

A STUDY OF THE EFFECTS OF VARIED STRAIN RATES
ON INTERNAL ENERGIES AND CORRESPONDING STRESSES
WITHIN STRUCTURAL MATERIALS

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SUMMARY

The purpose of this study has been to determine what influence rate of strain has upon stress-strain relations within the plastic range and to what extent these effects have been considered in the development of modern theories of plasticity.

This investigation was carried out in three steps. The effects of rate have been clearly pointed out in the experiments discussed in Part I. The effect of rate on the stress-strain relations within single crystals was pointed out in Part II and has been shown for the magnesium crystal in Figure 23. It was also seen that the rate of strain has little effect at normal temperatures for the case of single crystals; however, its effects are marked at higher temperatures. A general study of the mechanisms of plastic deformation within crystalline materials was also given in Part II. Part III was a study of existing theories of plastic deformations. Most of these theories fall into one of two general categories, that is, they are either of the flow type or the deformation type. One representative theory of each of these types was studied along with a third theory, of recent origin, which does not fall into either of the above categories.

INTRODUCTION

It is well known that the classical theories of elasticity are no longer considered adequate for application to certain types of structural problems. These theories are based on the law formulated by Hook in the seventeenth century wherein he states that if a force is applied to a body the resulting deformation is directly proportional thereto.

As stated by Dr. Clarence Zener¹:

Hook's Law, as originally formulated applies only to quasi-static forces, i.e., to forces very slowly applied. It is evident that the equilibrium deformation cannot be established in a time less than the time required for an elastic wave to travel from the region of application of the force to the most remote part of the body and back again.

From such a conclusion, it is seen that a force can only be classified as quasi-static when it is applied in a time which is relatively long as compared to the lowest natural frequency of the member. Under the application of rapidly applied forces anelastic* effects result.

It is also widely recognized that Hook's Law only applies up to the elastic limit of a material. This prevents its use not only in the special cases within the elastic range, as pointed out above, but also in the plastic range.

¹Melvin C. Zener, Elasticity and Anelasticity of Metals (Chicago: University of Chicago Press, 1948), p. vii

*Anelasticity denotes that property of a solid by virtue of which the functional relation between stress and strain is not single valued in the low stress range and no permanent set occurs. Not to be confused with nonelastic properties.

To date no completely satisfactory expressions for relating stress and strain within the plastic range have been worked out. In order to establish the authority of this statement, the following quotation is taken from a paper presented by Dr. S. B. Batdorf² at the January, 1949, meeting of the Institute of Aeronautical Sciences:

Present theories for the polyaxial stress-strain relation beyond the elastic range can be divided into two types often called flow and deformation theories. Theories of plastic buckling based on deformation theories are in better agreement with experiment than those based on flow theories. On the other hand, tests in which a material is compressed into the plastic range and then subjected to shear at constant compressive stress are in better agreement with flow than with deformation theories. Legitimate doubt therefore has existed as to the validity of any theory for the plastic buckling of plates.³

It is the writer's opinion that more satisfactory theories would result if the problem was attacked on the basis of the total energies present within the crystalline aggregate which make up a solid and the variations in the state of these energies under elastic and plastic deformations. This thesis is a study of the fundamental concepts of plastic deformations and their relations to existing theories. Particular attention will be given to the effects of these deformations on the state of the energies within a crystalline aggregate.

²S. B. Batdorf, "Theories of Plastic Buckling," Preprint No. 200, (New York: The Institute of Aeronautical Sciences, January, 1949), p. 1

³Loc. cit.

PART I

STRESS-STRAIN RELATIONS

The elastic properties of a material are completely specified by a set of elastic constants. On the other hand, the general mechanical properties, which include plastic deformation and fracture, cannot be so specified. In order that a practical analysis of the stresses beyond the elastic limit may be made several theories have been developed among which are:

1. Tangent modulus theory⁴
2. Secant modulus theory⁵
3. Handelman Pragner theory⁶
4. The Elasto-Plastic Stability of Plates⁷
5. Unified theory⁸

⁴Alfred S. Niles and Joseph S. Newell, Airplane Structures (Volume I, Third Edition; New York: John Wiley and Sons, Inc., 1947), pp. 332-33.

⁵George Gerard, "Secant Modulus Method for Determining Plate Instability Above the Proportional Limit," Journal of the Aeronautical Sciences, Vol. 13, No. 1: pp. 38-44 and 48, January, 1946.

⁶G. H. Handelman and W. Pragner, "Plastic Buckling of a Rectangular Plate under Edge Thrusts," U. S. National Advisory Committee for Aeronautics, Technical Note No. 1530: pp. 1-97, August, 1948.

⁷A. A. Ilyushin, "The Elasto-Plastic Stability of Plates," Translation, U. S. National Advisory Committee for Aeronautics, Technical Memorandum No. 1188: pp. 1-30, December, 1947.

⁸Elbridge Z. Stowell, "A Unified Theory of Plastic Buckling of Columns and Plates," U. S. National Advisory Committee for Aeronautics, Technical Note No. 1556: pp. 1-31, April, 1948.

6. Theory of Plastic Deformations by N. M. Bellean⁹
7. A Mathematical Theory of Plasticity Based on the Concept of Slip¹⁰

Through the use of the above mentioned theories it has been possible to design metal aircraft on a basis of ultimate loads. It has been found through practice and experiment that none of the above theories is exact. While one theory will give results which compare favorably for one type structure, the same theory may not be adequate for a different structure, even under similar loading conditions. Conclusive evidence of these variations is pointed out by Pride and Heimerl¹¹. The present theories of plastic deformation are subject to change, as they are in a state of development. Before an explanation of these theories can be given, it is necessary to make a study of the mechanisms influencing plastic deformations.

Since the elastic properties of a material are determined from the standard stress-strain diagram, it is reasonable to assume that some information regarding the plastic properties might be obtained from the same source. The ease with which reproducible results may be obtained in uniaxial loading at moderate temperatures has led to the

⁹L. N. Kachanov, and others, Plastic Deformation Principles and Theories (Brooklyn: Mapleton House, 1948), pp. 9-12.

¹⁰S. B. Batdorf and Bernard Budiansky, "A Mathematical Theory of Plasticity Based on the Concept of Slip," U. S. National Advisory Committee for Aeronautics, Technical Note No. 1871: pp. 1-33, April, 1949.

¹¹Richard A. Pride and George J. Heimerl, "Plastic Buckling of Simply Supported Compressed Plates," U. S. National Advisory Committee for Aeronautics, Technical Note No. 1817: pp. 1-22, April, 1949.

acceptance of such loading as the most common method of examining the mechanical properties of metals. Such an examination, known as the tensile test, is of such widespread use that a terminology has arisen which pertains to it alone; and standard methods of representing the data have been almost universally adopted.

The method of recording data in a standard tensile test is represented in Figure 1. Stress is regarded as the dependent variable; strain as the independent variable. This choice of independent and dependent variables is just as logical as the inverse, since in some testing machines, particularly of the older type, the strain is applied at a nearly constant rate and may, therefore, be justly regarded as the independent variable.

The recorded stress is defined as the load divided by the original cross section of the material, as in equation 1.

$$\text{nominal stress} = \frac{\text{Load}}{\text{Original Area}} \quad (1)$$

Wherever danger arises that the stresses defined may be confused with the actual stress--namely, load divided by actual area--it is called the engineering stress. The justifications for the use of the engineering rather than the actual stress are that it is measured directly and hence not subject to computational errors and that it may be recorded automatically with relatively simple instruments. It also offers the engineer a relative comparison of the overall strength properties of structural members.

The strain is defined in the conventional manner, as in

equation 2:

$$\text{Axial strain} = \frac{\text{change in length}}{\text{original length}} \quad (2)$$

The strain at fracture is known as the elongation of the specimen and is usually expressed as a percentage. From the dimensionless manner in which elongation is defined, it would appear as if a stress-strain curve would be independent of the initial gage length. Such is, however, not usually the case. Unless fracture sets in prematurely, deformation proceeds uniformly throughout the gage length until it becomes localized in a restricted region. This region becomes constricted and is hence known as the "neck" of the specimen. The contribution of "necking" to the elongation is of course less the longer the gage length. In the interest of uniformity, specifications have been established as to gage length and the size of specimens in commercial testing.

The standard plot of engineering stress versus conventional strain, as given in Figure 1, tells how the resistance to deformation of the specimen changes with continued deformation, but it does not tell how the material within the specimen changes. In order to show this information, it is necessary to plot true stress as defined in equation 3:

$$\text{true stress} = \frac{\text{load}}{\text{actual area}} \quad (3)$$

rather than engineering stress. Before discussing such a revised plot, it is in order to introduce a more natural definition of strain than

given in equation 2. Such a definition was introduced by Ludwik¹² in 1909. According to Ludwik, natural strain through its increment is defined by equation 4.

$$\text{natural strain} = d\epsilon = \frac{dL}{L} = \frac{dA}{A} \quad (4)$$

Gensamer¹³ has done some work with this type stress-strain diagram; and though it is only of general interest, he was able to show that necking commences at that strain where the tangent to the stress-strain curve intercepts the strain axis at the value $\epsilon = -1$, as illustrated in Figure 2. The essential features of the two type curves are compared in Figure 3. In contrast to the engineering stress, the actual stress continues to rise with increasing deformation. In other words, the material always strain hardens. With the standard stress-strain curve, the tensile load reaches its maximum value at the instant uniform extension of the test piece stops. It is at this point that necking begins and consequently the load on the test piece begins to decrease as indicated by the last part of the curve.

In summarizing these differences, it might be said that the lower curve gives the load-extension relationship while the upper the stress-extension relationship.

¹²A. Nadai, Plasticity (New York: McGraw-Hill Book Company, 1931), p. 81 citing P. Ludwik, Elemente der Technologischen Mechanik (Berling: Springer, 1909).

¹³M. Gensamer, "The Yield Point in Metals," Transactions of the American Institute of Mining and Metallurgical Engineers, 128: 104-117, 1938.

The above mentioned curves are based on tensile tests, but similar relations exist for compression except for the final drawing down before rupture. G. I. Taylor¹⁴ has shown that if true stress is plotted against natural strain for the cases of tension, compression, and torsion of the same metal the curves follow very closely.

Another property of plastic flow that is evidenced from this type test is that stress above the elastic limit varies with the rate of applied strain. There are three standard types of experiments which demonstrate some of the effects of rate of deformation. In one experiment, illustrated in Figure 4, stress-strain curves are made at various strain rates. It is found that the higher the strain rate, the higher is the stress-strain curve, the stress varying essentially logarithmically with strain rate.¹⁵

In a second type of experiment, illustrated in Figure 5, a load is suddenly applied and then maintained constant, the variation of elongation with time being observed. One initially observes a region in which the rate of strain diminishes rapidly. This region is called the region of primary creep. Next, one observes a region in which the creep rate is low and remains essentially constant for a long interval of time. This is known as secondary creep. The creep rate finally increases rapidly with time and gives rise to what is known as tertiary creep which continues until rupture takes place. In certain cases the

¹⁴G. I. Taylor and H. Quinney, "The Latent Energy Remaining in a Metal after Cold Working," Royal Society Proceedings, A 143: 307-326, January, 1934.

¹⁵M. Manjoine and A. Nadai, "High-Speed Tension Tests at Elevated Temperatures," Proceedings of the American Society for Testing Materials, 40: 835 (1940).

creep rate is initially zero, and a finite time elapses before the primary creep region is reached. Such incubation times have been observed in many metals containing alloying elements, but apparently are not present in pure metals.¹⁶

In the third type of standard test, a constant strain is suddenly imposed upon the specimen and is held constant while the variation of stress with time is measured. It is found that over a wide range of time, the stress necessary to maintain the strain constant diminishes logarithmically with time as illustrated in Figure 6. Such an experiment is known as stress relaxation.

The results of the first standard type experiment, i.e., variation of strain rate, will now be analyzed. These rates are relatively small as compared with those of the other two experiments mentioned above and, as is expected, the variations in the stresses are small, no variation whatsoever in the elastic range. This phenomenon recalls the viscous flow analogy, and one immediately thinks of the internal friction between the slipping particles during plastic deformation. Since the area under the stress-strain diagram is a measure of the work done in deforming the specimen, it is seen that the higher the rate of deformation, the more work required, i.e., to say a finite amount of time is required for the energies involved to change their state. If we regard the strain rate as a parameter, a parameter which is held constant throughout any one experiment, it cannot be concluded automatically

¹⁶M. Gensamer and R. F. Mehl, "Yield Point of Single Crystals of Iron under Static Loads," Transactions of the American Institute of Mining and Metallurgical Engineers, 131: 372-84 (1938)

that the stress is a unique function of the strain rate $\dot{\epsilon}$ and the strain ϵ . It is possible that if $\dot{\epsilon}$ be varied during a test, that the prior strain rates will have an effect on the continuing stress-strain curve.

With the second type experiment, it was seen that under instantaneous loads the strain may lag well behind the stress, even in the low stress ranges. As stated in the introduction, it is under this type condition that the classical theories of elasticity do not hold. As pointed out in Zener's discussion of anelasticity, attempts were made by O. Meyer and W. Voigt to generalize the theories of elasticity so as to include this phenomenon known as anelasticity.¹⁷ The essential of this generalization was to regard the stress as a linear function of both strain and strain rate. Under this assumption, the relation between the stress and strain for the case of uniaxial tension is expressed in equation 7, where α and β are arbitrary constants:

$$\sigma = \alpha \epsilon + \beta \dot{\epsilon} \quad (7)$$

Solids which obey this relation manifest the properties of internal friction and are known as "Voigt solids". The essentials of the mechanical properties of this type solid are illustrated in Figure 7. When a load is suddenly applied to this type solid, there is no instantaneous strain. Cork is known to possess this property. According to equation 7, when the stress is removed, there can be no instantaneous

¹⁷Zener, op. cit., p. 41.

recovery as is exhibited by structural metals. It is, therefore, evident that his simple generalization does not correspond to actualities.

If the properties of a solid are represented by a mechanical model, as was first done by Poynting and Thomson,¹⁸ and an equation of the motion of the model is then written, a better generalization may be obtained. This has been done by Dr. Zener.¹⁹ The mechanical model of the Voigt solid is shown in Figure 8, while the model assumed by Dr. Zener is shown in Figure 10. The model of Figure 10 will have an instantaneous displacement under a suddenly applied load, the magnitude of which will be determined by the spring constants. As the force is held constant, that portion of the reaction due to the dashpot is gradually released by its deformation. Under this arrangement when the force is suddenly released, there will be an instantaneous release of part of the energy stored in the springs. The complete release of all the stored energy must be a gradual process because of the dashpot. This more nearly approaches the actual condition in real metals. Such a property was first noted by Weber and called "Elastic After-effect".²⁰ The essentials of this type displacement are shown in Figure 9. The properties of the Voigt solid are represented in Figure 9. The equation corresponding to the mechanical properties of Figure 8 has the general form:

$$a_1 \sigma + a_2 \dot{\sigma} = b_1 \epsilon + b_2 \dot{\epsilon}$$

where a_1, a_2, b_1, b_2 are constants

(8)

¹⁸J. H. Poynting and J. J. Thomson, Properties of Matter, A Text-Book of Physics (London: C. Griffin and Co., 1902), p. 57.

¹⁹Zener, op. cit., p. 42

²⁰Loc. cit.

The general solutions of this equation with their physical interpretation are presented in reference one. In the third standard experiment, stress relaxation type, the total strain is held constant. The total strain may be regarded as the sum of an elastic and of a plastic strain. An increase of the plastic strain is then associated with a corresponding decrease of the elastic strain and therefore of the stress. As in the previous case, the plastic strain will increase logarithmically with time, and therefore the stress will decrease logarithmically with time, as represented in Figure 5.

In the types of deformation already discussed, the loads were applied until the metal fractured. It is now necessary to inquire into the nature of the stress-strain relations when the test is stopped before fracture, the stress removed and then re-applied. Tests of this nature are commonly called "fatigue tests".

If a periodic stress be applied to a solid and a plot of stress versus strain through a loading and unloading cycle be made, the result is a closed curve as indicated in Figure 11. This experiment shows that some of the energy of strain was lost, presumably as heat, during the cycle of loading and unloading. The energy lost would be measured by the area within the curve, and such a phenomenon is known as mechanical hysteresis or internal friction or damping.

In the case of a stress cycle of this sort, it is very difficult to detect variations in the stress-strain relation for low values of stress unless very accurate loadings are applied. It is practically impossible to achieve these results except by the use of dead weights.

Tests of this sort, carried out in the elastic range, produce no

permanent effects. Beyond the elastic limit, there is a permanent deformation; and when the load is removed, the metal does not return to its original form. This is demonstrated clearly if a tensile test is carried out in stages, the load being removed and repeated at the end of each stage. Figure 12 is a plot of the results obtained in this manner for a carbon steel specimen. There is in every case, on first loading, a part of the curve that is approximately straight. The curve on unloading should coincide with the loading curve if conditions were purely elastic, but some small deviation is generally found. If the load is removed after plastic deformation has begun, the metal will contract a small amount elastically; but the unloading curve varies considerably from the loading curve, the difference being a measure of the permanent deformation. On reloading, a new curve is traced which makes the hysteresis loop with the unloading curve. Such a loop will be found at every stage of the loading and unloading of the curve shown in Figure 12.

The same effect is more pronounced if a complete stress reversal is applied to the test specimen, as in the test for determining damping capacity. The damping capacity of metals has been shown to be identical with the "mechanical hysteresis effect" and has been defined by O. Foeppl as follows: "Damping Capacity is the amount of work dissipated into heat by a unit volume of a material during a completely reversed cycle of unit stresses".²¹ It is measured in inch-pounds per cubic inch

²¹ G. S. Von Heydekampf, "Damping Capacity of Metals," Proceedings of the American Society for Testing Materials, Vol. 31, Part II: p. 157, (1931).

per cycle. Figure 13 shows a hysteresis loop for a completely reversed cycle of stresses. The specific damping capacity has been defined as the damping capacity divided by the potential energy accumulated at the point of maximum stress for that cycle of loading.

That is,

p = specific damping factor

$$p = \frac{\text{Area in Loop}}{\text{Area A}} \quad (\text{See Figure 13})$$

where $A = \frac{f^2}{2E}$ approximately

where A = potential energy of maximum stress

f = maximum stress

E = modulus of elasticity

The specific damping capacity may be expressed as follows:

$$P = \frac{d}{A}$$

where d = damping capacity

p = specific damping capacity.

It has been shown by Von Heydekampf²² that p is equal to twice the logarithmic decrement of a damped vibration.

It would be logical to assume that rate of loading might have some effect on the area within the hysteresis loop for experiments of this type, since an increase in strain rate was found to increase the stress in experiments of type one which are illustrated in Figure 4.

²² Von Heydekampf, op. cit., p. 158.

Becker²³ has offered experimental evidence of the fact that the rate of loading does not affect the amount of work absorbed in the form of heat. He compared results obtained in a static machine requiring about five minutes to complete a cycle with those obtained in fatigue machines at speeds up to 2,600 cycles per minute. The higher speed tests were made in the following way. A specimen is placed in a fatigue machine which has been suitably arranged so that the temperature of the specimen might be measured. As the machine runs, the flexing of the specimen generates heat in amounts proportional to the value of the maximum stress. This heat is the energy given off as hysteresis losses.

If a plot of the temperature versus time is made, the resulting diagram would be similar to that shown in Figure 14.

At the beginning of the test, practically all the heat generated is absorbed by the machine and the surrounding atmosphere. Eventually a state of equilibrium is reached so that the specimen is generating the same amount of heat that it gives off. The equilibrium temperature is then " t_e " proportional to the damping capacity. This constant of proportionality may be determined from the cooling curve. Since the equilibrium temperature is known, the amount of heat given off per cycle can be calculated.

²³ E. Becker and O. Foepl, "Uber Dauerversuche," Forschungsheft, 304, Verin Deutscher Ingenieur Verlag, Berlin, 1928 cited by G. S. Von Heydekampf, op. cit., p. 160.

PART II

MECHANISMS OF DEFORMATION IN CRYSTALLINE SOLIDS

In order to understand the state of the energies involved and the mechanisms of plastic and elastic deformations within crystalline solids, it is necessary to begin with a few fundamentals of metallurgy.

Physically, metals can exist in the form of gases, liquids, or solids. The energy content of the individual atoms which make up the metal will vary in any one of the three states, depending upon the temperature and pressure conditions.

According to Dr. George Gamow²⁴, at extremely high temperatures and pressures (such as exist in certain of the gaseous stars), there is a complete breakdown of the atoms themselves; and the electrons, protons, and neutrons move about like the atoms within the familiar gases of our earth.

In this state of matter, most of the energy is in the form of radiation which, according to Einstein's theory of relativity, would be so dense that it possesses weight. Dr. Gamow has made calculations to determine the density and pressures that must have been present in order that our earth might have evolved from a completely disassociated gas into its present form with its known percentages of each of the elements. Similar work has been done by James H. Jeans.²⁵

²⁴ George Gamow, Atomic Energy in Cosmic and Human Life (Cambridge: The University Press, 1946), pp. 75-88.

²⁵ James Hopwood Jeans, Astronomy and Cosmogony (Cambridge: The University Press, 1929), p. 73.

Under atmospheric conditions when metals transform from a gaseous to a liquid state or from a liquid to a solid, there is always a discontinuous change in the energy content of the molecules or atoms. This discontinuity is best shown by considering the parameters temperature and time as the metal is heated or cooled. A plot of these variables is shown in Figures 15a and 15b. From an examination of these curves, it is seen that during a cooling cycle the temperature drops at a uniform rate, depending upon the specific heat and the thermal conductivity of the metal, until the transition temperature is reached. At this point, the temperature of the metal remains constant for a finite time, indicating that a new source of energy has been added to the process. This new energy source is the energy of vibration that is given up by the molecules as their freedom of motion is reduced by change from a gaseous to a liquid state or from a liquid to a solid state. The association which exists between the molecules in the different states of the metal are not completely understood; however, it is obvious that the attraction is greatest in the solid state.

At the transition temperature between the molten and the solid state, nuclei begin to form; and the atoms arrange themselves in an orderly fashion about these to form crystals. In order to bind themselves into this compact unit, the molecules must give up some of their energy of motion, thereby supplying heat to the process. More and more of these nuclei begin to form until an equilibrium state between the heat added by the molecules and the heat taken off by the surroundings is reached, thereby holding the temperature constant. The crystals

formed in this manner continue to grow until they are obstructed by neighboring grains or crystals. When all the metal has solidified, the temperature again begins to drop.

The particular patterns into which the atoms arrange themselves within the crystal are called space lattice, or space lattice formations.²⁶ In general, every metal or alloy has a particular space lattice formation, subject to conditions of temperature and pressure. The simplest of these lattice formations is the cubic lattice, which is illustrated in Figure 16, Part 12. In this figure only four atoms are shown, but a complete crystal is made up of a large number of atoms all in this same pattern. Each of the corner atoms is shared by each of the seven other cubes which form the eight octants in space. This means that only one-eighth of each of the corner atoms belongs to a space lattice.

The fundamental characteristic of the space lattice is that every point (or intersection of lines forming the space net) has identical surroundings. This is to say that the grouping of lattice points about any particular lattice point is identical to the grouping about any other lattice point. Because of symmetry required by the above characteristic, there are only fourteen possible space lattice formations, all of which are illustrated in Figure 16. This does not mean that there are only fourteen different crystalline structures, for in nature there exists an almost unlimited number of different types of

²⁶ Carl H. Samans, Engineering Metals and Their Alloys (New York: The MacMillan Company, 1949), pp. 76-87.

crystals; but whatever the crystalline structure, there is only one (sometimes two) of fourteen different ways in which it may be imagined sectioned off so that corresponding points of the section lines have identical surroundings.

All plastic deformations of solids are the accumulative effects of changing the relative positions of the atoms within the crystals. This relative shifting may take place in one of four ways²⁷:

- (a) Slip: Parallel displacement (translations) of the elements of the crystal.
- (b) Formation of twins: Shift, as a whole, of part of a crystal to a second position.
- (c) Change of position of atoms occurring because of agitation of atoms due to heat.
- (d) Breakdown of structure.

Deformation due to slip is the most common type in structural metals; however, some twinning is nearly always present. The mechanism of slip can best be understood by considering the geometrical arrangement of the atoms within the crystal. Because of the symmetry of the space lattice formation, it is possible to pass families of parallel planes through the crystal so that the atoms contained within these planes are packed closer together than they are for any other plane which might section the crystal. The number of directions within these families of close packed planes may have depends entirely upon the lattice formation. For instance, it is possible to pass these families of planes through the face centered cubic lattice in four different

²⁷ A. Nadai, Plasticity, op. cit., pp. 30-38.

directions so that the arrangements of the atoms within these planes are not only the same but have their most dense spacing. Figure 17 shows two of the four planes. Within these planes, the atoms are arranged in a symmetrical manner so that there are also certain close packed directions. Figure 18 shows the arrangement of the atoms within one of the close packed planes of the face-centered lattice. From an examination of the Figure, it is seen that there are three close-packed directions within each of the four dense planes, making a total of twelve directions in space along which the atoms are most densely spaced. For the body-centered lattice, there are forty-eight possibilities.

If an external force is to deform a crystal, it must overcome the force of mutual attraction between the atoms. Naturally, less distorting force is required if each atom moves toward its nearest neighbor. It is understandable then why slip takes place along the planes of greatest atomic density and in the direction of the close packed lines within these planes, i.e., a sliding or shearing effect causes the atoms to slip past one another over these close packed planes toward one of their nearest neighbors. This type deformation is illustrated in Figure 19, in which the entire test specimen is assumed to be a single crystal and only the operating dense planes (or slip planes) are shown. This figure also illustrates that there must be a rotation of these planes during the slipping process, i.e., the angle Θ must change. If the atoms are shifted along the slip planes to the extent that each atom within the plane occupies the position previously occupied by its neighbor, there is no tendency for it to

return to its original position leaving the crystal plastically deformed. For a cubic lattice, the atoms on the outer edges of these slip planes are now corners for only two space lattice instead of four, thus leaving the crystal slightly less stable. It is known that crystals which have been strained in this manner will recrystallize or their atoms will join themselves to other more stable crystals if their temperature is raised so as to make them active enough. This recrystallization type of relative movement would be classified under (c) as described above.

Since slipping is a shearing action, shearing stresses are required to produce it. With the normal tensile specimen under axial load, maximum shearing stresses will be developed on planes at forty-five degrees to the direction of the load. In a general way, then, it is seen that the greater the number of possible slip directions, the greater the likelihood that one of these directions will align with the direction of maximum shearing stress. Under the foregoing conclusions it should be clear that the cubic lattice metals, such as gold, aluminum, iron, and copper with twelve and forty-eight slip directions, should be more plastic than the close-packed hexagonal lattice metals, such as magnesium, cadmium, and zinc, having only one family of close-packed planes, i.e., the plane containing the bases of the hexagonal lattice.

Twinned crystals may result from crystallization from the melt, recrystallization, or from mechanical deformations. The mechanism of mechanical twinning is more difficult to understand than that of slip but may also be explained on the basis of the geometrical construction

of the crystal.²⁸ With this type deformation, there is a simultaneous shifting of every atom in one part of a crystal, with respect to the atoms in the remaining part, so that the lattice formation in the first part is a mirror image of that in the second part. For this to be true, there must necessarily be a plane of symmetry known as the twinning plane. Figure 20 shows the relative movements of the atoms in a body-centered cubic crystal. Parts (a) and (c) of the figure show the twinning plane normal to the paper while part (b) shows the plane in the paper. The twinning plane is indicated by the broken line. Each atom is numbered so as to represent its position in each view. Part (c) shows some of the atoms to be directly over one another, in which case the point has two numbers. The dotted portion of part (c) shows the position of the atoms after twinning has occurred and the arrows indicate their motion. It is seen that each atom moves by an amount proportional to its distance from the twinning plane, resulting in purely plastic deformation, since the equilibrium of the crystal is undisturbed. With large groups of atoms shifting simultaneously in this manner, the resulting deformation is instantaneous and usually creates an audible sound. Mechanical twinning is very prevalent in the deforming of pure tin, and some metallurgists explain the audible "cry" heard when a bar of this metal is rapidly bent as being due to a large number of twins formed in rapid succession.

In order to study the mechanisms of slip and twinning from an

²⁸ Charles S. Barrett, Structure of Metals (New York: McGraw-Hill Book Co., 1943), pp. 313-17.

experimental point of view, it is necessary to have single crystals large enough to use as test specimens. C. F. Elam²⁹ completely describes the following three methods of growing large crystals:

- (1) By solidifying from the molten state under proper conditions.
- (2) By a process of deforming (or straining) a fine grained aggregate and then heat treating so as to promote recrystallization and grain growth while in the solid state.
- (3) By a distillation or deposition from the vapor state.

Using method one, Birdgman³⁰ melted metal in a graphite or porcelain tube and instead of allowing the metal to solidify normally, he slowly lowered it through the bottom of a furnace so as to allow it to solidify from one end only. The bottom end of the tube was tapered so as to allow only a single nucleus to form. As the tube is progressively lowered out of the furnace, the single nucleus continues to grow into one large crystal.

The main objection to this method is that there is no control over the orientation of the crystallographic planes. In order to completely analyze data obtained from testing single crystal pieces, it is necessary to orientate the slip planes (or planes of greatest atomic density) at different angles with respect to the loading axis. Birdgman³¹ was able to overcome this difficulty to some extent by

²⁹C. F. Elam, Distortion of Metal Crystals, The Oxford Engineering Science Series, (Oxford: The Clarendon Press, 1935), pp. 3-6

³⁰P. W. Birdgman, "Certain Physical Properties of Single Crystals of Tungsten, Antimony, Bismuth, Tellurium, Cadmium, Zinc, and Tin," Proceedings of the American Academy of Arts and Sciences, 60: 305-83 (1925).

³¹Loc. cit.

making the crucible consist of a network of tubes entering the main body at different angles.

Another method, described by Elam³², in which large crystals are grown from molten metal, consists of attaching a small crystal to the end of a rod to use as a nucleus. The crystal is just dipped into molten metal and slowly withdrawn as a jet of carbon dioxide is played on it. If the rate of withdrawal is properly regulated to the rate of cooling, a rod consisting of a single crystal specimen will be formed and the direction of the crystallographic planes will be the same as those of the starting crystal.

As previously mentioned, crystals that have been plastically deformed are left in a slightly less stable condition and will recrystallize if the atoms are excited by heating. This phenomenon is used to grow large crystals from the solid state. The method is used as follows:

- (1) Anneal the piece in which the large crystals are to be grown.
- (2) Strain the piece to approximately two or three per cent. The smallest amount of strain that will induce recrystallization at all produces the largest crystals.
- (3) Reheat the piece slowly and evenly. The best temperature can be found only by trial and error. It is generally found that no growth exists below a certain minimum temperature, but the lowest temperature at which growth will take place is the best.

The deposition of crystals from the vapor phase is simplest for the case of zinc. Zinc is the most volatile of the commoner metals and large crystals can be deposited from these vapors.

³²Elam, op. cit., p. 3

Now that methods of growing single crystal test pieces have been briefly described, it is in order to discuss their deformation. If an isotropic material is strained in one direction, it will deform proportionally in all other directions. That is, if a circular test specimen is strained in an ordinary tensile test, its cross-section remains circular. This fact is not necessarily true with single crystal specimens, since it may and generally does deform more in one direction than another causing a circular cross-section to become elliptical. Mark, Polanyi, and Schmidt³³ found that a single crystal test specimen of zinc in the form of a wire could be elongated as much as seventeen times its original length at temperatures of 205° C and the cross-section changed from its originally circular form to an ellipse and finally into an almost flat or rectangular shape. It was found that this same type crystal (i. e. zinc) would be very brittle and practically no elongation could be measured prior to rupture if its dense planes were orientated at 90° to the axis of stress. This continues to uphold the shear theory, since zinc has a hexagonal space lattice formation and thereby only one set of slip planes. Under these conditions, the phenomenon of twinning sometimes takes place, allowing a rotation of the dense planes and thereby making slip more inducive.

As soon as slip begin, two new factors must be considered:

- (1) Because of the mechanisms of slip, the angle between the load axis and the slip planes will be continuously decreasing (see Figure 14) causing the components of the shearing stress on the slip planes to increase.

³³ H. Mark, M. Polanyi, E. Schmidt, Z. Physik, 12: 58 (1927) cited by Elam, op. cit., p. 9.

(2) The metal work hardens.

Since the shearing stress is continuously increasing because of rotation, there must be present the second factor of work hardening; or the crystal would continue to slide apart without any increase in load. Experiments show that as one plane begins to slip, it increases its resistance to shear and any further deformation must be taken by a new slip plane, which, as previously explained, is paralleled to the first. The distance between active slip planes is fairly uniform for some metals but irregular for others; however, the greater the number of slip surfaces for the same system, the more plastic is the crystal.

If two cubes cut from different crystals are acted on by equal shears both paralleled to the direction of slip, the resulting deformations can be different, since a different number of glide planes will result. The plasticity of the crystal is accordingly the result of the number "n" of active glide surfaces per unit length vertical to the direction of gliding. It has been found that this number "n" changes in the same crystal, depending upon which set of glide planes become active, and is also a function of the temperature.

The number of glide planes is always very much less than the number of atomic layers. If they approached the number of molecular layers, the grooves seen on the surface of a deformed crystal would hardly be perceivable.

It now seems conclusive that plastic deformations are, for the larger part, the results of slip due to shearing stresses. Consider a single crystal being subjected to a gradually applied load. No measurable deformations will occur until a certain critical value of stress

is reached, this critical value being a function of the crystal orientation. Many investigators have shown that differently orientated crystals of a given metal will begin to slip at different values of applied stress; but if this stress be resolved into components parallel and perpendicular to the direction of slip, it is found that the parallel component is always the same when slip begins. It is, therefore, termed the critical resolved shear stress.

Figure 21 shows a single crystal test piece sectioned along a slip plane. If ϕ is the angle between the applied force F and the normal to the plane and λ is the angle between the line of maximum slope on the plane and the direction of slip, then the resolved shear stress τ is given by the relation:

$$\tau = \frac{F}{A} \cos \phi \sin \phi \cos \lambda \quad (9)$$

Where A = the original cross-sectional area, F/A is then the axial stress σ . It is evident from this relation that the crystal orientation, number of possible slip plane orientations and slip directions available, and the crystal symmetry will have an important effect on experimental values obtained for τ critical.

There are eight octahedral planes in a cubic crystal, of which four are parallel to the other four, so that from the point of view of distortion there are four possible slip planes in metals such as aluminum having a face-centered cubic lattice. It is evident from a consideration of equation 9 that there will generally be one plane on which the shear stress is greater than on the other three because of its inclination to the axis of the applied stress. That plane will function

as the slip plane.

There are also three possible directions of slip with this type lattice, but the shear stress is greatest in the direction nearest to the direction of maximum slope, i.e., where λ is small.

If δ denotes the shear strain parallel to the direction of slip because of the shear stress τ then the strain in the direction of the applied stress (i.e., δ_σ) would be given by the relation:

$$\delta_\sigma = \delta \sin \phi \cos \lambda \quad (10)$$

as seen from Figure 21. As the resolved shear stress τ exceeds the critical value τ_c , then shear strain δ shows up in the form of a slip line in the crystal. If it were not for the ability of the crystal to work-harden, slip would continue with no further increase in F until the crystal ruptured. It is therefore evident that as slip continues, a progressively greater resolved shear stress τ is required. This is to say that δ is a function of τ once slip begins:

$$\delta = f(\tau) \quad (11)$$

This function has been determined experimentally for aluminum by G. I. Taylor³⁴. Taylor's results show the function f to be the same in tension or compression and to have the form of a parabola $\tau = C \sqrt{\delta}$

³⁴ G. I. Taylor, "The Distortion of Crystals of Aluminum under Compression," Royal Society Proceedings, Series A, 116: 16-72, (1927).

while for the case of zinc the relation is almost a straight line. Figure 22 shows a plot of these results. Taylor also found the shape of the strain-hardening curve to be a function of temperature and rate of deformation. Figure 23 shows the relation between stress and strain for magnesium crystals at different temperatures. The amount of strain hardening decreases markedly as the temperature rises. The dashed lines marked 100° and 250° are for a rate of test about a hundred times faster than the others. It is seen that while rate has little effect at a temperature of 100°C , it produces very great effects at temperatures of 250° .

The function as represented by the graph of Figure 22 is approximately:

$$\tau = 1870 \sqrt{\gamma}$$

or

$$\tau^2 = 3.5 \times 10^6 \gamma \quad (12)$$

Substituting the equation 12 into equation 10 gives:

$$\gamma_{\sigma} = \frac{\tau^2}{3.5 \times 10^6} \sin \phi \cos \lambda \quad (13)$$

Substituting the relation given by equation 9, we have:

$$\gamma_{\sigma} = \frac{\sigma^2}{3.5 \times 10^6} \sin^3 \phi \cos^5 \lambda \quad (14)$$

Equation 114 gives an approximate relation between axial stress and plastic strain for a single aluminum crystal provided its orientation is known.

Up to this point we have only considered the deformation of single crystals. Next in the order of the discussion is the deformation of polycrystalline aggregates.

If the polished face of a piece of metal that has crystallized from the melt is etched with appropriate etching agents, a network of almost straight lines arranged in a polyhedral pattern appears on the surface. These lines represent the outline of intersecting surfaces of the crystals. The formation of these crystal surfaces takes place in the following way. As a cooling melt reaches the transition temperature, nuclei begin to form; and from these nuclei, crystals grow until they intersect. By this process of growth, the orientation of the individual crystals with the aggregate is purely random; and likewise the boundary planes of the crystal might have any orientation with respect to the crystallographic planes.

Until about 1934, the means by which the crystals of an aggregate were bound together were not completely understood. Tammann³⁵ has advanced the theory that the space between the crystals is filled with the impurities within the metal which form a complex eutectic of non-isomorphous materials. As evidence of this interstitial substance, the following experiment was outlined. When a metal sheet was placed

³⁵ Gustav Tammann, A Textbook of Metallography, Translated from the Third German Edition by Reginald Scott Dean and Leslie Gerald Swenson (New York: Chemical Catalog Co., Inc., 1925), pp. 15-18.

in a transparent solvent by whose action no gas was evolved, there appeared on the surface of the sheet fine seams of the interstitial substance, forming a network which was easily shifted and broken apart by any motion of the solvent.

It has already been pointed out that slip bands appear on the surface of a deformed crystal, and this is true whether the crystal stands alone or is part of an aggregate. Just how the crystals are able to slip within an aggregate and not separate the grains is not completely understood, but one thing is certain. If a piece of metal is ruptured at room temperature, the break is across the grains and not around them. The forces which band the grains together seem to be stronger than the grains. At higher temperatures the break is through the interstitial substance.

In recent year, physicists have applied the mathematics of quantum mechanics to the problems of metallic solids. In his book on the theory of solids, Seitz³⁶ gives a detailed development of the electron theory of metals. Dr. Charles Barrett³⁷ says it is now well established that the atoms in metal crystals are ionized and that a metal should be considered as an arrangement of positive ions immersed in a cloud of electrons. The electrons of this cloud are relatively "free"; they are not bound by any particular ion but move rapidly through the metal in such a way that there is always an approximately

³⁶ Frederick Seitz, The Modern Theory of Solids (New York: McGraw-Hill Book Company, 1940), pp. 76-96.

³⁷ Barrett, op. cit., p. 251.

uniform density of them through the interior between the ions. Metal crystals are held together by the electrostatic attraction between this so-called "gas" of negative electrons and positively charged ions. The binding forces in metals are thus in sharp contrast to those in non-metallic substances, where the predominating forces are from atom to atom or from positive to negative ions.

PART III

ANALYSIS OF EXISTING THEORIES ON PLASTIC DEFORMATIONS

As previously mentioned in this paper, there are many theories of plasticity, none of which are completely satisfactory. All but one of these theories fall into two general categories. The first of these includes so-called deformation theories, and the latter are referred to as flow theories. Both type theories attempt to modify the elastic stress-strain relation:

$$\epsilon_x = \frac{\sigma_x}{E}$$

$$\epsilon_y = -\nu \frac{\sigma_x}{E}$$

$$\epsilon_z = -\nu \frac{\sigma_x}{E}$$

Where ϵ = unit strain,

σ = unit stress,

ν = Poisson's ratio,

and subscripts denote directions. (17)

All of the deformation type theories relate stress and strain directly by use of the Secant modulus, which varies with the stress, and are of the form:

$$\epsilon = \frac{\sigma}{E_s}$$

(18)

where E_s is the slope of the Secant line to the stress-strain curve shown in Figure 1.

Flow theories are based on the tangent modulus, which is an incremental relation expressed as follows:

$$d\epsilon = \frac{d\sigma}{E_t} \quad (19)$$

where E_t is the slope of the tangent to the stress strain curve also shown in Figure 1. Theories of this type have been found very satisfactory and are simple to use for the case of uniaxial loading such as one would find in columns and tension members; but when the loading becomes complex, such as that for a buckled plate, these theories do not lend themselves too readily. The various theories of plastic buckling which are applicable to plate problems seek to generalize equations 18 and 19 so as to formulate stress-strain relations for complex loading in the region beyond the elastic range. Both type theories assume that plastic laws hold while loads are increasing beyond the elastic limit, but that elastic relations must be used for conditions of unloading. This conforms closely to the physical relation shown in Figure 12, if the hysteresis losses are neglected as they might well be under quasi-static loadings. Whether the change in stress state constitutes a loading or an unloading condition is dependent upon whether or not it causes an increase or a decrease in a rationally invariant stress function upon which these theories are based. This function, which is sometimes called the stress intensity, is usually given in terms of the maximum shearing stress.

Stowell³⁸ has developed a theory for the buckling of columns and plates which falls under the deformation classification, while Handelman and Pragner³⁹ have developed a theory of the flow type. These theories are representative of the two types and will be the only ones considered here.

A third type theory which is based on neither of the above concepts, but on the physical concept of slip with the crystalline structure, has recently been developed by Batdorf and Budiansky⁴⁰. These theories will now be discussed in the order in which they have been mentioned.

A UNIFIED THEORY OF PLASTIC BUCKLING OF COLUMNS AND PLATES⁴¹

This theory is an attempt to eliminate the necessity of having to use one theory when investigating axially loaded members and a different one when investigating members under complex loadings. Beginning with the stress invariant given by Ilyushin⁴², i.e.,

$$\sigma_i = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau^2}$$

$$\Theta_i = \frac{2}{\sqrt{3}} \sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_x \epsilon_y + \frac{\gamma^2}{4}}$$

where τ = shear stress and γ = shear strain.

(20)

³⁸Stowell, op. cit., pp. 1-31.

³⁹Handelman and Pragner, op. cit., pp. 1-97.

⁴⁰Batdorf and Budiansky, op. cit., pp. 1-33.

⁴¹Stowell, loc. cit.

⁴²Ilyushin, op. cit., pp. 1-30.

Dr. Stowell has developed the equation of equilibrium for a buckled plate.

According to the fundamental hypothesis of the theory of plasticity, the stress intensity σ_i is a uniquely defined, single valued function of the intensity of strain ϵ_i if σ_i is increasing (i.e., under a loading condition). If σ_i is decreasing (unloading), the relation of σ_i to ϵ_i must be linear since this represents an elastic condition. (See Figure 12.) In equation 20, the material is taken to be incompressible, and Poisson's ratio is then one-half. The stress-strain relations compatible with equation 20 are then found to be

$$\begin{aligned}\epsilon_x &= \frac{\sigma_x - \frac{1}{2} \sigma_y}{E_s} \\ \epsilon_y &= \frac{\sigma_y - \frac{1}{2} \sigma_x}{E_s} \\ \gamma &= \frac{3 \tau}{E_s} \\ \epsilon_i &= \frac{\sigma_i}{E_s}\end{aligned}\tag{21}$$

These are the same as the elastic relations developed by Timoshenko⁴³, except that they are modified by the value of E_s .

By writing the differential equations of equilibrium for a

⁴³ S. Timoshenko, Theory of Elasticity (New York: McGraw-Hill Book Company, 1934), pp. 8-9.

differential element of a buckled plate and solving these with equation 21, the equations of equilibrium for the buckled plate are found to be

$$\begin{aligned}
 & \left[1 - \frac{3}{4} \left(\frac{\sigma_x}{\sigma_i} \right)^2 \left(1 - \frac{E_t}{E_s} \right) \right] \frac{\partial^4 \omega}{\partial x^4} - 3 \frac{\sigma_x \tau}{\sigma_i^2} \left(1 - \frac{E_t}{E_s} \right) \frac{\partial^4 \omega}{\partial x^3 \partial y} \\
 & + 2 \left[1 - \frac{3}{4} \frac{\sigma_x \sigma_y + 2\tau^2}{\sigma_i^2} \left(1 - \frac{E_t}{E_s} \right) \right] \frac{\partial^4 \omega}{\partial x^2 \partial y^2} - 3 \frac{\sigma_y \tau}{\sigma_i^2} \left(1 - \frac{E_t}{E_s} \right) \frac{\partial^4 \omega}{\partial x \partial y^3} \\
 & + \left[1 - \frac{3}{4} \left(\frac{\sigma_y}{\sigma_i} \right)^2 \left(1 - \frac{E_t}{E_s} \right) \right] \frac{\partial^4 \omega}{\partial y^4} = - \frac{h}{D'} \left(\sigma_x \frac{\partial^2 \omega}{\partial x^2} + \sigma_y \frac{\partial^2 \omega}{\partial y^2} + 2\tau \frac{\partial^2 \omega}{\partial x \partial y} \right)
 \end{aligned}
 \tag{22}$$

where w denoted deflection and $D' = \frac{E_t h^3}{9}$ is the flexural rigidity of the plate. It is interesting to note that when $E_{sec} = E$ as in the elastic range and ν is taken as $\frac{1}{2}$, this relation reduces to the form

$$\nabla^4 \omega = - \frac{h}{D} \left(\sigma_x \frac{\partial^2 \omega}{\partial x^2} + \sigma_y \frac{\partial^2 \omega}{\partial y^2} + 2\tau \frac{\partial^2 \omega}{\partial x \partial y} \right)
 \tag{24}$$

and is precisely the equation of an elastically buckled surface as

given by Timoshenko.⁴⁴

Dr. Stowell has applied the above equation to several basic types of structural members and a quantity η , the number by which the critical buckling stress computed from elastic equations must be multiplied to give the critical stress for the plastic case, has been determined. The values of η given in the table on the following page were obtained by dividing the critical stress of the structure in the plastic range by the critical stress that would be obtained on the assumption of perfect elasticity.

HANDELMAN, PRAGNER THEORY⁴⁵

Instead of using the stress invariant developed by Ilyushin⁴⁶ as used in most other theories, a new set of stress strain relations is worked out. The development of such relations is quite complicated because the material, which was originally in a state of simple compression, is not loaded in some regions and unloaded in other during the process of buckling. With this thought in mind, the authors of this theory arrived at the following relations on the basis of intuitive thinking and comparative symmetry:

⁴⁴ S. Timoshenko, Theory of Elastic Stability (New York: McGraw-Hill Book Company, 1936) p. 324.

⁴⁵ Handelman and Pragner, op. cit., pp. 1-33.

⁴⁶ Ilyushin, op. cit., pp. 1-30.

TABLE I

VALUES OF η FOR SEVERAL LOADING CONDITIONS

Structure	η
Long flange, one unloaded edge simply supported	$\frac{E_s}{E}$
Long flange, one unloaded edge clamped	$\frac{E_s}{E} \left(0.428 + 0.572 \sqrt{\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}} \right)$
Long plate, both unloaded edges simply supported	$\frac{E_s}{E} \left(\frac{1}{2} + \frac{1}{2} \sqrt{\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}} \right)$
Long plate, both unloaded edges clamped	$\frac{E_s}{E} \left(0.352 + 0.648 \sqrt{\frac{1}{4} + \frac{3}{4} \frac{E_t}{E_s}} \right)$
Short plate, loaded as a column	$\frac{1}{4} \frac{E_s}{E} + \frac{3}{4} \frac{E_t}{E}$
Square plate, loaded as a column	$0.114 \frac{E_s}{E} + 0.886 \frac{E_t}{E}$
Long column	$\frac{E_t}{E}$

$$E_o d\epsilon_x = \lambda d\sigma_x - \left(\nu + \frac{\lambda-1}{2} \right) d\sigma_y$$

$$E_o d\epsilon_y = - \left(\nu - \frac{\lambda-1}{2} \right) d\sigma_x + \frac{\lambda+3}{4} d\sigma_y$$

$$E_o d\epsilon_z = - \left(\nu + \frac{\lambda-1}{2} \right) d\sigma_x - \left(\nu - \frac{\lambda-1}{4} \right) d\sigma_y$$

$$E_o d\gamma_{xy} = 2 (1 + \nu) d\tau_{xy}$$

(25)

The significance of the symbols is as follows:

E_o = Young's Modulus

λ = $\frac{E_o}{E}$

$d\epsilon$ = Infinitesimal strain increment

τ = Shear stress

E = Tangent Modulus

ν = Poisson's Ratio

With the above stress strain relations known, the problem is again that of writing the differential equations for the equilibrium of a buckled plate. The procedure is practically the same as for the case of elastic buckling as outlined by Timoshenko except that the

neutral surface does not lie in the center of the plate.⁴⁷

The solution of the equation yields the following results:

$$D_{11} \frac{\partial^4 \omega}{\partial x^4} + 2D_{12} \frac{\partial^4 \omega}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 \omega}{\partial y^4} = \sigma_0 h \frac{\partial^2 \omega}{\partial x^2} \quad (26)$$

where ω = deflection

$$D_{11} = D [1 - C \delta (2 - \nu)^2]$$

$$D_{12} = D [1 - C \delta (2 - \nu)(2\nu - 1)^2]$$

$$D_{22} = D [1 - C \delta (2\nu - 1)^2]$$

$$D = \frac{E_0 h^3}{12(1 - \nu^2)}$$

h = plate thickness

$$\sigma_0 = -\sigma_x$$

$$C = \frac{(\lambda - 1)}{[(5 - 4\nu)\lambda - (1 - 2\nu)^2]}$$

$$\delta = \begin{cases} \frac{1}{2} \left[1 - \frac{3}{2} \mathcal{F}_0^+ + \frac{1}{2} (\mathcal{F}_0^+)^3 \right] & \text{for } k > 0 \\ \frac{1}{2} \left[1 + \frac{3}{2} \mathcal{F}_0^- - \frac{1}{2} (\mathcal{F}_0^-)^3 \right] & \text{for } k < 0 \end{cases}$$

⁴⁷ S. Timoshenko, Theory of Elastic Stability, op. cit., pp. 302-05.

$$\dot{K} = (2 - \nu) \dot{K}_1 + (2\nu - 1) \dot{K}_2$$

$$\dot{K}_1 = \frac{\partial^2 \omega}{\partial x^2}$$

$$\dot{K}_2 = \frac{\partial^2 \omega}{\partial y^2}$$

$$\dot{K}_{12} = 2 \frac{\partial^2 \omega}{\partial x \partial y}$$

$$\varphi_0^+ = \alpha + \sqrt{\alpha^2 - 1}$$

$$\varphi_0^- = -\alpha - \sqrt{\alpha^2 - 1}$$

α = ratio of reduced modulus to Young's modulus.

This equation resembles the equation for the buckling of an anisotropic plate given by Timoshenko⁴⁸. In the case of an anisotropic plate, the coefficients D_{11} , D_{12} , and D_{22} are constants for the material; for the plastic case D_{11} , D_{12} , and D_{22} are functions of the stress σ . In other words, the plate is anisotropic but his anisotropy is caused by and is a function of the compressive stress. Graphs for $\frac{D_{11}}{D}$, $\frac{D_{12}}{D}$, and $\frac{D_{22}}{D}$ as functions of λ and ν are given in NACA Technical Note No. 1530.⁴⁹

⁴⁸ Ibid., p. 380.

⁴⁹ Handelman and Pragner, op. cit.

A MATHEMATICAL THEORY OF PLASTICITY BASED ON THE CONCEPT OF SLIP⁵⁰

The plastic deformation of a polycrystalline mass is the accumulative results of the deformation of each individual crystal, the deformation of the individual crystal being the results of the mechanism of slip. It has already been pointed out that slip within an isolated crystal is independent of the normal stress and is only influenced by two factors: namely, (1) the resolved shearing stress on the active planes in the direction of slip, (2) the amount of previous work hardening, since the minimum value of the resolved shearing stress required to produce slip is a function of the amount of this previous workhardening. Although any deformation of isolated crystals is anisotropic, it is generally assumed that a polycrystalline material has isotropic properties and that any infinitesimal part of the substance may be treated as though it were a continuum. Accordingly, instead of considering a small slip along each of a large number of discrete planes, the theory contemplates an infinitesimal plastic shear strain in every possible direction within every possible plane that can be passed through each infinitesimal fraction of the continuum.

If this small fraction of the continuum is imagined to be made up of an infinite number of crystals having all possible orientations with respect to the direction of an applied stress, it is seen that each of these crystals contributes to the strain by an amount which is a function of the shear stress τ that resolves onto its active plane in

⁵⁰ Batdorf and Budiansky, op. cit., pp. 1-33.

the direction of slip.

If this small fraction of the continuum be enclosed within a sphere, it is convenient to represent the orientation of a particular plane by the co-ordinates of the point at which it would be tangent to the sphere. The radius to the point of tangency is the normal to the plane. The axis 1 shown in Figure 24 is taken in the direction of the normal to the plane, and the axis 2 denotes a particular direction of slip within the plane. An infinitesimal band of planes may be represented by the normals to the plane included in the solid angle $d\Omega$ and an infinitesimal band of slip directions may be represented by $d\beta$. In order to include all possible planes only once it is necessary to consider planes tangent to a hemisphere.

The theory now postulates that the slip along the planes $d\Omega$, in the increment $d\beta$ of slip directions, produces an infinitesimal plastic shear strain $d\gamma_{12}''$ associated with the 1,2 axis that is given by the relation

$$d\gamma_{12}'' = F(\tau_{12}) d\Omega d\beta$$

where F is a function depending only on the history of τ_{12} , the shear stress in the 2 direction on a plane perpendicular to the 1 axis.

It follows from the previous discussion concerning slip that, if τ_{12} is gradually increased on any planes within the continuum, the function F must remain zero until a limiting value τ_c is reached, if $d\gamma_{12}''$ is to represent only the portion of the deformation which is due to slip. This limiting value of τ and the manner in which it increases with work hardening has been determined experimentally for

certain types of single crystals by G. I. Taylor⁵¹ and others, but its value is also influenced by neighboring crystals when a polycrystalline aggregate is considered. It, therefore, seems logical to take one-half the elastic limit as determined from a standard uniaxial test for τ_L , since the maximum shear stress developed at this point is one half the normal stress. Beyond this limiting value of τ_L , the function F varies with τ_{12} in a manner which is characteristic of the material and is referred to in this theory as the "characteristic shear function".

It is now assumed that the characteristic shear curve can be represented by a power series of the form

$$F(\tau_{12}) = \sum_{n=1}^N a_n \left(\frac{\tau_{12}}{\tau_L} - 1 \right)^n \quad (28)$$

This function would have a value of zero for $0 < \tau_{12} < \tau_L$ as required.

The contribution of the infinitesimal shear strain $d\gamma''_{12}$ to the strains in the standard x, y, and z axis may be written in terms of the direction cosines of the 1-axis and 2-axis. Thus,

$$\begin{aligned} d\epsilon_x &= \lambda_{x1} \lambda_{x2} d\gamma''_{12} & ; & & d\gamma''_{xy} &= (\lambda_{x1} \lambda_{x2} + \lambda_{y1} \lambda_{y2}) d\gamma''_{12} \\ d\epsilon_y &= \lambda_{y1} \lambda_{y2} d\gamma''_{12} & ; & & d\gamma''_{xz} &= \dots \\ d\epsilon_z &= \lambda_{z1} \lambda_{z2} d\gamma''_{12} \end{aligned} \quad (29)$$

⁵¹ Taylor, op. cit., pp. 16-72.

where $\lambda_{x1}, \lambda_{y1}, \dots, \lambda_{z2}$ are the cosines of the angles between x and $1, y$ and $1, \dots, z$ and 2 directions.

In order to find the total plastic shear strains in the standard axes, the effects of the plastic shear-strain increments must be summed up over all possible directions and all possible planes. Thus

$$\begin{aligned} \epsilon_x'' &= \int_H \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(T_{12}) \lambda_{x1} \lambda_{x2} d\Omega d\beta \\ &\dots\dots\dots \\ \gamma_{xy}'' &= \int_H \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(T_{12}) (\lambda_{x1} \lambda_{y2} + \lambda_{y1} \lambda_{x2}) d\Omega d\beta \end{aligned} \quad (30)$$

where H denotes the entire hemisphere. The slip direction β is integrated through 180° so as to include all possible directions only once.

If the applied stresses are given in the standard co-ordinate system as $\sigma_x, \sigma_y, \sigma_z, \dots, \tau_{yz}$ the resolved shear stress in the $1,2$ direction is given by

$$\begin{aligned} T_{12} &= \lambda_{x1} \lambda_{x2} \sigma_x + \lambda_{y1} \lambda_{y2} \sigma_y + \lambda_{z1} \lambda_{z2} \sigma_z \\ &+ (\lambda_{x1} \lambda_{y2} + \lambda_{y1} \lambda_{x2}) \tau_{xy} + (\lambda_{x1} \lambda_{z2} + \lambda_{z1} \lambda_{x2}) \tau_{xz} \\ &+ (\lambda_{y1} \lambda_{z2} + \lambda_{z1} \lambda_{y2}) \tau_{yz} \end{aligned} \quad (31)$$

The next step in the development of this theory is to determine the characteristic shear function. Substituting equation 28 into equation 30 gives

$$\epsilon_x'' = \int_H \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=1}^N a_n \left(\frac{\tau_{12}}{\tau_L} - 1 \right)^n \lambda_{x1} \lambda_{x2} d\Omega d\beta$$

• • • •

(32)

Since τ_{12} is the shear stress component in the 1,2 direction of the particular applied stress system, it is a function of σ as given by equation 31. The next step is then to substitute the appropriate part of equation 31 into equation 32, and $\sigma_L/2$ for τ_L , since it has been pointed out that the limiting value of τ_{12} would be taken as one-half the yield stress σ_L .

In order to facilitate integration of equation 30, the authors of this theory developed a set of analytical expressions to transform the direction cosines $\lambda_{x1}, \lambda_{x2}, \dots, \lambda_{z2}$ into a polar-co-ordinate system. Before equations 30 are in order to integrate, it is necessary to select the stress system which is to be applied to the continuum. If the characteristic function is to be determined from the standard stress strain diagram, it is necessary to select uniaxial loading. Let this loading be applied in the direction of the z axis; then equation 31 reduces to

$$\tau_{12} = \lambda_{z1} \lambda_{z2} \sigma_z$$

Substituting this relation into equation 32 gives

$$\epsilon_z'' = \int_H \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sum_{n=1}^N a_n \left(2\lambda_{21} \lambda_{22} \frac{\sigma_z}{\sigma_L} - 1 \right)^n d\Omega d\beta \quad (34)$$

Integration of equation 34 gives

$$\epsilon_z'' = \sum_{n=1}^N a_n g_n \left(\frac{\sigma}{\sigma_L} \right) \quad (35)$$

The g functions are not characteristic of the material and have been calculated for several values of σ/σ_L and are given in NACA Report No. 1871.⁵²

The values of the a_n are now determined from the stress strain curve of the material in the following way: if it is assumed that the characteristic function can be represented by a fifth power series, then values of $g_1, g_2, \dots, g_5$ are taken from the above report. σ_L is next determined from the stress strain curve. From the ratios of σ/σ_L at which the g functions were taken, σ can be determined; that is, if the g function was calculated at a value of $\sigma/\sigma_L = k$, then $\sigma = k\sigma_L$. With this value of σ the corresponding value of ϵ can be found from the stress strain curve. This is done for each of the five g values, and substitution of these

⁵² Batdorf and Budiansky, Loc. cit.

values into equation 35 yields five simultaneous equations from which the value of $a_1, a_2, \dots, a_5$ may be determined.

Once the characteristic shear function has been found, it is only necessary to apply equations 30 and 31 to obtain the plastic strains resulting from any sequence of stresses.

CONCLUSIONS

As results of this study, the following conclusions have been reached:

1. No crystalline substance obeys Hook's Law. It has been pointed out that very accurate measurements of stress and strain always indicate hysteresis losses even in the low stress range. If this were not true there could be no internal damping of freely vibrating members.

2. All purely elastic deformations within crystalline substances are the result of:

- (a) a change in the dimensions of the space lattice
- (b) a change in the crystal spacing
- (c) or a combination of the two.

It was pointed out in this thesis that any deformation due to slip was permanent unless removed by application of externally applied shearing stresses, opposite in sense to those originally producing the deformation, or by recrystallization due to annealing. It was also stated that no measurable deformation prior to rupture was exhibited by a tin crystal when the stress was applied normal to its single set of slip planes.

3. There can be no definite line of demarkation between elastic and plastic deformations within polycrystalline substances. Since it is the resolved shear stress that induces slip, it is evident that, with a random orientation of the crystals, slip is more inducive in those crystals which are more favorably orientated. It may therefore be concluded that all measurable deformations are the sum of an elastic

and a plastic part. Elastic after-effect as previously discussed may be explained on this basis if it is assumed that the elastic part of the deformation produces enough energy to restore the plastically deformed crystals by a creep effect after the stress is removed.

4. The ionized theory of atomic binding within crystalline structures offers a better explanation of the following changes when a crystalline structure is deformed:

- (a) the density is slightly decreased
- (b) the electrical resistance is changed
- (c) the magnetic properties are changed
- (d) the thermal conductivity is changed
- (e) worked metal is more easily dissolved and more easily corroded.

5. The results of rate of strain on corresponding stresses were found to be very pronounced in single crystal test specimens at relatively high temperatures, but of small magnitude at normal temperatures. The same results, but of lesser magnitude, were found to exist in polycrystalline materials.

Up to the present time, very little study has been made of the effects of varying strain rates on internal energies, and no correlation has been established between internal energies and deformation in the plastic range.

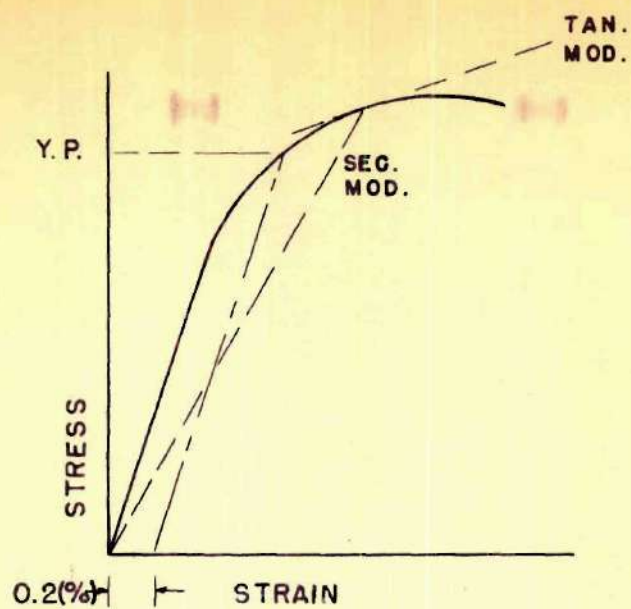
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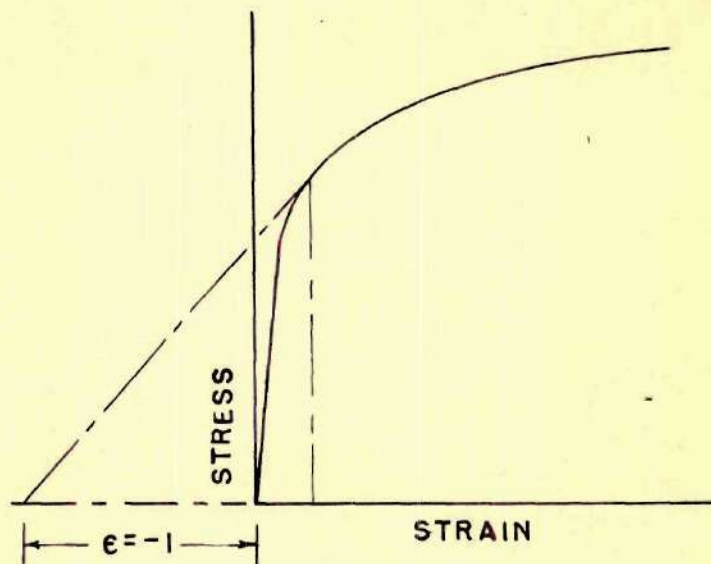
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APPENDIX



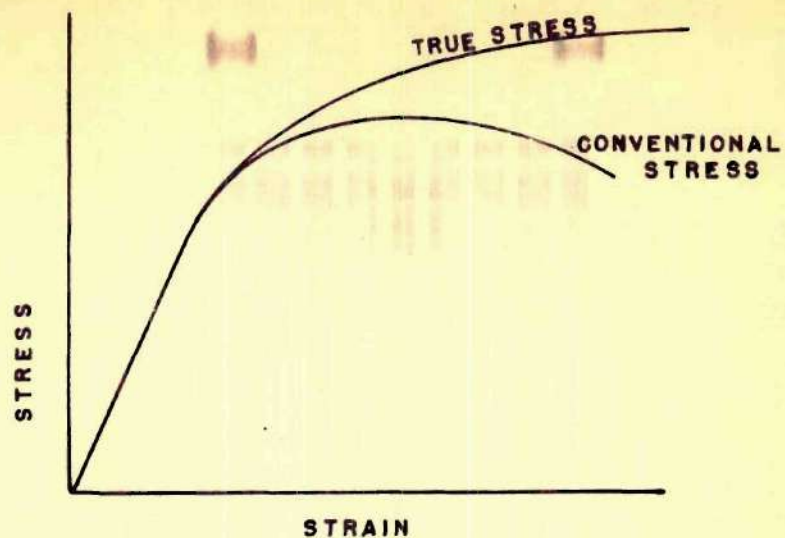
CONVENTIONAL STRESS-STRAIN DIAGRAM

FIGURE 1



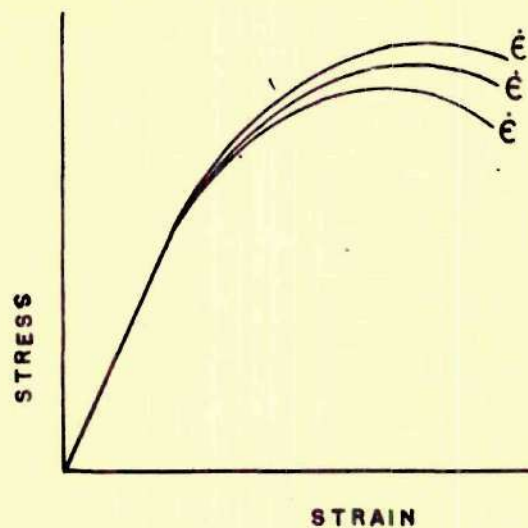
CURVE OF TRUE STRESS AND NATURAL STRAIN

FIGURE 2



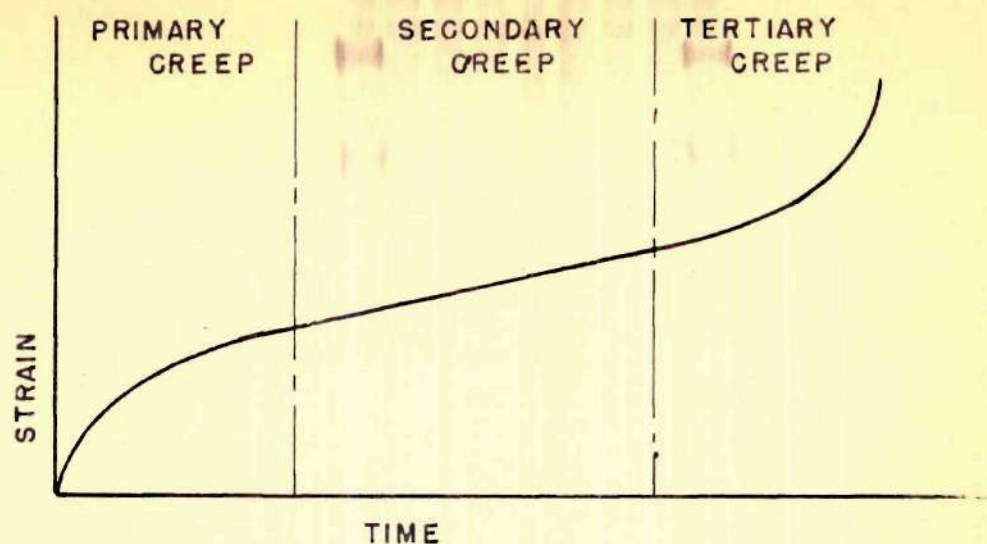
STRESS-STRAIN CURVE AND CURVE OF TRUE STRESS

FIGURE 3



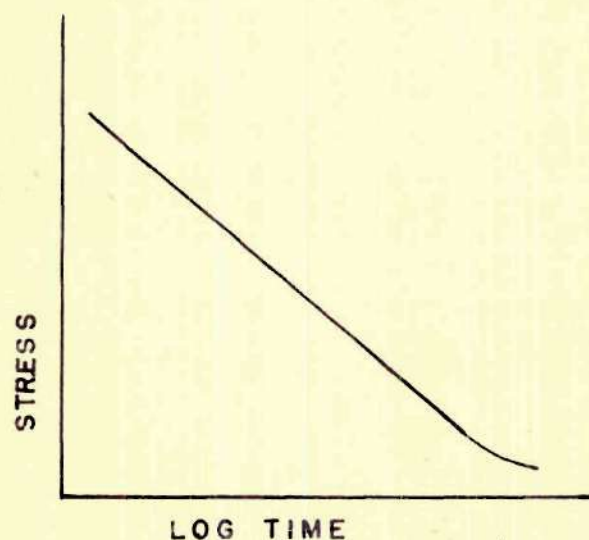
STRESS-STRAIN CURVE AS A
FUNCTION OF $\dot{\epsilon}$

FIGURE 4



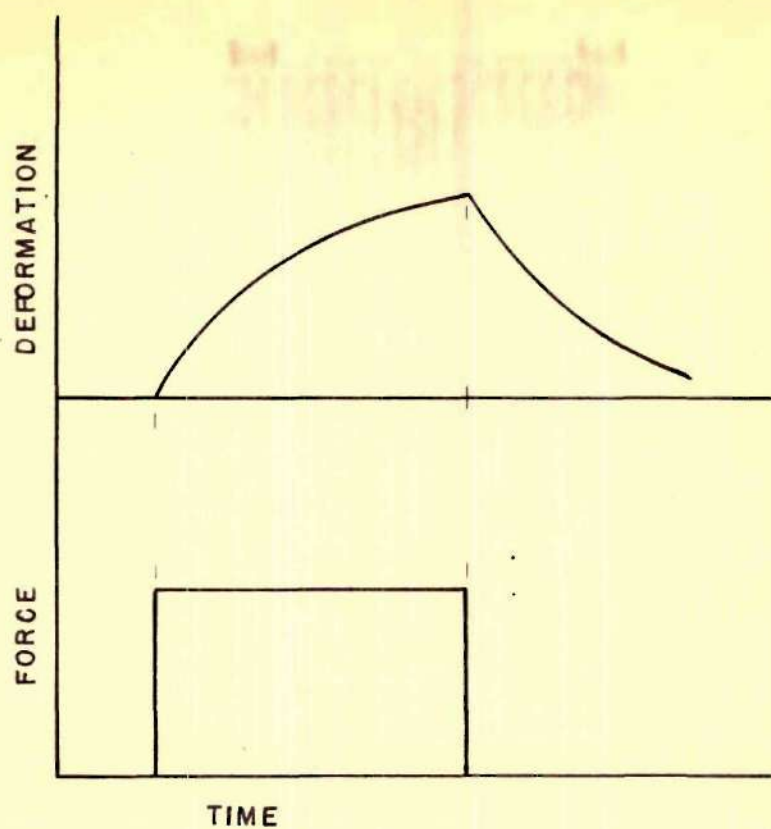
STRAIN-TIME PLOT FOR SUDDENLY APPLIED STRESS

FIGURE 5



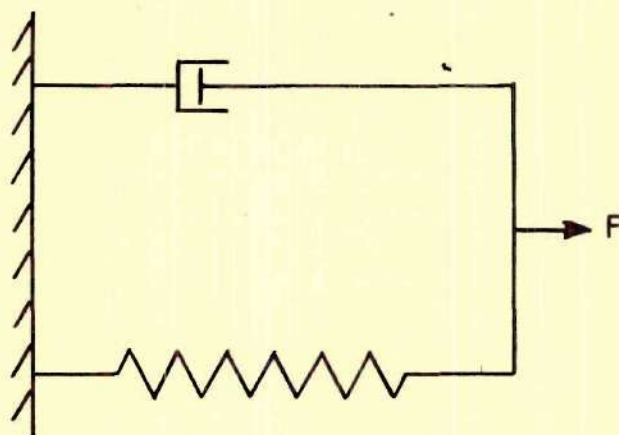
STRESS VERSUS LOGARITHMIC TIME
FOR SUDDENLY APPLIED STRAIN

FIGURE 6



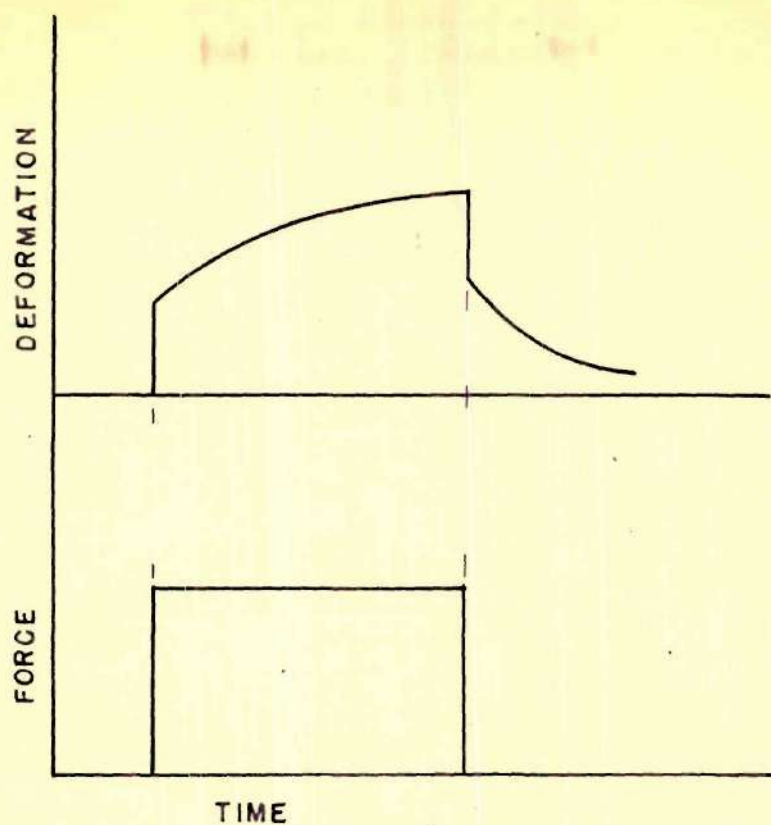
MECHANICAL BEHAVIOR OF THE VOIGOT SOLID

FIGURE 7



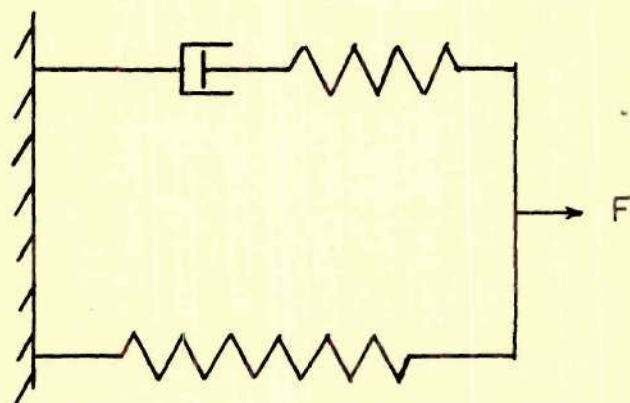
MECHANICAL MODEL OF THE VOIGOT SOLID

FIGURE 8



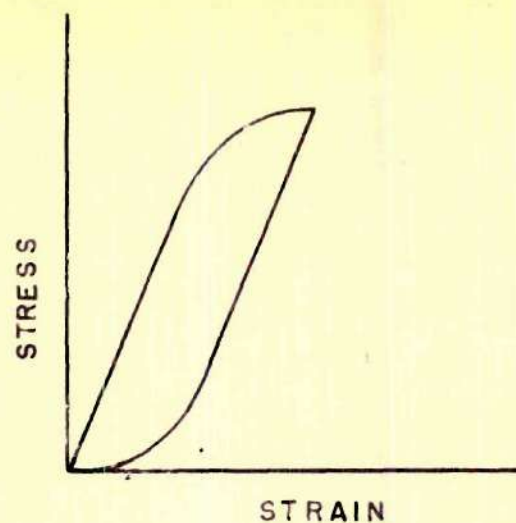
MECHANICAL BEHAVIOR OF THE LINEAR SOLID

FIGURE 9



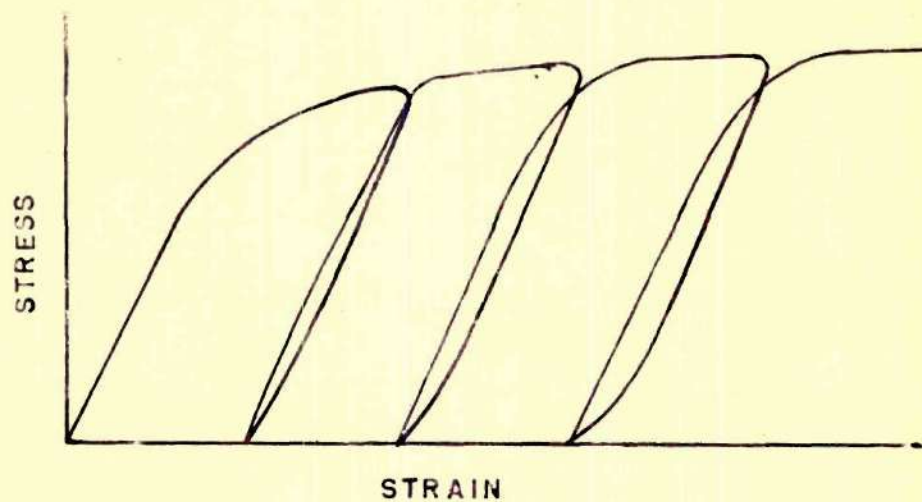
MECHANICAL MODEL OF THE LINEAR SOLID

FIGURE 10



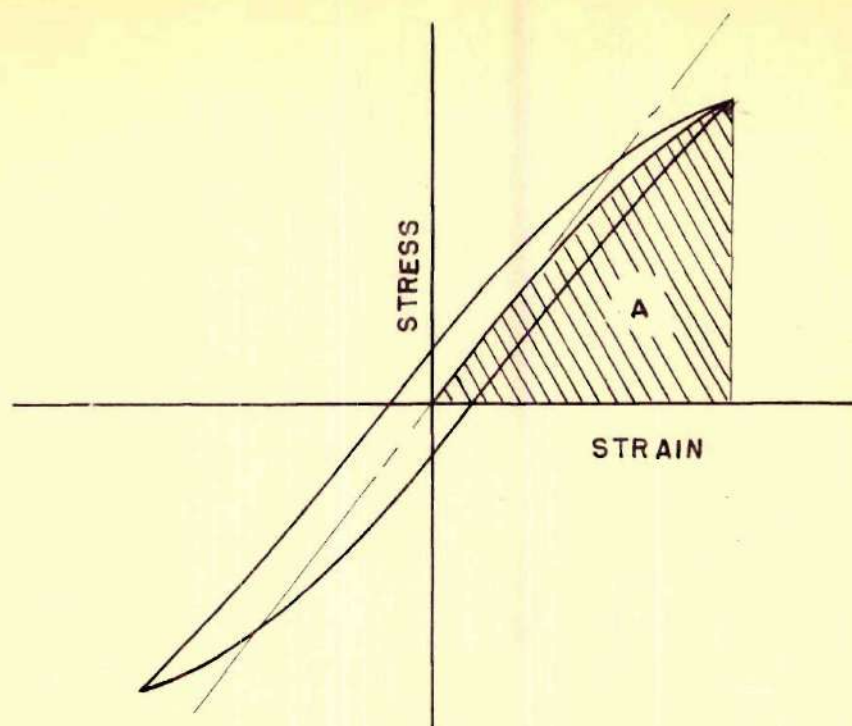
HYSTERESIS LOOP FOR A LOADING AND UNLOADING CYCLE

FIGURE 11

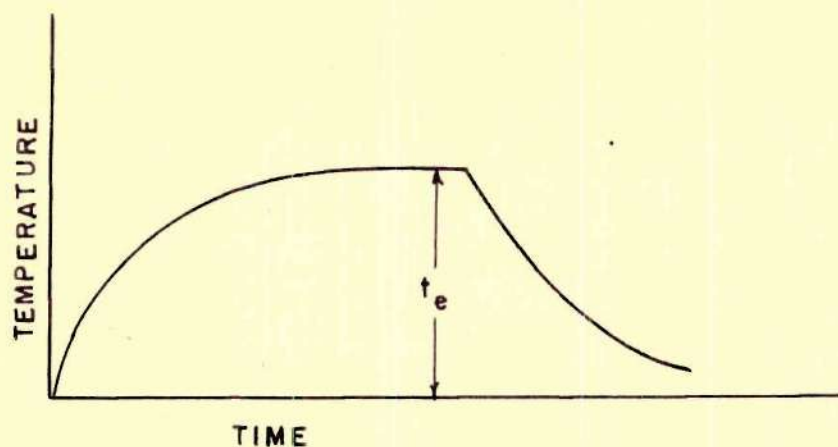


HYSTERESIS LOOP FOR REPEATED CYCLE

FIGURE 12

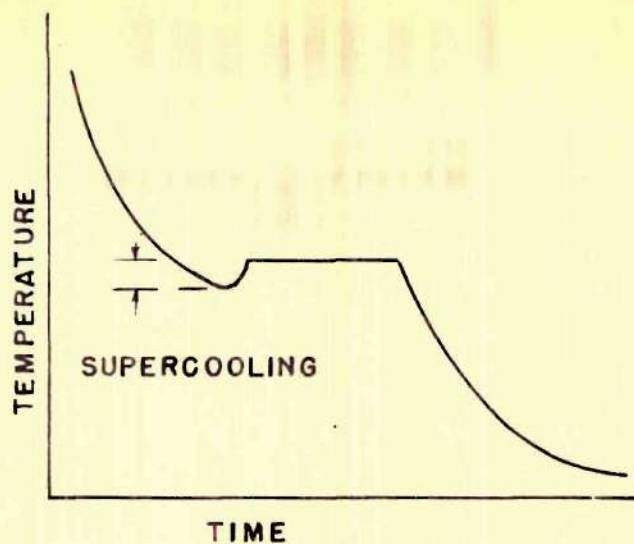


HYSTERESIS LOOP FOR A REVERSED LOADING CYCLE
FIGURE 13



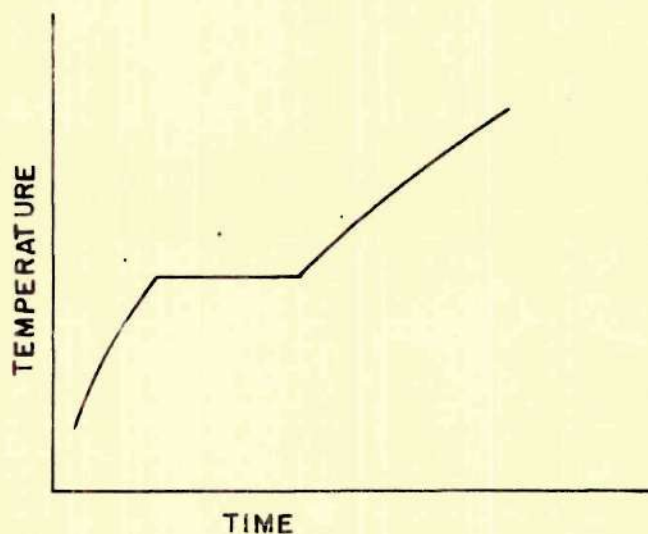
SCHEMATIC DIAGRAM SHOWING TEMPERATURE OF FATIGUE SPECIMEN AS A FUNCTION OF TIME AFTER START

FIGURE 14



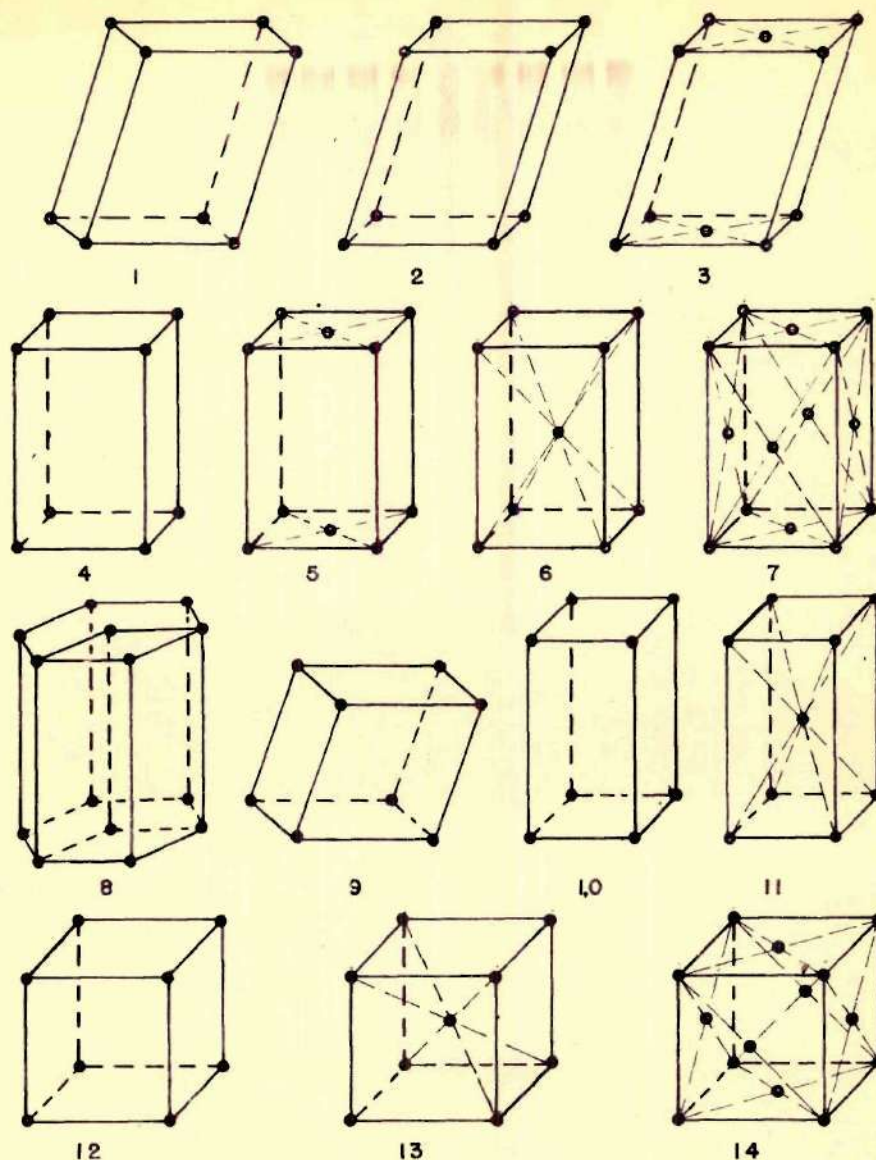
SOLIDIFICATION OF LIQUID METAL AS INDICATED ON A COOLING CURVE

FIGURE 15a



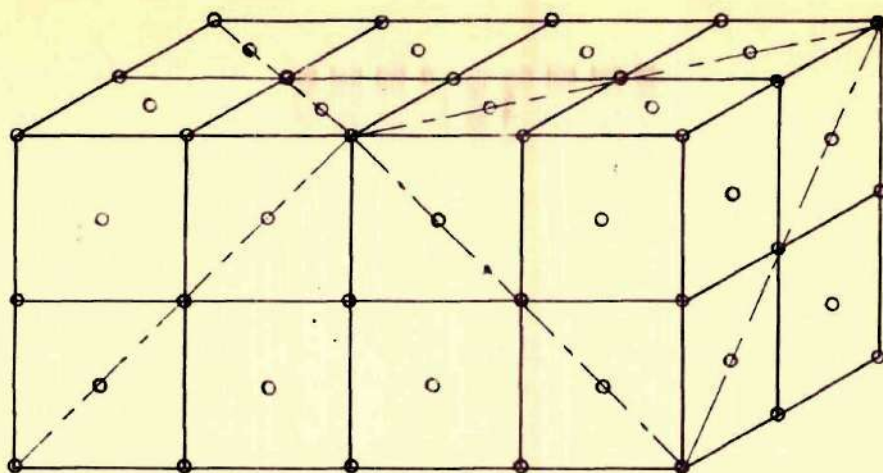
MELTING OF SOLID METAL AS INDICATED ON A HEATING CURVE

FIGURE 15b



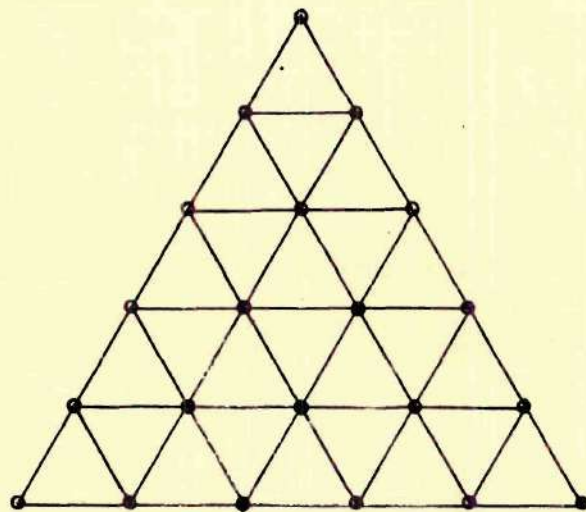
A UNIT CELL OF EACH OF THE FOURTEEN SPACE LATTICE FORMATIONS
 (1) TRICLINIC, SIMPLE ; (2) MONOCLINIC, SIMPLE ; (3) MONOCLINIC, BASE CENTERED ;
 (4) ORTHORHOMBIC, SIMPLE ; (5) ORTHORHOMBIC, BASE CENTERED ; (6) ORTHORHOMBIC,
 BODY CENTERED ; (7) ORTHORHOMBIC, FACE CENTERED ; (8) HEXAGONAL ; (9) RHOM-
 BOHEDRAL ; (10) TETRAGONAL, SIMPLE ; (11) TETRAGONAL, BODY CENTERED ;
 (12) CUBIC, SIMPLE ; (13) CUBIC, BODY CENTERED ; (14) CUBIC, FACE CENTERED.

FIGURE 16



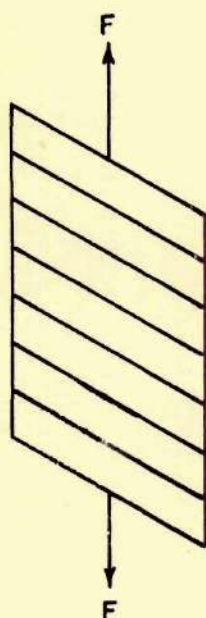
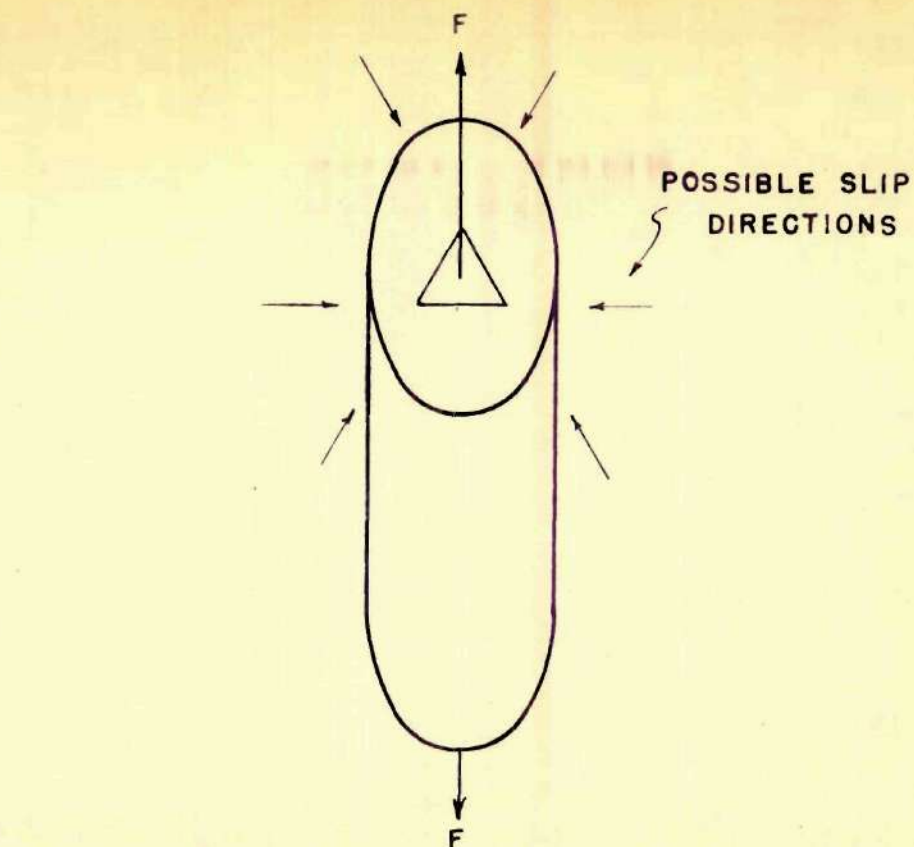
DASHED LINES INDICATE ORIENTATION OF TWO OF THE FOUR CLOSE-PACKED PLANES WITHIN THE FACE-CENTERED CUBIC.

FIGURE 17

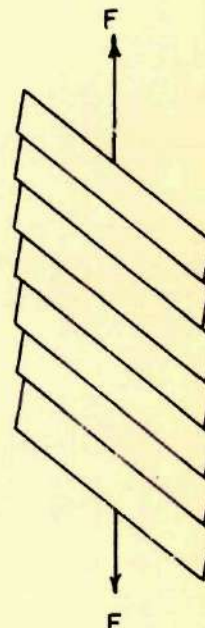


ATOM ARRANGEMENT ON ONE OF THE ABOVE CLOSE-PACKED PLANES. LINES INDICATE CLOSE-PACKED DIRECTIONS.

FIGURE 18



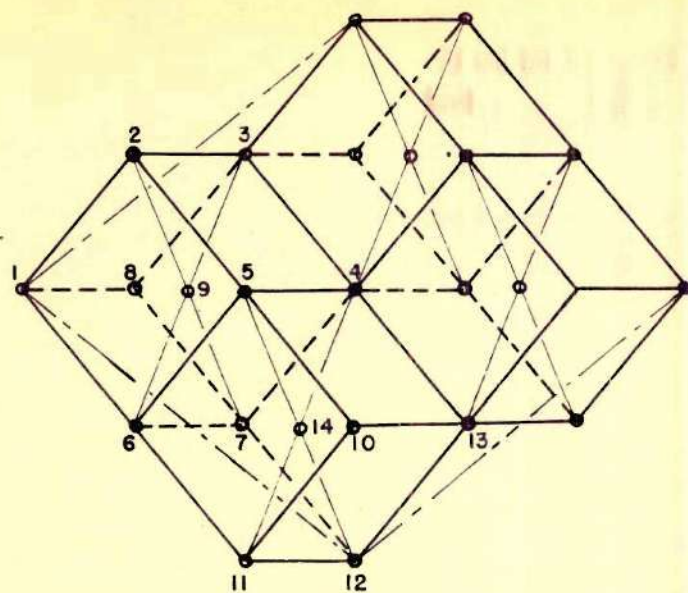
BEFORE DISTORTION



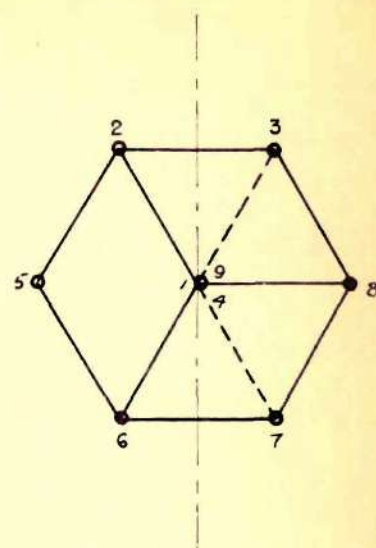
AFTER DISTORTION

DIAGRAM ILLUSTRATING SLIP IN CRYSTAL HAVING FACE-CENTERED CUBIC LATTICE

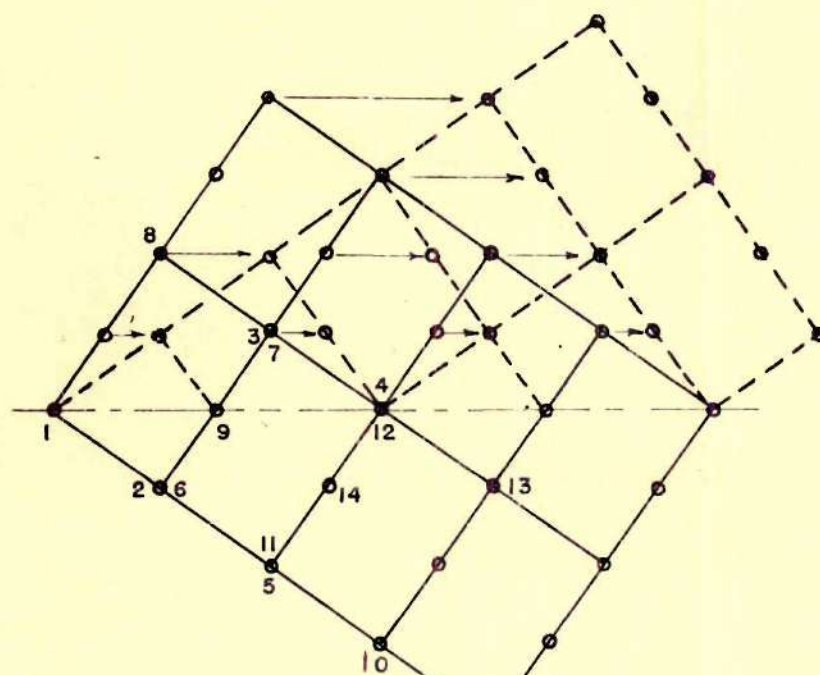
FIGURE 19



(b)



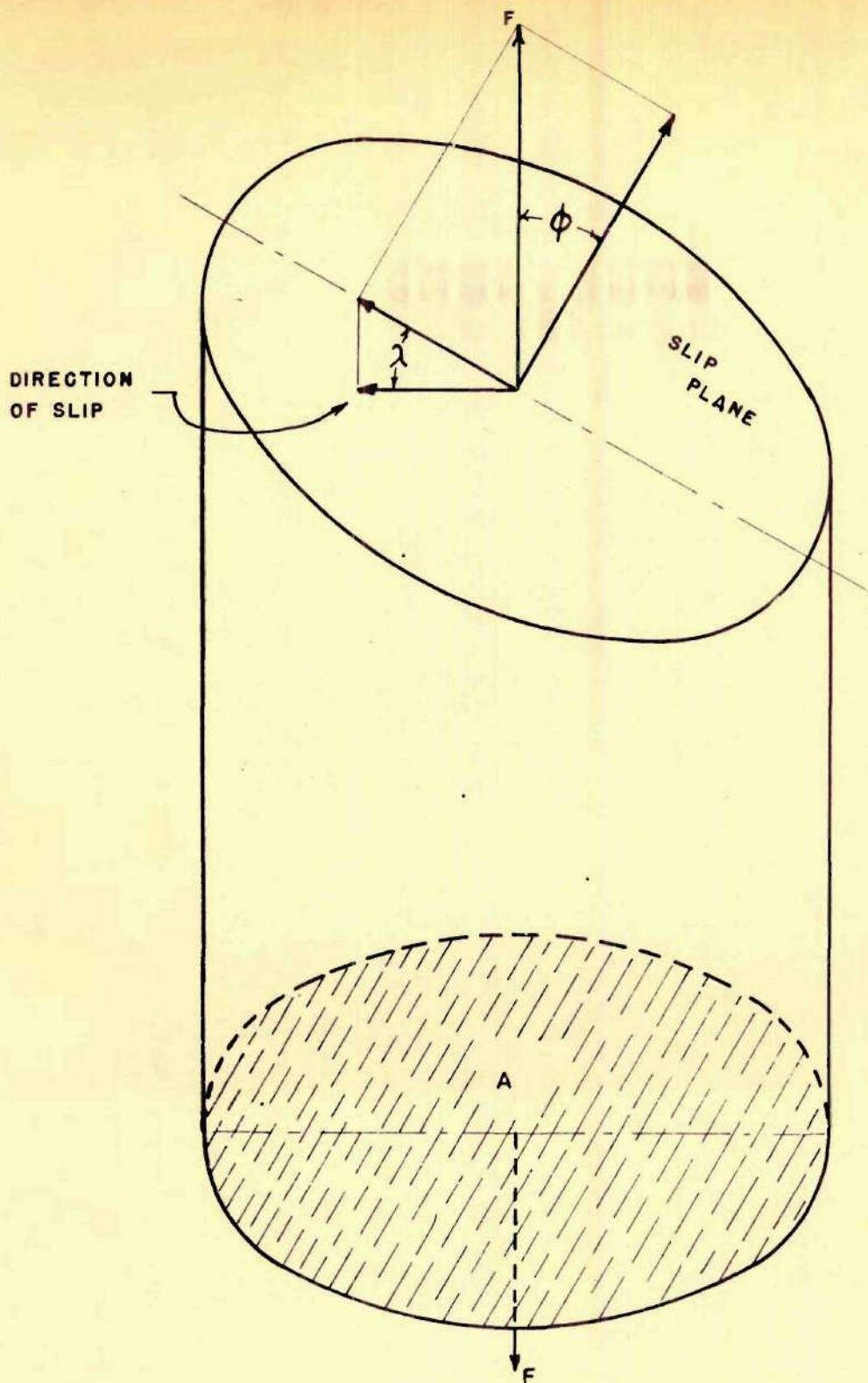
(a)



(c)

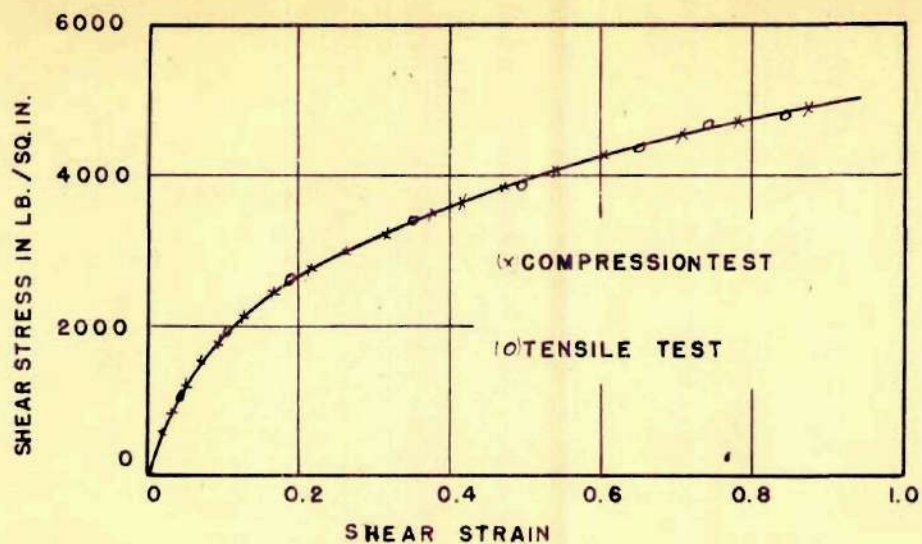
DIAGRAM SHOWING RELATIVE MOVEMENT OF ATOMS IN TWINNING OF A
BODY-CENTERED CUBIC CRYSTAL

FIGURE 20



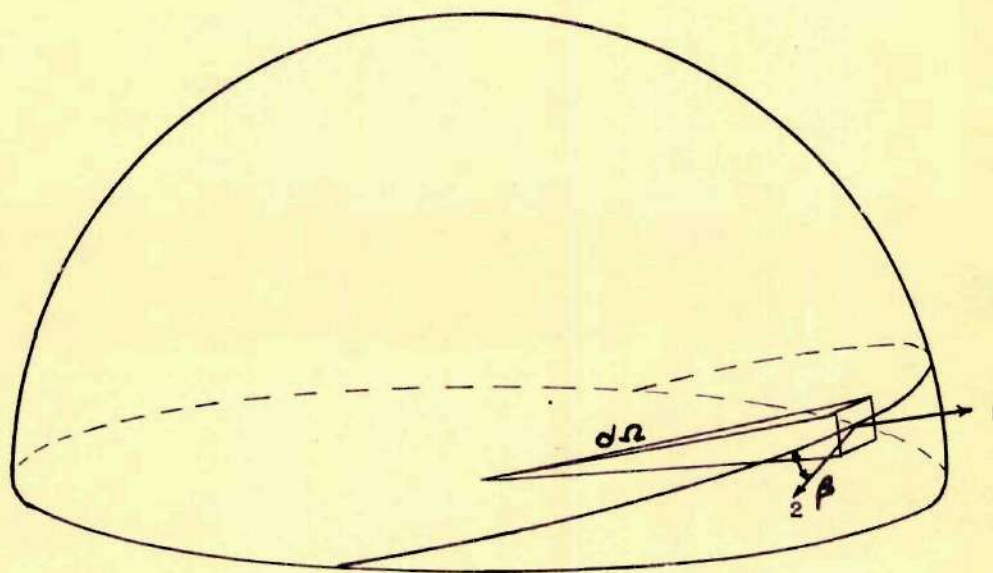
COORDINATES FOR CALCULATING RESOLVED SHEARING STRESSES

FIGURE 21



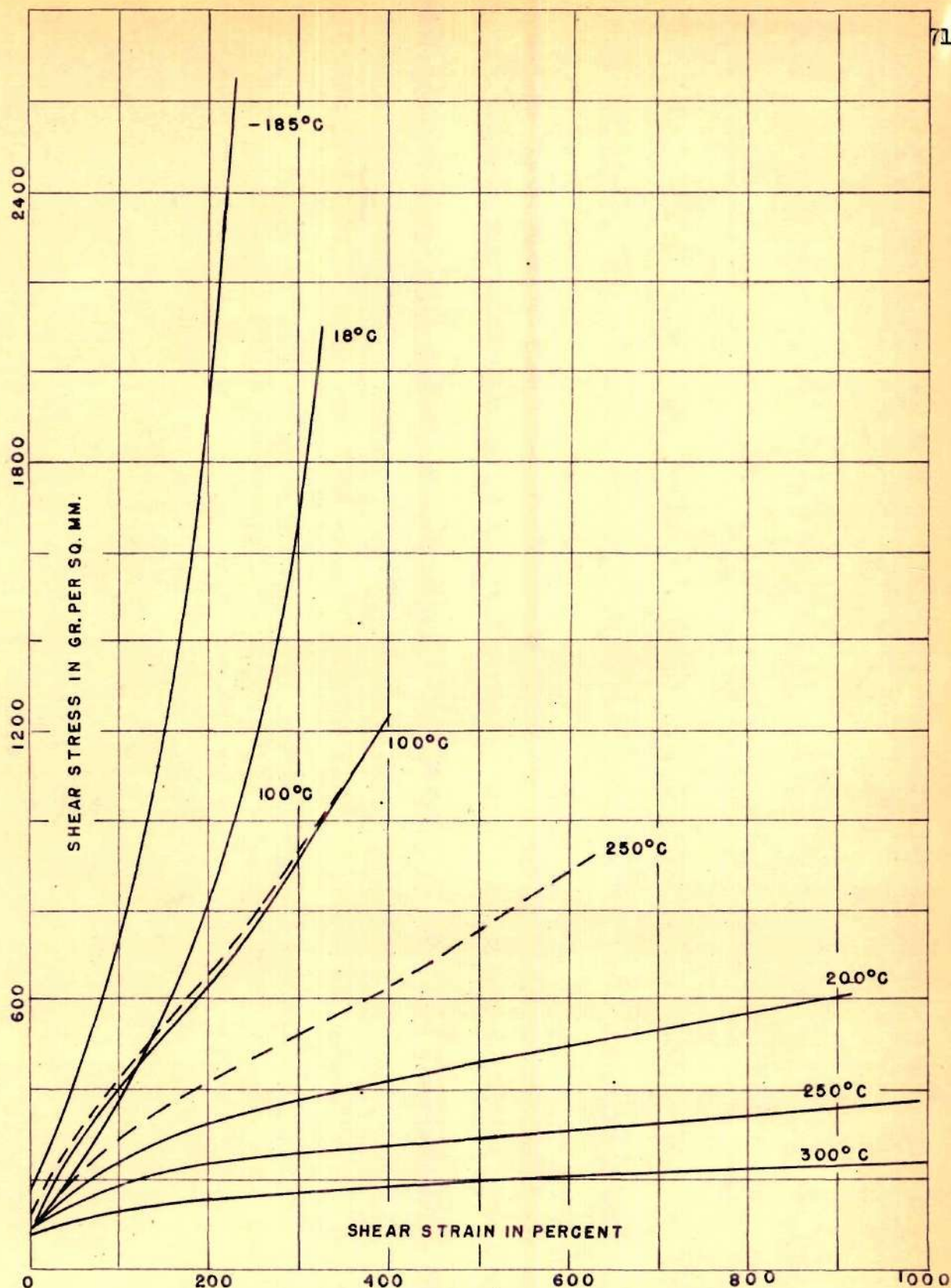
STRAIN HARDENING IN ALUMINUM SPECIMEN (G.I. TAYLOR)

FIGURE 22



VECTORS NORMAL TO PLANE OF SLIP AND IN DIRECTION OF SLIP

FIGURE 24



STRAIN HARDENING IN MAGNESIUM CRYSTAL AT DIFFERENT TEMPERATURES. THE DOT-
TED CURVES ARE FOR A RATE ONE HUNDRED TIMES FASTER THAN THE OTHERS.

FIGURE 23

(BY SCHMID & SIEBEL)