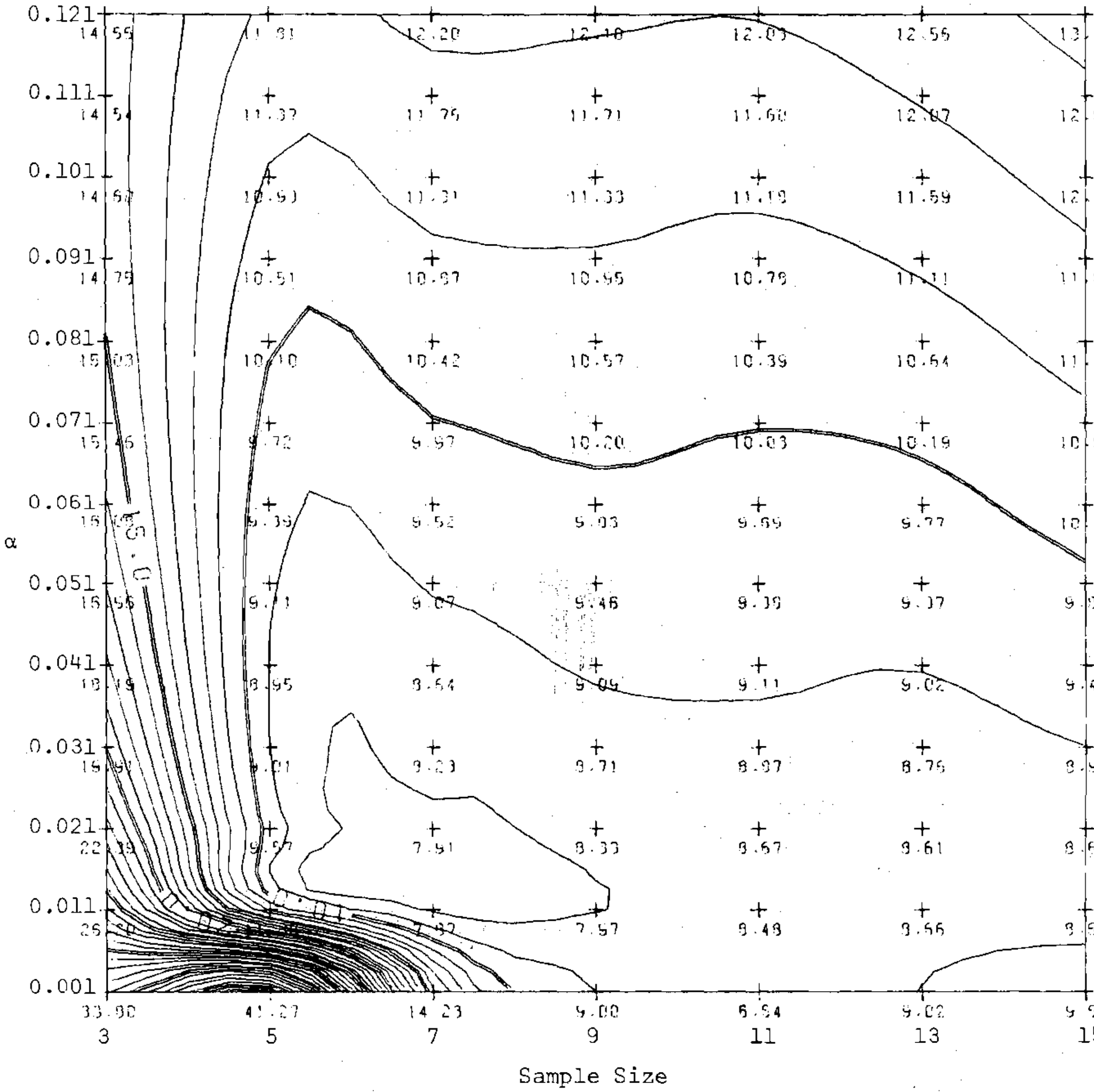

Figure 15. Contour for the Geometric Model (1 Unit Interval) (a_1=0.4; a_2=50; a_3=100; π=5; θ=0.025)

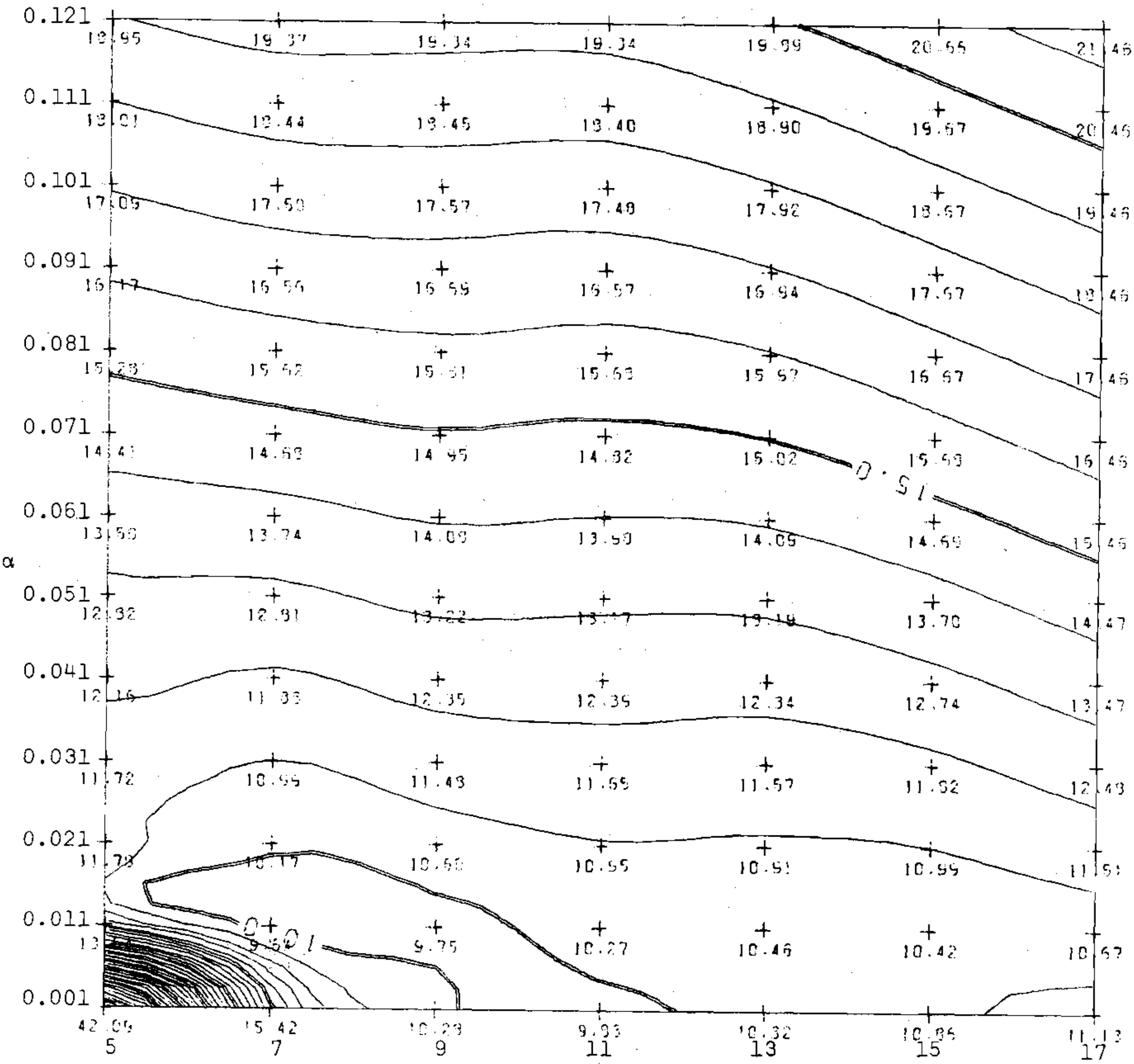

Figure 16. Contour for the Geometric Model (1 Unit Interval)
(a_1=0.4; a_2=100; a_3=100; π=5; θ=0.025)

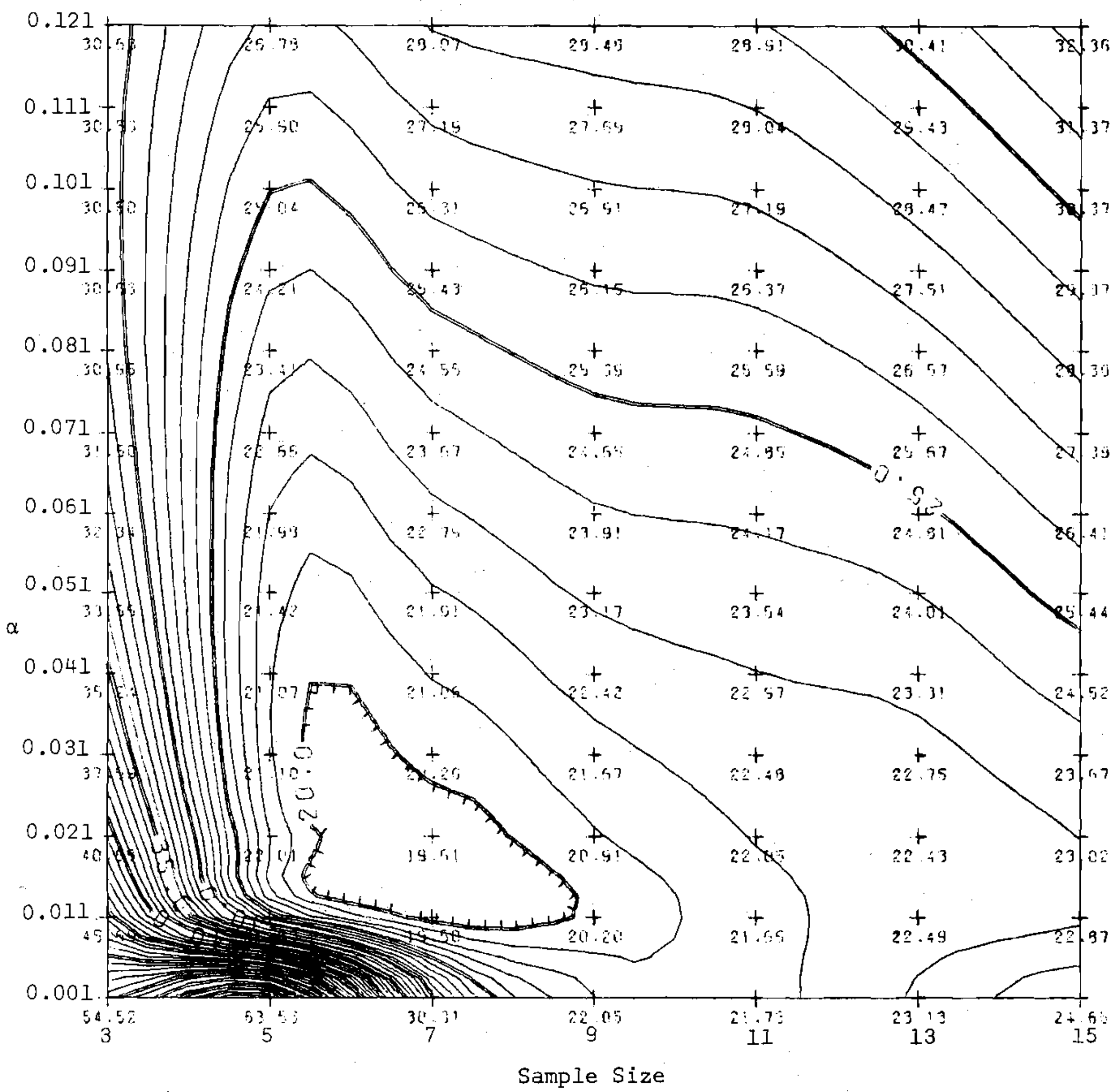

Figure 17. Contour for the Geometric Model (1 Unit Interval)
(a_1=1.0; a_2=100; a_3=100; π=5; π = 5; θ=0.05)

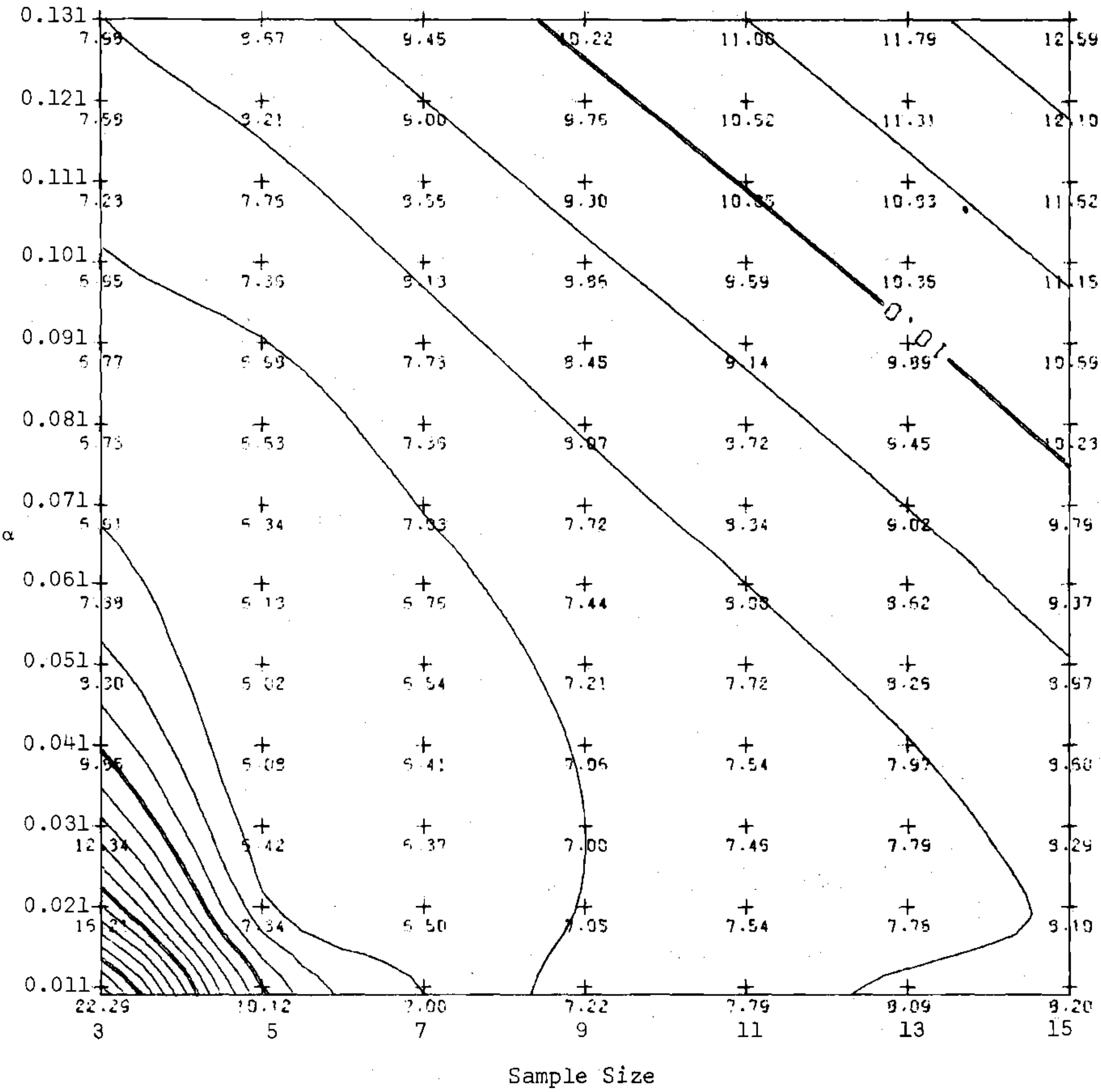

Figure 18. Contour for the Poisson Model (1 Unit Interval) (a_1=0.4; a_2=50; a_3=100; π=5; θ=39)

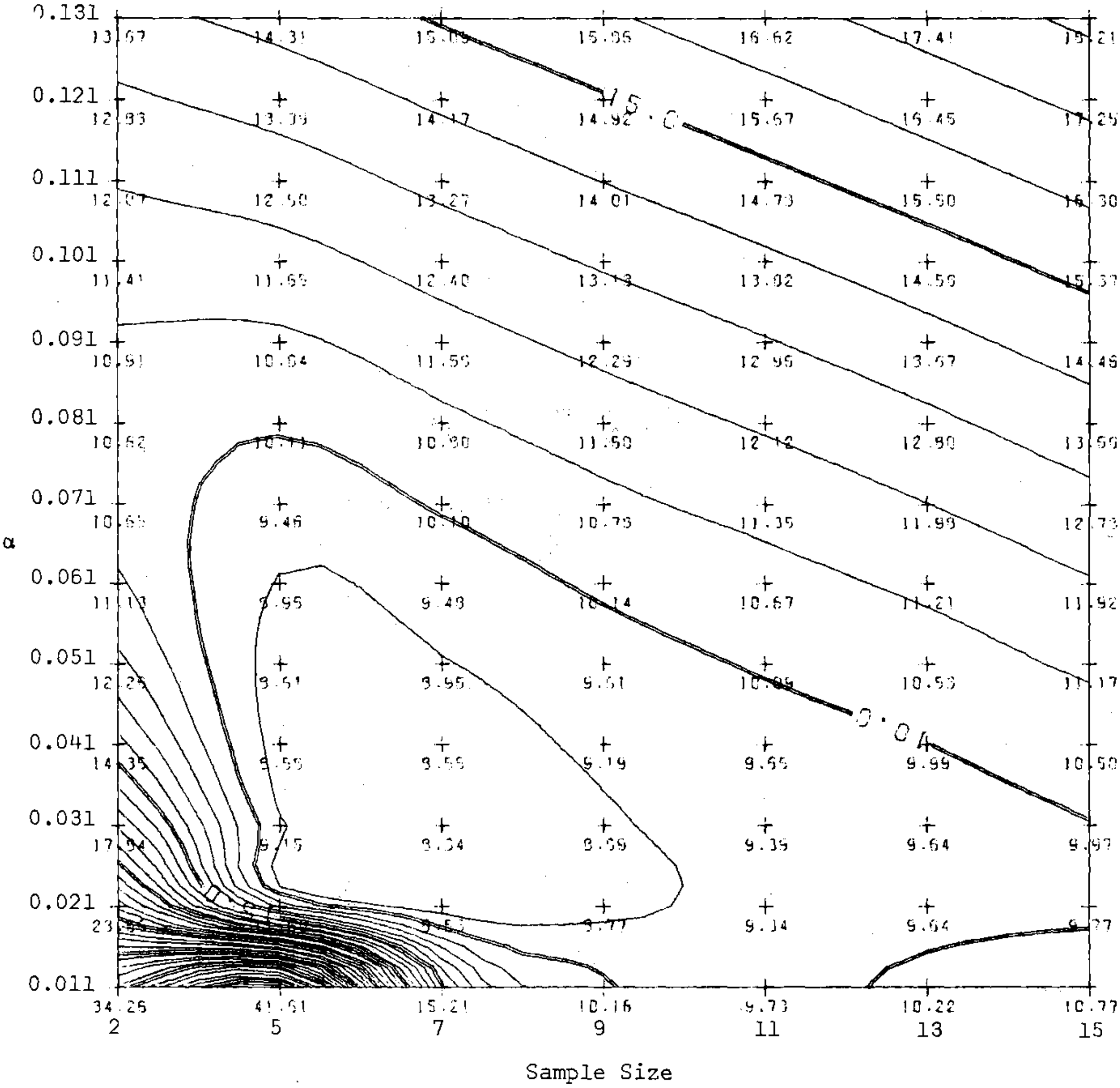

Figure 19. Contour for the Poisson Model (1 Unit Interval)
(a_1=0.4; a_2=100; a_3=100; π=5; θ=39)

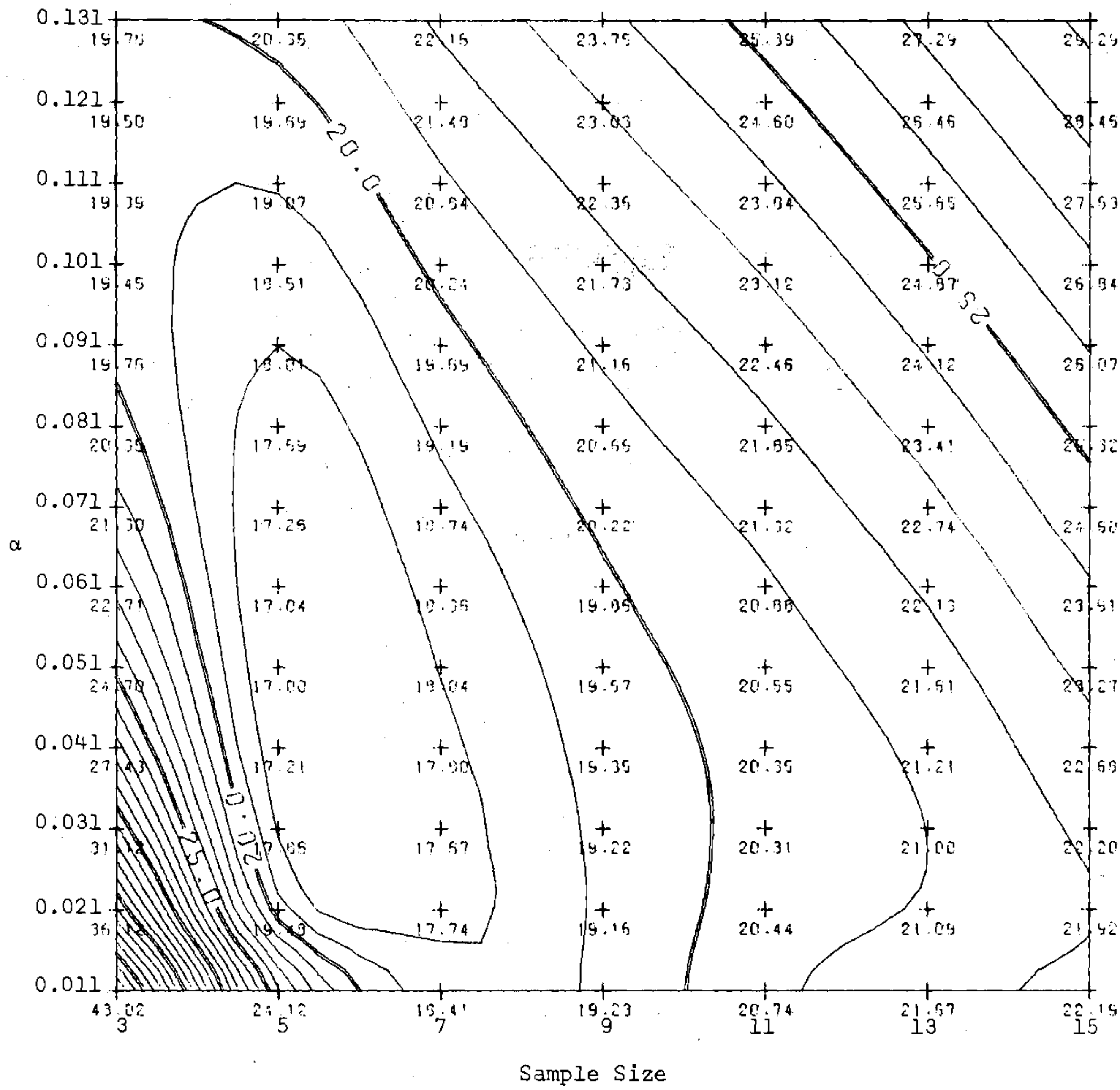

Figure 20. Contour for the Poisson Model (1 Unit Interval) (a_1=1.0; a_2=100; a_3=100; π=5; θ=19)

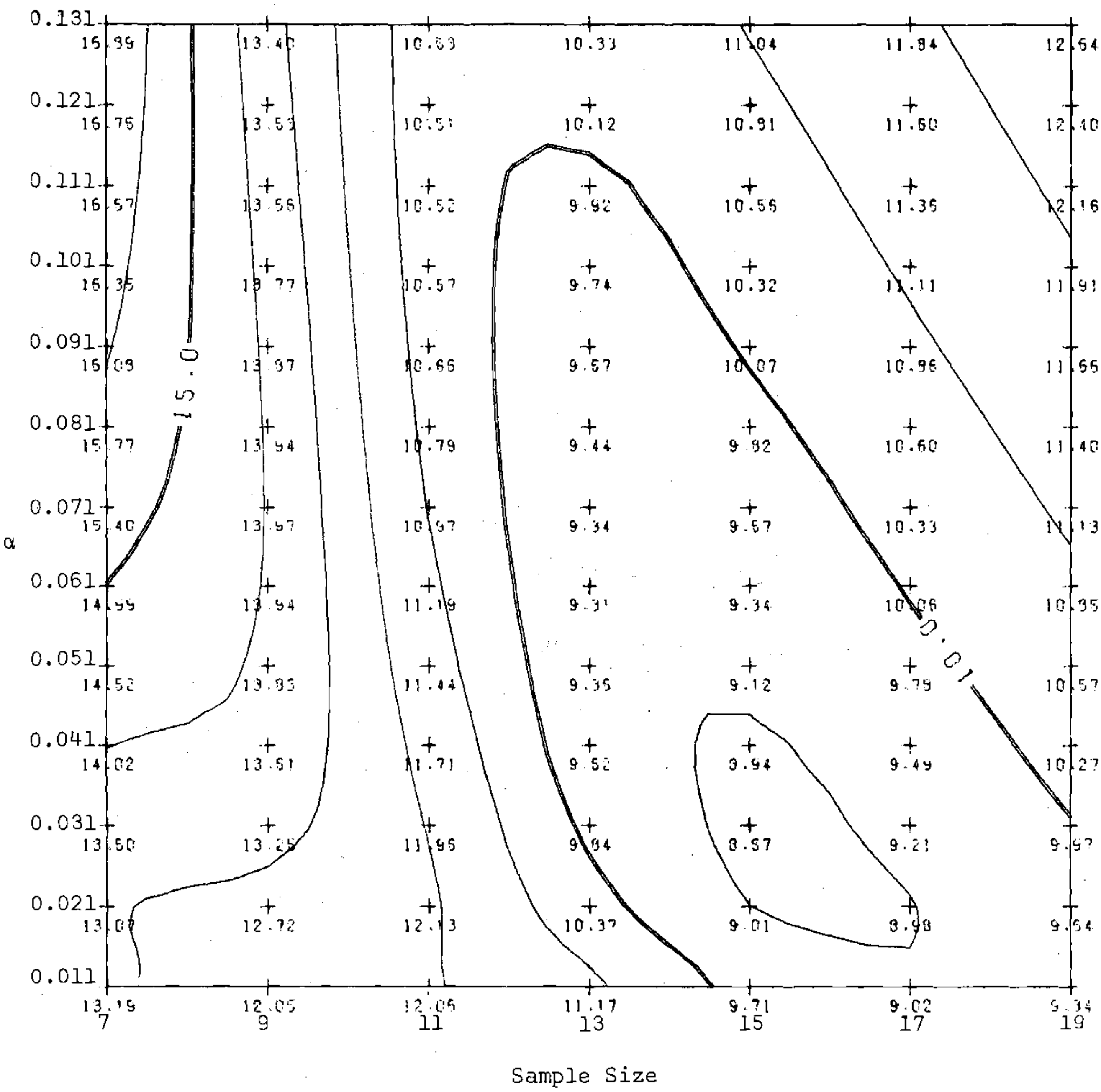

Figure 21. Contour for the Logseries Model (1 Unit Interval)
(a_1=0.4; a_2=50; a_3=100; π=5; P_0=0.025; θ=.98689)

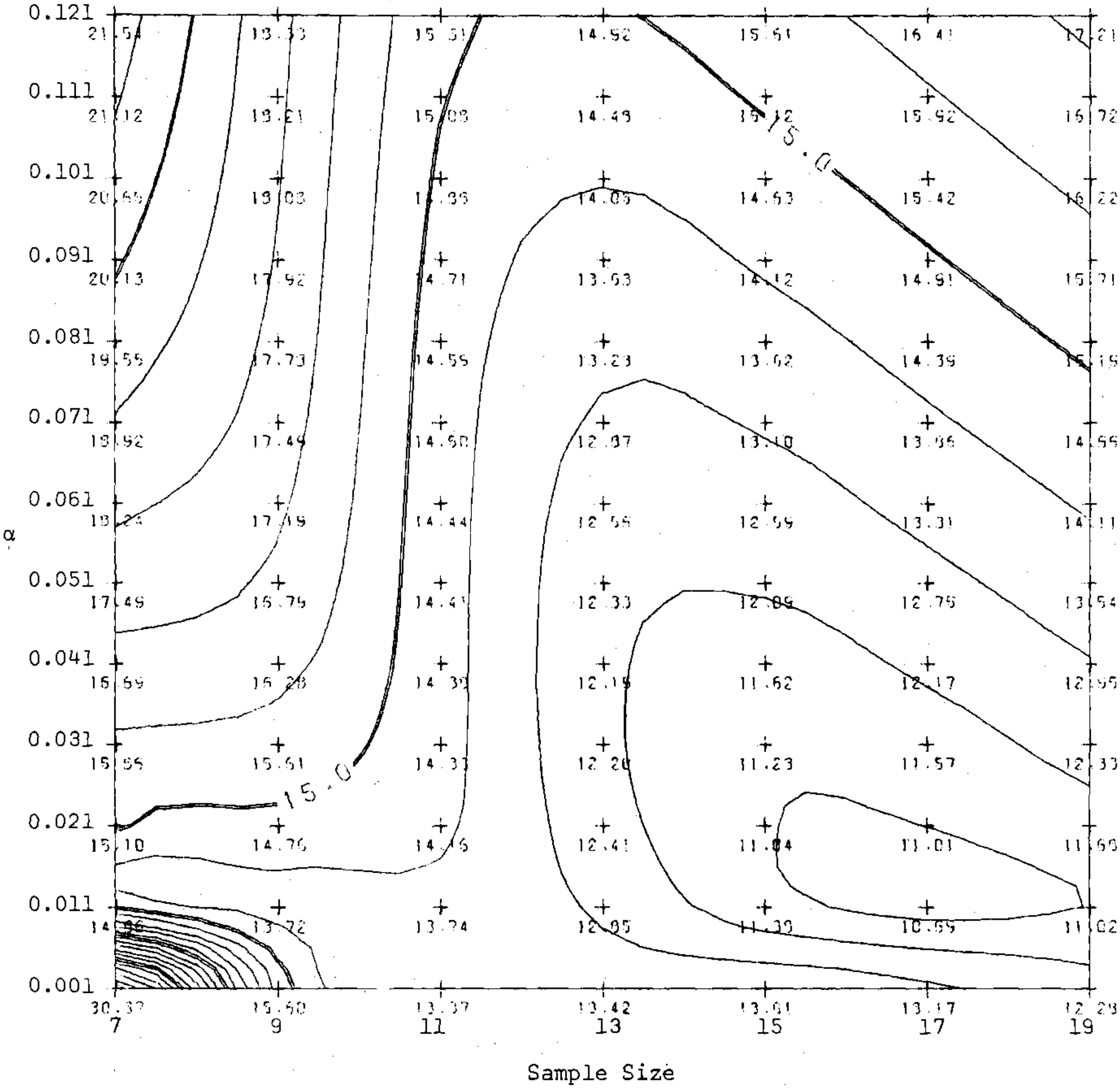

Figure 22. Contour for the Logseries Model (1 Unit Interval) (a_1=0.4; a_2=100; a_3=100; π=5; P_0=0.025; θ=0.98689)

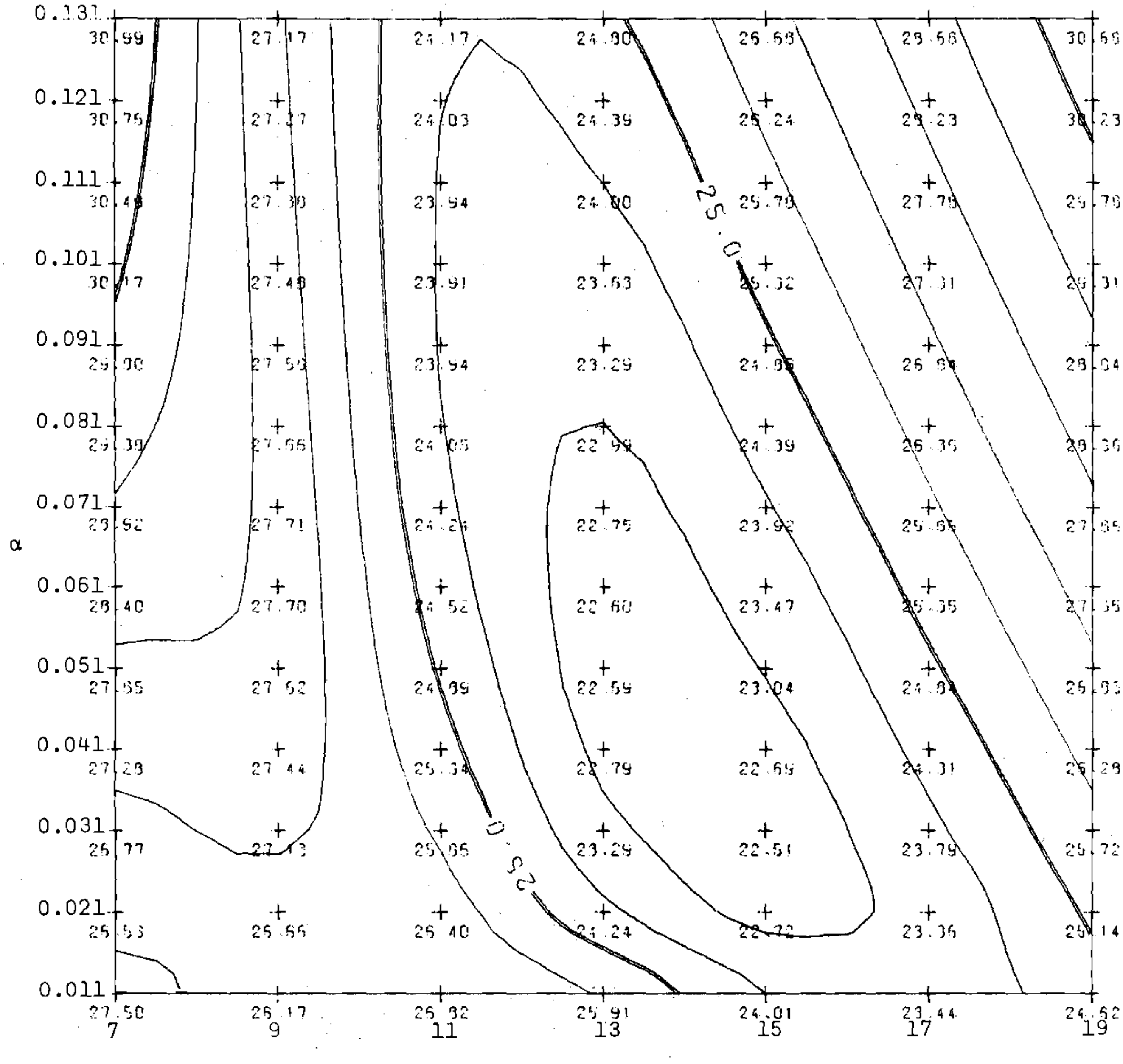

Figure 23. Contour for the Logseries Model (1 Unit Interval)
(a_1=1.0; a_2=100; a_3=100; π=5; P_0=0.05; θ=0.9675)

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

In the preceding chapters, the theory and development of three cost models for the economic design of T^2 control charts were presented. These models were developed with the assumption that a process shifts out of control at the start of a period and remains out of control for at least one period. The models are for a single assignable cause. These assumptions may not completely coincide with real world, but the models should be adequate for many situations.

A simple grid search technique was developed and programmed for a digital computer to determine the optimal sample size (n), critical region parameter (T^2_α), and the optimal average time cost (C). A total of 162 problems were solved.

The conclusions that can be drawn from this study are:

1. The optimal values of n decrease and the optimal averages time cost increases as a_1 increases.

2. The optimal values of n, T^2_α and C increase as a_2 increases.

3. The optimal values of n and C increase by an increase in the value of a_3, whereas T^2_α decreases by an increase in a_3.

4. Care must be exercised in making the Markov assumption regarding the time in the in-control state.

5. The optimal value of n and T^2_α decreases as the expected

number of periods the process remains in control decreases. The optimal value of C increases when the expected number of periods in control decreases.

6. The usual assumption of a process possessing the Markov property causes the optimal value of n, T_α^2, and C to be higher than for the Poisson model.

7. The T^2 control chart is less sensitive to the Markov assumption than the $\bar{X}$ control chart.

8. The cost functions are in general well behaved and appear to be unimodal. The geometric model has a steeper cost surface than does the Poisson model. The logseries model has a cost surface that is flatter than either the geometric or Poisson model.

9. If the optimal test parameters must be estimated, it is best to overestimate both n and T_α^2 for all three models.

Recommendations

Duncan (6) indicated that in the invariate case, a single assignable cost model is sufficient, but it is not known if this holds in the multivariate case. It would be of interest to extend this work to consider more than one assignable cause.

The computer program developed in this thesis can only handle two quality characteristics, as the numerical approximation for T_α^2 holds only for this case. Since it is very inconvenient to store all the values of T_α^2 in the memory of the computer, the development of a numerical approximation for T_α^2 when the number of quality characteristics is greater than two should be very useful.

The simple grid search technique used is not very efficient. To achieve higher numerical accuracy will require excessive computer time. It is recommended that a more efficient optimization technique be employed in future work.

APPENDIX

COMPUTER PROGRAM

```
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION AMIN(51),OPTALP(51),OPTTSQ(51)
      INTEGER YES,NO,ANSWER
      DATA YES,NO/'YES','NO '/
      COMMON A1,A2,A3,V1,PI,P
    1 FORMAT( )
      DEFINE TSQ(Y,X) = ((1.0/Y)**(2.0/(X-2.0))-1.0)*(X-1.0)
C READ IN COST PARAMETER
      WRITE(6,10)
   10 FORMAT(' THIS IS THE GEOMETRIC DISTRIBUTION MODEL'//' ALL'
     1' INPUT OF THIS PROGRAM ARE IN FREE FIELD'/ ' ENTRIES'
     2' MUST BE SEPARATED BY COMMAS'//)
    2 WRITE(6,11)
   11 FORMAT ('1',' INPUT VARIABLE COST OF SAMPLING,'/
     1' COST OF SEARCHING FOR ASSIGNABLE CAUSES,'/' COST OF OPE'
     2'RATING OUT OF CONTROL FOR 1 PERIOD'/' NUMBER OF QUALITY'
     3' CHARACTERISTICS'/' PARAMETER P OF THE DISTRIBUTION')
      READ (5,1) A1,A2,A3,V1,P
C READ IN RANGE OF ALPHA VALUES AND THE PI VALUE
    3 WRITE(6,12)
   12 FORMAT(' INPUT THE RANGE OF PROBABILITY FOR DETECTING OUT'
     1'-OF-CONTROL'/' STARTING WITH THE LOWER LIMIT, THEN THE '
     2'UPPER LIMIT'/' AND THE SEARCH INCREMENT FOR ALPHA')
      READ(5,1) QBE,QEN,QDEL
      WRITE(6,13)
   13 FORMAT(' INPUT THE PI VALUE')
      READ(5,1) PI
      WRITE(6,18)
   18 FORMAT(18X, ' GEOMETRIC DISTRIBUTION MODEL')
      WRITE(6,19) A1,A2,A3,V1,P,PI
   19 FORMAT(' COST PARAMETER'/' COST A1'
     120X,'COST A2',20X,'COST A3'/3( F9.4,17X)/' NO. OF QUALITY '
     2'CHAR.',6X,'PARA. OF DIST.',14X,' PI'/ 2(F10.2,15X),F8.2//)
      WRITE(6,20)
   20 FORMAT(' SAMPLE SIZE',2X,'ALPHA-VALUE',2X,'T-SQUARE-VALUE',
     12X,'AVG.-TOTAL-COST/CYCLE'//)
C SEARCH FOR OPTIMUM VALUE FOR A PARTICULAR SAMPLE SIZE
    4 DO 888 I = 3,25
      QB = QBE
      QE = QEN
      SN = I
      AMIN(I) = 9999999.
      AA = QBE
  111 TA = TSQ(AA,SN)
      CALL EVALUE(SN,TA,AA,ATC)
      IF(ATC.LE.AMIN(I)) GO TO 222
      GO TO 333
  222 AMIN(I) = ATC
      OPTALP(I) = AA
      OPTTSQ(I) = TA
  333 AA = AA+QDEL
      IF(AA.GT.QEN) GO TO 555
      GO TO 111
  555 WRITE(6,21) I,OPTALP(I),OPTTSQ(I),AMIN(I)
```

```
 21 FORMAT(' ',2X,I3,7X,F10.5,4X,F10.5,8X,F10.5)
    IF(I.EQ.3) GO TO 666
    IF(AMIN(I).GE.OPTMIN) GO TO 888
    OPTMIN = AMIN(I)
    GO TO 777
666 OPTMIN = AMIN(I)
777 J = I
888 CONTINUE
    WRITE(6,23)
 23 FORMAT(// ' THE OPTIMUM IS ')
    WRITE(6,24) J,OPTALP(J),OPTTSQ(J),OPTMIN
 24 FORMAT(10X,' A SAMPLE SIZE OF ' I3,/
   110X,' AN ALPHA VALUE OF ' F10.5,/
   210X,' A T-SQUARE VALUE OF ' F10.5,' AND'/
   310X,' AN AVERAGE TOTAL COST OF ' F10.5/)
    WRITE(6,25)
 25 FORMAT(' DO YOU HAVE ANOTHER PROBLEM TO SOLVE? ')
    READ(5,26) ANSWER
 26 FORMAT(A3)
    IF(ANSWER.EQ.NO ) GO TO 999
    GO TO 2
999 STOP
    END
```

- - - T H E   E N D - - -

```
      SUBROUTINE EVALUE (SN,TSQ,ALP,ATC)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON A1,A2,A3,V1,PI,P
      REAL LAMDA
      C1 = 0.0498673470
      C2 = 0.0211410062
      C3 = 0.0032776263
      C4 = 0.0000380036
      C5 = 0.0000488900
      C6 = 0.0000053830
      V2 = SN-V1
      LAMDA = SN*PI
      V1LA = V1+LAMDA
      V2TSQ = V2*TSQ
      X1 = (V2TSQ/(V1LA*(SN-1.0)))**(0.3333)*(1.0-2.0/(9.0*V2))
      X2 = 1.0-2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)
      X3 = 2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)+2.0/(9.0*V2)*
     1(V2TSQ/((V1+LAMDA)*(SN-1.0)))**(0.6667)
      X = (X1-X2)/SQRT(X3)
      ALPR =.5/(1.0+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)
     1**16
      ATC = A1*SN+(A2*ALPR*(P+ALP*(1.-P))+A3*P)/(ALPR*(1.-P)+P)
      RETURN
      END
```

- - - T H E   E N D - - -

```
      SUBROUTINE EVALUE (SN,TSQ,ALP,ATC)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON A1,A2,A3,V1,P0,PI,P
      REAL LAMDA
      C1 = 0.0498673470
      C2 = 0.0211410062
      C3 = 0.0032776263
      C4 = 0.0000380036
      C5 = 0.0000488900
      C6 = 0.0000053830
      V2 = SN-V1
      LAMDA = SN*PI
      V1LA = V1+LAMDA
      V2TSQ = V2*TSQ
      X1 = (V2TSQ/(V1LA*(SN-1.0)))**(0.3333)*(1.0-2.0/(9.0*V2))
      X2 = 1.0-2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)
      X3 = 2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)+2.0/(9.0*V2)*
     1(V2TSQ/((V1+LAMDA)*(SN-1.0)))**(0.6667)
      X = (X1-X2)/SQRT(X3)
      ALPR =.5/(1.0+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)
     1**16
      GAM = 1./-DLOG10(1.-P)
      COM = -(DLOG(1.-P*(1.-ALP)))
      ATC = A1*SN+(A2+A3*(1./ALPR*(P0+GAM*(1.-P0)*COM)))/
     1(1./ALPR*P0+(1.+P0)*(GAM/ALP*(-DLOG(1.-P)-COM)))
      RETURN
      END
```

- - - T H E   E N D - - -

```
      SUBROUTINE EVALUE (SN,TSQ,ALP,ATC)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON A1,A2,A3,V1,PI,P
      REAL LAMDA
      C1 = 0.0498673470
      C2 = 0.0211410062
      C3 = 0.0032776263
      C4 = 0.0000380036
      C5 = 0.0000488900
      C6 = 0.0000053830
      V2 = SN-V1
      LAMDA = SN*PI
      V1LA = V1+LAMDA
      V2TSQ = V2*TSQ
      X1 = (V2TSQ/(V1LA*(SN-1.0)))**(0.3333)*(1.0-2.0/(9.0*V2))
      X2 = 1.0-2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)
      X3 = 2.0*(V1+2.0*LAMDA)/(9.0*(V1+LAMDA)**2)+2.0/(9.0*V2)*
     1(V2TSQ/((V1+LAMDA)*(SN-1.0)))**(0.6667)
      X = (X1-X2)/SQRT(X3)
      ALPR =.5/(1.0+C1*X+C2*X**2+C3*X**3+C4*X**4+C5*X**5+C6*X**6)
     1**16
      ATC = A1*SN+(A2*ALPR*ALP+A3*ALP*DEXP(-ALP*P))/(ALPR*
     1(1.-DEXP(-ALP*P))+ALP*DEXP(-ALP*P))
      RETURN
      END
```

- - - T H E   E N D - - -

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