ACCURACY AND EFFICIENCY OF BARRIER-HITTING AND EXTREME EVENT SIMULATION

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To Ani and Amayi

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SUMMARY

In this thesis we consider the broad field of simulation of barrier-hitting and extreme events events of stochastic processes. Our focus is on the analysis of the *efficiency* and *accuracy* of these simulation methods; especially, we are interested in providing theoretical mathematical guarantees for efficiency and accuracy. Loosely speaking, *efficiency* of a simulation algorithm is related to the time the algorithm requires to produce an output sample; and *accuracy* relates to how close is the distribution of the output of the algorithm, compared to the "true" or "target" distribution of the event intended to sample. A third topic that permeates all aspects of this thesis, although mildly, is the simulation of reflected processes, which is usually done by simulating the extremes of a "free" or "unreflected" process. We now give a brief description of the parts of this thesis.

In Chapter 1 we give a general overview of Chapters 2 and 3 of this thesis. We argue that the overarching theme connecting them is the study of efficiency and accuracy of methods for the simulation of barrier-hitting events, extreme events, or events of reflected processes.

In Chapter 2 we consider heavy-tailed random walks with negative drift, and study the simulation of paths of the random walk up to the first time it crosses a fixed positive barrier. We consider the framework of *exact* sampling algorithms that use the change of measure technique; in particular, with this technique the simulated barrier-hitting paths are unbiased or completely accurate. As a consequence, our work in Chapter 2 is mostly concerned with the study of efficiency of these methods. For that, we propose a framework to evaluate how efficient is a change of measure for the simulation of barrier-hitting paths with the change of measure technique. We apply this framework to evaluate the efficiency of the change of measure of Blanchet and Glynn [1] when the random walk step sizes have regularly varying right tails, say with tail index $\alpha > 1$, and lighter left tails. We study this particular change of measure because it has been a successful measure for the sampling of rare events for heavy-tailed random walks with Importance Sampling. Our main contributions of Chap-

ter 2 are the following. First, for a requirement characterizing how much more likely the change of measure is to sample a barrier-hitting path, the Blanchet-Glynn change of measure is efficient only when the tail index α is in $(1,\ 3/2)$; in particular this is the regime of heaviest tails. Second, for a requirement quantifying the expected number of steps needed to sample a barrier-hit with the Blanchet-Glynn change of measure, we show that the expected number of steps is infinite for α in $(1,\ 3/2)$. In particular, the latter result corrects a non-central result of [1]. We also discuss that the value $\alpha = 3/2$ has arisen as a threshold for efficiency criteria in other related algorithms for random walks with regularly varying right tails, which raises the question of how precisely are these works connected. We remark that the work in Chapter 2 was originally motivated by the design of an algorithm for exact sampling of stationary reflected processes.

In Chapter 3 of this thesis we derive weak limits related to the discretization errors of sampling barrier-hitting and extreme events of Brownian motion by using the Euler discretization simulation method. In detail, we consider the Euler discretization approximation of Brownian motion to sample barrier-hitting events, i.e. hitting for the first time a deterministic "barrier" function; and to sample extreme events, i.e. attaining a minimum on a given compact time interval or unbounded closed time interval. For each case we study the discretization error between the actual time the event occurs versus the time the event occurs for the discretized path, and also the discretization error on the position of the Brownian motion at these times. We show that if the step-size of the time mesh is $1/n$ then in each of the three aforementioned events the discretization error for the times converges at rate $O(1/n)$ and the error for the position of the Brownian motion converges at rate $O(1/\sqrt{n})$. We show limits in distribution for the discretization errors normalized by their convergence rate, and give closed-form analytic expressions for the limiting random variables. Additionally, we use these limits to study the asymptotic behavior of Gaussian random walks in the following situations: (1.) the overshoot of a Gaussian walk above a barrier that goes to infinity; (2.) the minimum of a Gaussian walk compared to the minimum of the Brownian motion obtained when interpolating the Gaussian walk with Brownian bridges, both up to the same time horizon that goes to infinity; and (3.) the global minimum of a Gaussian walk compared to the global minimum of the Brownian motion obtained when interpolating the Gaussian walk with Brownian bridges, when both have the same positive drift decreasing to zero. In deriving these limits in distribution we provide a unified framework to understand the relation between several papers where the constant $-\zeta(1/2)/\sqrt{2\pi}$ has appeared, where ζ is the Riemann zeta function. In particular, we show that this constant is the mean of some of the limiting distributions we encounter.

CHAPTER 1
INTRODUCTION

In this thesis we consider a classical theme in the field of Applied Probability and Computational Probability: *the simulation of barrier-hitting events of stochastic processes,* and also *the simulation of extreme events of stochastic processes.* That is, algorithms that simulate the stochastic process up to the time where a minimum or maximum (local or global) is attained; or up to the first time it hits a certain deterministic barrier; respectively. Specifically, we are interested in mathematical guarantees for efficiency and accuracy of these type of simulation algorithms.

We focus on stochastic processes which evolve in time is as follows: the process transitions between "states" as time passes by; transitions between states consists of the addition of an uncertain quantity to the current state of the system; and each transition is independent of both the current state of the process and current time. Two fundamental cases can be distinguished: the *discrete time* case where the transitions between states are made at discrete time steps, i.e., there is a "first" transition time, a "second" one, and so on; and the *continuous time* case, where the transitions between states can occur at each of a set of "infinitesimally small" time steps.

To further clarify the aforementioned additive property of the stochastic processes we consider, we focus on processes such that, intuitively speaking, there is a random quantity that is added to the current state of the process at each elapsed time step. This property can be conceptually harder to conceive in the continuous case than in the discrete case, since in the former the transitions occur at infinitesimal time steps. Nonetheless, the so-called Functional Central Limit theorem and its generalizations show that the continuous time case can be obtained by properly "zooming-out" in time-space from a discrete time stochastic process satisfying the additive property.

We work with discrete- and continuous-time processes which, implicitly, can be efficiently simulated over finite time horizons. That is, we will assume that we have at hand an algorithm such that, given any finite time horizon, the algorithm *efficiently* produces sample paths over some or all of the specified finite time horizon. Moreover, we will either assume that the algorithm is *exact*, meaning that the output of the algorithm has the desired stochastic process distribution; or will assume that it is *arbitrarily accurate*, meaning the output is distributed arbitrarily close, in some sense, to the distribution of the stochastic process. In either case, we focus on analyzing the efficiency and accuracy *of simulating barrier-hitting or extreme events*, given a simulation algorithm to sample the process with a given accuracy.

Simulation of barrier-hitting events and of extreme events. In this thesis we study the efficiency and accuracy of simulation algorithms that simulate either barrier-hitting events or extreme events. We now give a loose description of such events.

We call *simulation of barrier-hitting events of a stochastic process* the simulation of the process up to the first time it crosses a certain deterministic barrier. Examples can be simulation of the wear of a production machine up to the first time it crosses a certain wear threshold determined by quality or security purposes; or simulation of a weather forecast model up to the first time the global temperature surpasses a critical ecological threshold.

On the other hand, we call *simulation of a stochastic process up to its extreme* the simulation of the process up to the first time it attains a minimum or maximum of the process, either on a bounded predefined time interval, or on an unbounded interval. Examples can be simulation of a stock price model up to the maximum price in a given time horizon; or simulation of a queue up to the first time the queue attains its maximum length.

1.1 Overarching themes.

In this section we discuss the three main themes overarching this thesis, which are the following:

1. the *efficient* and *accurate* simulation of barrier-hitting events and extreme events of stochastic processes;
2. the tight relation between *the simulation of extreme events* and *the simulation of barrier-hitting events;*
3. the tight relation between *the simulation of reflected processes* and *the simulation of extreme events.*

In the following sections we further explain each of these points.

1.1.1 Efficient and accurate simulation of barrier-hitting events and extreme events

The first overarching theme of our work is that we devise theoretical guarantees of the efficiency and accuracy of algorithms that simulate either barrier-hitting events or extreme events. We now go through the basic challenges that affect both the efficiency and accuracy of these type of algorithms.

A first challenge is related to the inability to simulate continuous-time stochastic processes, so hitting times or times of extremes may not be determined by simple inspection of a simulated path. Indeed, because of the discrete nature of computers, simulation of stochastic processes is limited to simulate and store only discrete objects. Therefore any simulation procedure can only output the state of the path at a finite number of time instants. Then, it can happen that with high probability the actual hitting time or time of extreme will never be sampled.

A second challenge is that a simulation procedure may be unable to detect that an event will ever occur in the simulation run it is producing. This can occur, for instance, when the barrier intended to cross is actually never going to be crossed in a path realization. If the simulation algorithm is unable to detect this then it may never stop and therefore not produce output.

A third challenge is that to sample extrema over unbounded time horizons the simulation procedure needs to take into account information of the infinite horizon. Therefore a naive simulation algorithm may introduce a bias in the output by stopping a simulation without having guarantees that further extremes will not be attained if the simulation run would instead continue.

A fourth challenge is that computer simulation is limited to work with floating point arithmetic, which presents its own limitations; see [2] for a classical survey on the topic. Nonetheless, in this thesis we will not consider this particular aspect of the problem and make the blunt assumption that computers actually have infinite precision to simulate real numbers.

All in all, the design of an accurate and efficient simulation algorithm to sample extrema or barrier-hitting events presents various non-trivial challenges.

1.1.2 Connection of simulation of barrier-hitting events and simulation of extreme events

The second overarching theme in this thesis is that the simulation of barrier-hitting events of a stochastic process is tightly related to the simulation of its extreme events. We now give a brief overview of how the simulation of these two events are connected, and how it appears in our work.

The most basic connection between barrier-hitting events and extreme events is the following loosely stated duality relation in the case of processes taking values in $\mathbb{R}$: the maximum of a process up to a certain time, say t, is greater than a certain value, say b, if and only if the barrier b has been hit (or *up-crossed*) by time t. This holds under fairly general conditions on the regularity of the paths of the stochastic process. Consequently, to simulate an event where the maximum of the process is greater than b by time t, it is enough to simulate an event where the process hits (or up-crosses) the barrier b by time t, and vice versa.

The aforementioned basic duality relation for simulation can be generalized to simulate more general extreme events and for more general processes. We now mention a few examples in the line of unbiased or *exact* sampling. Ensor and Glynn [3] simulate without bias the global maximum of a light-tailed random walk by simulating a certain barrier-crossing event. Blanchet and Sigman [4] extend the latter work to sample without bias not only the maximum but also the path of the process up to the time at which it attains the maximum; they do this by sampling without bias barrier-hitting paths for certain barriers. Blanchet and Chen [5] further extend this idea to sample paths up to certain extreme events in multiple dimensions by sampling a certain high-dimensional barrier. Overall, these examples suggest that, in some generality, the simulation of extreme-type events usually can be addressed through simulating barrier-hitting-type events.

We now highlight, in broad terms, how some parts in this thesis use, explore, or were motivated by the connection between simulation of barrier-hitting events and simulation of extreme events.

The initial motivation of the work in Chapter 2 was to devise an algorithm for exact sampling of a process up to its maximum. To do this we followed the lines of Blanchet and Sigman [4] and Blanchet and Chen [5], who devise algorithms for exact sampling of barrier-hitting-type events. The latter, in turn, was the essential motivation for our definition and study of the *efficiency for conditional sampling* of a *change of measure for exact sampling*; see Section 2.2 for further details.

In Chapter 3 we delve in a further relation between barrier-hitting and extreme events. There, we derive limiting distributions for the normalized Euler discretization error when estimating both barrier-hitting and extreme events. Even though both expressions are different, remarkably their mean agree and its value is a constant that has appeared several times in the literature; see Section 3.3.1 for further details.

1.1.3 Connection of simulation of extreme events and simulation of reflected processes

A third overarching theme in this thesis is that the simulation of extreme events is closely related to the simulation of reflected processes.

Loosely speaking, a reflected processes is obtained by applying to the stochastic process a reflection mechanism in the boundary of a pre-specified domain of the state space of the process. We remark, though, that the term "reflection" can be misleading and counterintuitive, since the reflection mechanism that is applied is, generally, different from the notion of normal light reflection that we physically observe in mirrors and the like. It usually holds that the reflected process can be obtained by applying a deterministic (non-random) mapping, the "reflection mapping", to an *unconstrained* or *free* stochastic process.

As an example, we now show the reflected process obtained by applying a normal reflection to a one-dimensional random walk when it gets to the negative orthant. Denote the random walk by $X = (X_n)_{n\in\mathbb{N}}$ and assume it starts from zero, $X_0 := 0$; we say that X is the *free* process. The *reflected* version of X when it gets to the negative orthant, say $\Gamma = (\Gamma_n)_{n\in\mathbb{N}}$, is defined as

$$\Gamma_n := X_n - \min_{0\leq k\leq n} X_k,$$

for all $n \in \mathbb{N}$. We remark that the latter formula is the first materialization of this part's theme: the joint simulation of X_n and its reflection Γ_n is equivalent to the joint simulation of X_n and the extreme $\min_{0\leq k\leq n} X_k$.

More generally, reflected processes are ubiquitous objects in Applied Probability. One of the most basic examples in this field is the waiting time of a customer in a single server queue. Indeed, the waiting time of a customer in a single server queue can be expressed using Lindley's recursion. Solving the recursion expresses the waiting time as the reflection mapping of a random walk; more specifically, the random walk of service times minus inter-arrival times. See Section 5.6 of Asmussen [6] for further details.

In simulation, the following are some examples of works using sampling of extreme events to sample reflected processes. Blanchet and Chen [5] obtain unbiased samples of stationary *reflected Brownian motion* in multiple dimensions by sampling paths up to an extreme event of the “free” Brownian motion. Blanchet and Sigman [4] and Blanchet and Wallwater [7] obtain unbiased samples of one-dimensional reflected processes in stationarity when the free process is a random walk, the former for the light-tailed version and the latter for the heavy-tailed.

On a slightly different vein, Asmussen et al. [8] study the error of sampling one-dimensional reflected Brownian motion using an Euler discretized approximation. They derive weak convergence results for the error normalized by its convergence rate. In Chapter 3 we extend the work of Asmussen et al. [8] to also study the discretization error of the drift-derivative of reflected Brownian motion. We show that studying the discretization error of the drift-derivative is equivalent to studying the discretization error of the time at which extremes are attained. See Section 3.2.2 for further details.

1.2 Summary and main contributions

In this section we give an overview of the research contained in this thesis and highlight their main contributions. In Section 1.2.1 we give an overview of Chapter 2, where we consider an algorithm for *exact sampling* of barrier-hitting events and study its efficiency. The stochastic process treated is on discrete time and the one-setp increments are *heavy-tailed*. Then, in Section 1.2.2 we give an overview of Chapter 3, where we consider the Euler discretization approach to simulate continuous-time stochastic processes, and study the accuracy of simulating barrier-hitting and extreme events using this approach. The stochastic process treated is Brownian motion, which is the canonical continuous-time stochastic process.

1.2.1 Efficiency of conditional sampling for heavy-tailed random walks

In this section we give a summary of Chapter 2. There we study the efficiency of *exact* or unbiased sampling of barrier-hitting events when using the change of measure technique. We propose a framework to study the efficiency of a specific change of measure and apply it to the particular case of random walks with heavy-tailed one-step increments.

Setup. In Chapter 2 we consider a random walk, which is a discrete-time stochastic process of the form $S = (S_n)_{n\in\mathbb{N}}$ where for all n

$$S_n := \sum_{k=1}^{n} X_k,$$

with $S_0 := 0$, where $(X_k)_{k\in\mathbb{N}}$ is a collection of independent real random variables having the same distribution. We assume that the mean $\mathbb{E}X_k$ is finite and strictly negative.

Given a fixed barrier-level $b > 0$ we consider the problem of simulating random walk paths up to the first time that the walk S goes above the barrier b, conditional on the barrier being eventually hit. Indeed, since $\mathbb{E}X_k < 0$ and $b > 0$ the probability that S goes above b is strictly positive but also strictly less than one. Moreover, the probability that S hits b decreases as the level b grows; thus, the event of hitting b is considered a *rare event*. For simulation purposes, this means that in a straightforward sequential simulation run the event may or may not occur.

Framework. We study the case where the random walk increments X_k are *heavy-tailed*, meaning that for all $\lambda > 0$ it holds $\mathbb{P}(X_k > t)/e^{-\lambda t} \to \infty$ as $t \to \infty$. That is, as t grows, the tail probability $\mathbb{P}(X_k > t)$ decays slower than any decaying exponential function. We focus in the particular case where the increments X_k have a *regularly varying* right tail; intuitively speaking this means that, for large t, the right tail $\mathbb{P}(X_k > t)$ roughly behaves like C/t^{α}, for some strictly positive constants α and C.

We consider simulation procedures for *exact sampling* of the rare event, that is, that produce unbiased samples of the rare event, and that attain this by using the *change of measure* technique. One such method is the *Acceptance-Rejection algorithm*; a related method but for rare-event probability estimation is *Importance Sampling*. The idea of the change of measure technique is that to sample an event of interest from a "target" distribution one instead considers an "alternative" or "proposal" distribution from which to sample from. In the Acceptance-Rejection algorithm, for instance, theoretical conditions, if satisfied, guarantee that the proposed samples are indeed distributed according to the target distribution.

We propose a framework to analyze whether a change of measure is *efficient* in producing samples of the rare event. This framework consists of three conditions on the new, proposed, change of measure: whether it samples the rare event with probability one; whether with the change of measure any measurable subset of the rare event is sampled with higher probability than with the original measure; and whether the number of step-sizes needed to sample the rare event grows "in a reasonable way" as the barrier to be crossed grows.

Objectives. The main objective of the work shown in Chapter 2 is to study the efficiency, according to our proposed framework, of the change of measure proposed by Blanchet and Glynn [1], when applied for sampling heavy-tailed random walks that cross over an arbitrary barrier b. We study this change of measure because it proved to be very important for the problem of rare-event probability-estimation with Importance Sampling.

Relevance. Rare events of heavy-tailed processes is a topic which has received much attention from the Applied Probability field and community in the last two decades. Reasons range from modeling to theoretical ones.

A first reason is that heavy-tailed processes are used to model several important phenomena in sciences, business and engineering. Some examples are traffic in telecommunications [9], stock options in finance [10], natural disasters in insurance and risk [11], to name just a few.

A second reason is that barrier-crossing events of heavy-tailed processes occur in structurally different forms than in the light-tailed case. Loosely speaking, in the light-tailed case a barrier cross occurs because all steps leading to the event make a small contribution to cross the barrier, see Heidelberger [12]. In contrast, in the heavy-tailed case it is known that a barrier cross occurs because only one or a few increments are very large, while the others make no special contribution to take the process above the barrier; see Rhee [13]. From a simulation perspective, the aforementioned structural difference between barrier-crossing events of light- and heavy-tailed processes translates into both processes needing very different approaches. While the simulation in the light-tailed case is pretty much solved by *exponential tilting*, see Heidelberger [12], in the heavy-tailed case it is not clear which is the best simulation approach.

A third reason for studying the simulation of barrier-crossing events is that they are key in simulating extreme events and reflected processes. Both these objects are of their own practical and theoretical interest, which reinforces the importance of studying barrier-crossing events.

A fourth reason that makes relevant our work is that the regularly varying distributions are, in a sense, the canonical choice of heavy-tailed distributions. Indeed, regularly varying distributions have a very rich structure which makes their study easier. Moreover, the clean and insightful treatment that they allow some times lead to extending results to broader families of heavy-tailed processes.

Main contributions. The main objective in Chapter 2 is to determine if the change of measure of Blanchet and Glynn [1] can be *efficient* for exact sampling of heavy-tailed random walks given a barrier-crossing rare event. The Blanchet-Glynn measure is designed to approximate such a conditional distribution, so one might therefore expect an answer to this question in the affirmative. Surprisingly, we answer this question by and large in the negative: this measure cannot be used for conditional sampling unless the tails are very heavy.

A central contribution in Chapter 2 is that we reveal an intriguing dichotomy on the efficiency of conditional sampling under the Blanchet-Glynn measure: this measure is efficient for the requirement of sampling all subsets of the barrier-crossing event with higher probability than the original measure, if and only if the tail index is below the threshold $3/2$. This roughly implies that only the heaviest tails can be efficient in this setting, which is counterintuitive as heavier tails typically make problems harder. Also, it is noteworthy that the value $3/2$ for the threshold of efficiency is not directly connected to the existence of integer moments for the step size distribution.

We highlight the fact that the critical tail index $3/2$ for regularly varying step-sizes has already appeared in other simulation works as a threshold for efficiency. Indeed, Blanchet and Liu [14], and Blanchet, Juneja and Murthy [15] show that their respective proposed changes of measure are efficient for Importance Sampling only for $\alpha > 3/2$. Remarkably, there is no unified explanation as to why the same threshold appears in these works and ours, especially since the three studies are different. Indeed, the changes of measure proposed in [14] and in [15] are not directly related to the Blanchet-Glynn change of measure we analyze; the problem they treat (probability estimation) is different from ours (exact sampling); their efficiency criterion is different from ours; and they show that heavier tails are inefficient, while we obtain that heavier tails are efficient for the criterion of sampling subsets of the rare event with higher probability than the original measure. Our work thus highlights the importance of studying what makes special the value $3/2$ for the tail index α.

A further contribution of our work is that it corrects Proposition 4 of Blanchet and Glynn [1]. This result establishes that expected hitting time to a barrier b time grows linearly in b as $b \to \infty$ for all $\alpha > 1$; in contrast we show that this is not true for $\alpha \in (1, 3/2)$ but is true for $\alpha > 2$.

1.2.2 Weak convergence of some Euler discretization errors of Brownian motion

In this section we give an overview of the work shown in Chapter 3. In short, Chapter 3 considers sampling barrier-hitting and extreme events of Brownian motion using a discretized version of the Brownian motion. We give the rates of convergence of the resulting errors and also give closed-form analytical expressions for the limiting normalized errors. We also translate the weak convergence results into weak limits of Gaussian random walks. In doing the latter, we clarify the relation between several loosely connected works in the literature where the constant $\zeta(1/2)/\sqrt{2\pi}$ has appeared, where ζ is the Riemann zeta function; see Janssen and van Leeuwaarden [16] for a survey on works where this constant appears.

Setup. In Chapter 3 we consider a Brownian motion B with constant drift and strictly positive variance. In the driftless case with unit variance, so-called standard Brownian motion in $\mathbb{R}$ can be defined as the only continuous-time process starting from zero at time zero, having almost surely continuous paths, having independent increments, and having $B(t) - B(s)$, any $t > s \geq 0$, distributed as a normal random variable with mean zero and variance $t - s$. We are broadly interested in simulating barrier-hitting events and extreme events of Brownian motion.

In general, simulation of Brownian motion is a challenging problem. Indeed, it is a continuous-time stochastic process characterized by its self-similar structure, which is called Brownian scaling. In contrast, computers can only simulate and store discrete and finite objects. Therefore, simulation of Brownian motion will almost always be inaccurate, except for a few especially structured collection of events where Brownian motion can be simulated without bias, see e.g. Devroye [17] for a survey on such special cases.

We consider the *Euler discretization approach* to simulate Brownian motion, which we describe next. For a strictly positive integer n we first consider the regular time mesh $\mathbb{Z}_+/n := \{0, 1/n, 2/n, \ldots\}$. The vector $\{B(0), B(1/n), B(2/n), \ldots\}$ is a Gaussian random walk, i.e., a random walk with Gaussian iid increments, so in particular it is easily sampled without bias. Consequently, a possible approximation of the Brownian motion of $B = (B(t) : t \in \mathbb{R}_+)$ is to take the piecewise constant path $B^n := (B^n(t) := B(\lfloor nt \rfloor / n) : t \in \mathbb{R}_+)$. We call B^n the *Euler discretization approximation of B on the mesh $\mathbb{Z}_+/n$.*

The Euler discretization approximation of Brownian motion is the easiest method to simulate Brownian motion, since it is straightforward to produce and to store finite-time samples of it. This makes it the most attractive approach for practitioners. Additionally, in some situations there are physical real-world constraints that makes it the most sensible approach to take, for instance in stock-pricing in finance, in situations where the stock price can only be monitored at regular time intervals, see e.g. Broadie et al. [18, 19].

Our work in Chapter 3 was initially motivated by the study of simulating multidimensional reflected Brownian motion (RBM) using an Euler discretization approach along the lines of Asmussen et al. [8]. In short, RBM can usually be considered as a deterministic pathwise mapping, a *reflection mapping*, acting on paths of an unconstrained, "free", Brownian motion. This reflection mapping in particular involves the local minima up to each time. With this, a possible simulation procedure of RBM is to take the reflection mapping of the Euler discretization of the "free" Brownian motion; however, such approach should inherit the inaccuracies of the Euler discretization of extremes of Brownian motion. An analogous situation occurs when simulating the *drift derivative of RBM*, which roughly consists on an *sensitivity analysis* on the drift of RBM. In this case, we show in Chapter 3, the Euler discretization error of the drift derivative of RBM is directly related to the discretization error of the time at which extremes are attained.

Problem studied. We broadly consider the problem of, for Brownian motion, simulating barrier-hitting times and events, and also simulation of extreme events and their times. More specifically, we consider a deterministic barrier function, say $b := (b(t) : t \in \mathbb{R}_+)$ with $b(0) > B(0)$, and consider the hitting-time

$$\tau_b := \inf \{t \geq 0 : B(t) \geq b(t)\}.$$

We wish to simulate τ_b in the case when $\tau_b < \infty$ holds. For that, we approximate the hitting time τ_b by taking its counterpart on the Euler discretization B^n, i.e.,

$$\tau_b^n := \inf\{t \geq 0 : B^n(t) \geq b(t)\}.$$

A similar approximation can be used for the time of minimum or maximum when it is finite, either over bounded or unbounded time intervals.

A caveat of the Euler discretization approach is that it is especially prone to error. Indeed, in the barrier hitting case, with probability one the barrier will not be hit on a point in the regular mesh $\mathbb{Z}_+/n$ for any n. The same problem occurs in the case of simulation of extremes. This motivates the study of the error incurred when sampling barrier-hitting and extreme events by taking their corresponding counterpart for the Euler discretization.

Our work is also motivated by the fact that results about the Euler discretization error of Brownian motion can be translated into convergence results of Gaussian random walks. Indeed, using the Brownian scaling property of Brownian motion it holds that for a standard Brownian motion W, for all strictly positive n the sequence $(\sqrt{n}W(t) : t \in \{0, 1/n, 2/n, \ldots\})$ has the same distribution as $(W(t) : t \in \{0, 1, 2, \ldots\})$. In Section 3.3 we exploit this relationship to derive convergence results of Gaussian walks.

Objective. Inspired by the inaccuracy issues of the Euler discretization approach to simulate Brownian motion, the objective of our work in Chapter 3 is to study the convergence of the errors $t^n - t^*$ and $B^n(t^n) - B(t^*)$, where t^* is either a barrier-hitting time or a time where an extreme is attained, and t^n is the corresponding time for the Euler discretization on the regular mesh $\mathbb{Z}_+/n$. We are interested in the convergence to zero of these errors, the rate at which they converge, and the convergence in distribution of the errors normalized by their convergence rate to zero.

Main contributions. The main contributions of Chapter 3 are the following.

We show that the error $B(t^*)-B(t^n)$ converges to zero at rate $O(1/\sqrt{n})$ as $n \to \infty$, and the error $t^* - t^n$ converge to zero at rate $O(1/n)$. Here, t^* is either a barrier hitting time for continuously differentiable and non-decreasing barrier; or a time of maxima over a compact time interval; or maxima over an unbounded time interval; and t^n is its corresponding time for the Euler discretization on the regular mesh $\mathbb{Z}_+/n$.

We also show that the normalized errors $\sqrt{n}(B(t^*) - B(t^n))$ and $n(t^* - t^n)$ converge in distribution, and we give analytical expressions in closed form for the limiting random variables. In the particular case where t^* is the time of minimum over a compact time interval, our work extends the one of Asmussen et al. [8], which analyzes the error of simulating one-dimensional RBM by using an Euler discretization scheme. In Section 3.2.1 we argue that our Theorem 3 extends [8, Theorem 1] to include the discretization error of the drift derivative of RBM.

We also obtain weak convergence results for Gaussian walks, related to barrier-hitting events and extreme events. These are derived as corollaries of the Euler discretization error results. This simple derivation allows in particular to clarify the connection between several works in the literature, theoretical [16] and applied [18], where the quantity $\zeta(1/2)/\sqrt{2\pi}$ has appeared by reasons that are unclear; see Janssen and van Leeuwaarden [16].

Another contribution of our work is that we establish convergence in distribution of Brownian motion when "zoomed-in" about either a barrier-hitting time or a time of extremes. The limiting two-sided process consists of either a Brownian motion and a Bessel(3) process attached end-to-end in the barrier-hitting case, or a two-sided Bessel(3) processes attached end-to-end in the extremes case. Moreover, we show that this weak convergence holds for the *weighted supremum* norm, see [20, Section 11.5.2], which is a stronger topology than the usual topology of uniform convergence over compact sets. This result has the potential of facilitating the derivation of further weak convergence results for Euler discretizations, "zoomed-in" operations, and random walks as in [21].

CHAPTER 2

EFFICIENCY OF CONDITIONAL SAMPLING FOR HEAVY-TAILED RANDOM WALKS

Abstract. We study the simulation of paths of heavy-tailed random walks up to the first time it crosses a fixed positive barrier, when the mean step-size is strictly negative. We consider the framework of *exact* sampling algorithms that use the change of measure technique. In particular, the simulated barrier-hitting paths are unbiased or completely accurate, so in this chapter we are mostly concerned with the study of efficiency of using a change of measure. For that, we propose a framework to evaluate how efficient is a change of measure to sample barrier-hitting paths. We apply this framework to evaluate the efficiency of the change of measure of Blanchet and Glynn [1] when the random walk step sizes have regularly varying right tails, say with tail index $\alpha > 1$, and lighter left tails. We study this particular change of measure because it has been a successful measure for the sampling of rare events for heavy-tailed random walks with Importance Sampling. The main contributions of this chapter are the following. First, for a requirement characterizing how much more likely the change of measure is to sample a barrier-hitting path, the Blanchet-Glynn change of measure is efficient only when the tail index α is in $(1,\, 3/2)$; in particular this is the regime of heaviest tails. Second, for a requirement quantifying the expected number of steps needed to sample a barrier-hit with the Blanchet-Glynn change of measure, we show that the expected number of steps is infinite for α in $(1,\, 3/2)$. In particular, the latter result corrects a non-central result of [1]. We also discuss that the value $\alpha = 3/2$ has arisen as a threshold for efficiency criteria in other related algorithms for random walks with regularly varying right tails, which raises the question of how precisely are these works connected.

2.1 Introduction

Barrier-crossing events of random walks appear in numerous engineering and science models. Examples range from stationary waiting times in queues to ruin events in insurance risk processes [11, 9, 10]. Random walks with regularly varying step size distributions are of particular interest, and their special analytic structure facilitates an increasingly complete understanding of associated rare events.

This chapter considers the problem of sampling a path of a random walk until it crosses a given fixed barrier in the setting of heavy-tailed step sizes with negative mean. The higher the barrier, the lower the likelihood of reaching it. This poses challenges for conditional sampling, since naive Monte Carlo sampling devotes much computational time to paths that never cross the barrier and must therefore be ultimately discarded.

The ability to sample up to the first barrier-crossing time plays a central role in several related problems, such as for sampling paths up to their maximum [4] or for sampling only the maximum itself [3]. In turn these have applications to perfect sampling from stationary distributions [5], [22] and to approximately solving stochastic differential equations [23].

Main contributions. The central question in this chapter is: Can the change of measure proposed in Blanchet and Glynn [1] be used to devise an algorithm for conditional sampling of heavy-tailed random walks given a barrier-crossing rare event? The Blanchet-Glynn measure is designed to approximate such a conditional distribution, so one might therefore expect an answer to this question in the affirmative. Surprisingly, we answer this question by and large in the negative: this measure cannot be used for conditional sampling unless the tails are very heavy. Our results are a consequence of a delicate second-order analysis of tail probabilities of a sum of heavy tailed random variables.

This chapter reveals an intriguing dichotomy on the efficiency of conditional sampling under the Blanchet-Glynn measure: this measure is 'efficient' if and only if the tail index is below the threshold $3/2$. It is worthwhile to stress two immediate consequences. First, our result roughly implies that only the heaviest tails can be efficient in this setting. This is counterintuitive, since heavier tails typically make problems harder. Second, the threshold is not directly connected to the existence of integer moments for the step size distribution.

The threshold $3/2$ also arises in the simulation literature involving barrier-crossing events with regularly varying step sizes [14, 15]. The nature of the threshold we obtain here is however different from these works for three reasons. First, these papers focus on estimating the rare event probability of exceeding a barrier; in contrast, our work focuses on sampling barrier-crossing paths. Second, these papers obtain that the heaviest tails are inefficient in their framework, while we obtain the opposite. Third, and perhaps most importantly, the threshold $3/2$ in the existing literature is a direct consequence of requiring second moment conditions of the estimator, while a direct relation with moments is absent for conditional sampling problems.

A by-product of our work is a counterexample for the statement of Proposition 4 of Blanchet and Glynn [1]. This proposition states that, for a broad class of heavy-tailed step sizes, the expected hitting time of the barrier grows linearly in the barrier level under the Blanchet-Glynn change of measure. We show, though, that this result does not always hold. This proposition is not central to the framework introduced in [1], and the issue we expose here can also be deduced from Corollary 1 in [14], but our result reopens the question of when the measure of Blanchet and Glynn induces a linear hitting time expectation.

Related literature. The primary means for sampling from heavy-tailed random walks is based on the change of measure technique. Simply put, this procedure consists in sampling from a distribution different from the desired one and determining (or computing) the output using the likelihood ratio. The essential idea is that the changed distribution should emphasize characteristics of barrier-crossing paths.

The literature of *simulating paths* that cross a barrier is closely related to the one of *estimating the probability* of exceeding the barrier. In the heavy-tailed setting, the latter problem has already been studied for two decades. In contrast, the path-sampling problem has only recently received attention, mostly driven by applications of *Dominated Coupling From the Past* when in presence of heavy tails; see [22].

For the probability estimating problem under heavy tails, early approaches are [24, 25, 26]. An important contribution for the current chapter is [1], which was later followed by [14, 15]. A recent new technique is [27], which uses *Markov Chain Monte Carlo* to estimate the multiplicative inverse of the probability of crossing the barrier.

The path-sampling problem with heavy tails, on the other hand, has only recently been tackled by [22]. The latter modifies the measure of [15], which focuses on the probability estimation problem, and builds on the scheme for *exact sampling* of paths introduced in Section 4 of [4]. The approach studied in this chapter is based on the Blanchet-Glynn change of measure, which is conceptually simpler than the approach proposed in [22]. The search for a simpler algorithm provided the motivation for the work in this chapter.

Outline. This chapter is organized as follows. In Section 2.2 we discuss the general preliminaries for our conditional sampling problem: a change of measure technique and a notion of efficiency. In Section 2.3 we state our main result of efficiency for conditional sampling when using the Blanchet-Glynn change of measure [1] and with regularly varying step sizes. In Section 2.4 we compare our threshold result of Section 2.3 with similar ones in the literature of rare event sampling. In Section 2.5 we give a proof of the main result of Section 2.4.

Notation. We denote by $\{S_n\}$ the infinite length paths of the random walk. Given a probability measure $\mathbb{Q}$ over $\{S_n\}$, we denote the expectation with respect to measure $\mathbb{Q}$ as $\mathbb{E}^{\mathbb{Q}}$. We write $\mathbb{E}_y^{\mathbb{Q}}[\cdot] := \mathbb{E}^{\mathbb{Q}}[\cdot|S_0 = y]$ and omit y when $y = 0$, as customary in the literature. Given two probability measures P and Q over the same space, we denote *absolute continuity* of P with respect to Q as $\mathsf{P} \ll \mathsf{Q}$, meaning that for all measurable B $\mathsf{Q}(B) = 0$ implies $\mathsf{P}(B) = 0$. For x, y real we denote $x^+ := \max\{x, 0\}$,

$x^- := -\min\{x, 0\}$, $x \wedge y := \min\{x, y\}$ and $x \vee y := \max\{x, y\}$. Also, for two functions f and g we write $f(t) \sim g(t)$ when $\lim_{t\to\infty} f(t)/g(t) = 1$; we write $f(t) = O(g(t))$ when $\limsup_{t\to\infty} |f(t)/g(t)| < \infty$, and $f(t) = o(g(t))$ when $\lim_{t\to\infty} |f(t)/g(t)| = 0$.

2.2 Preliminaries

This section gives the background necessary for the exposition of our main result. In Section 2.1, we describe techniques for conditional sampling using change of measure technique. In Section 2.2, we give an efficiency criterion for this problem. In Section 2.3, we briefly introduce the Blanchet-Glynn [1] change of measure.

General setting. We consider a random walk $S_n := \sum_{i=1}^n X_i$, where X_i are iid, $\mathbb{E}|X_i| < \infty$ and $S_0 = 0$ unless explicitly stated otherwise. We assume that $\{S_n\}$ has *negative drift*, meaning that $\mathbb{E}X_i < 0$. We also assume that X_i has unbounded right support; that is, $\mathbb{P}(X_i > t) > 0$ for all $t \in \mathbb{R}$.

Given a *barrier* $b \geq 0$, let $\tau_b := \inf\{n \geq 0 : S_n > b\}$ be the first barrier-crossing time. Since the random walk has negative drift, we have $S_n \to -\infty$ a.s. as $n \to \infty$, and also $\mathbb{P}(\tau_b = \infty) > 0$.

Our main goal is to study the *efficiency* of an algorithm to sample paths $(S_1, \ldots, S_{\tau_b})$ conditional on $\{\tau_b < \infty\}$.

We remark that for the sake of clarity of exposition we will abuse notation and write that '$(S_0, \ldots, S_{\tau_b})$ follows the distribution $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$' to mean that for all finite $n \in \mathbb{N}$ the random vector $(S_0, \ldots, S_{\tau_b})$ with $\tau_b = n$ has the distribution $\mathbb{P}(\,\cdot\,|\tau_b = n)$.

2.2.1 Conditional sampling via change of measure

We tackle the problem of conditional sampling using the Acceptance-Rejection algorithm, which uses the change of measure technique. Here we give a brief exposition of these two methods.

Change of measure technique. Let $\mathsf{P}(y,\ \mathrm{d}z)$ be the *transition kernel* of the random walk, i.e., $\mathsf{P}(y,\ \mathrm{d}z) = \mathbb{P}(S_1 \in\ \mathrm{d}z | S_0 = y)$. We consider a "new" or "changed" transition kernel $\mathsf{Q}(y,\ \mathrm{d}z)$, which may be chosen *state dependent*, meaning that $\mathsf{Q}(y_1, y_1 + \cdot)$ and $\mathsf{Q}(y_2, y_2 + \cdot)$ may be different measures for $y_1 \neq y_2$. We assume that $\mathsf{P}(y, \cdot) \ll \mathsf{Q}(y, \cdot)$ for all y, which implies that the *likelihood ratio* function $\mathrm{d}\mathsf{P}/\mathrm{d}\mathsf{Q}(y, \cdot)$ exists. Letting $\mathbb{Q}$ be the distribution of $\{S_n\}$ induced by the proposal kernel Q, we slightly abuse notation and denote by $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(S_n : 0 \leq n \leq T)$ the likelihood ratio of a finite path $(S_0, \ldots, S_T)$. More precisely, for T finite $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(S_n : 0 \leq n \leq T) := L_T$ where L_T is the nonnegative random variable satisfying $\mathbb{E}^{\mathbb{Q}}[\mathbf{1}_B L_T] = \mathbb{E}^{\mathbb{P}}[\mathbf{1}_B]$ for all B in the σ-algebra $\sigma(S_n : 0 \leq n \leq T)$. With this, it holds that

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n : 0 \leq n \leq T) = \frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{Q}}(S_0, S_1) \cdots \frac{\mathrm{d}\mathsf{P}}{\mathrm{d}\mathsf{Q}}(S_{T-1}, S_T),$$

for all T finite or $\mathbb{Q}$-a.s. finite stopping time. See Section XIII.3 of [6] for further details.

Acceptance-Rejection algorithm for conditional sampling. This procedure considers the situation of a distribution that is "difficult" to sample from, and another distribution that is "easy" to sample from; the aim is to simulate from the difficult distribution. The Acceptance-Rejection algorithm allows one to sample from the difficult distribution by repeatedly sampling from the easy, "proposal", distribution. Here we show a known specialization of this technique to the problem of sampling paths from the conditional distribution $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$, see [4].

Let P be the transition kernel of the random walk, and consider a "proposal" kernel Q, possibly state dependent, such that $\mathsf{P}(y, \cdot) \ll \mathsf{Q}(y, \cdot)$ for all y. Assume that for some computable constant $C > 0$ we have

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n : 0 \leq n \leq \tau_b) \cdot \mathbf{1}_{\{\tau_b < \infty\}} \leq C \qquad \mathbb{Q}\text{-a.s.}$$

If U is uniformly distributed on $[0,1]$ under $\mathbb{Q}$ and drawn independently from $\{S_n\}$, then it can be verified that

$$\mathbb{Q}\left(U \leq \frac{\mathbf{1}_{\{\tau_b<\infty\}}}{C}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n\!:\!0\!\leq\! n\!\leq\!\tau_b)\right) \;=\; \frac{\mathbb{P}(\tau_b<\infty)}{C}, \tag{2.1}$$

$$\mathbb{Q}\left(\{S_n\}\in\cdot \;\middle|\; U \leq \frac{\mathbf{1}_{\{\tau_b<\infty\}}}{C}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n\!:\!0\!\leq\! n\!\leq\!\tau_b)\right) \;=\; \mathbb{P}\left(\{S_n\}\in\cdot\,|\tau_b<\infty\right), \tag{2.2}$$

over events $B \in \mathcal{F}_{\tau_b}$ such that $B \subseteq \{\tau_b < \infty\}$.

The Acceptance-Rejection procedure consists on iterating the steps: (i) sample jointly $(U,(S_0,\ldots,S_{\tau_b}))$ from $\mathbb{Q}$, and (ii) check whether

$$U \;\;\leq\;\; \frac{\mathbf{1}_{\{\tau_b<\infty\}}}{C}\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n\!:\!0\!\leq\! n\!\leq\!\tau_b) \tag{2.3}$$

holds. The algorithm stops, "accepts", the first time inequality (2.3) is satisfied, and outputs the path $(S_0,\ldots,S_{\tau_b})$. Equation (2.1) states that a sample is "accepted" with probability $\mathbb{P}(\tau_b<\infty)/C$; and equation (2.2) assures that the distribution of the output is $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$.

2.2.2 Efficiency framework for conditional sampling

We now establish a framework for when an "alternative" transition kernel Q is useful in sampling paths up to τ_b from the conditional distribution $\mathbb{P}(\,\cdot\,|\tau_b<\infty)$. Simply put, it states that crossing events occur with higher probability under the new measure than under the original.

Definition 1 (Efficiency for conditional sampling)**.** *Let $\mathsf{Q}(y,\,\mathrm{d}z)$ be a transition kernel such that $\mathsf{P}(y,\cdot) \ll \mathsf{Q}(y,\cdot)$ for all y. Let $\mathbb{Q}$ be the distribution of $\{S_n\}$ on $\mathbb{R}^{\mathbb{N}}$ induced by Q. We say that $\mathbb{Q}$ is efficient for conditional sampling from $\mathbb{P}(\,\cdot\,|\tau_b<\infty)$ iff*

$$\mathbb{Q}\left(\{S_n\}\in B\right) \geq \mathbb{P}\left(\{S_n\}\in B\right),$$

for all events $B \in \mathcal{F}_{\tau_b}$ such that $B \subseteq \{\tau_b < \infty\}$, where $\mathcal{F}_{\tau_b}$ is the usual σ-algebra associated to the stopping time τ_b.

We remark that the previous notion does not require $\mathbb{Q}(\tau_b < \infty) = 1$, although that is true for the Blanchet-Glynn change of measure, as we will see in Proposition 2.

We remark that by the definition of likelihood ratio we have that for all $B \subseteq \{\tau_b < \infty\}$ it holds that $\mathbb{P}(\{S_n\} \in B) = \mathbb{E}^Q\left[\mathbf{1}_B \mathbf{1}_{\{\tau_b<\infty\}} \cdot \mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(S_n : 0 \le n \le \tau_b)\right]$. Together with Definition 1, this identity gives the following equivalent condition for efficiency.

Corollary 1. *The following statements are equivalent:*

1. *$\mathbb{Q}$ is efficient for conditional sampling from $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$*

2. *$\mathbf{1}_{\{\tau_b<\infty\}} \cdot \mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(S_n : 0 \le n \le \tau_b) \le 1$ holds $\mathbb{Q}$-a.s.*

Another characterization of efficiency for conditional sampling ensues. Consider the following procedure: sample a path $(S_0, \ldots, S_{\tau_b})$ from $\mathbb{Q}$; set $I := 1$ if $\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}(S_n : 0 \le n \le \tau_b) \le 1$, and $I := 0$ otherwise; output $(I, (S_0, \ldots, S_{\tau_b}))$. If $\mathbb{Q}(\tau_b < \infty) = 1$ then parts 1. and 2. of Corollary 1 are also equivalent to the following statement: I is distributed as a Bernoulli random variable with parameter $\mathbb{P}(\tau_b < \infty)$ and if $I = 1$ then the sample path $(S_0, \ldots, S_{\tau_b})$ follows the distribution $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$. Indeed, this characterization is direct from (2.1) and (2.2) using $C = 1$, by Corollary 1 part 2. Remarkably, this procedure does not require knowing the value of $\mathbb{P}(\tau_b < \infty)$. This property has been a key component of several recent simulation works, see e.g. [4, 23, 22, 5], and initially motivated the research presented in this chapter of the thesis.

2.2.3 The Blanchet-Glynn change of measure

We now present the essential ideas of the Blanchet-Glynn change of measure [1]. This measure proved efficient for estimating the probability $\mathbb{P}(\tau_b < \infty)$ as $b \to \infty$. In the current chapter, we are interested in its use for the conditional sampling problem.

The main idea motivating the Blanchet-Glynn change of measure is to approximate the transition kernel of the conditional distribution. Indeed, it is well-known that the one-step transition kernel of $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$, say P^*, satisfies

$$\mathsf{P}^*(y,\ \mathrm{d}z) = \mathsf{P}(y,\ \mathrm{d}z) \cdot \frac{u^*(z-b)}{u^*(y-b)},$$

where $u^*(x) := \mathbb{P}_x(\tau_0 < \infty)$ for all $x \in \mathbb{R}$ and P is the original transition kernel of $\{S_n\}$; see Section VI.7 of [28]. Here, $u^*(y-b)$ can be interpreted as a normalizing term, since $\int \mathsf{P}(y,\ \mathrm{d}z)u^*(z-b) = u^*(y-b)$ for all y. It is nevertheless impractical to simulate from this kernel because u^* is unknown.

The idea put forth by Blanchet and Glynn is to approximate u^* using the asymptotic approximation given by Pakes-Veraverbeke Theorem, see Chapter 5 of [29]. This result states that

$$u^*(x) = \mathbb{P}_x(\tau_0 < \infty) \sim \frac{1}{|\mathbb{E}X|} \int_{-x}^{\infty} \mathbb{P}(X > s)\,\mathrm{d}s \quad \text{as} \quad x \to -\infty$$

for random walks with negative drift and step sizes X which are (right) *strongly subexponential*. Inspired by this fact, the Blanchet-Glynn change of measure uses the following transition kernel:

$$\mathsf{Q}^{(c)}(y,\ \mathrm{d}z) := \mathsf{P}(y,\ \mathrm{d}z) \cdot \frac{v^{(c)}(z-b)}{w^{(c)}(y-b)}, \tag{2.4}$$

where

$$v^{(c)}(x) \quad := \quad \min\left\{1, \frac{1}{|\mathbb{E}X|} \int_{c-x}^{\infty} \mathbb{P}(X > s)\,\mathrm{d}s\right\}$$

and $w^{(c)}(y-b) := \int \mathsf{P}(y,\ \mathrm{d}z)v^{(c)}(z-b) = \int \mathsf{P}(y-b,\ \mathrm{d}z)v^{(c)}(z)$ is a normalizing term. The constant $c \in \mathbb{R}$ is a translation parameter, which in [1] and in our work, we will see, is eventually chosen sufficiently large.

In proving efficiency results for this transition kernel we heavily rely on the fact that the functions $v^{(c)}$ and $w^{(c)}$ are closely related to the *residual life tail distribution of* X. That is, a random variable Z with distribution given by

$$\mathbb{P}(Z > t) := \min\left\{1, \frac{1}{|\mathbb{E}X|}\int_t^\infty \mathbb{P}(X > s)\,\mathrm{d}s\right\} \qquad \text{for all } t. \tag{2.5}$$

We thus have $v^{(c)}(x) = \mathbb{P}(Z > -x + c)$ and $w^{(c)}(x) = \mathbb{P}(X + Z > -x + c)$ for all x. For further details we refer the reader to [1] and Section VI.7 of [28].

2.3 Main result: a threshold for efficiency

In this section we present our main result on the efficiency for conditional sampling of random walks with regularly varying step sizes using the Blanchet-Glynn change of measure [1]. We give two results from which our main result easily follows. The first characterizes efficiency for conditional sampling with the Blanchet-Glynn change of measure, and the second explores this characterization in the case of regularly varying step sizes. Lastly, we study the time at which the barrier is hit under the Blanchet-Glynn measure.

Main result. We now describe the main result of this chapter. We work under the following assumptions on the distribution of the step sizes, in addition to the assumption of negative drift, i.e., $\mathbb{E}X < 0$.

Assumptions:

(A1) *The right tail* $\mathbb{P}(X^+ > \cdot)$ *is regularly varying with tail index* $\alpha > 1$*; that is, for all* $u > 0$ *we have* $\mathbb{P}(X > ut) \sim u^{-\alpha}\mathbb{P}(X > t)$ *as* $t \to \infty$.

(A2) *The left tail* $\mathbb{P}(X^- > \cdot)$ *decays fast enough so that there exists a function* $h(t) = o(t)$ *such that* $h(t) \to \infty$ *and* $\int_{h(t)}^\infty \mathbb{P}(X^- > s)\,\mathrm{d}s = o(t \cdot \mathbb{P}(X^+ > t))$ *as* $t \to \infty$.

(A3) *The step size distribution has a continuous density which is regularly varying with tail index* $\alpha + 1$.

We note that the more natural condition $\mathbb{P}(X^- > t) = o(\mathbb{P}(X^+ > t))$ as $t \to \infty$ not necessarily implies Assumption (A2); although it does imply that $\int_{h(t)}^{\infty} \mathbb{P}(X^- > s)\,\mathrm{d}s = o\left(h(t) \cdot \mathbb{P}(X^+ > h(t))\right)$ for all h such that $h(t) \to \infty$. Nonetheless, Assumption (A2) is not overly restrictive. Indeed, a stronger condition that implies (A2) is that there exists $\delta > 0$ such that $t^{\delta} \cdot \mathbb{P}(X^- > t) = O(\mathbb{P}(X^+ > t))$ as $t \to \infty$; the latter holds for instance when $\mathbb{P}(X^- > \cdot)$ is light-tailed, or when $\mathbb{P}(X^- > \cdot)$ is regularly varying with tail index β satisfying $\beta > \alpha$. We also note that Assumption (A3) can be replaced by the less restrictive assumption that the step size distribution be ultimately absolutely continuous with respect to the Lebesgue measure, with continuous and regularly varying density. More precisely, it can be replaced by the assumption that there exists some t_0 such that on $[t_0, \infty)$ the step size distribution has a continuous density $f(\cdot)$ which is regularly varying with tail index $\alpha + 1$.

The main result of this chapter follows.

Theorem 1 (Efficiency for regularly varying right tails)**.** *Let* $\mathbb{Q}^{(c)}$ *be the distribution of* $\{S_n\}$ *induced by the transition kernel* $\mathsf{Q}^{(c)}$ *defined in* (2.4)*. Under Assumptions (A1)–(A3) the following hold:*

1. *If* $\alpha \in (1,\ 3/2)$ *then there exists some sufficiently large* c *so that* $\mathbb{Q}^{(c)}$ *is efficient for conditional sampling from* $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$ *for all* $b \geq 0$*.*

2. *If* $\alpha \in (3/2,\ 2)$ *then for all* $c \in \mathbb{R}$ *and all* $b \geq 0$ *it holds that* $\mathbb{Q}^{(c)}$ *is not efficient for conditional sampling from* $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$*.*

It is noteworthy that the change of measure is efficient for conditional sampling only for step sizes with very heavy tails. Indeed, recall that the tail index α is an indicator of how heavy a tail is, c.f. $\mathbb{E}\left[(X^+)^p\right] < \infty$ for $p \in (1, \alpha)$ and $\mathbb{E}\left[(X^+)^p\right] = \infty$ for $p > \alpha$.

Proof elements. We show here the main elements of the proof of Theorem 1, and start by investigating how the following statements are related. The proof is deferred to Section 2.6.1.

(S1_b^c) The distribution $\mathbb{Q}^{(c)}$ induced by the Blanchet-Glynn kernel (2.4) is efficient for conditional sampling from $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$.

(S2) We have $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ for all sufficiently large t, where Z has the residual life distribution (2.5) and is independent of X.

Proposition 1. *1. If (S2) holds, then there exists some sufficiently large c so that (S1_b^c) holds for all $b \geq 0$.*

2. Suppose that $\mathbb{P}(|X| \leq \delta) > 0$ for all $\delta > 0$. If (S1_b^c) holds for some $b \geq 0$ and some $c \in \mathbb{R}$, then (S2) also holds.

We remark that part 1. says that the same parameter c, chosen sufficiently large, works for *all* barriers $b \geq 0$; that is, b is independent of c in this case. We also remark that in the case of 2., applying 1. we get that (S1_b^c) actually holds *for all* $b \geq 0$, possibly after changing the constant c.

It is shown in [1] that $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) = o\left(\mathbb{P}(X > t)\right)$ as $t \to \infty$ for the family of strongly subexponential distributions, which includes regularly varying tails. Hence, the previous proposition shows that for efficiency of conditional sampling it is not enough to know that the difference decays faster than $\mathbb{P}(X > t)$, we actually need the sign of the difference as $t \to \infty$.

The following result shows that, in the case of step sizes satisfying Assumptions (A1)–(A2), the sign of $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)$ when $t \to \infty$ is fully determined by the tail index α of the right tail distribution. The proof is given in Section 2.5.

Theorem 2. *Assume that Assumptions (A1)–(A3) hold. Let Z be a random variable independent of X with the residual life distribution* (2.5)*. Then the following statements hold:*

1. *If* $\alpha \in (1,\ 3/2)$ *then* $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ *for all* $t > 0$ *sufficiently large.*

2. *If* $\alpha \in (3/2,\ 2)$ *then* $\mathbb{P}(X + Z > t) \geq \mathbb{P}(Z > t)$ *for all* $t > 0$ *sufficiently large.*

With this, Theorem 1 is a corollary of Proposition 1 and Theorem 2.

Hitting time analysis. We now investigate the finiteness and mean value of the hitting time τ_b under the Blanchet-Glynn change of measure $\mathbb{Q}^{(c)}$. The motivation is that τ_b gives a rough estimate of the computational effort of sampling a barrier-crossing path using the measure $\mathbb{Q}^{(c)}$.

The following result explores the hitting time in the regularly varying right tails setting of Assumptions (A1)–(A2). Its proof is deferred to Section 2.6.2.

Proposition 2 (Hitting time under $\mathbb{Q}^{(c)}$)**.** *Let* $\mathbb{Q}$ *be the distribution of* $\{S_n\}$ *induced by the transition kernel* $\mathsf{Q}^{(c)}$ *defined in* (2.4)*. Consider the setting of Assumptions (A1)–(A3). For any sufficiently large* c *the following hold for all* $b \geq 0$*:*

1. *If* $\alpha > 1$ *then* $\mathbb{Q}^{(c)}(\tau_b < \infty) = 1$.

2. *If* $\alpha \in (1, 3/2)$ *then* $\mathbb{E}^{\mathbb{Q}^{(c)}}\tau_b = \infty$ *for all* $b \geq 0$.

3. *If* $\alpha > 2$ *then* $\mathbb{E}^{\mathbb{Q}^{(c)}}\tau_b = O(b)$ *as* $b \to \infty$.

We remark that part 2. of the previous proposition, although a negative result, is actually independent of the Blanchet-Glynn measure $\mathbb{Q}^{(c)}$ and holds essentially because we have $\mathbb{E}^{\mathbb{P}}[\tau_b | \tau_b < \infty] = \infty$ when $\alpha \in (1, 2)$, see e.g. Theorem 1.1 of [11]. In other words, if $\alpha \in (1, 2)$ no algorithm — efficient or not — sampling paths $(S_0, \ldots, S_{\tau_b})$ from $\mathbb{P}(\,\cdot\,| \tau_b < \infty)$ can produce paths of finite expected length.

We also remark that part 2. of Proposition 2 is a counterexample for Proposition 4 of [1]. Indeed, the latter result claims that we have $\mathbb{E}^{\mathbb{Q}^{(c)}}\tau_b = O(b)$ as $b \to \infty$ when the step sizes satisfy $\mathbb{E}^{\mathbb{P}}[X^p;\ X > 0] < \infty$ for some $p > 1$. part 2. of Proposition 2 shows that the latter condition is not enough in general. Alternatively, this issue with Proposition 4 of [1] can also be derived from Corollary 1 of [14]. The latter result shows that if $\alpha \in (1, 3/2)$ no change of measure can be at the same time *strongly efficient for importance sampling* and have linear expected hitting time; in contrast, Proposition 4 of [1] states that for any $\alpha > 1$ the Blanchet-Glynn measure is both *strongly efficient for importance sampling* and has linear expected hitting time. Clearly both results are contradictory.

2.4 Threshold 3/2: a comparison

In this section we compare the threshold result of Theorem 1 with previous simulation works where, when using regularly varying step sizes, some form of efficiency of the method has given rise to the same threshold 3/2 for the tail index. We argue that the threshold arises in this existing literature for reasons unrelated to our work.

Review. Previous works in which the $3/2$ threshold appears in the context of efficiency are Blanchet and Liu [14] and Murthy, Juneja and Blanchet [15]. Both papers focus on solving the probability estimation problem via importance sampling; that is, their aim is to estimate the probability $\mathbb{P}(\tau_b < \infty)$ for arbitrarily large barriers b, using Monte Carlo sampling from another measure. Blanchet and Liu propose a parameterized and state-dependent change of measure, say $\mathbb{Q}^{\mathsf{BL}}$, which in the regularly varying case takes the form of a mixture between a *big-* and a *small-jump* transition kernel. Murthy *et al.* propose a similar big- and small-jump mixture kernel, say $\mathbb{Q}^{\mathsf{MJB}}$, however their change of measure is state-independent and additionally conditions on the time interval at which the barrier-crossing event occurs.

Both Blanchet and Liu [14] and Murthy *et al.* [15] have two requirements on their proposed measures: (i) the linear scaling $\mathbb{E}^{\mathbb{Q}}\tau_b = O(b)$ as $b \to \infty$ of the hitting time, and (ii) *strong efficiency* of the estimation procedure. In short, the latter means that, under the proposed change of measure $\mathbb{Q}$, the coefficient of variation of the random variable

$$\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}}(S_n : 0 \le n \le \tau_b) \cdot \mathbf{1}_{\{\tau_b < \infty\}} \tag{2.6}$$

stays bounded as $b \to \infty$; this is a second moment condition on (2.6). Both papers arrive at the same threshold result: for regularly varying step sizes, the proposed change of measure satisfies the previous two requirements for some combination of tuning parameters if and only if the tail index α is greater than 3/2.

Comparison. Given that the same threshold appears, it is natural to ask if there is a connection between our efficiency result in Corollary 1 and the results in prior work. We now argue why there is no clear or direct connection between these results.

In Blanchet and Liu [14] and Murthy *et al.* [15] the threshold 3/2 is strictly related to the second moment condition over the likelihood ratio (2.6) that is imposed by the requirement of efficiency for importance sampling. More precisely, in both these works if a moment condition is imposed on a different moment than the second, then we get a different threshold for the admissible tail indexes. In contrast, our result arises from imposing an almost sure condition on the Blanchet-Glynn change of measure. Indeed, by Corollary 1, efficiency for conditional sampling is a $\mathbb{Q}$-almost sure condition on the random variable (2.6). In contrast, and as said before, efficiency for importance sampling is a second moment condition on (2.6).

2.5 Proof of Theorem 2

In this section we prove Theorem 2, which is the main component of our main result of Theorem 1. That is, we prove that the tail index α completely determines the sign of the difference

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) \tag{2.7}$$

when t is large enough. We work under Assumptions (A1)–(A3), which state roughly speaking that the step sizes have regularly varying right tails with tail index α, and lighter left tails. In short, Theorem 2 establishes that if $\alpha \in (1, 3/2)$ then the difference (2.7) is negative for large t, and positive if $\alpha \in (3/2, 2)$.

The following is a roadmap for the main steps of the proof. First, in Lemma 1 we write the difference (2.7) as a sum of several terms. Second, in Lemma 2 we carry out an asymptotic analysis to determine which terms dominate when $t \to \infty$. It follows that the sign of the difference (2.7) when $t \to \infty$ can be reduced to the sign of the sum of dominant terms when $t \to \infty$. Finally, the latter is analyzed in Lemma 3, which reveals the dichotomy for α in $(1, 3/2)$ or $(3/2, 2)$.

Before embarking on the proof, some remarks on our notation are in order. Recall that we say that the random variable Z has the *residual life distribution of* X if its distribution is given by

$$\mathbb{P}(Z > t) \quad = \quad \min\left\{1, \frac{1}{|\mathbb{E}X|}\int_t^\infty \mathbb{P}(X > s)\,\mathrm{d}s\right\}, \quad \text{for all } t.$$

We write the left-most point of the support of Z as $z_0 := \inf\{t \,:\, \mathbb{P}(Z > t) < 1\}$, which is finite since $\mathbb{E}X$ is also finite. Additionally, we use that the density of Z is $\mathbb{P}(X > t)/|\mathbb{E}X|$ for all $t > z_0$ and that $\int_{z_0}^\infty \mathbb{P}(X > s)\,\mathrm{d}s = |\mathbb{E}X|$. We also use the notation $\overline{F}(t) := \mathbb{P}(X > t)$ for all t. Lastly, we recall that Assumption (A1) establishes that the right tail $\mathbb{P}(X > \cdot)$ is regularly varying with *tail index* $\alpha > 1$.

We start with a general decomposition of the difference (2.7).

Lemma 1. *Let X be a random variable with negative mean, and let Z be independent of X with the residual life distribution of X. Consider any function h such that* $\max\{z_0, 0\} < h(t) < t/2$ *for all* $t > \max\{2z_0, 0\}$. *Then the following holds for* $t > \max\{2z_0, 0\}$*:*

$$\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t) \quad = \quad p(t) - q(t) + \epsilon_1(t) - \epsilon_2(t),$$

where we define for $t > \max\{2z_0, 0\}$

$$\begin{aligned}
p(t) &:= \frac{1}{|\mathbb{E}X|}\int_{h(t)}^{t-h(t)} \overline{F}(t-s)\cdot\overline{F}(s)\,\mathrm{d}s \\
q(t) &:= \frac{\overline{F}(t)}{|\mathbb{E}X|}\int_{h(t)}^{\infty} \left[2\overline{F}(s) - F(-s)\right]\,\mathrm{d}s \\
\epsilon_1(t) &:= \frac{1}{|\mathbb{E}X|}\left[\left(\int_0^{h(t)} + \int_{z_0}^{h(t)}\right)\left[\overline{F}(t-s) - \overline{F}(t)\right]\cdot\overline{F}(s)\,\mathrm{d}s\right. \\
&\qquad\left. + \int_{-h(t)}^{0}\left[\overline{F}(t) - \overline{F}(t-s)\right]\cdot F(s)\,\mathrm{d}s\right] \\
\epsilon_2(t) &:= \frac{1}{|\mathbb{E}X|}\int_{-\infty}^{-h(t)} \overline{F}(t-s)\cdot F(s)\,\mathrm{d}s.
\end{aligned}$$

Proof. First note that

$$\mathbb{P}(X+Z>t) - \mathbb{P}(Z>t) = \mathbb{P}(X+Z>t, Z\le t) - \mathbb{P}(X+Z\le t, Z>t).$$

For t satisfying $\max\{z_0, 0\} < h(t) < t/2$ we decompose the first term on the right-hand side as follows:

$$\begin{aligned}
\mathbb{P}(X+Z>t, Z\le t) &= \int_{z_0}^{t}\overline{F}(t-s)\cdot\frac{1}{|\mathbb{E}X|}\overline{F}(s)\,\mathrm{d}s \\
= \frac{1}{|\mathbb{E}X|}&\left[\int_{z_0}^{h(t)}\overline{F}(t)\overline{F}(s)\,\mathrm{d}s + \int_{z_0}^{h(t)}\left[\overline{F}(t-s) - \overline{F}(t)\right]\overline{F}(s)\,\mathrm{d}s\right. \\
&+ \int_{h(t)}^{t-h(t)}\overline{F}(t-s)\overline{F}(s)\,\mathrm{d}s \\
&\left. + \int_0^{h(t)}\overline{F}(t)\overline{F}(s)\,\mathrm{d}s + \int_0^{h(t)}\left[\overline{F}(t-s) - \overline{F}(t)\right]\overline{F}(s)\,\mathrm{d}s\right].
\end{aligned}$$

A similar decomposition follows for the second term:

$$\begin{aligned}
\mathbb{P}(X+Z \le t, Z > t) &= \int_t^\infty F(t-s)\cdot \frac{1}{|\mathbb{E}X|}\overline{F}(s)\,\mathrm{d}s \\
&= \frac{1}{|\mathbb{E}X|}\left[\int_{-\infty}^{-h(t)} F(s)\overline{F}(t-s)\,\mathrm{d}s + \int_{-h(t)}^{0} F(s)\overline{F}(t)\,\mathrm{d}s \right. \\
&\qquad \left. + \int_{-h(t)}^{0} F(s)\left[\overline{F}(t-s)-\overline{F}(t)\right]\,\mathrm{d}s\right].
\end{aligned}$$

Subtracting both terms we obtain

$$\begin{aligned}
&|\mathbb{E}X|\cdot[\mathbb{P}(X+Z>t)-\mathbb{P}(Z>t)] = \\
&= \int_{h(t)}^{t-h(t)} \overline{F}(t-s)\overline{F}(s)\,\mathrm{d}s \\
&\quad -\overline{F}(t)\left[\int_{-h(t)}^{0} F(s)\,\mathrm{d}s - \int_{z_0}^{h(t)} \overline{F}(s)\,\mathrm{d}s - \int_0^{h(t)} \overline{F}(s)\,\mathrm{d}s\right] \\
&\quad + \left[\left(\int_{z_0}^{h(t)} + \int_0^{h(t)}\right)\left[\overline{F}(t-s)-\overline{F}(t)\right]\overline{F}(s)\,\mathrm{d}s \right. \\
&\quad \left. - \int_{-h(t)}^{0}\left[\overline{F}(t-s)-\overline{F}(t)\right]F(s)\,\mathrm{d}s\right] - \int_{-\infty}^{-h(t)} F(s)\overline{F}(t-s)\,\mathrm{d}s \\
&= |\mathbb{E}X|\cdot\left[p(t)-q(t)+\epsilon_1(t)-\epsilon_2(t)\right].
\end{aligned}$$

The last equality comes from using the definition of p, q, ϵ_1 and ϵ_2, and noting that $|\mathbb{E}X| = \int_{z_0}^{\infty}\overline{F}(s)\,\mathrm{d}s$ and $\mathbb{E}X < 0$, so we have that

$$\begin{aligned}
&\int_{-h(t)}^{0} F(s)\,\mathrm{d}s - \int_{z_0}^{h(t)} \overline{F}(s)\,\mathrm{d}s - \int_0^{h(t)} \overline{F}(s)\,\mathrm{d}s \\
&\quad = \mathbb{E}X^- - |\mathbb{E}X| - \mathbb{E}X^+ + 2\int_{h(t)}^{\infty}\overline{F}(s)\,\mathrm{d}s - \int_{-\infty}^{-h(t)} F(s)\,\mathrm{d}s \\
&\quad = \int_{h(t)}^{\infty}\left[2\overline{F}(s)-F(-s)\right]\,\mathrm{d}s.
\end{aligned}$$

This concludes the proof. □

The next step consists in determining which terms dominate when $t\to\infty$; this is done in the following result.

Lemma 2. *Let X be a random variable with negative mean satisfying Assumptions (A1)–(A3) for some index of regular variation $\alpha \in (1,2)$. Let Z be independent of X with the residual life distribution of X. In the definition of p, q, ϵ_1 and ϵ_2 consider a function h satisfying Assumption (A2); in particular h satisfies $\max\{z_0, 0\} < h(t) < t/2$ for all $t > \max\{2z_0, 0\}$, and it holds that $h(t) \to \infty$ and $h(t) = o(t)$ as $t \to \infty$. Then the following hold as $t \to \infty$:*

1.
$$p(t) - q(t) \sim \frac{t\overline{F}(t)^2}{|\mathbb{E}X|}\left(2\int_0^{1/2}[(1-u)^{-\alpha}-1]u^{-\alpha}\,\mathrm{d}u - \frac{2^\alpha}{\alpha-1}\right),$$

2.
$$\epsilon_1(t) = o\left(t\overline{F}(t)^2\right) \qquad \textit{and} \qquad \epsilon_2(t) = o\left(t\overline{F}(t)^2\right).$$

We remark that Lemmas 1 and 2 together establish that if X satisfies Assumptions (A1)–(A3) and $\alpha \in (1,2)$ then

$$\mathbb{P}(X+Z>t) - \mathbb{P}(Z>t) \;\sim\; K_\alpha \mathbb{P}(Z>t)\mathbb{P}(X>t) \tag{2.8}$$

as $t \to \infty$, where

$$K_\alpha := (\alpha-1)\int_0^1 \left((1-u)^{-\alpha}-1\right)\left(u^{-\alpha}-1\right)\mathrm{d}u - (\alpha+1).$$

This comes from $\mathbb{P}(Z>t) \sim t\overline{F}(t)/((\alpha-1)|\mathbb{E}X|)$ by Karamata's Theorem, see Theorem 1.6.1 of [30]. We note that, in contrast, Proposition 3 of [1] shows that $\mathbb{P}(X+Z > t) - \mathbb{P}(Z>t) = o\left(\mathbb{P}(X>t)\right)$, so the result (2.8) is much finer.

Proof. We start proving part 1. For that, first rewrite $|\mathbb{E}X| \cdot (p(t)-q(t))/(t\overline{F}(t)^2)$ as

$$2\int_{h(t)}^{t/2} \frac{\overline{F}(t-s)-\overline{F}(t)}{\overline{F}(t)} \cdot \frac{\overline{F}(s)}{t\overline{F}(t)}\,\mathrm{d}s - \frac{2\int_{t/2}^\infty \overline{F}(s)\,\mathrm{d}s}{t\overline{F}(t)} + \frac{\int_{h(t)}^\infty F(-s)\,\mathrm{d}s}{t\overline{F}(t)}.$$

The third term goes to zero by Assumption (A2), so we can ignore it for the proof of the statement. For the second term, note that since $\alpha > 1$ Karamata's Theorem, Theorem 1.6.1 of [30], yields

$$2\int_{t/2}^{\infty} \overline{F}(s)\,\mathrm{d}s \sim \frac{t\overline{F}(t/2)}{\alpha-1} \sim \frac{2^{\alpha}}{\alpha-1} t\overline{F}(t).$$

It remains to investigate the first term. To this end, we first rewrite the integral as

$$\int_{h(t)/t}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1\right] \frac{\overline{F}(tu)}{\overline{F}(t)}\,\mathrm{d}u; \tag{2.9}$$

we need to show that as $t \to \infty$ this integral converges to $\int_0^{1/2}[(1-u)^{-\alpha}-1]u^{-\alpha}\,\mathrm{d}u$. To this end, consider $\delta \in (0, 1/2)$ and note that since $h(t) = o(t)$ we can write (2.9) for all sufficiently large t as

$$\int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1\right] \frac{\overline{F}(tu)}{\overline{F}(t)}\,\mathrm{d}u + \int_{\delta}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1\right] \frac{\overline{F}(tu)}{\overline{F}(t)}\,\mathrm{d}u. \tag{2.10}$$

We start by analyzing the second term in (2.10). Since $\overline{F}$ is regularly varying with tail index α we get by the Uniform Convergence Theorem, Theorem 1.2.1 of [30], that

$$\sup_{u\in[\delta,\,1/2]} \left| \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1\right] \frac{\overline{F}(tu)}{\overline{F}(t)} - [(1-u)^{-\alpha}-1]u^{-\alpha} \right| \to 0 \quad \text{as } t \to \infty,$$

so

$$\int_{\delta}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1\right] \frac{\overline{F}(tu)}{\overline{F}(t)}\,\mathrm{d}u \to \int_{\delta}^{1/2} [(1-u)^{-\alpha}-1]u^{-\alpha}\,\mathrm{d}u, \tag{2.11}$$

as $t \to \infty$. We next analyze the first term in (2.10). Using Assumption (A3) we apply the mean value theorem on the interval $(0, u)$ to the function $s \mapsto \overline{F}(t(1-s))$ and establish that $\overline{F}(t(1-u))/\overline{F}(t) - 1 = f\left(t(1-\xi)\right) tu/\overline{F}(t)$ for some $\xi = \xi(t, u) \in (0, u)$, where f is the density of X. Additionally, we have that for all sufficiently large t it holds that $f\left(t(1-\xi)\right) t/\overline{F}(t) \le 2(1+2^{\alpha+1})\alpha$ for all $\xi \in (0, \delta)$. Indeed, since f is regularly varying with tail index $\alpha + 1$ then by Karamata's Theorem, Theorem 1.6.1 of [30] we get $tf(t)/\overline{F}(t) \to \alpha$; additionally, by Uniform Convergence Theorem, Theorem 1.2.1 of [30], we have that for any large enough t

$$\sup_{\xi \in (0,\delta)} \left| \frac{f(t(1-\xi))}{f(t)} - \frac{1}{(1-\xi)^{\alpha+1}} \right| \leq 1,$$

so $f(t(1-\xi))/f(t) \leq 1 + 1/(1-\xi)^{\alpha+1} \leq 1 + 2^{\alpha+1}$ for all sufficiently large t and for all $\xi \in (0,\delta) \subset (0, 1/2)$. We conclude that $\overline{F}(t(1-u))/\overline{F}(t) - 1 \leq 2(1+2^{\alpha+1})\alpha$ for all large enough t. We use this inequality to bound the term in the brackets of the first term of (2.10), obtaining that for all sufficiently large t

$$\int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u \leq 2(1+2^{\alpha+1})\alpha \int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} \, \mathrm{d}u.$$

We now argue that $\int_{h(t)/t}^{\delta} \left(tu\overline{F}(tu)\right) / \left(t\overline{F}(t)\right) \, \mathrm{d}u \leq 2\delta^{2-\alpha}/(2-\alpha)$ for all sufficiently large t. Indeed,

$$\int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} \, \mathrm{d}u = \left(\int_0^{\delta t} - \int_0^{h(t)} \right) \frac{s\overline{F}(s)}{t^2\overline{F}(t)} \, \mathrm{d}s,$$

so using that $t\overline{F}(t)$ is regularly varying with tail index $\alpha - 1 \in (0,1)$ we can apply Karamata's Theorem, Theorem 1.6.1 of [30], and that $h(t) \to \infty$ to get

$$\int_{h(t)/t}^{\delta} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} \, \mathrm{d}u \sim \frac{1}{2-\alpha} \left(\delta^2 \frac{\overline{F}(\delta t)}{\overline{F}(t)} - \left(\frac{h(t)}{t} \right)^2 \frac{\overline{F}(h(t))}{\overline{F}(t)} \right) \sim \frac{\delta^{2-\alpha}}{2-\alpha}$$

as $t \to \infty$. Indeed, note that since $h(t) = o(t)$ and $s \mapsto s^2\overline{F}(s)$ is regularly varying with tail index $\alpha - 2 \in (-1, 0)$ then $(h(t)/t)^2 \left(\overline{F}(h(t))/\overline{F}(t)\right) \to 0$. All in all, we obtain that the first term of (2.10) satisfies, for all large enough t,

$$\int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u \leq \frac{4(1+2^{\alpha+1})\alpha}{2-\alpha} \delta^{2-\alpha}.$$

Lastly, note that $\delta \in (0, 1/2)$ is arbitrary, so letting δ decrease to 0 in the latter inequality we get that

$$\lim_{\delta \searrow 0} \limsup_{t\to\infty} \int_{h(t)/t}^{\delta} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u = 0. \tag{2.12}$$

Similarly, letting δ decrease to 0 in (2.11) we obtain

$$\lim_{\delta \searrow 0} \lim_{t\to\infty} \int_{\delta}^{1/2} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u = \int_0^{1/2} [(1-u)^{-\alpha} - 1] u^{-\alpha} \, \mathrm{d}u. \tag{2.13}$$

From (2.12) and (2.13) and the decomposition (2.10) of (2.9) we get the desired result.

We now prove part 2. First we show that $\epsilon_1(t) = o\left(t\overline{F}(t)^2\right)$. To this end, it is sufficient to prove $\int_0^{h(t)} \left[\overline{F}(t-s) - \overline{F}(t)\right] \overline{F}(s) \, \mathrm{d}s = o\left(t\overline{F}(t)^2\right)$; that is, we want to prove that the expression

$$\int_0^{h(t)} \frac{\overline{F}(t-s) - \overline{F}(t)}{\overline{F}(t)} \frac{\overline{F}(s)}{t\overline{F}(t)} \, \mathrm{d}s = \int_0^{h(t)/t} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u$$

goes to zero as $t \to \infty$. We proceed by using the same line of reasoning used to prove (2.12), which is delineated in the following. First, apply the mean value theorem on the interval $(0, u)$ to the function $s \mapsto \overline{F}(t(1-s))$, and then use the Uniform Convergence Theorem , Theorem 1.2.1 of [30], to get that for all sufficiently large t we have

$$\int_0^{h(t)/t} \left[\frac{\overline{F}(t(1-u))}{\overline{F}(t)} - 1 \right] \frac{\overline{F}(tu)}{\overline{F}(t)} \, \mathrm{d}u \le 2(1 + 2^{\alpha+1})\alpha \int_0^{h(t)/t} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} \, \mathrm{d}u. \tag{2.14}$$

Second, apply Karamata's Theorem, Theorem 1.6.1 of [30], to obtain that

$$\int_0^{h(t)/t} \frac{tu\overline{F}(tu)}{t\overline{F}(t)} \, \mathrm{d}u = \int_0^{h(t)} \frac{s\overline{F}(s)}{t^2\overline{F}(t)} \, \mathrm{d}s \sim \frac{1}{2-\alpha} \left(\frac{h(t)}{t} \right)^2 \frac{\overline{F}(h(t))}{\overline{F}(t)}, \tag{2.15}$$

since $h(t) \to \infty$. Third, since the function $s \mapsto s^2\overline{F}(s)$ is regularly varying with $2 - \alpha \in (0, 1)$ and $h(t) = o(t)$ then $(h(t)/t)^2\, \overline{F}(h(t))/\overline{F}(t) \to 0$ when $t \to \infty$. The latter fact, together with (2.14) and (2.15), allows to conclude the desired result.

Lastly, $\epsilon_2(t) = o\left(t\overline{F}(t)^2\right)$ also holds because

$$\int_{-\infty}^{-h(t)} \frac{\overline{F}(t-s)}{\overline{F}(t)} \frac{F(s)}{t\overline{F}(t)}\, \mathrm{d}s = \int_{h(t)}^{\infty} \frac{\overline{F}(t+s)}{\overline{F}(t)} \frac{F(-s)}{t\overline{F}(t)}\, \mathrm{d}s \leq \int_{h(t)}^{\infty} \frac{F(-s)}{t\overline{F}(t)}\, \mathrm{d}s,$$

with the last term going to zero as $t \to \infty$ by Assumption (A2). $\square$

The previous result shows that when $t \to \infty$ the sign of the difference $\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)$ reduces to the sign of the term $p(t) - q(t)$. We now show that the latter is fully determined by the tail index α being either in $(1,\, 3/2)$ or in $(3/2,\, 2)$.

Lemma 3. *The quantity*

$$\int_0^{1/2} [(1-u)^{-\alpha} - 1]u^{-\alpha}\, \mathrm{d}u - \frac{2^{\alpha-1}}{\alpha - 1} \tag{2.16}$$

is negative for $\alpha \in (1,\, 3/2)$ *and positive for* $\alpha \in (3/2,\, 2)$.

Proof. First note that for fixed $u \in (0, 1/2)$ the function $[(1-u)^{-\alpha} - 1]u^{-\alpha}$ is strictly increasing in $\alpha > 0$, so $\int_0^{1/2}[(1-u)^{-\alpha} - 1]u^{-\alpha}\, \mathrm{d}u$ is as well. Also, since $2^\beta/\beta$ is strictly-decreasing for $\beta \in (0,1)$ then $-2^{\alpha-1}/(\alpha-1)$ is strictly increasing in α when $\alpha \in (1,2)$. Thus (2.16) is strictly increasing in α for $\alpha \in (1,2)$. Lastly, it is easy to verify that if $\alpha = 3/2$ then (2.16) is equal to zero. $\square$

With the previous lemmas the proof of Theorem 2 is straightforward.

Proof of Theorem 2. Consider in the definition of p, q, ϵ_1 and ϵ_2 a function h satisfying Assumption (A2); in particular it satisfies $\max\{z_0, 0\} < h(t) < t/2$ for all $t > \max\{2z_0, 0\}$, and $h(t) \to \infty$ and $h(t) = o\left(t\right)$ as $t \to \infty$. By Lemma 1 and Lemma 2 we have that as $t \to \infty$

$$\frac{\mathbb{P}(X + Z > t) - \mathbb{P}(Z > t)}{t\overline{F}(t)^2} = \frac{p(t) - q(t)}{t\overline{F}(t)^2} + o\left(1\right).$$

We conclude from Lemma 3 that as $t \to \infty$ the right-hand side is negative for $\alpha \in (1,\, 3/2)$ and positive for $\alpha \in (3/2,\, 2)$. $\square$

2.6 Proof of Propositions 1 and 2

2.6.1 Proof of Proposition 1

Proof of Proposition 1. For part 1., assume that (S2) holds; that is, that we have $\mathbb{P}(X + Z > t) \leq \mathbb{P}(Z > t)$ for all t sufficiently large. Take then c sufficiently large so that $\mathbb{P}(X + Z > c + t) \leq \mathbb{P}(Z > c + t)$ holds for all $t \geq 0$. Thus, we have by definition of $v^{(c)}$ and $w^{(c)}$ that

$$w^{(c)}(y-b)/v^{(c)}(y-b) \leq 1 \qquad \text{for all } y \leq b. \tag{2.17}$$

Using then the definition (2.4) of $\mathsf{Q}^{(c)}$, the following holds for $S_0 = 0$ and all $b \geq 0$:

$$\begin{aligned} \mathbf{1}_{\{\tau_b<\infty\}} \frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{Q}^{(c)}}(S_n : 0 \leq n \leq \tau_b) &= \mathbf{1}_{\{\tau_b<\infty\}} \frac{w^{(c)}(S_0-b)}{v^{(c)}(S_{\tau_b}-b)} \prod_{n=1}^{\tau_b-1} \frac{w^{(c)}(S_n-b)}{v^{(c)}(S_n-b)} \\ &\leq \mathbf{1}_{\{\tau_b<\infty\}} \frac{w^{(c)}(S_0-b)}{v^{(c)}(S_{\tau_b}-b)}. \end{aligned}$$

It follows that, conditional on $\tau_b < \infty$, we have $w^{(c)}(S_0-b) \leq v^{(c)}(S_0-b) \leq v^{(c)}(S_{\tau_b}-b)$, by inequality (2.17) and monotonicity of $v^{(c)}$. So then we obtain that $\mathbf{1}_{\{\tau_b<\infty\}}\mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}^{(c)}(S_n : 0 \leq n \leq \tau_b) \leq 1$. We conclude that statement (A_b^c) holds, by Corollary 1.

For part 2., assume that (A_b^c) holds, for some $b \geq 0$ and some $c \in \mathbb{R}$; i.e., that $\mathbb{Q}^{(c)}$ is efficient for conditional sampling from $\mathbb{P}(\,\cdot\,|\tau_b < \infty)$. We proceed by contradiction and assume that (S2) does not hold, i.e., that for all t we have that there exists a $t_0 > 0$ such that $\mathbb{P}(X + Z > t_0) > \mathbb{P}(Z > t_0)$. Using the fact that $w^{(c)}(y) = \mathbb{P}(X + Z > c - y)$ and $v^{(c)}(y) = \mathbb{P}(Z > c - y)$, we get that the previous hypothesis implies in particular that for all $y \leq b$ there exists $y_0 < y$ such that $w^{(c)}(y_0 - b)/v^{(c)}(y_0 - b) > 1$ holds.

With this, we will show that necessarily the following holds

$$\mathbb{Q}^{(c)}\left(\mathbf{1}_{\{\tau_b<\infty\}}\frac{w^{(c)}(S_0-b)}{v^{(c)}(S_{\tau_b}-b)}\prod_{n=1}^{\tau_b-1}\frac{w^{(c)}(S_n-b)}{v^{(c)}(S_n-b)}>1\right)>0, \tag{2.18}$$

i.e., that $\mathbb{Q}^{(c)}\left(\mathbf{1}_{\{\tau_b<\infty\}}\cdot \mathrm{d}\mathbb{P}/\mathrm{d}\mathbb{Q}^{(c)}(S_n:0\le n\le\tau_b)>1\right)>0$. The latter is a contradiction with hypothesis (A_b^c), by Corollary 1. Now, to prove (2.18) the main idea is to construct paths $(S_n:0\le n\le\tau_b)$ under $\mathbb{Q}^{(c)}$ that, before crossing the barrier b, spend a sufficiently large amount of time in the set

$$Y_{>1}^{c,b}:=\left\{y\le b\ :\ \frac{w^{(c)}(y-b)}{v^{(c)}(y-b)}>1\right\}.$$

For that we distinguish two cases: if $S_0=0\in Y_{>1}^{c,b}$ or not. We start with the former case.

Case 1: if c and $b\ge 0$ are such that $S_0=0\in Y_{>1}^{c,b}$. In this case we have that for all $C>0$ there exists $N>0$ such that

$$\mathbb{P}\left(\prod_{n=1}^{N-1}\frac{w^{(c)}(S_n-b)}{v^{(c)}(S_n-b)}>C\right)>0.$$

Indeed, this comes from the fact that $\mathbb{P}(|X|\le\delta)>0$ for all $\delta>0$ and that the function $w^{(c)}(\cdot)/v^{(c)}(\cdot)$ is continuous; hence, the random walk can stay for an arbitrary amount of steps in a small neighborhood of $S_0=0$ subset of $Y_{>1}^{c,b}$. It follows that we have, for a sufficiently large $N>0$,

$$\mathbb{P}\left(\tau_b=N\ ;\ \prod_{n=1}^{N-1}\frac{w^{(c)}(S_n-b)}{v^{(c)}(S_n-b)}>\frac{v^{(c)}(S_N-b)}{w^{(c)}(S_0-b)}\right)>0,$$

since X has unbounded right support and $v^{(c)}(\cdot) \leq 1$. Using then absolute continuity of $\mathbb{P}$ with respect to $\mathbb{Q}^{(c)}$ over paths with finite number of steps, we get that

$$\mathbb{Q}^{(c)}\left(\tau_b = N\ ;\ \prod_{n=1}^{N-1} \frac{w^{(c)}(S_n - b)}{v^{(c)}(S_n - b)} > \frac{v^{(c)}(S_N - b)}{w^{(c)}(S_0 - b)}\right) > 0. \tag{2.19}$$

We conclude then that (2.18) also holds, since the event in (2.19) is subset of the event in (2.18). This proves inequality (2.18), which is a contradiction with hypothesis (A_b^c).

Case 2: if c and $b \geq 0$ are such that $S_0 = 0 \notin Y_{>1}^{c,b}$. The idea for this case is to reduce it to the previous one, by constructing paths that, first, move to the set $Y_{>1}^{c,b}$, and second, spend a sufficiently large amount of time in $Y_{>1}^{c,b}$. For that, first define $\tau_{>1} := \inf\{n \geq 0 : S_n \in Y_{>1}^{c,b}\}$. Take then a compact set $A \subseteq \mathbb{R}$ and a large enough $M > 0$, so that they satisfy

$$\mathbb{P}\left(\tau_{>1} = M;\ S_n - b \in A \text{ for } n = 0, \ldots, M\right) > 0;$$

here, A is chosen to satisfy $S_0 - b \in \operatorname{int}(A)$ and $Y_{>1}^{c,b} \cap \operatorname{int}(A) \neq \emptyset$. Note now that it holds that $\sup\left\{\prod_{n=1}^{M} v^{(c)}(y_n - b)/w^{(c)}(y_n - b)\ :\ y_0, \ldots, y_M \in A + b\right\}$ is finite, since A is compact and $v^{(c)}(\cdot)/w^{(c)}(\cdot)$ is continuous. With this, and using the same arguments of Case 1, we obtain that there exists a large enough $N > 0$ such that

$$\mathbb{P}\left(\tau_b = N + M\ ;\ \frac{v^{(c)}(S_0 - b)}{w^{(c)}(S_{N+M} - b)} \prod_{n=1}^{M+N-1} \frac{w^{(c)}(S_n - b)}{v^{(c)}(S_n - b)} > 1\right) > 0. \tag{2.20}$$

Indeed, it is sufficient to condition the probability on the left hand side of (2.20) on the event $\{\tau_{>1} = M;\ S_0, \ldots, S_M \in A + b\}$ and use the strong Markov property. It follows that, by absolute continuity of $\mathbb{P}$ with respect to $\mathbb{Q}^{(c)}$ over paths with finite number of steps, we have that

$$\mathbb{Q}^{(c)}\left(\tau_b = N + M\ ;\ \frac{v^{(c)}(S_0 - b)}{w^{(c)}(S_{N+M} - b)} \prod_{n=1}^{M+N-1} \frac{w^{(c)}(S_n - b)}{v^{(c)}(S_n - b)} > 1\right) > 0. \tag{2.21}$$

Clearly then inequality (2.18) holds, since the event of the latter inequality contains the event in (2.21). We have arrived to a contradiction with hypothesis (A_b^c). $\square$

2.6.2 Proof of Proposition 2

Before showing the proof of Proposition 2 we establish the following lemma, which is a direct corollary of Lemma 2 of [14]. It will be used to prove part 3. of the latter result.

Lemma 4. *Let $\mathbb{Q}$ be a measure over paths of $\{S_n\}$. Assume that we have for any large enough $b > 0$ that*

$$\liminf_{y\to-\infty} \left\{ \mathbb{E}_y^{\mathbb{Q}} \left[S_1 - S_0\right] - \int_{b-y}^{\infty} \mathbb{Q}_y(S_1 - S_0 > u)\, \mathrm{d}u \right\} > 0. \tag{2.22}$$

Then $\mathbb{E}^{\mathbb{Q}}\tau_b = O(b)$ as $b \to \infty$.

Proof. The proof consists in showing that if (2.22) holds then the function $h(y) := (C + |b - y|)\mathbf{1}_{\{y\le b\}}$ satisfies the hypothesis of Lemma 2 of [14], for some $C > 0$. That is, $\mathbb{E}_y^{\mathbb{Q}^{(c)}}\left[h(S_1)\right] - h(y) < -\rho$ for all $y \le b$ and for some $\rho > 0$. For that, note that for $y < b$ we have

$$\begin{aligned} &\mathbb{E}_y^{\mathbb{Q}^{(c)}} h(S_1) - h(y) \\ &\quad = -\mathbb{E}_y^{\mathbb{Q}^{(c)}}\left[S_1 - S_0\right] + \int_{b-y}^{\infty} \mathbb{Q}_y^{(c)}\left(S_1 - S_0 > u\right)\, \mathrm{d}u - C\mathbb{Q}_y^{(c)}(S_1 > b) \end{aligned}$$

Therefore, inequality $\limsup_{y\to-\infty}\{\mathbb{E}_y^{\mathbb{Q}^{(c)}} h(S_1) - h(y)\} < 0$ is equivalent to inequality (2.22). Using then Lemma 2 of [14] we conclude that $\mathbb{E}^{\mathbb{Q}^{(c)}}\tau_b \le h(0)/\rho = C/\rho + \rho^{-1}b = O(b)$. $\square$

Proof of Proposition 2. For part 1., we have to show that $\mathbb{Q}^{(c)}(\tau_b < \infty | S_0 = 0) = 1$ holds for all $b \ge 0$. We will actually show that this is true for all $c \in \mathbb{R}$.

For that, first consider any $c \in \mathbb{R}$ and note that from Lemma 1 of [1] we have that $\lim_{y\to-\infty} \mathbb{E}^{\mathbb{Q}^{(c)}}\left[S_1 - S_0 | S_0 = y\right] > 0$. This result applies in our case because X is strongly subexponential, since X has regularly varying right tails with tail index $\alpha > 1$. Also, it can be checked that $\mathbb{E}^{\mathbb{Q}^{(c)}}\left[S_1 - S_0 | S_0 = y\right] = \mathbb{E}^{\mathbb{P}}\left[X | X + Z > c + b - y\right]$ holds, since

$$\mathbb{Q}^{(c)}\left(S_1 - S_0 \in \cdot | S_0 = y\right) = \mathbb{P}\left(X \in \cdot | X + Z > c + b - y\right), \tag{2.23}$$

where X and Z are independent and Z has the residual life distribution of X.

We have thus that there exists $\epsilon > 0$ and $y_0 \in \mathbb{R}$ such that for all $y \leq y_0$ we have $\mathbb{E}^{\mathbb{Q}^{(c)}}\left[S_1 - S_0 | S_0 = y\right] > \epsilon$. It follows that $\mathbb{Q}^{(c)}(\tau_{y_0} < \infty | S_0 = y) = 1$ holds for all $y \leq y_0$. We distinguish two cases now: if $b \leq y_0$ and if $y_0 < b$. In the former case, it is direct that $\mathbb{Q}^{(c)}(\tau_b < \infty | S_0 = 0) \geq \mathbb{Q}^{(c)}(\tau_{y_0} < \infty | S_0 = 0) = 1$ holds, since $0 \leq b \leq y_0$. In the latter case on the other hand, that is if $y_0 < b$, we can use a standard geometric trials argument to get that, for all $y \leq b$,

$$\mathbb{Q}^{(c)}(\tau_b < \infty | S_0 = y) \geq \mathbb{Q}^{(c)}(\text{Geom}(\rho) < \infty) = 1.$$

Here, Geom(ρ) is an independent geometric random variable with parameter

$$\rho := \inf_{\overline{y} \in [y_0, b]} \mathbb{Q}^{(c)}\left(S_1 - S_0 > b - y_0 | S_0 = \overline{y}\right),$$

which is finite by (2.23). In both cases we have shown that $\mathbb{Q}^{(c)}(\tau_b < \infty | S_0 = y) = 1$.

For part 2., we proceed by contradiction and assume that $\mathbb{E}^{\mathbb{Q}^{(c)}}\left[\tau_b\right] < \infty$. By Theorem 1.1 of [11], for $\alpha \in (1, 2)$ it holds $\mathbb{E}^{\mathbb{P}}\left[\tau_b \,| \tau_b < \infty\right] = \infty$. But since $\mathbb{Q}^{(c)}$ is efficient for conditional sampling when $\alpha \in (1, 3/2)$, then by Corollary 1 part 2. we have

$$\mathbb{E}^{\mathbb{Q}^{(c)}}\left[\tau_b\right] = \mathbb{E}^{\mathbb{Q}^{(c)}}\left[\sum_{n \geq 0} n \mathbf{1}_{\{\tau_b = n\}}\right] \geq \mathbb{E}^{\mathbb{P}}\left[\sum_{n \geq 0} n \mathbf{1}_{\{\tau_b = n\}}\right] = \mathbb{E}^{\mathbb{P}}\left[\tau_b \mathbf{1}_{\{\tau_b < \infty\}}\right] = \infty,$$

which is a contradiction.

For part 3., by Lemma 4 it is sufficient to show that (2.22) holds. For that, note that by definition of the Blanchet-Glynn kernel in (2.4) and the fact that $v^{(c)}(y) = \mathbb{P}(Z > -y + c)$ and $w^{(c)}(y) = \mathbb{P}(X + Z > -y + c)$ we have

$$\mathbb{Q}^{(c)}\left(S_1 - S_0 \in \cdot \mid S_0 = y\right) \;=\; \mathbb{P}\left(X \in \cdot \mid X + Z > -y + c\right), \tag{2.24}$$

where Z is independent of X and has the residual life distribution of X. With this we get that

$$\begin{aligned}
&\mathbb{E}_y^{\mathbb{Q}}\left[S_1 - S_0\right] - \int_{b-y}^{\infty} \mathbb{Q}_y(S_1 - S_0 > u)\,\mathrm{d}u \\
&= \;\mathbb{E}^{\mathbb{P}}\left[X|X + Z > -y + c\right] - \int_{b-y}^{\infty} \mathbb{E}^{\mathbb{P}}\left[\mathbf{1}_{\{X>u\}}|X + Z > -y + c\right]\,\mathrm{d}u \\
&= \;\mathbb{E}^{\mathbb{P}}\left[X|X + Z > -y + c\right] - \mathbb{E}^{\mathbb{P}}\left[[X - b + y]^{+}\,|X + Z > -y + c\right].
\end{aligned}$$

For the first term in the right-hand side of the previous display one obtains that

$$\liminf_{y\to-\infty}\; \mathbb{E}^{\mathbb{P}}\left[X|X + Z > -y + c\right] \geq (\alpha - 1)\cdot|\mathbb{E}^{\mathbb{P}}X|$$

by following the same arguments of the proof of Lemma 1 of [1] and using that $t\cdot\mathbb{P}(X > t) \sim |\mathbb{E}^{\mathbb{P}}X|(\alpha - 1)\mathbb{P}(Z > t)$ as $t \to \infty$, which is direct from Karamata's Theorem, see Theorem 1.6.1 of [30]. For the second term in the right-hand side one gets that if $b > 0$ is sufficiently large so that $z_0 > c - b$ then

$$\mathbb{E}^{\mathbb{P}}\left[[X - b + y]^{+}\,|X + Z > c - y\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[[X - b + y]^{+}\right]}{\mathbb{P}(X + Z > c - y)} = |\mathbb{E}^{\mathbb{P}}X|\frac{v^{(c)}(y - b)}{w^{(c)}(y - b)}.$$

It follows that

$$\lim_{y\to-\infty} \mathbb{E}^{\mathbb{P}}\left[[X - b + y]^{+}\,|X + Z > -y + c\right] = |\mathbb{E}^{\mathbb{P}}X|,$$

since $v^{(c)}(y - b)/w^{(c)}(y - b) \to 1$ as $y \to -\infty$ by Proposition 3 of [1].

We have thus obtained that

$$\liminf_{y\to-\infty}\left\{\mathbb{E}_y^{\mathbb{Q}}\left[S_1-S_0\right]-\int_{b-y}^{\infty}\mathbb{Q}_y(S_1-S_0>u)\,\mathrm{d}u\right\}\geq(\alpha-2)\cdot|\mathbb{E}^{\mathbb{P}}X|,$$

so applying Lemma 4 we conclude that if $\alpha>2$ then $\mathbb{E}^{\mathbb{Q}^{(c)}}\tau_b=O(b)$ as $b\to\infty$. $\square$

CHAPTER 3
ACCURACY OF SAMPLING BROWNIAN FUNCTIONALS THROUGH EULER APPROXIMATION

Abstract. In this chapter we derive weak limits for the discretization errors of sampling barrier-hitting and extreme events of Brownian motion by using the Euler discretization simulation method. Specifically, we consider the Euler discretization approximation of Brownian motion to sample barrier-hitting events, i.e. hitting for the first time a deterministic "barrier" function; and to sample extreme events, i.e. attaining a minimum on a given compact time interval or unbounded closed time interval. For each case we study the discretization error between the actual time the event occurs versus the time the event occurs for the discretized path, and also the discretization error on the position of the Brownian motion at these times. We show that if the step-size of the time mesh is $1/n$ then the discretization error for the times converges at rate $O(1/n)$ and the error for the position of the Brownian motion converges at rate $O(1/\sqrt{n})$. We show limits in distribution for the discretization errors normalized by their convergence rate, and give closed-form analytic expressions for the limiting random variables. Additionally, we use these limits to study the asymptotic behavior of Gaussian random walks in the following situations: (1.) the overshoot of a Gaussian walk above a barrier that goes to infinity; (2.) the minimum of a Gaussian walk compared to the minimum of the Brownian motion obtained when interpolating the Gaussian walk with Brownian bridges, both up to the same time horizon that goes to infinity; and (3.) the global minimum of a Gaussian walk compared to the global minimum of the Brownian motion obtained when interpolating the Gaussian walk with Brownian bridges, when both have the same positive drift decreasing to zero. In deriving these limits in distribution we provide a unified framework to understand the relation between several papers where the constant $-\zeta(1/2)/\sqrt{2\pi}$ has appeared, where ζ is the

Riemann zeta function. In particular, we show that this constant is the mean of some of the limiting distributions we derive.

3.1 Introduction

Brownian motion is arguably the most important continuous-time stochastic process in probability theory. Its relevance is only increased by its widespread use as a stochastic model in engineering, sciences and business. On the other hand, the simulation of Brownian motion raises fundamental challenges, since it is a continuous-time process characterized by its violent fluctuations and self-similar-type structure in time-space, whereas computers can only simulate and store discrete objects. This forces the simulation of Brownian motion to be inherently inaccurate, except for a few especially structured collection of events where Brownian motion can be simulated without bias, see e.g. Devroye [17] for a survey on such special cases.

In this chapter we consider the simulation of Brownian motion by approximating it on a constant, regularly spaced, time mesh; that is, we consider the *Euler discretization* of Brownian paths on a *regular* mesh. This approximation is arguably the easiest and simplest possible simulation method for Brownian motion, which makes it appealing from a practical point of view. Moreover, in some special practical applications there are exogenous conditions which makes the Euler discretization the most sensible simulation method to use; for example, in finance when a financial instrument can only be monitored at regular time intervals, see Broadie et al. [18].

We are particularly interested in the simulation of extreme events and of barrier-hitting events of Brownian motion. We call a barrier-hitting event an event where the Brownian motion “hits” or “crosses” for the first time a given “barrier”, either constant or non-constant. We call extreme event an event where a minimum (or maximum) of the Brownian motion over a closed time interval is attained; in this case the time interval can be bounded or unbounded. These two types of events are of fundamental importance, both in the theory of stochastic processes, see e.g. fluctuation theory; as well as in practice, where usually events of critical interest can be formulated as one of these events, e.g. a stock price ever reaching a certain value, or a natural disaster occurring in a certain time horizon.

In this chapter we study the accuracy of simulating barrier-hitting and extreme events of Brownian motion by instead simulating the respective event for the Euler discretized Brownian motion. The accuracy of the Euler discretization is a non-trivial issue, since, except for trivial cases, with probability one none of these two events will ever occur exactly on a regular time mesh of the form $\{0, T/n, 2T/n, \ldots\}$ for constant T. This raises the question of what theoretical conditions are there that guarantee accuracy of the Euler discretization. This question motivates the main objective of this paper, which is: the study of convergence of the discretization error for these two events, as well as their rate of convergence, and weak convergence of the normalized errors.

The work in this chapter was initially motivated by the study of the discretization error when sampling multidimensional *reflected Brownian motion* (RBM) using the Euler discretization method. Indeed, multidimensional RBM can be computed as a deterministic mapping, the *reflection mapping*, acting on paths of Brownian motion; therefore RBM can be simulated by instead applying the reflection mapping to the Euler discretized Brownian motion. In turn, the reflection mapping contains terms involving the running minimum of each dimension of the Brownian motion path, so usually events related to RBM are equivalent to extreme or barrier-hitting events of Brownian motion.

Our work is also motivated by the connection of Gaussian random walks with the Euler discretization of Brownian motion. More specifically, the constant $-\zeta(1/2)/\sqrt{2\pi}$, where ζ is the Riemann zeta function, has appeared in a number of papers working with Brownian motion and Gaussian walks. see e.g. [31, 21, 32, 16, 33, 8, 18]. Up to now it had been unknown how these works were precisely related; in this chapter we clarify this connection.

Main contributions. We consider barrier-hitting events of a Brownian motion B, where the barrier function is continuously differentiable and non-decreasing on $\mathbb{R}_+$, and which has initial position at time zero above the Brownian motion. We also consider the extreme events where the Brownian motion attains a global minimum over $[0, \infty)$, or attains its minimum over an interval of the type $[0, a]$ for a given $a > 0$ finite. In any of these three cases denote for now, if they are finite, by t^* the time at which the event occurs and $B(t^*)$ the position of the Brownian motion at such time; denote analogously by t^n and $B^n(t^n)$ the respective values for the Euler discretization approximation B^n of B on the mesh $\{0, 1/n, 2/n, \ldots\}$.

A first contribution is that we show the following convergence results conditional on t^* and $B(t^*)$ being finite. First, we show that the absolute errors $|t^n - t^*|$ and $|B^n(t^n) - B(t^*)|$ converge almost surely to zero at rates $O(1/n)$ and $O(1/\sqrt{n})$ respectively, when $n \to \infty$. Second, we show that the normalized errors $n\,(t^n - t^*)$ and $\sqrt{n}\,(B^n(t^n) - B(t^*))$ converge jointly in distribution, and we give a closed-form analytic expression for the limiting random variables. In the case of barrier-hitting events, the limiting random variable is related to the overshoot above zero of an "equilibrium" Gaussian random walk. In the case of extreme events, the joint limiting random variable is the same in both cases, and involves the minimum of a two-sided Bessel process over the integers displaced by a uniform random variable. Nonetheless, we remark that some of the derived limits in distribution were at least partially known in the literature. Specifically, our Theorem 3 contains [34, Lemma 10.10] in the Gaussian walk case and also contains [18, Lemma 4.2]; and our Theorem 4 contains [8, Theorem 1]. Our work contributes in augmenting these results and also in giving a unified derivation of them.

A second contribution of this chapter is that our Euler discretization limits in distribution allow to give rise to other limits in distribution for Gaussian random walks. These limits are related to (1.) the overshoot of the Gaussian walk above a barrier that goes to infinity; (2.) the minimum of the Gaussian walk compared to the minimum of a Brownian motion, both up to the same time horizon that goes to infinity; and (3.) the global minimum of the Gaussian walk compared to the global minimum of a Brownian motion, when both have the same positive drift decreasing to zero. The limiting distributions of these quantities are some of the same limiting random variables obtained for the Euler discretization errors, since the limits are derived as simple corollaries of the Euler discretization analysis. In particular, this allows us to give a crisper intuition of the limiting distributions obtained for the Euler discretization errors, since from the Gaussian walk perspective the limiting random variables can be understood as the equilibrium distributions of renewal processes.

A third contribution of this work is that we demystify the relationship between several papers in the literature where the constant $-\zeta(1/2)/\sqrt{2\pi}$ has appeared, where ζ is the Riemann zeta function. This constant has appeared in works about Gaussian walks, [31, 21, 32, 16]; discretized Brownian motion, [33, 8, 18]; and approximations of stochastic processes, [31, 21, 35]. As mentioned in [16], the precise connection that made the same constant appear in these works was unclear. In part, this is because the aforementioned constant showed up in expressions for convergence in mean, and usually this mean was derived either using Spitzer's identities for random walks, see e.g. [8, 33], or through cumbersome *ad hoc* methods, see [19]. We show that the constant $-\zeta(1/2)/\sqrt{2\pi}$ is the mean of two of the limiting distributions we derive. Indeed, [19, Theorem 1] gives convergence in mean for the second component in the triplet in our Theorem 4; [8, Theorem 2] and [33, Theorem 1] give convergence in mean for the second component in the triplet in our Corollary 3; and [16, Theorem 2], [21, Theorem 1] and [35, Equations (4) and (10)] give convergence in mean for the second component in the triplet in our Corollary 4. Our work allows to view the aforementioned papers from a unified perspective, clarifying the relationship between them.

An additional contribution of this chapter is that we show convergence in distribution of Brownian motion when "zoomed-in" in time-space about the times where the barrier is hit or an extreme is attained. We prove this convergence for a subspace of $\mathcal{C}(\mathbb{R})$, i.e. the continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and for the *weighted-supremum* metric. This metric generates a *stronger* or *coarser* topology than the usual metric of *uniform convergence over compact sets*, see e.g. Chapter 5 of Ganesh et al. [36] and Dieker [37].

Literature review. A variety of methods for simulation of Brownian motion exist, given its popularity and usefulness. Sophisticated methods include *exact simulation* of some special quantities, see Devroye [17], and the simulation of approximations of Brownian motion that have path-wise guarantees of accuracy, see Beskos et al. [38]. On the other hand, the Euler discretization approximation is a classic and simple approach to sample Brownian motion and more general diffusions, see e.g. Platen [39] for a comprehensive exposition. Nonetheless, the first approach along the lines of our work is Asmussen et al. [8], who study the error of the Euler discretization approach to sample one-dimensional reflected Brownian motion. We highlight the work of Broadie et al. [18, 19], which uses the Euler discretization approximation to study the pricing of barrier and lookback options, respectively. They justify their choice of the discretized approximation in that the financial instruments they study can only be monitored at a pre-specified regular time mesh and not in continuous-time. In greater generality, the last decade has seen the development of several sophisticated methods for exact sampling of more general diffusion processes, see e.g. Beskos et al. [40, 41]. In this line, a work related to ours is Etore et al. [42], who study exact simulation of diffusions that involve the local-time at zero; nonetheless, their work cannot be extended to ours since they do not treat reflected diffusions.

A central element of our work is the convergence in distribution of the Brownian motion when "zoomed-in" in time-space about random times. The study of this convergence heavily relies on path decompositions of Brownian motion; Williams [43] is the quintessential reference in this line. For our particular study we use decompositions of Brownian motion about its global minima as Bessel process, see Rogers and Pitman [44], Bertoin [45]; and also decomposition about its local minima, see Asmussen et al. [8] and Imhof [46].

A related work is Chaumont [47], who studies the pre- and post-minimum paths of Lévy processes, but when process is conditioned to stay positive. In our case we do not condition on the process staying positive.

In several works in the literature the constant $-\zeta(1/2)/\sqrt{2\pi}$ has appeared, where ζ is the Riemann zeta function. These works essentially deal with extreme events of discretized Brownian motion, e.g. Asmussen et al. [8], Calvin [33], Broadie et al. [18, 19]; with Gaussian random walks, e.g. Janssen and van Leeuwaarden [16], Chang and Peres [32]; and with approximations of stochastic processes, e.g. Chernoff [31], Siegmund [21], Comtet and Majumdar [35]. More generally, Biane et al. [48] gives an overview of multiple random variables related to Brownian motion where the Riemann zeta and Jacobi's theta functions appear.

In Section 3.5 we work with a *Polish* metric space which allows to derive stronger, in a sense, weak convergence results of stochastic processes over unbounded time horizons. This space uses the *weighted-supremum* metric, see e.g. Section 11.5.2 of Whitt [20]. This framework has mostly been used in the queueing theory literature to derive large-deviations-type limits, see e.g. Chapter 5 of Ganesh et al. [36] and Dieker [37].

Outline. In Section 3.2 we show the main results of this chapter concerning the Euler discretization error of Brownian motion about random times of interest. In detail, in Section 3.2.1 we treat the error for barrier-hitting events; in Section 3.2.2 the error for extreme events on compact time intervals, and in Section 3.2.3 we treat the error for extreme events on unbounded closed time intervals.

In Section 3.3 we transform the convergence in distribution of the normalized Euler errors into asymptotic results for Gaussian walks. More precisely, in Section 3.3.1 we analyze the overshoot above a barrier when the barrier goes to infinity; in Section 3.3.2 we study the minimum of a Gaussian walk compared to the minimum of a Brownian motion, both up to the same time horizon that goes to infinity; and in Section 3.3.3 we analyze the global minimum of the Gaussian walk compared to the global minimum of a Brownian motion, when both have the same positive drift decreasing to zero.

Section 3.4 is dedicated to the proofs of the main results in Section 3.2. For that, in Section 3.4.1 we write the discretization errors as mappings of Brownian "zoomed-in" about the random time of interest; then in Section 3.4.2 we prove that such zoomed-in processes converge in distribution; and in Section 3.4.3 we give the actual proofs of the main results of this paper, Theorems 3, 4 and 5 of Section 3.2.

Finally, in Section 3.5 we show the convergence of the Brownian motion "zoomed-in" about times of barrier-hit or times at which extremes are attained.

Notation. We denote as $\mathbb{R}_+$ and $\mathbb{Z}_+$ the nonnegative real numbers and integer numbers, respectively. The function $\lceil x \rceil$ denotes the integer part of the number x. Also, $\mathscr{C}(\mathbb{R})$ and $\mathscr{C}(\mathbb{R}_+)$ are the set of real functions defined on all real numbers and on the nonnegative real numbers, respectively. We call *standard Brownian motion* and *standard Bessel(3) process* as the Brownian motion and Bessel(3) process, respectively, with no drift and unit variance. Unless otherwise stated, we assume that all process start from zero at time zero.

3.2 Main results

In this section we show our main results about the error convergence when simulating barrier hitting and extreme value events of Brownian motion by using an Euler discretization approach. In Section 3.2.1 we focus on the error of estimating barrier hitting events with the Euler discretization. In Section 3.2.2 we consider the error when sampling extreme values of the Brownian motion over a fixed finite horizon. In Section 3.2.3 we extend the analysis to extreme values over an infinite horizon.

Consider a Brownian motion $B = (B(t) : t \in \mathbb{R}_+)$ with constant drift μ and variance σ^2. For $n > 0$ integer, let $B^n = (B^n(t) : t \in \mathbb{R}_+)$ be the piecewise constant process defined as

$$B^n(t) := B\left(\lfloor nt \rfloor / n\right) \quad \text{for all} \quad t \in \mathbb{R}_+; \tag{3.1}$$

that is, B^n is the Euler discretization of B on the mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$.

3.2.1 Error of barrier hitting times

We consider first the error of estimating barrier hitting times using the Euler discretization of the Brownian motion. Consider a deterministic barrier function $b = (b(t) : t \geq 0)$ satisfying the following assumptions:

(H$_b$) The function $b : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing on $\mathbb{R}_+$, continuously differentiable on $\mathbb{R}_+ \backslash \{0\}$, and $b(0) \geq 0$.

We want to estimate the time

$$\tau_b := \inf \left\{ t \in (0, \infty) : B(t) \geq b(t) \right\}; \tag{3.2}$$

for which we use the approximation

$$\tau_b^n := \inf \left\{ t \in (0, \infty) : B^n(t) \geq b(t) \right\}, \tag{3.3}$$

where B^n is the discretized version of B defined in (3.1). Here we use the usual convention that $\inf \emptyset = +\infty$. Our objective is to study the error of $\tau_b^n - \tau_b$ and $B^n(\tau_b^n) - B(\tau_b)$. The following result establishes the convergence rate and limiting distribution of these errors.

Theorem 3. *Consider a barrier function $b = (b(t) : t \geq 0)$ satisfying (H$_b$). Conditioned on the event $\{\tau_b < \infty\}$, as $n \to \infty$ the triplet*

$$\left(n\left(\tau_b^n - \tau_b\right), \quad \sqrt{n}\left(B^n(\tau_b^n) - B(\tau_b)\right), \quad \lceil n\tau_b \rceil - n\tau_b \right), \tag{3.4}$$

converges jointly in distribution to the triplet

$$\left(U + \min\{k \geq 0 : W(U+k) > 0\},\ \sigma W\left(U + \min\{k \geq 0 : W(U+k) > 0\}\right),\ U\right), \tag{3.5}$$

where $W = (W(t) : t \in \mathbb{R}_+)$ is a standard Brownian motion independent of B, and U is a uniformly distributed random variable on $(0,1)$ which is independent of B and W.

We remark that the limiting random variable (3.5) can be understood in the following way. Consider a uniform random variable U in $(0,1)$ and a sequence of iid standard normal random variables $(X_k^*)_{k\geq 0}$, and define a modified Gaussian walk $S^* = (S_k^* : k \geq 0)$ as $S_0^* := \sqrt{U} X_0^*$ and $S_k^* := S_{k-1}^* + X_k^*$ for $k \geq 1$. Denote the first strictly positive time τ_+ of S^* as

$$\tau_+ := \min\{k \geq 0 : S_k^* > 0\}.$$

Then the triplet

$$\big(U + \tau_+, \quad \sigma S_{\tau_+}^*, \quad U\big)$$

is equal in distribution to the limiting random variable (3.5).

3.2.2 Error of minimum on finite horizon

We consider now the error of estimating the minimum on bounded time intervals by using the Euler discretization of the Brownian motion. Consider a time interval $[0,a]$ such that $0 < a < \infty$. We want to estimate the time

$$T_{\min,a} := \inf\left\{t \in [0,a] : B(t) = \min_{s\in[0,a]} B(s)\right\}; \tag{3.6}$$

for which we use the approximation

$$T_{\min,a}^n := \inf\left\{t \in [0,a] : B^n(t) = \min_{s\in[0,a]} B^n(s)\right\}; \tag{3.7}$$

where the discretized version B^n is defined in (3.1). Note that $T_{\min,a}$ is the almost surely unique value satisfying $B(T_{\min,a}) = \min_{s\in[0,a]} B(s)$; however, because of how B^n is defined,

$$\left\{t \in [0,a] : B^n(t) = \min_{s\in[0,a]} B^n(s)\right\} = \left[T^n_{\min,a},\ T^n_{\min,a} + \frac{1}{n}\right).$$

We want to study the error of $T^n_{\min,a} - T_{\min,a}$ and $B^n(T^n_{\min,a}) - B(T_{\min,a})$. The following result establishes the limiting distribution and convergence rate of these errors.

Theorem 4. *The triplet*

$$\left(n\left(T^n_{\min,a} - T_{\min,a}\right),\quad \sqrt{n}\left(B^n(T^n_{\min,a}) - B(T_{\min,a})\right),\quad \lceil nT_{\min,a}\rceil - nT_{\min,a}\right), \tag{3.8}$$

converges jointly in distribution to the triplet in

$$\left(U + \arg\min_{k\in\mathbb{Z}} R(U+k),\quad \sigma \min_{k\in\mathbb{Z}} R(U+k),\quad U\right), \tag{3.9}$$

where $R = (R(t) : t \in \mathbb{R})$ is a two-sided Bessel(3) process and U is a uniformly distributed random variable on $(0,1)$ which is independent of R. Here we have abused notation by denoting as $\arg\min$ in (3.9) *the almost surely unique value k at which the minimum is attained.*

We remark that the convergence of the second component of the triplet in (3.8) corresponds to Theorem 1 of Asmussen et al. [8]. Their limit however contains an extra time term which in our framework would correspond to $\sqrt{a}$; this term appears because they consider a discretization with step size a/n, whereas we consider $1/n$. They also show convergence of all finite moments and show that

$$\beta := \mathbb{E}\left[\min_{k\in\mathbb{Z}} R(U+k)\right] = -\frac{\zeta(1/2)}{\sqrt{2\pi}}. \tag{3.10}$$

The objective of Asmussen et al. [8] is to study the Euler discretization error $\Gamma B(t) -$

$\Gamma B^n(t)$ of the *reflected Brownian motion* ΓB. Here, Γ is the *reflection mapping*, defined for all trajectory X with $X(0) = 0$ as

$$\Gamma X(t) := X(t) - \inf_{s \in [0,t]} X(s),$$

Our work extends theirs in the following way. Let the *busy period* mapping Γ' be

$$\Gamma' X(t) := t - \sup \{s \in [0,t] : \Gamma X(s) = 0\} = t - \sup \left\{ s \in [0,t] : X(s) = \inf_{u \in [0,s]} X(u) \right\}$$

for X trajectory with $X(0) = 0$. Alternatively, Γ' can be seen as the *drift derivative* of the reflection mapping; see Dieker and Gao [49] for further details on the latter process. It is easy to show that if B^n is the Euler discretization with stepsize $1/n$ as defined in (3.1) then

$$n \left(\Gamma' B(t) - \Gamma' B^n(t) \right) = 1 + n \left(T^n_{\min,t} - T_{\min,t} \right)$$

and

$$\sqrt{n} \left(\Gamma B(t) - \Gamma B^n(t) \right) = \sqrt{n} \left(B^n(T^n_{\min,t}) - B(T_{\min,t}) \right).$$

In particular, Theorem 4 gives the joint limit in distribution of these Euler discretization errors.

Another related work is Broadie et al. [18], which studies the problem of pricing barrier options that can only be monitored at regular time intervals. They are interested in expressing the event $\{B^n(t) > y,\ \tau^n_b \le t\}$ in terms of $B(t)$ and τ_b; here, B is a Brownian motion, τ_b is its hitting time to the barrier b, and B^n and τ^n_b are the analogous discretized versions. Essentially, they show the approximation

$$\mathbb{P}(B^n(t) > y,\ \tau^n_b \le t) = \mathbb{P}(B(t) > y,\ \tau_{b+\sigma\beta/\sqrt{n}} \le t) + o(1/\sqrt{n}),$$

for t in the discretization mesh, where β is as defined in (3.10). In view of Theorem 4 above, the latter approximation can be heuristically understood as

$$\begin{aligned}
&\{B^n(t) > y,\, \tau_b^n \le t\} \\
&\quad = \left\{ B(t) > y,\, \max_{s\in[0,t]} B(s) \ge b + \frac{1}{\sqrt{n}}\sqrt{n}\left(\max_{s\in[0,t]} B(s) - \max_{s\in[0,t]} B^n(s) \right) \right\} \\
&\quad \approx \left\{ B(t) > y,\, \max_{s\in[0,t]} B(s) \ge b + \frac{1}{\sqrt{n}}\mathbb{E}\left[\sqrt{n}\left(\max_{s\in[0,t]} B(s) - \max_{s\in[0,t]} B^n(s) \right) \right] \right\} \\
&\quad \approx \left\{ B(t) > y,\, \max_{s\in[0,t]} B(s) \ge b + \frac{1}{\sqrt{n}}\sigma\beta \right\} = \left\{ B(t) > y,\, \tau_{b+\sigma\beta/\sqrt{n}} \le t \right\},
\end{aligned}$$

for t in the discretization mesh. Dia and Lamberton [50] use this idea to extend the work of Broadie et al. [18, 19] to jump diffusions.

3.2.3 Error of minimum on an infinite horizon

We consider now the error of estimating the minimum on unbounded time intervals by using the Euler discretization of the Brownian motion. We want to estimate the time

$$T_{\min,\infty} := \inf\left\{ t \in [0,\infty) : B(t) = \min_{u\in[0,\infty)} B(u) \right\}; \tag{3.11}$$

which we approximate as follows by using the discretized path B^n defined in (3.1):

$$T^n_{\min,\infty} := \inf\left\{ t \in [0,\infty) : B^n(t) = \min_{u\in[0,\infty)} B^n(u) \right\}. \tag{3.12}$$

For that, we study the errors $T^n_{\min,\infty} - T_{\min,\infty}$ and $B^n(T^n_{\min,\infty}) - B(T_{\min,\infty})$. The following result establishes their convergence rate and limiting distribution.

Theorem 5. *As $n \to \infty$, the triplet*

$$\left(n\left(T^n_{min,\infty} - T_{min,\infty} \right),\quad \sqrt{n}\left(B^n(T^n_{min,\infty}) - B(T_{min,\infty}) \right),\quad \lceil nT_{min,\infty} \rceil - nT_{min,\infty} \right), \tag{3.13}$$

converges jointly in distribution to the same triplet in (3.9).

We remark that we are *not* claiming that both triplets (3.8) and (3.13) converge *jointly* together to the same limit (3.9).

3.3 Extension to Gaussian walks

In this section we extend our weak convergence error results to the setting of Gaussian walks. A *Gaussian walk* is a discrete time stochastic process $S = (S_k : k \geq 0)$ where $S_k := \sum_{i=1}^{k} X_i$ and the increments X_i are iid normal random variables. We will see that several important phenomena of these processes can be described using the limiting distributions found in the Theorems 3, 4 and 5.

3.3.1 Corollary 1: limiting overshoot for increasing barrier

Consider a Gaussian walk $S = (S_n : n \geq 0)$ starting at zero, with nonnegative drift $\mathbb{E}S_1 = \nu \geq 0$ and with variance $\mathbb{E}(S_1 - \nu)^2 = \sigma^2$. For any fixed "barrier" $m > 0$ consider its barrier-hitting time τ_m:

$$\tau_m = \min\{k \geq 0 : S_k \geq m\}.$$

We want to analyze the distribution of the *overshoot* $S_{\tau_m} - m$ as m grows.

We use the modified Gaussian walk $S^* = (S_k^* : k \geq 0)$ defined as $S_0^* := \sqrt{U}X_0^*$ and $S_k^* := \sum_{i=0}^{k} X_k^*$ for $k \geq 0$, where X_k^* are iid standard normal random variables for $k \geq 0$, and U is a uniform random variable on $(0,1)$ independent of $(X_k^*)_{k\geq 0}$. Define its first strictly positive time τ_+ as:

$$\tau_+ := \min\{k \geq 0 : S_k^* > 0\}.$$

Corollary 2. *As $m \to \infty$, the overshoot $S_{\tau_m} - m$ converges in distribution to $\sigma S^*_{\tau_+}$.*

The proof is a straightforward corollary of Theorem 3. Indeed, using Brownian scaling the overshoot $S_{\tau_m} - m$ is equal in distribution to the second component of (3.4) for $\mu = \nu$, barrier $b = 1$ and $n = m^2$.

Broadie et al. [18] compute the mean of the limiting overshoot $S^*_{\tau_+}$ in the Gaussian walk case by matching the expression for the mean limiting overshoot for general random walks in Theorem 10.55 of [34] to the expression shown in Corollary 1 of [31]. They obtain that the mean in the Gaussian walk case is

$$\mathbb{E}\left[S^*_{\tau_+}\right] = \mathbb{E}\left[W\left(U + \min\{k \geq 0 : W(U+k) > 0\}\right)\right] = -\frac{\zeta(1/2)}{\sqrt{2\pi}}, \tag{3.14}$$

which we have already denoted as β. Interestingly, recall that in equation (3.10) we have already seen that $\beta = \mathbb{E}\left[\min_{k\in\mathbb{Z}} R(U+k)\right]$. In other words, two limiting random variables, $W\left(U + \min\{k \geq 0 : W(U+k) > 0\}\right)$ and $\min_{k\in\mathbb{Z}} R(U+k)$, coming from different but related problems have the same mean; c.f. Theorems 3 and 4. A question then is if both distribution are the same. Simulating these random variables, however, suggest that the answer is negative; see the Appendix of [8] for an algorithm to simulate $\min_{k\in\mathbb{Z}} R(U+k)$.

3.3.2 Corollary 2: running minimum asymptotics

Consider a Gaussian walk $S = (S_n : n \geq 0)$ starting at zero having drift $\mathbb{E}S_1 = 0$ and variance $\mathbb{E}(S_1)^2 = \sigma^2$. We want to study the random variables $\arg\min_{k=0,\ldots,n} S_k$ and $\min_{k=0,\ldots,n} S_k$ by comparing them to their Brownian counterparts $\arg\min_{t\in[0,n]} B(t)$ and $\min_{t\in[0,n]} B(t)$, respectively, where in both cases we refer to $\arg\min$ as the almost sure unique values at which the minimums are attained. More precisely, assume that the probability space is sufficiently rich so that there exists a Brownian motion $B = (B(t) : t \geq 0)$ starting from zero, with zero drift and variance σ^2, and such that $S_n = B(n)$ for all $n \in \mathbb{Z}_+$.

Corollary 3. *As $n \to \infty$, the pair*

$$\left(\arg\min_{k=0,\ldots,n} S_k - \arg\min_{t\in[0,n]} B(t), \quad \min_{k=0,\ldots,n} S_k - \min_{t\in[0,n]} B(t)\right) \tag{3.15}$$

converges in distribution to the pair

$$\left(U + \arg\min_{k\in\mathbb{Z}} R(U+k), \quad \sigma \min_{k\in\mathbb{Z}} R(U+k)\right), \tag{3.16}$$

where $R = (R(t) : t \in \mathbb{R})$ *is a two-sided Bessel(3) process and* U *a uniformly distributed random variable on* $(0,1)$ *that is independent of* R*. Here we have abused notation and in all cases the* $\arg\min$ *operations corresponds to the almost surely unique value at which the minimum is attained in each case.*

The proof is straightforward when noting that, using Brownian scaling, the pair (3.15) is equal in distribution to the first two terms of the triplet in (3.8) of Theorem 4.

Comtet and Majumdar [35] study the mean of the running maximum for general random walks and compare it to the mean of the running maximum of Brownian motion, i.e.,

$$\mathbb{E}\left[\max_{t\in[0,n]} B(t)\right] - \mathbb{E}\left[\max_{k=0,\ldots,n} S_k\right],$$

as $n \to \infty$, where S is a general random walk. In the Gaussian walk case they obtain the limiting mean $\beta = -\zeta(1/2)/\sqrt{2\pi}$, which is equal to $\mathbb{E}[\min_{k\in\mathbb{Z}} R(U+k)]$, as we have already discussed in (3.10).

3.3.3 Corollary 3: minimum as drift vanishes

Consider a Gaussian walk $S = (S_n : n \geq 0)$ starting from zero, with strictly positive drift $\mathbb{E}S_1 = \nu > 0$ and variance $\mathbb{E}(S_1 - \nu)^2 = \sigma^2$. We want to study the values $\arg\min_{k\in\mathbb{Z}_+} S_k$ and $\min_{k\in\mathbb{Z}_+} S_k$ as the drift ν decreases to zero, and we do it by comparing them to their Brownian counterparts $\arg\min_{t\in\mathbb{R}_+} B(t)$ and $\min_{t\in\mathbb{R}_+} B(t)$, respectively, where in both cases we refer to $\arg\min$ as the almost sure unique values at which the minimums are attained. More precisely, assume that the probability space is sufficiently rich so that there exists a Brownian motion $B = (B(t) : t \geq 0)$ starting from zero, having drift $\nu > 0$ and variance σ^2, and such that $S_n = B(n)$ for all $n \in \mathbb{Z}_+$.

Corollary 4. *As the drift decreases to zero, $\nu \searrow 0$, the pair*

$$\left(\arg\min_{k\in\mathbb{Z}_+} S_k - \arg\min_{t\in\mathbb{R}_+} B(t), \quad \min_{k\in\mathbb{Z}_+} S_k - \min_{t\in\mathbb{R}_+} B(t) \right) \tag{3.17}$$

converges in distribution to the pair in (3.16)*. Here we have abused notation and in all cases the* $\arg\min$ *operations corresponds to the almost surely unique value at which the minimum is attained in each case.*

The proof is direct from Theorem 5, since using Brownian scaling we obtain that (3.17) is equal in distribution to the first two terms of the triplet in (3.13) with drift $\mu = 1$ and $n = \nu^{-2}$.

The convergence of the moments of $\min_{k\in\mathbb{Z}_+} S_k - \min_{t\in\mathbb{R}_+} B(t)$ has already been studied. Janssen and van Leeuwaarden [16] gave an exact expansion of the first moment in terms of the drift ν and found that the leading term was $-\zeta(1/2)/\sqrt{2\pi}$, which is equal to $\mathbb{E}[\min_{k\in\mathbb{Z}} R(U+k)]$, see (3.10). Later in the follow up paper [51] they extend their approach to express higher moments in a similar way. On a related vein, Siegmund [21] studies approximating the expected maximum of general random walks with mean $-\nu$, $\nu > 0$, by comparing it to the expected maximum of Brownian motion with drift $-\nu$, i.e.,

$$\mathbb{E}\left[\max_{t\in\mathbb{R}_+} B(t)\right] - \mathbb{E}\left[\max_{k\in\mathbb{Z}_+} S_k\right].$$

He obtains an expansion of the latter difference when $\nu \searrow 0$ and the leading term is the mean limiting overshoot $\mathbb{E}[S^*_{\tau_+}]$, see Theorem 1 of [21]. In the Gaussian case this value is $\beta = -\zeta(1/2)/\sqrt{2\pi}$, see equation (3.14), and in turn this value is equal to $\mathbb{E}[\min_{k\in\mathbb{Z}} R(U+k)]$, c.f. (3.10).

3.4 Proof of Theorems 3, 4 and 5

In this section we show the proof of Theorems 3, 4 and 5. The main idea is to write the discretization errors as mappings of the original Brownian motion and then apply the continuous mapping theorem, or an argument of that type, to show weak convergence of the errors. We do this by first, in Section 3.4.1, writing the discretization errors as mappings of the original Brownian motion "zoomed-in" about the random time of interest. Then, in Section 3.4.2 we show that the zoomed-in processes converge in distribution. Lastly, in Section 3.4.3 we tie all things together and prove Theorems 3, 4 and 5.

For the sake of clarity of exposition, we briefly recall the notation used. For a Brownian motion path B with drift μ and variance σ^2 we consider its Euler discretization B^n on the mesh $\{0,\ 1/n,\ 2/n,\ \ldots\}$ as $B^n(t) := B(\lfloor nt \rfloor/n)$ for all $t \geq 0$. Also, recall that the times τ_b, $T_{\min,a}$ and $T_{\min,\infty}$ are defined as follows:

$$\begin{aligned}
\tau_b &:= \inf\left\{t \in (0,\infty) : B(t) \geq b(t)\right\}, \\
T_{\min,a} &:= \inf\left\{t \in [0,a] : B(t) = \inf_{u\in[0,a]} B(u)\right\}, \\
T_{\min,\infty} &:= \inf\left\{t \in [0,\infty) : B(t) = \inf_{u\in[0,\infty)} B(u)\right\};
\end{aligned}$$

and also that their discretized counterparts τ_b^n, $T_{\min,a}^n$ and $T_{\min,\infty}^n$ are defined analogously by replacing B by B^n in the previous definitions. Recall too that by Assumption (H_b) the function $b = (b(t) : t \geq 0)$ is continuous and nondecreasing on $\mathbb{R}_+$, continuously differentiable on $\mathbb{R}_+\backslash\{0\}$, and with $b(0) \geq 0$. We onwards assume that B is actually a two-sided Brownian motion $B = (B(t) : t \in \mathbb{R})$ with $B(0) = 0$.

3.4.1 Discretization errors as mappings of zoomed-in processes

In this section we show that the discretization error expressions in Theorems 3, 4 and 5 can be rewritten as mappings of certain centerings and scalings of the original Brownian motion B; we call these the *zoomed-in* processes. We remark that these processes are separate entities from the Euler discretization B^n of the Brownian motion B.

Definition 2. *1. Under $\{\tau_b < \infty\}$, for $m > 0$ define*

$$U^{(m)}_{\mathrm{hit},b} := \lceil m\tau_b \rceil - m\tau_b, \tag{3.18}$$

the zoomed-in process $Z^{(m)}_{\mathrm{hit},b} = \left(Z^{(m)}_{\mathrm{hit},b}(s) : s \in \mathbb{R}\right)$ as

$$Z^{(m)}_{\mathrm{hit},b}(s) := \sqrt{m}\left(B\left(\tau_b + \tfrac{s}{m}\right) - B\left(\tau_b\right)\right), \qquad s \in \mathbb{R}, \tag{3.19}$$

and

$$b^{(m)}_{\mathrm{hit},B}(s) := \sqrt{m}\left(b\left(\tau_b + \tfrac{s}{m}\right) - b\left(\tau_b\right)\right), \qquad s \in \mathbb{R}_+. \tag{3.20}$$

2. For $m > 0$ define

$$U^{(m)}_{\mathrm{min},a} := \lceil mT_{\mathrm{min},a} \rceil - mT_{\mathrm{min},a} \tag{3.21}$$

and the zoomed-in process $Z^{(m)}_{\mathrm{min},a} = \left(Z^{(m)}_{\mathrm{min},a}(s) : s \in \mathbb{R}\right)$ as

$$Z^{(m)}_{\mathrm{min},a}(s) := \sqrt{m}\left(B\left(T_{\mathrm{min},a} + \tfrac{s}{m}\right) - B\left(T_{\mathrm{min},a}\right)\right), \qquad s \in \mathbb{R}. \tag{3.22}$$

3. Under $\{T_{\mathrm{min},\infty} < \infty\}$, for $m > 0$ define

$$U^{(m)}_{\mathrm{min},\infty} := \lceil mT_{\mathrm{min},\infty} \rceil - mT_{\mathrm{min},\infty} \tag{3.23}$$

and the zoomed-in process $Z^{(m)}_{\mathrm{min},\infty} = \left(Z^{(m)}_{\mathrm{min},\infty}(s) : s \in \mathbb{R}\right)$ as

$$Z^{(m)}_{\mathrm{min},\infty}(s) := \sqrt{m}\left(B\left(T_{\mathrm{min},\infty} + \tfrac{s}{m}\right) - B\left(T_{\mathrm{min},\infty}\right)\right), \qquad s \in \mathbb{R}. \tag{3.24}$$

Intuitively, as m grows, the processes (3.19), (3.22) and (3.24) can be understood as "zooming-in's" of the Brownian path in time-space respectively about the points $(\tau_b, B(\tau_b))$, $(T_{\min,a}, B(T_{\min,a}))$ and $(T_{\min,\infty}, B(T_{\min,\infty}))$.

We now rewrite the discretization errors in Theorems 3, 4 and 5 as mappings of the processes $Z^{(m)}_{\text{hit},b}$, $Z^{(m)}_{\min,a}$ and $Z^{(m)}_{\min,\infty}$, the times $T_{\min,a}$ and $T_{\min,\infty}$, and the random variables $U^{(m)}_{\text{hit},b}$, $U^{(m)}_{\min,a}$ and $U^{(m)}_{\min,\infty}$.

Lemma 5. *1. Under $\{\tau_b < \infty,\ \tau_b^n < \infty\}$, almost surely we have*

$$\begin{pmatrix} n\left(\tau_b^n - \tau_b\right) \\ \sqrt{n}\left(B^n(\tau_b^n) - B(\tau_b)\right) \end{pmatrix} = E_{\text{hit}}\left(U^{(n)}_{\text{hit},b},\ Z^{(n)}_{\text{hit},b},\ b^{(n)}_{\text{hit},B}\right), \tag{3.25}$$

where the mapping $E_{\text{hit}} : \mathbb{R}\times\mathscr{C}(\mathbb{R})\times\mathscr{C}(\mathbb{R}_+) \to (\mathbb{R}\cup\{+\infty\})\times(\mathbb{R}\cup\{\partial\})$ is defined as

$$E_{\text{hit}}(u,f,g) := \begin{pmatrix} u + \min\left\{k\in\mathbb{Z}_+ : f(u+k) > g(u+k)\right\} \\ f\left(u + \min\left\{k\in\mathbb{Z}_+ : f(u+k) > g(u+k)\right\}\right) \end{pmatrix}, \tag{3.26}$$

where for completeness we use the convention $\min\emptyset := +\infty$ and for $f\in\mathscr{C}(\mathbb{R})$ we define $f(+\infty) := \partial$.

2. It holds that

$$\begin{pmatrix} n\left(T^n_{\min,a} - T_{\min,a}\right) \\ \sqrt{n}\left(B^n(T^n_{\min,a}) - B(T_{\min,a})\right) \end{pmatrix} = E^{(n)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right) \tag{3.27}$$

where for all $m\in\mathbb{Z}_+$ the mapping $E^{(m)}_{\min,a} : \mathbb{R}\times\mathscr{C}(\mathbb{R})\times(0,a)\to\mathbb{R}^2$ is defined as

$$E^{(m)}_{\min,a}(u,f,t) := \begin{pmatrix} u + \arg\inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil, ma-\lceil mt\rceil]} f(u+k) \\ \inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil, ma-\lceil mt\rceil]} f(u+k) \end{pmatrix}, \tag{3.28}$$

where we have abused notation and actually denote $\arg\inf_{s\in N} g(s) := \inf\{s \in N : g(s) = \inf_{u\in N} g(u)\}$ *for* $g \in \mathscr{C}(\mathbb{R})$*,* $N \subseteq \mathbb{R}$ *compact and* $\inf \emptyset := +\infty$*.*

3. *Under* $\{T_{\min,\infty} < \infty\}$ *we have that*

$$\begin{pmatrix} n\left(T^n_{\min,\infty} - T_{\min,\infty}\right) \\ \sqrt{n}\left(B^n(T^n_{\min,\infty}) - B(T_{\min,\infty})\right) \end{pmatrix} = E^{(n)}_{\min,\infty}\left(U^{(n)}_{\min,\infty},\ Z^{(n)}_{\min,\infty},\ T_{\min,\infty}\right) \tag{3.29}$$

where for all $m \in \mathbb{Z}_+$ *the mapping* $E^{(m)}_{\min,\infty} : \mathbb{R} \times \mathscr{C}_{\inf}(\mathbb{R}) \times \mathbb{R} \to (\mathbb{R} \cup \{+\infty\}) \times \mathbb{R}$ *is defined as*

$$E^{(m)}_{\min,\infty}(u, f, t) := \begin{pmatrix} u + \arg\inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil,\infty)} f(u+k) \\ \inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil,\infty)} f(u+k) \end{pmatrix}, \tag{3.30}$$

where, for completeness, we denote $\mathscr{C}_{\inf+}(\mathbb{R}) := \{f \in \mathscr{C}(\mathbb{R}) : \lim_{t\to\infty} f(t) = \infty\}$ *and* $\arg\inf_{s\in N} g(s) := \inf\{t \in N : g(t) = \inf_{s\in N} g(s)\}$ *for* $g \in \mathscr{C}_{\inf+}(\mathbb{R})$ *and* $N \subset \mathbb{R}$ *closed, with the convention* $\inf \emptyset = \infty$*.*

The proof of the previous result is a straightforward but non-illuminating calculation, so it is deferred to the Section 3.6.

3.4.2 Weak convergence of the zoomed-in processes

In this section we show that the zoomed-in processes defined in Definition 2 converge in distribution. We endow the space $\mathscr{C}(\mathbb{R})$ of continuous real functions on $\mathbb{R}$ with the metric *of uniform convergence over compact sets*, say $\|\cdot\|_{\mathsf{K}}$, defined for all f in $\mathscr{C}(\mathbb{R})$ as $\|f\|_{\mathsf{K}} := \sum_{A=1}^{\infty} 2^{-A}\min\{1, \sup_{t\in[-A,A]}|f(t)|\}$. It holds that $\|f\|_{\mathsf{K}} = 0$ if and only if $\sup_{s\in[-A,A]}|f(s)| = 0$ for all $A > 0$ finite, which motivates the name of the metric. The space $\mathscr{C}(\mathbb{R})$ endowed with the topology generated by this metric is a *Polish* space, i.e., a complete and separable topological space; see e.g. [52] for further details. We also use a one-sided version of the metric space $(\mathscr{C}(\mathbb{R}), \|\cdot\|_{\mathsf{K}})$, which is defined analogously to the two-sided version $(\mathscr{C}(\mathbb{R}), \|\cdot\|_{\mathsf{K}})$.

Lemma 6. *1. Given $t > 0$ the process $Z^{(m)}_{\text{hit},b}$ conditioned on $\{\tau_b = t\}$ converges in distribution on $(\mathscr{C}(\mathbb{R}),\ \|\cdot\|_{\mathsf{K}})$ to $(\sigma R(-s) : s \le 0\,;\ \sigma W(s) : s \ge 0)$ as $m \to \infty$, where W is a standard Brownian motion and R is a standard Bessel(3) process independent of W.*

2. Given $t \in (0, a)$, $l < 0$ and $y > 0$, conditioned on $\{T_{\min,a} = t,\ B(T_{\min,a}) = l,\ B(a) = l + y\}$ the process $Z^{(m)}_{\min,a}$ converges in distribution on $(\mathscr{C}(\mathbb{R}),\ \|\cdot\|_{\mathsf{K}})$ to σR as $m \to \infty$, where $R = (R(s) : s \in \mathbb{R})$ is a two-sided standard Bessel(3) process.

3. Given $t > 0$ and $l < 0$, conditioned on the event $\{T_{\min,\infty} = t,\ B(T_{\min,\infty}) = l\}$ the process $Z^{(m)}_{\min,\infty}$ converges in distribution on $(\mathscr{C}(\mathbb{R}),\ \|\cdot\|_{\mathsf{K}})$ to σR as $m \to \infty$, where $R = (R(s) : s \in \mathbb{R})$ is a standard two-sided Bessel(3) process.

We remark that in all three parts of Lemma 6 the convergence in distribution also holds unconditionally, since the limiting processes do not depend on the values of the conditioning. Note that none of these results explicitly depend on the drift μ of the Brownian motion B.

Proof of Lemma 6. For the sake of clarity of the exposition, we always assume that the treatment is conditioned on the events $\{\tau_b < \infty,\ \tau^n_b < \infty\}$ and $\{T_{\min,\infty} < \infty\}$ when dealing with the processes $Z^{(m)}_{\text{hit},b}$ and $Z^{(m)}_{\min,\infty}$ respectively.

We start by proving (i). By the strong Markov property the process $(Z^{(1)}_{\text{hit},b}(s) = B(t + s) - b : s \ge 0)$ conditioned on $\{\tau_b = t\}$ is distributed as a Brownian motion with drift μ and variance σ^2. In particular, $(Z^{(1)}_{\text{hit},b}(s) : s \ge 0)$ is independent of $(Z^{(1)}_{\text{hit},b}(s) : s \le 0)$. Then, by Brownian scaling $(Z^{(m)}_{\text{hit},b}(s) : s \ge 0)$ is equal in distribution to a Brownian motion with drift $\mu/\sqrt{m}$ and variance σ^2. Since it converges almost surely to σW on $(\mathscr{C}(\mathbb{R}_+),\ \|\cdot\|_{\mathsf{K}})$ as $m \to \infty$, where W is a standard Brownian motion, then it also converges weakly on $(\mathscr{C}(\mathbb{R}_+),\ \|\cdot\|_{\mathsf{K}})$.

On the other hand, the process $(-Z^{(1)}_{\mathrm{hit},b}(-s) = b - B(\tau_b - s) : s \in [0,t])$ conditioned on $\{\tau_b = t\}$ is distributed as a Bessel(3) process conditioned on being at b at time t, see Theorem 3.4 of Williams [43]. Applying Lemma 1 of [8] we obtain that $(-\sqrt{m}Z^{(1)}_{\mathrm{hit},b}(-s/m) : s \geq 0)$ converges weakly to σR on $(\mathscr{C}(\mathbb{R}_+), \|\cdot\|_{\mathsf{K}})$ as $m \to \infty$, where R is a Bessel(3) process independent of W. This proves part (i).

Part (ii) corresponds to Lemma 1 of [8].

Finally, we prove (iii). The process $(Z^{(1)}_{\min,\infty}(s) = B(T_{\min,\infty} + s) - B(T_{\min,\infty}) : s \geq 0)$ conditioned on the event $\{T_{\min,\infty} = t,\ B(T_{\min,\infty}) = l\}$ is distributed as a Bessel(3) process with drift μ, see Corollary 3 of [44]. Then, by Brownian scaling of the Bessel processes, $(\sqrt{m}Z^{(1)}_{\min,\infty}(s/m) = Z^{(m)}_{\min,\infty}(s) : s \geq 0)$ is distributed as a Bessel(3) process with drift $\mu/\sqrt{m}$. Such a process converges almost surely in $(\mathscr{C}(\mathbb{R}_+), \|\cdot\|_{\mathsf{K}})$ to a Bessel(3) process with no drift, say to R_+, so it also converges in distribution to R_+.

On the other hand, by Theorems 2.1 and 3.4 of Williams [43] we have that $(Z^{(1)}_{\min,\infty}(-s) : s \in [0,t])$ conditioned on $\{T_{\min,\infty} = t,\ B(T_{\min,\infty}) = l\}$ is distributed as a Bessel(3) process conditioned on being at $-l$ at time t. We conclude using Lemma 1 of [8] that $(\sqrt{m}Z^{(1)}_{\min,\infty}(-s/m) = Z^{(m)}_{\min,\infty}(-s) : s \geq 0)$ converges weakly to σR_- on $(\mathscr{C}(\mathbb{R}_+), \|\cdot\|_{\mathsf{K}})$ as $m \to \infty$, where R_- is a Bessel(3) process independent of R_+. This proves (iii). □

3.4.3 Proofs

In this section we prove Theorems 3, 4 and 5. The idea is to apply the continuous mapping theorem, see Chapter 3.4 of Whitt [20]. We do this inspired by Lemma 5, which shows that the errors are mappings of the zoomed-in processes $Z^{(n)}_{\mathrm{hit},b}$, $Z^{(n)}_{\min,a}$ and $Z^{(n)}_{\min,\infty}$ and other random variables, and Lemma 6, which shows weak convergence of the zoomed-in processes.

The following result shows the weak convergence of the random variables $U^{(n)}_{\mathrm{hit},b}$, $U^{(n)}_{\mathrm{hit},b}$ and $U^{(n)}_{\mathrm{hit},b}$ to uniform random variables. This result motivates the weak convergence of the pairs $(U^{(n)}_{\mathrm{hit},b},\ Z^{(n)}_{\mathrm{hit},b})$, $(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a})$ and $(U^{(n)}_{\min,\infty},\ Z^{(n)}_{\min,\infty})$ as $n \to \infty$, with all the limiting distributions independent of the random variables τ_b, $T_{\min,a}$ and $T_{\min,\infty}$. We defer its proof to the Section 3.6.

Lemma 7. *Consider a nonnegative random variable T which has a distribution that is absolutely continuous with respect to the Lebesgue measure. Then as $n \to \infty$, $\lceil nT \rceil - nT$ converges in distribution to a uniformly distributed random variable on $(0,1)$, which moreover is independent of T.*

With the previous result we are now able to prove the main results of this paper.

Proof of Theorem 3. First recall that by Lemma 5, conditioned on $\{\tau_b < \infty,\ \tau_b^n < \infty\}$ we have that

$$\left(n\left(\tau_b^n - \tau_b\right),\ \sqrt{n}\left(B^n(\tau_b^n) - B(\tau_b)\right)\right) = E_{\mathrm{hit}}\left(U_{\mathrm{hit},b}^{(n)},\ Z_{\mathrm{hit},b}^{(n)},\ b_{\mathrm{hit},B}^{(n)}\right).$$

The plan of the proof is to first show that the triplet

$$\left(U_{\mathrm{hit},b}^{(n)},\ Z_{\mathrm{hit},b}^{(n)},\ b_{\mathrm{hit},B}^{(n)}\right) \tag{3.31}$$

converges in distribution, then show that the function E_{hit} is continuous, and conclude the desired convergence of the errors using the continuous mapping theorem. We will use the metric of uniform convergence on compact sets $\|\cdot\|_{\mathsf{K}}$ for the weak convergence of $Z_{\mathrm{hit},b}^{(n)}$ and for the continuity of the mapping E_{hit}.

We first argue that the triplet in (3.31) converges in distribution on $(\mathbb{R} \times \mathscr{C}(\mathbb{R}) \times \mathscr{C}(\mathbb{R}_+), |\cdot| \times \|\cdot\|_{\mathsf{K}} \times \|\cdot\|_{\mathsf{K}})$ where, recall, $\|\cdot\|_{\mathsf{K}}$ is the metric of uniform convergence on compact sets. Indeed, by Lemmas 6 and 7 conditional on $\{\tau_b < \infty\}$ the pair $(U_{\mathrm{hit},b}^{(n)},\ Z_{\mathrm{hit},b}^{(n)})$ converges in distribution to $(U,\ (-\sigma R(-s) : s \le 0; \sigma W(s) : s \ge 0))$, where the weak convergence of $Z_{\mathrm{hit},b}^{(n)}$ is on $(\mathscr{C}(\mathbb{R}), \|\cdot\|_{\mathsf{K}})$. Here, R and W are standard Bessel(3) and Brownian motion processes, respectively; U is uniformly distributed on $(0,1)$; and R, W, U and τ_b are all independent. Additionally, the function $b_{\mathrm{hit},B}^{(n)}$ almost surely converges to 0 on $(\mathscr{C}(\mathbb{R}_+), \|\cdot\|_{\mathsf{K}})$, i.e., to the zero function. Indeed, $b_{\mathrm{hit},B}^{(n)}$ is continuously differentiable on $\mathbb{R}_+\backslash\{0\}$ by Assumption (H_b), and $\tau_b > 0$ almost surely, so for all $k > 0$ it holds almost surely that

$$\sup_{t\in[0,k]}\left|b^{(n)}_{\mathrm{hit},B}\right| = \sqrt{n}\sup_{t\in[0,k]}\left|b\left(\tau_b+\frac{t}{n}\right)-b(\tau_b)\right| = \sqrt{n}\sup_{t\in[0,k]}\left|b'\left(\tau_b+\frac{\xi}{n}\right)\frac{t}{n}\right| \to 0$$

as $n\to\infty$, where the last equality holds for some $\xi\in(0,k)$ by the mean value theorem. With this, we conclude that the triplet (3.31) converges in distribution as $n\to\infty$ to the triplet

$$(U,\ (-R(-s) : s\le 0; W(s) : s\ge 0),\ 0) \tag{3.32}$$

on $(\mathbb{R}\times\mathscr{C}(\mathbb{R})\times\mathscr{C}(\mathbb{R}_+), |\cdot|\times\|\cdot\|_{\mathsf{K}}\times\|\cdot\|_{\mathsf{K}})$.

The next step is to show that the mapping E_{hit} defined in (3.26) is continuous on a measurable set, say S_{hit}, containing the support of the limiting random variable (3.32). We consider the set S_{hit} defined as

$$\left\{(u,w,0)\in(0,1)\times\mathscr{C}(\mathbb{R}_+)\times\{0\} : \sup_{k\in\mathbb{Z}_+} w(u+k)>0,\ w(u+k)\neq 0 \text{ for all } k\in\mathbb{Z}_+\right\}.$$

Clearly it is measurable and the support of (3.32) is contained in S_{hit}. In particular, the mapping E_{hit} on S_{hit} takes values in $\mathbb{R}^2$. Note now that by composition of continuous functions it is sufficient to only show the continuity of the mapping

$$(u,w,0)\in S_{\mathrm{hit}}\mapsto\min\{k\in\mathbb{Z}_+ : w(u+k)>0\}. \tag{3.33}$$

To see that the function (3.33) is continuous, first take any $(u,w,0)\in S_{\mathrm{hit}}$ and a sequence $((U^{(n)}_{\min,a}, w_n) : n\ge 0)$ in $\mathbb{R}\times\mathscr{C}(\mathbb{R}_+)$ such that $\|w_n-w\|_{\mathsf{K}}+|U^{(n)}_{\min,a}-u|\to 0$ as $n\to\infty$. Denote $K^* := \min\{k\in\mathbb{Z}_+ : w(u+k)>0\}$ and $\delta^* := \min\{|w(u+k)| : k=0,\dots,K^*\}$, and note that $\delta^*>0$ by definition of S_{hit}. Since $(U^{(n)}_{\min,a}, w_n)\to(u,w)$ on $|\cdot|\times\|\cdot\|_{\mathsf{K}}$ then in particular $\sup_{k\in\mathbb{Z}_+\cap[0,K^*]}\left|w(u+k)-w_n(U^{(n)}_{\min,a}+k)\right|<\delta^*$ for all $n\geq n^*$, for some n^* sufficiently large. In particular, for all $n\geq n^*$ the values $w_n(U^{(n)}_{\min,a}+k)$, $k=0,\dots,n^*$, are all different from zero and have the same sign of $w(u+k)$, $k=0,\dots,n^*$, respectively. This implies that $\min\{k\in\mathbb{Z}_+ : w_n(U^{(n)}_{\min,a}+k)>0\}=K^*=\min\{k\in\mathbb{Z}_+ : w(u+k)>0\}$ for all $n\geq n^*$. We have shown that the mapping (3.33), and thus E_{hit}, is continuous on S_{hit}, which contains the support of the limiting random variable (3.32).

It follows that applying the continuous mapping theorem, see Theorem 3.4.3 of [20], the random variable $E_{\text{hit}}(U^{(n)}_{\text{hit},b},\ Z^{(n)}_{\text{hit},b},\ b^{(m)}_{\text{hit},B})$ converges to $E_{\text{hit}}(U,\ (-\sigma R,\sigma W),\ 0)$ in distribution as $n\to\infty$. Lastly, note that $\mathbb{P}(\tau^n_b<\infty|\tau_b<\infty)\to 1$ as $n\to\infty$, so we can drop the condition $\tau^n_b<\infty$. This concludes the proof of the Theorem 3. $\square$

Proof of Theorems 4 and 5. We show only the proof of Theorem 4, since the proof of Theorem 5 is analogous.

To prove the joint convergence of the normalized discretization errors $n\left(T^n_{\min,a}-T_{\min,a}\right)$ and $\sqrt{n}\left(B^n(T^n_{\min,a})-B(T_{\min,a})\right)$ recall first that by Lemma 5 they can be written as

$$\begin{pmatrix} n\left(T^n_{\min,a}-T_{\min,a}\right) \\ \sqrt{n}\left(B^n(T^n_{\min,a})-B(T_{\min,a})\right) \end{pmatrix} = E^{(n)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right).$$

Inspired on this we first show that the following vector converges in distribution as $n\to\infty$:

$$\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right). \tag{3.34}$$

We would then like to conclude the joint convergence of the normalized errors by using the generalized continuous mapping theorem, see Theorem 3.4.4 of [20]. This considers showing that, in a sense, the mapping $\lim_{n\to\infty}E^{(n)}_{\min,a}$ is continuous for the metric $\|\cdot\|_{\mathsf{K}}$ on the support of the limiting distribution of (3.34); however this is not true. Nonetheless, we show that this problem can be circumvented by first restricting to compact time intervals, then using there the continuous mapping theorem, and then increasing the size of the compact time interval.

We start by arguing that as $n \to \infty$ the random variable in (3.34) converges in distribution to $(U,\ \sigma R,\ T_{\min,a})$, where R is a standard Bessel(3) process, U is uniformly distributed on $(0,1)$, and R, U and $T_{\min,a}$ are all independent. Indeed, by Lemmas 6 and 7, as $n \to \infty$ the pair $(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a})$ converges in distribution to $(U,\ \sigma R)$, where the weak convergence of $Z^{(n)}_{\min,b}$ is on $(\mathscr{C}(\mathbb{R}), \|\cdot\|_{\mathsf{K}})$, and moreover U and R are independent of $T_{\min,a}$.

The rest of the proof consists on showing that the following limit in distribution holds

$$\lim_{n\to\infty} E^{(n)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right) = E^{(\infty)}_{\min,a}\left(U,\ \sigma R,\ T_{\min,a}\right), \tag{3.35}$$

where the mapping $E^{(\infty)}_{\min,a}$ is defined as

$$E^{(\infty)}_{\min,a}(u,f,t) := \left(u + \arg\inf_{k\in\mathbb{Z}} f(u+k),\ \inf_{k\in\mathbb{Z}} f(u+k)\right), \tag{3.36}$$

where f takes values in $\mathscr{C}_{\inf\pm}(\mathbb{R}) := \{f \in \mathscr{C}(\mathbb{R}) : \lim_{t\to\pm\infty} f(t) = \infty\}$. By Lemma 5 this would conclude the proof of Theorem 4. To prove the limit (3.35) we first restrict, in a sense, the mappings $E^{(m)}_{\min,a}$ to compact time intervals of the form $[-A,A]$, then prove the weak convergence there, and then take the limit $A \to \infty$.

Define for all $m \in \mathbb{Z}_+$ and $A > 0$ the mapping $E^{(m,A)}_{\min,a} : (0,1)\times\mathscr{C}(\mathbb{R})\times(0,a) \to \mathbb{R}^2$ as

$$E^{(m,A)}_{\min,a}(u,f,t) := \begin{pmatrix} u + \arg\inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil,\ ma-\lceil mt\rceil]\cap[-A,A]} f(u+k) \\ \inf_{k\in\mathbb{Z}\cap[-\lceil mt\rceil,\ ma-\lceil mt\rceil]\cap[-A,A]} f(u+k) \end{pmatrix}, \tag{3.37}$$

where we have abused notation and actually denote $\arg\inf_{s\in N} g(s) := \inf\{s \in N : g(s) = \inf_{u\in N} g(u)\}$ for $g \in \mathscr{C}(\mathbb{R})$, $N \subseteq \mathbb{R}$ compact and $\inf\emptyset := +\infty$. By the generalized continuous mapping theorem, see Theorem 3.4.4 of [20], as $n \to \infty$ the vector

$E^{(n,A)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right)$ converges in distribution to $E^{(\infty,A)}_{\min,a}\,(U,\ \sigma R,\ T_{\min,a})$, where

$$E^{(\infty,A)}_{\min,a}\,(u,\ r,\ t) \ := \ \left(u + \operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-A,A]} r(u+k),\ \min_{k\in\mathbb{Z}\cap[-A,A]} r(u+k)\right). \qquad (3.38)$$

Indeed, for all $(u,r,t) \in (0,1)\times\mathscr{C}(\mathbb{R})\times(0,a)$ such that all the values $\{r(u+k) : k\in\mathbb{Z}\}$ are different it holds that $E^{(n,A)}_{\min,a}\,(u_n,\ r_n,\ t_n) \to E^{(\infty,A)}_{\min,a}\,(u,\ r,\ t)$ for any sequence (u_n, r_n, t_n) in $(0,1)\times\mathscr{C}(\mathbb{R})\times(0,a)$ such that $u_n \to u$, $r_n \to r$ and $t_n \to t$, where the convergence of r_n is with the norm $\|\cdot\|_{\mathsf{K}}$. This implies that, with probability one, the limiting random variable $(U,\ \sigma R,\ T_{\min,a})$ does not take values on the set where $E^{(\infty,A)}_{\min,a}$ is discontinuous; therefore the generalized continuous mapping theorem applies and we obtain the desired convergence in distribution.

Now note that the following limit in distribution holds

$$\lim_{A\to\infty} E^{(\infty,A)}_{\min,a}\,(U,\ \sigma R,\ T_{\min,a}) = E^{(\infty)}_{\min,a}\,(U,\ \sigma R,\ T_{\min,a})\,,$$

with $E^{(\infty)}_{\min,a}$ is as defined in (3.36), because the convergence actually holds almost surely. Therefore, abusing notation, we conclude that

$$\lim_{A\to\infty}\lim_{n\to\infty} E^{(n,A)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right) = E^{(\infty)}_{\min,a}\,(U,\ \sigma R,\ T_{\min,a})\,, \qquad (3.39)$$

where both limits and the equality are in distribution.

On the other hand, for all n

$$\lim_{A\to\infty} E^{(n,A)}_{\min,a}\left(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}) = E^{(n)}_{\min,a}(U^{(n)}_{\min,a},\ Z^{(n)}_{\min,a},\ T_{\min,a}\right) \qquad (3.40)$$

in distribution, since the convergence holds almost surely.

We now show that, in a sense, the following interchange of limits holds

$$\lim_{n\to\infty}\lim_{A\to\infty} E^{(n,A)}_{\min,a}\left(U^{(n)}_{\min,a},\, Z^{(n)}_{\min,a},\, T_{\min,a}\right) = \lim_{A\to\infty}\lim_{n\to\infty} E^{(n,A)}_{\min,a}\left(U^{(n)}_{\min,a},\, Z^{(n)}_{\min,a},\, T_{\min,a}\right),$$

where the limits and the equality are in distribution; note that this would conclude the limit in distribution (3.35). The latter interchange of limits, we will see, is a consequence of the limit

$$\lim_{A\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]\cap[-A,A]^{\mathsf{C}}} Z^{(n)}_{\min,a}\left(k+U^{(n)}_{\min,a}\right)<\xi\right) = 0, \quad (3.41)$$

which holds for all $\xi\in\mathbb{R}$. Indeed, (3.41) is just Lemma 4 of [8], which is easily checked by using the definitions of $Z^{(n)}_{\min,a}$ and $U^{(n)}_{\min,a}$ in Definition 2. The limit (3.41) in turn implies that the $\limsup_{A\to\infty}$ of the $\limsup_{n\to\infty}$ of the following probability converges to zero:

$$\mathbb{P}\left(U^{(n)}_{\min,a}+\operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}\left(k+U^{(n)}_{\min,a}\right)\in[-A,A]^{\mathsf{C}}\right), \qquad (3.42)$$

since

$$\begin{aligned}
&\limsup_{A\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(U^{(n)}_{\min,a}+\operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}(k+U^{(n)}_{\min,a})\in[-A,A]^{\mathsf{C}}\right)\\
&\quad=\limsup_{A\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]\cap[-A,A]^{\mathsf{C}}} Z^{(n)}_{\min,a}(k+U^{(n)}_{\min,a})\right.\\
&\qquad\qquad\left.<\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]\cap[-A,A]} Z^{(n)}_{\min,a}(k+U^{(n)}_{\min,a})\right)\\
&\quad\le\mathbb{E}\left[\limsup_{A\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]\cap[-A,A]^{\mathsf{C}}} Z^{(n)}_{\min,a}(k+U^{(n)}_{\min,a})<\xi\right.\right.\\
&\qquad\qquad\left.\left.\Bigg|\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\, na-\lceil nT_{\min,a}\rceil]\cap[-A,A]} Z^{(n)}_{\min,a}(k+U^{(n)}_{\min,a})=\xi\right)\right]\\
&\quad=0,
\end{aligned}$$

where in the inequality we used the reverse Fatou's lemma. It follows that for all s,t,u

in $\mathbb{R}$ we have

$$\begin{aligned}
&\Bigg|\mathbb{P}\Bigg(U_{\min,a}^{(n)} + \operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \le s,\\
&\qquad\qquad \min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \le t,\ U_{\min,a}^{(n)} \le u\Bigg)\\
&\quad - \mathbb{P}\Bigg(U_{\min,a}^{(n)} + \operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]\cap[-A,A]} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \le s,\\
&\qquad\qquad \min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]\cap[-A,A]} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \le t,\ U_{\min,a}^{(n)} \le u\Bigg)\Bigg|\\
&\quad \le \mathbb{P}\Bigg(U_{\min,a}^{(n)} + \operatorname*{arg\,min}_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \in [-A,A]^{\mathsf{C}}\Bigg)\\
&\qquad\qquad + \mathbb{P}\Bigg(\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]\cap[-A,A]^{\mathsf{C}}} Z_{\min,a}^{(n)}(k+U_{\min,a}^{(n)}) \le t\Bigg).
\end{aligned}$$

Using the limits (3.41) and (3.42) one obtains that the difference in the previous display goes to zero when taking $\limsup_{n\to\infty}$ and then $\limsup_{A\to\infty}$. Together with the limit (3.39) we conclude the limit in distribution

$$\lim_{n\to\infty} E_{\min,a}^{(n)}\left(U_{\min,a}^{(n)},\ Z_{\min,a}^{(n)},\ T_{\min,a}\right) = E_{\min,a}^{(\infty)}\left(U,\ \sigma R,\ T_{\min,a}\right),$$

i.e., the limit (3.35), which is what we wanted to prove.

This proves Theorem 4. The proof of Theorem 5 is analogous. □

3.5 Strengthened convergence of the zoomed-in processes

In this section we give an alternate approach to tackle the proof of Theorems 4 and 5. Recall that in Section 3.4 we proved these two theorems by using an ad-hoc continuous mapping theorem-type result where we first restricted to compact time-horizons of the form $[-A, A]$ and then made $A \to \infty$. In this section we show a different approach to prove Theorems 4 and 5 where the aim is to be able to directly apply the continuous mapping theorem to conclude the convergence of the normalized discretization errors. This approach consists on using another metric on the space $\mathscr{C}(\mathbb{R})$, the *weighted-supremum* metric, which is briefly introduced in Section 3.5.1, and which is intended to make continuous the errors mapping $E^{(\infty)}_{\min,a}$ defined in (3.36). In Section 3.5.2 we depict to which extent we can show, with this alternative metric space, the convergence in distribution of the normalized Euler discretization errors. Lastly in Section 3.5.3 we show that the zoomed-in processes actually converge in distribution for this alternative metric space. We focus on showing the alternate approach only for Theorem 4; the case of Theorem 5 is completely analogous.

3.5.1 Weighted-supremum metric

In this section we introduce the *weighted-supremum* metric and show its basic properties.

We consider the *weighted-supremum* metric, denoted $\|\cdot\|_1$, defined as

$$\|f\|_1 := \sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|}. \tag{3.43}$$

We work on the subspace $\mathscr{C}_1(\mathbb{R})$ of $\mathscr{C}(\mathbb{R})$, defined as

$$\mathscr{C}_1(\mathbb{R}) := \left\{ f \in \mathscr{C}(\mathbb{R}) \ : \ f(0) = 0,\ \lim_{t\to+\infty} \left|\frac{f(t)}{t}\right| \text{ and } \lim_{t\to-\infty} \left|\frac{f(t)}{t}\right| \text{ exist and are finite} \right\}.$$

We also consider a one-sided version of the metric space $(\mathscr{C}_1(\mathbb{R}_+), \|\cdot\|_1)$, which is defined analogously to the two-sided version $(\mathscr{C}_1(\mathbb{R}), \|\cdot\|_1)$.

It holds that $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ is a Polish space, i.e., a complete and separable metric space, since it is isometric to the space $(\mathscr{C}_0(-1,1),\ \|\cdot\|_\infty)$, where $\mathscr{C}_0(-1,1) := \{f \in \mathscr{C}(-1,1) : f(0) = 0\}$ and $\|\cdot\|_\infty$ is the usual supremum norm[1]. Moreover, the topology generated by $\|\cdot\|_1$ is stronger or finer than the one generated by the metric of uniform convergence on compact sets $\|\cdot\|_{\mathsf{K}}$, that is, the former topology contains the latter one. In particular, mappings on $\mathscr{C}_1(\mathbb{R})$ that are continuous for the metric $\|\cdot\|_{\mathsf{K}}$ are also continuous

[1]Note that $x \mapsto x/(1-|x|)$ is a bijective mapping from $(-1,1)$ to $\mathbb{R}$. Thus, the mapping $(f(y) : y \in \mathbb{R}) \mapsto \left(\frac{f(\frac{x}{1-|x|})}{1+\left|\frac{x}{1-|x|}\right|} : x \in (-1,1) \right)$ is a bijective isometry from $(\mathscr{C}_1(\mathbb{R}), \|\cdot\|_1)$ to $(\mathscr{C}_0(-1,1), \|\cdot\|_\infty)$.

for $\|\cdot\|_1$; or equivalently, sequences that converge for the metric $\|\cdot\|_1$ also converge for $\|\cdot\|_{\mathsf{K}}$. As a consequence, weak convergence on $(\mathscr{C}_1(\mathbb{R}_+), \|\cdot\|_1)$ implies weak convergence on $(\mathscr{C}_1(\mathbb{R}_+), \|\cdot\|_{\mathsf{K}})$. See Chapter 5 of Ganesh et al. [36] and Dieker [37] for further details.

We consider the metric space $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ because it makes continuous some mappings that are not continuous for the metric $\|\cdot\|_{\mathsf{K}}$ of uniform convergence over compact sets; this is a direct consequence of $\|\cdot\|_1$ inducing a stronger topology than the one induced by $\|\cdot\|_{\mathsf{K}}$. Consider for example the mapping

$$r \in \mathscr{C}(\mathbb{R}) \mapsto \min_{t\in\mathbb{R}} r(t) + \delta|t|, \tag{3.44}$$

which by Lemma 2 of [37] is continuous for the metric $\|\cdot\|_1$ when restricted to the space

$$\mathscr{C}_{1,0}(\mathbb{R}) := \left\{ f \in \mathscr{C}_1(\mathbb{R}) \ : \ \lim_{t\to\pm\infty} \frac{f(t)}{t} = 0 \right\}.$$

In contrast, the mapping is not continuous for the metric $\|\cdot\|_{\mathsf{K}}$ of uniform convergence over compact sets; see the counterexample in Section 3.4 of [37].

3.5.2 Discretization errors with the weighted-supremum metric

In this section we show how to apply the metric spaces $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ and $(\mathscr{C}_{1,0}(\mathbb{R}),\ \|\cdot\|_1)$, defined in the previous section, to derive weak convergence of the normalized discretization errors by directly using the continuous mapping theorem. We focus the discussion only on an alternate proof for Theorem 4; the discussion for Theorem 5 is analogous.

We are motivated by the fact that the mapping $E^{(\infty)}_{\min,a}$ stands a chance to be continuous for the metric $\|\cdot\|_1$, where we defined $E^{(\infty)}_{\min,a}$ in equation (3.36) as

$$E^{(\infty)}_{\min,a}(u,f,t) := \left(u + \arg\inf_{k\in\mathbb{Z}} f(u+k),\ \inf_{k\in\mathbb{Z}} f(u+k) \right), \tag{3.45}$$

for f in $\mathscr{C}_{\inf\pm}(\mathbb{R}) := \{f \in \mathscr{C}(\mathbb{R}) : \lim_{t\to\pm\infty} f(t) = \infty\}$. Indeed, $E^{(\infty)}_{\min,a}$ is of the form of the mapping in equation (3.44), which is continuous for the metric $\|\cdot\|_1$. We remark that in Section 3.4.3 we showed that the generalized continuous mapping theorem can be applied when, essentially, with probability one the limiting random variable does not take values in the set of discontinuity points of $E^{(\infty)}_{\min,a}$. On the other hand, as $n \to \infty$ the zoomed-in process $Z^{(n)}_{\min,a}$ converges in distribution on $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ to σR, as we will see in Proposition 4 of Section 3.5.3. Together with Lemma 7, this implies that the pair $(Z^{(n)}_{\min,a},\ U^{(n)}_{\min,a})$ converges in distribution to $(\sigma R,\ U)$, with U a random variable uniformly distributed on $(0,1)$ and independent of R. Therefore, to be able to apply the generalized continuous mapping theorem to conclude Theorem 4 it is only left to prove that the mapping $E^{(\infty)}_{\min,a}$ is continuous for the metric $\|\cdot\|_1$. In the following we will see that this is actually not true; nonetheless in Proposition 3 we show a slightly weaker version of Theorem 4 that we are able to conclude with the framework shown in this section.

We start by analyzing the continuity of the mapping $E^{(\infty)}_{\min,a}$ for the metric $\|\cdot\|_1$. First note that for all $\delta > 0$ the modified mapping

$$(u,\ r) \in (0,1) \times \mathscr{C}_{1,0}(\mathbb{R}) \mapsto \begin{pmatrix} u + \arg\inf_{k\in\mathbb{Z}} r(u+k) + \delta|u+k| \\ \inf_{k\in\mathbb{Z}} r(u+k) + \delta|u+k| \end{pmatrix} \tag{3.46}$$

is continuous for the metric $\|\cdot\|_1$ on the support of the random variable $(U, \sigma R)$; this is a simple consequence of the continuity of the mapping (3.44) for the metric $\|\cdot\|_1$. Here, $(U, \sigma R)$ is the limiting random variable in Proposition 4 part (ii) of Section 3.5.3, which we assume true for the moment. The continuity of the mapping (3.46) implies the following limit in distribution for all $\delta > 0$ as $n \to \infty$

$$\begin{pmatrix} U^{(n)}_{\min,a} + \arg\min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}(U^{(n)}_{\min,a}+k) + \delta|U^{(n)}_{\min,a}+k| \\ \min_{k\in\mathbb{Z}\cap[-\lceil nT_{\min,a}\rceil,\ na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}(U^{(n)}_{\min,a}+k) + \delta|U^{(n)}_{\min,a}+k| \end{pmatrix}$$
$$\to \begin{pmatrix} U + \arg\min_{k\in\mathbb{Z}} \sigma R(U+k) + \delta|U+k| \\ \min_{k\in\mathbb{Z}} \sigma R(U+k) + \delta|U+k| \end{pmatrix},$$

by the generalized continuous mapping theorem, see Theorem 3.4.4 of [20]. Using the definition of $E^{(m)}_{\min,a}$ in Definition 2 it can be checked that the latter limit corresponds to the following convergence in distribution.

Proposition 3. *The following limit in distribution holds for all $\delta > 0$ as $n \to \infty$*

$$\begin{pmatrix} n\left(T^{n,\delta}_{\min,a} - T_{\min,a}\right) \\ \sqrt{n}\left(B^{n,\delta}(T^{n,\delta}_{\min,a}) - B(T_{\min,a})\right) \end{pmatrix} \to \begin{pmatrix} U + \arg\min_{k\in\mathbb{Z}} R(U+k) + \delta|U+k| \\ \min_{k\in\mathbb{Z}} R(U+k) + \delta|U+k| \end{pmatrix},$$

where U is a uniform random variable and R a standard two-sided Bessel(3) process independent of U. Here, $B^{n,\delta}$ is the piecewise-constant Euler discretization of the process $(B(t) + \sqrt{n}\delta|t - T_{\min,a}| : t \geq 0)$ on the mesh $\mathbb{Z}_+/n = \{0,\, 1/n,\, 2/n,\, \ldots\}$, i.e.,

$$B^{n,\delta}(t) := B\left(\frac{\lfloor nt\rfloor}{n}\right) + \sqrt{n}\delta\left|\frac{\lfloor nt\rfloor}{n} - T_{\min,a}\right|,$$

and $T^{n,\delta}_{\min,a}$ is the minimum time in $[0,a]$ where $B^{n,\delta}(t)$ attains its minimum, i.e.

$$T^{n,\delta}_{\min,a} := \min\left\{t \in [0,a] \ : \ B^{n,\delta}(t) = \min_{s\in[0,a]} B^{n,\delta}(s)\right\}.$$

Note that Theorem 3 states that in Proposition 3 the result also holds for $\delta = 0$. Nonetheless, the case $\delta = 0$ cannot be deduced using the generalized continuous mapping theorem as in the proof of Proposition 3. Essentially, because the limiting random variable $(U,\, \sigma R)$ of part (ii) of Proposition 4 with probability zero takes values on the set where the mapping

$$(u,\ r) \in (0,1) \times \mathscr{C}_{1,0}(\mathbb{R}) \mapsto \begin{pmatrix} u + \arg\inf_{k\in\mathbb{Z}} r(u+k) \\ \inf_{k\in\mathbb{Z}} r(u+k) \end{pmatrix} \tag{3.47}$$

is discontinuous for the metric $\|\cdot\|_{\mathsf{K}}$. Indeed, consider any pair $(u,\ r) \in (0,1) \times \mathscr{C}_{1,0}(\mathbb{R})$ such that $\lim_{t\to\pm\infty} r(t) = \infty$ and such that $\inf_{k\in\mathbb{Z}} r(u+k) > 0$; clearly with probability one $(U,\ R)$ lies in the set of these type of pairs $(u,\ r)$. Now define the sequence $(r_n)_n$ as follows: for all $n \geq 1$, r_n is identical to r, except that on $(n-1, n+1)$ the function r is modified continuously so as to make $r_n(n) := n^{-1} \inf_{k\in\mathbb{Z}} r(u+k)$. The sequence $(r_n)_n$ is in $\mathscr{C}_{1,0}(\mathbb{R})$ and $\|r_n - r\|_1 \to 0$ as $n \to \infty$; however $\inf_{k\in\mathbb{Z}} r_n(u+k) \to 0 < \inf_{k\in\mathbb{Z}} r(u+k)$ as $n \to \infty$. This shows that Theorem 3 cannot be deduced by directly applying Proposition 4 and the generalized continuous mapping theorem, see Theorem 3.4.4 of [20], using the Polish metric space $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$.

3.5.3 Strengthened convergence of the zoomed-in processes

In this section we show that the zoomed-in processes $Z^{(m)}_{\mathrm{hit},b}$, $Z^{(m)}_{\mathrm{min},a}$ and $Z^{(m)}_{\mathrm{min},\infty}$, defined in Defintion 2 converge in distribution on the space $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$. This is done in Proposition 4. This result relies heavily on Lemma 8, which roughly speaking analyzes how local behavior as Bessel process propagates into a global behavior when "zooming in" in time-space by using Brownian scaling.

Proposition 4. *1. Under $\{\tau_b < \infty\}$, almost surely for all $m > 0$ the zoomed-in process $Z^{(m)}_{\mathrm{hit},b}$ takes values in $\mathscr{C}_1(\mathbb{R})$. Moreover, given $t > 0$ the process $Z^{(m)}_{\mathrm{hit},b}$ conditioned on $\{\tau_b = t\}$ converges in distribution to $(\sigma R(-s) : s \leq 0\,;\ \sigma W(s) : s \geq 0)$ on $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ as $m \to \infty$, where W is a standard Brownian motion and R is a standard Bessel(3) process independent of W.*

2. Almost surely for all $m > 0$ the zoomed-in process $Z^{(m)}_{\mathrm{min},a}$ is in $\mathscr{C}_1(\mathbb{R})$. Moreover, given $t \in (0, a)$, $l < 0$ and $y > 0$, conditioned on $\{T_{\mathrm{min},a} = t,\ B(T_{\mathrm{min},a}) = l,\ B(a) = l + y\}$ the process $Z^{(m)}_{\mathrm{min},a}$ converges in distribution on $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ to σR as $m \to \infty$, where $R = (R(s) : s \in \mathbb{R})$ is a two-sided standard Bessel(3) process.

3. Under $\{T_{\mathrm{min},\infty} < \infty\}$, almost surely for all $m > 0$ the zoomed-in process $Z^{(m)}_{\mathrm{min},\infty}$ is in $\mathscr{C}_1(\mathbb{R})$. Moreover, given $t > 0$ and $l < 0$, conditioned on the event $\{T_{\mathrm{min},\infty} =$

$t,\ B(T_{\min,\infty}) = l\}$ the process $Z^{(m)}_{\min,\infty}$ converges in distribution on $(\mathscr{C}_1(\mathbb{R}),\ \|\cdot\|_1)$ to σR as $m \to \infty$, where $R = (R(s) : s \in \mathbb{R})$ is a standard two-sided Bessel(3) process.

We remark that in all three parts of Proposition 4 the convergence in distribution also holds unconditionally. Also note that these results do not depend on the value of the drift μ of the Brownian motion B. We also remark that Proposition 4 implies Lemma 6, since the topology generated by $\|\cdot\|_1$ is stronger than the one generated by $\|\cdot\|_{\mathsf{K}}$.

The proof of Proposition 4 relies on the following result, which analyzes how local behavior as Bessel process propagates into a global one when "zooming in" using Brownian scaling. Its proof delves into the technicalities of weak convergence of processes on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$, see e.g. [37] for further details.

Lemma 8. *Let $X = (X(s) : s \geq 0)$ be a stochastic process almost surely taking paths in $\mathscr{C}_1(\mathbb{R}_+)$. Assume that there is a strictly positive random variable T such that, given $u, v > 0$, conditioned on $\{T = u,\ X(T) = v\}$ the process $(X(s) : s \in [0, u])$ has the same distribution as a standard Bessel(3) process conditioned on being at v at time u. Then as $m \to \infty$ the process $(\sqrt{m}X(s/m) : s \geq 0)$ converges in distribution on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ to a standard Bessel(3) process.*

Proof of Lemma 8. We follow the usual approach of showing consistency and tightness to prove convergence of probability measures over the space of continuous functions $\mathscr{C}(\mathbb{R}_+)$; see Section 7 of Billingsley [53] for a classical reference on this approach. However, the method has to be strengthened due to the fact that we want weak convergence on the space $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$. Indeed, Lemma 4 of [37] shows that in this case the proof consists of two parts. The first is to restrict the process to $(\mathscr{C}_1[0,k],\ \|\cdot\|_\infty)$, any $k > 0$ and where $\|\cdot\|_\infty$ is the supremum norm, and prove its weak convergence on this space. The second step is to prove the following tail limit

$$\lim_{k\to\infty} \limsup_{m\to\infty} \mathbb{P}\left(\sup_{s\geq k} \frac{|\sqrt{m}X(s/m)|}{1+s} \geq \xi \,\middle|\, T = u, X(T) = v \right) = 0, \tag{3.48}$$

for all $\xi > 0$.

For clarity of exposition, we write $\mathbb{P}^{(u,v)}(\cdot) := \mathbb{P}\left(\cdot \,|\, T = u, X(u) = v\right)$ in the remainder.

We start with the weak convergence on $(\mathscr{C}_1[0,k],\, \|\cdot\|_\infty)$ for $k > 0$ arbitrary. First recall that under $\mathbb{P}^{(u,v)}$ by hypothesis $(X(s) : s \in [0,u])$ is distributed as a standard Bessel(3) conditioned on being at v at time u. Hence, by Brownian scaling, $(\sqrt{m}X(s/m) : s \in [0, mu])$ is distributed as a standard Bessel(3) conditioned on being at $\sqrt{m}v$ at time mu. In particular, for all $m \geq k/u$ we have that $(\sqrt{m}X(s/m) : s \in [0,k])$ is distributed as a standard Bessel(3) process conditioned on being at $\sqrt{m}v$ at time mu. Weak convergence on $(\mathscr{C}_1[0,k],\, \|\cdot\|_\infty)$ is then concluded by using e.g. Lemma 1 of [8].

We now prove the tail limit (3.48). We actually show that the following holds

$$\lim_{m\to\infty} \mathbb{P}^{(u,v)}\left(\sup_{s\geq k} \frac{|\sqrt{m}X(s/m)|}{1+s} \geq \xi\right) = \mathbb{P}\left(\sup_{s\geq k} \frac{R(s)}{1+s} \geq \xi\right), \tag{3.49}$$

where, recall, we write $\mathbb{P}^{(u,v)}(\cdot) = \mathbb{P}\left(\cdot \,|\, T = u, X(u) = v\right)$, and where R is a standard Bessel(3) process under $\mathbb{P}$. In this case (3.48) would easily follow. To prove the limit (3.49) we start by decomposing the probability on the left hand side as follows:

$$\mathbb{P}^{(u,v)}\left(\sup_{s\geq k} \frac{|\sqrt{m}X\left(\frac{s}{m}\right)|}{1+s} \geq \xi\right)$$

$$= \mathbb{P}^{(u,v)}\left(\frac{|\sqrt{m}X\left(\frac{k}{m}\right)|}{1+k} \geq \xi\right) \tag{3.50}$$

$$+\mathbb{P}^{(u,v)}\left(\sup_{s\in[k,mu]} \frac{|\sqrt{m}X\left(\frac{s}{m}\right)|}{1+s} < \xi,\ \sup_{s>mu} \frac{|\sqrt{m}X\left(\frac{s}{m}\right)|}{1+s} \geq \xi\right) \tag{3.51}$$

$$+\mathbb{P}^{(u,v)}\left(\frac{|\sqrt{m}X\left(\frac{k}{m}\right)|}{1+k} < \xi,\ \sup_{s\in[k,mu]} \frac{|\sqrt{m}X\left(\frac{s}{m}\right)|}{1+s} \geq \xi\right). \tag{3.52}$$

The proof essentially consists of showing that (3.50) converges to $\mathbb{P}\left(R(k)/(1+k) \geq \xi\right)$ as $m \to \infty$, (3.51) vanishes to zero, and (3.52) converges to

$$\mathbb{P}\left(R(k)/(1+k) < \xi,\ \sup_{s\geq k} R(s)/(1+s) \geq \xi\right).$$

In that case it is clear that (3.49) holds.

We start by showing that (3.50) converges to $\mathbb{P}\left(R(k)/(1+k)\geq\xi\right)$ as $m\to\infty$. By hypothesis, $(X(s):s\in[0,u])$ is distributed under $\mathbb{P}^{(u,v)}$ as a standard Bessel(3) process, so for all $m>k/u$ we have

$$\mathbb{P}^{(u,v)}\left(|\sqrt{m}X(k/m)|\geq\xi(1+k)\right)=\mathbb{P}\left(R\left(k\right)\geq\xi(1+k)\,\middle|\,\frac{1}{\sqrt{m}}R(mu)=v\right),$$

where we used Brownian scaling. It follows that

$$\lim_{m\to\infty}\mathbb{P}\left(R\left(k\right)\geq\xi(1+k)\,\middle|\,\frac{1}{\sqrt{m}}R(mu)=v\right)=\mathbb{P}\left(R\left(k\right)\geq\xi(1+k)\right).$$

Indeed, using that the transition kernel density of the Bessel(3) process is

$$\mathbb{P}_x\left(R\left(s\right)\in\mathrm{d}v\right)/\mathrm{d}v=\sqrt{\frac{2}{\pi s}}\frac{v}{x}e^{-\frac{x^2+v^2}{2s}}\sinh\left(\frac{xv}{s}\right),\tag{3.53}$$

which is continuous and bounded for $(s,x,v)\in(0,\infty)\times[0,\infty)^2$, we obtain from Bayes' theorem that

$$\begin{aligned}
&\mathbb{P}\left(R\left(k\right)\geq\xi(1+k)\,\middle|\,\frac{1}{\sqrt{m}}R(mu)=v\right)\\
&=\int_{\xi(1+k)}^{\infty}\frac{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\,\middle|\,R(k)=x\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\mathbb{P}\left(R\left(k\right)\in\mathrm{d}x\right)\qquad(3.54)\\
&=\int_{\xi(1+k)}^{\infty}\frac{\mathbb{P}_x\left(\frac{1}{\sqrt{m}}R(mu-k)\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\mathbb{P}\left(R\left(k\right)\in\mathrm{d}x\right)\\
&=\int_{\xi(1+k)}^{\infty}\frac{\mathbb{P}_{x/\sqrt{m}}\left(R(u-k/m)\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(R(u)\in\mathrm{d}v\right)/\mathrm{d}v}\mathbb{P}\left(R\left(k\right)\in\mathrm{d}x\right)\\
&\to\mathbb{P}\left(R\left(k\right)\geq\xi(1+k)\right)\qquad\text{as } m\to\infty,
\end{aligned}$$

where $\mathbb{P}_x(\cdot):=\mathbb{P}(\cdot|R(0)=x)$, and where we used dominated convergence and also Brownian scaling of the Bessel(3) process.

We now show that (3.51) converges to 0 as $m \to \infty$. Indeed, it is bounded by

$$\begin{aligned}
&\mathbb{P}^{(u,v)}\left(\frac{|\sqrt{m}X(mu/m)|}{1+mu} < \xi,\ \sup_{s>mu}\frac{|\sqrt{m}X(s/m)|}{1+s} \geq \xi\right)\\
&\quad = \mathbf{1}_{\left\{\frac{v}{1/\sqrt{m}+\sqrt{m}u}<\xi\right\}}\mathbb{P}^{(u,v)}\left(\sup_{s>u}\frac{|X(s)|}{1/\sqrt{m}+\sqrt{m}s} \geq \xi\right)\\
&\quad = \mathbf{1}_{\left\{\frac{v}{m^{-1}+u}<\sqrt{m}\xi\right\}}\mathbb{P}^{(u,v)}\left(\sup_{s>u}\frac{|X(s)|}{m^{-1}+s} \geq \sqrt{m}\xi\right)\\
&\quad \to 0 \qquad \text{as } m \to \infty,
\end{aligned}$$

where we used that X almost surely takes paths in $\mathscr{C}_1(\mathbb{R}_+)$.

Lastly, it remains to prove that as $m \to \infty$ the term (3.52) converges to

$$\mathbb{P}\left(R(k)/(1+k) < \xi,\ \sup_{s\geq k} R(s)/(1+s) \geq \xi\right).$$

Again, by hypothesis, under $\mathbb{P}^{(u,v)}$ the process $(X(s) : s \in [0,u])$ is distributed as a standard Bessel(3) process, say R. Thus, for all $m > k/u$ we have

$$\begin{aligned}
&\mathbb{P}^{(u,v)}\left(\frac{|\sqrt{m}X(k/m)|}{1+k} < \xi,\ \sup_{s\in[k,mu]}\frac{|\sqrt{m}X(s/m)|}{1+s} \geq \xi\right)\\
&\quad = \mathbb{P}\left(\frac{R(k)}{1+k} < \xi,\ \sup_{s\in[k,mu]}\frac{R(s)}{1+s} \geq \xi \,\middle|\, \frac{1}{\sqrt{m}}R(mu) = v\right),
\end{aligned}$$

where we used Brownian scaling.

We now argue that as $m \to \infty$

$$\mathbb{P}\left(\frac{R(k)}{1+k} < \xi,\ \sup_{s\in[k,mu]}\frac{R(s)}{1+s} \geq \xi \,\middle|\, \frac{1}{\sqrt{m}}R(mu) = v\right) \to \mathbb{P}\left(\frac{R(k)}{1+k} < \xi,\ \sup_{s\geq k}\frac{R(s)}{1+s} \geq \xi\right).$$

To prove that, we again use that the transition kernel of the Bessel(3) process has the continuous and bounded density (3.53); similar to (3.54) we obtain that

$$
\begin{aligned}
&\mathbb{P}\left(\frac{R(k)}{1+k}<\xi,\ \sup_{s\in[k,mu]}\frac{R(s)}{1+s}\geq\xi\,\middle|\,\frac{1}{\sqrt{m}}R(mu)=v\right)\\
&\quad=\int_0^{\xi(1+k)}\frac{\mathbb{P}\left(\sup_{s\in[k,mu]}\frac{R(s)}{1+s}\geq\xi,\ \frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\,\middle|\,R(k)=x\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\mathbb{P}\left(R(k)\in\mathrm{d}x\right)\\
&\quad=\int_0^{\xi(1+k)}\frac{\mathbb{P}_x\left(\sup_{s\in[0,mu-k]}\frac{R(s)}{1+k+s}\geq\xi,\ \frac{1}{\sqrt{m}}R(mu-k)\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\\
&\qquad\cdot\mathbb{P}\left(R(k)\in\mathrm{d}x\right),
\end{aligned}
\tag{3.55}
$$

where we used the Markov property in the last equality. It remains to show that, for all $x\in(0,\xi(1+k)]$, as $m\to\infty$ it holds that

$$
\frac{\mathbb{P}_x\left(\sup_{s\in[0,mu-k]}\frac{R(s)}{1+k+s}\geq\xi,\ \frac{1}{\sqrt{m}}R(mu-k)\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\to\mathbb{P}_x\left(\sup_{s\geq0}\frac{R(s)}{1+k+s}\geq\xi\right)
\tag{3.56}
$$

Proving the latter would conclude the proof, since it would imply that as $m\to\infty$ we have

$$
\mathbb{P}\left(\frac{R(k)}{1+k}<\xi,\ \sup_{s\in[k,mu]}\frac{R(s)}{1+s}\geq\xi\,\middle|\,\frac{1}{\sqrt{m}}R(mu)=v\right)\to\mathbb{P}\left(\frac{R(k)}{1+k}<\xi,\ \sup_{s\geq k}\frac{R(s)}{1+s}\geq\xi\right).
$$

The interchange of limit and integral in (3.55) holds by bounded convergence, since

$$
\begin{aligned}
&\frac{\mathbb{P}_x\left(\sup_{s\in[0,mu-k]}\frac{R(s)}{1+k+s}\geq\xi,\ \frac{1}{\sqrt{m}}R(mu-k)\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu)\in\mathrm{d}v\right)/\mathrm{d}v}\\
&\quad=\mathbb{P}_x\left(\sup_{s\in[0,mu-k]}\frac{R(s)}{1+k+s}\geq\xi\,\middle|\,\frac{1}{\sqrt{m}}R(mu-k)=v\right)\\
&\qquad\cdot\frac{\mathbb{P}_{x/\sqrt{m}}\left(R(u-\frac{k}{m})\in\mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(R(u)\in\mathrm{d}v\right)/\mathrm{d}v}
\end{aligned}
$$

with the transition kernel (3.53) being bounded.

To prove the limit (3.56) we define for $a, b \in \mathbb{R}$ the stopping time

$$H_{a,b} := \inf\{s \geq 0 : R(s) \geq a + bs\}$$

and use that by Theorem 1.6 of [54] it has a density with respect to the Lebesgue measure. We have thus that for all $x \in (0, \xi(1+k)]$ it holds

$$\begin{aligned}
&\frac{\mathbb{P}_x\left(\sup_{s\in[0,mu-k]} \frac{R(s)}{1+k+s} \geq \xi,\ \frac{1}{\sqrt{m}}R(mu-k) \in \mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu) \in \mathrm{d}v\right)/\mathrm{d}v} \\
&\quad = \frac{\mathbb{P}_x\left(H_{\xi(1+k),\xi} \in [0, mu-k],\ \frac{1}{\sqrt{m}}R(mu-k) \in \mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu) \in \mathrm{d}v\right)/\mathrm{d}v} \\
&\quad = \int_0^{mu-k} \frac{\mathbb{P}_x\left(\frac{1}{\sqrt{m}}R(mu-k) \in \mathrm{d}v \,\middle|\, H_{\xi(1+k),\xi} = y\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu) \in \mathrm{d}v\right)/\mathrm{d}v} \mathbb{P}_x\left(H_{\xi(1+k),\xi} \in \mathrm{d}y\right) \\
&\quad = \int_0^{mu-k} \frac{\mathbb{P}_x\left(\frac{1}{\sqrt{m}}R(mu-k) \in \mathrm{d}v \,\middle|\, R(y) = \xi(1+k+y)\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu) \in \mathrm{d}v\right)/\mathrm{d}v} \mathbb{P}_x\left(H_{\xi(1+k),\xi} \in \mathrm{d}y\right) \\
&\quad = \int_0^{mu-k} \frac{\mathbb{P}_{\xi(1+k+y)}\left(\frac{1}{\sqrt{m}}R(mu-k-y) \in \mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(\frac{1}{\sqrt{m}}R(mu) \in \mathrm{d}v\right)/\mathrm{d}v} \mathbb{P}_x\left(H_{\xi(1+k),\xi} \in \mathrm{d}y\right) \\
&\quad = \int_0^{mu-k} \frac{\mathbb{P}_{\xi(1+k+y)/\sqrt{m}}\left(R(u - \frac{k+y}{m}) \in \mathrm{d}v\right)/\mathrm{d}v}{\mathbb{P}\left(R(u) \in \mathrm{d}v\right)/\mathrm{d}v} \mathbb{P}_x\left(H_{\xi(1+k),\xi} \in \mathrm{d}y\right) \\
&\quad \to \int_0^{\infty} \mathbb{P}_x\left(H_{\xi(1+k),\xi} \in \mathrm{d}y\right) = \mathbb{P}_x\left(\sup_{s\geq 0} \frac{R(s)}{1+k+s} \geq \xi\right) \qquad \text{as } m \to \infty,
\end{aligned}$$

where we used bounded convergence and continuity of the transition kernel (3.53) in the limit equation; we used Brownian scaling in the last equality; and the strong Markov property in the third equality. This concludes the proof. □

We are ready to prove Proposition 4.

Proof of Proposition 4. For the sake of clarity of the exposition, in what follows we always assume that the treatment is conditioned on the events $\{\tau_b < \infty,\ \tau_b^n < \infty\}$ and $\{T_{\min,\infty} <$

$\infty\}$ when dealing with the processes $Z^{(m)}_{\text{hit},b}$ and $Z^{(m)}_{\min,\infty}$ respectively.

First we note that for all $m > 0$ the processes $Z^{(m)}_{\text{hit},b}$, $Z^{(m)}_{\min,a}$ and $Z^{(m)}_{\min,\infty}$ almost surely take paths in $\mathscr{C}_1(\mathbb{R})$. Indeed, by definition,

$$Z^{(m)}_{\text{hit},b}(0) = Z^{(m)}_{\min,a}(0) = Z^{(m)}_{\min,\infty}(0) = 0,$$

and by the strong law of large numbers we have that

$$\lim_{|t|\to\infty} \frac{\left|Z^{(m)}_{\text{hit},b}(t)\right|}{1+|t|} = \lim_{|t|\to\infty} \frac{\left|Z^{(m)}_{\min,a}(t)\right|}{1+|t|} = \lim_{|t|\to\infty} \frac{\left|Z^{(m)}_{\min,\infty}(t)\right|}{1+|t|} = |\mu|$$

holds almost surely.

We now prove (i). For that, note that conditioned on $\{\tau_b = t\}$, by the strong Markov property the process $(Z^{(1)}_{\text{hit},b}(s) = B(t+s) - b : s \geq 0)$ is distributed as a Brownian motion with drift μ and variance σ^2. In particular, $(Z^{(1)}_{\text{hit},b}(s) : s \geq 0)$ is independent of $(Z^{(1)}_{\text{hit},b}(s) : s \leq 0)$. By Brownian scaling, $(Z^{(m)}_{\text{hit},b}(s) : s \geq 0)$ is thus equal in distribution to a Brownian motion with drift $\mu/\sqrt{m}$ and variance σ^2. Since it converges almost surely to σW on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ as $m \to \infty$, where W is a standard Brownian motion, then it also converges weakly on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$.

On the other hand, the process $(-Z^{(1)}_{\text{hit},b}(-s) : s \geq 0)$ almost surely takes paths in $\mathscr{C}_1(\mathbb{R}_+)$, and $(-Z^{(1)}_{\text{hit},b}(-s) = b - B(\tau_b - s) : s \in [0,t])$ conditioned on $\{\tau_b = t\}$ is distributed as a Bessel(3) process conditioned on being at b at time t, see Theorem 3.4 of Williams [43]. Applying Lemma 8 we obtain that $(-\sqrt{m}Z^{(1)}_{\text{hit},b}(-s/m) : s \geq 0)$ converges weakly to σR on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ as $m \to \infty$, where R is a Bessel(3) process independent of W. This proves part (i).

We now prove (ii). The process $(Z^{(1)}_{\min,a}(s) = B(t+s) - l : s \geq 0)$ conditioned on $\{T_{\min,a} = t,\ B(T_{\min,a}) = l,\ B(a) = l + y\}$ almost surely takes paths in $\mathscr{C}_1(\mathbb{R}_+)$, and also $(Z^{(1)}_{\min,a}(s) : s \in [0, a-t])$ is distributed as a Bessel(3) process conditioned on being at y at time $a - t$, see Proposition 2 of [8]. By Lemma 8, $(\sqrt{m}Z^{(1)}_{\min,a}(s/m) = Z^{(m)}_{\min,a}(s) : s \geq 0)$

converges weakly to σR_+ on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ as $m \to \infty$, where R_+ is a Bessel(3) process.

Analogously, $(Z^{(1)}_{\min,a}(-s) : s \geq 0)$ almost surely takes paths in $\mathscr{C}_1(\mathbb{R}_+)$, and also conditioned on $\{T_{\min,a} = t,\ B(T_{\min,a}) = l,\ B(a) = l + y\}$ the process $(Z^{(1)}_{\min,a}(-s) = B(T_{\min,a} - s) - B(T_{\min,a}) : s \in [0, t])$ is distributed as a Bessel(3) process conditioned on being at $-l$ at time t, see Proposition 2 of [8]. Applying Lemma 8, $(\sqrt{m} Z^{(1)}_{\min,a}(-s/m) = Z^{(m)}_{\min,a}(-s) : s \geq 0)$ converges weakly to σR_- on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ as $m \to \infty$, where R_- is a Bessel(3) process independent of R_+. This proves (ii).

Finally, we prove (iii). The process $(Z^{(1)}_{\min,\infty}(s) = B(T_{\min,\infty} + s) - B(T_{\min,\infty}) : s \geq 0)$ conditioned on the event $\{T_{\min,\infty} = t,\ B(T_{\min,\infty}) = l\}$ is distributed as a Bessel(3) process with drift μ, see Corollary 3 of [44]. Then, by Brownian scaling of the Bessel processes, $(\sqrt{m} Z^{(1)}_{\min,\infty}(s/m) = Z^{(m)}_{\min,\infty}(s) : s \geq 0)$ is distributed as a Bessel(3) process with drift $\mu/\sqrt{m}$. Such a process converges almost surely in $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ to a Bessel(3) process with no drift, so it also converges in distribution, say to R_+.

On the other hand, $(Z^{(1)}_{\min,\infty}(-s) : s \geq 0)$ almost surely takes values in $\mathscr{C}_1(\mathbb{R})$, and by Theorems 2.1 and 3.4 of Williams [43] we have that $(Z^{(1)}_{\min,\infty}(-s) : s \in [0, t])$ conditioned on $\{T_{\min,\infty} = t,\ B(T_{\min,\infty}) = l\}$ is distributed as a Bessel(3) process conditioned on being at $-l$ at time t. By Lemma 8 we conclude that $(\sqrt{m} Z^{(1)}_{\min,\infty}(-s/m) = Z^{(m)}_{\min,\infty}(-s) : s \geq 0)$ converges weakly to σR_- on $(\mathscr{C}_1(\mathbb{R}_+),\ \|\cdot\|_1)$ as $m \to \infty$, where R_- is a Bessel(3) process independent of R_+. This proves (iii). $\square$

3.6 Proofs of Lemmas 5 and 7

Proof of Lemma 5. We first prove (i). Using that $\tau_b^n \geq \tau_b$ almost surely, because $B^n(t) = B(\lfloor nt \rfloor / n)$ is piecewise constant as a function of t and b is nondecreasing, we obtain that

$$
\begin{aligned}
n\left(\tau_b^n - \tau_b\right) &= n \min\{q \in \mathbb{Z}_+/n \cap [\tau_b, \infty) : B(q) \geq b(q)\} - n\tau_b \\
&= n \min\{q \in \mathbb{Z}_+/n \cap [\tau_b, \infty) : B(q) > b(q)\} - n\tau_b,
\end{aligned}
$$

where the second equality holds almost surely for the Wiener measure. By $B(\tau_b) = b(\tau_b)$ it follows that

$$
\begin{aligned}
&= \min\left\{k \in \mathbb{Z}_+ \cap [\lceil n\tau_b \rceil, \infty) : \sqrt{n}\left(B\left(\frac{k}{n}\right) - B\left(\tau_b\right)\right) > \sqrt{n}\left(b\left(\frac{k}{n}\right) - b\left(\tau_b\right)\right)\right\} \\
&\qquad -n\tau_b \\
&= \min\left\{k \in \mathbb{Z}_+ : \sqrt{n}\left(B\left(\frac{k + \lceil n\tau_b \rceil}{n}\right) - B(\tau_b)\right) > \sqrt{n}\left(b\left(\frac{k + \lceil n\tau_b \rceil}{n}\right) - b(\tau_b)\right)\right\} \\
&\qquad + \lceil n\tau_b \rceil - n\tau_b \\
&= \min\left\{k \in \mathbb{Z}_+ : Z^{(n)}_{\mathrm{hit},b}\left(k + U^{(n)}_{\mathrm{hit},b}\right) > \sqrt{n}\left(b\left(\tau_b + \frac{k + U^{(n)}_{\mathrm{hit},b}}{n}\right) - b(\tau_b)\right)\right\} + U^{(n)}_{\mathrm{hit},b},
\end{aligned}
$$

where we used that $U^{(n)}_{\mathrm{hit},b} = \lceil n\tau_b \rceil - n\tau_b$ by definition. Now noting that $\tau^n_b \in \mathbb{Z}_+/n$ we can use the previous identity for τ^n_b to obtain that

$$
\begin{aligned}
&\sqrt{n}\left(B^n(\tau^n_b) - B(\tau_b)\right) = \sqrt{n}\left(B\left(\tau^n_b\right) - B(\tau_b)\right) \\
&\quad = \sqrt{n}\left(B\left(\tau_b + \frac{n\left(\tau^n_b - \tau_b\right)}{n}\right) - B(\tau_b)\right) \\
&\quad = Z^{(n)}_{\mathrm{hit},b}\left(n\left(\tau^n_b - \tau_b\right)\right) \\
&\quad = Z^{(n)}_{\mathrm{hit},b}\left(\min\{k \in \mathbb{Z}_+ : Z^{(n)}_{\mathrm{hit},b}\left(k + U^{(n)}_{\mathrm{hit},b}\right) > \sqrt{n}(b(\tau_b + \frac{k + U^{(n)}_{\mathrm{hit},b}}{n}) - b(\tau_b))\} + U^{(n)}_{\mathrm{hit},b}\right).
\end{aligned}
$$

We now prove (ii). Using the definition $U^{(n)}_{\min,a} = \lceil nT_{\min,a} \rceil - nT_{\min,a}$ it holds

$$
\begin{aligned}
n\left(T^n_{\min,a} - T_{\min,a}\right) &= n\left(-T_{\min,a} + \operatorname*{arg\,min}_{q\in\mathbb{Z}_+/n\cap[0,a]} B(q)\right) \\
&= -nT_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[0,na]} B\left(\frac{k}{n}\right) \\
&= -nT_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} B\left(\frac{k+\lceil nT_{\min,a}\rceil}{n}\right) + \lceil nT_{\min,a}\rceil \\
&= U^{(n)}_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} B\left(T_{\min,a} + \frac{k+U^{(n)}_{\min,a}}{n}\right) \\
&= U^{(n)}_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} \sqrt{n}\left(B\left(T_{\min,a} + \frac{k+U^{(n)}_{\min,a}}{n}\right) - B\left(T_{\min,a}\right)\right) \\
&= U^{(n)}_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}\left(k+U^{(n)}_{\min,a}\right),
\end{aligned}
$$

and since $T^n_{\min,a} \in \mathbb{Z}_+/n$ then using the previous identity for $T^n_{\min,a}$ we obtain that

$$
\begin{aligned}
\sqrt{n}\left(B^n\left(T^n_{\min,a}\right) - B\left(T_{\min,a}\right)\right) &= \sqrt{n}\left(B\left(T^n_{\min,a}\right) - B\left(T_{\min,a}\right)\right) \\
&= \sqrt{n}\left(B\left(T_{\min,a} + \frac{n\left(T^n_{\min,a} - T_{\min,a}\right)}{n}\right) - B\left(T_{\min,a}\right)\right) \\
&= Z^{(n)}_{\min,a}\left(n\left(T^n_{\min,a} - T_{\min,a}\right)\right) \\
&= Z^{(n)}_{\min,a}\left(U^{(n)}_{\min,a} + \operatorname*{arg\,min}_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}\left(k+U^{(n)}_{\min,a}\right)\right) \\
&= \min_{k\in\mathbb{Z}_+\cap[-\lceil nT_{\min,a}\rceil, na-\lceil nT_{\min,a}\rceil]} Z^{(n)}_{\min,a}\left(k+U^{(n)}_{\min,a}\right).
\end{aligned}
$$

The proof of (iii) is analogous to the two previous ones. □

Proof of Lemma 7. It will be sufficient to prove that for all $u \in (0,1)$ and all t such that $\mathbb{P}(T \leq t) > 0$ we have $\mathbb{P}\left(\lceil nT\rceil - nT \leq u \mid T \leq t\right) \to u$ as $n \to \infty$.

For that, note that for all n we have

$$\begin{aligned}\mathbb{P}\left(\lceil nT\rceil - nT < u \,|\, T \le t\right) &= \sum_{k=1}^{\infty} \mathbb{P}\left(\lceil nT\rceil - nT \le u,\ \lceil nT\rceil = k \,|\, T \le t\right) \\ &= \sum_{k=1}^{\infty} \mathbb{P}\left(T \in \left[\frac{k-u}{n}, \frac{k}{n}\right] \Big| \, T \le t\right) = \sum_{k=1}^{\infty} \int_{(k-u)/n}^{k/n} f_t(v)\mathrm{d}v,\end{aligned}$$

where f_t is the density with respect to the Lebesgue measure of T conditioned on $\{T \le t\}$. On the other hand, it also holds that

$$\begin{aligned}\sum_{k=1}^{\infty} \int_{(k-u)/n}^{k/n} f_t(k/n)\mathrm{d}v &= \sum_{k=1}^{\infty} f_t(k/n)\left(\frac{k}{n} - \frac{k-u}{n}\right) = \sum_{k=1}^{\infty} f_t(k/n)\frac{u}{n} \\ &= \sum_{k=1}^{\infty} f_t(k/n)u\left(\frac{k}{n} - \frac{k-1}{n}\right) = u\sum_{k=1}^{\infty} \int_{(k-1)/n}^{k/n} f_t(k/n)\mathrm{d}v,\end{aligned}$$

and additionally

$$u\sum_{k=1}^{\infty} \int_{(k-1)/n}^{k/n} f_t(v)\mathrm{d}v = u\int_0^1 f_t(v)\mathrm{d}v = u.$$

It follows that since f_t is Riemann-integrable then as $n \to \infty$

$$\left|\sum_{k=1}^{\infty} \int_{(k-u)/n}^{k/n} f_t(v)\mathrm{d}v - \sum_{k=1}^{\infty} \int_{(k-u)/n}^{k/n} f_t(k/n)\mathrm{d}v\right| \to 0$$

and

$$\left|\sum_{k=1}^{\infty} \int_{(k-1)/n}^{k/n} f_t(k/n)\mathrm{d}v - \sum_{k=1}^{\infty} \int_{(k-1)/n}^{k/n} f_t(v)\mathrm{d}v\right| \to 0.$$

Thus, $\mathbb{P}\left(\lceil nT\rceil - nT < u \middle| T \leq t\right) \to u$ as $n \to \infty$, since

$$\begin{aligned}
&\left|\mathbb{P}\left(\lceil nT\rceil - nT < u \middle| T \leq t\right) - u\right| \\
&\quad \leq \left|\mathbb{P}\left(\lceil nT\rceil - nT < u \middle| T \leq t\right) - \sum_{k=1}^{\infty}\int_{(k-u)/n}^{k/n} f_t(k/n)\mathrm{d}v\right| \\
&\qquad\qquad + \left|u\sum_{k=1}^{\infty}\int_{(k-1)/n}^{k/n} f_t(k/n)\mathrm{d}v - u\right| \\
&\quad = \left|\sum_{k=1}^{\infty}\int_{(k-u)/n}^{k/n} f_t(v)\mathrm{d}v - \sum_{k=1}^{\infty}\int_{(k-u)/n}^{k/n} f_t(k/n)\mathrm{d}v\right| \\
&\qquad\qquad + u\left|\sum_{k=1}^{\infty}\int_{(k-1)/n}^{k/n} f_t(k/n)\mathrm{d}v - \sum_{k=1}^{\infty}\int_{(k-1)/n}^{k/n} f_t(v)\mathrm{d}v\right|,
\end{aligned}$$

which concludes the proof. $\square$

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